ON A PROBLEM OF M. TALAGRAND

KEITH FRANKSTON, JEFF KAHN, AND JINYOUNG PARK

ABSTRACT. We address a special case of a conjecture of M. Talagrand relating two notions of "threshold" for an increasing family $\mathcal{F}$ of subsets of a finite set V. The full conjecture implies equivalence of the "Fractional Expectation-Threshold Conjecture," due to Talagrand and recently proved by the authors and B. Narayanan, and the (stronger) "Expectation-Threshold Conjecture" of the first author and G. Kalai. The conjecture under discussion here says there is a fixed J such that if, for a given $\mathcal{F}$, $p \in [0,1]$ admits $\lambda : 2^V \to \mathbb{R}^+$ with

$$\sum_{S \subseteq F} \lambda_S \geq 1 \;\; \forall F \in \mathcal{F}$$

and

$$\sum_S \lambda_S p^{|S|} \leq 1/2$$

(a.k.a. $\mathcal{F}$ is *weakly p-small*), then p/J admits such a λ taking values in $\{0,1\}$ ($\mathcal{F}$ is (p/J)*-small*). Talagrand showed this when λ is supported on singletons and suggested, as a more challenging test case, proving it when λ is supported on pairs. The present work provides such a proof.

1. INTRODUCTION

Given a finite set V, write 2^V for the power set of V and, for $p \in [0,1]$, μ_p for the product measure on 2^V given by $\mu_p(S) = p^{|S|}(1-p)^{|V \setminus S|}$. An $\mathcal{F} \subseteq 2^V$ is *increasing* if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. For $\mathcal{G} \subseteq 2^V$ we use $\langle \mathcal{G} \rangle$ for the increasing family generated by $\mathcal{G}$, namely $\{B \subseteq V : \exists A \in \mathcal{G}, B \supseteq A\}$.

We assume throughout that $\mathcal{F} \subseteq 2^V$ is increasing and not equal to $2^V, \emptyset$. Then $\mu_p(\mathcal{F})(:= \sum\{\mu_p(S) : S \in \mathcal{F}\})$ is strictly increasing in p, and we define the *threshold*, $p_c(\mathcal{F})$, to be the unique p for which $\mu_p(\mathcal{F}) = 1/2$. (This is finer than the original Erdős–Rényi notion, according to which $p^* = p^*(n)$ is ***a*** threshold for $\mathcal{F} = \mathcal{F}_n$ if $\mu_p(\mathcal{F}) \to 0$ when $p \ll p^*$ and $\mu_p(\mathcal{F}) \to 1$ when $p \gg p^*$. That $p_c(\mathcal{F})$ *is* always an Erdős–Rényi threshold follows from [2].)

Thresholds have been a—maybe *the*—central concern of the study of random discrete structures (random graphs and hypergraphs, for example) since its initiation by Erdős and Rényi [4], with much of that effort concerned with identifying (Erdős–Rényi) thresholds for specific properties (see [1, 6])—though it was not observed until [2] that *every* sequence of increasing properties admits such a threshold.

The main concern of this paper is the relation between the following two notions of M. Talagrand [8, 9, 10]. (Our focus is Conjecture 1.4 and our main result is Theorem 1.6; we will come to these following some motivation.)

Say $\mathcal{F}$ is p*-small* if there is a $\mathcal{G} \subseteq 2^V$ such that

$$\langle \mathcal{G} \rangle \supseteq \mathcal{F} \tag{1}$$

(that is, each member of $\mathcal{F}$ contains a member of $\mathcal{G}$) and

$$\sum_{S \in \mathcal{G}} p^{|S|} \leq 1/2, \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 05C80; Secondary 60C05, 82B26, 06E30.

and set $q(\mathcal{F}) = \max\{p : \mathcal{F} \text{ is } p\text{-small}\}$. Say $\mathcal{F}$ is *weakly p-small* if there is a $\lambda : 2^V \to \mathbb{R}^+$ such that

$$\sum_{S \subseteq F} \lambda_S \geq 1 \ \ \forall F \in \mathcal{F} \tag{3}$$

and

$$\sum_{S} \lambda_S p^{|S|} \leq 1/2, \tag{4}$$

and set $q_f(\mathcal{F}) = \max\{p : \mathcal{F} \text{ is weakly } p\text{-small}\}$. As in [5] we refer to $q(\mathcal{F})$ and $q_f(\mathcal{F})$ (respectively) as the *expectation-threshold* and *fractional expectation-threshold* of $\mathcal{F}$. (Note the former is used slightly differently in [7].) Notice that

$$q(\mathcal{F}) \leq q_f(\mathcal{F}) \leq p_c(\mathcal{F}). \tag{5}$$

(The first inequality is trivial and the second holds since, for λ as in (3), (4) and Y drawn from μ_p,

$$\mu_p(\mathcal{F}) \leq \sum_{F \in \mathcal{F}} \mu_p(F) \sum_{S \subseteq F} \lambda_S \leq \sum_S \lambda_S \mu_p(Y \supseteq S) = \sum_S \lambda_S p^{|S|} \leq 1/2.) \tag{6}$$

In particular, each of q, q_f is a lower bound on p_c, and these turn out to be easily understood (and to agree up to constant) in many cases of interest; see [5]. The next two conjectures—respectively the main conjecture (Conjecture 1) of [7] and a sort of LP relaxation thereof suggested by Talagrand [10, Conjecture 8.3]—say that these bounds are never far from the truth.

Conjecture 1.1. *There is a universal K such that for every finite V and increasing $\mathcal{F} \subseteq 2^V$,*

$$p_c(\mathcal{F}) \leq K q(\mathcal{F}) \log |V|.$$

Conjecture 1.2. *There is a universal K such that for every finite V and increasing $\mathcal{F} \subseteq 2^V$,*

$$p_c(\mathcal{F}) \leq K q_f(\mathcal{F}) \log |V|.$$

Talagrand [10, Conjecture 8.5] also proposes the following strengthening of Conjecture 1.2, in which $\ell(\mathcal{F})$ is the maximum size of a minimal member of $\mathcal{F}$.

Conjecture 1.3. *There is a universal $K > 0$ such that for every finite V and increasing $\mathcal{F} \subseteq 2^V$,*

$$p_c(\mathcal{F}) < K q_f(\mathcal{F}) \log \ell(\mathcal{F}).$$

Conjecture 1.3 is proved in [5], to which we also refer for discussion of the very strong consequences that originally motivated Conjecture 1.1, but follow just as easily from Conjecture 1.2.

Turning, finally, to the business at hand, we are interested in the following conjecture of Talagrand [10, Conjecture 6.3], which says that the parameters q and q_f are in fact not very different.

Conjecture 1.4. *There is a fixed L such that, for any $\mathcal{F}$, $q(\mathcal{F}) \geq q_f(\mathcal{F})/L$.*

(That is, weakly p-small implies (p/L)-small.) This of course implies *equivalence of* Conjectures 1.2 and 1.1, as well as of Conjecture 1.3 and the corresponding strengthening of Conjecture 1.1, so in particular, in view of [5], Conjecture 1.4 would now supply a proof of Conjecture 1.1. (Post-[5] this implication is probably the best motivation for Conjecture 1.4, but the authors have long been interested in the conjecture for its own sake.)

The following mild reformulation of Conjecture 1.4 will be convenient.

Conjecture 1.5. *There is a fixed J such that for any V, p and $\lambda : 2^V \setminus \{\emptyset\} \to \mathbb{R}^+$,*

$$\{A \subseteq V : \sum_{S \subseteq A} \lambda_S \geq \sum_S \lambda_S (Jp)^{|S|}\} \tag{7}$$

is p-small.

As Talagrand observes, even simple instances of Conjecture 1.4 are not easy to establish. He suggests two test cases, which in the formulation of Conjecture 1.5 become:

(i) $V = \binom{[n]}{2} = E(K_n)$ and (for some k) λ is the indicator of {copies of K_k in K_n};

(ii) λ is supported on 2-element sets.

(He does prove Conjecture 1.5 for λ supported on singletons; see Proposition 2.1 for a quantified version that will be useful in what follows.)

The very specific (i) above was treated in [3]. Here we dispose of the broader (ii):

Theorem 1.6. *Conjecture 1.5 holds when $\text{supp}(\lambda) \subseteq \binom{V}{2}$; in other words, there is a J such that for any graph $G = (V, E)$, $p \in [0, 1]$ and $\lambda : E \to \mathbb{R}^+$,*

$$\{U \subseteq V : \lambda(G[U]) \geq J^2 \lambda(G) p^2\}$$

is p-small (where $G[U]$ is the subgraph induced by U).

It seems not impossible that the ideas underlying Theorem 1.6 can be extended to give Conjecture 1.4 in full, but we don't yet see this.

The rest of the paper is devoted to the proof of Theorem 1.6. The most important part of this turns out to be (a quantified version of) the "unweighted" case, where λ takes values in $\{0, 1\}$, though deriving Theorem 1.6 from this still needs some ideas. Section 2 collects a few preliminaries and gives an overview of our proof strategies. In Section 3 we prove Theorem 1.6 modulo a result on the unweighted case, Theorem 3.4, whose proof is given in Section 4.

2. Orientation

Usage. We use $[n]$ for $\{1, 2, \ldots, n\}$, 2^X for the power set of X, and $\binom{X}{r}$ for the family of r-element subsets of X, and recall from above that $\langle \mathcal{A} \rangle$ is the increasing family generated by $\mathcal{A} \subseteq 2^X$. For a set X and $p \in [0, 1]$, X_p is the "p-random" subset of X in which each $x \in X$ appears with probability p independent of other choices. We assume throughout that p has been specified and often omit it from our notation.

Graphs here are always simple and are mainly thought of as sets of edges; thus $|G|$ is $|E(G)|$. We use $\nabla_G(v)$ or ∇_v for $\{e \in E(G) : v \in e\}$; so the degree of v is $d_v = |\nabla_v|$. (We also use $N_G(v)$ for the neighborhood of v in G.)

Case of singletons. We first introduce a quantified notion of p-small. For $\mathcal{A} \subseteq 2^V$, the *cost* of $\mathcal{A}$ (w.r.t. our given p) is $\mathrm{C}(\mathcal{A}) = \sum_{S \in \mathcal{A}} p^{|S|}$. We say $\mathcal{A}$ *covers* $\mathcal{B} \subseteq 2^V$ if $\langle \mathcal{A} \rangle \supseteq \mathcal{B}$, set

$$\mathrm{C}^*(\mathcal{B}) = \min\{\mathrm{C}(\mathcal{A}) : \mathcal{A} \text{ covers } \mathcal{B}\},$$

and say $\mathcal{B}$ can be *covered at cost* γ if $\mathrm{C}^*(\mathcal{B}) \leq \gamma$. (So $\mathcal{B}$ being p-small means $\mathrm{C}^*(\mathcal{B}) \leq 1/2$.) Talagrand's observation that Conjecture 1.4 holds for λ supported on singletons may now be stated as:

Proposition 2.1. *For all $\zeta : V \to \mathbb{R}^+$ and $J > 2e$,*

$$\mathrm{C}^*(\{U \subseteq V : \zeta(U) \geq J\zeta(V)p\}) < 2e/(J-2e). \tag{8}$$

(The dependence on J is best possible up to constants: e.g. take $|V| = J$, $p = 1/J^2$ and $\zeta \equiv 1$.)

Proof. We may take $V = [n]$ and assume ζ is non-increasing (and positive) and $Jp \leq 1$ (since the statement is trivial when $Jp > 1$). Define R by

$$\frac{1}{Rp} = \left\lceil \frac{1}{Jp} \right\rceil =: a.$$

We claim that the collection

$$\mathcal{A} = \bigcup_{k \geq 1} \binom{[ak]}{k}$$

covers the family in (8); this gives the proposition since the l.h.s. of (8) is then at most

$$\mathrm{C}(\mathcal{A}) = \sum_{k \geq 1} \binom{ak}{k} p^k < \sum_{k \geq 1} \left(\frac{e}{R}\right)^k < \frac{e}{R-e} < \frac{2e}{J-2e}$$

(the last inequality holding since $Jp \leq 1$ implies $R > J/2$.)

To see that the claim holds, observe that its failure implies the existence of some $U = \{u_1 < u_2 < \cdots < u_\ell\} \subseteq [n]$ with $\zeta(U) \geq J\zeta(V)p$ such that $|U \cap [ak]| < k$ for all $k > 0$. But then $u_i > ia$ for all $i \in [\ell]$, yielding the contradiction

$$\zeta(V) > \sum_{i=0}^{\ell-1} \sum_{j \in [a]} \zeta(j + ia) \geq a\zeta(U) \geq \zeta(V). \qquad \square$$

Towards doubletons. As the proof of Proposition 2.1 illustrates, one way to show a collection is p-small is to construct an explicit "cheap" cover. We use a similar strategy in the proof of Theorem 1.6. Observe that what helps us to find the cover in Proposition 2.1 is the fact that the set U satisfies

$$\zeta(U) \geq J\zeta(V)p,$$

i.e. U is "heavier" than what it should be. (Note that if $U \sim V_p$ then $\mathbb{E}[\zeta(U)] = \zeta(V)p$.)

Roughly, the proof of Theorem 1.6 consists of two steps: in the first step, we decompose G into subgraphs $G_1, G_2, \ldots$ so that the edges in G_i have roughly the same value of λ. The point here is that since our set U is heavy (again, if $U \sim V_p$ then $\mathbb{E}[\lambda(G[U])] = \lambda(G)p^2$), there must exist some G_i which, again, contains a heavy part (see (19) for a precise description).

In the second step (Theorem 3.4), which is the heart of our argument, we give a way to construct a (cheap) cover for the heavy part from the first step. The main difficulty in implementing this idea is that specifying a heavy part in *each* heavy U is too expensive. To manage this issue, we rather produce a generic structure which all heavy sets must contain some form of, and then explicitly cover all of these structures without regard to whether our actual sets of interest contain any particular one.

Weighted subsets. The following convention will be helpful. Given a graph G on V, we associate with each $U \subseteq V$ a "weighted subset" $D(U) = D_G(U)$ of $E(G)$ that assigns to each e the weight $|e \cap U|/2$. (We also use D_v or $D_G(v)$ for $D(\{v\})$.) We then have, for any $\lambda : G \to \mathbb{R}^+$,

$$\lambda(D(U)) = \tfrac{1}{2} \textstyle\sum_{v \in U} \lambda(\nabla_v)$$

(e.g. $|D(U)| := \frac{1}{2}\sum_{v\in U} d_v$). To see why this is natural, notice that

$$\mathbb{E}\lambda(G[V_p]) = \mathbb{E}\lambda(D(V_p))p$$

(e.g. $\mathbb{E}|G[V_p]| = \mathbb{E}|D(V_p)|p$), so that $\lambda(D(U))p$ is a natural benchmark against which to measure $\lambda(G[U])$.

3. PROOF OF THEOREM 1.6

We actually prove the following quantified version of Theorem 1.6.

Theorem 3.1. *For any graph G on V, $\lambda : G \to \mathbb{R}^+$ and*

$$R \geq 4096\sqrt{2}e, \tag{9}$$

the set

$$\mathcal{U}_0 = \{U \subseteq V : \lambda(G[U]) \geq R^2\lambda(G)p^2\}$$

can be covered at cost $O(1/R)$.

(Here and throughout we don't worry about getting good constants, and try instead to keep the argument fairly clean.)

Proof. We take G, λ, R to be as in the theorem, use $D(U)$ for $D_G(U)$ (defined in Section 2), and assume throughout that

$$U \in \mathcal{U}_0.$$

We first observe that it is enough to prove the theorem assuming

$$\lambda \text{ takes only values } \theta_i := 2^{-i}, \ i = 1, 2, \ldots \quad , \tag{10}$$

with (9) slightly weakened to

$$R \geq 4096e. \tag{11}$$

Then for a general λ (which we may of course scale to take values in $[0, 1]$) and λ' given by

$$\lambda'_S = \max\{\theta_i : \theta_i \leq \lambda_S\},$$

$\mathcal{U}_0$ as in the theorem is contained in the corresponding collection with λ and R^2 replaced by λ' and $R^2/2$ (which supports (11)), since $U \in \mathcal{U}_0$ implies $2\lambda'(G[U]) > \lambda(G[U]) \geq R^2\lambda(G)p^2 \geq R^2\lambda'(G)p^2$. So we assume from now on that λ and R are as in (10) and (11) (respectively).

Note also that Proposition 2.1, with $\zeta(v) = \lambda(D_v)$ (for which we have $\zeta(V) = \sum\zeta(v) = \frac{1}{2}\sum\lambda(\nabla_v) = \lambda(G)$ and $\zeta(U) = \lambda(D(U))$), says that the set

$$\{U \subseteq V : \lambda(D(U)) \geq R\lambda(G)p\}$$

admits a cover of cost less than $6/R$. So we specify such a cover as a first installment on $\mathcal{G}$ and it then becomes enough to show that

$$\mathcal{U}^* := \{U \in \mathcal{U}_0 : \lambda(D(U)) < R\lambda(G)p\}$$

can be covered at cost $O(1/R)$; in fact we will show

$$\mathrm{C}^*(\mathcal{U}^*) = O(R^{-2}). \tag{12}$$

As sketched in Section 2, our goal is to specify a (cheap) collection of sets which covers $\mathcal{U}^*$. We crucially use the fact that $\lambda(G[U])$ is quite larger than $\lambda(G)p^2$ (i.e. "what it should be") while $\lambda(D(U))$ is not very far from the expectation, $\lambda(G)p$. This idea is implemented in (16), (17), and their consequence, Claim 3.2.

Set $G_i = \{e \in G : \lambda(e) = \theta_i\}$ and write $D_i(U)$ for $D_{G_i}(U)$. We then observe, for any $H \subseteq G$,

$$\lambda(H) = \sum_i \theta_i |H \cap G_i|,$$

and abbreviate

$$\mathsf{w}_i = \lambda(G_i) = \theta_i |G_i|, \quad \mathsf{w} = \lambda(G) = \sum \mathsf{w}_i.$$

Given U, define $L = L(U)$, $K = K(U)$, $L_i = L_i(U)$ and $K_i = K_i(U)$ by

$$\lambda(D(U)) = L\mathsf{w}p,$$

$$\lambda(G[U]) = KL\mathsf{w}p^2,$$

$$|D_i(U)| = L_i|G_i|p, \tag{13}$$

and

$$|G_i[U]| = K_iL_i|G_i|p^2. \tag{14}$$

Then

$$L\mathsf{w}p = \sum \theta_i |D_i(U)| = \sum L_i \mathsf{w}_i p \tag{15}$$

and

$$KL\mathsf{w}p^2 = \sum \theta_i |G_i[U]| = \sum K_iL_i\mathsf{w}_ip^2.$$

Since $U \in \mathcal{U}_0$, we have

$$\sum K_iL_i\mathsf{w}_i \geq R^2\mathsf{w}, \tag{16}$$

while $U \in \mathcal{U}^*$ gives

$$L < R. \tag{17}$$

Note also that, with

$$I = I(U) = \{i : K_i > R/2\},$$

we have

$$\sum \{K_iL_i\mathsf{w}_i : i \in I\} > R^2\mathsf{w}/2, \tag{18}$$

as follows from (16) and (using (15) and (17))

$$\sum \{K_iL_i\mathsf{w}_i : i \notin I\} \leq (R/2)L\mathsf{w} < R^2\mathsf{w}/2.$$

Now let $\mathsf{E}_i = |G_i|p^2$. (Note that this is the expected number of edges in the graph G_i induced on a p-random subset of $V(G_i)$.) For integer α, define

$$\mathcal{E}_\alpha = \{i : \mathsf{E}_i \in (2^{\alpha-1}, 2^\alpha]\}.$$

We arrange the i's in an array, with columns indexed by α's (in increasing order) and column α consisting of the indices in $\mathcal{E}_\alpha$, again in increasing order. (So w_i's within a column *decrease* as we go down. Note column lengths may vary.) Define $\mathcal{B}_\beta$ to be the set of indices in row β.

To keep the computation cleaner (or less messy) we use sort of discretization of w_i: for $i \in \mathcal{E}_\alpha$ set $\mathsf{y}_i = \theta_i 2^\alpha/p^2$ and $\mathsf{y} = \sum_{i \geq 1} \mathsf{y}_i$, noting that

$$\mathsf{y}_i/2 < \mathsf{w}_i \leq \mathsf{y}_i.$$

	$\cdots \quad \alpha-1$	α	$\alpha+1 \quad \cdots$
1			
$\vdots$			
β		i	
$\vdots$			

TABLE 1. i is the βth smallest index in $\mathcal{E}_\alpha$ (when $|\mathcal{E}_\alpha| \geq \beta$).

Set

$$c^*_\beta = (3/2)^{\beta-1}R^2/16 \quad (\beta \geq 1)$$

and $c_i = c^*_\beta$ if $i \in \mathcal{B}_\beta$. Let w^*_β and y^*_β be (respectively) the sums of the w_i's and y_i's over $i \in \mathcal{B}_\beta$, and notice that

$$\mathsf{y}^*_{\beta+1} \leq \mathsf{y}^*_\beta/2 \quad \text{for } \beta \geq 1$$

(since $i = \mathcal{B}_{\beta+1} \cap \mathcal{E}_\alpha$ (where we abusively use i for $\{i\}$) implies $i > j := \mathcal{B}_\beta \cap \mathcal{E}_\alpha$, whence $2\mathsf{y}_i \leq \mathsf{y}_j$).

Claim 3.2. *For each $U \in \mathcal{U}^*$ there is an $i \in I(U)$ with $K_i(U)L_i(U) > c_i$.*

Proof. With $\sum^\star$ denoting summation over I, we have (using (18) at the end)

$$\begin{aligned}\sum\nolimits^\star c_i\mathsf{w}_i \leq \sum c^*_\beta \mathsf{w}^*_\beta &\leq \sum c^*_\beta \mathsf{y}^*_\beta \\ &\leq \mathsf{y}^*_1(c^*_1 + c^*_2/2 + c^*_3/2^2 + \cdots) \\ &\leq \mathsf{y}(c^*_1 + c^*_2/2 + c^*_3/2^2 + \cdots) \\ &\leq (R^2/4)\mathsf{y} < (R^2/2)\mathsf{w} < \sum\nolimits^\star K_i(U)L_i(U)\mathsf{w}_i.\end{aligned}$$

□

It follows that if, for each i, $\mathcal{G}_i$ covers

$$\mathcal{U}_i := \{U \subseteq V : i \in I(U);\ K_i(U)L_i(U) > c_i\}, \tag{19}$$

then $\cup\mathcal{G}_i$ covers $\mathcal{U}^*$; so we have

$$\mathrm{C}^*(\mathcal{U}^*) \leq \sum_i \mathrm{C}^*(\mathcal{U}_i). \tag{20}$$

To bound $\mathrm{C}^*(\mathcal{U}_i)$, we introduce the following definition.

Definition 3.3. *Define*

$$\mathrm{C}^*_J(\mu, T)$$

to be the infimum of those γ's for which, for every p and (simple) graph G (on V) with $|G|p^2 \leq \mu$,

$$\{U \subseteq V : |G[U]| \geq \max\{T, J|D_G(U)|p\}\} \tag{21}$$

can be covered at cost γ.

Now, if (α,β) is the pair corresponding to i (that is, i is the βth entry in column α of our array), then

$$\mathrm{C}^*(\mathcal{U}_i) \le \mathrm{C}^*_{R/2}(2^\alpha, T_{\alpha,\beta}),$$

where $T_{\alpha,\beta} = \max\{c^*_\beta 2^{\alpha-1}, 1\}$; namely, $|G_i|p^2 = \mathsf{E}_i \le 2^\alpha$, while $U \in \mathcal{U}_i$ implies (using (13), (14) and $i \in I(U)$)

$$|G_i[U]| = K_i(U)L_i(U)|G_i|p^2 \begin{cases} > c_i|G_i|p^2 > c^*_\beta 2^{\alpha-1} \\ = K_i|D_i(U)|p > (R/2)|D_i(U)|p. \end{cases}$$

So it is enough to show that

$$\sum_{\alpha\in\mathbb{Z},\beta\in\mathbb{Z}^+} \mathrm{C}^*_{R/2}(2^\alpha, T_{\alpha,\beta}) = O(R^{-2}). \tag{22}$$

We use the following theorem to show (22), whose proof is postponed to Section 4.

Theorem 3.4. *For any μ and $T = cJ^2\mu$ with*

$$c \ge 256e/J \text{ and } J \ge 8e, \tag{23}$$

and $J_1 = J/(8e)$,

$$\mathrm{C}^*_J(\mu, T) \le 32c^{-1}\min\{J_1^{-2}, J_1^{-\sqrt{T}/16}\}. \tag{24}$$

(Note that we don't use the notion of the weight function λ in Definition 3.3, because all the edges in G_i have the same weight. So Theorem 3.4 is a reduction of Theorem 1.6 to an unweighted case.)

Proof of (22) *via Theorem 3.4.* For $T_{\alpha,\beta} = 1$ we bound $\mathrm{C}^*_{R/2}(2^\alpha, T_{\alpha,\beta})$ by the trivial

$$\mathrm{C}^*_J(\mu, 1) \le \mu \tag{25}$$

(simply take $\{\{x,y\} : xy \in G\}$ as a cover), which—since $T_{\alpha,\beta} = 1$ iff $2^\alpha \le 32R^{-2}(2/3)^{\beta-1}$—bounds the contribution of such pairs (to the sum in (22)) by

$$\sum_{\beta\in\mathbb{Z}^+} \sum_{\{\alpha:2^\alpha\le 2/c^*_\beta\}} 2^\alpha \le 64R^{-2}\sum_{\beta\ge1}(2/3)^{\beta-1} = 3\cdot 64R^{-2}. \tag{26}$$

For $T_{\alpha,\beta} > 1$ we use Theorem 3.4 with $T = T_{\alpha,\beta}(= 2^{\alpha-1}c^*_\beta)$, $\mu = 2^\alpha$, $J = R/2$, and (thus)

$$c = T/(\mu J^2) = c^*_\beta/(2J^2) = (3/2)^{\beta-1}/8.$$

Note that (11) gives $J \ge 8e$ and $c \ge 256e/J$, so (23) holds.

For each integer $s \ge 0$ let $\mathcal{T}_s = \{(\alpha,\beta) : T_{\alpha,\beta} \in (2^s, 2^{s+1}]\}$. For each $\beta \ge 1$ there is a unique α such that $(\alpha,\beta) \in \mathcal{T}_s$, and every (α,β) with $T_{\alpha,\beta} > 1$ is in some $\mathcal{T}_s$. Let $f(s) = \min\{J_1^{-2}, J_1^{-2^{s/2-4}}\}$. Then for fixed s, we have (see (24))

$$\sum_{(\alpha,\beta)\in\mathcal{T}_s} \mathrm{C}^*_J(2^\alpha, T_{\alpha,\beta}) \le \sum_{\beta\ge1} 32c^{-1}f(s) = \sum_{\beta\ge1} 256\left(\frac{2}{3}\right)^{\beta-1} f(s) < 3\cdot 256 f(s), \tag{27}$$

and summing over all s we get

$$\sum_{T_{\alpha,\beta}>1} \mathrm{C}^*_J(2^\alpha, T_{\alpha,\beta}) < \sum_{s\ge0} 768 f(s) = \sum_{s\ge0} 768\min\{J_1^{-2}, J_1^{-2^{s/2-4}}\} = O(J_1^{-2}). \tag{28}$$

Finally, combining (28) and (26) gives (22). □

4. Proof of Theorem 3.4

Aiming for simplicity, we first bound the cost in (24) assuming

$$T = 2^{2k+3}$$

for some positive integer k and

$$c = T/(\mu J^2) \geq 64e/J, \tag{29}$$

showing that in this case

$$\mathrm{C}_J^*(\mu, T) \leq 8c^{-1} J_1^{-2^{k-1}-1}. \tag{30}$$

Before proving this, we show it implies Theorem 3.4, which, since $\mathrm{C}_J^*(\mu, t)$ is decreasing in t, just requires showing that the r.h.s. of (24) bounds $\mathrm{C}_J^*(\mu, T_0)$ for some $T_0 \leq T$.

If $T < 32$ this follows from the trivial (25), since $\mu = T/(cJ^2) < 32c^{-1}J_1^{-2}$, matching the bound in (24). Suppose then that $T \geq 32$ and let $T_0 = c_0 J^2 \mu$ be the largest integer not greater than T of the form 2^{2k+3} (with positive integer k). We then have $c_0 > c/4$ (supporting (29)) and $2^{k-1} > \sqrt{T_0}/8 > \sqrt{T}/16$, and it follows that the bound on $\mathrm{C}_J^*(\mu, T_0)$ given by (30) is less than the bound in (24).

Proof of (30). We have $|G|p^2 \leq \mu$, $T = 2^{2k+3}$ ($= cJ^2\mu$ with J as in (23) and c as in (29)), and, with

$$\mathcal{U} := \{U \subseteq V : |G[U]| > \max\{T, J|D_G(U)|p\}\}, \tag{31}$$

want to show that $\mathrm{C}^*(\mathcal{U})$ is no more than the bound in (30).

A basic challenge for Conjecture 1.4 in general is identifying a suitable covering set $\mathcal{G}$. In the present instance, each member of $\mathcal{G}$ will be a disjoint union of stars, where for present purposes a *star at v in W* ($\subseteq V$) is some $\{v\} \cup S \subseteq W$ with $S \subseteq N_G(v)$. (Where convenient we will also refer to this as the "star (v, S).") We say such a star is *good* if

$$|S| \geq J d_v p/4. \tag{32}$$

Given a positive integer L, we define

$$L^v = \max\{L, \lceil J d_v p/4 \rceil\} \tag{33}$$

and say a star $(v,\, S)$ is *L-special* if $|S| = L^v$.

For positive integers b and L, let $\mathcal{G}(b, L)$ ($\subseteq 2^V$) consist of all disjoint unions of b L-special stars in G. We will specify a particular collection $\mathcal{C}$ of pairs (b, L) and set

$$\mathcal{G} = \cup\{\mathcal{G}(b, L) : (b, L) \in \mathcal{C}\}.$$

Theorem 3.4 is then given by the following two assertions.

Claim 4.1. *$\mathcal{G}$ covers $\mathcal{U}$.*

Claim 4.2. *$\mathrm{C}(\mathcal{G})$ is at most the bound in* (30).

Set (with $i \in [k]$ throughout) $L_i = 2^{i-1}$ and

$$\delta_i = \max\{2^{-(i+2)}, 2^{i-k-3}\} \geq 1/(8L_i), \tag{34}$$

and notice that

$$\sum \delta_i \leq \sum 2^{-(i+2)} + \sum 2^{i-k-3} \leq 1/2. \tag{35}$$

Let

$$b_i = \delta_i 4^{-i} T \geq 2^{k-i}. \tag{36}$$

Finally, set

$$\mathcal{C} = \{(b_i, L_i) : i \in [k]\}.$$

Proof of Claim 4.1. We are given $U \in \mathcal{U}$ and must show it contains a member of $\mathcal{G}$. Let $U_0 = U$ and for $j = 1, \ldots$ until no longer possible do: let (v_j, S_j), with $S_j = N_G(v_j) \cap U_{j-1}$, be a largest good star in U_{j-1}, and set $d_j = |S_j|$ and $U_j = U_{j-1} \setminus (\{v_j\} \cup S_j)$.

The passage from U_{j-1} to U_j deletes at most d_j^2 edges containing vertices of S_j of U_{j-1}-degree at most d_j; any other edge deleted in this step contains $u \in S_j$ with U_{j-1}-degree less than $Jd_u p/4$ (or u, having U_{j-1}-degree greater than d_j, would have been chosen in place of v_j); and of course each vertex u of the final U_j has U_j-degree less than $Jd_u p/4$. We thus have

$$|G[U]| \leq \sum_j d_j^2 + \sum_{v \in U} Jd_v p/4 \leq \sum_j d_j^2 + |G[U]|/2$$

(using the second bound in (31)), so

$$\sum_j d_j^2 \; (\geq |G[U]|/2) \; \geq T/2. \tag{37}$$

Set

$$B_i = \begin{cases} \{j : d_j \in [2^{i-1}, 2^i)\} & \text{if } i \in [k-1], \\ \{j : d_j \geq 2^{k-1}\} & \text{if } i = k. \end{cases}$$

(It may be worth noting that, while the d_j's are decreasing, the degrees corresponding to B_i increase with i.) In view of (37), either $|B_k| \geq 1$ or

$$\sum_{i \in [k-1]} |B_i| 4^i \geq T/2 \geq \sum_{i \in [k-1]} \delta_i T = \sum_{i \in [k-1]} b_i 4^i$$

(using (35)). Recalling that $b_k = 1$, it follows that for some $i \in [k]$ we have

$$|B_i| \geq b_i. \tag{38}$$

On the other hand, since $|S_j| \geq L_i^v (= \max\{L_i (= 2^{i-1}), \lceil Jd_v p/4 \rceil\})$ for $j \in B_i$, the set $\bigcup\{S_j \cup \{v_j\} : j \in B_i\}$ contains some $W \in \mathcal{G}(b_i, L_i) (\subseteq \mathcal{G})$ whenever i is as in (38). This completes the proof of Claim 4.1. □

Proof of Claim 4.2. We first bound the cost, say $\mathrm{C}(b, L)$, of the collection $\mathcal{G}(b, L)$ for any given b and L. Set

$$q_v = p \left(\frac{ed_v p}{L^v} \right)^{L^v}.$$

Then q_v bounds the total cost of the set of L-special stars at v (using $\binom{d_v}{L^v} \leq (ed_v/L^v)^{L^v}$), and it follows that

$$\mathrm{C}(b, L) \leq \sum \Big\{ \prod_{v \in B} q_v : B \in \tbinom{V}{b} \Big\}. \tag{39}$$

For a given value of $\varphi := \sum_{v \in V} q_v$, the r.h.s. of (39) is largest when the q_v's are all equal (this just uses $xy \leq [(x+y)/2]^2$), whence

$$\mathrm{C}(b, L) \leq \binom{|V|}{b} \left(\frac{\varphi}{|V|} \right)^b \leq \left(\frac{e\varphi}{b} \right)^b. \tag{40}$$

Recalling (33), we have

$$q_v \leq d_v p^2 \cdot \frac{e}{L}\left(\frac{4e}{J}\right)^{L-1},$$

so (since $|G|p^2 \leq \mu$)

$$\varphi \leq 2\mu \cdot \frac{e}{L}\left(\frac{4e}{J}\right)^{L-1}. \tag{41}$$

Now using (40) and (41), recalling that $T = cJ^2\mu$, $L_i = 2^{i-1}$, $b_i = \delta_i 4^{-i}T = \delta_i T/(4L_i^2)$ and $J_1 = J/(8e)$, and for the moment omitting the subscript i, we have (with the final inequality (42) justified below)

$$\begin{aligned} \mathrm{C}(b,L) &\leq \left[\frac{2e^2\mu}{L}\frac{4L^2}{\delta T}\left(\frac{4e}{J}\right)^{L-1}\right]^b \\ &= \left[8e^2L \cdot \frac{1}{cJ^2\delta}\left(\frac{4e}{J}\right)^{L-1}\right]^b \\ &= \left[c^{-1}\frac{L}{2\delta}\left(\frac{4e}{J}\right)^{L+1}\right]^b \\ &\leq \left[\frac{c}{4}\cdot J_1^{L+1}\right]^{-b}. \end{aligned} \tag{42}$$

For (42), or the equivalent

$$2^{L+4}\delta \geq L, \tag{43}$$

it is enough to show $2^{L+1} \geq L^2$ (since $\delta \geq 1/(8L)$; see (34)), which is true for positive integer L.

Finally, returning to Claim 4.2 (and recalling that L and b in the display ending with (42) are really L_i and b_i), we have

$$\mathrm{C}(\mathcal{G}) = \sum_{i=1}^{k} \mathrm{C}(b_i, L_i) \leq \sum_{i=1}^{k}\left[\frac{c}{4}\cdot J_1^{L_i+1}\right]^{-b_i}. \tag{44}$$

We use $b_i \geq 2^{k-i}$ from (36) to bound the r.h.s. of (44) by (recall that $L_i = 2^{i-1}$)

$$\sum_{i=1}^{k}\left[\frac{cJ_1^{2^{i-1}+1}}{4}\right]^{-2^{k-i}} = \sum_{i=1}^{k} J_1^{-2^{k-1}}\left[\frac{cJ_1}{4}\right]^{-2^{k-i}} = \sum_{j=0}^{k-1}\left(\frac{c}{4}J_1^{2^{k-1}+1}\right)^{-1}\left[\frac{cJ_1}{4}\right]^{1-2^j}, \tag{45}$$

and, since (29) implies that $\frac{cJ_1}{4} \geq 2$, the last expression in (45) is at most $8c^{-1}J_1^{-2^{k-1}-1}$, matching (30) as desired.

□

References

[1] B. Bollobás, *Random graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. 1

[2] B. Bollobás and A. Thomason, *Thresholds functions*, Combinatorica **7** (1987), 35-38. 1

[3] B. DeMarco and J. Kahn, *Note on a problem of M. Talagrand*, Random Structures & Algorithms **47** (2015), no. 4, 663-668. 3

[4] P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 17–61. 1

[5] K. Frankston, J. Kahn, B. Narayanan, and J. Park, *Thresholds versus fractional expectation-thresholds*, arXiv:1910.13433 [math.CO]. 2

[6] S. Janson, T. Łuczak, and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. 1

[7] J. Kahn and G. Kalai, *Thresholds and expectation thresholds,* Combin. Probab. Comput. **16** (2007), 495–502. 2

[8] M. Talagrand, *Are all sets of positive measure essentially convex?*, Geometric Aspects of Functional Analysis, Springer, 1995, pp. 295–310. 1

[9] M. Talagrand, *The generic chaining,* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. 1

[10] M. Talagrand, *Are many small sets explicitly small?,* Proceedings of the 2010 ACM International Symposium on Theory of Computing, (2010), 13–35. 1, 2

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854, USA

E-mail address: `kmf196@scarletmail.rutgers.edu`

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854, USA

E-mail address: `jkahn@math.rutgers.edu`

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NJ 08540, USA

E-mail address: `jpark@math.ias.edu`