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Smooth generalized symmetries of quantum field theories

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ABSTRACT

Dynamical quantum field theories (QFTs), such as those in which spacetimes are equipped with a metric and/or a field in the form of a smooth map to a target manifold, can be formulated axiomatically using the language of ∞-categories. According to a geometric version of the cobordism hypothesis, such QFTs collectively assemble themselves into objects in an ∞-topos of smooth spaces. We show how this allows one to define and study generalized global symmetries of such QFTs. The symmetries are themselves smooth, so the 'higher-form' symmetry groups can be endowed with, *e.g.*, a Lie group structure.

Among the more surprising general implications for physics are, firstly, that QFTs in spacetime dimension d, considered collectively, can have d-form symmetries, going beyond the known $(d-1)$-form symmetries of individual QFTs and, secondly, that a global symmetry of a QFT can be anomalous even before we try to gauge it, due to a failure to respect either smoothness (in that a symmetry of an individual QFT does not smoothly extend to QFTs collectively) or locality (in that a symmetry of an unextended QFT does not extend to an extended one).

Smoothness anomalies are shown to occur even in 2-state systems in quantum mechanics (here formulated axiomatically by equipping $d = 1$ spacetimes with a metric, an orientation, and perhaps some unitarity structure). Locality anomalies are shown to occur even for invertible QFTs defined on $d = 1$ spacetimes equipped with an orientation and a smooth map to a target manifold. These correspond in physics to topological actions for a particle moving on the target and the relation to an earlier classification of such actions using invariant differential cohomology is elucidated.

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1. Introduction

Previously [23], we gave an interpretation of generalized symmetries of topological quantum field theories (TQFTs) using homotopy theory. In a nutshell, the idea there was that TQFTs with a given tangential structure assemble themselves, according to the cobordism hypothesis [28], into a space X and that a generalized global[1] symmetry of a particular TQFT can be understood as a fixed point of an external group G acting on X, provided that all of these notions are interpreted in a suitably homotopy-theoretic sense. So by a ‘space’ we mean a homotopy type, and by a ‘group’ we mean not a group in the usual sense of a set with an associative and invertible multiplication law, but rather a space with a notion of associative invertible multiplication up to homotopy.

As noted in [23], one upshot of this interpretation is that there is rather more structure associated to a generalized symmetry than physicists had previously observed. The known tower of abelian ‘n-form symmetry’ groups, for $n \geq 1$, lying above the usual ‘0-form symmetry’ group corresponds here to the homotopy groups $\pi_n(G)$, for $n \geq 0$. So we get, for free, an action of $\pi_0(G)$ on $\pi_n(G)$ for each n, a Samelson product $\pi_n(G) \times \pi_m(G) \to \pi_{n+m}(G)$ for each n and m, &c. In fact, every G is equivalent to the loop space of some pointed connected space, so the structure associated to a generalized symmetry group G is completely described by its delooping BG. Moreover, a group action of G on X is equivalent to a morphism to BG whose fibre over the basepoint is X and a homotopy fixed point is equivalent to a section of that morphism; in favorable cases, such as TQFTs in spacetime dimension one or two, this enabled us to give a more-or-less explicit description of all possible actions and their homotopy fixed points.

To give a concrete example of the power of this approach, consider the global symmetries of gauge theory on oriented spacetimes with dimension $d = 2$ and gauge group a discrete group K. A gauge field is a connection on a principal K-bundle over the spacetime manifold, but, since K is discrete, such a connection is unique, so we can instead think of the gauge fields simply as maps from spacetime to BK. A classical action can then be formed, following Dijkgraaf and Witten [15], by taking a class in $H^2(BK, \mathbb{C}^\times)$, pulling back to spacetime along the gauge field, and pairing with the fundamental class. In the case where $K = (\mathbb{Z}/p)^2$, such that the classical action is specified by $q \in H^2(B(\mathbb{Z}/p)^2, \mathbb{C}^\times) \simeq \mathbb{Z}/p$, semiclassical arguments were used in [21] to argue that the quantized theory has 0- and 1-form symmetry groups both isomorphic to $(\mathbb{Z}/r)^2$, where r is the greatest common divisor of p and q. In fact, it was shown in [23] that these are mere subgroups of much larger 0- and 1-form symmetries, isomorphic to S_{r^2} and $(\mathbb{C}^\times)^{r^2}$, respectively. The underlying reason for this larger symmetry is duality: given an arbitrary classical gauge theory as above the resulting quantum theory depends, up to equivalence, only on the dimensions of the irreducible projective representations of K corresponding to the class in $H^2(BK, \mathbb{C}^\times)$. (In our specific example, the theories specified by (p, q) have r^2 inequivalent irreducible projective representations, each of which has dimension p/r.) To spell it out, many theories that are classically distinct are quantumly equivalent and so the

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[1] We also discussed generalized gauge symmetries in [23], but will not do so here.

symmetries of any one QFT are consequently larger. This is, presumably, hard to see using semi-classical arguments, but is child's play once one knows the corresponding X: one needs only to find the $x \in X$ that corresponds to the given classical action. Once one has done so, one knows the full generalized symmetry structure (a connected homotopy 2-type); in our specific example, it is specified by the further information that S_{r^2} acts on $(\mathbb{C}^\times)^{r^2}$ by permutation with Postnikov invariant given by the trivial class in $H^3\left(BS_{r^2}, (\mathbb{C}^\times)^{r^2}\right)$.

In this work, we wish to play a similar game in a set-up that is rather closer to the real world in two ways. Firstly, we shall generalize from TQFTs to genuine quantum field theories (QFTs), equipped with geometric structures such as maps to some target manifold (*i.e.* quantum fields), spacetime metrics, and so on. Secondly, we shall ask that the generalized symmetries of these theories be equipped with a suitable smooth structure. This allows us, for example, to consider the n-form symmetry groups not just as abstract groups, but as Lie groups, as physicists are wont to do. In a nutshell, the idea here will be that QFTs with a given geometric structure assemble themselves, via a geometric version of the cobordism hypothesis [22], into an object in a certain ∞-topos whose objects we generically call smooth spaces. Many of the relevant constructions from homotopy theory can be transferred to any ∞-topos [29,33]. In particular, not only do we have analogous notions of a group object G, an action of G on another object X, and a fixed point thereof (up to a notion of homotopy), but we also have equivalences (via a notion of delooping) sending these, respectively, to a pointed connected object BG, a morphism to BG with fibre X over the basepoint, and a section thereof. As a result, we are able to carry over the discussion in [23], *mutatis mutandis*, to the ∞-topos of smooth spaces. Here, a group object G is equipped with a smooth structure, as desired, giving us a new, and hopefully powerful, perspective on smooth generalized symmetries of dynamical (*i.e.* non-topological) QFTs.

Among the juicier titbits, we find that QFTs in spacetime dimension d have a non-trivial notion of an n-form symmetry for $n \leq d$ (rather than $n < d$, as is the common lore) and that a global symmetry can be anomalous even before one tries to gauge it, either due to a failure to respect smoothness, or due to a failure to respect locality. To be explicit, a smoothness anomaly (which occurs even in quantum mechanics) arises when a symmetry of a particular QFT fails to extend smoothly to the smooth space of QFTs, while a locality anomaly arises when a symmetry of an unextended QFT fails to extend when we extend the QFT. These anomalies reinforce the view we espoused in [23] that anomalies should be considered not so much as a problem associated to gauging a global symmetry, but rather to a failure to define global symmetries properly in the first place, namely in a way that is fully consistent with the sacred physical principles of locality and smoothness. Such anomalies arise frequently in practice, because physicists have a nasty habit of defining QFTs by writing them down not only one at a time, so smoothness is obscured, but also by specifying their values only on closed spacetime manifolds (and often on only one, such as the d-dimensional sphere) so locality is obscured as well.

We illustrate all of this using known examples of smooth spaces of QFTs. Unfortunately, our collective ignorance as to which higher category to take as the target for QFTs in $d > 1$ means that most known examples of X are in $d = 1$, but the principles apply quite generally. The holy grail would be, given X and G, to classify all possible G-actions on the whole of X and their corresponding homotopy fixed points, as we did for TQFTs in [23], but it will become clear that this is a daunting task in general, even in $d = 1$. For example, for quantum mechanics with a 1-dimensional state space, the hamiltonian describing the time evolution along an interval is given by a real number; if G corresponds to a Lie group, there is a G-action for every smooth action of the Lie group on $\mathbb{R}$ and part of the data of a homotopy fixed point is the fixed point subset of the smooth action.

Though dispiriting, it is important to remark that finding all actions on X and their homotopy fixed points goes way beyond what the typical physicist does. As we have already remarked, such a physicist, is, by-and-large, ignorant of what X is and instead contents themself with studying symmetries of QFTs considered one at a time. In our framework, a QFT is represented by a point x of X and the problem of finding all physical symmetries of that QFT considered in isolation can be phrased here as the problem of finding all G-actions and homotopy fixed points of the connected component of X at x. As we shall see, connectedness makes the problem of finding G-actions and homotopy fixed points more straightforward. It is much less straightforward to establish whether or not actions on a connected component extend to actions on all of X; if not, we have the aforementioned smoothness anomaly.

Explicitly, our main examples are QFTs in $d = 1$ equipped with an orientation and either a smooth map to a target manifold or a metric. The most striking result in the latter case (which upon addition of a suitable unitarity structure leads us to quantum mechanics) is that one can have a smoothness anomaly already for a quantum system whose state space is two-dimensional. In the former case, we find a locality anomaly even for a system whose state space is one-dimensional. Such a QFT is a so-called invertible QFT, and we can think of it as a semi-classical version of the mechanics of a point particle moving on the target manifold. Each such QFT defines a topological physics action, by evaluating the QFT on closed spacetime manifolds (*i.e.* disjoint unions of circles). For group actions that are induced by a smooth action of a Lie group on the target manifold, we can compare with the classification of invariant topological actions proposed in [13] in terms of invariant differential cohomology. The fact that a QFT here is defined in a fully-extended fashion, *i.e.* we specify its value not just on closed spacetime manifolds (*i.e.* disjoint unions of circles), but also on manifolds with boundary (*i.e.* disjoint unions of intervals and circles), leads to very different classifications. Indeed, the map from the extended symmetric theories to the unextended ones defined by restriction is neither surjective (meaning that a symmetry of the physics action does not extend, giving a locality anomaly, as defined above) nor injective (meaning that a symmetry of the physics action can be extended in multiple ways), in general. Two simple examples from physics may serve to illustrate the general situation:

firstly, for a particle moving in a plane in the presence of a uniform magnetic field, the action for motion in a loop is invariant under translations of the plane, but the fact that the lagrangian shifts by a total derivative means that it cannot be extended to an interval; secondly, the physics action describing motion of a particle moving on a point is obviously symmetric under the trivial action of any group on the point. But this symmetry can be extended to the theory defined on an interval in different ways, by assigning different characters of the group to the one-dimensional state space assigned by the theory to a spacetime point.

In case the basic point gets lost amidst all the mathematical heavy machinery, let us consider further the example of quantum mechanics. This is certainly a 'real world' example (even though we consider here only the simplified case where the state space is finite-dimensional). The smooth space X that we obtain in this case corresponds to (for each finite dimension n of the state space) the smooth manifold of $n \times n$ hermitian matrices together with the smooth action on it by conjugation of the Lie group of $n \times n$ unitary matrices. Physically, the hermitian matrices represent the possible hamiltonians of a quantum-mechanical system. Every physicist knows that hamiltonians related by conjugation by unitary matrices are equivalent, but identifying them and choosing one from each equivalence class (*e.g.* by diagonalizing and quotienting out by permutations), as a physicist usually does, not only results in a set which is not a smooth manifold, but also entails a loss of information. It is better, in general, to keep track manifestly of the ways in which hamiltonians are equivalent and this is what the smooth space X does. It is better, in particular, for describing symmetries of quantum mechanical theories because it allows us to keep track of transformations that send a hamiltonian not to itself, but to a conjugate one, in a smooth fashion.

In contrast to the physics, we lay no great claim to mathematical novelty. (In particular, the notion of an ∞-topos is now well-established, largely thanks to [29], and the notion of group actions therein was developed in [33].) But one or two minutiae may be of mathematical interest. To give one example, we introduce in §2.3 notions, for each non-negative integer m, of the m-orbit and m-stabilizer of a point under a group action in an ∞-topos. In addition, there are a number of new results regarding group actions in ∞-topoi and their associated homotopy fixed points.

A brief review of existing literature. It is worth highlighting a few papers relevant to our discussion here. Locality anomalies have previously been discussed in [31,42]. Anomalies associated with smoothness are discussed in [11,12], although it is unclear how they are related to the smoothness anomalies discussed here. The way we enforce the condition of a field theory having a d-form symmetry has been used to enforce the spin-statistics relation in [32].

A note on style and notation. Since our target audience consists of physicists, we have opted for an informal presentation, emphasing pedagogy over details. To this end, we eschew definitions, theorems, and proofs whenever possible; such impedimenta, when not supplied in the Appendix or given an explicit reference, can mostly be found in [29]. We mostly follow the nomenclature and notation used in [29] with one significant exception: an '∞-category' there will henceforth be called simply a 'category' here (and denoted, as there, using calligraphic typeface); the same goes for all ∞-categorical paraphernalia, such as functor, adjoint, (co)limit, topos, &c. On the rare occasions where we need to refer to a bog-standard category, we will call it a '1-category', &c.[2] As usual, by a pointed object, we mean one equipped with the datum of a point, *i.e.* a morphism $x : * \to X$ from a terminal object. As in [29], by 'space' we mean '∞-groupoid' and denote the archetypal topos in which they live by $\mathcal{S}$. Similarly, we too allow ourselves to use 'the' or 'an essentially unique' to mean 'a' in situations where the gamut of possibilities forms a contractible space. When we come to consider the specific topos $\mathcal{S}\mathrm{m}$ of smooth spaces in §3, we shall frequently need to pass back and forth between classical constructs in differential geometry, such as manifolds and Lie groups, and their corresponding objects in the topos of smooth spaces. To distinguish the two, we use sans serif typeface (*e.g.* M, G or R) for the former and serif typeface for the latter (*e.g.* M, G or $\mathbb{R}$).

The outline is as follows. In the next Section, we recall some relevant details of topos theory and describe notions of groups, actions and homotopy fixed points in an arbitrary topos. In §3, we describe the particular topos of smooth spaces. In §4, we describe some examples in smooth spaces that are relevant to physics.

2. Group actions in a topos

Our goal in this Section is to sketch the relevant details of topos theory and introduce various notions and constructions related to group actions therein. The notion of a topos and a group object therein were given in [29] and group actions were discussed in [33]; the notions of orbits, stabilizers, and various results that will be useful in our later analysis of examples are perhaps new. To keep in the spirit of things, we mostly express these in a functorial fashion, which requires a certain amount of abstraction. Happily, unwinding the definitions at the level of objects usually results in something straightforward.

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[2] Lest readers who know their 1-categorical onions be lulled into a false sense of security, we stress that their intuition may fail spectacularly on occasion. For example, consider a point $x : * \to X$ within an object X, where $*$ is a terminal object. The pullback of this point over itself is not a terminal object, end of story. On the contrary, it is the very beginning of the story of homotopy theory in a topos, defining as it does the object of loops in X based at x. Those who are *au fait* with classical homotopy theory should be on their guard too. For example, an object X being connected is neither a necessary nor a sufficient condition for it to be what we shall call path-connected, namely such that any two points $x, x' : * \to X$ in it are homotopic to one another. In the topos of smooth spaces, being connected implies being path-connected, but not the converse. The classifying space of principal bundles with connection, described in §3, provides a counterexample.

A topos is a special kind of category, which in turn is designed to be a homotopy-theoretic generalization of a 1-category. So let us begin by describing how the latter is achieved.

A category is a special kind of simplicial set. Like a 1-category, it has objects and morphisms (given respectively by the 0-simplices, a.k.a. vertices, and 1-simplices, a.k.a. edges, a.k.a. 1-morphisms, of the simplicial set), along with an identity morphism $\mathrm{id}_X : X \to X$ for each object X (given by its image under the degeneracy map), but unlike a 1-category it also has n-morphisms for every $n > 1$ (given by the n-simplices). The rôle of these is to provide a notion of the associative composition of morphisms defined in a 1-category that is weaker in that it holds only up to coherent homotopy. To wit, given a morphism into some object and a morphism out of that same object, there is a contractible space of 2-simplices that are sent by the appropriate face maps to the input morphisms. Each such 2-simplex gives the data of a possible composition (the image under the remaining face map) along with a coherent homotopy witnessing it (the 2-simplex itself). The 3-simplices provide a homotopy-coherent notion of associativity, and so on. Unlike a 1-category, where the morphisms between two objects form a set, in a category the morphisms form a space, whose 0-simplices are the 1-morphisms.

A functor is simply a map of simplicial sets, so boils down to a map of sets for each n, preserving the relations between the face and degeneracy maps.

The archetypal category $\mathcal{S}$ has objects given by spaces up to weak homotopy equivalence. A topos $\mathcal{X}$ is a generalization of $\mathcal{S}$ which maintains many of its desirable properties, admitting many notions akin to those in homotopy theory. There are (small) limits and colimits, so we have the terminal object $*$ along with the terminal morphism $*_X : X \to *$ for each object X. In addition, we have notions of n-truncated and n-connected morphisms, for each $n \geq -2$, defined as follows. Firstly, recall that a space is called n-truncated if its i-th homotopy group vanishes for every $i > n$ and every choice of basepoint.[3] An object X of the topos $\mathcal{X}$ is called n-truncated if for any $Y \in \mathcal{X}$ the mapping space $\mathcal{X}(Y, X)$ is n-truncated. The inclusion of the full subcategory $\tau_{\leq n}\mathcal{X}$ of n-truncated objects in $\mathcal{X}$ has a left adjoint $\tau_{\leq n}$ and we say that an object $X \in \mathcal{X}$ is n-connected if $\tau_{\leq n}X \simeq *$. A morphism $f : Y \to Z$ in $\mathcal{X}$ is called n-truncated (respectively, n-connected) if it is an n-truncated (n-connected) object in $\mathcal{X}_{/Z}$. For each n, the n-connected/n-truncated morphisms form an orthogonal factorization system. Among other things, this implies that any morphism $f : X \to Y$ in $\mathcal{X}$ canonically factorizes as the composition of an n-connected morphism followed by an n-truncated one and that, given a commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "n-\mathrm{conn}"'] & Y \arrow[d, "n-\mathrm{trun}"] \\
Z \arrow[r] \arrow[ur, dashed] & W
\end{tikzcd}$$

in which the morphism on the left is n-connected and the morphism on the right is n-truncated, there is an essentially unique dashed arrow that makes the diagram commute. For $n = -2$, these notions are not particularly exciting, since any morphism is -2-connected and a morphism is -2-truncated iff it is an equivalence; for $n = -1$ we have the notions of monomorphism (-1-truncated) and effective epimorphism (-1-connected), generalizing the notions in $\mathcal{S}$ of morphisms that induce respectively an injection on π_0 (and bijections on higher homotopy groups) and a surjection on π_0. The reader is encouraged to think of an effective epimorphism $X \twoheadrightarrow Y$ as a cover of Y by X.

Along similar lines, given a pointed object, *i.e.* an object $X \in \mathcal{X}$ and a morphism $x : * \to X$ in $\mathcal{X}$, we define its k-connected cover to be the object $\tau_{\geq(k+1)}(X, x)$ defined via the cartesian square

$$\begin{tikzcd}
\tau_{\geq(k+1)}(X,x) \arrow[r] \arrow[d] & X \arrow[d] \\
* \arrow[r] & \tau_{\leq k}X
\end{tikzcd} \tag{1}$$

where $X \to \tau_{\leq k}X$ is the k-connected morphism in the k-connected/k-truncated factorization of $X \to *$ and where $* \to \tau_{\leq k}X$ is the composition of x and $X \to \tau_{\leq k}X$. Since k-connectedness is stable under pullback, $\tau_{\geq(k+1)}(X, x)$ is indeed k-connected, as the name suggests. It is, moreover, canonically pointed, via the morphism x and the universal property of (1). As is traditional in spaces, we call $\tau_{\geq 1}(X, x)$ the connected component of X at x.

Before going on to groups and their actions, let us define one finial bit of notation. For each topos $\mathcal{X}$ there is a special functor which takes an object $X \in \mathcal{X}$ to the mapping space $\mathcal{X}(*, X)$. We denote this functor $\Gamma : \mathcal{X} \to \mathcal{S}$, and will call the object ΓX the underlying space of X.

2.1. Groups and group actions as groupoids

Group objects and group actions in a topos are both special cases of groupoid objects; the latter may be viewed as the appropriate homotopical generalization of the run-of-the-mill notion of an equivalence relation on a set. This generalization is achieved in three steps. Firstly, since the very essence of homotopy theory is that we care not just about whether two

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[3] A space is -2-truncated iff it is equivalent to the point, while a space is -1-truncated iff it is equivalent to the point or to the empty space.

things are equivalent, but rather about all the ways in which they are equivalent, we generalize from an equivalence relation on a set to a groupoid. Thus, the elements of the set become objects in a 1-category and the equivalence relations between them become morphisms that are isomorphisms. (To go back, we demand that there be at most one isomorphism between any two objects.) Secondly, we generalize to groupoids valued in any 1-category by asking for a 1-simplicial object $U_\bullet$ in that 1-category obeying certain conditions due to Segal [40] (and earlier Grothendieck [24]). (To go back, we demand that the 1-category be the 1-category of sets.) Thus, denoting by U_n the object of n-simplices, our original set and equivalences are replaced respectively by U_0 and U_1 and the identities are determined by the degeneracy map $s_0 : U_0 \to U_1$. The Segal conditions for U_2, for example, require that the three squares

$$\begin{tikzcd}
U_2 \arrow[r, "d_0"] \arrow[d, "d_1"'] & U_1 \arrow[d, "d_0"] & U_2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & U_1 \arrow[d, "d_0"] & U_2 \arrow[r, "d_2"] \arrow[d, "d_1"'] & U_1 \arrow[d, "d_1"] \\
U_1 \arrow[r, "d_0"'] & U_0, & U_1 \arrow[r, "d_1"'] & U_0, & U_1 \arrow[r, "d_1"'] & U_0,
\end{tikzcd}$$

are 1-cartesian. Collectively, these define composition along with left- and right- inverses. Associativity is enforced by the Segal conditions for U_3. Thirdly, we generalize from 1-categories to categories by removing the '1-'.

Having done this legwork, we may now easily define a group object as a groupoid object whose U_0 is the terminal object $*$. As a result of the Segal conditions, we have that U_n is the n-fold product of U_1. We shall sometimes abuse notation, by denoting both the simplicial object $U_\bullet$ and the underlying object U_1 of a group object by, *e.g.*, G.

Now we come to a first miracle of topos theory: there is an equivalence of categories between groupoid objects in $\mathcal{X}$ and effective epimorphisms in $\mathcal{X}$, obtained in one direction by taking the colimit of the groupoid object and in the other direction by taking the Čech nerve. Under the colimit, a group object G is thus sent to an effective epimorphism out of the terminal object and into an object that we call the delooping of G and denote BG. Since $* \twoheadrightarrow BG$ is -1-connected and is a section of $BG \to *$, it follows [29, Prop. 6.5.1.20] that BG is 0-connected, a.k.a. connected. In short, there is an equivalence of categories between group objects in $\mathcal{X}$ and pointed, connected objects in $\mathcal{X}$. Going in the other direction, the first step of the Čech nerve construction sends a morphism to its fibre product with itself and so we define, for any object X with basepoint $x : * \to X$, the object $\Omega_x X$ (we write simply ΩX if the basepoint is clear) of loops in X based at x via the cartesian square

$$\begin{tikzcd}
\Omega_x X \arrow[r] \arrow[d] & * \arrow[d, "x"] \\
* \arrow[r, "x"'] & X.
\end{tikzcd}$$

Using this equivalence, we may define, following [33, Defn. 3.1] and [37, Prop. 3.2.76], a group action of a group object G on an object X as a morphism to BG together with an identification of the fibre over the basepoint $* \twoheadrightarrow BG$ with X. In other words, it is the data of an object $X/\!\!/G$ and a cartesian square

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d] & X/\!\!/G \arrow[d] \\
* \arrow[r, two heads] & BG.
\end{tikzcd}$$

Since effective epimorphisms are stable under pullback, the morphism $X \twoheadrightarrow X/\!\!/G$ is an effective epimorphism, as indicated. To see that this is a sensible definition of a group action, take the Čech nerve of $X \twoheadrightarrow X/\!\!/G$. This results upstairs in a groupoid, in which moreover U_n is equivalent to $X \times G^n$. We have a diagram of the schematic form

$$\begin{tikzcd}
\cdots & X \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times G \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X \arrow[r, two heads] & X/\!\!/G,
\end{tikzcd}$$

where the face morphisms $d_0, d_1 : X \times G \to X$ correspond to projection on the right hand factor and the group action respectively, but the latter satisfies the usual properties only up to higher coherent homotopies specified by the subsequent stages.

The holy grail for the purposes of physics would be to classify all possible actions by G and to find the corresponding homotopy fixed points. To do so, we need a notion of when two G-actions are equivalent and it is convenient to do so by assembling a category of G-actions on arbitrary objects in $\mathcal{X}$. A suitable category (which is, in fact, a topos) is given by the slice category $\mathcal{X}_{/BG}$ (see *e.g.* [37, Prop 3.2.76]). Given an object therein, *i.e.* an object E in $\mathcal{X}$ along with a morphism $E \to BG$ in $\mathcal{X}$, we recover the object in $\mathcal{X}$ on which G acts by pulling back $E \to BG$ along the basepoint $* \twoheadrightarrow BG$ of BG. In cases where there is little risk of confusion, we will often write simply E to denote the object $E \to BG$ in $\mathcal{X}_{/BG}$ in the following.

As usual when studying group actions, it is useful to consider the notion of an equivariant morphism, which we define to be a morphism in $\mathcal{X}_{/BG}$. As we shall see in the next Subsection, this notion is convenient for discussing homotopy fixed points. Occasionally, we abuse nomenclature and use the same term to refer to the underlying nonequivariant morphism in $\mathcal{X}$ induced by pullback along $* \twoheadrightarrow BG$.

2.2. *Homotopy fixed points*

We now wish to define a homotopy-theoretic version of a fixed point for a group object G acting on an object X in a topos $\mathcal{X}$. As we have seen, a group action is a fibre sequence and so it is natural to guess that a homotopy fixed point should be a section of the corresponding morphism $X/\!\!/G \to BG$. To see that this makes sense, we note that we may pull back a section $s : BG \to X/\!\!/G$ to get a commutative diagram

$$\begin{tikzcd}
* \arrow[r, two heads] \arrow[d, "x"'] & BG \arrow[d, "s"] \\
X \arrow[r, two heads] \arrow[d] & X/\!\!/G \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

in which all squares are cartesian and the vertical composite morphisms are identities. In this way, we associate to every section the data of a morphism $x : * \to X$ in $\mathcal{X}$, *i.e.* a point in X, that is moreover equivariant with respect to the given action of G on X and the trivial action on $*$, reproducing the usual 1-categorical notion of a fixed point of a group action. We say that s is a homotopy fixed point through the underlying fixed point x. Note that there may be more than one section through a given x, meaning that there is more data associated to a homotopy fixed point than its underlying fixed point. This reflects the fact that to be a *homotopy* fixed point is not merely a property, but a structure: to be fixed means to be fixed up to homotopy, and the structure records the possible homotopies that do the job.

Since we shall be interested for physics in the question of finding all homotopy fixed points, it is behooves us to find a home where homotopy fixed points can live. As in [23], a useful strategy for physics (we shall discuss why shortly) is to assemble the homotopy fixed points into an object X^{hG} in $\mathcal{X}$, which we furthermore equip with a morphism to X, corresponding to sending a homotopy fixed point to its underlying fixed point and forgetting that it is a fixed point.[4] This is achieved as follows. Given any morphism $f : Y \to Z$ in a topos $\mathcal{X}$ we have a triple adjunction [29]

$$\begin{tikzcd}
\mathcal{X}_{/Y} \arrow[r, bend left=40, "f_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"'] \arrow[r, bend right=40, "f_*"'] & \mathcal{X}_{/Z} \arrow[l, "f^*" description] .
\end{tikzcd}$$

Given $X \in \mathcal{X}$, we construct the object X^{hG} by noting that when we set $Y = X$, $Z = *$ and f to be a terminal morphism $*_X : X \to *$ in the above, the right-adjoint $(*_X)_*$ sends an object in $\mathcal{X}_{/X}$, *viz.* a morphism $g : W \to X$, to an object in $\mathcal{X}$ whose points (*i.e.* its morphisms from $*$) correspond to morphisms from $\mathrm{id} : X \to X$ to $g : W \to X$, *i.e.* to sections of g. Thus, it is reasonable to make the definition $X^{hG} := (*_{BG})_*(X/\!\!/G \to BG)$. As a check, the adjunction implies that $\mathcal{X}_{/BG}(BG, X/\!\!/G) \simeq \mathcal{X}(*, X^{hG})$, so points of X^{hG} indeed correspond to sections of $X/\!\!/G \to BG$.

To get the morphism $X^{hG} \to X$, we need to use the basepoint of BG, which we write explicitly as $b_G : * \to BG$. The unit of the adjunction $b_G^* \dashv (b_G)_*$ defines a functor $\Delta^1 \times \mathcal{X}_{/BG} \to \mathcal{X}_{/BG}$ (or, in other words, a natural transformation of functors $\mathcal{X}_{/BG} \to \mathcal{X}_{/BG}$) sending the morphism $(\{0,1\}, \mathrm{id}_{X/\!\!/G})$ to $X/\!\!/G \to (b_G)_* b_G^*(X/\!\!/G)$. By postcomposing this with $(*_{BG})_* : \mathcal{X}_{/BG} \to \mathcal{X}$ we get a functor $\Delta^1 \times \mathcal{X}_{/BG} \to \mathcal{X}$ which sends the morphism $(\{0,1\}, \mathrm{id}_{X/\!\!/G})$ to a morphism $X^{hG} \to X$. (Alternatively, we may think of this as a functor $\mathcal{X}_{/BG} \to \mathcal{F}\mathrm{un}(\Delta^1, \mathcal{X})$ sending $X/\!\!/G \to BG$ to $X^{hG} \to X$.)

The construction of the morphism $X^{hG} \to X$ enjoys the following functoriality properties. Firstly, given G-actions on two objects X and Y and a morphism $f : X/\!\!/G \to Y/\!\!/G$ in $\mathcal{X}_{/BG}$ we get a commutative square

$$\begin{tikzcd}
X^{hG} \arrow[r, "(*_{BG})_* f"] \arrow[d] & Y^{hG} \arrow[d] \\
X \arrow[r, "b_G^* f"'] & Y,
\end{tikzcd}$$

where $b_G^* f$ is a G-equivariant morphism from X to Y.

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[4] Again, because to be a homotopy fixed point is a structure not a property, we cannot expect the morphism $X^{hG} \to X$ to be a monomorphism, *i.e.* -1-truncated, unlike the case of a common or garden group action on a set (where it corresponds to the inclusion of the fixed point subset).

Secondly, given a 2-simplex

$$\begin{tikzcd}
& BG \arrow[dr, "h"] & \\
* \arrow[ur, "b_G"] \arrow[rr, "b_H"'] & & BH
\end{tikzcd}$$

in $\mathcal{X}$ and a $X/\!\!/H \to BH \in \mathcal{X}_{/BH}$ representing an H-action on X, we may form $X/\!\!/G = h^*(X/\!\!/H) \to BG \in \mathcal{X}_{/BG}$ as the pullback of $X/\!\!/H \to BH$ along $h : BG \to BH$, which represents the restriction of the H-action on X to a G-action, and we get another 2-simplex

$$\begin{tikzcd}
& X^{hG} \arrow[dr] & \\
X^{hH} \arrow[ur] \arrow[rr] & & X.
\end{tikzcd} \tag{2}$$

in $\mathcal{X}$, expressing functoriality with respect to the group objects. The construction goes as follows. Letting η_h, η_{b_H}, and η_{b_G} denote, respectively, the units of the adjunctions $h^* \dashv h_*$, $b_H^* \dashv (b_H)_*$, and $b_G^* \dashv (b_G)_*$, we get a functor $\Delta^2 \times \mathcal{X}_{/BH} \to \mathcal{X}_{/BH}$ (or, in other words, a modification) that, regarded as a 2-simplex in $\mathcal{F}\mathrm{un}(\mathcal{X}_{/BH}, \mathcal{X}_{/BH})$, takes the form

$$\begin{tikzcd}
& h_*h^* \arrow[dr, "\mathrm{id}_{h_*}\eta_{b_G}\mathrm{id}_{h^*}"] & \\
\mathrm{id} \arrow[ur, "\eta_h"] \arrow[rr, "\eta_{b_H}"'] & & (b_H)_*b_H^*,
\end{tikzcd}$$

Postcomposing with $(*_{BH})_* : \mathcal{X}_{/BH} \to \mathcal{X}$ gives a functor $\Delta^2 \times \mathcal{X}_{/BH} \to \mathcal{X}$ which, evaluated at $X/\!\!/H \to BH \in \mathcal{X}_{/BH}$, returns the 2-simplex in Eq. (2).

Before going further, let us pause to discuss how these notions should be applied to physics. The discussion will be general, but we will use the familiar example of an Lie group symmetry of quantum mechanics, discussed in detail in §4.3, to illustrate.

The idea will be that the object X represents a collection of QFTs with some specified structure, with a particular QFT given by a point $x : * \to X$. In the case of quantum mechanics, an x is specified by a unitary representation of $\mathbb{R}$, or equivalently a hamiltonian.

The group object G represents a possible symmetry group of a QFT; to see if it really is a symmetry, however, we must first specify more data, namely an action of G on X. In quantum mechanics, for example, the trivial action of on X will give rise to symmetries acting via a smooth unitary representation on the state space, while non-trivial actions will give rise to smooth representations that are twisted in some way, such as projective or antiunitary representations.

Having specified the data of a group object G and an action of it on X, we are now in a position to ask if there is a homotopy fixed point through x; if there is, we say that the QFT x admits a G-symmetry (the action of G is left implicit in the notation). The associated homotopy fixed point is to be interpreted as the QFT x equipped with the symmetry structure; we call it a G-symmetric QFT. In quantum mechanics, for example, with the trivial action of G, a homotopy fixed point is given by a unitary representation of $\mathrm{G} \times \mathbb{R}$ that restricts along $\mathbb{R} \hookrightarrow \mathrm{G} \times \mathbb{R}$ to the representation specifying x (equivalently, the unitary operators assigned to each $\mathrm{g} \in \mathrm{G}$ commute with the hamiltonian). We stress again that a symmetry of a QFT is a structure rather than a property, because (for fixed G and fixed action thereof) there may be multiple homotopy fixed points through x. In quantum mechanics, for example, there may be multiple inequivalent representations of $\mathrm{G} \times \mathbb{R}$ that restrict suitably. The G-symmetric QFTs in X assemble themselves into the object X^{hG} and the morphism $X^{hG} \to X$ sends a G-symmetric QFT to its underlying QFT, forgetting the G-symmetry structure.

Since our construction is functorial, it follows that group actions that are isomorphic in $\mathcal{X}_{/BG}$ lead to isomorphic $X^{hG} \to X$ in $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{X})$, so are physically equivalent.

It is to be stressed here that G represents a symmetry group that is extrinsic rather than intrinsic to X (in particular, G can be chosen freely). There also exists a notion of an intrinsic symmetry group of a particular QFT $x : * \to X$, given by the group object $\Omega_x X$ of loops based at x. Furthermore, there is even a notion [33] of an intrinsic symmetry group $\mathrm{Aut}\, X$ of the object X of QFTs, at least when the object X, or rather the morphism $X \to *$ is suitably compact. In practice, actually giving an explicit description of $\mathrm{Aut}\, X$ starting from an explicit description of X is a difficult task. Moreover, the interplay between $\mathrm{Aut}\, X$ and $\Omega_x X$ is subtle. Passing to an extrinsic notion of symmetry is convenient in that it allows us to avoid having to do any of this.

To remove some of the mystery of these assertions, consider the simple example of a quantum mechanical system whose dynamics is invariant under rotations of 3-dimensional space. There is an intrinsic $\mathsf{SO}(3)$ symmetry, and one possibility is for the states of the system to form a representation thereof. But the fact that $\operatorname{Aut} X$ is non-trivial for the object of quantum-mechanical theories (cf. §4.2) means that one can also have projective representations of $\mathsf{SO}(3)$.

Carrying on our discussion of the physics interpretation, an important remark is that physicists are presumably interested only in objects in X that admit a point (since otherwise there is no QFT in the 'object of QFTs') and only in G-actions on X that admit a homotopy fixed point (since otherwise there will be no QFT that admits a G-symmetry under the given action). Moreover, given a G and an X, it would be desirable to have a means of considering, in one go, all possible actions admitting fixed points, along with their homotopy fixed points. We can achieve this by considering not just the topos $\mathcal{X}_{/BG}$ in which an object corresponds to a G-action, but also the related category of pointed objects.

Given any topos $\mathcal{X}$, we define the corresponding category of pointed objects $\mathcal{X}_*$ as the slice category $\mathcal{X}_{*/}$.[5] We claim that $(\mathcal{X}_{/BG})_*$ is a category whose objects are equivalent to homotopy fixed points of G-actions. Indeed, the terminal object $*$ in $\mathcal{X}_{/BG}$ is id_{BG}, so a section is manifestly an object in $(\mathcal{X}_{/BG})_*$. To recover the G-action (which is necessarily one that admits a homotopy fixed point), we simply follow the functor $(\mathcal{X}_{/BG})_* \to \mathcal{X}_{/BG}$ that forgets the point, while to recover the underlying fixed point $* \to X$, we follow the functor $(b_G^*)_* : (\mathcal{X}_{/BG})_* \to \mathcal{X}_*$ induced by $b_G^* : \mathcal{X}_{/BG} \to \mathcal{X}$ (*i.e.* pullback along the basepoint $b_G : * \to BG$).

As well as recovering the individual homotopy fixed point corresponding to an object in $(\mathcal{X}_{/BG})_*$, we can recover the space (not smooth space) of homotopy fixed points of a given action as the fibre of the functor $(\mathcal{X}_{/BG})_* \to \mathcal{X}_{/BG}$ above a G-action $X /\!\!/ G \to BG$. Indeed, this is equivalent via §2.1.2 of [29] to the space $\mathcal{X}(*, X^{hG})$.

It is sometimes useful to exploit a relation, described in Appendix B.1, between the category $(\mathcal{X}_{/BG})_*$ and the topos $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{X})_{/\mathrm{id}_{BG}}$. Since the latter is evidently equivalent to the topos of actions on objects given by arrows in $\mathcal{X}$ by the group given by looping the arrow id_{BG}, the relation shows that the problem of finding group actions with homotopy fixed points is really no different to the problem of finding bare group actions – it simply takes place in a different topos. The problem of finding bare group actions will be discussed in §2.4.

2.3. Orbits and stabilizers

For better or for worse, some physicists seem to be not so much interested in (or perhaps are entirely unaware of) the space X of QFTs, but rather tend to focus on their pet theory $x : * \to X$ within it. We shall have frequent need to refer to such a physicist in the sequel, so we shall give them the allegorical moniker Simplicio.[6] Now Simplicio, parochial as they are, is presumably not (or not yet, at least) interested in arbitrary G-actions on X, nor even G-actions on X admitting (or equipped with) a homotopy fixed point as above, but rather only on G-actions on X admitting (or equipped with) a homotopy fixed point through their favorite x.

For Simplicio's benefit, it is convenient to formulate certain notions associated to a point $x : * \to X$. Firstly, by the orthogonal factorization property we have a Postnikov decomposition (*cf.* the Whitehead tower of a space)

$$x : * \to \cdots \to \tau_{\geq m}(X, x) \to \cdots \to \tau_{\geq 2}(X, x) \to \tau_{\geq 1}(X, x) \to X,$$

where $\tau_{\geq m}(X, x)$ is the $(m-1)$-connected cover of (X, x) defined by Eq. (1). To see this, observe that since $\tau_{\geq m}(X, x)$ is $(m-1)$-connected, [29, Prop. 6.5.1.20] tells us that the morphism $* \to \tau_{\geq m}(X, x)$ is $(m-2)$-connected. The morphism $\tau_{\geq m}(X, x) \to X$ appears at the top of the diagram in Eq. (1), and is the pullback of $* \to \tau_{\leq m-1} X$. This latter morphism is $(m-2)$-truncated, a consequence of [29, Lem. 5.5.6.15] and the fact that looping increases truncatedness. Since truncatedness is preserved by pullback, $\tau_{\geq m}(X, x) \to X$ is $(m-2)$-truncated too.

Although it is not true in a general topos, in the topos $\mathcal{S}\mathrm{m}$ of smooth spaces being 0-connected implies[7] being path-connected, *i.e.* any two points are homotopic to one another, meaning physically that the theories are equivalent. Thus we can sensibly interpret, in our framework, what Simplicio does as studying homotopy fixed points of group actions on the connected component $\tau_{\geq 1}(X, x)$, rather than X itself, (with the QFT x replaced by the point $* \to \tau_{\geq 1}(X, x)$ defined by the decomposition above.) This modus operandi, which has both features and a bug, will be of some significance in the sequel. The first feature, as we shall see in the next Subsection, is that all symmetries of $\tau_{\geq 1}(X, x)$ (*i.e.* all possible homotopy fixed points of all possible actions) can be found, as we shall see, in a way analogous to what is done in spaces. The second feature, is that every G-action on X equipped with a homotopy fixed point, s_x, through x descends to a G-action on

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[5] Note that $\mathcal{X}_*$ is not a topos (unless it is trivial), so the usual theorems for topoi do not apply.

[6] We hope that no physicist (Galileo included) will be unduly offended by our doing so.

[7] The converse does not hold, a counterexample being the classifying space of bundles with connections introduced in §3.

$\tau_{\geq m}(X, x)$, for each m, so that Simplicio will not miss any of the possible symmetries of the QFT x. To see this, consider the Postnikov decomposition of s_x,

$$\begin{tikzcd}
* \arrow[r, two heads] \arrow[d] \arrow[ddddd, bend right=60, "x"'] & BG \arrow[d] \arrow[ddddd, bend left=60, "s_x"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\tau_{\geq m}(X, x) \arrow[r, two heads] \arrow[d] & \tau_{\geq m}(X /\!\!/ G, s_x) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\tau_{\geq 1}(X, x) \arrow[r, two heads] \arrow[d] & \tau_{\geq 1}(X /\!\!/ G, s_x) \arrow[d] \\
X \arrow[r, two heads] \arrow[d] & X /\!\!/ G \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

in which all squares are cartesian. Thus $\tau_{\geq m}(X /\!\!/ G, s_x) \to BG$ defines a G-action on $\tau_{\geq m}(X, x)$.

The bug is that not every G-action on $\tau_{\geq 1}(X, x)$ admitting a homotopy fixed point extends to a G-action on X. Thus we exhibit a new kind of anomaly that is associated purely to global symmetries, arising because there are group actions of individual QFTs that do not respect the smooth structure of the whole space of QFTs. We call these anomalies ‘smoothness anomalies’. As we shall see in §4.3, an example arises even when X corresponds to the smooth space of unitary quantum mechanical theories, so these smoothness anomalies seem likely to play a rôle in the real world.

The other notions that we wish to formulate generalize the usual notions of the orbit and stabilizer of a group acting on a set. Given a point $x : * \to X$ and an action on X by a group object G, we can form the morphism $[x] : * \to X /\!\!/ G$ obtained by composing x with the inclusion $X \twoheadrightarrow X /\!\!/ G$ of the fibre. We call $[x]$ the stabilizer morphism of x. The Postnikov decomposition of the stabilizer morphism for each $m \geq 1$ gives the pointed, $(m-1)$-connected object $\tau_{\geq m}(X /\!\!/ G, [x])$, whose mth looping $\Omega^m \tau_{\geq m}(X /\!\!/ G, [x])$ we call the $(m-1)$-stabilizer of x under the G-action. In particular, $\Omega \tau_{\geq 1}(X /\!\!/ G, [x])$ generalizes the usual notion of the stabilizer subgroup at a point of a group acting on a set. Pulling the stabilizer morphism $[x]$ back along $X \to X /\!\!/ G$ we obtain a commutative diagram of the form

$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[d, "o_x"] & * \arrow[d, "{[x]}"] \\
X \arrow[r, two heads] \arrow[d] & X /\!\!/ G \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

in which both squares are cartesian; we call the resulting morphism $o_x : G \to X$ the orbit morphism, since it generalizes the usual orbit map for a group acting on a set. Its $(m-2)$-connected/$(m-2)$-truncated factorization for each $m \geq 1$ defines an object $\mathrm{im}_m\, o_x$ which we call the $(m-1)$-orbit of x under the G-action. The object $\mathrm{im}_1\, o_x$ generalizes the usual notion of the orbit of a point in a set under a group action.[8]

Universality of the pullbacks implies, furthermore, that we have a diagram of the form

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[8] As an aside, the usual notions of transitive and free actions of group actions on sets are encoded, respectively, by -1-connectedness or -1-truncatedness of the orbit morphism, leading to obvious (though x-dependent) generalizations of m-transitive and m-free group actions in any topos, but we make no use of them here.

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & * \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{im}_m\, o_x \arrow[r, two heads] \arrow[d] & \tau_{\geq m}(X/\!\!/G, [x]) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{im}_1\, o_x \arrow[r, two heads] \arrow[d] & \tau_{\geq 1}(X/\!\!/G, [x]) \arrow[d] \\
X \arrow[r, two heads] \arrow[d] & X/\!\!/G \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

in which all squares are cartesian.

Let us now discuss what can be said about the m-orbits and m-stabilizers at x when we have a homotopy fixed point at x, *i.e.* a section s_x through x. A first observation is that when we have a homotopy fixed point through x, the orbit morphism $G \to X$ factors through $x : * \to X$, as can be seen by considering the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & * \arrow[d] \arrow[dd, bend left, "{[x]}"] \\
* \arrow[r] \arrow[d, "x"'] & BG \arrow[d, "s_x"] \\
X \arrow[r] \arrow[d] & X/\!\!/G \arrow[d] \\
* \arrow[r] & BG
\end{tikzcd}$$

in which all squares are cartesian. (An action having an orbit morphism that factors through $x : * \to X$ does not necessarily imply the existence of a homotopy fixed point through x, however.)

The observation that the orbit morphism $G \to X$ factors through $x : * \to X$ when we have a homotopy fixed point s_x, naturally leads us to compare the Postnikov decompositions of the morphisms s_x and $[x]$. The sources of these morphisms, respectively BG and $*$ disagree, but their targets agree, so we can hope that their Postnikov decompositions agree far from the sources but close to the targets, and moreover that the region in which they coincide will be larger the more BG resembles $*$, *i.e.* the more connected it is. This intuition leads to the following theorem (Appendix B.2): when BG is p-connected and $0 \leq m \leq p+1$, we have that $\tau_{\geq m}(X/\!\!/G, s_x) \simeq \tau_{\geq m}(X/\!\!/G, [x])$, that $s_x : BG \to X/\!\!/G$ has a $(m-2)$-connected/$(m-2)$-truncated factorization $BG \to \tau_{\geq m}(X/\!\!/G, [x]) \to X/\!\!/G$, where the morphism $\tau_{\geq m}(X/\!\!/G, [x]) \to X/\!\!/G$ is as above, and that consequently $\tau_{\geq m}(X, x) \simeq \mathrm{im}_m\, o_x$.

The usefulness of this result for our purposes is that the Postnikov decomposition of any section $BG \to X/\!\!/G$ depends, toward its far end, only on its underlying fixed point x. So if we try to reconstruct sections by piecing together their decompositions, part of our work can be done in a wholesale fashion. Since BG is always 0-connected, we can make use of this at least for $\tau_{\geq 1}(X, x)$, which, as we have argued, is Simplicio's *domus*.

Even more is true when the object X is path-connected. For then the morphism of spaces $\mathcal{X}_{/BG}(BG, \tau_{\geq 1}(X/\!\!/G, [x])) \to \mathcal{X}_{/BG}(BG, X/\!\!/G)$, which formalizes the construction of sections of $X/\!\!/G \to BG$ from sections of $\tau_{\geq 1}(X/\!\!/G, [x]) \to BG$ described above, is not only -1-truncated, but is also -1-connected, as will be shown momentarily. Thus it is an equivalence and we can find the whole space of homotopy fixed points of the G-action on path-connected X from the space of homotopy fixed points of the corresponding action on a single $\tau_{\geq 1}(X, x)$. This will be of use when we consider group actions on the smooth space classifying principal bundles with connection. To see that the morphism $\mathcal{X}_{/BG}(BG, \tau_{\geq 1}(X/\!\!/G, [x])) \to \mathcal{X}_{/BG}(BG, X/\!\!/G)$ is -1-connected, *i.e.* essentially surjective, we note that, since X is path-connected, for any section s in $\mathcal{X}_{/BG}(BG, X/\!\!/G)$, the composite $* \to BG \xrightarrow{s} X/\!\!/G$ is homotopic to $[x]$. Thus, from the discussion above, s factors through $\tau_{\geq 1}(X/\!\!/G, [x])$, and is consequently in the essential image of $\mathcal{X}_{/BG}(BG, \tau_{\geq 1}(X/\!\!/G, [x])) \to \mathcal{X}_{/BG}(BG, X/\!\!/G)$.

2.4. *Finding group actions and homotopy fixed points*

We now discuss the practical business of finding group actions and homotopy fixed points.

We have already introduced the notions of n-truncated and n-connected morphism, for $n \geq -2$. For QFTs in spacetime dimension d and typical choices of target space,[9] the corresponding smooth space X is d-truncated and we will see in physical examples that X sometimes enjoys various connectedness properties as well. As for the extrinsic symmetry group object G, choosing BG to be n-connected amounts to insisting that the 'symmetry be at least n-form or higher', in the physics lingo, while we shall shortly show that it can be taken to be d-truncated without loss of generality. This nearly, but not quite, corresponds to the physics lore that QFTs in dimension d can have at most $(d-1)$-form symmetries. In fact, we will see that d-form symmetries do occur, but they manifest themselves in a more subtle way.

These facts about truncatedness and connectedness motivate us to study their interplay with the notions of fibre sequences and sections that we used earlier to define group actions and homotopy fixed points. As we shall see, they dovetail nicely. As ever, we shall try to emphasize the physical motivations underlying the mathematical results. All of those results are known in homotopy theory, but the fact that they generalize to any topos is sometimes new.

Let us begin by showing that when X is n-truncated, one can replace any action of a group on X by an action of the n-truncation of that group, without loss of generality, but that truncating further implies a loss of generality. This immediately implies the claim above that QFTs in spacetime dimension d can have d-form symmetries, but not $(d+1)$-form symmetries.

The first part follows immediately from the fact [34, Cor. 3.62] that the functor $\tau_{\leq n}(\mathcal{X}_{/B\tau_{\leq n}G}) \to \tau_{\leq n}(\mathcal{X}_{/BG})$ induced by pullback along $BG \to B\tau_{\leq n}G$ is an equivalence of categories. On objects this implies that, given an action of G on an n-truncated X, we have a diagram

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d] & X /\!/ G \arrow[r, two heads] \arrow[d] & \tau_{\leq n+1}(X /\!/ G) \arrow[d] \\
* \arrow[r, two heads] & BG \arrow[r, two heads] & B\tau_{\leq n}G,
\end{tikzcd}$$

where the left cartesian square is the fibre sequence corresponding to the given action and the right square is also cartesian. This tells us that any G-action can be obtained from an action by the nth truncation of G, $\tau_{\leq n}G$. Moreover, the object X^{hG} is isomorphic over X to the object $X^{h\tau_{\leq n}G}$, in the sense of Eq. (2).

That truncating further implies a loss of generality follows by observing an ordinary group (a.k.a. a 0-truncated group object in spaces) can act non-trivially on a set (a.k.a. a 0-truncated object in spaces).

So how do we recover the physics lore that one can at most have $(d-1)$-form symmetries? As we have already remarked, Simplicio is interested in a single QFT $x : * \to X$, and their study of symmetries of x amounts here to finding group actions on the 0-connected object $\tau_{\geq 1}(X, x)$. Moreover, Simplicio is interested in G-actions that admit a homotopy fixed point through x. As such, they find themself in the situation of the following theorem, which we prove in Appendix B.3: if X is both n-truncated and 0-connected, then every G-action on X equipped with a homotopy fixed point is in the essential image of the functor $(\mathcal{X}_{/B\tau_{\leq n-1}G})_* \to (\mathcal{X}_{/BG})_*$, and thus every G-action on X admitting a homotopy fixed point is in the essential image of the functor $\mathcal{X}_{/B\tau_{\leq n-1}G} \to \mathcal{X}_{/BG}$.

A weaker statement holds when X is not connected. Let $\eta : BG \to B\tau_{\leq n-1}G$ be the unit of the truncation adjunction and let $\iota : \tau_{\leq n}(\mathcal{X}_{B\tau_{\leq n-1}G}) \to \mathcal{X}_{/B\tau_{\leq n-1}G}$ be the inclusion of truncated objects. Unlike the connected case, not every G-action on X that admits a homotopy fixed point is in the essential image of η^*. Nor is $\eta^* \circ \iota$ an equivalence, as it is for $B\tau_{\leq n}G$. However, for those $X /\!/ G$ that are in the essential image of η^*, *i.e.*, of the form $\eta^*(X /\!/ \tau_{\leq n-1}G)$ (with $X /\!/ \tau_{\leq n-1}G$ corresponding to a $\tau_{\leq n-1}G$ action on X), then X^{hG} is equivalent to $X^{h\tau_{\leq n-1}G}$ over X. This follows since the counit of the adjunction

$$\begin{tikzcd}
\tau_{\leq n}(\mathcal{X}_{/B\tau_{\leq n-1}G}) \arrow[r, shift right=1.5, "\iota"'] \arrow[r, phantom, "\perp"] & \mathcal{X}_{/B\tau_{\leq n-1}G} \arrow[l, shift right=1.5, "\tau_{\leq n}"'] \arrow[r, shift right=1.5, "\eta^*"'] \arrow[r, phantom, "\perp"] & \mathcal{X}_{/BG}, \arrow[l, shift right=1.5, "\eta_!"']
\end{tikzcd}$$

is an equivalence of functors. Since both η^* and η_* preserve n-truncated objects and morphisms, they induce an adjunction

$$\begin{tikzcd}
\tau_{\leq n}(\mathcal{X}_{/B\tau_{\leq n-1}G}) \arrow[r, shift right=1.5, "\eta_*"'] \arrow[r, phantom, "\perp"] & \tau_{\leq n}(\mathcal{X}_{/BG}), \arrow[l, shift right=1.5, "\eta^*"']
\end{tikzcd}$$

in which η^* is full and faithful. Thus, the unit of this adjunction is an equivalence. The image of the components of this unit under $(*_{B\tau_{\leq n-1}G})_*$ appears in (2), manifesting the equivalence of X^{hG} and $X^{h\tau_{\leq n-1}G}$ over X in this case.

The upshot is that Simplicio sees only $(d-1)$-form symmetries, but only because of their myopic focus on individual QFTs. Once one accepts that the smooth space X of QFTs is not connected (as it rarely is), d-form symmetries can occur.

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[9] Namely, the target category should be d-discrete, meaning that it is equivalent to a category in which the k-morphisms for $k > d$ are identities. Other choices of target can lead to spaces of field theories that are not d-truncated; in these cases we expect that even-higher-form symmetries could occur.

We remark here that it does not seem reasonable to dismiss these d-form symmetries as irrelevant for physics. Indeed, starting from some QFT x in a smooth space X, one can often reach an inequivalent QFT y (which therefore belongs to a distinct connected component), by means of a smooth change in the coupling constants. (That this is so will become obvious when we describe the smooth space of quantum-mechanical theories in §4.2, but for now let us give an even simpler example: the smooth space representing a manifold M has a distinct connected component for every distinct point in M.[10]) So d-form symmetries could play a rôle in any phenomena where a smooth family or ensemble of QFTs are considered, such as Berry phases.

A related remark is that there is a significant difference between the topos of spaces and a generic topos when it comes to connectedness. In the former, every space is a coproduct (*i.e.* a disjoint union) of connected spaces. So while there certainly exist non-connected spaces, they are generated in a certain sense by connected ones. In a generic topos this is not so. (For the example of the smooth space representing a manifold M given above, the coproduct of the connected components returns not M with its original smooth structure, but the set underlying M equipped with the smooth structure corresponding to the discrete topology.)

This difference will cause headaches when we start trying to actually find group actions on smooth spaces. Since actions correspond to fibre sequences and since the Postnikov decomposition can be carried out fibrewise, for any action of a group object G on an object X we have an associated diagram

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d] & X/\!\!/G \arrow[d] \\
\vdots \arrow[r, two heads] \arrow[d] & \vdots \arrow[d] \\
\tau_{\leq m}X \arrow[r, two heads] \arrow[d] & \tau_{\leq m}X/\!\!/G \arrow[d] \\
\tau_{\leq m-1}X \arrow[r, two heads] \arrow[d] & \tau_{\leq m-1}X/\!\!/G \arrow[d] \\
\vdots \arrow[r, two heads] \arrow[d] & \vdots \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

in which all squares are cartesian. The left vertical arrows denote the Postnikov decomposition of $X \to *$, whereas the right vertical arrows denote that of $X/\!\!/G \to BG$. For the latter, we adopt the notation $\tau_{\leq m}X/\!\!/G$, indicating that $\tau_{\leq m}X/\!\!/G \to BG$ represents a G-action on $\tau_{\leq m}X$. By pasting of pullbacks, and noting that for physics the object X will be n-truncated for some n (so we may as well take G to be n-truncated as well), we see that one can find G-actions via a finite bootstrap algorithm, provided one can find the individual cartesian squares of the form

$$\begin{tikzcd}
Y \arrow[r, dashed] \arrow[d] & ? \arrow[d, dashed] \\
V \arrow[r] & W,
\end{tikzcd}$$

where the morphism $Y \to V$ comes as close as it can to being an isomorphism, in that it is both m-truncated and $m-1$-connected, for some m.

As we show in Appendix B.4, one can do this provided one makes certain technical assumptions about the objects X, $X/\!\!/G$ and the morphism $X \to X/\!\!/G$. These assumptions involve no loss of generality in the topos $\mathcal{S}$, where every object is isomorphic to a coproduct of connected objects, but they severely hamstring us in smooth spaces, because the smooth spaces X representing QFTs rarely take such a form, and even if they do we are only able to find the fibre sequences satisfying the other assumptions in this way. So we cannot generally hope to be able to find all G-actions in this way, as we did for TQFTs in [23].

Happily for Simplicio, the assumptions hold in any topos when X is connected and admits a point (as $\tau_{\geq 1}(X, x)$ surely is and does). Indeed, suppose that in the process of constructing the possible $X/\!\!/G$ via Postnikov decomposition we have succeeded in constructing the possible $\tau_{\leq m-1}X/\!\!/G$. The latter object is also connected and also admits a point, so there exists an effective epimorphism $* \twoheadrightarrow \tau_{\leq m-1}X$. To try to construct the possible $\tau_{\leq m}X/\!\!/G$, we consider the diagram

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[10] Mathematically, this is best understood that there are two distinct notions of connectedness in the topos of smooth spaces. A connected manifold in $\mathcal{S}\mathrm{m}$ is disconnected in the sense of connectedness described here, but is connected in the other sense [38].

$$\begin{tikzcd}
F_m \arrow[r, two heads] \arrow[d] & \tau_{\leq m} X \arrow[r, dashed, two heads] \arrow[d] & ? \arrow[d, dashed] \\
* \arrow[r, two heads] & \tau_{\leq m-1} X \arrow[r, two heads] & \tau_{\leq m-1} X /\!\!/ G,
\end{tikzcd}$$

in which the left square is cartesian and in which we wish to find the possible right-hand cartesian squares (for each of which the entry indicated by '?' corresponds to a cromulent $\tau_{\leq m} X /\!\!/ G$). The reverse pasting lemma, [1, Lem. 3.7.3], shows that finding the right-hand cartesian squares is equivalent to finding the cartesian rectangles, which have the special form of a fibre sequence and so are hopefully easier to find.[11]

We remark for later purposes that an m-truncated and $(m-1)$-connected object such as F_m with $m \geq 1$ is called an m-gerbe in [29], or an Eilenberg-Mac Lane object of degree m if it is pointed. Since any point in a connected object is homotopic to any other in smooth spaces, it will do us little harm to blur the distinction between the two.

We now discuss the business of finding the homotopy fixed points corresponding to a given group action. Again, we cannot expect this to be an easy endeavor in a general topos and will have to rely on applying *ad hoc* results to favorable cases. Our main useful result exploits the notions of the m-orbit and m-stabilizer of a given point $x : * \to X$ of X that we introduced in §2.3 and simplifies the finding of the homotopy fixed points through x. Indeed, we saw in §2.3 that when BG is $(m-1)$-connected, any such section factorizes as $s : BG \to \tau_{\geq m}(X /\!\!/ G, [x]) \to X /\!\!/ G$ (and furthermore this pulls back along $X \to X /\!\!/ G$ to $x : * \to \tau_{\geq m}(X, x) \to X$). Thus, instead of finding the sections of $X /\!\!/ G \to BG$ over x, we can attack the problem of finding sections of $\tau_{\geq m}(X /\!\!/ G, [x]) \to BG$. Since the fibre of $\tau_{\geq m}(X /\!\!/ G, [x]) \to BG$, *viz.* $\tau_{\geq m}(X, x)$, is $(m-1)$-connected, finding its sections is usually easier. We still have to find the morphism $\tau_{\geq m}(X /\!\!/ G, [x]) \to X /\!\!/ G$, of course. We stress that this trick always works for $m = 1$, so is of general applicability.

To introduce our final box of tricks, we recall that we have already seen that finding fibre sequences becomes simpler when X is connected and admits a point. We close this section by discussing further simplifications that occur in finding both fibre sequences and sections when X is k-connected, for large enough k.

To discuss this will take us into the topos-theoretic version of stable homotopy theory. This is most conveniently done in the category $\mathcal{X}_*$ of pointed objects. Since for physics we are interested in X admitting a point (*i.e.* a QFT), and since X will be connected here all points are homotopic to one another, there is no loss of generality in doing so.

We have already described the looping operation Ω in §2.4 on a pointed object as the pullback of the point along itself. We now regard this as a functor $\mathcal{X}_* \to \mathcal{X}_*$. It is right-adjoint to the suspension functor $\Sigma : \mathcal{X}_* \to \mathcal{X}_*$, which on an object in $\mathcal{X}_*$ is given by the pushout along the terminal morphism. When X is pointed and k-connected, the unit $X \to \Omega\Sigma X$ (which is defined on objects by the universality of the pullback defining the looping) is $2k$-connected as a morphism in $\mathcal{X}$ (this generalizes the Blakers-Massey theorem to any topos [1]), as is the counit $\Sigma\Omega X \to X$.

Letting $\mathcal{X}_*^{n,k}$ denote the full subcategory of $\mathcal{X}_*$ on the pointed, n-truncated and k-connected objects of $\mathcal{X}$, it follows that we induce an adjunction $\tau_{\leq n+1}\Sigma : \mathcal{X}_*^{n,k} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{X}_*^{n+1,k+1} : \Omega$. We claim that the unit and counit of this adjunction are n-truncated and $2k$-connected as morphisms in $\mathcal{X}$. In particular, the adjunction defines an equivalence of categories for $2k \geq n$ (see [30, Thm. 5.1.2] for a similar result).

To see that this is so, observe that the unit, considered as a morphism in $\mathcal{X}$, is a morphism of n-truncated objects, so is itself n-truncated. It factors, moreover, as $X \to \Omega\Sigma X \to \Omega\tau_{\leq n+1}\Sigma X$. where we have seen that the first morphism is $2k$-connected and where the second morphism is n-connected as discussed in §2.4 of [3]. So either $2k < n$ and the unit is n-truncated and manifestly $2k$-connected, or $2k \geq n$ and the unit is n-truncated and n-connected, in which case it is an isomorphism and so $2k$-connected as well. It is, therefore, also an isomorphism in $\mathcal{X}_*$. A similar argument holds for the counit.

The power of this result, for our purposes, is that when X is sufficiently connected, we can find the group actions on it and sections thereof from the group actions and sections of either its looping or its suspension and that this process can be iterated provided we remain in the stable range $2k \geq n$. To be explicit, let us discuss the most favorable case where X is an Eilenberg-Mac Lane object of degree n in $\mathcal{X}$, meaning a pointed, n-truncated and $(n-1)$-connected object in $\mathcal{X}$.

We have already seen in §2.2 that a group action of G on any X with a homotopy fixed point corresponds to an object $BG \to X /\!\!/ G$ in $(\mathcal{X}_{/BG})_*$. It will come as no great surprise that when X is an Eilenberg-Mac Lane object of degree n in $\mathcal{X}$, then $BG \to X /\!\!/ G$ is itself an Eilenberg-Mac Lane object in $\mathcal{X}_{/BG}$. Indeed, the morphism $X \to *$ in $\mathcal{X}$ is n-truncated and $(n-1)$-connected, so the same is true of any morphism $X /\!\!/ G \to BG$ defining a group action via §2.1. But this means that $X /\!\!/ G \to BG$ is an n-truncated and $(n-1)$-connected object in $\mathcal{X}_{/BG}$ and thus, as above, the section $BG \to X /\!\!/ G$ defines an Eilenberg-Mac Lane object in $\mathcal{X}_{/BG}$.

Letting $\Omega_{BG} : (\mathcal{X}_{/BG})_* \to (\mathcal{X}_{/BG})_*$ and $\Sigma_{BG} : (\mathcal{X}_{/BG})_* \to (\mathcal{X}_{/BG})_*$ denote looping and suspension in the topos $\mathcal{X}_{/BG}$, then since b_G^* preserves colimits and limits we have that $b_G^*\Omega_{BG}(BG \to X /\!\!/ G)$ is equivalent to $\Omega(* \to X)$, where $(* \to X) \simeq b_G^*(BG \to X /\!\!/ G)$ is an object in $\mathcal{X}_*$, and likewise $b_G^*\tau_{\leq n+1}\Sigma_{BG}(BG \to X /\!\!/ G)$ is equivalent to $\tau_{\leq n+1}\Sigma(* \to X)$. So looping sends a section of an action on X to a section of an action on ΩX, &c.

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[11] In particular, they are classified by the space $\mathcal{X}(\tau_{\leq m-1} X /\!\!/ G, B\mathrm{Aut}\, F_m)$ [33], which may be interpreted [33] as degree-one cohomology of $\tau_{\leq m-1} X /\!\!/ G$ with local coefficients in $\mathrm{Aut}\, F_m$.

3. The topos of smooth spaces

According to the geometric version of the cobordism hypothesis [22], QFTs assemble themselves into objects in the topos $\mathcal{S}\mathrm{m}$ of smooth spaces, given by sheaves on the site $\mathcal{C}\mathrm{art}$ whose objects are nonnegative integers, with p representing $\mathbb{R}^p$, whose morphisms are smooth maps $\phi : \mathbb{R}^p \to \mathbb{R}^q$, and whose covers are given by good open covers (meaning a cover for which every finite non-empty intersection is diffeomorphic to some $\mathbb{R}^p$).[12] A sheaf here is a presheaf, that is a functor $\mathcal{C}\mathrm{art}^{op} \to \mathcal{S}$ satisfying descent (See Defn. 6.2.2.6 of [29]).[13]

The data of an object in $\mathcal{S}\mathrm{m}$ are specified by a map $\mathcal{C}\mathrm{art}^{op} \to \mathcal{S}$ of simplicial sets, so consist of: a space for each p, a morphism of spaces for each smooth map $\phi : \mathbb{R}^p \to \mathbb{R}^q$, a 2-morphism of spaces for each pair $(\phi : \mathbb{R}^p \to \mathbb{R}^q, \phi' : \mathbb{R}^q \to \mathbb{R}^r)$ of composable smooth maps, and so on.

The terminal object $*$ in $\mathcal{S}\mathrm{m}$ corresponds to the locally constant sheaf with value $*$. Perhaps the next simplest objects in $\mathcal{S}\mathrm{m}$ correspond to smooth manifolds. Given a smooth manifold M,[14] we define a corresponding smooth space M by assigning: to p, the 0-truncated space corresponding to the set $\mathrm{Map}(\mathbb{R}^p, \mathsf{M})$ of smooth maps from $\mathbb{R}^p$ to M; to ϕ, the morphism of spaces corresponding to the map of sets $\mathrm{Map}(\mathbb{R}^q, \mathsf{M}) \to \mathrm{Map}(\mathbb{R}^p, \mathsf{M}) : \alpha \mapsto \alpha \circ \phi$; and so on. The smooth space M is 0-truncated, but not 0-connected (unless M is a point): the first follows since a sheaf is truncated precisely when its underlying presheaf is and truncatedness of presheaves can be tested objectwise, and the second then follows since $\tau_{\leq 0} M = M \neq *$ if M is not a point.

The data of a morphism in $\mathcal{S}\mathrm{m}$ are specified by giving a map $\Delta^1 \times \mathcal{C}\mathrm{art}^{op} \to \mathcal{S}$ of simplicial sets. Given two manifolds M and N and a smooth map $\mathsf{f} : \mathsf{M} \to \mathsf{N}$, there is a morphism $f : M \to N$ that restricts to M and N, respectively, on the two non-degenerate 0-simplices $\{0\}, \{1\} \in \Delta^1_0$ and that sends the 1-simplex $(\{0,1\}, \phi) \in \Delta^1_1 \times \mathcal{C}\mathrm{art}^{op}_1$ to $\alpha \mapsto \mathsf{f} \circ \alpha \circ \phi$. Every morphism from M to N is homotopic to one of this form and in fact something stronger is true (*cf.* Example 1.3.32 of [38]): the full subcategory of $\mathcal{S}\mathrm{m}$ on such objects is equivalent to the category given by the nerve of the familiar 1-category of manifolds and smooth maps.

Just like any other topos, $\mathcal{S}\mathrm{m}$ has group objects and these are equivalent via delooping to pointed connected smooth spaces. Perhaps the most basic group object corresponds to a Lie group G. We define the associated delooping BG by assigning: to p, the 1-truncated space corresponding to the 1-groupoid with a single object whose morphisms are smooth maps $\mathbb{R}^p \to \mathsf{G}$, composed by multiplying pointwise in G; to ϕ, the morphism of spaces that on morphisms in the aforementioned 1-groupoid is given by $\mathrm{Map}(\mathbb{R}^q, \mathsf{G}) \to \mathrm{Map}(\mathbb{R}^p, \mathsf{G}) : f \mapsto f \circ \phi$; and so on. The smooth space BG is pointed in an obvious way and the fact that it is 0-connected and 1-truncated follows from an objectwise observation. The underlying object G is obtained via the pullback

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & BG,
\end{tikzcd}$$

and an objectwise computation shows that it corresponds to the manifold underlying the Lie group G.[15]

We have defined BG as the delooping of the group object G, but it is useful to see explicitly that it is sensible to regard it as a smooth space analogue of the usual classifying space of smooth principal G-bundles. Indeed, if it does play such a rôle, then we expect the G-bundles over any smooth space X to be classified (as a space) by $\mathcal{X}(X, BG)$. But by the Yoneda lemma, when $X = \mathbb{R}^p$, $\mathcal{X}(X, BG)$ is just the space assigned by BG to the object $\mathbb{R}^p$ on the site, which when G corresponds to G is the nerve of the 1-groupoid with a single object and (iso)morphisms given by smooth maps from $\mathbb{R}^p$ to G. But this 1-groupoid is equivalent to the 1-groupoid whose objects are principal G-bundles on $\mathbb{R}^p$ and whose (iso)morphisms are the usual morphisms of principal G-bundles: every such bundle is trivializable and the morphisms of the trivial bundle are in bijective correspondence with smooth maps $\mathbb{R}^p \to \mathsf{G}$. So indeed BG plays the rôle of a classifying space, at least for G-bundles on $\mathbb{R}^p$.

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[12] The topos $\mathcal{S}\mathrm{m}$ is, in fact, equivalent to the topos of sheaves on the site of manifolds with open covers [7].

[13] We remark that, as in 1-categories, limits in sheaves can be computed in presheaves, which in turn can be computed objectwise.

[14] We remind the reader that we use sans serif typeface for classical notions in differential geometry and serif typeface for the corresponding notions in $\mathcal{S}\mathrm{m}$.

[15] In more detail, this follows from the observation that for any 0-truncated group K in spaces we can work in the 2-category whose objects are 1-groupoids, whose 1-morphisms are 1-functors, and whose 2-morphisms are 1-natural transformations. We then have a commuting square

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & BK,
\end{tikzcd}$$

in which K is the 1-groupoid whose objects are elements of K and whose morphisms are identities and BK is the 1-groupoid with one object whose morphisms are elements of K. The filling 2-simplices here are both given by the 1-natural transformation from the unique 1-functor $K \to BK$ to itself whose component on the object in K given by $k \in K$ is the morphism in BK given by k. Since any cone from any space X into $* \to BK \leftarrow *$ is specified by a map of sets from $\tau_{\leq 0} A$ to K, this square is a universal cone.

For later purposes, it will be convenient to describe both the pointed and unpointed versions of the full subcategories on the smooth spaces corresponding to deloopings of Lie groups. In the pointed case, we have that the full subcategory of $\mathcal{S}\mathrm{m}_*$ is equivalent to the nerve of the usual 1-category of Lie groups and smooth homomorphisms. In the unpointed case, the full subcategory of $\mathcal{S}\mathrm{m}$ (which for later purposes we denote $\mathcal{L}$) is equivalent to the category in which a 0-simplex is a Lie group G, a 1-simplex is a smooth homomorphism $\mathsf{f} : \mathsf{G} \to \mathsf{H}$, and a 2-simplex

$$\begin{tikzcd}
& \mathsf{H} \arrow[dr, "\mathsf{f}'"] & \\
\mathsf{G} \arrow[ur, "\mathsf{f}"] \arrow[rr, "\mathsf{f}''"'] & \mathsf{k} & \mathsf{K},
\end{tikzcd}$$

is an element $\mathsf{k} \in \mathsf{K}$ such that $\mathsf{f}'' = \mathsf{k}(\mathsf{f}' \circ \mathsf{f})\mathsf{k}^{-1}$. Given 2-simplices specified by $\mathsf{k} \in \mathsf{K}$ and $\mathsf{l}_{1,2,3} \in \mathsf{L}$ forming a tetrahedron

$$\begin{tikzcd}
& \mathsf{K} \arrow[ddr, bend left, "\mathsf{f}"] \arrow[ddr, Rightarrow, "\mathsf{l}_2"] \arrow[dd, Rightarrow, bend left, "\mathsf{l}_3"] & \\
& \mathsf{H} \arrow[u] \arrow[dr] \arrow[l, Rightarrow, "\mathsf{k}"'] \arrow[d, Rightarrow, "\mathsf{l}_1"] & \\
\mathsf{G} \arrow[uur, bend left] \arrow[ur] \arrow[rr, bend right] & & \mathsf{L},
\end{tikzcd}$$

we have a 3-simplex if and only if $\mathsf{l}_1\mathsf{l}_2 = \mathsf{l}_3\mathsf{f}(\mathsf{k})$, in which case the 3-simplex is unique.

The notions of manifold and Lie group are subsumed in the notion of a Lie groupoid L [16], namely a 1-groupoid in which both the objects and morphisms form manifolds, which we denote L_0 and L_1, respectively, such that the source s and target t maps are smooth surjective submersions and the multiplication m, the unit u and inverse i are all smooth maps.[16] We define a corresponding object L in $\mathcal{S}\mathrm{m}$ by assigning: to p, the 1-truncated space corresponding to the 1-groupoid $\mathrm{Map}(\mathbb{R}^p, \mathsf{L}_1) \rightrightarrows \mathrm{Map}(\mathbb{R}^p, \mathsf{L}_0)$ (where the source, target, multiplication, unit, and inversion are induced by $\mathsf{s}, \mathsf{t}, \mathsf{m}, \mathsf{u}$, and i in the obvious way), and so on.

The simplest example of a Lie groupoid that is neither a manifold nor a Lie group arises when we act with a Lie group on a manifold and will be described in the next Subsection.

A description of the full subcategory of $\mathcal{S}\mathrm{m}$ on smooth spaces corresponding to Lie groupoids is somewhat complicated and not needed in what follows. Thus, we content ourselves with a description of how to construct *some* morphisms in $\mathcal{S}\mathrm{m}$ between Lie groupoids. In particular, given a 1-functor $\mathsf{f} : \mathsf{L} \to \mathsf{J}$ inducing smooth maps, f_0 and f_1, on the manifolds of objects and morphisms, respectively, we construct a 1-morphism $f : L \to J$ in $\mathcal{S}\mathrm{m}$ as follows. For each p, f assigns to the morphism $(0 \to 1, \mathrm{id}_{\mathbb{R}^p})$ in $\Delta^1 \times \mathcal{C}\mathrm{art}$ the functor $L(\mathbb{R}^p) \to J(\mathbb{R}^p)$ sending the object $\mathsf{h} \in \mathrm{Map}(\mathbb{R}^p, \mathsf{L}_0)$ of $L(\mathbb{R}^p)$ to the object $\mathsf{f}_0 \circ \mathsf{h} \in \mathrm{Map}(\mathbb{R}^p, \mathsf{J}_0)$ of $J(\mathbb{R}^p)$ and the morphism $\mathsf{r} \in \mathrm{Map}(\mathbb{R}^p, \mathsf{L}_1)$ of $L(\mathbb{R}^p)$ to the morphism $\mathsf{f}_1 \circ \mathsf{r} \in \mathrm{Map}(\mathbb{R}^p, \mathsf{J}_1)$ of $J(\mathbb{R}^p)$. The remaining data of f are taken such that the homotopies involved (which in this case are natural transformations of functors) are in fact strict identities. Every morphism in between objects in $\mathcal{S}\mathrm{m}$ corresponding to manifolds is homotopic to such a morphism, and likewise for morphisms between objects corresponding to Lie groups. However, not every morphism in $\mathcal{S}\mathrm{m}$ from an object corresponding to a manifold to an object corresponding to a Lie group can be constructed in this way up to homotopy.

Finally, we wish to describe two other smooth spaces constructed from a Lie group G that will play an important rôle in examples. These smooth spaces, which we will denote $B_\nabla G$ and $G\mathrm{Conn}(M)$ for a manifold M, are close relatives of the smooth space BG that classifies smooth principal G-bundles on a smooth space M corresponding to a manifold.

The first, $B_\nabla G$, classifies smooth principal G-bundles with the added data of a connection. For discussions in the literature of this and related objects, see [19,38,39]. Generalizing our discussion of principal bundles without connection above, we define $B_\nabla G$ to be the sheaf that assigns to p the nerve of the 1-groupoid whose objects are smooth principal G-bundles with connection on $\mathbb{R}^p$ and whose morphisms are connection-preserving morphisms of principal G-bundles. (On $\phi : \mathbb{R}^p \to \mathbb{R}^q$, we get the usual pullbacks of the bundles and connections.) Equivalently, since any such bundle on $\mathbb{R}^p$ is trivializable and since a connection on the trivial bundle on $\mathbb{R}^p$ is given by a $\mathfrak{g}$-valued 1-form A on $\mathbb{R}^p$, we get the 1-groupoid in which an object is an A and an isomorphism from A to A' is a smooth map $\mathsf{g} : \mathbb{R}^p \to \mathsf{G}$ such that $\mathsf{A}' = \mathsf{g}^{-1}\mathsf{A}\mathsf{g} + \mathsf{g}^{-1}d\mathsf{g}$.

The smooth space $B_\nabla G$ has some curious properties, which will be relevant for our discussion of the symmetries of QFTs. Since its value on p is the nerve of a 1-groupoid, $B_\nabla G$ is 1-truncated, just like BG. Unlike BG, however, it is not connected, so it cannot be the delooping of a group object in $\mathcal{S}\mathrm{m}$. This follows from the fact that there is more than one equivalence class (under bundle morphisms) of connections on any $\mathbb{R}^p$ for $p > 0$. Nevertheless, like any connected smooth

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[16] We recover a manifold when $\mathsf{L}_1 = \mathsf{L}_0$ and a Lie group when L_0 is a point.

space (such as BG), $B_\nabla G$ admits a point $* \to B_\nabla G$ and moreover any one point is homotopic to any other one, so $B_\nabla G$ is path-connected. Indeed, via the Yoneda lemma, the space of points $\mathcal{S}\mathrm{m}(*, B_\nabla G) \in \mathcal{S}$ is given by the value of $B_\nabla G$ on the site object $\mathbb{R}^0$, and is connected, since every bundle over a point is trivializable and the trivial bundle over a point has a unique connection.

To give a description of the connected component of $B_\nabla G$ at any point, we need to introduce the notion of the discretization X^δ of a smooth space X. The functor $\Gamma : \mathcal{S}\mathrm{m} \to \mathcal{S}$ that sends a smooth space X to its space $\mathcal{S}\mathrm{m}(*, X)$ of points has a left adjoint, Disc, that sends a space S to the smooth space given by the locally constant sheaf with value S. We define $X^\delta = \mathrm{Disc}\,\Gamma X$ and note that the counit of the adjunction gives us a natural morphism $X^\delta \to X$. Because both Disc and Γ preserve finite products this construction extends naturally to group objects in $\mathcal{S}\mathrm{m}$ [8, Prop. 3.5]. Furthermore, since Disc and Γ also preserve colimits (they are both left-adjoints), they commute with delooping B [8, Prop. 3.5]. When G is a group object in smooth spaces corresponding to a Lie group G, we have that G^δ corresponds to the discrete group, which we denote G^δ, obtained by replacing the given smooth structure on G with the discrete one.

We now show that the connected component of $B_\nabla G$ at any point is equivalent to the smooth space BG^δ, which can be thought of as classifying principal G-bundles with flat connection. Since BG^δ is the delooping of a group object, $* \to BG^\delta$ is (-1)-connected. There is a canonical morphism $BG^\delta \to B_\nabla G$ that for each p is the inclusion of the 1-groupoid {principal G-bundles on $\mathbb{R}^p$ with flat connection} into the 1-groupoid {principal G-bundles on $\mathbb{R}^p$ with connection}. This is an inclusion of a path-connected component in spaces, so is a (-1)-truncated morphism. Truncatedness may be tested objectwise, so $BG^\delta \to B_\nabla G$ is -1-truncated in $\mathcal{S}\mathrm{m}$. Since any point $* \to B_\nabla G$ is unique up to homotopy, it must factor through BG^δ, exhibiting $* \to BG^\delta \to B_\nabla G$ as the $(-1$-connected,-1-truncated) factorization of the point. Thus BG^δ is indeed the connected component. There is, of course, also a canonical morphism $B_\nabla G \to BG$ that forgets the connection.

The second smooth space $G\mathrm{Conn}(M)$ [18] classifies smooth families of principal bundles over M. We define it analogously to $B_\nabla G$, except that we assign to p the (nerve of the) 1-groupoid whose objects are G-principal bundles on $\mathsf{M} \times \mathbb{R}^p$ together with a fibrewise connection with respect to the projection $\mathsf{M} \times \mathbb{R}^p \to \mathbb{R}^p$.

The space $\mathcal{S}\mathrm{m}(*, G\mathrm{Conn}(M))$ is equivalent to the space of G-principal bundles on M, or $\mathcal{S}\mathrm{m}(M, B_\nabla G)$ if one wishes. That is, $G\mathrm{Conn}(M)$ and $(B_\nabla G)^M$ are equivalent as spaces $(\Gamma(G\mathrm{Conn}(M)) \simeq \Gamma(B_\nabla G)^M)$ (but not as smooth spaces). The connected component of $G\mathrm{Conn}(M)$ at each point corresponds to the smooth family of fibrewise connection preserving gauge transformations of that bundle. As an example, when M is taken as the point $*$, we recover $G\mathrm{Conn}(*) \simeq BG$.

3.1. *Group actions in smooth spaces*

3.1.1. *Lie groups acting on manifolds*

Having defined the smooth spaces corresponding to the simplest kinds of (group) objects $\mathcal{S}\mathrm{m}$, namely manifolds and Lie groups, we can now define the simplest kind of group action in $\mathcal{S}\mathrm{m}$, which corresponds to a smooth action of a Lie group on a manifold. Namely, given a Lie group G acting on a smooth manifold M, we define the action Lie groupoid $\mathsf{M}/\!\!/\mathsf{G}$ by taking $(\mathsf{M}/\!\!/\mathsf{G})_0 = \mathsf{M}$, $(\mathsf{M}/\!\!/\mathsf{G})_1 = \mathsf{M} \times \mathsf{G}$, s to be the projection on the left-hand factor, t to be the group action, and defining m, &c, using the corresponding maps in G. Denoting the corresponding smooth space by $M/\!\!/G$, we have a morphism $M/\!\!/G \to BG$ in $\mathcal{S}\mathrm{m}$, induced by the morphism of Lie groupoids that on $(\mathsf{M}/\!\!/\mathsf{G})_1 = \mathsf{M} \times \mathsf{G}$ is projection on the right-hand factor, whose fibre over the basepoint is M, exhibiting a G-action on M in $\mathcal{S}\mathrm{m}$.

The observation that a Lie group action on a manifold defines a group action in $\mathcal{S}\mathrm{m}$ can be elevated into the following equivalence of categories [38, Prop. 6.4.44]. Let G be a Lie group and BG the corresponding delooped object in $\mathcal{S}\mathrm{m}$. Then the full subcategory of $\mathcal{S}\mathrm{m}_{/BG}$ on objects whose fibre is a smooth space corresponding to a manifold is equivalent to the nerve of the usual 1-category of G-manifolds.

So every G-action on M in $\mathcal{S}\mathrm{m}$ is equivalent to a G-action on the corresponding M. The associated object M^{hG} and the morphism $M^{hG} \to M$ realize the fixed point subset of the G-action on M as a smooth space along with its inclusion in M. The points of M^{hG} are given by $\mathcal{S}\mathrm{m}_{/BG}(BG, M/\!\!/G)$. But by the equivalence of categories given above, this is the set of equivariant maps $* \to \mathsf{M}$, with the trivial G-action on the point manifold $*$, giving us the expected fixed points.

3.1.2. *Lie groups acting on deloopings of Lie groups*

The next simplest action to consider is that of a G which corresponds to a Lie group G acting on BK, the delooping of another Lie group K. As we will now see, such an action, which will play a prominent rôle in our examples, corresponds to a short exact sequence of Lie groups.

To show it, recall that such an action corresponds to a cartesian square

$$\begin{tikzcd}
BK \arrow[r, two heads] \arrow[d] & E \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

for some smooth space E. Let us begin by showing that E is 0-connected, 1-truncated, and admits a point, so that we can legitimately call it BH for some 0-truncated group object H. The morphism $BK \to *$ is 0-connected and 1-truncated and therefore so is $E \to BG$. Now BG is also 0-connected and 1-truncated, so E is 0-connected (from [29, Prop. 6.5.1.16]) and 1-truncated too (from [29, Lem. 5.5.6.14]). The smooth space E certainly admits a point, since BK does, and thus is equivalent to a BH for some 0-truncated group H.

We next want to show that the 0-truncated group H is equivalent to a Lie group. Looping, we get a cartesian square

$$\begin{tikzcd}
H \arrow[r, two heads, "p"] \arrow[d] & G \arrow[d] \\
* \arrow[r, two heads] & BK,
\end{tikzcd}$$

in $\mathcal{S}\mathrm{m}$ exhibiting $p : H \to G$ as an principal K-bundle (of smooth spaces), as defined in [33]. Then [38, Prop. 6.4.39] tells us that H is equivalent as an object in $\mathcal{S}\mathrm{m}$ to a (smooth space corresponding to a) manifold.[17] The maps corresponding to multiplication and inversion defined by the Segal conditions on a manifold correspond to smooth maps of that manifold, so H is equivalent to a group object corresponding to a Lie group H.

The group action $E \to BG$ can therefore be replaced by

$$\begin{tikzcd}
BK \arrow[r, two heads, "Bi"] \arrow[d] & BH \arrow[d, "Bp"] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

where $\mathsf{i} : \mathsf{K} \to \mathsf{H}$ and $\mathsf{p} : \mathsf{H} \to \mathsf{G}$ are smooth homomorphisms and so, in particular, have constant rank. Looping again, we get the cartesian square

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, two heads] & H \arrow[d, two heads, "p"] \\
* \arrow[r] & G,
\end{tikzcd}$$

in $\mathcal{S}\mathrm{m}$. Since smooth spaces corresponding to manifolds form a full-subcategory of $\mathcal{S}\mathrm{m}$, it follows that

$$\begin{tikzcd}
\mathsf{K} \arrow[r, two heads] \arrow[d, "\mathsf{i}"'] & * \arrow[d] \\
\mathsf{H} \arrow[r, two heads, "\mathsf{p}"'] & \mathsf{G},
\end{tikzcd}$$

is a 1-pullback in the 1-category of manifolds. The smooth map p is surjective and therefore, by the Global Rank Theorem of [27], also a submersion. Furthermore, the image of i, *viz.* $\ker \mathsf{p}$, is embedded in H, so the restriction of i to it is both smooth [27] and bijective, *ergo* (again by the Global Rank Theorem) a diffeomorphism.

It follows that every action of G on BK corresponds to a Lie group extension

$$\mathsf{K} \xrightarrow{\mathsf{i}} \mathsf{H} \xrightarrow{\mathsf{p}} \mathsf{G},$$

as defined, *e.g.*, in [13]. (Equivalently, we have that $\mathsf{p} : \mathsf{H} \to \mathsf{G}$ is a smooth principal K-bundle in the usual sense with respect to the action of K on H induced via i and multiplication in H.) Again, this lifts to an equivalence between the full subcategory of $\mathcal{S}\mathrm{m}_{/BG}$ on objects whose fibre is a smooth space corresponding to the delooping of a Lie group and the full subcategory of $\mathcal{L}_{/\mathsf{G}}$ on those objects for which the smooth homomorphism $\mathsf{p} : \mathsf{H} \to \mathsf{G}$ is surjective (and, *ergo*, a submersion). Let us denote this last category as $\mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}}$.

Unsurprisingly, the homotopy fixed points of such actions correspond to sections of the surjection (or equivalently smooth splittings of the smooth short exact sequences). To be precise, the category $(\mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}})_*$ is equivalent to the nerve of the 1-category whose objects are pairs of Lie group homomorphisms $(\mathsf{p} : \mathsf{H} \to \mathsf{G}, \mathsf{s} : \mathsf{G} \to \mathsf{H})$ such that $\mathsf{p} \circ \mathsf{s}$ is the identity homomorphism on G, and whose morphisms from $(\mathsf{p} : \mathsf{H} \to \mathsf{G}, \mathsf{s} : \mathsf{G} \to \mathsf{H})$ to $(\mathsf{p}' : \mathsf{H}' \to \mathsf{G}, \mathsf{s}' : \mathsf{G} \to \mathsf{H}')$ are smooth homomorphisms $\mathsf{q} : \mathsf{H} \to \mathsf{H}'$ such that $\mathsf{p}' \circ \mathsf{q} = \mathsf{p}$ and $\mathsf{s}' = \mathsf{q} \circ \mathsf{s}$. The forgetful functor $(\mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}})_* \to \mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}}$ sends (p, s) to p and sends q to the 2-simplex whose edges are q, p and p' and whose filler is the identity of the group G.

Making the Lie group K explicit, we have that the full subcategory of $(\mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}})_*$ on objects p whose kernel is isomorphic to K is equivalent to the nerve of the 1-category whose objects are smooth homomorphisms $\mathsf{e} : \mathsf{G} \times \mathsf{K} \to \mathsf{K}$ that are automorphisms for all $g \in \mathsf{G}$, and whose morphisms from e to e' are smooth automorphisms $\mathsf{f} : \mathsf{K} \to \mathsf{K}$ such that $\mathsf{f}(\mathsf{e}(g, k)) = \mathsf{e}'(g, \mathsf{f}(k))$. The equivalence sends $\mathsf{s} : \mathsf{G} \to \mathsf{H}$ to $\mathsf{e} : \mathsf{G} \times \mathsf{K} \to \mathsf{K} : (g, k) \mapsto \mathsf{i}^{-1}\left(\mathsf{s}(g)\mathsf{i}(k)\mathsf{s}(g^{-1})\right)$.

The corresponding space of homotopy fixed points, *i.e.* the fibre of the forgetful functor $(\mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}})_1 \to \mathcal{L}^{\twoheadrightarrow}_{/\mathsf{G}}$, is then equivalent to the nerve of the 1-groupoid whose objects are smooth maps $\mathsf{r} : \mathsf{G} \to \mathsf{K}$ such that

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[17] In fact, [38, Prop. 6.4.39] tells us more, namely that $p : H \to G$ is equivalent to a smooth principal K-bundle in the usual sense, which allows for an alternative proof of what follows here.

$$\mathsf{r}(\mathsf{g}_2)\mathsf{e}(\mathsf{g}_2, \mathsf{r}(\mathsf{g}_1)) = \mathsf{r}(\mathsf{g}_2\mathsf{g}_1), \tag{3}$$

for all $\mathsf{g}_1, \mathsf{g}_2 \in \mathsf{G}$, and whose morphisms are elements $\mathsf{k} \in \mathsf{K}$ such that

$$\mathsf{r}'(\mathsf{g}) = \mathsf{k}\mathsf{r}(\mathsf{g})\mathsf{e}(\mathsf{g}, \mathsf{k}^{-1}),$$

for all $\mathsf{g} \in \mathsf{G}$.

In some of our examples, the Lie group K will be connected. This enables us to give its group of automorphisms the structure of a Lie group,[18] $\mathrm{Aut}\,\mathsf{K}$, such that we can equivalently view e as a smooth homomorphism $\mathsf{G} \to \mathrm{Aut}\,\mathsf{K}$ and f as a conjugation in $\mathrm{Aut}\,\mathsf{K}$. This construction goes as follows. The derivative defines a functor from the 1-category of Lie groups to the 1-category of Lie algebras. This functor induces a homomorphism from the group of automorphisms of any Lie group K to those of its Lie algebra $\mathfrak{k}$. This map is a bijection if the Lie group K is simply-connected. It follows that the map is an injection if the Lie group K is merely connected, since any such K is a quotient of its simply-connected universal cover $\tilde{\mathsf{K}}$ by some discrete subgroup Z of its center. (To be explicit, the map lands on the automorphisms of $\tilde{\mathsf{K}}$ that induce automorphisms on Z, or equivalently that map Z onto itself.) Now, giving the group of automorphisms of K the compact-open topology makes it into a closed subgroup of the group of automorphisms of $\mathfrak{k}$ which is itself a closed subgroup of the general linear group on $\mathfrak{k}$, with its usual Lie group structure. So we have a Lie group $\mathrm{Aut}\,\mathfrak{k}$ and a Lie subgroup $\mathrm{Aut}\,\mathsf{K}$ thereof.

This construction also gives us a means to give an explicit description of $\mathrm{Aut}\,\mathsf{K}$, provided we can do the same for $\mathrm{Aut}\,\mathfrak{k}$. The latter task is complicated by the fact that the field $\mathbb{R}$ over which we are working is not algebraically closed. These difficulties can be circumvented if K is compact as well as connected, and a description of $\mathrm{Aut}\,\mathsf{K}$ in that case in terms of root diagrams can be found in *e.g.* §IX.4.10 of [5].

In other examples, K will be a discrete Lie group. Then we can replace G by its group of path-components, since the smooth maps e and r both have targets that are discrete, so factor through $\pi_0\mathsf{G}$. Thus the analysis of actions on BK and their homotopy fixed points collapses to the analysis of short exact sequences of groups and their splittings, with G replaced by $\pi_0\mathsf{G}$.

With these results in hand, we are ready to consider some examples of group actions on smooth spaces corresponding to QFTs.

3.1.3. *Lie groups acting on non-connected smooth spaces*

In the examples, we will frequently encounter examples of smooth spaces X that are not connected. It is useful to relate group actions on these to actions on related connected smooth spaces. As shown in §2.3, one useful result here is that every homotopy fixed point through x of a group action on X descends to a homotopy fixed point of a corresponding group action on the connected component $\tau_{\geq 1}(X, x)$. Assuming one can find the latter actions and homotopy fixed points, the remaining difficulties are, firstly, that one needs to consider all x, and, secondly, that one needs to determine which group actions on $\tau_{\geq 1}(X, x)$ extend to the whole of X (note that the homotopy fixed points then necessarily extend).

We are unable to surmount these difficulties for general X, but we can make some progress in special cases. One in particular is where $X = B_\nabla K$ for some Lie group K. We discuss this example further in Appendix A.1.

3.2. *Comparison with the topological case*

It is worth pausing here to compare with what happens in the case of topological quantum field theories, discussed in [23].

The two cases are similar in that, in both cases, a symmetry of a QFT is a homotopy fixed point of a group object acting on some object in an underlying topos.

The only difference is that, in the topological case, the underlying topos is that of spaces, $\mathcal{S}$, whilst in the smooth case we have the topos $\mathcal{S}\mathrm{m}$ of smooth spaces.

So how do the topoi $\mathcal{S}\mathrm{m}$ and $\mathcal{S}$ differ? Given any topos $\mathcal{X}$, as introduced above, there is a functor Γ to $\mathcal{S}$, defined by evaluating $\mathcal{X}(*, -)$. This functor forms the left half of an adjunction (whose right adjoint produces 'discrete' objects in $\mathcal{X}$) and the failure of this adjunction to be an equivalence allows us to characterize, in a canonical way, the difference between $\mathcal{X}$ and $\mathcal{S}$.

Now, the topos $\mathcal{S}\mathrm{m}$ is rather special, in that this adjunction is much closer to being an equivalence than it is for a generic topos $\mathcal{X}$. To wit, the adjunction extends to a quadruple adjunction, making $\mathcal{S}\mathrm{m}$ into what is called in [38] a cohesive topos. In particular, Γ is also a right adjoint, whose left adjoint produces 'concrete' objects in $\mathcal{S}\mathrm{m}$, of which manifolds (and more generally diffeological spaces) are examples.

Via these adjunctions, one can often translate statements about smooth symmetries of QFTs into statements about symmetries of TQFTs and *vice versa*, though some care is needed. Suppose, for example, that one finds smooth G-actions on some smooth space X in $\mathcal{S}\mathrm{m}$. Any such action leads to an action in $\mathcal{S}$ of the underlying discrete group ΓG on ΓX, but not every action of ΓG on ΓX can be got in this way, since some may not act on ΓX smoothly. So even if we are considering topological quantum field theories, the physical requirement that things be smooth (*i.e.* that we work in $\mathcal{S}\mathrm{m}$), leads to a

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[18] More generally, this can be done if the group of components is finitely generated [25].

more restrictive notion of symmetry. Explicitly, as we shall see in §4.1.1, some of the symmetries of TQFTs that we found in [23] do not correspond to smooth actions.

The change from $\mathcal{S}$ in [23] to $\mathcal{S}\mathrm{m}$ here leads to another important new feature, namely that of a smoothness anomaly. To see that these cannot occur in the topological case, it is enough to observe, as we did in §2.4, that in $\mathcal{S}$ every space is a coproduct of connected components. Thus, every action (with or without a homotopy fixed point) on a connected component of a space X can be extended to an action on the whole of X by demanding that it act trivially on the other components.

4. Examples from physics

4.1. QFTs in $d = 1$ with a smooth map to a target manifold

For a first example, consider QFTs in spacetime dimension $d = 1$ in which the spacetimes in the bordism category are equipped with both an orientation and a smooth map to a manifold M and the target category is the nerve of the 1-category whose objects are complex vector spaces and whose morphisms are linear maps. These can be interpreted physically as 1-dimensional QFTs with a background field. In the invertible case, these QFTs correspond to topological actions (see *e.g.* [13]) in classical mechanics formed using only the smooth map and the orientation (which allows us to do integrals, and so obtain actions by integrating local lagrangians). An example is the coupling of a particle (which in 3-d space would correspond to choosing $\mathsf{M} = \mathbb{R}^3$) to an external magnetic field. We note however that here such actions are not required to be real (a.k.a. hermititan), since we have imposed no requirement of unitarity yet.[19]

According to [4] such QFTs correspond to smooth complex vector bundles with connection. The explicit correspondence is as follows. The 0-dimensional manifold given by a point equipped with a smooth map to M defines a point $\mathsf{m} \in \mathsf{M}$; to this we associate the fibre of the vector bundle over m. The 1-dimensional manifold with boundary given by an interval equipped with a smooth map to M defines a smooth path in M; to this we associate the parallel transport of the connection.

So what is the smooth space X representing such QFTs? The results of [4] only give us a description of the underlying space, so there is a possible ambiguity. Let us describe two likely candidates. Instead of dealing with vector bundles with connection, we can consider (for each $n \geq 0$) the principal GL_n-bundles with connection to which they are associated. These, as we have seen in §3, can be assembled either into the smooth space $(B_\nabla GL_n)^M$, where by Z^Y we denote the smooth space (*i.e.* an object in $\mathcal{S}\mathrm{m}$) of morphisms from a smooth space Y to a smooth space Z, or into the smooth space $GL_n\mathrm{Conn}(M)$. Both of these have the same underlying space, so are consistent with the results of [4]. But now consider a path in either smooth space, which ought to correspond physically to a smooth 1-parameter family of theories. A path in $(B_\nabla GL_n)^M$ is precisely a morphism $M \times [0, 1] \to B_\nabla GL_n$, *i.e.* a GL_n-principal bundle on $M \times [0, 1]$ with connection. But on physical grounds, what we want is a GL_n-principal bundle on $M \times [0, 1]$ with a notion of connection that is fibrewise over $[0, 1]$, in that it only allows parallel transport along vectors tangent to M. This corresponds to a path in $GL_n\mathrm{Conn}(M)$.

As such, we shall proceed here with the assumption that $GL_n\mathrm{Conn}(M)$ is the correct smooth space. Putting everything together, we take X to be given by $\coprod_n GL_n\mathrm{Conn}(M)$. In Appendix A.2 we discuss what would happen if we had taken $B_\nabla GL_n$, and show the presence of a smoothness anomaly in that case.

4.1.1. Smooth TQFTs in $d = 1$

As a warm up, let us consider the case where M is a point, such that $X \simeq \coprod_n BGL_n$. The general arguments in §3.1.2 thus show that, at least when the group G acting corresponds to a Lie group G, the problem reduces to finding smooth splittings of short exact sequences of Lie groups extending G by GL_n. The (splittable) short exact sequences are given by smooth homorphisms $\mathsf{G} \to \mathsf{Aut}\,\mathsf{GL}_n$ as in §3.1.2 and the corresponding homotopy fixed points are given by smooth maps $\mathsf{G} \to \mathsf{GL}_n$ as in Eq. (3).

The physics interpretation is as follows. Such a TQFT is specified by the dimension n (necessarily finite) of the state space, which is a mere vector space since we have no notion of unitarity yet. The hamiltonian is zero, as one expects for a TQFT, corresponding to the fact that the linear operator on the state space associated to time evolution along an interval is the identity operator. A symmetry of the TQFT is naïvely a linear (not unitary, since this notion makes no sense) operator on the state space that commutes with the hamiltonian, *i.e.* any linear operator. A G-symmetry of the TQFT is an assignment of a linear operator to each element in G. This assignment need not form a *bona fide* representation of G, because a symmetry need not send a TQFT to itself, but rather could send it to an equivalent one. The twisted representations as defined above give a complete description of the ways in which this can happen.

4.1.2. The general case

Now let us return to the case of general M. The smooth space $GL_n\mathrm{Conn}(M)$ is then neither connected nor path-connected, which complicates our analysis of symmetries.

To make things tractable, let us consider just a special class of group actions on $GL_n\mathrm{Conn}(M)$, namely those where a group object G corresponding to a Lie group G acts on $GL_n\mathrm{Conn}(M)$ via the usual notion of a smooth action of G on M.

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[19] We will do when we consider quantum mechanics in the next Subsection.

The induced action on $GL_n\mathrm{Conn}(M)$ is given by pulling smooth families of bundles back along a smooth family of elements in G. Doing so allows us to make contact with the way in which physicists usually study symmetries of QFTs, namely by considering transformations of fields leaving some classical action (or strictly speaking, the exponentiated action) invariant.

So let us compare with what physicists do. To begin with, we ignore symmetry considerations. Since $d = 1$, the physics here is that of a particle moving on the smooth manifold M. In the case where $n = 1$ (*i.e.* invertible QFTs), evaluating a QFT on the closed spacetime manifold S^1 gives a complex number, which represents a possible contribution to the exponentiated action governing the dynamics. Since this action depends only on the orientation and the smooth map to M with which S^1 is equipped, a physicist calls this a 'topological' action term.

Now, it has long been known (for a recent discussion, see [13]) that in any spacetime dimension d, a source of such action terms is given by the differential cohomology [9] in degree $d + 1$ of the manifold M, which refines the ordinary integral cohomology of M with information about the differential forms on M. (Here, since no requirement of unitarity is imposed on the physics side, we should consider differential forms valued in $\mathbb{C}$ rather than $\mathbb{R}$.)

Since this differential cohomology in degree 2 is known to correspond to the set of equivalence classes of principal $\mathbb{C}^\times$-bundles with connection on M [2], which in turn is given by $\pi_0\mathcal{S}\mathrm{m}(*, \mathbb{C}^\times\mathrm{Conn}(M))$, we see that in $d = 1$, every such topological action term (that is compatible with locality in that it can be extended to an invertible QFT) takes this form. So differential cohomology classifies topological action terms in $d = 1$.[20]

Now let us consider symmetry. What the physicist does is to consider a smooth action of some Lie group G on M and declares a physical theory to be symmetric if the physics action is invariant under the G action. This requirement corresponds to the mathematical notion of invariant differential cohomology defined in [13]. Our definition here of a G-symmetric QFT as a homotopy fixed point of a group action on a smooth space of fully-extended QFTs is rather more sophisticated, even in $d = 1$, since it requires the symmetry to be consistently defined not just on closed spacetime manifolds, *i.e.* on disjoint unions of circles, but also on compact spacetime manifolds with boundary, *i.e.* on disjoint unions of circles and intervals.

The upshot is that there is no reason to expect that the classification of invariant topological action terms given by invariant differential cohomology to agree with the more refined classification based on fully-extended QFTs given here and indeed we shall now see that, in general, they do not agree. In more detail, we expect that there should be a map from the latter to the former obtained by restricting a QFT to closed spacetimes, but we will show that this map is neither surjective nor injective, in general. In physics terms, a failure to be surjective represents a locality anomaly, in that a symmetry of the classical action defined on closed spacetimes cannot be extended consistently to spacetimes with boundary, while a failure to be injective implies that there exist many ways to extend a symmetry of the classical action on closed spacetimes to spacetimes with boundary.

To construct the map, we begin by considering an action on an object X by a group object G in a general topos $\mathcal{X}$. We may pull this back along the morphism $G^\delta \to G$ to get an action of G^δ on X, along with a morphism $X^{hG} \to X^{hG^\delta}$. Applying Γ to the latter to pass to spaces (recall that $\Gamma : X \mapsto \mathcal{X}(*, X)$) and making use of the adjunctions $\mathrm{Disc} \dashv \Gamma$ and those of the form $(*_X)^* \dashv (*_X)_*$ results in a morphism

$$\Gamma(X^{hG}) \to (\Gamma X)^{h\Gamma G}, \tag{4}$$

in $\mathcal{S}$.

Taking the Postnikov decomposition of the ΓG-action on ΓX

$$\begin{tikzcd}
\Gamma X \arrow[r, two heads] \arrow[d] & (\Gamma X)/\!\!/\Gamma G \arrow[d] \\
\tau_{\leq 0}\Gamma X \arrow[r, two heads] \arrow[d] & (\tau_{\leq 0}\Gamma X)/\!\!/\Gamma G \arrow[d] \\
* \arrow[r, two heads] & B\Gamma G,
\end{tikzcd}$$

implies that we have a morphism $((\Gamma X)^{h\Gamma G}) \to (\tau_{\leq 0}\Gamma X)^{h\Gamma G}$ in $\mathcal{S}$ which, since the target is 0-truncated, factors through $\tau_{\leq 0}((\Gamma X)^{h\Gamma G}) \to (\tau_{\leq 0}\Gamma X)^{h\Gamma G}$. Combining this with the 0-truncation of Eq. (4) then yields a sequence

$$\tau_{\leq 0}\Gamma\left(X^{hG}\right) \to \tau_{\leq 0}((\Gamma X)^{h\Gamma G}) \to (\tau_{\leq 0}\Gamma X)^{h\Gamma G}, \tag{5}$$

of set maps.

Now specialize to the topos $\mathcal{S}\mathrm{m}$ of smooth spaces, setting $X = \mathbb{C}^\times\mathrm{Conn}(M)$, where M corresponds to a smooth manifold M and suppose that G corresponds to a Lie group G acting smoothly on M. We claim that the composition of the two maps

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[20] In arbitrary $d > 1$, it is clear that differential cohomology cannot capture all topological action terms; rather some differential refinement of cobordism is needed.

in Eq. (5) gives the desired map. Indeed, the left-hand term in Eq. (5) is then the set of equivalence classes of G-symmetric invertible QFTs in $d = 1$ equipped with an orientation and a smooth map to M. In the right hand term, $\tau_{\leq 0}\Gamma\mathbb{C}^\times\mathrm{Conn}(M)$ is the differential cohomology in degree two of M. The fact that G acts on $\mathbb{C}^\times\mathrm{Conn}(M)$ via M implies that G^δ also acts via M. Thus $\left(\tau_{\leq 0}\Gamma\mathbb{C}^\times\mathrm{Conn}(M)\right)^{h\Gamma G}$ is the invariant differential cohomology in degree two of the G-manifold M, as defined in [2]. Moreover, since there is a 1-1 correspondence between differential characters in degree two and holonomy maps of bundles with connection [2], we see that the composition of the two maps indeed corresponds to restricting the QFT to closed spacetime manifolds and is the map we seek.

To see that the map is not surjective, in general, consider the case where the abelian Lie group $\mathbb{R}^2$ acts on itself by left (say) translation. A calculation of the invariant differential cohomology in degree two, as in [13], shows that $(\tau_{\leq 0}\Gamma(\mathbb{C}^\times\mathrm{Conn}(\mathbb{R}^2)))^{h\Gamma\mathbb{R}^2} \cong \mathbb{C}$. Physically this corresponds to the classical mechanics of a particle moving in a plane in the presence of a uniform magnetic field (which, since there is no requirement of unitarity, is allowed to take complex values), whose physics action on a spacetime S^1 is indeed invariant under translations of the plane. One of these fixed points corresponds to the bundle with connection $A = ydx$ (where x and y parameterize $\mathbb{R}^2$). However, the action of $\Gamma\mathbb{R}^2$ on the corresponding connected component of $\Gamma(\mathbb{C}^\times\mathrm{Conn}(\mathbb{R}^2)))$ corresponds to the non-trivial central extension of $\mathbb{R}^2$ by $\mathbb{C}^\times$ represented by the cocycle $\epsilon((x_1, y_1), (x_2, y_2)) = e^{-iy_1x_2}$. This central extension does not split, meaning there is no fixed point in $\tau_{\leq 0}((\Gamma\mathbb{C}^\times\mathrm{Conn}(\mathbb{R}^2)))^{h\Gamma\mathbb{R}^2})$ corresponding to that in $(\tau_{\leq 0}\Gamma(\mathbb{C}^\times\mathrm{Conn}(\mathbb{R}^2)))^{h\Gamma\mathbb{R}^2} \cong \mathbb{C}$. In other words, the right-hand map in (5) is not surjective, so the composite of the two maps in (5) cannot be surjective either. This corresponds to the well-known fact that the (exponentiated) physics action defined on an interval is not invariant under translations, but rather shifts by a boundary term.

To see that the map is no injective, in general, it suffices to consider the action of a Lie group G on the manifold consisting of a single point, for which the (invariant) differential cohomology in degree two is a singleton set. Since $\mathbb{C}^\times\mathrm{Conn}(*) \simeq \mathbb{C}^\times$ the set $\tau_{\leq 0}\Gamma\left(X^{hG}\right)$ is the set of equivalence classes of smooth splittings of the trivial extension of G by $\mathbb{C}^\times$. Thus choosing G to be $\mathbb{Z}/2$ already provides us with an example where the map fails to be injective.

The physical interpretation is as follows. The structure associated to the homotopy fixed point is a 1-dimensional representation of G, which is carried by the state space attached to a spacetime point. This structure is invisible when we evaluate the theory only on closed spacetimes of dimension one.

4.2. Non-unitary quantum mechanics

Now let us consider QFTs in $d = 1$ on spacetimes equipped with both an orientation and a Riemannian metric. These theories are close to what a physicist would call quantum mechanics, in that an oriented spacetime interval has associated to it a length and an orientation allowing non-trivial time evolution via a hamiltonian. They have, however, no notion yet of unitarity, so we call them ‘non-unitary quantum mechanical theories’.

According to [22], if we force the target category defining our theories to be the nerve of the 1-category of finite-dimensional vector spaces and linear maps to avoid technical difficulties, the relevant smooth space is $X = \coprod_n BGL_n^{B\mathbb{R}}$. In what follows we shall consider just one component, so $X = BGL_n^{B\mathbb{R}}$.

So a point in X is a morphism from $B\mathbb{R}$ to BGL_n. As we saw in §3, every such morphism is homotopic to a morphism corresponding to a smooth homomorphism from $\mathbb{R}$ to GL_n, *i.e.* to a smooth representation of $\mathbb{R}$ of dimension n. This corresponds physically to the time evolution.

In fact, the smooth space $BGL_n^{B\mathbb{R}}$ is equivalent to the smooth space $Mat_n /\!/ GL_n$ corresponding to the action Lie groupoid $\mathrm{Mat}_n /\!/ \mathrm{GL}_n$, where Mat_n is the smooth manifold of $n \times n$-matrices with values in $\mathbb{C}$ and the action of the Lie group GL_n is by conjugation. To see this, let us construct a morphism $Mat_n /\!/ GL_n \to BGL_n^{B\mathbb{R}}$ as follows: For every $p \in \mathbb{N}$, we must give a functor from $(Mat_n /\!/ GL_n)(\mathbb{R}^p)$ to $BGL_n^{B\mathbb{R}}(\mathbb{R}^p)$. The source groupoid $(Mat_n /\!/ GL_n)(\mathbb{R}^p)$ has as objects smooth maps $\mathrm{h}_1 : \mathbb{R}^p \to \mathrm{Mat}_n$ and has as morphisms from h_1 to h_2 smooth maps $\mathrm{m} : \mathbb{R}^p \to \mathrm{GL}_n$ such that $\mathrm{h}_2(\mathrm{x}) = \mathrm{m}(\mathrm{x})\mathrm{h}_1(\mathrm{x})\mathrm{m}(\mathrm{x})^{-1}$. The target groupoid $BGL_n^{B\mathbb{R}}(\mathbb{R}^p)$ has as objects smooth maps $\mathrm{r} : \mathbb{R}^p \times \mathbb{R} \to \mathrm{GL}_n$ which are homomorphisms for all $\mathrm{x} \in \mathbb{R}^p$ and has as morphisms from r_1 to r_2 smooth maps $\mathrm{m} : \mathbb{R}^p \to \mathrm{GL}_n$ such that $\mathrm{r}_2(\mathrm{x}) = \mathrm{m}(\mathrm{x})\mathrm{r}_1(\mathrm{x})\mathrm{m}(\mathrm{x})^{-1}$. We choose the functor that takes objects h in $(Mat_n /\!/ GL_n)(\mathbb{R}^p)$ to $(\mathrm{x}, \mathrm{t}) \mapsto e^{\mathrm{ih}(\mathrm{x})\mathrm{t}}$ in $BGL_n^{B\mathbb{R}}(\mathbb{R}^p)$, and which takes morphisms m in $(Mat_n /\!/ GL_n)(\mathbb{R}^p)$ to the corresponding morphism described by the same smooth map m in $BGL_n^{B\mathbb{R}}(\mathbb{R}^p)$. The factor of i we introduced will be convenient when we come to study unitary quantum mechanics. The remaining unspecified data of $Mat_n /\!/ GL_n \to BGL_n^{B\mathbb{R}}$ are taken such that the homotopies involved are in fact strict identities between compositions of 1-functors. This morphism is an equivalence of smooth spaces whose inverse can be constructed by differentiating the smooth maps $\mathbb{R}^p \times \mathbb{R} \to \mathrm{GL}_n :$ $(\mathrm{x}, \mathrm{t}) \mapsto e^{\mathrm{ith}(\mathrm{x})}$ with respect to t. Physically, we can think of Mat_n as the smooth manifold of hamiltonians of quantum-mechanical theories with a state space of dimension n. For now, since we have no notion of unitarity yet, the state space is a mere vector space, and there is no requirement for a hamiltonian to be hermitian. The action encodes the fact that conjugation by GL_n sends a hamiltonian to one that is physically equivalent.

We remark that it is crucial that we retain the information about the physical equivalences between hamiltonians. If we simply naïvely pass to the set $\mathrm{Mat}_n/\mathrm{GL}_n$ of equivalence classes of hamiltonians (which Simplicio does when they 'diagonalize the hamiltonian'[21]), we lose the smooth structure. Even if we consider $\tau_{\leq 0}(Mat_n /\!\!/ GL_n)$, which is the smooth space corresponding to $\mathsf{Mat}_n/\mathsf{GL}_n$, we will not be able to give a complete description of the smooth symmetries.

Let us now make some relevant remarks about the smooth space $Mat_n /\!\!/ GL_n$. Firstly, it is not connected and has many inequivalent points, for its 0-truncation is, as a space, that corresponding to the set of equivalence classes of hamiltonians, of which there are many. (For $n = 1$, for example, we get the set $\mathbb{R}$; more generally we get the set of possible elementary divisors, *i.e.* the characteristic polynomials of the Jordan blocks.)

Secondly, given a point h in $Mat_n /\!\!/ GL_n$ corresponding to a hamiltonian $\mathsf{h} \in \mathsf{Mat}_n$, the connected component at h is given by the delooping of the group object S_h corresponding to the Lie group S_h given by the stabilizer of h under the action of GL_n.

From here, it is easy to satisfy Simplicio. For G-actions with homotopy fixed point through a point corresponding to a hamiltonian h, we may replace G by $\tau_{\leq 0}G$, following the discussion in §2.4. Taking $G_0 := \tau_{\leq 0}G$ to be Lie for simplicity, we find that the possible G-actions with homotopy fixed point through h are described by split short exact sequences of Lie groups extending G_0 by S_h. Suppose the automorphism $\mathsf{e} : \mathsf{G}_0 \times \mathsf{S}_h \to \mathsf{S}_h$ describes the splitting (see §3.1.2). Then the homotopy fixed points are given by smooth maps $\mathsf{r} : \mathsf{G}_0 \to \mathsf{S}_h$ satisfying Eq. (3). So, for example, if we take the trivial extension $\mathsf{S}_h \times \mathsf{G}_0$, we get a *bona fide* representation of G_0 that commutes with h, but if G_0 acts on S_h by inner automorphisms, we get a projective representation of G_0 that commutes with h. So physically, we get back the usual expectation that symmetries correspond to linear (not unitary here) operators that commute with the hamiltonian, in the sense that $\mathsf{r}(\mathsf{g})\mathsf{h} = \mathsf{h}\mathsf{r}(\mathsf{g})$, but they need only form a representation of a group up to twisting via e.

Here an explicit description of the automorphisms of S_h, is far from straightforward, not least because S_h, though connected, is not compact.[22]

To give just the simplest example, if $\mathsf{S}_h = \mathsf{GL}_1 \simeq \mathbb{C}^\times$, all automorphisms are necessarily outer and the Lie group $\mathsf{Aut}\,\mathsf{GL}_1$ is isomorphic to the subgroup of GL_2 consisting of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & \pm 1 \end{pmatrix}$, which, in terms of the connected component at the identity (isomorphic to the affine group of the real line) and the group of components (isomorphic to $\mathbb{Z}/2$), can be expressed as a short exact sequence of Lie groups that is right- (but not left-) split.

Since the smooth space X of non-unitary quantum-mechanical theories is not connected, we cannot expect to be able to classify all group actions on the full smooth space easily. We shall content ourselves with describing some specific examples. Moreover, there can be smoothness anomalies, in that a group action defined on the connected component at h may not extend to all of X and indeed the example we give for unitary quantum mechanics in §4.3 provides an example for non-unitary quantum mechanics as well.

4.2.1. *Smooth TQFTs in $d = 1$ - take two*

If we focus on the special case of theories corresponding to the point $h = 0$ (of which up to equivalence there is one for each n), we get another smooth incarnation of the oriented TQFTs in $d = 1$ whose symmetries we discussed in [23].

To be precise, we should like to consider the homotopy fixed points through $h = 0$ of group actions on $Mat_n /\!\!/ GL_n$. According to the general arguments in §2.3, we can do this by first studying the group actions with homotopy fixed points on $\tau_{\geq 1}(X, x)$, which here is equivalent to BGL_n (since every element of GL_n stabilizes $h = 0$), and then worrying about which actions on $\tau_{\geq 1}(X, x)$ extend to actions on X. As we show in the Appendix B.5, there is in fact nothing to worry about here, in that all actions extend.

The problem, and the physical interpretation is now identical, as one would hope, to what we found in §4.1.1.

4.2.2. *Invertible non-unitary quantum mechanics*

Next consider the special case where $n = 1$, for which $Mat_1 /\!\!/ GL_1 = \mathbb{C} \times B\mathbb{C}^\times$. While this is still not connected, its 0-truncation is simply the smooth space corresponding to the manifold $\mathbb{C}$, which will allow us to make some progress.

For a generic group G acting on $\mathbb{C} \times B\mathbb{C}^\times$, we have a diagram

$$\begin{tikzcd}
\mathbb{C} \times B\mathbb{C}^\times \arrow[r, two heads] \arrow[d] & (\mathbb{C} \times B\mathbb{C}^\times) /\!\!/ G \arrow[d] \\
\mathbb{C} \arrow[r, two heads] \arrow[d] & \mathbb{C} /\!\!/ G \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

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[21] We caution that in the situation without unitarity, one cannot in fact necessarily diagonalize the hamiltonian, *cf. e.g.* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathsf{Mat}_2$; but one can pass to some canonical normal form, such as the Jordan normal form.

[22] To see that S_h is connected, let $\mathsf{s}, \mathsf{s}' \in \mathsf{S}_h$ and consider the function $\mathsf{f} : \mathbb{C} \to \mathsf{Mat}_n : \mathsf{t} \mapsto \mathsf{st} + \mathsf{s}'(1 - t)$. By the fundamental theorem of algebra, the zero set Z of $\det \mathsf{f} : \mathbb{C} \to \mathbb{C}$ consists of at most n isolated points so $\mathbb{C} - Z$ is path connected. Moreover, $0, 1 \in \mathbb{C} - Z$. Since $\mathsf{fh} = \mathsf{hf}$, evaluating f along any path from $\mathsf{t} = 0$ to $\mathsf{t} = 1$ in $\mathbb{C} - Z$ exhibits a path in S_h connecting s to s'. To see that S_h is not compact, observe that it has a Lie subgroup isomorphic to $\mathbb{C}^\times$ given by the non-zero diagonal matrices.

in $\mathcal{S}\mathrm{m}$ in which all squares are cartesian. The bottom square defines an action of G on the smooth space corresponding to the manifold $\mathbb{C}$, whose homotopy quotient we denote $\mathbb{C}/\!\!/G$. The top square shows that the morphism $(\mathbb{C} \times B\mathbb{C}^\times)/\!\!/G \to \mathbb{C}/\!\!/G$ has the structure of a $B\mathbb{C}^\times$-fibre bundle over $\mathbb{C}/\!\!/G$, as defined in [33], in which the effective epimorphism $\mathbb{C} \to \mathbb{C}/\!\!/G$ provides a cover that trivializes the bundle. As shown there, $B\mathbb{C}^\times$-fibre bundles over $\mathbb{C}/\!\!/G$ are classified by morphisms from $\mathbb{C}/\!\!/G \to B\mathrm{Aut}(B\mathbb{C}^\times)$, though we must take care that not all such bundles will be trivialized by $\mathbb{C} \to \mathbb{C}/\!\!/G$. Going in the other direction, we can classify all G-actions on $\mathbb{C} \times B\mathbb{C}$ by first finding all G-actions on $\mathbb{C}$. For each of these, we must then find the possible $B\mathbb{C}^\times$-bundles over the resulting homotopy quotient $\mathbb{C}/\!\!/G$, subject to the condition that they trivialize under $\mathbb{C} \to B\mathbb{C}/\!\!/G$. Such a bundle always exists, since we have the trivial bundle $B\mathbb{C}^\times \times \mathbb{C}/\!\!/G$.

The richness of possible G-actions on smooth spaces of QFTs is now clear, since even when G corresponds to a Lie group, in general there will be many ways that it can act smoothly on $\mathbb{C}$, and each of these will induce at least one action on X. So it is hard to say anything else in full generality. Thus, let us turn to constructing explicit examples of group actions.

Suppose, as a first example, that G acts trivially on $\mathbb{C}$, but that we have a non-trivial action of G on $B\mathbb{C}^\times$, denoted $B\mathbb{C}^\times/\!\!/G$. Then we have a G-action on $\mathbb{C} \times B\mathbb{C}^\times$ given by

$$\begin{tikzcd}
\mathbb{C} \times B\mathbb{C}^\times \arrow[r, two heads] \arrow[d] & (\mathbb{C} \times B\mathbb{C}^\times)/\!\!/G \arrow[d] \\
\mathbb{C} \arrow[r, two heads] \arrow[d] & \mathbb{C} \times BG \arrow[d] \\
* \arrow[r, two heads] & BG,
\end{tikzcd}$$

This example is of interest because it shows that we have no smoothness anomalies. Indeed, the connected component of any point $* \to \mathbb{C} \times B\mathbb{C}^\times$ is simply $B\mathbb{C}^\times$ and the above action leads to the action corresponding to $B\mathbb{C}^\times/\!\!/G$ on each connected component.

For our next example, we consider actions on $\mathbb{C} \times B\mathbb{C}^\times$ of a group object that is itself a delooping, such that we may write it as $G = B\hat{G}$. These are of interest, since they justify our earlier claim that QFTs in spacetime dimension d can have non-trivial d-form symmetries. Moreover, they illustrate another surprising feature of actions of such groups: their possible homotopy fixed points can pick out a 'proper subspace' of $\tau_{\leq 0}X$, even though they necessarily act trivially on $\tau_{\leq 0}X$.

We consider a specific class of actions of such groups that are obtained by pullback of a canonical example, in which $\hat{G}$ is the abelian group object $(\mathbb{C}^\times)^{\mathbb{C}}$ corresponding to the abelian group whose elements are smooth maps from $\mathbb{C}$ to $\mathbb{C}^\times$, with multiplication given by pointwise multiplication in the target, made into a smooth space in the obvious way.[23]

Evaluation on the target defines a morphism $\mathbb{C} \times (\mathbb{C}^\times)^{\mathbb{C}} \to \mathbb{C}^\times$ and we can use this to build a morphism $\mathbb{C} \times B^2(\mathbb{C}^\times)^{\mathbb{C}} \to B^2\mathbb{C}^\times$ as, for each $\mathbb{R}^p$, the functor $(\mathbb{C} \times B^2(\mathbb{C}^\times)^{\mathbb{C}})(\mathbb{R}^p) \to B^2\mathbb{C}^\times(\mathbb{R}^p)$ that acts trivially on 0- and 1-simplices and by evaluation on 2-simplices. Using this morphism, we can form the diagram

$$\begin{tikzcd}
\mathbb{C} \times B\mathbb{C}^\times \arrow[r] \arrow[d] & (\mathbb{C} \times B\mathbb{C}^\times)/\!\!/B(\mathbb{C}^\times)^{\mathbb{C}} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \times B^2(\mathbb{C}^\times)^{\mathbb{C}} \arrow[r] \arrow[d] & B^2\mathbb{C}^\times \\
* \arrow[r] & B^2(\mathbb{C}^\times)^{\mathbb{C}}, &
\end{tikzcd}$$

in which all squares are cartesian and the top rectangle is the trivial principal $B\mathbb{C}^\times$-bundle over $\mathbb{C}$. The left-hand rectangle then exhibits an action of $B(\mathbb{C}^\times)^{\mathbb{C}}$ on $\mathbb{C} \times B\mathbb{C}^\times$.[24]

Given any abelian group object $B\hat{G}$ and a morphism $B^2\hat{G} \to B^2(\mathbb{C}^\times)^{\mathbb{C}}$, we can construct a group action of $B\hat{G}$ on $B(\mathbb{C}^\times)^{\mathbb{C}}$ by pulling back. We note that we get a similar diagram for $B\hat{G}$, namely

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[23] The reader will observe that the construction works for any X of the form $Y \times BA$, provided BA is deloopable.

[24] More explicitly, the object $(\mathbb{C} \times B\mathbb{C}^\times)/\!\!/B(\mathbb{C}^\times)^{\mathbb{C}}$ can be described as follows. To $\mathbb{R}^p$, we assign the 2-groupoid whose objects are smooth maps $\mathbb{R}^p \to \mathbb{C}$, while a 1-simplex $f_0 \to f_1$ exists if and only if $f_0 = f_1$ and is given by a smooth map $\mu : \mathbb{R}^p \to \mathbb{C}^\times$, and a 2-simplex with all 0-vertices f and 1-vertices μ_{01}, μ_{12} and μ_{02} is a smooth map $\eta : \mathbb{C} \times \mathbb{R}^p \to \mathbb{C}^\times$ such that $\mu_{01}(x)\mu_{12}(x) = \mu_{02}(x)\eta(f(x), x)$.

$$\begin{tikzcd}
\mathbb{C} \times B\mathbb{C}^{\times} \arrow[r] \arrow[d] & (\mathbb{C} \times B\mathbb{C}^{\times})/\!\!/B\hat{G} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \times B^2\hat{G} \arrow[r] \arrow[d] & B^2\mathbb{C}^{\times} \\
* \arrow[r] & B^2\hat{G}. &
\end{tikzcd}$$

While the bottom square imposes no constraint on the action (since $B\hat{G}$ always acts trivially on the 0-truncation $\mathbb{C}$ of $\mathbb{C} \times B\mathbb{C}^{\times}$), the action is strongly constrained by the top squares. Namely, $(\mathbb{C} \times B\mathbb{C}^{\times})/\!\!/B\hat{G} \to \mathbb{C} \times B^2\hat{G}$ must not only have the structure of a principal $B\mathbb{C}^{\times}$-bundle, but also must trivialize under pullback along $\mathbb{C} \to \mathbb{C} \times B\mathbb{C}^{\times}$.

Moreover, our original action of $B(\mathbb{C}^{\times})^{\mathbb{C}}$ is apparently anyway uninteresting for physics, since it has no homotopy fixed points. Indeed, $\tau_{\geq 2}((\mathbb{C} \times B\mathbb{C}^{\times})/\!\!/B(\mathbb{C}^{\times})^{\mathbb{C}}, [z]) =: B^2S_z$ at a point z represented by $\mathrm{z} \in \mathbb{C}$ assigns to $\mathbb{R}^0$ the subgroup of smooth maps $\mathrm{f} : \mathbb{C} \to \mathbb{C}^{\times}$ for which $\mathrm{f}(\mathrm{z})$ is the identity in $\mathbb{C}^{\times}$. In spaces $B^2S_z(\mathbb{R}^0) \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}(\mathbb{R}^0)$ does not have a section for any z, thus neither does the morphism $B^2S_z \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}$ in smooth spaces. Hence, the action of $B(\mathbb{C}^{\times})^{\mathbb{C}}$ on $\mathbb{C} \times B\mathbb{C}^{\times}$ has no sections *i.e.* no homotopy fixed points.

Nevertheless, after pulling back we can obtain a plentiful supply of interesting actions that do have homotopy fixed points using this construction, as the following examples show.

Consider a $B\hat{G}$-action of the above type. Then, $\tau_{\geq 2}((\mathbb{C} \times B\mathbb{C}^{\times})/\!\!/B\hat{G}, [z]) =: B^2\hat{S}_z$ for a point $z : * \to \mathbb{C}$ is related to B^2S_z by the cartesian square

$$\begin{tikzcd}
B^2\hat{S}_z \arrow[r] \arrow[d] & B^2S_z \arrow[d] \\
B^2\hat{G} \arrow[r] & B^2(\mathbb{C}^{\times})^{\mathbb{C}}.
\end{tikzcd}$$

For this $B\hat{G}$-action to have a homotopy fixed point, the left vertical morphism must have a section. This is equivalent to saying that $B^2\hat{G} \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}$ must factor, up to homotopy, through $B^2S_z \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}$. Looking at this result in spaces, *i.e.* after applying Γ, tells us that we only have homotopy fixed points through z if, under the double looping of $B^2\Gamma\hat{G} \to B^2\Gamma(\mathbb{C}^{\times})^{\mathbb{C}}$, every object in $\Gamma\hat{G}$ is taken to a smooth map $\mathbb{C} \to \mathbb{C}^{\times}$ in $\Gamma(\mathbb{C}^{\times})^{\mathbb{C}}$ that evaluates to $1 \in \mathbb{C}^{\times}$ at z. (If $\hat{\mathrm{G}}$ happens to be discrete, this condition is both necessary and sufficient.)

For a first example, let $B\hat{G} = B\mathbb{R}$, corresponding to the abelian group of the real numbers under addition, considered as a discrete group. Then let $B^2\mathbb{R} \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}$ be the morphism corresponding to the homomorphism sending $\mathrm{x} \in \mathbb{R}$ to the map $\mathbb{C} \to \mathbb{C}^{\times} : \mathrm{z} \mapsto e^{\mathrm{zx}}$. The induced action of $B\mathbb{R}$ on $\mathbb{C} \times B\mathbb{C}^{\times}$ has a single homotopy fixed point through $\mathrm{z} = 0$. More generally, one could take x to the map $\mathbb{C} \to \mathbb{C}^{\times} : \mathrm{z} \mapsto e^{\mathrm{f(z)x}}$ for some smooth map $\mathrm{f} : \mathbb{C} \to \mathbb{C}$; the homotopy fixed points are then through the zeroes of f.

As a second example, take $B\hat{G} = B\mathbb{Z}$, the abelian group of integers under addition. Then let $B^2\mathbb{Z} \to B^2(\mathbb{C}^{\times})^{\mathbb{C}}$ be the morphism corresponding to the map taking $\mathrm{n} \in \mathbb{Z}$ to $\mathbb{C} \to \mathbb{C}^{\times} : \mathrm{z} \mapsto e^{\mathrm{z\,n}}$. The points $z \in \mathbb{C}$ which are homotopy fixed points of this action are those corresponding to $\mathrm{z} \in 2\pi\mathrm{i}\mathbb{Z} \subset \mathbb{C}$.

These group actions are analogous, in terms of the homotopy fixed points of the induced action on $\tau_{\leq 0}X$, to the standard notion of group actions on sets. But in terms of the actions themselves, they are very much unlike standard group actions on sets, because they produce non-trivial homotopy fixed points of $\tau_{\leq 0}X$ even though the action on $\tau_{\leq 0}X$ is trivial. The 'explanation' for this is that although every group action on X with a homotopy fixed point induces a group action on $\tau_{\leq 0}X$ with homotopy fixed point, there is no guarantee that all homotopy fixed points of the latter arise in this way.

We stress that this phenomenon arises only because X is not connected; poor Simplicio will never observe such things.

4.3. *Unitary quantum mechanics*

We now discuss how to add a unitary structure to the theories described, to get something which could genuinely be described as quantum mechanics. In fact this can be done using symmetry considerations,[25] making it all the more suitable for discussion here.

A key point is that we in fact need both a structure and a property (corresponding respectively to reflection and postivity, respectively, in the euclidean context [20,32]), which we can implement here by means of 0- and 1-form symmetries, respectively.

We begin by observing that there is an action of the group object corresponding to the Lie group $\mathbb{Z}/2$ on $\coprod_n BGL_n$ that sends the non-trivial element of $\mathbb{Z}/2$ to the automorphism $M \mapsto (M^{\dagger})^{-1}$ of GL_n. According to the discussion in §3.1.2, a homotopy fixed point of this action is given by an element $\mathrm{A} \in \mathrm{GL}_n$ such that $\mathrm{A} = \mathrm{A}^{\dagger}$, *i.e.* a non-degenerate hermitian form.

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[25] For an alternative implementation, which generalizes to QFTs in higher d, see [26].

This A specifies where the non-trivial element of $\mathbb{Z}/2$ is sent under the twisted representation as defined in Eq. (3). The morphisms between homotopy fixed points correspond to conjugation by GL_n, so every homotopy fixed point is equivalent to a diagonal matrix whose entries are ± 1. More generally, the smooth space of homotopy fixed points assigns to p the groupoid with objects smooth maps $\mathsf{A} : \mathbb{R}^p \to \mathsf{GL}_n$ taking values in non-degenerate hermitian forms and morphisms from A_1 to A_2 given by smooth maps $\mathsf{m} : \mathbb{R}^p \to \mathsf{GL}_n$ such that $\mathsf{A}_2(\mathsf{x}) = \mathsf{m}(\mathsf{x})\mathsf{A}_1(\mathsf{x})\mathsf{m}(\mathsf{x})^\dagger$. This smooth space is equivalent to $\coprod_n \coprod_{p+q=n} BU_{p,q}$, since the Gram-Schmidt procedure can be carried out for smooth families.

This $\mathbb{Z}/2$-action on BGL_n extends, of course, to an action on the smooth space $\coprod_n (BGL_n)^{B\mathbb{R}}$ of non-unitary quantum mechanics (where it corresponds to sending objects in the target category of the QFT to their duals).[26] The corresponding homotopy quotient is given (for each n) by the mapping object $(BGL_n /\!\!/ \mathbb{Z}/2)^{B\mathbb{R}\times B\mathbb{Z}/2}$ in the slice topos $\mathcal{S}\mathrm{m}_{/B\mathbb{Z}/2}$ and the space of homotopy fixed points of this action can be written in terms of the semidirect product associated to the $\mathbb{Z}/2$ action on BGL_n as $\mathcal{S}\mathrm{m}_{/B\mathbb{Z}/2}(B(\mathbb{R}\times\mathbb{Z}/2), B(GL_n \rtimes \mathbb{Z}/2))$. A vertex in this space corresponds, via the arguments in §3.1.2, to a smooth homomorphism $\mathbb{R}\times\mathbb{Z}/2 \to \mathsf{GL}_n \rtimes \mathbb{Z}/2$, such that the diagram

$$\begin{tikzcd}
& \mathsf{GL}_n \rtimes \mathbb{Z}/2 \arrow[dr] & \\
\mathbb{R}\times\mathbb{Z}/2 \arrow[ur] \arrow[rr] & & \mathbb{Z}/2,
\end{tikzcd}$$

commutes (on the nose, since $\mathbb{Z}/2$ is abelian).

More prosaically, a homotopy fixed point of this action is, for some n, an n-dimensional representation $e^{\mathrm{i}th}$ of $\mathbb{R}$ together with a non-degenerate hermitian form A of dimension n such that $Ae^{\mathrm{i}th} = e^{\mathrm{i}th^\dagger}A$.

Using the mapping space and $(*_{B\mathbb{Z}/2})^* \dashv (*_{B\mathbb{Z}/2})_*$ adjunctions, we find that the smooth space of homotopy fixed points is equivalent to $\coprod_{p+q=n}(BU_{p,q})^{B\mathbb{R}}$. This smooth space of QFTs, which we might call hermitian quantum mechanics, suffers from the fact that probabilities computed using Born's rule can exceed 1. To remedy that, we need an extra property that singles out the $p=n$, $q=0$ component of $\coprod_{p+q=n}(BU_{p,q})^{B\mathbb{R}}$. One can do this by hand or via an action of a connected group object in smooth spaces ('a 1-form symmetry').[27]

The smooth space corresponding to unitary QM is then $X = BU_n^{B\mathbb{R}}$. Similar to the non-unitary case, this is equivalent to the smooth space $Her_n /\!\!/ U_n$ corresponding to the action Lie groupoid $\mathsf{Her}_n /\!\!/ \mathsf{U}_n$, where Her_n denotes the smooth manifold of $n\times n$-hermitian matrices with values in $\mathbb{C}$ and the action of the Lie group U_n is by conjugation. The smooth space $Her_n /\!\!/ U_n$ is not connected; by diagonalizing, we see that its 0-truncation, as a space, corresponds to the set in which an element is a choice of n unordered real numbers. Labeling the degeneracies of these by $q_i \in \{0, 1, 2, \dots\}$ (so that $\sum_{i\in I} q_i = n$), we find that the connected component is given by the delooping of the stabilizer under the U_n action, *i.e.* by the Lie group $\prod_{i\in I}\mathsf{U}_{q_i}$.

Ideally, Simplicio would now like to know the possible smooth automorphisms of $\mathsf{S}_h = \prod_{i\in I}\mathsf{U}_{q_i}$ corresponding to the h of interest. We note that S_h is both connected and compact, (though neither simply-connected nor semisimple unless $n=0$). It should therefore be possible to give an explicit general description, but we refrain from doing so here. To give just the simplest case, when $\mathsf{S}_h = \mathsf{U}_1$, we get that $\mathrm{Aut}\,\mathsf{U}_1 \simeq \mathbb{Z}/2$, corresponding to complex conjugation.

Let us instead, finally, give an explicit example of a smoothness anomaly. Recall that in this case, a smoothness anomaly is a G-action on BS_h which does not extend to a G-action on X. We have already seen that there are no smoothness anomalies in non-unitary quantum mechanics when $n=1$ and a similar argument applies here. So to make things as simple as possible, we will consider the case $n=2$. For the same reason, we take $G = \mathbb{Z}/2$ and $\mathsf{h} = \mathrm{diag}(1,-1)$, so that $\mathsf{S}_h \simeq \mathsf{U}_1^2 := \mathsf{U}_1 \times \mathsf{U}_1$ and $\mathrm{Aut}\,\mathsf{S}_h \simeq \mathsf{GL}_2(\mathbb{Z})$. We take the action in which the non-trivial element of $\mathbb{Z}/2$ is sent to the smooth automorphism $\mathsf{c} : (e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi}) \mapsto (e^{\mathrm{i}\theta}, e^{-\mathrm{i}\phi})$ of S_h. A homotopy fixed point of this $\mathbb{Z}/2$-action on BU_1^2 corresponds to a twisted representation, as per Eq. (3), which assigns an element of the form $(\pm 1, e^{\mathrm{i}\theta}) \in \mathsf{U}_1^2$ to the non-trivial element of $\mathbb{Z}/2$. We denote by $j : BU_1^2 \to Her_2 /\!\!/ U_2$ the inclusion of BU_1^2 into $Her_2 /\!\!/ U_2$ which (following our discussion of Lie groupoids in §3) is induced by the smooth map $\mathsf{j} : \mathsf{U}_1^2 \to \mathsf{Her}_2 \times \mathsf{U}_2 : (e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi}) \mapsto \big(\mathsf{h}, (e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi})\big)$.

Now let us suppose that there is an action on $Her_2 /\!\!/ U_2$ that leads to this action on BS_h and show that it implies a contradiction. Taking the putative action, we have a diagram (analogous to the diagram (A.1))

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[26] To get a reflection structure, as is required for reflection-positivity in euclidean QFTs, one should instead take the homotopy fixed points of the action which simultaneously sends $t \in \mathbb{R}$ to $-t$, *cf.* [20].

[27] For example, the $B\mathbb{Z}/2$ action that is induced by the trivial action on $BU_{n,0}$ and the $B\mathbb{Z}/2$-action on $BU_{p,q}$, for which $BU_{p,q} /\!\!/ B\mathbb{Z}/2$ assigns to, *e.g.*, $\mathbb{R}^0$ the 2-groupoid with a single 0-simplex, 1-simplices elements of $\mathsf{U}_{p,q}$, and 2-simplices with edges $U_{01}, U_{12}, U_{02} \in U_{p,q}$ that are labeled by ± 1 such that $U_{01}U_{12} = \pm U_{02}$.

$$\begin{tikzcd}
BU_1^2 \arrow[rr, "j"] \arrow[dd] & & Her_2/\!/U_2 \arrow[rr] \arrow[dd] & & * \arrow[dd] \arrow[dr, Rightarrow] & \\
& BU_1^2 \arrow[ul, "Ba"'] \arrow[dl] \arrow[rr, "j"] & & Her_2/\!/U_2 \arrow[ul, "f"'] \arrow[dl] \arrow[rr] & & * \arrow[ul] \arrow[dl] \\
BU_1^2/\!/\mathbb{Z}/2 \arrow[rr] & & (Her_2/\!/U_2)/\!/\mathbb{Z}/2 \arrow[rr] & & B\mathbb{Z}/2, &
\end{tikzcd}$$

in which all squares are cartesian and the morphisms denoted Ba and f must be equivalences, since they are obtained by pulling back the equivalence $* \to *$. There are two possible choices for the filler of the horn $* \to B\mathbb{Z}/2 \leftarrow *$ denoted by the double arrow, corresponding to the two elements of $\mathbb{Z}/2$; we shall show that for the non-trivial element, no suitable f exists.

To do so, we consider first the constraints on a imposed by the outer prism in the diagram. Since this prism only involves objects that are deloopings of Lie groups, we may replace it by the equivalent diagram

$$\begin{tikzcd}
\mathsf{U}_1^2 \arrow[rr] \arrow[dd] & & * \arrow[dd] \arrow[dr, Rightarrow] & \\
& \mathsf{U}_1^2 \arrow[ul, "\mathsf{a}"'] \arrow[dl] \arrow[rr] & & * \arrow[ul] \arrow[dl] \\
\mathsf{U}_1^2 \rtimes \mathbb{Z}/2 \arrow[rr] & & \mathbb{Z}/2, &
\end{tikzcd}$$

in the category $\mathcal{L}$ defined in §3, where the semidirect product $\mathsf{U}_1^2 \rtimes \mathbb{Z}/2$ is defined by the given action of $\mathbb{Z}/2$ on U_1^2 in the usual way. In order for this to commute, we must have that $\mathsf{a} = \mathsf{b} \circ \mathsf{c} \circ \mathsf{b}^{-1}$ for some automorphism b of U_1^2.

Having constrained the allowed forms a, our job will be done if we can show that there is no valid a and no equivalence f making a commutative square

$$\begin{tikzcd}
BU_1^2 \arrow[r, dashed, "Ba"] \arrow[d, "j"'] & BU_1^2 \arrow[d, "j"] \\
Her_2/\!/U_2 \arrow[r, dashed, "f"'] & Her_2/\!/U_2.
\end{tikzcd}$$

To show it, we turn this putative commutative diagram into a putative commutative triangle involving Lie groups, for which we can use the explicit description of the full sub-category $\mathcal{L}$ of $\mathcal{S}\mathrm{m}$ in §3 to show that such a triangle cannot exist.

As an intermediate step, we form the commutative diagram

$$\begin{tikzcd}
& BU_1^2 \arrow[r, dashed, "Ba"] \arrow[d, "j"'] \arrow[dl, "\{1\}\times\mathrm{id}"'] \arrow[drr, dashed, bend left, "BR"] & BU_1^2 \arrow[d, "j"] & \\
\mathbb{R} \times BU_1^2 \arrow[r] & Her_2/\!/U_2 \arrow[r, dashed, "f"'] & Her_2/\!/U_2 \arrow[r] & BU_1. \\
BU_1^2 \arrow[u, "\{0\}\times\mathrm{id}"] \arrow[r, "Bd"'] & BU_2 \arrow[u] \arrow[urr, dashed, "BT"'] & &
\end{tikzcd}$$

Here, the morphism Bd corresponds to the homomorphism $\mathsf{d} : \mathsf{U}_1^2 \to \mathsf{U}_2 : (e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi}) \mapsto \mathrm{diag}(e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi})$. The morphism $\mathbb{R} \times BU_1^2 \to Her_2/\!/U_2$ is that constructed, in the manner of §3, from the smooth functor of Lie groupoids $\mathsf{IR} \times \mathsf{BU}_1^2 \to \mathsf{Her}_2/\!/\mathsf{U}_2$ that maps morphisms $(\mathsf{x}, \mathsf{g}) \in \mathsf{IR} \times \mathsf{U}_1^2$ to $(\mathsf{x}\,\mathsf{h}, \mathsf{d}(\mathsf{g})) \in \mathsf{Her}_2 \times \mathsf{U}_2$. The morphism $BU_2 \to Her_2/\!/U_2$ is that constructed from the smooth functor $\mathsf{BU}_2 \to \mathsf{Her}_2/\!/\mathsf{U}_2$ that maps morphisms $\mathsf{g} \in \mathsf{U}_2$ to $(0, \mathsf{g}) \in \mathsf{Her}_2 \times \mathsf{U}_2$. The morphism $Her_2/\!/U_2 \to BU_1$ is the composition of the morphism $Her_2/\!/U_2 \to BU_2$ defining the conjugation action of U_2 on Her_2 and the morphism $Bdet : BU_2 \to BU_1$ corresponding to the determinant homomorphism $\mathsf{det} : \mathsf{U}_2 \to \mathsf{U}_1$. The morphism $BR : BU_1^2 \to BU_1$ corresponds to the homomorphism

$$\mathsf{R} : (e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi}) \mapsto \exp\left\{\mathrm{i}\begin{pmatrix}1 & 1\end{pmatrix} b \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} b^{-1}\begin{pmatrix}\theta\\ \phi\end{pmatrix}\right\}, \tag{6}$$

where $b \in \mathsf{GL}_2(\mathbb{Z})$ is the element in $\mathrm{Aut}\,\mathsf{U}_1^2$ corresponding to the automorphism b. Lastly, taking a composite of the morphism $BU_2 \to Her_2/\!\!/U_2$, the morphism f, and the morphism $Her_2/\!\!/U_2 \to BU_1$ we can construct a $BT : BU_2 \to BU_1$ making the diagram commute.

Any morphism $\mathbb{R} \to (BU_1)^{BU_1^2} \simeq BU_1 \times \mathbb{Z} \times \mathbb{Z}$ factors through $*$; consequently the composite horizontal morphism in the above diagram $\mathbb{R} \times BU_1^2 \to BU_1$ must factor through the projection $p_2 : \mathbb{R} \times BU_1^2 \to BU_1^2$. The result is the commutative diagram

$$\begin{tikzcd}
 & BU_1^2 \arrow[dl, "\{1\}\times\mathrm{id}"'] \arrow[dr, dashed, bend left, "BR"] & \\
\mathbb{R} \times BU_1^2 \arrow[r, "p_2"] & BU_1^2 \arrow[r, dashed] & BU_1, \\
BU_1^2 \arrow[u, "\{0\}\times\mathrm{id}"] \arrow[r, "Bd"'] & BU_2 \arrow[ur, dashed, "BT"'] &
\end{tikzcd}$$

which implies the existence of a commutative triangle

$$\begin{tikzcd}
BU_1^2 \arrow[rr, dashed, "BR"] \arrow[dr, "Bd"'] & & BU_1. \\
 & BU_2 \arrow[ur, dashed, "BT"'] &
\end{tikzcd}$$

We may equivalently consider this diagram in $\mathcal{L}$, in which case it becomes

$$\begin{tikzcd}
\mathsf{U}_1^2 \arrow[rr, dashed, "\mathsf{R}"] \arrow[dr, "\mathsf{d}"'] & & \mathsf{U}_1. \\
 & \mathsf{U}_2 \arrow[ur, dashed, "\mathsf{T}"'] &
\end{tikzcd}$$

Since U_1 is abelian, this diagram must commute on the nose. The universality of abelianization tells us that T must factor through $\det$, and thus $\mathsf{T} \circ \mathsf{d}$ must take the form $(e^{\mathrm{i}\theta}, e^{\mathrm{i}\phi}) \mapsto e^{\mathrm{i}\mathsf{m}(\theta+\phi)}$ for some $\mathsf{m} \in \mathbb{Z}$. There is, however, no choice of $b \in \mathsf{GL}_2(\mathbb{Z})$ in Eq. (6) such that R takes this form, so our assumption that the action extends to $Her_2/\!\!/U_2$ must be false. We thus have a smoothness anomaly, in the familiar world of unitary quantum mechanics.

4.4. Invertible TQFTs in $d = 2$

In [23], we gave a classification of the group actions and homotopy fixed points on the space of framed or oriented TQFTs in $d = 2$ taking values in the bicategory whose objects are algebras over $\mathbb{C}$, 1-morphisms are bimodules and 2-morphisms are intertwiners.

We have already seen in the case of $d = 1$ that there is a freedom in choosing how to realize smooth versions of TQFTs, namely by thinking of them as QFTs on spacetimes equipped with either a smooth map to a point or a metric, but with trivial dynamical evolution. We expect the same in higher d, but unfortunately no explicit description of a smooth space of such QFTs is yet available.

Neverthless, it seems reasonable to assume that in the former case we get the same pattern of symmetries as in [23] (*i.e.* we can assume that any group is discrete) and that in the latter case we endow the groups with their usual smooth structure. Considering the latter case, the permutation group S_n can act on $B(\mathbb{C}^\times)^{\oplus n}$ via permutation, and this action has a natural homotopy fixed point derived from the inclusion $\mathsf{S}_n \to (\mathbb{C}^\times)^{\oplus n} \rtimes \mathsf{S}_n$. Following the discussion in Section 2.4, this induces an S_n-action with homotopy fixed point on $B^2(\mathbb{C}^\times)^{\oplus n}$ which we denote $B^2(\mathbb{C}^\times)^{\oplus n}/\!\!/S_n$. This provides us with a natural guess for the smooth space X, which we note is connected and therefore (in this topos) path-connected.

Before delving into the nitty-gritty, let us make some remarks that may help with the physics interpretation. Very roughly, given a 0-form group G_0, a homotopy fixed point assigns to each $g_0 \in G_0$ a natural transformation from the functor (from the bordism category to some target category) defining the QFT to itself. A natural transformation in a 1-category assigns to each object in the source a morphism in the target, but in a higher category it assigns in addition 2-morphisms to 1-morphisms and so on. Here, a QFT assigns to the closed 1-manifold S^1 the $(\mathbb{C}, \mathbb{C})$-bimodule, *i.e.* the $\mathbb{C}$-vector space $\mathbb{C}^n$, which we interpret here as the state space associated to S^1, while we assign to g_0 a linear map from $\mathbb{C}^n$ to itself, which we can think of as the usual action of the group on the state space.

Continuing, given a group object G_1 whose underlying smooth space is connected (a '1-form group'), a homotopy fixed point assigns to each g_1 in G_1 a 2-natural transformation, which assigns to an object in the source a 2-morphism in the target, and so on. So here, a QFT assigns to a closed 0-manifold (*i.e.* a point) the commutative $\mathbb{C}$-algebra $\mathbb{C}^n$ over $\mathbb{C}$, while to g_1 we assign an intertwiner from the identity 1-morphism on $\mathbb{C}^n$, *i.e.* the bimodule ${}_{\mathbb{C}^n}(\mathbb{C}^n)_{\mathbb{C}^n}$, to itself.

Now for the nitty-gritty. Given a G-action on $X := B^2(\mathbb{C}^\times)^{\oplus n}/\!/S_n$, we factorize to obtain the diagram

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d] & X/\!/G \arrow[d] \\
BS_n \arrow[r, two heads] \arrow[d] & BS_n/\!/G \arrow[d] \\
* \arrow[r, two heads] & BG.
\end{tikzcd}$$

The bottom square defines a G-action on BS_n. From §2.4, we know that for this action we can replace G with $\tau_{\leq 0}G$. Supposing this to be a Lie group, we get homotopy fixed points given by smooth maps from $\pi_0\mathsf{G}$ to S_n that are twisted homomorphisms, in the sense of Eq. (3). Since the state space associated to S^1 is $\mathbb{C}^n$, we interpret these as twisted representations of the state space that are special in that the structure of the TQFT forces them to act by permuting the states.

Although every homotopy fixed point of a G-action on X induces one on $\tau_{\leq 0}X$, it is not the case that the former are completely determined by the latter, even when G is 0-truncated. This is most easily seen by considering the invertible case where $n = 1$ and the only action on $\tau_{\leq 0}X = BS_1 = *$ is the trivial one. We then get an action of the form

$$\begin{tikzcd}
B^2\mathbb{C}^\times \arrow[r, two heads] \arrow[d] & B^2\mathbb{C}^\times/\!/G \arrow[d] \\
* \arrow[r, two heads] & BG.
\end{tikzcd}$$

The space of homotopy fixed points $\mathcal{S}\mathrm{m}_{/BG}(BG, B^2\mathbb{C}^\times/\!/G)$ is in [33] interpreted as the degree-two $\mathbb{C}^\times$-cohomology of BG with coefficients in a local system given by the action of G on $\mathbb{C}^\times$. In general, this cohomology is not related to any traditional cohomology theory, even when G is a Lie group. However, in the case of a trivial action, the story is somewhat simplified. For instance, when G is a compact Lie group, we have, following [38, Thm. 6.4.38],

$$\pi_0\mathcal{S}\mathrm{m}_{/BG}(BG, B^2\mathbb{C}^\times \times BG) = H^2_{\mathrm{Segal}}(\mathsf{G}, \mathbb{R}_+) \times H^3_{\mathrm{Segal}}(\mathsf{G}, \mathbb{Z}),$$

where $H^*_{\mathrm{Segal}}(-,-)$ represents Lie group cohomology as defined by Segal [6,41]. The first factor simply corresponds to smooth cocycles *i.e.* smooth maps $\chi : \mathsf{G} \times \mathsf{G} \to \mathbb{R}_+$ satisfying

$$\chi(\mathsf{g}_2, \mathsf{g}_3) + \chi(\mathsf{g}_1, \mathsf{g}_2\mathsf{g}_3) = \chi(\mathsf{g}_1\mathsf{g}_2, \mathsf{g}_3) + \chi(\mathsf{g}_1, \mathsf{g}_2).$$

We have that $e^{\chi(\mathsf{g}_1, \mathsf{g}_2)} \in \mathbb{C}^\times$ represents a self-intertwiner of the bimodules ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ associated to the 1-dimensional interval.

Sticking with the invertible case $n = 1$, let us study pure 1-form symmetries. So we take G to be 1-connected and 2-truncated, so that we may write it as, $B^2\hat{G}$, and we furthermore take $\hat{G}$ to correspond to a Lie group. From the discussion in §2.4 on Eilenberg-Maclane objects, it follows that, since there is only one group action of G with homotopy fixed point on $B\mathbb{C}^\times$, there is also only one on $B^2\mathbb{C}^\times$, and this corresponds to the trivial action. The homotopy fixed point space is given by $\mathcal{S}\mathrm{m}(BG, B^2\mathbb{C}^\times)$. There is an essentially surjective functor $\mathcal{S}\mathrm{m}_*(BG, B^2\mathbb{C}^\times) \to \mathcal{S}\mathrm{m}(BG, B^2\mathbb{C}^\times)$ induced by 'forgetting the point'. We have, from [29, Prop. 7.2.2.12], that $\mathcal{S}\mathrm{m}_*(B^2\hat{G}, B^2\mathbb{C}^\times)$ is equivalent to the set of homomorphisms r from $\hat{\mathsf{G}}$ to $\mathbb{C}^\times$. Thus, every homotopy fixed point can be traced back to a 1-dimensional $\hat{\mathsf{G}}$ representation r with values in $\mathbb{C}^\times$. For each $\mathsf{g} \in \hat{\mathsf{G}}$ we can again have that $\mathsf{r}(\mathsf{g}) \in \mathbb{C}^\times$ is a self-intertwiner of the bimodules ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ associated to the 1d-interval.

5. Closing words

We have described how the language of ∞-topoi, in general, and smooth spaces, in particular, can be used to formulate the notion of smooth generalized symmetries of dynamical quantum field theories. Though we have tried to put the basic concepts in place, it is clear that we have barely scratched the surface in terms of what could be done. Indeed, comparing to [23], we see that we have only generalized the very first notion there, namely that of generalized global symmetries. It remains, for example, to study gauge symmetries and associated anomalies, for which we uncovered a rich story in [23], tied to the cobordism hypothesis. We expect the same to be true in the smooth context.

Elevating symmetries to smooth symmetries also offers us the hope of being able to differentiate. There are tools for doing this in the topos of smooth spaces [36,38] and this presumably furnish us with the means of deriving a generalized version of Noether's theorem relating smooth symmetries and conserved charges of QFTs, as we briefly sketched in [23].

Part of the reason that we have not been able to do very much in terms of applications is certainly the intrinsic mathematical difficulty of working in an ∞-topos. But another part is that we currently lack many examples of smooth spaces of QFTs. This in turn is partly due to the lack of understanding of what higher category, generalizing the 1-category of vector spaces and linear maps, should be used as the target for QFTs. We have reasons to hope, however, that progress will be made on this question in the near future; we hope that this will allow the power of the ∞-categorical methods described here to be brought to bear on physics. “To infinity and beyond!”

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Appendix A. Group actions on $B_\triangledown K$

A.1. Lie groups acting on $B_\triangledown K$

As we have seen in §2.3, any homotopy fixed point of any action of any G corresponds to a homotopy fixed point of a corresponding action on some connected component, which is equivalent to BK^{δ}, and for these we can take G to be 0-truncated, without loss of generality.

If we further assume that G corresponds to a Lie group G, then, as we have argued in §3.1.2, the analysis of actions on BK^{δ} and their homotopy fixed points collapses to the analysis of short exact sequences of groups and their splittings.

We can also say something about the presence of smoothness anomalies in this case, at least if we also assume that the Lie group K is abelian. This will enable us to give another concrete example of a smoothness anomaly in Appendix A.2, albeit in a case that is most likely of mathematical, rather than physical, interest.

The arguments are somewhat involved, so let us first sketch the essence of the idea before dotting the is and crossing the $\hbar$s to make everything precise. Suppose the action of G on BK^{δ} corresponds to assigning the automorphism $\mathsf{a} : \mathsf{K}^{\delta} \to \mathsf{K}^{\delta}$ to some element of G. If there is no smoothness anomaly, this must lift to some autoequivalence $f : B_\triangledown K \to B_\triangledown K$, which in turn induces an autoequivalence on the space of bundles with connection for every manifold M.

Now, given a principal K-bundle with connection on a manifold M, we can compute its holonomy. When the Lie group K is abelian, this allows us to associate an element in K to a loop in M; since a small deformation of a loop leads to a small change in the holonomy, we expect that this leads to a suitably defined morphism in $\mathcal{S}\mathrm{m}$.

The holonomy depends only on restrictions of the principal bundle with connection to S^1, where the connection is always flat. Since we have already seen that BK^{δ} can be interpreted as a smooth space classifying principal bundles with flat connections, the induced effect of f on holonomy is to post-compose the holonomy morphism of the domain bundle with connection on M with the map a. But if a, which is certainly smooth with respect to K^{δ}, is not smooth with respect to the smooth structure on K, then it seems unlikely that the resulting holonomy morphism of the codomain bundle will be smooth; if it isn't, f cannot exist and so we have a smoothness anomaly.

Now for the gory details. Let G be a Lie group and K be an abelian Lie group. Starting from a G-action on $B_\triangledown K$ admitting a homotopy fixed point, we get a G-action on BK^{δ}. Choosing a horn filler of $* \to BG \leftarrow *$ (which corresponds to choosing an element of G), we get a commutative diagram

$$\begin{tikzcd}
BK^{\delta} \arrow[rr, "j"] \arrow[dd] & & B_\triangledown K \arrow[rr] \arrow[dd] & & * \arrow[dd] \arrow[dr, Rightarrow] & \\
& BK^{\delta} \arrow[ul, "Ba"'] \arrow[rr, "j"] \arrow[dl] & & B_\triangledown K \arrow[ul, "f"'] \arrow[rr] \arrow[dl] & & * \arrow[ul] \arrow[dl] \\
BK^{\delta}/\!\!/G \arrow[rr] & & B_\triangledown K/\!\!/G \arrow[rr] & & BG, &
\end{tikzcd} \tag{A.1}$$

in which all squares are cartesian, the bottom and front rectangles are equivalent to one another, and in which the morphisms denoted Ba and f must be equivalences, since they are pullbacks of the equivalence $* \to *$.

Supposing that we instead start from a G-action on BK^δ admitting a homotopy fixed point, described by the homomorphism $\mathsf{e} : \mathsf{G} \times \mathsf{K}^\delta \to \mathsf{K}^\delta$, we construct the allowed outer prisms in this diagram. All the objects in this prism are deloopings of Lie groups, so we can replace by an equivalent diagram in the category $\mathcal{L}$ of the form

$$\begin{tikzcd}
\mathsf{K}^\delta \arrow[rr] \arrow[dd] & & * \arrow[dd] \arrow[dr, Rightarrow, shorten >=8pt] & \\
& \mathsf{K}^\delta \arrow[ul, "\mathsf{a}"] \arrow[rr] \arrow[dl] & & * \arrow[ul] \arrow[dl] \\
\mathsf{K}^\delta \rtimes \mathsf{G} \arrow[rr] & & \mathsf{G}. &
\end{tikzcd}$$

Choosing the 2-simplex filler indicated in the diagram to be $\mathsf{g} \in \mathsf{G}$, the most general allowed form of a, based on different manifestations of the pullback squares, is given by $\mathsf{b} \circ \mathsf{e}_{\mathsf{g}}^{-1} \circ \mathsf{b}^{-1}$, where b is an automorphism of K^δ.[28]

A smoothness anomaly will result if there exists $\mathsf{g} \in \mathsf{G}$ such that it is not possible to find a corresponding allowed form of a and an equivalence f making a commutative diagram of the form

$$\begin{tikzcd}
BK^\delta \arrow[r, dashed, "Ba"] \arrow[d] & BK^\delta \arrow[d] \\
B_\nabla K \arrow[r, dashed, "f"'] & B_\nabla K.
\end{tikzcd}$$

Now we simplify the problem using holonomy, which defines a morphism in $\mathcal{S}\mathrm{m}$, denoted $\mathrm{hol} : (B_\nabla K)^{S^1} \to K$,[29] where $(-)^{(-)} : \mathcal{S}\mathrm{m} \times \mathcal{S}\mathrm{m}^{op} \to \mathcal{S}\mathrm{m}$ denotes the internal hom in $\mathcal{S}\mathrm{m}$ and S^1 is the smooth space corresponding to the circle manifold S^1. This is described as follows [17]. Let X be a manifold. Then $\mathcal{S}\mathrm{m}(X, (B_\nabla K)^{S^1})$ is the groupoid of principal K-bundles with connection over $\mathsf{X} \times \mathsf{S}^1$. From such a principal bundle we get a smooth map $\mathsf{X} \to \mathsf{K}$, *i.e.* an object of $\mathcal{S}\mathrm{m}(X, K)$, by taking for each $\mathsf{x} \in \mathsf{X}$ the holonomy along $\{\mathsf{x}\} \times \mathsf{S}^1$. This construction is natural in X and so gives rise to a morphism $(B_\nabla K)^{S^1} \to K$.

Consider now the addition of the following data: A smooth manifold M and a principal K-bundle P on $\mathsf{M} \times \mathsf{S}^1$ with connection. Such a bundle is an object in $\mathcal{S}\mathrm{m}(M \times S^1, B_\nabla K)$ or equivalently an object in $\mathcal{S}\mathrm{m}(M, (B_\nabla K)^{S^1})$; we denote the latter object by $P : M \to (B_\nabla K)^{S^1}$. With this data, we can form the following commutative diagram

$$\begin{tikzcd}
& (BK^\delta)^{S^1} \arrow[r, dashed, "(Ba)^{S^1}"] \arrow[d] & (BK^\delta)^{S^1} \arrow[d] \\
M \arrow[r, "P"] & (B_\nabla K)^{S^1} \arrow[r, dashed, "f^{S^1}"'] \arrow[d, "\mathrm{hol}"'] & (B_\nabla K)^{S^1} \arrow[d, "\mathrm{hol}"] \\
& K & K.
\end{tikzcd}$$

From this diagram we get two morphisms $r : M \to K$ and $r' : M \to K$ defined, respectively, as $\mathrm{hol} \circ P$ and $\mathrm{hol} \circ f^{S^1} \circ P$. Since r and r' correspond to smooth maps between manifolds, they are determined by their underlying maps of sets, or equivalently by the morphisms Γr and $\Gamma r'$ in $\mathcal{S}$.

On applying $\Gamma : \mathcal{S}\mathrm{m} \to \mathcal{S}$ to the above diagram, we can make use of the fact that $\mathcal{S}\mathrm{m}(S^1, BK^\delta) \to \mathcal{S}\mathrm{m}(S^1, B_\nabla K)$ is an equivalence. Indeed it is -1-truncated, since $BK^\delta \to B_\nabla K$ is, and it is -1-connected, since every bundle with connection over S^1 is flat. Since the space $\mathcal{S}\mathrm{m}(S^1, BK^\delta)$ corresponds to the groupoid of principal K^δ-bundles on S^1, an object of $\mathcal{S}\mathrm{m}(S^1, BK^\delta)$ corresponds to a homomorphism $\mathsf{h} : \mathbb{Z} \to \mathsf{K}^\delta$. The morphism $\Gamma\mathrm{hol} : \mathcal{S}\mathrm{m}(S^1, BK^\delta) \to \Gamma K$ corresponds to evaluating h at $1 \in \mathbb{Z}$ and the morphism $\Gamma(Ba)^{S^1}$ corresponds to post-composing with a. Thus, we have a commutative diagram in $\mathcal{S}$ of the form

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[28] This statement remains true if we replace K^δ by any abelian Lie group, a fact we will use for our second example of a smoothness anomaly in §4.3.

[29] When K is non-abelian, we instead get a morphism to the smooth space $K /\!\!/ K$, where the action of K on itself is by conjugation.

$$\begin{tikzcd}
\mathcal{S}\mathrm{m}(S^1, BK^\delta) \arrow[r, "\Gamma(Ba)^{S^1}"] \arrow[d, "\Gamma\mathrm{hol}"'] & \mathcal{S}\mathrm{m}(S^1, BK^\delta) \arrow[d, "\Gamma\mathrm{hol}"] \\
\Gamma K \arrow[r, "\Gamma a"'] & \Gamma K,
\end{tikzcd}$$

where Γa is the morphism in $\mathcal{S}$ corresponding to the map of sets underlying the smooth (but trivially so) homomorphism $\mathsf{a} : \mathsf{K}^\delta \to \mathsf{K}^\delta$. From this, we can deduce that in $\mathcal{S}$ we have $\Gamma r' = \Gamma a \circ \Gamma r$, or equivalently that $\mathsf{r}'^\delta = \mathsf{a} \circ \mathsf{r}^\delta$ as maps of sets. If we can construct r and a such that the set map $\mathsf{a} \circ \mathsf{r}^\delta$ does not descend from a smooth map r', then the group action we started with will have a smoothness anomaly. We will see an example of this in Appendix A.2.

A.2. Smooth TQFTs in $d = 1$ – take three

In this appendix we repeat the analysis of §4.1.1, taking as X the smooth space $\coprod_n B_\nabla GL_n$. This is a possible (though admittedly unlikely) candidate for the smooth space version of TQFTs in $d = 1$.

We find that the space called BGL_n there is replaced not by the connected smooth space BGL_n, as one might naïvely have guessed, but rather by the non-connected smooth space $B_\nabla GL_n$. This seems perhaps surprising at first glance, but the arguments of §A.1 show that the upshot for physical symmetries, at least from Simplicio's point of view, is entirely reasonable: for a group object G corresponding to a Lie group G, say, we can replace $B_\nabla GL_n$ by BGL_n^δ and G by $\pi_0\mathsf{G}$. In other words, the symmetries of the smooth version are unchanged compared to those of the original: the relevant group actions are specified by homomorphisms $\pi_0\mathsf{G} \to \mathrm{Aut}\,\mathsf{GL}_n^\delta$, where, as before, GL_n^δ is GL_n considered as a discrete group, and an associated homotopy fixed point is given by a set map $\mathsf{r} : \pi_0\mathsf{G} \to \mathsf{GL}_n^\delta$ satisfying the corresponding version of Eq. (3). The group of automorphisms of GL_n^δ, as described in [14], is generated by inner automorphisms, automorphisms involving determinants, inversion-transposition and automorphisms involving field automorphisms.

So a smooth 1-d oriented G-symmetric TQFT is, at least in this incarnation, a representation of $\pi_0\mathsf{G}$, but twisted in the sense of Eq. (3); the morphism that forgets the G-symmetry sends such a twisted representation to its underlying vector space.

The general discussion on smoothness anomalies for $B_\nabla K$ for abelian K in §A.1 can be used to demonstrate that we can have a smoothness anomaly in the $n = 1$ case (so $\mathsf{K} = \mathbb{C}^\times$).

We begin by asserting that there exists (at least assuming the axiom of choice) an automorphism of order three of the discrete group $(\mathbb{C}^\times)^\delta$. For instance, given $\mathbb{C}^\times \simeq \mathbb{R} \times \mathsf{U}_1$, where $\mathbb{R}$ is the group of additive reals, one can form such an automorphism using an order-three permutation of a basis for $\mathbb{R}^\delta$ (constructed using the axiom of choice), considered as a vector space over the rationals.

It will be crucial for what follows that such an automorphism cannot lie in the same conjugacy class as an automorphism of $(\mathbb{C}^\times)^\delta$ that is smooth with respect to the usual Lie group structure on $\mathbb{C}^\times$, since such an automorphism must have an order valued in $\{1, 2, \infty\}$.

Proceeding, consider the action of $\mathbb{Z}/3$ on $B(\mathbb{C}^\times)^\delta$ described by the homomorphism $\mathsf{e} : \mathbb{Z}/3 \times \mathbb{C}^\times \to \mathbb{C}^\times$ that sends the generator of $\mathbb{Z}/3$ to such an automorphism of order three. Let us assume that this action extends to an action on $B_\nabla\mathbb{C}^\times$ and show that it leads to a contradiction. Following the argument in §A.1, we choose $\mathsf{M} = \mathbb{R}^2$, and P to be the trivial principal $\mathbb{C}^\times$-bundle on $\mathbb{R}^2 \times \mathsf{S}^1$ with connection 1-form A given, at $((\mathsf{x}, \mathsf{y}), \theta, \mathsf{k}) \in \mathbb{R}^2 \times \mathsf{S}^1 \times \mathbb{C}^\times$, by

$$\mathsf{A} = \mathsf{k}^{-1}\mathsf{dk} + \frac{1}{\pi}(\mathsf{x} + \mathsf{iy})\cos^2\theta \mathsf{d}\theta.$$

The morphism r described in §A.1 for this choice of M and P corresponds to the smooth surjective local diffeomorphism $\mathsf{r} : \mathbb{R}^2 \to \mathbb{C}^\times : (\mathsf{x}, \mathsf{y}) \mapsto e^{\mathsf{x}+\mathsf{iy}}$ implying (since r' is also smooth) that the automorphism a described in §A.1 is smooth. But the automorphism a is also an automorphism of order three, being in the same conjugacy class as the original automorphism, so cannot be smooth, which is a contradiction. Thus the assumption that the action extends must be false, and we have another example of a smoothness anomaly.

Appendix B. Proofs and other results

B.1. Actions equipped with homotopy fixed points as actions in the arrow topos

§2.2, p. 9.

The identity morphism id_{BG} of BG, as an object in $\mathcal{O}_{\mathcal{X}} := \mathcal{F}\mathrm{un}(\Delta^1, X)$, is connected and inherits a basepoint from BG in an obvious way. Equipped as such, id_{BG} describes a group object in $\mathcal{O}_{\mathcal{X}}$ that we denote $\Omega\mathrm{id}_{BG}$. The topos $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ can then be viewed as the category of $\Omega\mathrm{id}_{BG}$-actions.

The canonical functor $\mathcal{O}_{\mathcal{X}_{/BG}} \to (\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ is an equivalence [10, Prop. 4.2.12]. We want to prove the following theorem.

Theorem. *An object of $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ is in the essential image of the canonical functor $\mathcal{O}_{\mathcal{X}_{/BG}} \to (\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ restricted to the full subcategory $(\mathcal{X}_{/BG})_* \subseteq \mathcal{O}_{\mathcal{X}_{/BG}}$ if and only if it corresponds to a $\Omega\mathrm{id}_{BG}$-action on an object of the form $* \to X$.*

Proof. The canonical functor $\mathcal{O}_{\mathcal{X}_{/BG}} \to (\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ takes objects in the full-subcategory $(\mathcal{X}_{/BG})_*$ to objects in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ of the form

$$\begin{tikzcd}
BG \arrow[r] \arrow[d] \arrow[dr] & E \arrow[d] \\
BG \arrow[r] & BG,
\end{tikzcd}$$

where the bottom 2-simplex is the degenerate 2-simplex on id_{BG}. Furthermore, it is easy to see that every object in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ of this form is the image of some object in $(\mathcal{X}_{/BG})_*$. We denote by $(\mathcal{O}_{\mathcal{X}})^{deg}_{/\mathrm{id}_{BG}}$ the full subcategory of $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ spanned by such objects.

What remains is to show that an object of $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ is equivalent to an object in $(\mathcal{O}_{\mathcal{X}})^{deg}_{/\mathrm{id}_{BG}}$ if and only if it describes a group action on some $x : * \to X$ in $\mathcal{X}_* \subseteq \mathcal{O}_{\mathcal{X}}$. The ‘only if’ direction holds since limits are calculated pointwise [29, Cor. 5.1.2.3]. We thus focus on the ‘if’ direction.

Let $\mathcal{M} : \Delta^2 \times \Delta^1 \to \mathcal{X}$ be a diagram in $\mathcal{X}$. We use the following notion for the vertices of $\mathcal{M}$,

$$\begin{tikzcd}
0' \arrow[rr] \arrow[dd] & & 1' \arrow[dd] & \\
& 0 \arrow[lu] \arrow[ld] \arrow[rr] & & 1 \arrow[lu] \arrow[ld] \\
2 \arrow[rr] & & 3, &
\end{tikzcd}$$

and use subscripts to indicate the subdiagrams obtained by restricting to the specified vertices. A 1-simplex in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ is a diagram $\mathcal{M}$ subject to the condition that $\mathcal{M}_{23} = \mathrm{id}_{BG}$. Such a diagram $\mathcal{M}$ is a morphism in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ from $\mathcal{M}_{0123}$ to $\mathcal{M}_{0'1'23}$ (considered as objects in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$) and so it makes sense to ask when it is an equivalence. It turns out that it is an equivalence if $\mathcal{M}_{00'} : \Delta^1 \to \mathcal{X}$ and $\mathcal{M}_{11'} : \Delta^1 \to \mathcal{X}$, considered as morphisms in $\mathcal{X}$ (through the image of the morphism $0 \to 1$ of Δ^1), are equivalences.

Let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{X}$ be an object $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ representing an $\Omega\mathrm{id}_{BG}$-action on $x : * \to X$. We prove, by explicit construction, the existence of an equivalence $\mathcal{M}$ from an object in $(\mathcal{O}_{\mathcal{X}})^{deg}_{/\mathrm{id}_{BG}}$ to σ. We construct $\mathcal{M}$ as follows. For $\mathcal{M}$ to be a 1-simplex in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$ we require $\mathcal{M}_{23} = \mathrm{id}_{BG}$. For $\mathcal{M}$ to be a morphism into σ we require $\mathcal{M}_{0'1'23} = \sigma$. For $\mathcal{M}$ to be a morphism from an object in $(\mathcal{O}_{\mathcal{X}})^{deg}_{/\mathrm{id}_{BG}}$ we require $\mathcal{M}_{023}$ to be the degenerate simplex on id_{BG}. We now use that σ is a $\Omega\mathrm{id}_{BG}$-action on x to conclude that $\mathcal{M}_{0'2}$ is an equivalence. It follows that a filler exists for the Λ^2_2 horn of $\mathcal{M}_{00'2}$; we choose such a filler, which also fixes $\mathcal{M}_{00'}$. For any choice of filler $\mathcal{M}_{00'}$ is an equivalence, as a morphism in $\mathcal{X}$. For $\mathcal{M}$ itself to be an equivalence (as a morphism in $(\mathcal{O}_{\mathcal{X}})_{/\mathrm{id}_{BG}}$) we take $\mathcal{M}_1 = \mathcal{M}_{1'}$ and $\mathcal{M}_{11'}$ to be the identity of $\mathcal{M}_1$. We now fill in the remaining data. We take sequentially $\mathcal{M}_{00'1'}$, $\mathcal{M}_{011'}$, $\mathcal{M}_{11'3}$, $\mathcal{M}_{00'23}$, $\mathcal{M}_{00'1'3}$, and $\mathcal{M}_{011'3}$ as fillers of the horns Λ^2_1, Λ^2_2, Λ^2_1, Λ^3_2, Λ^3_1, and Λ^3_2. These fix, respectively, $\mathcal{M}_{01'}$, $\mathcal{M}_{01}$, $\mathcal{M}_{13}$, $\mathcal{M}_{00'3}$, $\mathcal{M}_{01'3}$, and $\mathcal{M}_{013}$. Note that we can fill the Λ^2_2 horn of $\mathcal{M}_{011'}$ since $\mathcal{M}_{11'}$ is an equivalence, as a morphism in $\mathcal{X}$.

The existence of the equivalence $\mathcal{M}$ proves our theorem. □

The conclusion drawn from this theorem is that G-actions equipped with homotopy fixed points can be thought of as $\Omega\mathrm{id}_{BG}$-actions on objects in $\mathcal{X}_*$. In addition to this, such an $\Omega\mathrm{id}_{BG}$-action has an underlying object $x : 1 \to X$ if its preimage in $(\mathcal{X}_{/BG})_*$ lands on x under the functor $(\mathcal{X}_{/BG})_* \to \mathcal{X}_*$ induced by b^*_{BG}.

B.2. Relation between m-connected covers of $X/\!\!/G$ for connected group objects

§2.3, p. 11.

Theorem. *If BG is p-connected, then any section $s : BG \to X/\!\!/G$ through $x : * \to X$ has a $(m-2)$-connected/$(m-2)$-truncated factorization through $\tau_{\geq m}(X/\!\!/G, [x]) \to X/\!\!/G$ for $0 \leq m \leq p+1$.*

Proof. We have $(m-2)$-connected/$(m-2)$-truncated factorizations $[x] : * \to \tau_{\geq m}(X/\!\!/G, [x]) \to X/\!\!/G$ and $s : BG \to \tau_{\geq m}(X/\!\!/G, [x]) \to X/\!\!/G$. Since we also have a factorization of $[x]$ through the basepoint of BG and s we have a commutative square formed by the solid lines of

$$\begin{tikzcd}
* \arrow[rr] \arrow[d] & & \tau_{\geq m}(X/\!\!/G, s) \arrow[d] \\
\tau_{\geq m}(X/\!\!/G, [x]) \arrow[rru, dashed, "f"] \arrow[rr] & & X/\!\!/G,
\end{tikzcd}$$

which by the $(m-2)$-connected/$(m-2)$-truncated factorization system defines a canonical map $f : \tau_{\geq m}(X/\!\!/G, [x]) \to \tau_{\geq m}(X/\!\!/G, s)$ as shown. Our theorem is equivalent to the statement that f is an equivalence under the given conditions.

Since the bottom map in the above square is $(m-2)$-truncated, f is itself $(m-2)$-truncated. Thus f is an equivalence if and only if it is $(m-2)$-connected. We now show, through a series of implications, that f is $(m-2)$-connected if BG is p-connected for some $p \geq m-1$. Since $* \to \tau_{\geq m}(X/\!\!/G, [x])$ is $(m-2)$-connected, from [29, Prop. 6.5.1.16] and the top left triangle in the above diagram, f is $(m-2)$-connected if and only if the morphism $* \to \tau_{\geq m}(X/\!\!/G, s)$ is. But, $* \to \tau_{\geq m}(X/\!\!/G, s)$ is a composition of $* \to BG$ and the $(m-2)$-connected morphism $BG \to \tau_{\geq m}(X/\!\!/G, s)$. Thus, from [29, Prop. 6.5.1.16], $* \to \tau_{\geq m}(X/\!\!/G, s)$ is $(m-2)$-connected if $* \to BG$ is $(m-2)$-connected. From [29, Prop. 6.5.1.20], $* \to BG$ is $(m-2)$-connected if and only if BG is $(m-1)$-connected. But BG is $(m-1)$-connected if it is p-connected for some $p \geq m-1$. □

We remark that this argument could equally be viewed as being made in $\mathcal{X}_{/BG}$ as in $\mathcal{X}$.

B.3. Truncating actions on connected objects equipped with homotopy fixed points

§2.4, p. 12.

Theorem. *If X is both n-truncated and 0-connected, then every G-action on X equipped with a homotopy fixed point is in the essential image of the functor $(\mathcal{X}_{/B\tau_{\leq n-1}G})_* \to (\mathcal{X}_{/BG})_*$.*

Proof. Given a G-action on X equipped with a homotopy fixed point, we can form a commutative diagram in $\mathcal{X}$ of the form

$$\begin{tikzcd}
* \arrow[r, two heads] \arrow[d] & BG \arrow[r, two heads] \arrow[d] & B\tau_{\leq n-1}G \arrow[d] \\
X \arrow[r, two heads] \arrow[d] & X/\!\!/G \arrow[r, two heads] \arrow[d] & \tau_{\leq n}(X/\!\!/G) \arrow[d] \\
* \arrow[r, two heads] & BG \arrow[r, two heads] & B\tau_{\leq n-1}G,
\end{tikzcd}$$

where the left-hand side squares are cartesian, and the right-hand squares commute because of naturality. We have used that $\tau_{\leq n}BG \simeq B\tau_{\leq n-1}G$. We must show that the right-hand squares are, in addition, cartesian when X is n-truncated and 0-connected, since then, under the functor $(\mathcal{X}_{/B\tau_{\leq n-1}G})_* \to (\mathcal{X}_{/BG})_*$, the object of $(\mathcal{X}_{/B\tau_{\leq n-1}G})_*$ described by the 2-simplex formed by the right-hand vertical arrows would be mapped into the object of $(\mathcal{X}_{/BG})_*$ described by the 2-simplex formed by the middle vertical arrows.

Now, the right-hand total rectangle is manifestly cartesian. The map $X \to *$ is 0-connected and thus $* \to X$ is an effective epimorphism. Thus so is $BG \to X/\!\!/G$, and so is its truncation $B\tau_{\leq n-1}G \to \tau_{\leq n}(X/\!\!/G)$ by successive use of [29, Prop. 6.5.1.16 (5)]. Thus, by reverse pasting, [1, Lem. 3.7.3], if we can show that the top right square is cartesian, the bottom right square will be too.

By [34, Rem. 3.64] the top right square will be cartesian if we can show that $BG \to X/\!\!/G$ is $(n-1)$-truncated. To see that it is, first note that $* \to X$ is $(n-1)$-truncated, which follows from the fact that looping increases truncatedness. Then note that, since $X \to X/\!\!/G$ is an effective epimorphism, we can use [29, Prop. 6.2.3.17] to conclude that $BG \to X/\!\!/G$ must be $(n-1)$-truncated. □

B.4. A recipe for finding certain group actions

§2.4, p. 13.

It was stated in the main text that we can sometimes find group actions in an arbitrary topos in much the same way as in the topos of spaces, provided that certain technical assumptions are satisfied.

We begin by spelling these assumptions out. Firstly, we must assume that the object X on which we act can be written as $\coprod_{i \in A} X_i$, where A is an indexing set and all X_i are 0-connected. Secondly, we must assume that every 0-connected object in the topos (or at least every object that appears in the recipe below) has only one point up to homotopy. Thirdly, we must assume that the homotopy quotient $X/\!\!/G$ corresponding to the action of G on X takes the form $\coprod_\alpha (\coprod_{i \in A_\alpha} X_i/\!\!/G)$, where $\{A_\alpha\}$ is a partition of A and all $(\coprod_{i \in A_\alpha} X_i/\!\!/G)$ are 0-connected. In the topos of spaces, these assumptions hold (for any object and action thereon), so the recipe gives us an algorithm to find all possible actions on any object.

Now we describe the recipe.

Step 1: The first step is to choose a partition $\{A_\alpha\}$ of the set A subject to the condition that, for each $i, j \in A_\alpha$, we have $X_i \simeq X_j$. We can see the need for this condition as follows. We want to find G-actions on $X_\alpha := \coprod_{i \in A_\alpha} X_i$ for each A_α such that $X_\alpha /\!\!/ G$ is connected. Suppose we are given such an action $X_\alpha /\!\!/ G$. As suggested in the main text, we can take its 0-connected/0-truncated factorization, giving the diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, two heads] \arrow[d] & X_\alpha /\!\!/ G \arrow[d] \\
\coprod_{i \in A_\alpha} *_i \arrow[r, two heads] \arrow[d] & \tau_{\leq 0} X_\alpha /\!\!/ G \arrow[d] \\
* \arrow[r, two heads] & BG.
\end{tikzcd}$$

(We have put subscripts on the terminal objects $*$, to indicate which X_i they descend from.) By choosing a $j \in A_\alpha$, we can form the diagram

$$\begin{tikzcd}
X_j \arrow[r] \arrow[d] & X_\alpha \arrow[r, two heads] \arrow[d] & X_\alpha /\!\!/ G \arrow[d] \\
*_j \arrow[r] & \coprod_{i \in A_\alpha} *_i \arrow[r, two heads] & \tau_{\leq 0} X_\alpha /\!\!/ G,
\end{tikzcd}$$

in which the right square is cartesian by definition and the left square is cartesian by the fact that in any topos we have descent over coproducts [35]. By pasting, the rectangle formed from the two squares is also cartesian. We now make use of our second assumption, namely that connected objects only have one point up to homotopy. Under this assumption, since $\tau_{\leq 0} X_\alpha /\!\!/ G$ is connected, the diagram above shows that we must have $X_i \simeq X_j$ for each $i, j \in A_\alpha$ if $X_\alpha /\!\!/ G$ is to be connected.

Step 2: The second step is to choose a G-action on $\tau_{\leq 0} X_\alpha \simeq \coprod_{i \in A_\alpha} *_i$ for which $\tau_{\leq 0} X_\alpha /\!\!/ G$ is connected.

Choosing a $j \in A_\alpha$, we can now form the diagram

$$\begin{tikzcd}
X_j \arrow[r] \arrow[d] & X_\alpha \arrow[r, dashed, two heads] \arrow[d] & ? \arrow[d, dashed] \\
*_j \arrow[r] & \coprod_{i \in A_\alpha} *_i \arrow[r, two heads] & \tau_{\leq 0} X_\alpha /\!\!/ G,
\end{tikzcd}$$

in which the left-hand square is cartesian, and the rectangle and right-hand square want to be cartesian too. Here $\tau_{\leq 0} X_\alpha /\!\!/ G$ is connected, and pointed by $*_j$, thus we may associate to it a group object $\Omega \tau_{\leq 0} X_\alpha /\!\!/ G$.

Step 3: For all α and a chosen $j \in A_\alpha$, choose an action of $\Omega \tau_{\leq 0} X_\alpha /\!\!/ G$ on X_j. This gives us an object E, which sits in the commutative diagram

$$\begin{tikzcd}
X_j \arrow[r] \arrow[d] & Y \arrow[r, two heads] \arrow[d] & E \arrow[d] \\
*_j \arrow[r] & \coprod_i *_i \arrow[r, two heads] \arrow[d] & \tau_{\leq 0} X_\alpha /\!\!/ G \arrow[d] \\
 & * \arrow[r, two heads] & BG.
\end{tikzcd}$$

where all squares are cartesian, which determines Y. In fact, since it does not matter which j we chose, Y must be $\coprod_{i \in A_\alpha} X_i$ by the universality of coproducts [35]. Thus every E is in fact a group action on X_α, so we are entitled to call it $X_\alpha /\!\!/ G$.

Step 4: To construct the final action on X, we take the coproduct in $\mathcal{X}_{/BG}$ of $\coprod_\alpha X_\alpha /\!\!/ G$.

B.5. Absence of smoothness anomalies for TFTs in the smooth space of QM

§4.2.1, p. 23.

Theorem. *Let G be a group object in $\mathcal{S}\mathrm{m}$ corresponding to a Lie group G. For every G-action on BGL_n that admits a homotopy fixed point, there is a G-action on $X = Mat_n /\!\!/ GL_n$ with homotopy fixed point s through $h = 0$, such that $\tau_{\geq 1}(X /\!\!/ G, s) \to BG$ corresponds to the original G-action on BGL_n.*

Proof. A G-action on BGL_n admitting a homotopy fixed point is specified by a G-action $\mathsf{e} : \mathsf{G} \times \mathsf{GL}_n \to \mathsf{GL}_n$ in the traditional sense. The corresponding G-action as an object in $\mathcal{S}\mathrm{m}_{/BG}$ is the morphism $B\hat{G} \to BG$ corresponding to the projection of the semi-direct product $\hat{\mathsf{G}} := \mathsf{GL}_n \rtimes_{\mathsf{e}} \mathsf{G}$ onto G.

There exists an $\hat{G}$-action on Mat_n corresponding to the smooth map $\hat{\mathsf{G}} \times \mathsf{Mat}_n \to \mathsf{Mat}_n : ((\mathsf{M}, \mathsf{g}), \mathsf{h}) \mapsto \mathsf{M} d\mathsf{e}_{\mathsf{g}}(\mathsf{h})\mathsf{M}^{-1}$, where $d\mathsf{e}_{\mathsf{g}} : \mathsf{Mat}_n \to \mathsf{Mat}_n$ is the differential of $\mathsf{e}_{\mathsf{g}} : \mathsf{GL}_n \to \mathsf{GL}_n$. This action on Mat_n has a section through $h = 0$. We can combine the G-action on BGL_n that yields $\hat{G}$ and the $\hat{G}$-action on Mat_n, along with their sections, to form the commutative diagram

$$\begin{tikzcd}
 & * \arrow[r] \arrow[d] & BG \arrow[d] \\
 & BGL_n \arrow[r] \arrow[d] & B\hat{G} \arrow[d] \\
Mat_n \arrow[r] \arrow[d] & Mat_n /\!/ GL_n \arrow[r] \arrow[d] & Mat_n /\!/ \hat{G} \arrow[d] \\
* \arrow[r] & BGL_n \arrow[r] \arrow[d] & B\hat{G} \arrow[d] \\
 & * \arrow[r] & BG,
\end{tikzcd}$$

in which all squares are cartesian. By pasting, all rectangles are cartesian too, so we can read off that $Mat_n /\!/ \hat{G} \to BG$ is an action on $Mat_n /\!/ GL_n$ with section s through $h = 0$. Moreover, since the morphism $B\hat{G} \to Mat_n /\!/ \hat{G}$ is (-1)-truncated and the morphism $BG \to B\hat{G}$ is (-1)-connected, the factorization $s : BG \to B\hat{G} \to Mat_n /\!/ \hat{G}$ exhibits $B\hat{G}$ as $\tau_{\geq 1}(Mat_n /\!/ \hat{G}, s)$. □

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