



RESEARCH ARTICLE

Lim Ulrich sequences and Boij–Söderberg cones

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Abstract

This paper extends the results of Boij, Eisenbud, Erman, Schreyer and Söderberg on the structure of Betti cones of finitely generated graded modules and finite free complexes over polynomial rings, to all finitely generated graded rings admitting linear Noether normalizations. The key new input is the existence of lim Ulrich sequences of graded modules over such rings.

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1. Introduction

Let k be a field and $A := k[x_1, \dots, x_d]$ a polynomial ring in d -variables, each of degree one. The *Betti table* of a finitely generated graded A -module M is the matrix of integers, denoted $\beta(M)$, that records the homological positions and the twists of the graded free modules appearing in the minimal free resolution of M . Boij and Söderberg [1] conjectured an explicit description of the collection of all Betti tables of graded Cohen–Macaulay A -modules ‘up to rational scaling’. This conjecture was proved by Eisenbud and Schreyer [6] and subsequently extended to all graded A -modules by Boij and Söderberg [2]. In detail, for $0 \leq c \leq d$, let $\mathbf{B}^c(A)$ denote the cone inside the $\mathbb{Q}$ -vector space $\bigoplus_{0 \leq i \leq d, j \in \mathbb{Z}} \mathbb{Q}$ spanned by the Betti tables of all graded A -modules of codimension at least c . The results of Eisenbud and Schreyer, and Boij and Söderberg, give an equality

$$\mathbf{B}^c(A) = \mathbf{B}_d^c,$$

where $\mathbf{B}_d^c$ is the $\mathbb{Q}_{\geq 0}$ -span of certain explicit Betti tables $\beta(\underline{a})$, where $\underline{a} = (a_0, \dots, a_l)$ ranges over all ‘degree sequences’, that is to say, $(l+1)$ -tuples of increasing integers, for $l \geq c$; see Equation (5.2) below for definition of $\beta(\underline{a})$. A Cohen–Macaulay module M whose Betti table is a rational multiple of $\beta(\underline{a})$ for some choice of degree sequence $\underline{a}$ is called a ‘pure module’.

In this language, the theorem asserts that there exists a pure module for each possible degree sequence and $\mathbf{B}^c(A)$ is spanned by the Betti tables of pure modules. Roughly speaking, this means that the Betti tables of all modules are governed by those of the pure modules. In particular, the multiplicity conjecture of Huneke and Srinivasan is a direct consequence [1].

The main result of this paper extends these results to all $\mathbb{N}$ -graded k -algebras that admit linear Noether normalizations, that is to say, a linear system of parameters; see Section 4.4. We assume k is infinite throughout this paper, for the sake of simplicity, so this class of k -algebras includes the standard graded ones. For any graded k -algebra R and integer $0 \leq c \leq d := \dim(R)$, let $\mathbf{B}^c(R)$ be the cone spanned by Betti tables of modules of codimension at least c and of finite projective dimension. We prove:

Theorem 1.1 (Theorem 6.2). *For R and $\mathbf{B}_d^c$ as above, we have a containment $\mathbf{B}^c(R) \subseteq \mathbf{B}_d^c$, and equality holds when R is Cohen–Macaulay.*

As in [1], one can deduce the following bounds on the multiplicity of a perfect module; that is to say, a finitely generated module whose projective dimension equals its codimension. When R is Cohen–Macaulay, perfect modules are precisely those that are Cohen–Macaulay and of finite projective dimension. Thus, the statement below generalizes the theorem on multiplicities over polynomial rings.

Corollary 1.2 (Corollary 6.3). *With R as above, any perfect R -module M satisfies*

$$e(R) \frac{t_1 \cdots t_c}{c!} \leq e(M) \leq e(R) \frac{T_1 \cdots T_c}{c!},$$

where c is the codimension of M , and the t_i ’s and T_i ’s are the minimal and maximal shifts occurring in the minimal free resolution of M .

Eisenbud and Erman [4] generalized the results in [1, 6] to finite free complexes over A , and we extend their results to rings R as above; see Theorem 6.2.

The proof of Eisenbud and Schreyer uses a pairing between Betti tables of graded A -modules and cohomology tables of coherent sheaves on $\mathrm{Proj}(A) = \mathbb{P}_k^{d-1}$, along with a description of the rational cone spanned by the cohomology tables of such coherent sheaves. Although we do not use such a pairing in this paper, we do establish a result concerning the cone of cohomology tables for coherent sheaves over $\mathrm{Proj}(R)$ with R as in Theorem 1.1. When k is an infinite perfect field of positive characteristic, we prove $\overline{\mathbf{C}}(\mathbb{P}_k^{d-1}) = \overline{\mathbf{C}}(\mathrm{Proj}(R))$, where $\overline{\mathbf{C}}$ is the component-wise limit closure of cohomology tables of coherent sheaves; see Theorem 3.4.

The results stated above are known when R admits a graded Ulrich module or, equivalently, when $\mathrm{Proj}(R)$ admits an Ulrich sheaf; see [4]. But the existence of such modules is known only in a small number of cases. Instead, we use ‘lim Ulrich sequences’ of graded modules, which are sequences of graded modules that asymptotically approximate Ulrich modules. They exist over any graded k -algebra admitting a linear Noether normalization and where k is a infinite perfect field of positive characteristic. The relevant results are established in Sections 2 and 4.

2. Lim Ulrich sequences of sheaves

For any Noetherian scheme X , let $\mathbf{D}^b(X)$ be its derived category of coherent sheaves. The tensor product of coherent sheaves on X is denoted $- \otimes -$; its derived version is $- \otimes^{\mathbf{L}} -$. Throughout this manuscript, k is a field and $m \geq 1$ an integer. Let $\mathbb{P}_k^m$ be the projective space over k of dimension m ; we write $\mathbb{P}_k^m$ when the field k needs emphasis.

2.1. For each coherent sheaf $\mathcal{F}$ (or a bounded complex of such) on $\mathbb{P}^m$ set

$$\gamma_{i,t}(\mathcal{F}) := \text{rank}_k H^i(\mathbb{P}^m, \mathcal{F}(t)) \quad \text{for } i, t \text{ in } \mathbb{Z}.$$

With $\varepsilon: \mathbb{P}^m \rightarrow \text{Spec}(k)$ the structure map, for any coherent sheaf on $\mathcal{F}$ the counit of the adjoint pair $(\varepsilon^*, \varepsilon_*)$ is a natural map

$$\varepsilon^* \varepsilon_* \mathcal{F} \longrightarrow \mathcal{F}. \quad (2.2)$$

The following result is well-known; see [7, Proposition 2.1].

Lemma 2.3. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m$. The conditions below are equivalent:*

1. *The map (2.2) is an isomorphism.*
2. *$\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^m}^r$ for some integer $r \geq 0$.*
3. *$\gamma_{i,t}(\mathcal{F}) = 0$, except possibly when $i = 0$ and $t \geq 0$, or $i = m$ and $t \leq -m - 1$.*

When these conditions hold, $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^m}^r$ for $r = \gamma_{0,0}(\mathcal{F})$.

Any nonzero sheaf $\mathcal{F}$ on $\mathbb{P}^m$ satisfying the equivalent condition of the result above is said to be an *Ulrich sheaf*. Condition (3) is used to define Ulrich sheaves over arbitrary projective varieties.

Lim Ulrich sequences

We introduce the notion of lim Ulrich sequences of sheaves on $\mathbb{P}^m$, following [15, §6.6].

2.4. A *lim Ulrich sequence* of sheaves on $\mathbb{P}^m$ is a sequence $(\mathcal{F}_n)_{n \geq 0}$ of coherent sheaves on $\mathbb{P}^m$ for which the following properties hold:

1. $\gamma_{0,0}(\mathcal{F}_n) \neq 0$ for each $n \geq 0$;
2. There exists an integer t_0 such that $\gamma_{0,t}(\mathcal{F}_n) = 0$ for $t \leq t_0$ and all n ;
3. There exists an integer t_1 such that $\gamma_{\geq 1,t}(\mathcal{F}_n) = 0$ for $t \geq t_1$ and all n ;
4. Except possibly when $i = 0$ and $t \geq 0$, or $i = m$ and $t \leq -m - 1$, one has

$$\lim_{n \rightarrow \infty} \frac{\gamma_{i,t}(\mathcal{F}_n)}{\gamma_{0,0}(\mathcal{F}_n)} = 0.$$

The range of values of i and t arising in (4) is precisely the one from Lemma 2.3. For geometric applications, the key properties are (1) and (4); see, in particular, Theorem 2.8 below. The other conditions become important when considering the corresponding sequence of graded modules; see the proof of Theorem 4.17.

Clearly, if each $\mathcal{F}_n$ is an Ulrich sheaf, then the sequence $(\mathcal{F}_n)_{n \geq 0}$ is lim Ulrich. However, there are other lim Ulrich sequences on $\mathbb{P}_k^m$, at least when k has positive characteristic, and these will be used to construct lim Ulrich sequences on a much larger family of schemes; see Theorem 2.19.

Next, we present a characterization of lim Ulrich sequences on $\mathbb{P}^m$ in the spirit of the original definition of Ulrich sheaves.

2.5. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m$. Consider $\text{cone}(\mathcal{F})$, the mapping cone of the counit map (2.2) and the induced exact triangle

$$\varepsilon^* \varepsilon_* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \text{cone}(\mathcal{F}) \longrightarrow \quad (2.6)$$

in $\mathbf{D}^b(\text{coh } \mathbb{P}^m)$. Observe that this is natural in $\mathcal{F}$, but it is not compatible with twists. Keeping in mind that $\varepsilon^* \varepsilon_* \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^m}^r$, tensoring the exact triangle defining $\text{cone}(\mathcal{F})$ with $\mathcal{O}_{\mathbb{P}}(t)$ yields an exact triangle

$$\mathcal{O}_{\mathbb{P}^m}(t)^r \longrightarrow \mathcal{F}(t) \longrightarrow \text{cone}(\mathcal{F})(t) \longrightarrow .$$

The resulting exact sequence in cohomology reads

$$H^*(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(t))^r \rightarrow H^*(\mathbb{P}^m, \mathcal{F}(t)) \rightarrow H^*(\mathbb{P}^m, \text{cone}(\mathcal{F})(t)) \rightarrow H^{*+1}(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(t))^r. \quad (2.7)$$

The gist of the next result is that for a lim Ulrich sequence $(\mathcal{F}_n)_{n \geq 0}$, the counit maps $\varepsilon^* \varepsilon_* \mathcal{F}_n \rightarrow \mathcal{F}_n$ are asymptotically isomorphisms. If $\mathcal{F}$ is an Ulrich sheaf, applying it to the sequence $\mathcal{F}_n := \mathcal{F}$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^m}$, recovers Lemma 2.3, but of course the latter result, and its proof, are models for the one below.

Theorem 2.8. *Let $(\mathcal{F}_n)_{n \geq 0}$ be a lim Ulrich sequence of sheaves on $\mathbb{P}^m$. For any $\mathcal{G}$ in $\mathbf{D}^b(\text{coh } \mathbb{P}^m)$ and integers i, j , one has*

$$\lim_{n \rightarrow \infty} \frac{\gamma_{i,j}(\text{cone}(\mathcal{F}_n) \otimes^{\mathbf{L}} \mathcal{G})}{\gamma_{0,0}(\mathcal{F}_n)} = 0.$$

The proof only uses conditions (1) and (4) in Section 2.4.

Proof. In what follows, we write $\mathbb{P}$ instead of $\mathbb{P}^m$ and $\mathcal{O}$ for the structure sheaf on it. Set $r_n := \gamma_{0,0}(\mathcal{F}_n)$. It helps to consider the collection $\mathbf{S}$ of sequences $(\mathcal{C}_n)_{n \geq 0}$ with $\mathcal{C}_n$ in $\mathbf{D}^b(\text{coh } \mathbb{P})$, with the property that

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H^*(\mathbb{P}, \mathcal{C}_n)}{r_n} = 0. \quad (2.9)$$

The desired result is that the sequence with $\mathcal{C}_n := \text{cone}(\mathcal{F}_n) \otimes^{\mathbf{L}} \mathcal{G}$ is in $\mathbf{S}$. In the proof, we repeatedly use the following elementary observation: Given sequences $(\mathcal{C}'_n)_{n \geq 0}$, $(\mathcal{C}_n)_{n \geq 0}$ and $(\mathcal{C}''_n)_{n \geq 0}$, of bounded complexes of coherent sheaves such that for each n there is an exact triangle

$$\mathcal{C}'_n \rightarrow \mathcal{C}_n \rightarrow \mathcal{C}''_n \rightarrow$$

if two of the three sequences are in $\mathbf{S}$, then so is the third.

It follows from this observation that given an exact triangle $\mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow$ in $\mathbf{D}^b(\text{coh } \mathbb{P})$, if the desired result holds for any two of $\mathcal{G}'$, $\mathcal{G}$ and $\mathcal{G}''$, then it holds for the third. Since each object in the derived category is equivalent to a bounded complex consisting of twists of $\mathcal{O}$, a standard induction using these observations reduces checking the desired result to the case when $\mathcal{G} = \mathcal{O}(t)$, for $t \in \mathbb{Z}$; that is to say that the sequence $(\text{cone}(\mathcal{F}_n)(t))_{n \geq 0}$ is in $\mathbf{S}$.

When $-m \leq t \leq -1$, one has $H^*(\mathbb{P}, \mathcal{O}(t)) = 0$ so the exact sequence (2.7) with $\mathcal{F} := \mathcal{F}_n$ gives the first equality below:

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H^*(\mathbb{P}, \text{cone}(\mathcal{F}_n)(t))}{r_n} = \lim_{n \rightarrow \infty} \frac{\text{rank}_k H^*(\mathbb{P}, \mathcal{F}_n(t))}{r_n} = 0.$$

The second one is by condition 2.4(4). This is the desired result.

Suppose $t \geq 0$. We argue by induction on t that the sequence $(\text{cone}(\mathcal{F}_n)(t))_{n \geq 0}$ is in $\mathbf{S}$. The base case is $t = 0$. Then since $H^i(\mathbb{P}, \mathcal{O}(t)) = 0$ for all $i \geq 1$ the exact sequence (2.7), with $\mathcal{F} := \mathcal{F}_n$, reduces to the exact sequence

$$0 \rightarrow H^0(\mathbb{P}, \mathcal{O}(t))^{r_n} \rightarrow H^*(\mathbb{P}, \mathcal{F}_n(t)) \rightarrow H^*(\mathbb{P}, \text{cone}(\mathcal{F}_n)(t)) \rightarrow 0.$$

When $t = 0$ the map on the left is an isomorphism in degree 0, by the definition of r_n . Thus, we get

$$\text{rank}_k H^*(\mathbb{P}, \text{cone}(\mathcal{F}_n)) = \sum_{i \geq 1} \text{rank}_k H^i(\mathbb{P}, \mathcal{F}_n)$$

and the desired limit is again immediate from Section 2.4(4).

Suppose $t \geq 1$ and consider the Koszul resolution of $\text{cone}(\mathcal{F}_n)(t)$:

$$0 \rightarrow \text{cone}(\mathcal{F}_n)(t-m-1) \rightarrow \cdots \rightarrow \text{cone}(\mathcal{F}_n)^{m+1}(t-1) \rightarrow \text{cone}(\mathcal{F}_n)(t) \rightarrow 0.$$

Since $t-m-1 \geq -m$, from the already established part of the result and the induction hypotheses we get that the sequences $(\text{cone}(\mathcal{F}_n)(j))_{n \geq 0}$ are in $\mathbf{S}$ for $t-m-1 \leq j \leq t-1$. The exact sequence above implies that the same holds for $j = t$.

This completes the discussion of the case $t \geq 0$.

Given this one can use the Koszul resolution above and a descending induction on t to cover also the case $t \leq -m-1$. $\square$

We record a construction of $\lim$ Ulrich sequences of sheaves on $\mathbb{P}^m$ from [15, §7].

2.10. Let k be an infinite perfect field of positive characteristic p . Set $Z := (\mathbb{P}_k^1)^m$. Let $\rho: Z \rightarrow \mathbb{P}_k^m$ be the quotient by the action of the symmetric group S_m on Z or any finite flat map. For each $n \geq 0$ set

$$\mathcal{L}_n := \mathcal{O}_Z(p^n, 2p^n, \dots, mp^n) \quad \text{and} \quad \mathcal{E}_n := \rho_*(\mathcal{L}_n).$$

The $\mathcal{L}_n$ are line bundles on Z , and the $\mathcal{E}_n$ are vector bundles on $\mathbb{P}_k^m$. The following result is a special case of [15, Theorem 7.15]. Here, $\varphi: \mathbb{P}_k^m \rightarrow \mathbb{P}_k^m$ is the Frobenius map; it is a finite map since k is perfect.

Theorem 2.11. *Let k be an infinite perfect field of positive characteristic. Let $\mathcal{N}$ be a coherent sheaf on $\mathbb{P}_k^m$ of positive rank and $\gamma_{0,t}(\mathcal{N}) = 0$ for $t \ll 0$. With $\mathcal{E}_n$, the sheaf defined above, set*

$$\mathcal{G}_n := \varphi_*^n(\mathcal{N} \otimes \mathcal{E}_n) \quad \text{for } n \geq 0.$$

The sequence $(\mathcal{G}_n)_{n \geq 0}$ of sheaves is $\lim$ Ulrich.

2.12. Fix a proper k -scheme X and $\mathcal{L}$ a line bundle on X . For any coherent sheaf $\mathcal{F}$ on X and integer t , set

$$\mathcal{F}(t) := \mathcal{F} \otimes \mathcal{L}^{\otimes t}.$$

Recall that $\mathcal{F}$ is *globally generated* if there is a surjection $\mathcal{O}_X^r \rightarrow \mathcal{F}$ for some $r \geq 0$. The line bundle $\mathcal{L}$ is *ample* if for each coherent sheaf $\mathcal{F}$, the sheaf $\mathcal{F}(t)$ is globally generated for $t \gg 0$. The statement below is well-known.

Proposition 2.13. *Let k be a field, X a proper k -scheme and $\mathcal{L}$ a line bundle on X . The following statements are equivalent.*

1. $\mathcal{L}$ is ample and globally generated;
2. There exists a finite map $\rho: X \rightarrow \mathbb{P}_k^n$ for some integer n with $\mathcal{L} \cong \rho^*\mathcal{O}_{\mathbb{P}^n}(1)$.

If in addition k is infinite, with $m := \dim X$, these statements are equivalent to

- (3) *There exists a finite map $\pi: X \rightarrow \mathbb{P}_k^m$ and $\mathcal{L} \cong \pi^*\mathcal{O}_{\mathbb{P}^m}(1)$.*

Proof. The equivalence of (1) and (2) is proved in [9, Proposition I.4.4 and (the proof of) Proposition 1.4.6].

Suppose now that k is infinite. The implication (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): With ρ as in (2), we can find a linear projection $\pi': \mathbb{P}^n \rightarrow \mathbb{P}^m$ defined on all of $f(X)$ since k is infinite. The map $\pi' \circ \rho$ has the desired properties. $\square$

2.14. In what follows, we say $(X, \mathcal{L})$ is a *Noether pair* to mean that X is a proper scheme over an infinite field k , and $\mathcal{L}$ is an ample and globally generated line bundle on X . The basic example is the pair $(\mathbb{P}^m, \mathcal{O}(1))$ which we identify with $\mathbb{P}^m$. More generally, if R is a graded k -algebra admitting a linear system of parameters, then the pair $(\text{Proj}(R), \mathcal{O}(1))$ is a Noether pair; see Lemma 4.13.

A morphism $f: (X, \mathcal{L}) \rightarrow (X', \mathcal{L}')$ of Noether pairs is a map of k -schemes $f: X \rightarrow X'$ such that $f^*(\mathcal{L}') \cong \mathcal{L}$. By Proposition 2.13, given a Noether pair $(X, \mathcal{L})$ there is a finite dominant linear map of Noether pairs

$$\pi: (X, \mathcal{L}) \rightarrow \mathbb{P}_k^m \quad \text{where } m := \dim X.$$

Thus, by the projection formula, for any sheaf $\mathcal{F}$ on X one has

$$H^i(X, \mathcal{F}(t)) \cong H^i(\mathbb{P}_k^m, \pi_* \mathcal{F}(t)) \quad \text{for all } i, t. \quad (2.15)$$

This allows one to introduce a notion of *lim Ulrich sequence* of sheaves on X ; see Section 2.17. First, we record the result below, which is immediate from the equivalence (1) $\Leftrightarrow$ (2) in Proposition 2.13.

Lemma 2.16. *Let $(X, \mathcal{L})$ be a Noether pair and $f: X' \rightarrow X$ a finite morphism of k -schemes. Then $(X', f^* \mathcal{L})$ is also a Noether pair.*

2.17. Let $(X, \mathcal{L})$ be a Noether pair. An *Ulrich sheaf* on X is a coherent sheaf $\mathcal{F}$ such that the coherent sheaf $\pi_*(\mathcal{F})$ on $\mathbb{P}_k^m$ is Ulrich for some finite dominant map $\pi: (X, \mathcal{L}) \rightarrow \mathbb{P}_k^m$ of Noether pairs; see Section 2.1. Given Lemma 2.3 and Equation (2.15), one can characterize this property purely in terms of the cohomology modules of twists of $\mathcal{F}$; in particular, it is independent of the choice of π . In the same vein, a *lim Ulrich sequence* of sheaves on X is a sequence $(\mathcal{F}_n)_{n \geq 0}$ of coherent sheaves on X for which the sequence $(\pi_* \mathcal{F}_n)_{n \geq 0}$ is *lim Ulrich* in the sense of Section 2.4. Once again, this can be expressed purely in terms of the cohomology of the twists of $\mathcal{F}_n$, which reconciles the definition given here with that in [15, 6.6].

The result below serves to clarify that the *lim Ulrich* property is, to a certain extent, independent of the domain of definition of the sheaves involved.

Lemma 2.18. *Let $f: (X, \mathcal{L}) \rightarrow (X', \mathcal{L}')$ be a morphism of Noether pairs where f is finite and dominant. A sequence $(\mathcal{F}_n)_{n \geq 0}$ of coherent sheaves on X is *lim Ulrich* if and only if the sequence $(f_* \mathcal{F}_n)_{n \geq 0}$ of coherent sheaves on X' is *lim Ulrich*.*

Proof. The stated claim is immediate from the observation that if $\pi': X' \rightarrow \mathbb{P}^m$ is a finite dominant linear map, then so is the composition $\pi' \circ f$. $\square$

The result below is a minor extension of [15, Theorem 7.15].

Theorem 2.19. *Let $(X, \mathcal{L})$ be a Noether pair with $\dim X \geq 1$. Assume furthermore that the field k is infinite and perfect of positive characteristic. There exists a *lim Ulrich sequence* of sheaves $(\mathcal{F}_n)_{n \geq 0}$ on X such that for each $n \geq 0$ one has*

$$\text{depth}(\mathcal{F}_n)_x \geq \begin{cases} 1 & \text{when } x \in X \text{ is a closed point} \\ \text{depth } \mathcal{O}_{X,x} & \text{for all } x. \end{cases}$$

Proof. Set $m := \dim X$, and let $\pi: X \rightarrow \mathbb{P}_k^m$ be a finite dominant linear map. We begin by choosing a coherent sheaf $\mathcal{M}$ on X satisfying the following conditions

1. $H^0(\mathbb{P}_k^m, \pi_* \mathcal{M}(t)) = H^0(X, \mathcal{M}(t)) = 0$ for $t \ll 0$;
2. The sheaf $\pi_* \mathcal{M}$ on $\mathbb{P}_k^m$ has positive rank;
3. $\text{depth } \mathcal{M}_x \geq \text{depth } \mathcal{O}_{X,x}$ for each $x \in X$.

To construct such an $\mathcal{M}$, consider the closed subset W of X consisting of closed points $x \in X$ such that $\text{depth } \mathcal{O}_{X,x} = 0$. Let $\mathcal{J}$ be the sheaf of local sections of $\mathcal{O}_X$ supported on W , and set $\mathcal{M} := \mathcal{O}_X / \mathcal{J}$. This construction ensures $\mathcal{M}_x$ is isomorphic to $\mathcal{O}_{X,x}$ for $x \notin W$, and $\text{depth } \mathcal{M}_x \geq 1$ for $x \in W$. This justifies both (1) and (3); see Lemma 2.20 below. Moreover, $\dim \mathcal{M} = \dim X = \dim \mathbb{P}_k^m$, so $\pi_* \mathcal{M}$ has positive rank and (2) holds.

With $\mathcal{E}_n$ the sheaves on $\mathbb{P}_k^m$ defined in Section 2.10, set

$$\mathcal{F}_n := \varphi_*^n(\mathcal{M} \otimes \pi^* \mathcal{E}_n),$$

where φ is the Frobenius morphism on X . Since the Frobenius map preserves depth and $\pi^* \mathcal{E}_n$ is locally free, $\text{depth}(\mathcal{F}_n)_x = \text{depth} \mathcal{M}_x$ for each n and $x \in X$. In particular, $\text{depth}(\mathcal{F}_n)_x$ has the stated properties.

Using the fact that Frobenius commutes with pushforward of sheaves, and the projection formula, one gets

$$\pi_* \mathcal{F}_n = \pi_* \varphi_*^n(\mathcal{M} \otimes \pi^* \mathcal{E}_n) \cong \varphi_*^n(\pi_* \mathcal{M} \otimes \mathcal{E}_n).$$

Theorem 2.11 yields that the sequence $(\pi_* \mathcal{F}_n)_{n \geq 0}$ of sheaves on $\mathbb{P}_k^m$ is lim Ulrich. Thus, by definition, the sequence $(\mathcal{F}_n)_{n \geq 0}$ is lim Ulrich; see also Lemma 2.18. $\square$

Lemma 2.20. *Let X be as in Equation (2.14) and $\mathcal{F}$ a coherent sheaf on X . If $\mathcal{F}_x$ has positive depth at every closed point $x \in X$, then $H^0(X, \mathcal{F}(t)) = 0$ for $t \ll 0$.*

Proof. By pushing forward to $\mathbb{P}^m$, we may reduce to the case where $X = \mathbb{P}^m$. In this case, Grothendieck–Serre duality gives

$$H^0(X, \mathcal{F}(t))^* \cong \text{Ext}^m(\mathcal{F}, \mathcal{O}_X(-t - m - 1)),$$

where $(-)^*$ denotes k -linear duals and Ext is computed in the abelian category of coherent sheaves. We have a spectral sequence

$$E^{p,q} := H^p(X, \underline{\text{Ext}}^q(\mathcal{F}, \mathcal{O}_X(-t - m - 1))) \implies \text{Ext}^{p+q}(\mathcal{F}, \mathcal{O}_X(-t - m - 1)),$$

where $\underline{\text{Ext}}$ denotes sheafified Ext . Since $\mathcal{F}_x$ has positive depth at all closed points and the local rings $\mathcal{O}_{X,x}$ are regular, we have $\underline{\text{Ext}}^m(\mathcal{F}, \mathcal{O}_X) = 0$; thus, $E^{0,m} = 0$. By Serre vanishing, $E^{p,m-p} = 0$ for all $p \geq 1$ for $t \ll 0$. This proves

$$\text{Ext}^m(\mathcal{F}, \mathcal{O}_X(-t - m - 1)) = 0 \quad \text{for } t \ll 0. \quad \square$$

3. Cones of cohomology tables

Throughout this section, k is an infinite field and $(X, \mathcal{L})$ a Noether pair over k , as in Section 2.14 and $\pi: X \rightarrow \mathbb{P}_k^m$ a finite dominant map with $\mathcal{L} \cong \pi^* \mathcal{O}_{\mathbb{P}_k^m}(1)$; thus, $m = \dim X$. We prove that the closure of the cohomology tables of coherent sheaves on X and $\mathbb{P}_k^m$ coincide; see Theorem 3.4.

3.1. Extending the notation from Section 2.1, for any $\mathcal{G}$ in $\mathbf{D}^b(\text{coh } X)$ set

$$\gamma_{i,j}(\mathcal{G}) := \text{rank}_k H^i(X, \mathcal{G}(j)) \quad \text{and} \quad \gamma(\mathcal{G}) := (\gamma_{i,j})_{i,j}.$$

By the projection formula (2.15), which applies also to objects in $\mathbf{D}^b(\text{coh } X)$, any $\gamma(\mathcal{G})$ occurs as the cohomology table of a complex in $\mathbf{D}^b(\text{coh } \mathbb{P}^m)$. The result below means that the converse also holds, up to limits.

Lemma 3.2. *Let $(\mathcal{F}_n)_{n \geq 0}$ be a lim Ulrich sequence of sheaves on X . For each $\mathcal{G}$ in $\mathbf{D}^b(\text{coh } \mathbb{P}_k^m)$, there exists a sequence $(\mathcal{G}_n)_{n \geq 0}$ in $\mathbf{D}^b(\text{coh } X)$ such that*

$$\lim_{n \rightarrow \infty} \frac{\gamma_{i,j}(\mathcal{G}_n)}{\gamma_{0,0}(\mathcal{F}_n)} = \gamma_{i,j}(\mathcal{G}) \quad \text{for all } i, j.$$

When $\mathcal{G}$ is a vector bundle on $\mathbb{P}^m$, the $\mathcal{G}_n$ can be chosen in $\text{coh } X$.

Proof. For each $n \geq 0$, set

$$\mathcal{G}_n := \mathcal{F}_n \otimes^{\mathbf{L}} \mathbf{L}\pi^* \mathcal{G}.$$

In particular, when $\mathcal{G}$ is a vector bundle, each $\mathcal{G}_n$ is a coherent sheaf on X . The projection formula yields

$$\pi_*(\mathcal{G}_n) \cong \pi_*(\mathcal{F}_n) \otimes^{\mathbf{L}} \mathcal{G},$$

so Theorem 2.8 applied to the lim Ulrich sequence $(\pi_*(\mathcal{F}_n))_{n \geq 0}$ yields

$$\lim_{n \rightarrow \infty} \frac{\gamma_{i,j}(\mathcal{G}_n)}{\gamma_{0,0}(\mathcal{F}_n)} = \lim_{n \rightarrow \infty} \frac{\gamma_{i,j}(\varepsilon^* \varepsilon_* \pi_*(\mathcal{F}_n) \otimes^{\mathbf{L}} \mathcal{G})}{\gamma_{0,0}(\mathcal{F}_n)}.$$

Since one has equalities

$$\gamma_{i,j}(\varepsilon^* \varepsilon_* \pi_*(\mathcal{F}_n) \otimes^{\mathbf{L}} \mathcal{G}) = \gamma_{0,0}(\mathcal{F}_n) \gamma_{i,j}(\mathcal{G}),$$

the desired result follows. $\square$

3.3. Fix an integer m , and consider the $\mathbb{Q}$ -vector space

$$W := \bigoplus_{i=0}^m \prod_{j \in \mathbb{Z}} \mathbb{Q},$$

endowed with the topology defined by pointwise convergence.

For any coherent sheaf $\mathcal{G}$ on X , we view $\gamma(\mathcal{G})$, defined in Section 3.1, as an element in W . This is the *cohomology table* of $\mathcal{G}$. The *cone* of cohomology tables is the subspace

$$\mathbf{C}(X) := \sum_{\mathcal{G} \in \text{coh } X} \mathbb{Q}_{\geq 0} \gamma(\mathcal{G})$$

of W . Observe that the definition involves only the coherent sheaves on X , and not all of $\mathbf{D}^b(\text{coh } X)$. We write $\overline{\mathbf{C}}(X)$ for the closure of $\mathbf{C}(X)$ in W .

Theorem 3.4. *Let k be an infinite perfect field of positive characteristic and $(X, \mathcal{L})$ a Noether pair over k . There is an equality $\overline{\mathbf{C}}(X) = \overline{\mathbf{C}}(\mathbb{P}^m)$ for $m := \dim X$.*

Proof. Let $\pi: (X, \mathcal{L}) \rightarrow \mathbb{P}_k^m$ be a finite dominant linear map of Noether pairs. It is clear from Equation (2.15) that $\mathbf{C}(X) \subseteq \mathbf{C}(\mathbb{P}^m)$. We have only to prove that the cohomology table of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^m$ is in $\overline{\mathbf{C}}(X)$. By [5, Theorem 0.1], one has a convergent sum

$$\gamma(\mathcal{F}) = \sum_{i \geq 1} a_i \gamma(\mathcal{E}_i),$$

where each $\mathcal{E}_i$ is a coherent sheaf obtained as a push-forward of a vector bundle on a linear subset of $\mathbb{P}^m$ and each a_i is a positive real number. We claim that it suffices to verify that each $\gamma(\mathcal{E}_i)$ is in $\overline{\mathbf{C}}(X)$.

Indeed, assume this is so and for each $n \geq 1$ consider the sum

$$\gamma(n) := a_1 \gamma(\mathcal{E}_1) + \cdots + a_n \gamma(\mathcal{E}_n).$$

Since each $\gamma(\mathcal{E}_i)$ is in $\overline{\mathbf{C}}(X)$ approximating the a_i by positive rational numbers, we can write

$$\gamma(n) = \lim_{s \rightarrow \infty} \gamma(n)(s),$$

where each $\gamma(n)(s)$ is a positive rational combination of $\gamma(\mathcal{E}_1), \dots, \gamma(\mathcal{E}_n)$ and hence in $\overline{\mathbf{C}}(X)$. For each n pick an integer $s_n \geq 1$ such that

$$|\gamma(n)_{a,b} - \gamma(n)(s_n)_{a,b}| < 1/2^n$$

for all a, b with $|a|, |b| < n$. Evidently,

$$\lim_{n \rightarrow \infty} \gamma(n)(s_n) = \gamma(\mathcal{F})$$

and each $\gamma(n)(s_n)$ is in $\overline{\mathbf{C}}(X)$, so $\gamma(\mathcal{F})$ is in $\overline{\mathbf{C}}(X)$, as claimed.

It thus remains to consider a coherent sheaf of the form $\iota_*(\mathcal{E})$, where $\iota: \mathbb{P}^c \subseteq \mathbb{P}^m$ is a linear subspace and $\mathcal{E}$ is a vector bundle on $\mathbb{P}^c$ and verify that $\gamma(\mathcal{E})$ is in the closure of $\mathbf{C}(X)$.

Consider the scheme X' obtained by the pull-back of π along ι .

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \pi' \downarrow & & \downarrow \pi \\ \mathbb{P}^c & \xrightarrow{\iota} & \mathbb{P}^m. \end{array}$$

With $\mathcal{L}' := (\pi')^* \mathcal{O}_{\mathbb{P}^c}(1)$, the pair $(X', \mathcal{L}')$ is a Noether pair; see Lemma 2.16. Using the projection formula we identify the cohomology tables of coherent sheaves on X' and on $\mathbb{P}^c$ as elements of W , from Section 3.3. Thus, $\overline{\mathbf{C}}(X') \subseteq \overline{\mathbf{C}}(X)$.

The scheme X' admits a lim Ulrich sequence. This is clear when $c = 0$; for $c \geq 1$ it is contained in Theorem 2.19. Thus, keeping in mind that $\mathcal{E}$ is a vector bundle, Lemma 3.2 yields that $\gamma(\mathcal{E})$ is in $\overline{\mathbf{C}}(X')$, and hence in $\overline{\mathbf{C}}(X)$, as desired. $\square$

4. Lim Ulrich sequences of graded modules

As before k is a field. Let $R := \{R_i\}_{i \geq 0}$ be a finitely generated, graded k -algebra with $R_0 = k$. Set $\mathfrak{m} := R_{\geq 1}$; this is the unique homogeneous maximal ideal of R . We write $\text{grmod } R$ for the category of finitely generated graded R -modules, with morphisms the degree preserving R -linear maps. The component in degree j of a graded R -module M is denoted M_j .

The full subcategory of the derived category of $\text{grmod } R$ consisting of complexes C with $H_i(C) = 0$ for $i \ll 0$ is denoted $\mathbf{D}^+(\text{grmod } R)$. It contains $\mathbf{D}^b(\text{grmod } R)$ the bounded derived category of finitely generated graded R -modules.

4.1. Let $\mathbf{u} := (u_n)_{n \geq 1}$ be a sequence of positive integers. We say that a sequence of objects $(C^n)_{n \geq 1}$ of $\mathbf{D}^+(\text{grmod } R)$ is $\mathbf{u}$ -trivial if for all integers i, j one has

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(C^n)_j}{u_n} = 0.$$

A sequence of morphisms $(f^n: C^n \rightarrow D^n)$ in $\mathbf{D}^+(\text{grmod } R)$ is a $\mathbf{u}$ -equivalence if the sequence of their mapping cones, $\text{cone}(f^n)$, is $\mathbf{u}$ -trivial. It is easy to verify that a composition of $\mathbf{u}$ -equivalences is also a $\mathbf{u}$ -equivalence. This observation will be used often in the sequel.

The proof of the result below is straightforward; see also [15, Lemma 5.3].

Proposition 4.2. *If a sequence of morphisms $(f^n: C^n \rightarrow D^n)$ in $\mathbf{D}^+(\text{grmod } R)$ is a $\mathbf{u}$ -equivalence, then for any i, j there is an equality*

$$\limsup_{n \rightarrow \infty} \frac{\text{rank}_k H_i(C^n)_j}{u_n} = \limsup_{n \rightarrow \infty} \frac{\text{rank}_k H_i(D^n)_j}{u_n}.$$

The corresponding statements involving $\liminf$ also holds. In particular, if one of the limits exists, so does the other and the two limits coincide.

We say that a complex of graded R -modules is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated free R -modules.

Proposition 4.3. *Let $(f^n: C^n \rightarrow D^n)$ be a $\mathbf{u}$ -equivalence. For P in $\mathbf{D}^b(\text{grmod } R)$, the induced sequence*

$$(\text{id} \otimes f_n: P \otimes_R^{\mathbf{L}} C^n \rightarrow P \otimes_R^{\mathbf{L}} D^n)$$

is also an $\mathbf{u}$ -equivalence under either of the following conditions:

1. *The complex P is perfect;*
2. *There exists an integer s such that $H_i(\text{cone}(f_n)) = 0$ for all $i < s$ and all n .*

Proof. Replacing P by its minimal free resolution, we get that P_i , the component in homological degree i , is finite free with $P_i = 0$ for $i \ll 0$. The desired conclusion is that the sequence $(P \otimes_R \text{cone}(f_n))$ is $\mathbf{u}$ -trivial under either of the two conditions.

When P is perfect, $P_i = 0$ also for $i \gg 0$, and the $\mathbf{u}$ -triviality can be checked by a straightforward induction on the number of nonzero components of P .

Suppose condition (2) holds, and let s be as given; we may assume $s = 0$. Fix an integer i . It is easy to see that for each integer n , the inclusion $P_{\leq i+1} \subseteq P$ of subcomplexes induces an isomorphism

$$H_i(P_{\leq i+1} \otimes_R \text{cone}(f_n)) \xrightarrow{\cong} H_i(P \otimes_R \text{cone}(f_n)).$$

Since $P_{\leq i+1}$ is perfect, one can invoke the already established part (1). □

In the sequel, we need the theory of multiplicities for graded modules. Since the rings we work with are not necessarily standard-graded, we begin by recalling the basic definitions and results in the form we need.

4.4. Let R be a graded k -algebra as above. For any finitely generated graded R -module M and integer $q \geq \dim_R M$, set

$$e_q(M) := q! \lim_{n \rightarrow \infty} \frac{\ell_R(M/\mathfrak{m}^{n+1}M)}{n^q}.$$

This is an integer, and it is equal to 0 when $q \geq \dim_R M + 1$. The *multiplicity* of M is the integer

$$e(M) := e_q(M) \quad \text{for } q = \dim_R M.$$

This coincides with the multiplicity of $M_{\mathfrak{m}}$ as a module over the local ring $R_{\mathfrak{m}}$.

In what follows, the focus is on graded k -algebras R admitting a *linear* system of parameters; that is to say, a system of parameters in R_1 . It is convenient to call the k -subalgebra generated by a linear system of parameters a *linear Noether normalization* of R . Linear Noether normalizations exist, for instance, when R is standard-graded (homogeneous, in the language of [3, §4.1]) and k is infinite. The next results allow one to reduce computing multiplicities over rings admitting linear Noether normalizations to the case of standard-graded rings.

Lemma 4.5. *Let R be a finitely generated graded k -algebra, and set $\mathfrak{m} := R_{\geq 1}$. For any set of elements $\mathbf{x} \subseteq R_1$ satisfying $\sqrt{(\mathbf{x})} = \mathfrak{m}$, the ideal $(\mathbf{x})$ is a reduction of $\mathfrak{m}$; it is a minimal reduction when $\mathbf{x}$ is a system of parameters for R .*

Proof. The claim about minimality follows from [13, Corollary 8.3.6] once we prove that $(\mathbf{x})$ is a reduction of $\mathfrak{m}$, for the analytic spread of $\mathfrak{m}$ equals $\dim R$.

The desired result is clear when R is standard graded because $\mathfrak{m}^n = (\mathbf{x})\mathfrak{m}^{n-1}$ for any n such that $\mathfrak{m}^n \subseteq (\mathbf{x})$, and such an n exists since $\mathfrak{m}$ is the radical of $(\mathbf{x})$.

In what follows, given a homogenous ideal I in R such that $\ell_R(M/IM)$ is finite, we write $e(I, M)$ for the multiplicity of M with respect to I . Namely, the limit of the sequence in Section 4.4 with $\mathfrak{m}$ replaced by I ; see also [3, Definition 4.6.1].

The goal is to prove that $\mathfrak{m}$ is the integral closure of $(\mathbf{x})$; see [13, Corollary 1.2.5]. It suffices to verify that this property holds modulo each minimal prime of R ; see [13, Proposition 1.1.5]. Thus, we may assume R is a domain, and then the desired result is equivalent to: $e(\mathbf{x}, R) = e(R)$; this is by Rees' Theorem [13, Theorem 11.3.1].

There exists an integer s such that the Veronese subring $R^{(s)}$ of R is standard-graded, after rescaling. Thus, $\mathbf{x}^s := x_1^s, \dots, x_d^s$ is a reduction of the maximal ideal $(R_s)R^{(s)}$ of $R^{(s)}$, and hence one gets that $e(\mathbf{x}^s, R^{(s)}) = e(R^{(s)})$. This justifies the second equality below:

$$e(\mathbf{x}^s, R) = s \cdot e(\mathbf{x}^s, R^{(s)}) = s \cdot e((R_s), R^{(s)}) = e((R_s), R)$$

The first and the third equalities hold as the rank of R over $R^{(s)}$ is s ; see [3, Corollary 4.7.9]. If $f_1, \dots, f_n$ are homogeneous generators of the ideal $\mathfrak{m}$, then $\mathfrak{m}^s$ is integral over $(f_1^s, \dots, f_n^s)$, which is contained in the ideal (R_s) , so we get that

$$e((R_s), R) \leq e((f_1^s, \dots, f_n^s), R) = e(\mathfrak{m}^s, R).$$

We conclude that $e(\mathbf{x}^s, R) \leq e(\mathfrak{m}^s, R)$ and hence that $e(\mathbf{x}, R) \leq e(\mathfrak{m}, R)$. The reverse inequality is clear. $\square$

Lemma 4.6. *Let R be a graded k -algebra admitting a linear system of parameters $\mathbf{x}$. For each M in $\text{grmod } R$, there is an equality*

$$e(M) = (\dim_R M)! \lim_{n \rightarrow \infty} \frac{\sum_{i \leq n} \text{rank}_k(M_i)}{n^{\dim_R M}}.$$

When $\dim_R M = \dim R$, one has also an equality

$$e(M) = \sum_{i \geq 0} (-1)^i \text{rank}_k H_i(\mathbf{x}; M).$$

Proof. By Lemma 4.5, the ideal $(\mathbf{x})$ is a minimal reduction of the maximal ideal of R , so the multiplicity of M as an R -module coincides with its multiplicity when viewed as a module over the k -subalgebra of R generated by $\mathbf{x}$. Therefore, in the remainder of the proof, we assume that R is a standard-graded polynomial ring, over indeterminates $\mathbf{x}$. Then the fact the multiplicity is the Euler characteristic of the Koszul homology of M with respect to $\mathbf{x}$ is [3, Theorem 4.7.6].

The first equality is also well-known, but we could not find a suitable reference, so we outline a proof: Set $q := \dim_R M$, and let $e'_q(-)$ be the function defined on the subcategory of finitely generated R -modules of dimension at most q by

$$e'_q(N) := q! \lim_{n \rightarrow \infty} \frac{\sum_{i \leq n} \text{rank}_k(N_i)}{n^q}.$$

The desired result is that $e'_q(-) = e_q(-)$ on this subcategory of R -modules. It is clear that equality holds when the module is generated in a single degree. Since any finitely generated module admits a finite filtration by R -submodules such that the subquotients in the filtrations are generated in a single degree, it remains to note that both invariants are additive on short, exact sequences of modules of dimension at most q ; this is clear for $e'_q(-)$, and for $e_q(-)$ it is [3, Corollary 4.7.7]. $\square$

4.7. Let R be a graded k -algebra admitting a linear Noether normalization and U a finitely generated graded R -module. We write $\nu_R(U)$ for its minimal number of generators. The R -module U is *Ulrich* if it has the following properties:

1. U is maximal Cohen–Macaulay and nonzero;
2. $e(U) = \nu_R(U)$;
3. U is generated in degree 0.

When R is a standard graded polynomial ring, the only Ulrich modules are R^n , for some integer n ; no twists are allowed.

Following [15, 16], we introduce a notion *lim Ulrich sequences* of graded R -modules. To that end, we recall some basic properties of finite free complexes.

4.8. Consider a complex

$$F := 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

of finite free R -modules with $\text{rank}_k H(F)$ finite and nonzero. The New Intersection Theorem [19] yields $n \geq \dim R$; if equality holds we say F is a *short* complex. For such an F , any maximal Cohen–Macaulay R -module M satisfies

$$H_i(F \otimes_R M) = 0 \quad \text{for } i \geq 1.$$

This is by the Acyclicity Lemma [3, Exercise 1.4.24]. This property characterizes maximal Cohen–Macaulay modules, by the depth sensitivity of Koszul complexes.

Lim Ulrich sequences

Let R be a graded k -algebra admitting a linear system of parameters. Compare the definition below with that of an Ulrich module, Section 4.7, and also with that of *lim Ulrich sheaves*, Section 2.4.

4.9. A sequence $(U^n)_{n \geq 1}$ in $\text{grmod } R$ is *lim Ulrich* if each U^n is nonzero and the following properties hold:

1. For each finite free short complex F with $\text{rank}_k H(F)$ finite one has

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(F \otimes_R U^n)}{\nu_R(U^n)} = 0 \quad \text{for } i \geq 1.$$

2. $\lim_{n \rightarrow \infty} e_d(U^n)/\nu_R(U^n) = 1$ for $d := \dim R$.
3. With $\mathfrak{m} := R_{\geq 1}$ the unique homogeneous maximal ideal of R , one has

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k (U^n/\mathfrak{m}U^n)_0}{\nu_R(U^n)} = 1.$$

It suffices that condition (1) hold for F the Koszul complex on a *single* system of parameters for R ; this can be proved along the lines of [15, Lemma 5.7]. Moreover, with $F := K(\mathbf{x})$ the Koszul complex on a linear system of parameters $\mathbf{x}$, parts (1) and (2) above can be expressed more succinctly as follows:

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(\mathbf{x}; U^n)}{\nu_R(U^n)} = \begin{cases} 1 & \text{when } i = 0 \\ 0 & \text{when } i \neq 0. \end{cases} \quad (1')$$

Here is an expression of these conditions in the language introduced in Section 4.1.

Lemma 4.10. *Let R be a graded k -algebra with a linear system of parameters $\mathbf{x}$, and $(U^n)_{n \geq 1}$ a lim Ulrich sequence in $\text{grmod } R$. For $\mathbf{u} := (v_R(U^n))$, the following sequences of canonical surjections are $\mathbf{u}$ -equivalences:*

$$K(\mathbf{x}; U^n) \twoheadrightarrow U^n / \mathfrak{m}U^n \quad \text{and} \quad U^n / \mathfrak{m}U^n \twoheadrightarrow (U^n / \mathfrak{m}U^n)_0.$$

Proof. Consider the canonical morphisms

$$f^n: K(\mathbf{x}; U^n) \twoheadrightarrow U^n / \mathbf{x}U^n \quad \text{and} \quad g^n: U^n / \mathbf{x}U^n \twoheadrightarrow U^n / \mathfrak{m}U^n.$$

We prove that the sequences (f^n) and (g^n) are $\mathbf{u}$ -equivalences, which implies that the composition $(g^n f^n)$ is also a $\mathbf{u}$ -equivalence, as claimed.

It is easy to see that the homology of the cone of f^n satisfies

$$H_{i+1}(\text{cone}(f^n)) \cong \begin{cases} H_i(\mathbf{x}; U^n) & \text{for } i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, (f^n) is a $\mathbf{u}$ -equivalence if and only if

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(\mathbf{x}; U^n)_j}{u_n} = 0 \quad \text{for } i \geq 1 \text{ and } j \in \mathbb{Z}.$$

It remains to observe that the lim Ulrich property expressed in condition (1'), in Section 4.9, implies the limit above is 0, that is to say, (f^n) is a $\mathbf{u}$ -equivalence, as desired.

For each n , one has that

$$\text{rank}_k(\mathfrak{m}U^n / \mathbf{x}U^n) = \text{rank}_k(U^n / \mathbf{x}U^n) - u_n$$

so condition (2) defining the lim Ulrich property of $(U^n)_{n \geq 1}$ yields

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k(\mathfrak{m}U^n / \mathbf{x}U^n)}{u_n} = 0.$$

The cone of the morphism $g^n: U^n / \mathbf{x}U^n \rightarrow U^n / \mathfrak{m}U^n$ satisfies

$$H_i(\text{cone}(g^n)) \cong \begin{cases} (\mathfrak{m}U^n) / (\mathbf{x}U^n) & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the sequence of maps (g^n) is a $\mathbf{u}$ -equivalence.

This completes the proof that the sequence $(g^n f^n)$ is a $\mathbf{u}$ -equivalence.

Condition (3) in Section 4.9 is equivalent to the condition that

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k(U^n / \mathfrak{m}U^n)_{i \neq 0}}{u_n} = 0.$$

Since the homology of the mapping cone of the surjection

$$(U^n / \mathfrak{m}U^n) \rightarrow (U^n / \mathfrak{m}U^n)_0$$

is $(U^n / \mathfrak{m}U^n)_{i \neq 0}$ in degree 1 and zero otherwise, it follows that the sequence above is a $\mathbf{u}$ -equivalence, as desired. $\square$

4.11. It is clear from the proof of Lemma 4.10 that when $(U^n)_{n \geq 1}$ is a lim Ulrich sequence, the sequences (f^n) and (g^n) are $\mathbf{u}$ -equivalent in the stronger sense that

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(\text{cone}(f^n))}{v_R(U^n)} = 0 = \lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(\text{cone}(g^n))}{v_R(U^n)}.$$

That is to say, the ranks of the homology modules, and not just their graded pieces, of the cones are asymptotically zero with respect to $(v_R(U^n))$. These conditions are equivalent to the lim Ulrich property of the sequence $(U^n)_{n \geq 1}$.

Moreover, it is not hard to verify that if the sequence $(g^n f^n)$ is a $\mathbf{u}$ -equivalence (in either sense), then so are the sequences (f^n) and (g^n) ; the converse is clear.

Lemma 4.12. *Let R be a graded k -algebra with a linear system of parameters $\mathbf{x}$, and $(U^n)_{n \geq 1}$ a lim Ulrich sequence in $\text{grmod } R$. For any P in $\mathbf{D}^+(\text{grmod } R)$ and integers i, j , there is an equality*

$$\lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(P \otimes_R^{\mathbf{L}} K(\mathbf{x}; U^n))_j}{v_R(U^n)} = \text{rank}_k H_i(P \otimes_R^{\mathbf{L}} k)_j.$$

Proof. Set $u_n := v_R(U^n)$ and $\mathbf{u} := (u_n)$. Set

$$F^n := K(\mathbf{x}; U^n) \quad \text{and} \quad G^n := (U^n / \mathfrak{m}U^n)_0,$$

and let $h^n : F^n \rightarrow G^n$ be the composition of maps

$$K(\mathbf{x}; U^n) \rightarrow U^n / \mathfrak{m}U^n \rightarrow (U^n / \mathfrak{m}U^n)_0.$$

A composition of $\mathbf{u}$ -equivalences is a $\mathbf{u}$ -equivalence, so it follows from Lemma 4.10 that the sequence (h^n) is a $\mathbf{u}$ -equivalence. Evidently, for $i < 0$ and all n one has

$$H_i(K(\mathbf{x}; U^n)) = 0 \quad \text{and} \quad H_i(G^n) = 0.$$

Thus, Proposition 4.3(2) applies to yield that the sequence $(P \otimes_R^{\mathbf{L}} h^n)$ is also a $\mathbf{u}$ -equivalence. Since R acts on G^n via the surjection $R \rightarrow k$, one gets isomorphisms

$$H_i(P \otimes_R^{\mathbf{L}} G^n) \cong H_i((P \otimes_R^{\mathbf{L}} k) \otimes_k G^n) \cong H_i(P \otimes_R^{\mathbf{L}} k) \otimes_k G^n.$$

Since G^n lives in internal degree 0, this justifies the second equality below.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(P \otimes_R^{\mathbf{L}} G^n)_j}{u_n} &= \lim_{n \rightarrow \infty} \frac{\text{rank}_k H_i(P \otimes_R^{\mathbf{L}} G^n)_j}{\text{rank}_k(G^n)} \\ &= \text{rank}_k H_i(P \otimes_R^{\mathbf{L}} k)_j. \end{aligned}$$

The first one holds because the surjection $U^n / \mathfrak{m}U^n \rightarrow G^n$ is a $\mathbf{u}$ -equivalence, by Lemma 4.10, so Proposition 4.2 applies. $\square$

Lemma 4.13. *Let k be a field and R a graded k -algebra admitting a linear system of parameters. Set $X := \text{Proj}(R)$ and $\mathcal{L} := \widetilde{R(1)}$. The coherent sheaf $\mathcal{L}$ is invertible, ample, globally generated and satisfies $\mathcal{L}^{\otimes t} = \widetilde{R(t)}$ for each t .*

Proof. Let $\mathbf{x} := x_1, \dots, x_d$ be a linear system of parameters for R . To see that $\mathcal{L}$ is invertible and $\mathcal{L}^{\otimes t} = \widetilde{R(t)}$, one can use the same argument as in the proof of [10, Proposition 5.12 (a) (b)]. The point is that the affine open sets $\{D_+(x_i)\}_{i=1}^d$ cover X since $\mathbf{x}$ is a linear system of parameters for R . The sheaf $\mathcal{L}$ is globally generated since the $\mathbf{x}$ are global sections of $\mathcal{L}$ that have no common zero locus, again since $\mathbf{x}$ is a system of parameters. For $t \gg 0$, the t 'th Veronese of R is standard graded, so $\mathcal{L}^{\otimes t}$ is very ample, and hence $\mathcal{L}$ is ample; see [10, Theorem II.7.6]. $\square$

4.14. Let R be a graded k -algebra admitting a linear Noether normalization, with irrelevant maximal ideal $\mathfrak{m}$, and M a finitely generated R -module. We write $H_{\mathfrak{m}}^i(M)$ for the i th local cohomology module of M supported on $\mathfrak{m}$; see [3, p. 143]. These are graded R -modules, with $H_{\mathfrak{m}}^i(M)_t = 0$ for $t \gg 0$. The Castelnuovo–Mumford *regularity* of M is the integer

$$\operatorname{reg}_R M := \max\{i + t \mid H_{\mathfrak{m}}^i(M)_t \neq 0\}.$$

If $P \rightarrow R$ is a finite map of graded k -algebras, one has an isomorphism

$$H_{P_{\geq 1}}^i(M) \cong H_{\mathfrak{m}}^i(M)$$

so $\operatorname{reg}_R M = \operatorname{reg}_P M$. Thus, the regularity of M can be computed with respect any linear Noether normalization for R .

Set $X := \operatorname{Proj}(R)$, and let $\mathcal{F}$ be a coherent sheaf on X such that $\operatorname{depth} \mathcal{F}_x \geq 1$ at each closed point $x \in X$. Set

$$M := \Gamma_*(\mathcal{F}) := \bigoplus_{t \in \mathbb{Z}} \mathcal{F}(t).$$

The condition on depth of $\mathcal{F}$ ensures that this is a finitely generated R -module; see Lemma 2.20. One then has an equality

$$\operatorname{reg}_R M = \max\{i + t \mid H^{i-1}(X, \mathcal{F}(t)) \neq 0 \text{ and } i \geq 2\}. \quad (4.15)$$

This holds because

$$\begin{aligned} H_{\mathfrak{m}}^0(M) &= 0 = H_{\mathfrak{m}}^1(M), \quad \text{and} \\ H_{\mathfrak{m}}^i(M)_t &\cong H^{i-1}(X, \mathcal{F}(t)) \quad \text{for } i \geq 2. \end{aligned} \quad (4.16)$$

Next, we apply the Theorem 2.19 to construct a lim Ulrich sequences in $\operatorname{grmod} R$.

Theorem 4.17. *Let k be an infinite perfect field of positive characteristic. Let R be a graded k -algebra with $\dim R \geq 2$ and admitting a linear Noether normalization. There exists a lim Ulrich sequence $(U^n)_{n \geq 1}$ in $\operatorname{grmod} R$ satisfying the following conditions:*

1. *For each n , $\operatorname{depth}_R(U^n) \geq 2$ and $\operatorname{depth}_{R_{\mathfrak{p}}}(U_{\mathfrak{p}}^n) \geq \operatorname{depth} R_{\mathfrak{p}}$ for $\mathfrak{p}$ in $\operatorname{Proj}(R)$.*
2. *There exist integers t_0 and t_1 such that for each n one has*

$$U_j^n = 0 \text{ for } j < t_0 \quad \text{and} \quad \operatorname{reg} U^n \leq t_1.$$

Proof. Set $X := \operatorname{Proj}(R)$ and $\mathcal{L}$ the coherent sheaf on X defined by $R(1)$. Since R admits a linear system of parameters, Lemma 4.13 yields that $(X, \mathcal{L})$ is a Noether pair. Let $(\mathcal{F}_n)_{n \geq 1}$ be the lim Ulrich sequence of sheaves on X given by Theorem 2.19, and set

$$U^n := \Gamma_*(\mathcal{F}_n) \quad \text{for } n \geq 1.$$

This is an R -module, and it is finitely generated because $\operatorname{depth}(\mathcal{F}_n)_x \geq 1$ at all closed points $x \in X$, by construction; see Lemma 2.20.

(1) From Equation (4.16), one gets $\operatorname{depth}_R U^n \geq 2$. For any $x \in X$, one has an isomorphism $((\mathcal{F}_n)_x) \otimes_k k(t) \cong (U^n)_{\mathfrak{p}}$, where $\mathfrak{p} \in \operatorname{Proj}(R)$ corresponds to x . Thus, the claim about depths follows from the corresponding property for the sheaves $\mathcal{F}_n$.

(2) Since $U_t^n = H^0(X, \mathcal{F}_n(t))$, property (2) in Section 2.4 implies $U_j^n = 0$ for $j < t_0$. The claim about regularity is immediate from Equation (4.16), given property (3) in Section 2.4. $\square$

Given the preceding theorem, one can argue as in the proof of [16, Theorem A] (see also [16, Theorem B]) to obtain the following result.

Corollary 4.18. *With R as above, if $(R_{\mathfrak{m}}, \mathfrak{m}R_{\mathfrak{m}}) \rightarrow S$ is a flat local map of Noetherian local rings, then $e(R) \leq e(S)$.*

5. Boij–Söderberg theory

In this section, we recall basics of Boij–Söderberg theory, due mainly to Boij, Eisenbud, Erman, Schreyer and Söderberg [2, 1, 4, 5, 6].

Codimension sequences and degree sequences

Throughout, we fix a nonnegative integer d , soon to be the dimension of a graded ring. By a *codimension sequence* (for d), we mean a nondecreasing sequence $\mathbf{c} := (c_i)_{i \in \mathbb{Z}}$, where each c_i is an element of the set $\{\emptyset, 0, \dots, d, \infty\}$, with ordering

$$\emptyset < 0 < 1 < \dots < d < \infty.$$

For any nonnegative integer c , the constant sequence $(c)_{i \in \mathbb{Z}}$ is also denoted c . We consider the collection of codimension sequences with the natural partial order

$$(c_i)_{i \in \mathbb{Z}} \leq (c'_i)_{i \in \mathbb{Z}} \quad \text{if } c_i \leq c'_i \text{ for each } i.$$

A *degree sequence* is an increasing sequence of integers $t_a < t_{a+1} < \dots < t_{a+l}$, with $0 \leq l \leq d$. It is convenient to view it as an infinite sequence

$$\mathbf{t} := (\dots, -\infty, -\infty, t_a, \dots, t_{a+l}, \infty, \infty, \dots)$$

with t_i in position i . The integer l is the *codimension* of $\mathbf{t}$, denoted $\text{codim}(\mathbf{t})$; we define $\inf(\mathbf{t}) := a$. The sequence $\mathbf{t}$ is *compatible* with a codimension sequence $\mathbf{c}$ if

$$0 \leq c_a \leq \text{codim}(\mathbf{t}) \leq c_{a+1} \quad \text{for } a = \inf(\mathbf{t}).$$

In what follows, we consider the $\mathbb{Q}$ -vector space

$$V := \mathbb{Q}^{(\mathbb{Z} \times \mathbb{Z})} = \bigoplus_{i, j \in \mathbb{Z}} \mathbb{Q}. \quad (5.1)$$

Each degree sequence $\mathbf{t}$ determines an element $\beta(\mathbf{t})$ of V given by

$$\beta(\mathbf{t})_{i, j} := \begin{cases} \frac{\prod_{n \neq a} |t_n - t_a|}{\prod_{n \neq i} |t_n - t_i|} & \text{for } a \leq i \leq a + l \text{ and } j = t_i, \\ 0 & \text{for all other values of } i, j. \end{cases} \quad (5.2)$$

Here, $a := \inf(\mathbf{t})$ and $l = \text{codim}(\mathbf{t})$. Given a codimension sequence $\mathbf{c}$, we set

$$\mathbf{B}_d^{\mathbf{c}} := \left\{ \begin{array}{l} \text{cone in } V \text{ spanned by } \beta(\mathbf{t}) \text{ as } \mathbf{t} \text{ ranges} \\ \text{over degree sequences compatible with } \mathbf{c} \end{array} \right\}. \quad (5.3)$$

This is the smallest subset of V containing the $\beta(\mathbf{t})$, for the permitted $\mathbf{t}$, and closed under addition and multiplication by nonnegative rational numbers.

The finite topology on V

Given a finite subset $Y \subset \mathbb{Z} \times \mathbb{Z}$, we write V_Y for the subspace of V spanned by the corresponding coordinate vectors. Thus, a vector $v \in V$ is in V_Y if and only if $v_{i, j} = 0$ for $(i, j) \notin Y$. We topologize V

by giving it the finite topology: A subset U of V is open if and only if its intersection with any finite-dimensional subspace of V is open in the Euclidean topology. This is equivalent to the condition that $U \cap V_Y$ is open in V_Y , equipped with its Euclidean topology, for all finite subsets Y of $\mathbb{Z} \times \mathbb{Z}$.

Observe that the cone spanned by vectors $v^1, \dots, v^n$ in the positive orthant of V is contained in V_Y if and only if each v^i is in V_Y . This has the following consequence.

Lemma 5.4. *With Y as above, for any codimension sequence $\mathbf{c}$, the cone $\mathbf{B}_d^{\mathbf{c}} \cap V_Y$ is spanned by vectors $\beta(\mathbf{t})$, where $\mathbf{t}$ is any degree sequence compatible with $\mathbf{c}$ and satisfying the condition*

$$(n, t_n) \in Y \quad \text{for } \inf(\mathbf{t}) \leq n \leq \inf(\mathbf{t}) + \text{codim}(\mathbf{t}).$$

Thus, $\mathbf{B}_d^{\mathbf{c}} \cap V_Y$ is the cone spanned by a finite collection of vectors, and hence the subset $\mathbf{B}_d^{\mathbf{c}} \subset V$ is closed in the finite topology.

The result above means that if a sequence of elements $(\beta_n)_{n \geq 0}$ in $\mathbf{B}_d^{\mathbf{c}}$ converges in the finite topology to an element β of V , then β must also belong to $\mathbf{B}_d^{\mathbf{c}}$.

Graded modules over graded rings

As in Section 4, let k be a field and $R = \{R_i\}_{i \geq 0}$ a finitely generated graded k -algebra with $R_0 = k$. Set $\mathfrak{m} := R_{i \geq 1}$.

5.5. Let M be an R -complex in $\mathbf{D}^+(\text{grmod } R)$. For any pair of integers (i, j) , the *Betti number* of M in degree (i, j) is

$$\beta_{i,j}^R(M) := \text{rank}_k \text{Tor}_i^R(k, M)_j.$$

The *Betti table* of M is the array $\beta^R(M) := (\beta_{i,j}^R(M))_{i,j}$.

Finite free complexes

With R as above, we write $\mathcal{F}(R)$ for the class of finite free complexes of graded R -modules, that is to say, complexes of the form

$$0 \longrightarrow F_b \longrightarrow \cdots \longrightarrow F_a \longrightarrow 0,$$

where each F_i is a finitely generated graded free R -module, and the differential is homogeneous of degree zero. Each finite free complex is quasi-isomorphic to a *minimal* one: a finite free complex F whose differential d satisfies $d(F) \subseteq \mathfrak{m}F$. When F is minimal, one has

$$F_i \cong \bigoplus_j R(-j)^{\beta_{i,j}^R(F)}.$$

Since F is finite free, $\beta_{i,j}^R(F)$ is nonzero only for finitely many pairs (i, j) , so $\beta^R(F)$, the Betti table of F , is an element in the $\mathbb{Q}$ -vector space V from Equation (5.1).

Codimension

Let $\text{codim}_R M$ be the *codimension* of a finitely generated graded R -module M , namely, the height of its annihilator ideal. Thus, $\text{codim}_R M = \infty$ if and only if $M = 0$. Let $\mathbf{c}$ be a codimension sequence, as defined above. Set

$$\mathcal{F}^{\mathbf{c}}(R) := \left\{ F \in \mathcal{F}(R) \left| \begin{array}{l} \beta_{i,j}^R(F) = 0 \text{ when } c_i = \emptyset, \text{ and} \\ \text{codim } H_i(F) \geq c_i \text{ when } c_i > \emptyset \end{array} \right. \right\}. \quad (5.6)$$

For instance, for the constant sequence $\dim R$ the objects in $\mathcal{F}^{\dim R}(R)$ are precisely the finite free complexes with homology of finite length. Moreover, $\mathcal{F}^0(R) = \mathcal{F}(R)$. Clearly, if $\mathbf{c}'$ is another codimension sequence with $\mathbf{c} \leq \mathbf{c}'$, then $\mathcal{F}^{\mathbf{c}}(R) \supseteq \mathcal{F}^{\mathbf{c}'}(R)$.

Consider the cone in the $\mathbb{Q}$ -vector space V from Equation (5.1), spanned by the Betti tables of the finite free complexes F in $\mathcal{F}^{\mathbf{c}}(R)$:

$$\mathbf{B}^{\mathbf{c}}(R) := \sum_{F \in \mathcal{F}^{\mathbf{c}}(R)} \mathbb{Q}_{\geq 0} \beta^R(F). \quad (5.7)$$

Pure complexes

A finite free complex F is *pure* if there exists a degree sequence $\mathbf{t}$ as above such that for $a := \inf(\mathbf{t})$ and $l := \text{codim } \mathbf{t}$, the following conditions hold:

1. $\beta_{i,j}(F) \neq 0$ if and only if $j = t_i$, that is to say, the complex F is quasi-isomorphic to minimal complex of the form

$$0 \longrightarrow R(-t_{a+l})^{b_{a+l}} \longrightarrow \cdots \longrightarrow R(-t_a)^{b_a} \longrightarrow 0;$$

2. The R -module $H_a(F)$ is Cohen–Macaulay with $\text{codim } H_a(F) \geq l$;
3. $H_n(F) = 0$ for $n \neq a$.

In particular, $\Sigma^{-a}F$ is the free resolution of a finitely generated R -module, namely, $H_a(F)$. It follows from the result of Herzog and Kühl [11] that the Betti table of such an F has the form

$$\beta^R(F) = b \cdot \beta(\mathbf{t}),$$

where $b := \beta_a^R(F)$ and $\beta(\mathbf{t})$ is as in Equation (5.2).

Boij–Soderberg theory

Let $S := k[x_1, \dots, x_d]$ be a standard graded polynomial ring; thus the degree of each x_i is 1. The following result is due to Eisenbud and Erman and builds on work of Eisenbud and Schreyer and Boij and Söderberg. The cone $\mathbf{B}_d^{\mathbf{c}}$ is defined in Equation (5.3).

Theorem 5.8. [4, Theorem 3.1] *Each degree sequence has an associated pure object in $\mathcal{F}(S)$. For any codimension sequence $\mathbf{c}$ for S , one has $\mathbf{B}^{\mathbf{c}}(S) = \mathbf{B}_d^{\mathbf{c}}$.*

By a codimension sequence $\mathbf{c}$ for a graded ring R , we mean a codimension sequence for the integer $\dim R$. Combining the preceding theorem with Lemma 5.4 gives the following result.

Corollary 5.9. *For a standard graded polynomial ring S and any codimension sequence $\mathbf{c}$ for S , the subset $\mathbf{B}^{\mathbf{c}}(S) \subset V$ is closed in the finite topology.*

6. Cones of Betti tables

6.1. Throughout this section, k is an infinite perfect field (possibly of characteristic zero) and R a finitely generated graded k -algebra admitting a linear Noether normalization, say A .

Theorem 6.2. *Let R be as in Section 6.1. If $\mathbf{c}$ is a codimension sequence for R , then $\mathbf{B}^{\mathbf{c}}(R) \subseteq \mathbf{B}_d^{\mathbf{c}}$. Equality holds if $c_i \neq \infty$ for all i or R is Cohen–Macaulay.*

The proof of the theorem when k has positive characteristic is given in Section 6.13. The characteristic zero case is handled by reduction to positive characteristic; see Section 6.15. We record a couple of expected consequences.

When R is a standard graded polynomial ring, the bounds given below on the multiplicities (whose definition is recalled in Section 4.4) were conjectured by Huneke and Srinivasan and proved by Eisenbud and Schreyer [6] as a consequence of their proof of the conjectures of Boij and Söderberg [1].

Corollary 6.3. *Let R be as in Section 6.1, and let M be a nonzero finitely generated, graded, R -module generated in degree zero and of finite projective dimension. Then*

$$e(M) \leq \frac{e(R)\beta_0(M)}{c!} \prod_{i=1}^c \max\{j \mid \beta_{i,j}^R(M) \neq 0\},$$

for $c := \text{codim } M$. When M is perfect, one has also the lower bound

$$e(M) \geq \frac{e(R)\beta_0(M)}{c!} \prod_{i=1}^c \min\{j \mid \beta_{i,j}^R(M) \neq 0\}.$$

Each of the inequalities above is strict unless M has a pure resolution.

Proof. We deduce these bounds from Theorem 6.2 following the argument in [1].

There is nothing to check if $\dim R = 0$, so we can assume $d := \dim R \geq 1$. Let A be a Noether normalization of R as in Section 6.1. It follows from Lemma 4.6 that the multiplicity of a graded R -module coincides with its multiplicity as an A -module.

Fix an integer c with $0 \leq c \leq d$. We look at Betti tables of R -modules of finite projective dimension and codimension at least c . So let $\mathbf{c} := (\dots, \emptyset, c, \infty, \dots)$ with c in position 0 and $\mathbf{B}_d^{\mathbf{c}}$ the corresponding rational cone; see Equation (5.3). It is thus the positive rational cone spanned by $\beta(\mathbf{t})$, where $\mathbf{t}$ is any degree sequence of the form

$$(\dots - \infty, t_0, \dots, t_l, \infty, \dots)$$

with $c \leq l \leq \dim R$. We know that $\mathbf{B}^{\mathbf{c}}(R) \subseteq \mathbf{B}_d^{\mathbf{c}}$, by Theorem 6.2.

Let $\text{Hilb}_R(t)$ be the Hilbert series of R , and for $\beta \in \mathbf{B}_d^{\mathbf{c}}$, set

$$\text{Hilb}_\beta(t) := \text{Hilb}_R(t)g_\beta(t), \quad \text{where} \quad g_\beta(t) := \sum_{i,j} (-1)^i \beta_{i,j} t^j \in \mathbb{Z}[t, t^{-1}].$$

Since R admits a linear system of parameters, $\text{Hilb}_R(t) = f_R(t)/(1-t)^d$ for some polynomial $f_R(t)$ and $e(R) = f_R(1)$. If $M \in \text{grmod } R$ has finite projective dimension, then $\text{Hilb}_\beta(t) = \text{Hilb}_M(t)$, where $\beta = \beta^R(M)$; see [3, Lemma 4.1.13].

For $\beta := \beta(\mathbf{t})$ with $\mathbf{t}$ a degree sequence as above, we have that $g_\beta(t)$ is divisible by $(1-t)^l$, with $l = \text{codim}(\mathbf{t})$ and no higher power. In particular, we have that $g_\beta(t)$ is divisible by $(1-t)^c$ for all $\beta \in \mathbf{B}_d^{\mathbf{c}}$. It thus makes sense to define

$$e(\beta) := e(R) \left(\frac{g_\beta(t)}{(1-t)^c} \right)_{t=1} \quad \text{for } \beta \in \mathbf{B}_d^{\mathbf{c}}.$$

The function $e(-)$ depends on the chosen c and $e(\beta(\mathbf{t})) = 0$ if $\text{codim}(\mathbf{t}) \geq c+1$.

Recall that one has

$$\text{codim}_R M = \dim R - \dim_R M = d - \dim_R M$$

for any $M \in \text{grmod } R$ of finite projective dimension; see [18, Théorème 2].

Claim. *If $M \in \text{grmod } R$ has finite projective dimension and codimension at least c , then $e(\beta^R(M)) = e_{d-c}(M)$; see Section 4.4.*

Indeed, since M has dimension at most $d - c$, one has that

$$\mathrm{Hilb}_M(t) = \frac{f_M(t)}{(1-t)^{d-c}}$$

for some polynomial $f_M(t)$ and then $e_{d-c}(M) = f_M(1)$. On the other hand, for $\beta = \beta^R(M)$, the discussion above yields

$$\begin{aligned} \mathrm{Hilb}_M(t) &= \mathrm{Hilb}_\beta(t) \\ &= \mathrm{Hilb}_R(t)g_\beta(t) \\ &= \frac{f_R(t)}{(1-t)^d}g_\beta(t) \\ &= \frac{(f_R(t)g_\beta(t))/(1-t)^c}{(1-t)^{d-c}}. \end{aligned}$$

Thus, $f_M(t) = f_R(t)g_\beta(t)/(1-t)^c$; evaluating this equality at $t = 1$ yields the claim.

Once we know that the multiplicity of a module can be extracted from its Hilbert series, the rest of the argument is as in [1, Theorem 4.6]. $\square$

We write $\mathbf{B}^{\mathrm{short}}(R)$ for the positive cone in V , defined in Equation (5.1), spanned by $\beta^R(F)$ as F ranges over the short complexes over R ; see Section 4.8.

Corollary 6.4. *With R and A as in Section 6.1, there is an equality $\mathbf{B}^{\mathrm{short}}(R) = \mathbf{B}^{\mathrm{short}}(A)$.*

Proof. Let $\mathbf{c}$ be the codimension sequence $(\dots, \emptyset, \dim R, \dim R, \dots)$ with the first occurrence of $\dim R$ is in degree 0. Then $\mathbf{B}^{\mathbf{c}}(R) = \mathbf{B}^{\mathbf{c}}(A)$, by Theorem 6.2. Intersecting these cones with the cone $V^{[0,d]}$ yields the stated equality. $\square$

In proving Theorem 6.2 it will be expedient to use the notion of depth for complexes, and some results from [14]. In what follows we state the necessary definitions and results for local rings, with the implicit understanding that the corresponding statements in the graded context also hold.

6.5. Let R be a local ring with residue field k . By the *depth* of an R -complex M of R -modules, we mean the least integer i , possibly $\pm\infty$ such that $\mathrm{Ext}_R^i(k, M) \neq 0$; see [8] for alternative descriptions. Given a bounded complex M , let $\sup H_*(M)$ denote the supremum of integers i such that $H_i(M) \neq 0$. For such an R -complex M , there is an inequality

$$\mathrm{depth}_R M \geq -\sup H_*(M). \quad (6.6)$$

Equality holds if and only if $\mathrm{depth}_R H_s(M) = 0$ for $s := \sup H_*(M)$. These claims are contained in [14, Proposition 2.7(2) and Theorem 2.3].

One version of the Auslander–Buchsbaum formula for complexes reads: If F is finite free R -complex with $H(F) \neq 0$, then for any R -complex M one has

$$\begin{aligned} \mathrm{depth}_R(F \otimes_R M) &= \mathrm{depth}_R M - \mathrm{proj\,dim}_R F \\ &= \mathrm{depth}_R M - \mathrm{depth}_R R + \mathrm{depth}_R F. \end{aligned} \quad (6.7)$$

The first equality is [14, Corollary 2.2], and the second follows from the first applied with $M = R$.

Lemma 6.8. *Fix a codimension sequence $\mathbf{c}$ for R , an R -complex F in $\mathcal{F}^{\mathbf{c}}(R)$ and a finitely generated R -module U satisfying $\mathrm{depth}_{R_{\mathfrak{p}}} U_{\mathfrak{p}} \geq \mathrm{depth}_R U_{\mathfrak{p}}$ for all $\mathfrak{p}$ in $\mathrm{Spec} R \setminus \{\mathfrak{m}\}$. For any integer i , if $c_i < \infty$, or $\mathrm{depth}_R U \geq \mathrm{depth}_R R$, then*

$$\mathrm{codim}_R H_i(F \otimes_R U) \geq c_i.$$

When R is a finitely generated graded k -algebra and F and U are also graded, then it suffices that the condition on depth holds for $\mathfrak{p}$ in $\text{Proj}(R)$.

Proof. The key observation is the following:

Claim. Assume $\text{depth}_R U \geq \text{depth } R$ as well. Then $\sup H_*(F \otimes_R U) \leq \sup H_*(F)$.

Indeed, since $\sup H_*(F)$ does not increase on localization, we can further localize at prime minimal in the support of $H_s(F \otimes_R U)$, for $s := \sup H_*(F \otimes_R U)$, and assume it has nonzero finite length. Then the claim about equality in Equation (6.6) yields the first equality below:

$$\begin{aligned} -\sup H_*(F \otimes_R U) &= \text{depth}_R(F \otimes_R U) \\ &= \text{depth}_R U - \text{depth } R + \text{depth}_R F \\ &\geq \text{depth}_R F \\ &\geq -\sup H_*(F). \end{aligned}$$

The second equality is Equation (6.7). The first inequality holds by our hypotheses on U , and another application of Equation (6.6) yields the last one. This settles the claim.

Fix an i . The case $c_i = \infty$ and $\text{depth}_R U \geq \text{depth } R$ follows from the claim.

Suppose $c_i < \infty$, and choose $\mathfrak{q} \in \text{Proj}(R)$ such that $\text{height } \mathfrak{q} < c_i$. Since $c_i \leq \dim R$, it follows that $\mathfrak{q} \neq \mathfrak{m}$. The choice of $\mathfrak{q}$ means that $\sup H_*(F_{\mathfrak{q}}) < i$, because F is in $\mathcal{F}^c(R)$, and we have to verify that

$$\sup H_*((F \otimes_R U)_{\mathfrak{q}}) < i.$$

This is covered by the claim, applied to the local ring $R_{\mathfrak{q}}$, the $R_{\mathfrak{q}}$ -complex $F_{\mathfrak{q}}$ and $U_{\mathfrak{q}}$. This applies since $\mathfrak{q}$ is a nonmaximal prime.

In the graded context, it suffices to observe that the conditions on depth for $\mathfrak{p}$ in $\text{Proj}(R)$ imply the condition for all primes in $\text{Spec } R \setminus \{\mathfrak{m}\}$ by [3, Theorem 1.5.9]. $\square$

6.9. Let $\mathbf{c}$ be a codimension sequence of the form

$$\mathbf{c} := (\dots, \emptyset, \emptyset, c_a, \dots, c_b, \infty, \infty, \dots)$$

with $0 \leq c_a \leq c_b \leq \dim R$. Consider the sequence

$$\mathbf{c}' := (\dots, \emptyset, \emptyset, c_a, \dots, c_b, d, d, \dots),$$

where $d = \dim R$. Evidently, $\mathbf{c}' \leq \mathbf{c}$.

Lemma 6.10. With $\mathbf{c}$ and $\mathbf{c}'$ as above, if $\mathbf{B}^{\mathbf{c}'}(R) \subseteq \mathbf{B}_d^{\mathbf{c}'}$, then $\mathbf{B}^{\mathbf{c}}(R) \subseteq \mathbf{B}_d^{\mathbf{c}}$.

Proof. Pick $F \in \mathcal{F}^{\mathbf{c}}(R)$. Then one has

$$\text{proj dim}_R F = \text{depth } R - \text{depth}_R F \leq d + b.$$

Consequently, one has

$$\beta_{i,j}^R(F) = 0 \quad \text{for } i \geq d + b + 1 \text{ and all } j.$$

This fact will be used further below.

Since $\mathbf{c}' \leq \mathbf{c}$, we have $\mathbf{B}^{\mathbf{c}}(R) \subseteq \mathbf{B}^{\mathbf{c}'}(R)$, and also for A . Thus, by hypothesis, $\beta^R(F)$ is in $\mathbf{B}^{\mathbf{c}'}(A)$. Theorem 5.8 gives a decomposition

$$\beta^R(F) = \sum_n r_n \beta^A(G^n)$$

as a finite sum, with $r_n > 0$ and $G^n \in \mathcal{F}^{c'}(A)$, with G^n the shifted resolution of an A -module. That is, $H_i(G^n) = 0$ for all but one value, say i_n , of i . Moreover, by the definition of c' , we have

$$\operatorname{codim}_A H_{i_n}(G^n) = \begin{cases} c_{i_n} \text{ or more if } a \leq i_n \leq b \\ \dim A \text{ if } i_n > b. \end{cases}$$

In particular, when $i_n > b$ the Auslander–Buchsbaum formula yields

$$\beta_{d+b+1,j}(G^n) \neq 0 \quad \text{for some } j.$$

But then $\beta_{d+b+1,j}^R(F) \neq 0$ as well, a contradiction. Thus, $a \leq i_n \leq b$ for each n , that is to say, G^n is in $\mathcal{F}^c(A)$. $\square$

The last part of Theorem 6.2 holds over any field k , so we record it separately.

Proposition 6.11. *Let R be as in Section 6.1 and c a codimension sequence for R . If $c_i \neq \infty$ for all i or R is Cohen–Macaulay, then $\mathbf{B}^c(R) \supseteq \mathbf{B}_d^c$.*

Proof. With A as in Section 6.1, one has $\mathbf{B}^c(A) = \mathbf{B}_d^c$, by Theorem 5.8, so the desired inclusion is that $\mathbf{B}^c(R) \supseteq \mathbf{B}^c(A)$. When $c_i < \infty$, this is verified in the course of the proof of [4, Theorem 9.1]; see in particular, [4, Lemma 9.5, 9.6] and [17]. We give the argument, for one key equality in the proof was not justified in *op. cit.*

One can assume $c_i = \emptyset$ for $i < 0$ and c_0 is finite. Since $\mathbf{B}^c(A)$ is spanned by (shifts) of the Betti tables of finitely generated graded A -modules, it suffices to prove that if M is a finitely generated with $\operatorname{codim}_A M \geq c_0$, then $\beta^A(M)$ is in $\mathbf{B}^c(R)$. By [17], pulling back M along a k -algebra automorphism of A , one can suppose that the R -modules $\operatorname{Tor}_i^A(R, M)$ have finite length for each $i \geq 1$. Let F be the minimal free resolution of M over A , so the finite free complex of R -modules $R \otimes_A F$ satisfies

$$\beta^R(R \otimes_A F) = \beta^A(F) = \beta^A(M).$$

Since $H_i(R \otimes_A F) \cong \operatorname{Tor}_i^A(R, M)$ has finite length for $i \geq 1$, and $c_i < \infty$, one gets

$$\operatorname{codim}_R H_i(R \otimes_A F) \geq \dim R \geq c_i \quad \text{for } i \geq 1.$$

It remains to observe that

$$\operatorname{codim}_R H_0(R \otimes_A F) = \operatorname{codim}_R(R \otimes_A M) = \operatorname{codim}_A(M).$$

The equality on the right is by Lemma A.1; this is glossed over in the proof of [4, Theorem 9.1]. This completes the proof in the case when each c_i is finite.

Suppose that R is Cohen–Macaulay. Fix an A -complex G in $\mathbf{B}^c(A)$ and consider the R -complex $F := R \otimes_A G$. Evidently $\beta^R(F) = \beta^A(G)$, so for the desired inclusion one has only to verify that F is in $\mathbf{B}^c(R)$. Since R is Cohen–Macaulay and finite as an A -module for any finitely generated R -module M one has

$$\operatorname{codim}_R M = \dim R - \dim_R M = \dim A - \dim_A M = \operatorname{codim}_A M.$$

Thus, to verify that F is in $\mathbf{B}^c(R)$ it suffices to verify that

$$\operatorname{codim}_A H_i(R \otimes_A G) \geq c_i \quad \text{for each } i.$$

This follows from Lemma 6.8, applied to the A -complex G and for $U = R$. $\square$

6.12. With A as in Section 6.1, the regularity of a finitely generated graded R -module M (see Section 4.14) can be read from the Betti table of M over A , as follows:

$$\operatorname{reg}_R M = \max\{j \mid \beta_{i,i+j}^A(M) \neq 0\}.$$

See [3, Theorem 4.3.1].

6.13. (Proof of Theorem 6.2 when the characteristic of k is nonzero). Let A be as in Section 6.1. Given Theorem 5.8, the desired inclusion is $\mathbf{B}^c(R) \subseteq \mathbf{B}^c(A)$. By Lemma 6.10, we can assume the codimension sequence $\mathbf{c}$ satisfies $c_i \leq \dim R$ for each i .

Let $(U^n)_{n \geq 1}$ be a lim Ulrich sequence of graded R -modules provided by Theorem 4.17. This is where we need the hypothesis that the characteristic of k is positive. Fix an R -complex F in $\mathcal{F}^c(R)$, and consider the R -complexes $F \otimes_R U^n$.

For any M in $\operatorname{grmod} R$, one has $\operatorname{codim}_A M \geq \operatorname{codim}_R M$, so Lemma 6.8 yields that, viewed as an A -complex by restriction of scalars (the finite free resolution of) the complex $F \otimes_R U^n$ is in $\mathbf{B}^c(A)$. Set $u_n := \nu_R(U^n)$. Since A is regular, the canonical surjection $K(\mathbf{x}; A) \xrightarrow{\sim} k$ is a free resolution over A , so for all integers i, j one has equalities

$$\begin{aligned} \beta_{i,j}^A(F \otimes_R U^n) &= \operatorname{rank}_k H_i(F \otimes_R U^n \otimes_A K(\mathbf{x}; A))_j \\ &= \operatorname{rank}_k H_i(F \otimes_R U^n \otimes_R K(\mathbf{x}; R))_j \\ &= \operatorname{rank}_k H_i(F \otimes_R K(\mathbf{x}; U^n))_j. \end{aligned}$$

These equalities and Lemma 4.12 yield

$$\lim_{n \rightarrow \infty} \frac{\beta_{i,j}^A(F \otimes_R U^n)}{u_n} = \beta_{i,j}^R(F).$$

This proves that $\beta^R(F)$ is in the closure, in the point-wise topology on V , of the subcone of $\mathbf{B}^c(A)$ spanned by all the $\beta^A(F \otimes_R U^n)$, for $n \geq 0$. We claim that that the same is true in the finite topology on V , and to that end it suffices to verify:

Claim. For all but finitely many pairs (i, j) , one has $\beta_{i,j}^A(F \otimes_R U^n) = 0$ for all n .

Let t_0 and t_1 be as in Theorem 4.17. Given that $\operatorname{reg}_R(U^n)$ can be computed in terms of $\beta_{i,j}^A(U^n)$ – see Section 6.12 – we get that

$$\beta_{i,j}^A(U^n) = 0 \quad \text{unless } i \in [0, \dim A] \text{ and } j \in [t_0, t_1 + \dim A].$$

The stated claim follows from the usual change of rings spectral sequence

$$E_{p,q}^2 := \operatorname{Tor}_p^R(F, k) \otimes_k \operatorname{Tor}_q^A(U^n, k) \implies \operatorname{Tor}_{p+q}^A(F \otimes_R U^n, k).$$

Since $\mathbf{B}^c(A)$ is closed in the finite topology on V , by Corollary 5.9, we deduce that $\mathbf{B}^c(R) \subseteq \mathbf{B}^c(A)$, as desired.

The last part of the statement follows from Proposition 6.11. □

Characteristic zero

Next, we explain how to deduce the characteristic zero case of Theorem 6.2 from that of positive characteristic. The key is the following result, whose proof is a standard argument.

Lemma 6.14. Let k be a field of characteristic zero, R a finitely generated graded k -algebra, and fix F in $\mathcal{F}(R)$. There exists a field k_p of positive characteristic p , a finitely generated graded k_p -algebra R_p

and an R_p -complex F_p in $\mathcal{F}(R_p)$ such that

$$\beta^{R_p}(F_p) = \beta^R(F) \quad \text{and} \\ \text{codim}_{R_p} H_i(F_p) = \text{codim}_R H_i(F) \quad \text{for each } i.$$

Moreover, if the k -algebra R admits a linear normalization, then the k_p -algebra R_p can be chosen to admit one as well.

Proof. Write $R := k[x_1, \dots, x_n]/J$, where $J = (f_1, \dots, f_t)$ is a homogeneous ideal. Pick a finitely generated $\mathbb{Z}$ -subalgebra W of k such that the coefficients of the f_i and the coefficients in the entries in the differentials of F are in W . Thus, we have a graded ring $R_W := W[x_1, \dots, x_n]/(f_1, \dots, f_t)$ over W viewed as a ring concentrated in degree 0 and such that $R = k \otimes_W R_W$.

When R admits a linear Noether normalization, then by [12, (2,1,8)(h)], one can invert elements in W to ensure that the inclusion $W[x_1, \dots, x_d] \subseteq R_W$ is module-finite, with cokernel W -free. It follows that for all maximal ideal $\mathfrak{m}$ of W base-change along $W \rightarrow W/\mathfrak{m}$ induces a map

$$(W/\mathfrak{m})[x_1, \dots, x_d] \rightarrow R_{W/\mathfrak{m}}$$

that is a linear Noether normalization of $R_{W/\mathfrak{m}}$.

Write $\beta^R(F) = (\beta_{i,j})$. Evidently, there exists a diagram (not necessarily a complex) of finite free R_W -modules

$$F_W := \cdots \rightarrow \bigoplus R_W(-j)^{\beta_{i+1,j}} \rightarrow \bigoplus R_W(-j)^{\beta_{i,j}} \rightarrow \bigoplus R_W(-j)^{\beta_{i-1,j}} \rightarrow \cdots$$

such that $F = k \otimes_W F_W$. With Q the field of fractions of W , one has an isomorphism

$$k \otimes_W F_W \cong k \otimes_Q (Q \otimes_W F_W).$$

In particular, $Q \otimes_W F_W$ is a complex of R_Q -modules and the codimension of the R_Q -module $H_i(Q \otimes_W F_W)$ equals that of the R -module $H_i(F)$ for each i .

Inverting further elements in W , one can ensure that F_W is complex and for each maximal ideal $\mathfrak{m}$ of W and integer i one has

$$\text{codim}_{R_{W/\mathfrak{m}}} H_i(F_{W/\mathfrak{m}}) = \text{codim}_{R_W} H_i(F_W) = \text{codim}_R H_i(F).$$

Indeed, this follows from [12, (2.1.14)(g) last sentence, and (2.3.9)(c)]; the first reference tells us we can localize at elements in W to preserve the annihilators of the homology modules (note that $k \otimes_W H_i(F_W) = H_i(F)$), and the second one says that by inverting more elements in W the height of an ideal is preserved; see (2.3.3) and (2.3.4) in *op. cit.* for notation.

Finally, pick a maximal ideal $\mathfrak{m}$ of W such that $H(F_W) \otimes_{R_W} W/\mathfrak{m}W$ is nonzero and set $F_p := W/\mathfrak{m} \otimes_W F_W$. It is clear the R_p -complex F_p has the same Betti numbers as F ; the codimensions of their homology modules is also the same, by construction. It remains to note that $W/\mathfrak{m}$ has positive characteristic p . $\square$

6.15. (*Proof of Theorem 6.2 when the characteristic of k is zero*). This follows from Lemma 6.14 and the case of positive characteristic, settled in Section 6.13. $\square$

We do not know whether Theorem 3.4 also holds over fields of characteristic zero. The method of reduction to positive characteristics exploited above does not appear to work in the context of cohomology tables.

A. Codimension

In this section, $A \hookrightarrow R$ is an integral extension of Noetherian rings. For any finitely generated A -module M , it is easy to see that there is an inequality

$$\operatorname{codim}_R(R \otimes_A M) \leq \operatorname{codim}_A M.$$

Equality holds when, for instance, the going-down theorem holds for the extension $A \rightarrow R$, but not always; see Example A.2. Here is one other situation in which equality holds; this comes up in the proof of Proposition 6.11.

Lemma A.1. *Let $A \rightarrow R$ as above and M a finitely generated A -module with $\operatorname{proj dim}_A M$ finite and such that*

$$\ell_R \operatorname{Tor}_i^A(R, M) < \infty \quad \text{for all } i \geq 1.$$

There is an equality $\operatorname{codim}_R(R \otimes_A M) = \operatorname{codim}_A M$.

Proof. One has to only to verify that $\operatorname{codim}_R(R \otimes_A M) \geq \operatorname{codim}_A M$, that is to say

$$\operatorname{height} \mathfrak{q} \geq \operatorname{codim}_A M \quad \text{for all } \mathfrak{q} \text{ in } \operatorname{Supp}_R(R \otimes_A M).$$

Let $f: \operatorname{Spec} R \rightarrow \operatorname{Spec} A$ be the map induced by $A \rightarrow R$. Then

$$\operatorname{Supp}_R(R \otimes_A M) = f^{-1}(\operatorname{Supp}_A M).$$

Fix $\mathfrak{q}$ minimal in $f^{-1}(\operatorname{Supp}_A M)$. If $\mathfrak{q}$ is the maximal ideal of R , then

$$\operatorname{height} \mathfrak{q} = \dim R = \dim A \geq \operatorname{codim}_A M,$$

as desired. We can thus assume that $\mathfrak{q}$ is not maximal in R .

Set $\mathfrak{p} := \mathfrak{q} \cap A$; this is in $\operatorname{Supp}_A M$ and so contains a minimal element, say $\mathfrak{p}'$, in $\operatorname{Supp}_A M$. It suffices to verify

$$\operatorname{height} \mathfrak{q} \geq \operatorname{height} \mathfrak{p}'.$$

Since $\mathfrak{q}$ is not maximal in $\operatorname{Spec} R$, the hypotheses yield

$$\operatorname{Tor}_i^A(R, M)_{\mathfrak{q}} \cong \operatorname{Tor}_i^{A_{\mathfrak{p}}}(R_{\mathfrak{q}}, M_{\mathfrak{p}}) = 0 \quad \text{for } i \geq 1.$$

Thus, the $R_{\mathfrak{q}}$ -module $(R \otimes_A M)_{\mathfrak{q}}$ has finite projective dimension; it is also of nonzero finite length because $\mathfrak{q}$ is minimal in $\operatorname{Supp}_R(R \otimes_A M)$. This gives the first and the second equalities below:

$$\begin{aligned} \dim R_{\mathfrak{q}} &= \operatorname{depth} R_{\mathfrak{q}} \\ &= \operatorname{proj dim}_{R_{\mathfrak{q}}}(R \otimes_A M)_{\mathfrak{q}} \\ &= \operatorname{proj dim}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}) \\ &= \operatorname{proj dim}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\geq \operatorname{proj dim}_{A_{\mathfrak{p}'}} M_{\mathfrak{p}'} \\ &= \dim A_{\mathfrak{p}'} . \end{aligned}$$

The fourth one holds as $A_{\mathfrak{p}} \rightarrow R_{\mathfrak{q}}$ is local and, as $A_{\mathfrak{p}}$ -modules, $R_{\mathfrak{q}}$ and $M_{\mathfrak{p}}$ are Tor-independent. The inequality is clear. The last equality holds because $M_{\mathfrak{p}'}$ has finite length since $\mathfrak{p}'$ is minimal in $\operatorname{Supp}_A M$. Thus, $\operatorname{height} \mathfrak{q} \geq \operatorname{height} \mathfrak{p}'$. $\square$

Example A.2. Let $R := k[x, y, z]/(x, y) \cap (z)$ and A the subring $k[x + z, y]$. Then for $M := A/yA$ one has

$$\operatorname{codim}_A M = 1 \quad \text{whereas} \quad \operatorname{codim}_A(R \otimes_A M) = \operatorname{codim}_R(R/yR) = 0.$$

In geometric language, $\operatorname{Spec}(R)$ is the union of a line and a plane, both passing through the origin; $\operatorname{Spec}(A)$ is a plane and $\operatorname{Spec}(R) \rightarrow \operatorname{Spec}(A)$ any linear map that is finite (any generic such map would do). Then for one ‘bad’ line in $\operatorname{Spec}(A)$, its inverse image in $\operatorname{Spec}(R)$ is a union of the exceptional line in $\operatorname{Spec}(R)$ and another line passing through the origin. This gives an example where the codimension in A is 1 but the codimension in R drops to 0.

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