



Differential Equations for the KPZ and Periodic KPZ Fixed Points

Jinho Baik¹, Andrei Prokhorov^{1,2}, Guilherme L. F. Silva³

¹ Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.
 E-mail: baik@umich.edu

² St. Petersburg State University, Universitetskaya emb. 7/9, St. Petersburg, Russia 199034.
 E-mail: andreip@umich.edu

³ Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, SP 13566-590, Brazil. E-mail: silvag@usp.br

Received: 12 September 2022 / Accepted: 15 February 2023

Published online: 13 April 2023 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract: The KPZ fixed point is a 2d random field, conjectured to be the universal limiting fluctuation field for the height function of models in the KPZ universality class. Similarly, the periodic KPZ fixed point is a conjectured universal field for spatially periodic models. For both fields, their multi-point distributions in the space-time domain have been computed recently. We show that for the case of the narrow-wedge initial condition, these multi-point distributions can be expressed in terms of so-called integrable operators. We then consider a class of operators that include the ones arising from the KPZ and the periodic KPZ fixed points, and find that they are related to various matrix integrable differential equations such as coupled matrix mKdV equations, coupled matrix NLS equations with complex time, and matrix KP-II equations. When applied to the KPZ fixed points, our results extend previously known differential equations for one-point distributions and equal-time, multi-position distributions to multi-time, multi-position setup.

Contents

1. Introduction	1754
2. Statement of Main Results	1757
2.1 Cubic integrable operators and differential equations	1757
2.2 Fredholm determinants of cubic integrable operators	1761
2.3 The (periodic) KPZ fixed point and cubic integrable operators	1762
2.4 Organization of the rest of the paper	1764
3. Discussions on the Differential Equations and Comparison with Existing Works	1764
3.1 The case $m = 1$: self-similar solutions	1764
3.2 The case of equal times: reduction of KP-II	1765
3.3 The case of equal times: matrix ODE system	1765
3.4 Differential equations related to the Adler and van Moerbeke PDE	1766
3.4.1 A scalar PDE for cubic admissible determinants.	1766

3.4.2	Invariance of strongly cubic admissible determinants	1767
3.4.3	Scalar PDEs for the two-time and one-time cases - Adler and van Moerbeke PDE	1768
4.	Derivation of Differential Equations	1769
4.1	A review of integrable operators and RHP	1769
4.2	Lax equations	1770
4.3	NLS and mKdV: proof of Lemma 2.4 and Theorem 2.5	1771
4.4	The matrix KP equation: proof of Theorem 2.6	1774
4.5	Multi-component KP hierarchy	1775
4.6	Matrix ODE system à la Tracy and Widom: proof of Theorem 2.7 and Corollary 2.8	1780
4.7	Deformation formulas for cubic admissible determinants: proof of Proposition 2.10	1781
5.	A Class of Fredholm Determinants	1783
5.1	A class of operators	1784
5.2	A reformulation of a class of Fredholm determinants	1785
5.3	Proof of Theorem 5.3	1787
5.4	A lemma on trace class operators	1792
6.	Cubic Integrable Operators for the KPZ and the Periodic KPZ Fixed Points	1793
6.1	KPZ fixed point - proof of Theorem 2.11 (i)	1793
6.2	Periodic KPZ fixed point - proof of Theorem 2.11 (ii)	1797
7.	Large Height Limits and the Proof of Proposition 2.12	1799
8.	Adler and van Moerbeke PDE and a Non-equal Time Extension	1801

1. Introduction

The KPZ fixed point

$$\mathcal{H}(\gamma, \tau), \quad (\gamma, \tau) \in \mathbb{R} \times \mathbb{R}_+,$$

is a 1+1 dimensional random field that is conjectured to be the universal limit for the height fluctuations of the random growth models belonging to the KPZ universality class. Many remarkable properties of the KPZ fixed point were obtained over the last two decades, and several models are proved to converge to it in various senses. The field itself was constructed relatively recently by Matetski et al. [24]. Another construction was also obtained by Dauvergne et al. [11]. In this paper we focus on the multi-point distribution functions and study their connections to deterministic (integrable) differential equations. Such connections may depend on the initial condition of the KPZ fixed point. We consider the most well-studied initial condition, the narrow wedge initial condition given by $\mathcal{H}(\gamma, \tau = 0) = 0$ for $\gamma = 0$ and $\mathcal{H}(\gamma, \tau = 0) = -\infty$ for $\gamma \neq 0$. In this case, the 1 : 2 : 3 invariance property,

$$\left(\epsilon^{-1} \mathcal{H}(\epsilon^2 \gamma, \epsilon^3 \tau) \right)_{\gamma, \tau} \stackrel{d}{=} (\mathcal{H}(\gamma, \tau))_{\gamma, \tau},$$

holds for all $\epsilon > 0$.

Many differential equations are known for the one-time distributions and the equal-time, multi-position distributions. Some of them (for the narrow-wedge initial condition) are the following.

- The one-point distribution reduces to the GUE Tracy-Widom distribution F_{GUE} . For each fixed γ, τ ,

$$\mathbb{P}(\mathcal{H}(\gamma, \tau) \leq h) = F_{\text{GUE}}\left(\frac{h}{\tau^{1/3}} + \frac{\gamma^2}{\tau^{4/3}}\right), \quad (1.1)$$

where F_{GUE} is the GUE Tracy-Widom distribution. It satisfies [29]

$$\frac{d^2}{d\xi^2} \log F_{\text{GUE}}(\xi) = -u(\xi)^2,$$

where u solve the Painlevé II equation

$$u'' = \xi u^2 + 2u^3 \quad (1.2)$$

with the boundary condition $u(\xi) \sim \text{Ai}(\xi)$ as $\xi \rightarrow +\infty$.

- The equal-time, multi-position distribution is a scaled and shifted version of the Airy₂ process¹ introduced by Prähofer and Spohn [25]. In 2005, two differential equations were obtained for the multi-point distribution of the Airy₂ process.
 - Tracy and Widom [31] found a system of matrix ordinary differential equations (ODEs) with respect to a combination of the height variables.
 - On the other hand, Adler and van Moerbeke [3] obtained a nonlinear third order partial differential equation (PDE) for the case of the 2-point distribution with respect to the (shifted and scaled) height and spatial variables. Their result was extended to multi-point distribution by Wang [32].

In these two results, the time variable was kept as a constant and did not appear in the equations.

- Quastel and Remenik [26] also considered the equal-time, multi-point distribution, but included the time τ as an evolution variable instead of a fixed parameter. Thus, they regarded the equal-time m -point multi-position distribution as a function on $2m + 1$ variables. They derived a matrix version of the second Kadomtsev-Petviashvili (KP-II) equation with respect to these $2m + 1$ variables. If we specialize their result to the $m = 1$ case, it implies that (1.1), viewed as a function of three scalar variables h, γ, τ , is related to the scalar KP-II equation. Remarkably, the result of Quastel and Remenik applies to general initial conditions far beyond the narrow wedge initial condition.

One of the goals of this paper is to extend the above results to multi-time, multi-position distributions of the KPZ fixed point with the narrow-wedge initial condition. Fix a positive integer m and consider the function of $3m$ variables

$$\mathfrak{F}^{(\text{KPZ})}(h, \gamma, \tau) := \mathbb{P}\left(\bigcap_{i=1}^m \{\mathcal{H}(\gamma_i, \tau_i) \leq h_i\}\right), \quad (1.3)$$

where $\tau = (\tau_1, \dots, \tau_m)^T \in \mathbb{R}_+^m$, $\gamma = (\gamma_1, \dots, \gamma_m)^T \in \mathbb{R}^m$, and $h = (h_1, \dots, h_m)^T \in \mathbb{R}^m$ represent time, position, and height, respectively. We prove that this function is related to the following differential equations.

¹ The Airy₂ process is often regarded as an evolution. The time in the Airy₂ process corresponds to the position of the KPZ fixed point.

1. As a function of γ and h , we obtain a system of coupled $m \times m$ matrix nonlinear Schrödinger (NLS) equations with complex time $i\gamma$.
2. As a function of τ and h , we obtain a system of coupled matrix modified Korteweg-de Vries (mKdV) equations.
3. As a function of all $3m$ variables, we obtain matrix Kadomtsev-Petviashvili (KP) equations that generalizes the matrix KP-II equation obtained by Quastel and Remenik [26] for the case $\tau_1 = \dots = \tau_m$. We also show a connection to the multi-component KP hierarchy.
4. As a function of h , we obtain a matrix ODE system that generalizes the matrix ODE system of Tracy and Widom [31] for the Airy_2 process.

For the case $m = 2$, we also derive several PDEs with the aid of symbolic computations. If we take the special case $\tau_1 = \tau_2$, then a combination of two of them becomes the PDE of Adler and van Moerbeke [3] for the Airy_2 process.

We also study a periodic version of the KPZ fixed point. If we consider random growth models in the KPZ universality class on a periodic domain (instead of the infinite line) and take a large time, large period limit in a certain critical way, then a new (1+1) dimensional random field emerges. This field, which we call the periodic KPZ fixed point, is expected to interpolate between the Brownian motion and the KPZ fixed point [5, 7]. This interpolation property was proved for the one-point distribution in [8]. Even though the periodic KPZ fixed point is not yet constructed as a field, its multi-point distributions were obtained recently as limiting distributions for the totally asymmetric simple exclusion process on a ring [7]. It was shown in [8] that the one-point distribution of the periodic KPZ fixed point with the narrow step initial condition is related to a coupled NLS (with complex time), a coupled mKdV, and a KP-II equation. In this paper, we extend this result to the multi-point distributions and show that they are related to the same equations as the KPZ fixed point mentioned in 1–3 above. However, the Tracy-Widom type ODE system is not expected to hold for the periodic KPZ fixed point. In particular, the one-point distribution is not the Tracy-Widom distribution; it depends non-trivially on the time.

The multi-time, multi-position distributions of both the KPZ fixed point and the periodic KPZ fixed point have been computed explicitly and are expressible in terms of integrals of Fredholm determinants [7, 19, 23]. We will show that the Fredholm determinants for both fields are equal to the Fredholm determinants of so-called IKS integrable operators [12, 15]. These integrable operators have a certain common structure which we call cubic integrable. We derive the differential equations 1–3, and also 4 in some cases, for general cubic integrable operators. Such cubic integrable operators can be either defined on contours, which is the case of the KPZ fixed point, or on discrete sets in the complex plane, which is the case of the periodic fixed point.

IKS integrable operators have a special algebraic structure that associates them to Riemann-Hilbert problems in a canonical way. In turn, Riemann-Hilbert problems are deeply integrated in the theory of integrable differential equations, providing inverse transform methods for the latter [1, 13, 16]. In particular, if a Riemann-Hilbert problem depends on parameters and the jump/residue matrix for the Riemann-Hilbert problem has a certain decomposition structure, then one can derive linear matrix differential equations, called Lax equations, for its solution. The compatibility of a pair of Lax equations often yields nonlinear differential equations in those parameters.

For the case of cubic integrable operators, we find Lax equations for each of the parameters, h , γ , τ . The compatibility between the Lax equations in the variables h and γ yields the coupled NLS equation mentioned in 1 above, the compatibility between τ and

h yields the coupled mKdV mentioned in 2, and the compatibility between the equations on the three variables yield the KP-II equation and multi-component KP hierarchy from 3. If we impose an additional property on the cubic integrable operators, that we call strong cubic integrability, we obtain an additional Lax equation with respect to the so-called spectral variable. This additional equation allows us to obtain the equation in 4. We will see that the cubic integrable operator for the KPZ fixed point is strongly cubic integrable hence the equation 4 holds.

In the next section we introduce the cubic integrable operators formally, and state our main results. See Sect. 2.4 for the organization of the rest of the paper.

2. Statement of Main Results

In the first subsection, we define cubic integrable operators and state various differential equations associated to them. In the next subsection we introduce the Fredholm determinants of cubic integrable operators. The following subsection then discusses how the multi-time, multi-position distributions of the KPZ and the periodic KPZ fixed points are related to cubic integrable operators. We finish this section with a roadmap of the organization of the rest of the paper.

2.1. Cubic integrable operators and differential equations. Let $\Omega \subset \mathbb{C}$ be either a finite union of disjoint simple contours or a discrete set without accumulations points. If it is a union of contours, we further assume that each contour is either closed or extends to infinity, so in particular none of these contours have finite endpoints. Let

μ be the counting measure if Ω is discrete and $d\mu = dz$ if Ω consists of contours, (2.1)

so that

$$L^2(\Omega, \mu) = \begin{cases} \ell^2(\Omega) & \text{if } \Omega \text{ is discrete,} \\ L^2(\Omega, dz) & \text{if } \Omega \text{ is a union of contours.} \end{cases} \quad (2.2)$$

Fix an integer $m \geq 1$ and consider $3m$ parameters

$$\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m, \quad \mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m, \quad \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m.$$

In the (periodic) KPZ fixed point, they will correspond to time, spatial location, and height variables. Define the cubic exponential functions

$$\mathbf{m}_j(z) := \exp\left(t_j z^3 + y_j z^2 + x_j z\right), \quad z \in \mathbb{C}, \quad (2.3)$$

and introduce the $(m+1) \times (m+1)$ diagonal matrix-valued function

$$\Delta(z) = \Delta(z \mid \mathbf{x}, \mathbf{y}, \mathbf{t}) := \text{diag}(\mathbf{m}_1(z), \mathbf{m}_2(z), \dots, \mathbf{m}_m(z), 1). \quad (2.4)$$

Let, for $i \in \{1, \dots, m+1\}$,

$$\mathbf{E}_i \text{ be the } (m+1) \times (m+1) \text{ matrix whose } (i, i) \text{ entry is } 1 \text{ and all other entries are } 0. \quad (2.5)$$

An integral operator $H : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$ is called *integrable* in the sense of Its, Izergin, Korepin and Slavnov (or, shortly, an IKS operator, or an integrable operator) if its kernel is of the form²

$$H(u, v) = \frac{f(u)^T g(v)}{u - v} \quad \text{for } u \neq v \quad \text{and} \quad H(u, u) = 0 \quad \text{for } u, v \in \Omega, \quad (2.6)$$

for some vector-valued functions f and g (see [12, 15]). The operators of relevance to us possess an additional structure on the vector functions f and g .

Definition 2.1 (*Cubic integrable operator*). A bounded integral operator $H : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$ is called *cubic integrable* if it is integrable in the sense of (2.6) and the vector functions are of the form

$$\begin{aligned} f(u) &= f(u \mid x, y, t) := c(u) \Delta(u \mid x, y, t) U(u), \\ g(u) &= g(u \mid x, y, t) := \frac{1}{c(u)} \Delta(u \mid x, y, t)^{-1} V(u) \end{aligned} \quad (2.7)$$

for $u \in \Omega$, where Δ is as in (2.4), with the following additional structures:

- U and V are $(m + 1)$ -dimensional column vector functions which do not depend on x, y, t and satisfy

$$U(u)^T E_i V(u) = 0 \quad \text{for every } u \in \Omega \text{ and } i \in \{1, \dots, m + 1\}; \quad (2.8)$$

- c is a non-vanishing scalar function on Ω (which may depend on x, y, t);
- $f, g \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)$.

If, in addition, the set Ω is a union of disjoint contours and the vectors U, V are constants on each connected component of Ω , then we say that H is a *strongly* cubic integrable operator.

If we change the scalar c , the operator is a conjugate of the original operator H . Since we will be mostly interested in properties of the operators that are invariant under such conjugations, such as the Fredholm determinant and the trace, the scalar c does not play a major role in our results. However, including it in the definition as a free parameter is useful, as it provides an extra layer of flexibility that we explore in examples, to prove that H is not only bounded but also a trace class operator.

The term *cubic* comes from the cubic dependence of the kernel of H on the parameters x, y, t through the factor Δ in (2.7).

For a cubic integrable operator, we define the following matrix function.

Definition 2.2. Let H be a cubic integrable operator on $L^2(\Omega, \mu)$ and assume that $1 - H$ is invertible. Define the $(m + 1) \times (m + 1)$ matrix

$$\Phi_1 = \Phi_1(x, y, t) := \int_{\Omega} ((1 - H)^{-1} f)(s) g(s)^T d\mu(s) \quad (2.9)$$

where recall (2.1) for μ .

² The condition $H(u, u) = 0$ is not strictly necessarily. We included it here to simplify technical details in the theory, especially when Ω is a discrete set. This simplification is enough for our purposes regarding the (periodic) KPZ fixed points.

The subscript 1 is used in (2.9) since Φ_1 is the first element of a sequence of matrix functions; see (4.5) below. Observe that the integral (2.9) is unchanged even if we change the scalar c in the definition of f, g to a different scalar.

A non-empty set $S \subset \{1, \dots, m\}$ induces a permutation π on $\{1, \dots, m+1\}$ that fixes the element $m+1$; this permutation is given by $\pi(i) = k_i$ for $1 \leq i \leq m$ and $\pi(m+1) = m+1$, where $k_1 < \dots < k_{|S|}$ are elements of S and $k_{|S|+1} < \dots < k_m$ are elements of $\{1, \dots, m\} \setminus S$. For a non-empty set $S \subset \{1, \dots, m\}$, let $\Pi^S = (\delta_{\pi(i),j})_{i,j=1}^{m+1}$ denote the $(m+1) \times (m+1)$ permutation matrix for π . Note that the $(m+1, m+1)$ entry of the matrix Π^S is 1.

Definition 2.3. Let Φ_1 be the matrix in (2.9). For a non-empty set $S \subset \{1, \dots, m\}$, introduce matrix-valued functions q, p, r, s through the formula

$$\Pi^S \Phi_1 (\Pi^S)^T = \begin{pmatrix} q & p \\ r & s \end{pmatrix}, \quad (2.10)$$

with q and s being of sizes $|S| \times |S|$ and $(m+1-|S|) \times (m+1-|S|)$, respectively. Also, introduce the differential operators

$$\partial_t = \partial_t^S := \sum_{k \in S} \partial_{t_k}, \quad \partial_y = \partial_y^S := \sum_{k \in S} \partial_{y_k} \quad \text{and} \quad \partial_x = \partial_x^S := \sum_{k \in S} \partial_{x_k}. \quad (2.11)$$

The matrices q, p, r, s depend on the permutation S , but we do not display this dependence explicitly to simplify the notations. We note that the functions q, p, r, s are complex-valued matrices. The following are basic relations between them.

Lemma 2.4. Under the notations of Definition 2.3,

$$\text{Tr}(q) + \text{Tr}(s) = 0, \quad \partial_x q = -pr, \quad \partial_x s = rp.$$

We now state three main theorems on differential equations. The first theorem concerns the pair p and r , which are matrices of sizes $|S| \times (m+1-|S|)$ and $(m+1-|S|) \times |S|$, respectively.

Theorem 2.5 (Coupled matrix NLS with complex time and mKdV). *Let H be a cubic integrable operator and assume that $1 - H$ is invertible for the parameters (x, y, t) in an open set of \mathbb{R}^{3m} . Then, the following results hold.*

- (a) As functions of x and y , the matrices p and r satisfy a system of coupled matrix nonlinear Schrödinger (NLS) equations with complex time $y \mapsto iy$,

$$\partial_y p = \partial_x^2 p + 2prp, \quad \partial_y r = -\partial_x^2 r - 2rpr. \quad (2.12)$$

- (b) As functions of t and x , the matrices p and r satisfy a system of coupled matrix modified Korteweg-de Vries (mKdV) equations

$$\partial_t p = \partial_x^3 p + 3(\partial_x p)rp + 3pr(\partial_x p), \quad \partial_t r = \partial_x^3 r + 3(\partial_x r)pr + 3rp(\partial_x r). \quad (2.13)$$

These equations were obtained for the (periodic) KPZ fixed point when $m = 1$ in [8]. The coupled NLS equations with complex time with $m = 1$ also appeared in the recent work [21] on the weak noise theory of the KPZ equation.

The next differential equations involve all $3m$ variables t, y and x .

Theorem 2.6 (Matrix KP). *Let H be a cubic integrable operator and assume that $1 - H$ is invertible for the parameters $(\mathbf{x}, \mathbf{y}, \mathbf{t})$ in an open set of \mathbb{R}^{3m} . Then, the following result holds.*

(a) *The $|\mathbf{S}| \times |\mathbf{S}|$ matrices $\mathbf{u} := \mathbf{p}\mathbf{r}$ and \mathbf{q} satisfy the matrix KP-II equation³*

$$-4\partial_t \mathbf{u} + \partial_x^3 \mathbf{u} + 6\partial_x(\mathbf{u}^2) - 3\partial_y^2 \mathbf{q} + 6[\mathbf{u}, \partial_y \mathbf{q}] = 0, \quad \partial_x \mathbf{q} = -\mathbf{u}. \quad (2.14)$$

(b) *The $(m+1 - |\mathbf{S}|) \times (m+1 - |\mathbf{S}|)$ matrices $\mathbf{v} := \mathbf{r}\mathbf{p}$ and \mathbf{s} satisfy the matrix KP-II equation*

$$-4\partial_t \mathbf{v} + \partial_x^3 \mathbf{v} + 6\partial_x(\mathbf{v}^2) + 3\partial_y^2 \mathbf{s} + 6[\mathbf{v}, \partial_y \mathbf{s}] = 0, \quad \partial_x \mathbf{s} = \mathbf{v}. \quad (2.15)$$

If we insert $\mathbf{u} = -\partial_x \mathbf{q}$, (2.14) becomes a fourth order differential equation for the matrix \mathbf{q} only. If we take $\mathbf{S} = \{1, \dots, m\}$ and set $\mathbf{t}_1 = \dots = \mathbf{t}_m$, Equation (2.14) becomes the matrix KP-II equation obtained in [26] for the KPZ fixed point; see Sect. 3 below for further discussion on this end.

We also obtain a connection to the so-called multi-component KP hierarchy in Proposition 4.5 below.

For the next result, define $(m+1) \times (m+1)$ matrices

$$\mathbf{M}_t := \text{diag}(\mathbf{t}_1, \dots, \mathbf{t}_m, 0), \quad \mathbf{M}_y := \text{diag}(\mathbf{y}_1, \dots, \mathbf{y}_m, 0), \quad \mathbf{M}_x := \text{diag}(\mathbf{x}_1, \dots, \mathbf{x}_m, 0), \quad (2.16)$$

and the differential operator

$$\partial := \sum_{k=1}^m \mathbf{t}_k \partial_{\mathbf{x}_k}. \quad (2.17)$$

We take $\mathbf{S} = \{1, \dots, m\}$ in Definition 2.3 so that $\Pi^{\mathbf{S}}$ is the identity matrix, and $\mathbf{q}, \mathbf{p}, \mathbf{r}$, and \mathbf{s} are matrices of sizes $m \times m$, $m \times 1$, $1 \times m$, and 1×1 , respectively.

Theorem 2.7 (ODE system). *Let H be a strongly cubic integrable operator and assume that $1 - H$ is invertible for the parameters $(\mathbf{x}, \mathbf{y}, \mathbf{t})$ in an open set. Then, Φ_1 in (2.9) satisfies*

$$3\partial^2 \Phi_1 = 2[\partial \Phi_1, \mathbf{M}_y] + [[\Phi_1, \mathbf{M}_t], 3\partial \Phi_1 - 2[\Phi_1, \mathbf{M}_y] - \mathbf{M}_x]. \quad (2.18)$$

Furthermore, the additional relation

$$\partial \mathbf{s} = \mathbf{r}\mathbf{p}\mathbf{t} \quad (2.19)$$

also holds, where \mathbf{t} is the matrix defined in (2.20) below.

If we write the block form (2.18) (recall that $\Pi^{\mathbf{S}} = \mathbf{I}$ in this case) and use (2.19), then we find the following matrix ODE system for $\mathbf{q}, \mathbf{p}, \mathbf{r}$. Introduce the $m \times m$ matrices

$$\mathbf{t} := \text{diag}(\mathbf{t}_1, \dots, \mathbf{t}_m), \quad \mathbf{y} := \text{diag}(\mathbf{y}_1, \dots, \mathbf{y}_m), \quad \mathbf{x} := \text{diag}(\mathbf{x}_1, \dots, \mathbf{x}_m). \quad (2.20)$$

³ The usual KP-II equation is $\partial_x (\partial_t w + 6w\partial_x w + \partial_{xxx} w) + 3\partial_{yy} w = 0$ for a scalar function w . When $m = 1$, the x -derivative of Equation (2.14) becomes this equation if we set $w(t, y, x) = 2u(-4t, y, x)$.

Corollary 2.8. *Under the same conditions of Theorem 2.7, q, p, r satisfy the matrix ODE system*

$$\begin{aligned} 3\partial^2 q - 2[\partial q, y] - 3[[q, t], \partial q] + 3tp\partial r + 3(\partial p)rt + 2[[q, t], [q, y]] \\ - 2tpry + 2yprt + [[q, t], x] = 0, \\ 3\partial^2 p + 2y\partial p - 3(\partial q)tp - 3[q, t]\partial p + 3tp\partial p - 2(yqt - tqy)p + xtp = 0, \\ 3\partial^2 r - 2\partial ry - 3rt\partial q + 3\partial r[q, t] + 3rtpr - 2r(yqt - tqy) + rtx = 0. \end{aligned} \quad (2.21)$$

In the particular case when $t_1 = \dots = t_m =: t$, they satisfy the reduced ODE system

$$\begin{aligned} \partial_x q + pr = 0, \\ 3t\partial_x^2 p + 2y\partial_x p + 6tpr - 2[y, q]p + xp = 0, \\ 3t\partial_x^2 r - 2(\partial_x r)y + 6tpr - 2r[y, q] + rx = 0. \end{aligned} \quad (2.22)$$

where $\partial_x := \sum_{i=1}^m \partial_{x_i}$.

As we discuss in Sect. 3.3 below, the system (2.22) is equivalent to the ODE system found by Tracy and Widom for the Airy_2 process [30].

We also derive a scalar PDE of Adler-van Moerbeke [3] when $m = 2$ for strongly cubic integrable operators (see Sect. 3.4 and Sect. 8), and find a connection with the multi-component KP hierarchy (see Sect. 4.5).

2.2. Fredholm determinants of cubic integrable operators. When the operator H is trace class, the Fredholm determinant of H is related to the matrix Φ_1 in (2.9), and thus to the differential equations of the last subsection.

Definition 2.9 (*Cubic admissible determinant*). For a (strongly) cubic integrable operator H that is trace class, we call its Fredholm determinant

$$\mathfrak{D}(x, y, t) := \det(1 - H)$$

a (strongly) cubic admissible determinant.

Recall the definition (2.5) of E_i .

Proposition 2.10. *Let H be a cubic integrable operator. Let $\mathfrak{D}(x, y, t) = \det(1 - H)$ be the associated cubic admissible determinant and Φ_1 be the matrix from (2.9). Then,*

$$\partial_{x_i} \log \mathfrak{D}(x, y, t) = -\text{Tr}(\Phi_1 E_i), \quad i \in \{1, \dots, m\}.$$

Recalling Definition 2.2 and Lemma 2.4, this result implies that

$$\partial_x \log \mathfrak{D} = -\text{Tr}(q) = \text{Tr}(s) \quad \text{and} \quad \partial_x^2 \log \mathfrak{D} = \text{Tr}(pr) = \text{Tr}(rp),$$

where $\partial_x = \sum_{k \in S} \partial_{x_k}$ is as in (2.11). Thus, for example, we find from Theorem 2.6 that if

we take $S = \{1, \dots, m\}$, then $s = \partial_x \log \mathfrak{D}$ and $v = \partial_x^2 \log \mathfrak{D}$ satisfy the matrix KP-II equation (2.15), see Sect. 3.4.1.

In (4.7) and Proposition 4.7 below, we also find the derivatives of $\log \mathfrak{D}$ with respect to y and t .

2.3. The (periodic) KPZ fixed point and cubic integrable operators. We are now discuss how the multi-point distributions of the KPZ and the periodic KPZ fixed points with the narrow wedge initial condition are related to cubic admissible determinants.

Johansson and Rahman [19] and also independently Liu [23] obtained explicit formulas for the multi-point distributions of the KPZ fixed point. These results were preceded by two-time distribution formulas by Johansson [17, 18]. The formula of Liu is the following. Let $\gamma = (\gamma_1, \dots, \gamma_m)$ be position parameters, $\tau = (\tau_1, \dots, \tau_m)$ time parameters, and $\mathbf{h} = (h_1, \dots, h_m)$ height parameters. Arrange the time parameters so that $\tau_1 \leq \dots \leq \tau_m$ and arrange the location parameters further so that $\gamma_i < \gamma_{i+1}$ whenever $\tau_i = \tau_{i+1}$. Then, [23, Theorem 2.20 and Definition 2.23] states that the multi-point distribution in (1.3) for the KPZ fixed point with the narrow wedge initial condition is given by

$$\mathfrak{F}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau) = \frac{1}{(2\pi i)^{m-1}} \oint \dots \oint \mathfrak{D}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau | \zeta) \prod_{i=1}^{m-1} \frac{d\zeta_i}{(1 - \zeta_i)\zeta_i}. \quad (2.23)$$

Here, the integrals are over disjoint circles centered at the origin and of radii smaller than 1, and the term $\mathfrak{D}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau | \zeta)$ is a Fredholm determinant that depends on the parameters \mathbf{h} , γ and τ , and also on $\zeta := (\zeta_1, \dots, \zeta_{m-1})$. We will recall its precise definition in Sect. 6.1.

In [7], the authors computed the so-called relaxation time limit of the multi-point distributions of the periodic totally asymmetric simple exclusion process, extending their earlier work [6] on the one-point distribution. The limit obtained in [7, Theorem 2.1, Definition 2.5] is expected to be the multi-point distribution of the conjectured periodic KPZ fixed point. Due to the spatial periodicity, we may assume that the location variables satisfy $\gamma = (\gamma_1, \dots, \gamma_m)^T \in [0, 1)^m$. As before, arrange the parameters so that $\tau_1 \leq \dots \leq \tau_m$. If $\tau_j = \tau_{j+1}$, we assume that $\mathbf{h}_j < \mathbf{h}_{j+1}$. Then, for the step initial condition, it was shown that

$$\mathfrak{F}^{(\text{per})}(\mathbf{h}, \gamma, \tau) := \frac{1}{(2\pi i)^m} \oint \dots \oint \mathfrak{C}^{(\text{per})}(\zeta) \mathfrak{D}^{(\text{per})}(\mathbf{h}, \tau, \gamma | \zeta) \prod_{i=1}^m \frac{d\zeta_i}{\zeta_i} \quad (2.24)$$

where the integration contours are circles about the origin satisfying $0 < |\zeta_1| < \dots < |\zeta_m| < 1$, the function $\mathfrak{C}^{(\text{per})}(\zeta)$ is simple and explicit, and $\mathfrak{D}^{(\text{per})}(\mathbf{h}, \tau, \gamma | \zeta)$ is again a Fredholm determinant. We discuss the precise formula in Sect. 6.2.

One main difference between $\mathfrak{D}^{(\text{KPZ})}$ and $\mathfrak{D}^{(\text{per})}$ is that the operator for the former acts on an L^2 space of contours while the one for the latter acts on an ℓ^2 space of a discrete set. The underlying operators, as originally obtained, share some similarities but the formulas are somewhat involved. It turns out that we can recast them to cubic admissible determinants, as we state as our next result.

Theorem 2.11 (KPZ fixed point).

(i) For each $\zeta = (\zeta_1, \dots, \zeta_{m-1}) \in \mathbb{C}^{m-1}$ satisfying $0 < |\zeta_1|, \dots, |\zeta_{m-1}| < 1$, the function $(\mathbf{x}, \mathbf{y}, \mathbf{t}) \mapsto \mathfrak{D}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau | \zeta)$ is a strongly cubic admissible determinant with the identification

$$t_i = -\tau_i/3, \quad y_i = \gamma_i, \quad x_i = h_i. \quad (2.25)$$

Furthermore, the associated matrix $\Phi_1^{(\text{KPZ})} = \Phi_1$ satisfies the symmetry relation

$$L\Phi_1^{(\text{KPZ})}(\mathbf{x}, \mathbf{y}, \mathbf{t})L^{-1} = \Phi_1^{(\text{KPZ})}(\mathbf{x}, -\mathbf{y}, \mathbf{t})^T \quad (2.26)$$

where $L := \text{diag}(L_1, \dots, L_m, 1)$ with $L_1 = -(1 - \zeta_1)$ and for $j \geq 2$,

$$L_j := (-1)^j \frac{(1 - \zeta_1)(1 - \zeta_2) \cdots (1 - \zeta_j)}{(1 - \frac{1}{\zeta_1})(1 - \frac{1}{\zeta_2}) \cdots (1 - \frac{1}{\zeta_{j-1}})}. \quad (2.27)$$

(ii) (*Periodic KPZ fixed point*) For each $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ satisfying $0 < |\zeta_1| < \dots < |\zeta_m| < 1$, the function $(\mathbf{x}, \mathbf{y}, \mathbf{t}) \mapsto \mathfrak{D}^{(\text{per})}(\mathbf{h}, \gamma, \tau \mid \zeta)$ is a cubic admissible determinant with the identification

$$\mathbf{t}_i = -\tau_i/3, \quad \mathbf{y}_i = \gamma_i/2, \quad \mathbf{x}_i = \mathbf{h}_i. \quad (2.28)$$

We will describe the cubic integrable operators explicitly in Sect. 6. This theorem implies that Proposition 2.10, Lemma 2.4, Theorems 2.5, and 2.6 apply to (the cubic integrable operators $\mathbf{H}^{(\text{KPZ})}$ and $\mathbf{H}^{(\text{per})}$ for) $\mathfrak{D}^{(\text{KPZ})}$ and $\mathfrak{D}^{(\text{per})}$. The operator $\mathbf{H}^{(\text{KPZ})}$ also satisfies Theorem 2.7 since it is strongly admissible. These differential equation results assume that the Fredholm determinant is non-zero. Since the Fredholm determinant for the (periodic) KPZ fixed point is analytic in the parameters, this condition holds in an open set of the space of parameters. For the case of the equal-time distributions of the KPZ fixed point, Quastel and Remenik [26] obtained the same matrix KP-II equation as in Theorem 2.6 although the quantities that solve the equation are different than ours.⁴ When $m = 1$, Theorems 2.5, 2.6 and 2.6 for the periodic KPZ fixed point and the KPZ fixed point were obtained in [8].⁵

The equal-time slice of the KPZ fixed point is a simple change of the Airy_2 process. However, due to the integrals in the formula (2.23), the function $\mathfrak{D}^{(\text{KPZ})}$ is not directly related to the multi-point distribution of the Airy_2 process. Nonetheless, a different Fredholm determinant formula for the Airy_2 process is known and it can also be turned into the cubic admissible determinant formalism. Thus, the matrix NLS system with complex time in Theorem 2.5 will hold for the Airy_2 process. This discussion will be carried out in a separate paper.

For the periodic KPZ fixed point, when $m = 1$ it was shown in [8, Theorem 1.2] that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Phi_1^{(\text{per})}(\mathbf{x}, \mathbf{y}, \mathbf{t}) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi_1^{(\text{per})}(\mathbf{x}, -\mathbf{y}, \mathbf{t})^T.$$

However, it is not clear if a symmetry relation holds for $m > 1$.

Proposition 2.10 relates log derivatives of $\mathfrak{D}^{(\text{KPZ})}$ and $\mathfrak{D}^{(\text{per})}$ with Φ_1 which, in turn, satisfies differential equations. Integrating the log derivatives, we obtain the next formula.

Proposition 2.12. *Set*

$$\mathbf{a} := (1, 2, \dots, m). \quad (2.29)$$

Let \mathfrak{D} denote either $\mathfrak{D}^{(\text{KPZ})}$ or $\mathfrak{D}^{(\text{per})}$. Assume that the parameters $(\mathbf{x}, \mathbf{y}, \mathbf{t}) \in \mathbb{R}^{3m}$ satisfy $\mathfrak{D}(\mathbf{x} + \xi \mathbf{a}, \mathbf{y}, \mathbf{t}) \neq 0$ for every $\xi \geq 0$. Then,

$$\begin{aligned} \mathfrak{D}(\mathbf{x}, \mathbf{y}, \mathbf{t}) &= \exp \left[- \int_0^\infty \sum_{i=1}^m \text{Tr} (q_i(\mathbf{x} + \xi \mathbf{a}, \mathbf{y}, \mathbf{t})) d\xi \right] \\ &= \exp \left[\int_0^\infty \sum_{i=1}^m \text{Tr} (s_i(\mathbf{x} + \xi \mathbf{a}, \mathbf{y}, \mathbf{t})) d\xi \right] \end{aligned} \quad (2.30)$$

where q_i and s_i denote the matrices \mathbf{q} and \mathbf{s} in (2.10) with the choice of the index set $\mathbf{S} = \mathbf{S}_i := \{1, \dots, i\}$ for each $i = 1, \dots, m$.

⁴ The paper also obtained the result for general initial conditions.

⁵ In the paper [8], the roles of \mathbf{r} and \mathbf{p} are switched.

2.4. Organization of the rest of the paper. In Sect. 3 we discuss a few special cases of the differential equations and compare with previous works in the literature. The proofs of the differential equations are given in Sect. 4 using a connection between integrable operators and Riemann-Hilbert problems, applied to cubic integrable operators. In Sect. 5, we introduce operators of a certain structure, which is not restricted to but does arise in the multi-point distributions for the KPZ and periodic KPZ fixed points. We then show that the corresponding Fredholm determinants can be recast as cubic admissible determinants. A special case of this result recasts Theorem 2.11 for the KPZ and periodic KPZ fixed points, and Sect. 6 is devoted to this application. The remaining two sections are short. Section 7 is about simple asymptotic properties of cubic integrable operators and a proof of Proposition 2.12. In Sect. 8, we discuss scalar PDEs that extend a work of Adler and van Moerbeke, which we first state in Sect. 3.4.

3. Discussions on the Differential Equations and Comparison with Existing Works

We compare the differential equations in Sect. 2.1 with various results in the literature.

3.1. The case $m = 1$: self-similar solutions. When $m = 1$, the equations (2.12), (2.13), (2.14) and (2.22) become equations for scalar-valued functions. We show that they admit self-similar solutions constructed out of Painlevé II transcendents.

Suppose that $w(\xi)$ is a single-variable function that satisfies the Painlevé II equation (1.2)

$$w'' = \xi w + 2w^3.$$

Then, it is straightforward to check that for any non-zero constant α , the functions

$$\begin{aligned} p(t, y, x) &= \frac{\alpha}{(-3t)^{1/3}} \exp\left(-\frac{1}{3t}xy + \frac{2}{27t^2}y^3\right) w(\xi), \\ r(t, y, x) &= -\frac{1}{\alpha(-3t)^{1/3}} \exp\left(\frac{1}{3t}xy - \frac{2}{27t^2}y^3\right) w(\xi) \end{aligned}$$

with

$$\xi = \frac{x}{(-3t)^{1/3}} + \frac{y^2}{(-3t)^{4/3}}$$

satisfy the NLS system with complex time (2.12) and the mKdV system (2.13). They also satisfy the Tracy-Widom ODE system (2.22). It is also straightforward to check that the function

$$q(t, y, x) = -(-3t)^{-1/3} Q(\xi) \quad \text{and} \quad u(t, y, x) = (-3t)^{-2/3} w(\xi)^2$$

solve the scalar KP equation (2.14) (observe that the commutator drops out $[u, \partial_y q] = 0$, since the functions are scalar-valued). The functions $s(t, y, x) = -q(t, y, x)$ and $v(t, y, x) = u(t, y, x)$ also solve the equation (2.15). Thus, these equations have solutions constructed out of a Painlevé II transcendent.

For the KPZ fixed point, we can check that the functions p, r, q, s for $m = 1$ are indeed given by above forms with w chosen to be the Hastings-McLeod solution to the Painlevé II equation. However, the functions for the periodic KPZ fixed point are not

expected to be given by the above forms associated to the Painlevé II equation due to the fact that the space for the operator is discrete, which then implies that the relevant Riemann-Hilbert problem is of discrete type instead of of continuous type (see Sect. 4), the solutions are associated to solitons, and no self-similarity can be recast.

3.2. The case of equal times: reduction of KP-II. If we take $\mathbf{S} = \{1, \dots, m\}$, set $\tau_1 = \dots = \tau_m = -t/3$, and change the notations y_i, x_i, u, q to $x_i, r_i, q, -Q$, then (2.14) becomes

$$12\partial_t q + \partial_r^3 q + 6\partial_r(q^2) + 3\partial_x^2 Q + 6[q, \partial_x Q] = 0, \quad \partial_r Q = q,$$

where $\partial_r = \sum_{i=1}^m \partial_{r_i}$ and $\partial_x = \sum_{i=1}^m \partial_{x_i}$. This is the same matrix KP equation obtained in [26, (1.6)] by Quastel and Remenik. However, the functions q, Q considered in [26] are different from what we considered in this paper. Namely, the functions in [26] are directly related to the multi-point distributions $\mathfrak{F}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau)$ while in this paper, the functions are related to $\mathfrak{D}^{(\text{KPZ})}(\mathbf{h}, \gamma, \tau \mid \zeta)$ which is a part of the formula (2.23) for the multi-point distribution. It is puzzling that they satisfy the same KP-II equation. If we compare the result of [26] and Eq. (2.30) with formula (2.23) in mind, we obtain the identity

$$\begin{aligned} \text{Tr}(q(\mathbf{h}, \gamma, t)) = \partial_{\mathbf{h}}^2 \log & \left[\frac{1}{(2\pi i)^{m-1}} \oint \cdots \oint \exp \left[\int_0^\infty \sum_{i=1}^m \text{Tr}(\mathbf{S}_i(\mathbf{x} + \xi \mathbf{a}, \mathbf{y}, \mathbf{t})) d\xi \right] \right. \\ & \left. \prod_{i=1}^{m-1} \frac{d\zeta_i}{(1 - \zeta_i)\zeta_i} \right] \end{aligned} \quad (3.1)$$

between a single solution q to the KP equation on the left-hand side and a superposition on the right-hand side of families of solutions $\mathbf{S}_i = \mathbf{S}_i^{(\text{KPZ})}$ to different KP-II equations corresponding to different permutations $\mathbf{S} = \mathbf{S}_i = \{1, \dots, i\}$. It is unclear whether this identity holds because the KP-II solutions related to the KPZ fixed point are very special, or if such an identity holds for a broad class of solutions. We note that q on the left-hand side is a real solution while \mathbf{S}_i are complex-valued solutions.

3.3. The case of equal times: matrix ODE system. Tracy and Widom obtained in [30] a matrix ODE system for the Airy₂ process, which is a simple transformation of the equal-time slice of the KPZ fixed point. We show here that the ODE system (2.22) for the equal-time case of the cubic integrable operator is the same as the matrix ODE system of Tracy and Widom.

The ODE system (2.22) involves matrix functions $\mathbf{p}, \mathbf{r}, \mathbf{q}$, and derivatives with respect to the variables \mathbf{x}_k . We rename t to $-\frac{t}{3}$ so that $t_1 = \dots = t_m =: -\frac{t}{3}$. The ODE system contains t and \mathbf{y}_k as parameters. Change the variables \mathbf{x}_k to ξ_k by

$$\xi_k := \frac{\mathbf{x}_k}{t^{1/3}} + \frac{\mathbf{y}_k^2}{t^{4/3}}.$$

Set $\boldsymbol{\xi} := \text{diag}(\xi_1, \dots, \xi_m) = \frac{1}{t^{1/3}} \mathbf{x} + \frac{1}{t^{4/3}} \mathbf{y}^2$, where $\mathbf{y} := \text{diag}(\mathbf{y}_1, \dots, \mathbf{y}_m)$ and $\mathbf{x} := \text{diag}(\mathbf{x}_1, \dots, \mathbf{x}_m)$, and also $\boldsymbol{\xi} := (\xi_1, \dots, \xi_m)$, $\partial_{\boldsymbol{\xi}} := \sum_{j=1}^m \partial_{\xi_j}$. Define $\mathbf{P}, \mathbf{R}, \mathbf{Q}$ from $\mathbf{p}, \mathbf{r}, \mathbf{q}$ by setting

$$\begin{aligned}
p(t, y, x) &= it^{-1/3} \exp\left(\frac{1}{t}xy + \frac{2}{3t^2}y^3\right) P(\xi), \\
r(t, y, x) &= it^{-1/3} R(\xi) \exp\left(-\frac{1}{t}xy - \frac{2}{3t^2}y^3\right), \\
q(t, y, x) &= -it^{-1/3} \exp\left(\frac{1}{t}xy + \frac{2}{3t^2}y^3\right) Q(\xi) \exp\left(-\frac{1}{t}xy - \frac{2}{3t^2}y^3\right),
\end{aligned}$$

Then, with $\tilde{y} := t^{-2/3}y$, the system (2.22) becomes

$$\partial_{\xi} Q = -[\tilde{y}, Q] - PR, \quad \partial_{\xi}^2 P = \xi P + 2PRP + 2[\tilde{y}, Q]P, \quad \partial_{\xi}^2 R = R\xi + 2RPR + 2R[\tilde{y}, Q]. \quad (3.2)$$

Equations (1)–(3) of [30]⁶ form an ODE system for three $m \times m$ matrices r, q, \tilde{q} . For the m -dimensional vector $\mathbf{e} := (1, \dots, 1)^T$ consider the $m \times m$, $m \times 1$, and $1 \times m$ matrices $r, q\mathbf{e}$, and $\mathbf{e}^T \tilde{q}$. Then, equations (1)–(3) of [30] changed for these functions are precisely (3.2) with the identification

$$Q = r, \quad P = q\mathbf{e}, \quad R = \mathbf{e}^T \tilde{q} \quad \text{and} \quad \tau_k = t^{-2/3}y_k = \tilde{y}_k.$$

3.4. Differential equations related to the Adler and van Moerbeke PDE. For the equal-time, 2-position distributions of the KPZ fixed point (i.e. the Airy₂ process), Adler and van Moerbeke also obtained a PDE [3]. We discuss how this PDE arises from our analysis.

3.4.1. A scalar PDE for cubic admissible determinants. Assume that the cubic integrable operator H is trace class and consider the log determinant

$$M := \log \mathfrak{D}(\mathbf{x}, y, t) = \log \det(1 - H).$$

The KP-II equation in Theorem 2.6, which is for matrix functions p, r, q, s , yields a scalar partial differential equation for M in the following way. Choose the set $S = \{1, \dots, m\}$ so that \mathbf{s} and $\mathbf{v} = r\mathbf{p}$ become scalar functions. Then, the commutator in (2.15) disappears, and from Proposition 2.10 and Lemma 2.4, we find that $\partial_x M = \mathbf{s}$ and $\partial_x^2 M = \mathbf{v}$. Thus, the Eq. (2.15) becomes

$$\partial_x(-4\partial_x \partial_t M + \partial_x^4 M + 6((\partial_x^2 M)^2) + 3\partial_y^2 M) = 0.$$

If M decays fast enough as $x_i \rightarrow \infty$, which is the case for the (periodic) KPZ fixed points, integrating the above equation we obtain a fourth order nonlinear differential equation for M ,

$$-4\partial_x \partial_t M + \partial_x^4 M + 6((\partial_x^2 M)^2) + 3\partial_y^2 M = 0. \quad (3.3)$$

⁶ The published version of the system from [30] has a typo which was later fixed in the ArXiv version (arXiv:ArXiv:0302033v4, see Equations (1)–(3) and the last two displayed equations in page 3 therein). Our discussion here uses the latest Arxiv version.

3.4.2. Invariance of strongly cubic admissible determinants Let Ω be a union of contours, and suppose that H is a strongly cubic integrable operator that is trace class with parameters $\mathbf{x}, \mathbf{y}, \mathbf{t}$. Let Ω' be a scaled and translated contour of Ω given by

$$\Omega' = (-3\mathbf{t}_1)^{\frac{1}{3}}\Omega - \frac{\mathbf{y}_1}{(-3\mathbf{t}_1)^{2/3}}.$$

Using the fact that the functions U, V are constants on each connected component of Ω (since H is strongly cubic integrable), it is straightforward to check that

$$\frac{1}{(-3\mathbf{t}_1)^{\frac{1}{3}}}H\left(\frac{1}{(-3\mathbf{t}_1)^{\frac{1}{3}}}u - \frac{\mathbf{y}_1}{3\mathbf{t}_1}, \frac{1}{(-3\mathbf{t}_1)^{\frac{1}{3}}}v - \frac{\mathbf{y}_1}{3\mathbf{t}_1} \mid \mathbf{x}, \mathbf{y}, \mathbf{t}\right) = H(u, v \mid x, y, t), \quad u, v \in \Omega', \quad (3.4)$$

for the parameters $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, $t = (t_1, \dots, t_m)$ given by

$$\begin{aligned} x_i &= \frac{1}{(-3\mathbf{t}_1)^{\frac{1}{3}}}x_i + \frac{2\mathbf{y}_1}{(-3\mathbf{t}_1)^{\frac{4}{3}}}y_i - \frac{\mathbf{y}_1^2}{(-3\mathbf{t}_1)^{\frac{4}{3}}\mathbf{t}_1}t_i, \\ y_i &= \frac{1}{(-3\mathbf{t}_1)^{\frac{2}{3}}}y_i - \frac{\mathbf{y}_1}{(-3\mathbf{t}_1)^{\frac{2}{3}}\mathbf{t}_1}t_i, \quad t_i = \frac{1}{-3\mathbf{t}_1}t_i \end{aligned} \quad (3.5)$$

for $i = 1, \dots, m$. Note that for $i = 1$,

$$y_1 = 0, \quad t_1 = -\frac{1}{3}.$$

The left-hand side of (3.4) is a change of variables of the kernel $H(u, v \mid \mathbf{x}, \mathbf{y}, \mathbf{t})$, $u, v \in \Omega$, and hence its Fredholm determinant is equal to $\det(1 - H(\mathbf{x}, \mathbf{y}, \mathbf{t}))_{L^2(\Omega)}$. On the other hand, the Fredholm determinant of the right-hand side of (3.4), $\det(1 - H(x, y, t))_{L^2(\Omega')}$, is equal to $\det(1 - H(x, y, t))_{L^2(\Omega)}$ by the Cauchy's theorem using the fact that restrictions of $H(u, v \mid x, y, t)$ for u and v on connected components of Ω' extends to entire functions in u and v . Thus, we find a relationship

$$M = \det(1 - H(\mathbf{x}, \mathbf{y}, \mathbf{t})) = \det(1 - H(x, y, t)).$$

Under the change of variables given by (3.5), the Eq. (3.3) evaluated at $\mathbf{t}_1 = -1/3$ and $y_1 = 0$ becomes the following equation in x, y, t :

$$\begin{aligned} &12 \sum_{i=2}^m (1 - t_i) \partial_x \partial_{t_i} M - 8 \sum_{i=2}^m y_i \partial_x \partial_{y_i} M \\ &- 4 \sum_{i=1}^m t_i^{-2} \left(x_i t_i^2 + 3y_i^2 \right) \partial_x \partial_{x_i} M - 4 \partial_x M + \partial_x^4 M + 6(\partial_x^2 M)^2 \\ &+ 3 \sum_{j=2}^m \sum_{i=2}^m (1 - t_i)(1 - t_j) \partial_{y_j} \partial_{y_i} M + 12 \sum_{j=2}^m \sum_{i=1}^m y_i t_i^{-1} (1 - t_j) \partial_{y_j} \partial_{x_i} M \\ &+ 12 \sum_{i,j=1}^m y_i y_j t_i^{-1} t_j^{-1} \partial_{x_j} \partial_{x_i} M + 6 \sum_{i=1}^m t_i^{-1} \partial_{x_i} M = 0. \end{aligned} \quad (3.6)$$

Note that if $\mathbf{t}_1 = -1/3$ and $y_1 = 0$, then $(\mathbf{x}, \mathbf{y}, \mathbf{t}) = (x, y, t)$. Thus, we find that $M(\mathbf{x}, \mathbf{y}, \mathbf{t})$ with $\mathbf{t}_1 = -1/3$ and $y_1 = 0$ satisfies the differential Eq. (3.6) with respect to the original variables $(\mathbf{x}, \mathbf{y}, \mathbf{t})$.

3.4.3. Scalar PDEs for the two-time and one-time cases - Adler and van Moerbeke PDE
 Let H be a strongly cubic integrable operator that is trace class, and consider the case when $m = 2$ and $t_1 = -1/3$ and $y_1 = 0$. Then, $M = \det(1 - H)$ satisfies the Eq. (3.6) with $m = 2$ with respect to the variables (x, y, t) . Motivated by the work of [3] on the Airy_2 process, introduce the change of variables

$$t_2 = -\frac{t}{3}, \quad x_1 = \frac{E + W}{2}, \quad x_2 = \frac{E - W}{2} - \frac{y^2}{t}, \quad y_2 = y. \quad (3.7)$$

Then (3.6) with $m = 2$ becomes

$$\begin{aligned} & 24(1-t)t^2\partial_E\partial_t M + 16t^2\partial_E^4 M + 96t^2(\partial_E^2 M)^2 + 3(1-t)^2t^2\partial_y^2 M + 4yt(3-7t)\partial_E\partial_y M \\ & + 12yt(t-1)\partial_y\partial_W M - 4(2Et^2 + 3y^2)\partial_E^2 M \\ & - 8Wt^2\partial_E\partial_W M + 12y^2\partial_W^2 M + 2t(3-t)\partial_E M \\ & + 6t(t-1)\partial_W M = 0. \end{aligned} \quad (3.8)$$

In the further special case when $t = 1$ so that $t_1 = t_2$, the above equation becomes simpler. If we consider $\frac{1}{4}\partial_E$ (3.8), then we arrive at

$$\left(-4\partial_E^5 + 2W\partial_E^2\partial_W + (3y^2 + 2E)\partial_E^3 - 3y^2\partial_E\partial_W^2 - \partial_E^2 + 4y\partial_E^2\partial_y \right) M - 48\partial_E^2 M \partial_E^3 M = 0.$$

This PDE is the same as the one derived by Quastel and Remenik [26, Theorem 2.5] for the 2-point distribution of the KPZ fixed point with equal time⁷, except for a flipped sign in the term ∂_E^2 .

In Sect. 8, we derive the Eq. (3.8) in a different way using arguments on the underlying Lax pairs, supported by symbolic calculations in linear algebra. As we also indicate therein, such arguments allow us to obtain three additional PDEs. One of them is

$$\begin{aligned} & 3(t-1)\partial_E^4 M - 8(t+1)\partial_E^3\partial_W M + 6(t-1)\partial_E^2\partial_W^2 M \\ & + (1-t)\partial_W^4 M + 12W\partial_E^2 M + 8E\partial_E\partial_W M \\ & - 4W\partial_W^2 M + 3t(1-t)\partial_y^2 M - 8y\partial_W\partial_y M - 4\partial_W M \\ & + 18(t-1)(\partial_E^2 M)^2 + 24(t-1)(\partial_E\partial_W M)^2 \\ & + 6(1-t)(\partial_W^2 M)^2 + 12(t-1)\partial_E^2 M \partial_W^2 M - 48(t+1)\partial_E^2 M \partial_E\partial_W M = 0. \end{aligned} \quad (3.9)$$

If we set $t = 1$ in this equation, the equation again simplifies, and $\frac{1}{12}\partial_W$ (3.8) + $\frac{1}{12}\partial_E$ (3.9) becomes

$$\begin{aligned} & \left(y^2(\partial_W^3 - \partial_E^2\partial_W) - W(\partial_E\partial_W^2 - \partial_E^3) - 2y\partial_E\partial_W\partial_y \right) M \\ & - 8(\partial_E\partial_W M)(\partial_E^3 M) + 8(\partial_E^2 M)(\partial_E^2\partial_W M) = 0. \end{aligned} \quad (3.10)$$

This is the same PDE obtained by Adler and van Moerbeke [3, Corollary 1.3] for the 2-point distribution of the Airy_2 process. Thus, Eq. (3.9) is a non-equal time extension of the Adler-van Moerbeke PDE. We note that the Eq. (3.10) was also re-derived in [9] using a Riemann-Hilbert approach, and the Adler-van Moerbeke PDE was extended to general $m > 2$ (single time, multi-point) in [32].

⁷ The correspondence of terms between [26, Theorem 2.5] and here is $\psi = \partial_x M$, $\Psi = M$, $r = x$, and the variables y coincide.

As we just discussed, the multi-time multi-location differential equations we obtained reduce in the one-time multi-location case to several different equations obtained previously. We stress that although the differential equations admit such reductions, the solutions themselves are not directly comparable: all the previous works we mentioned obtain equations for the (log derivatives of the) distributions themselves, whereas in our case the solutions to the equations enter into the distribution through the integration in the ζ -variables in (2.23) (recall for instance the discussion on (3.1)). It is interesting that the underlying equations are nevertheless the same.

4. Derivation of Differential Equations

In this section we prove the results stated in Sects. 2.1 and 2.2. In order to derive differential equations, we use a known connection between IKS integrable operators and Riemann-Hilbert problems (RHPs), and then exploit the structure of the Riemann-Hilbert problem for the case of cubic integrable operators. We follow the general methodology, often called the dressing method, which derives so-called Lax equations from a Riemann-Hilbert problem when the jump/residue matrix does not depend on parameters. The Lax equations are then combined appropriately to derive our claimed differential equations.

4.1. A review of integrable operators and RHP. An operator H of the form (2.6) for general vectors \mathbf{f} and \mathbf{g} is called an (IKS) integrable operator, after its introduction by Its et al. [15]. Their general theory is discussed in [12, 15] for integrable operators acting on continuous contours and [10] for those on discrete sets. In this subsection we review some of their properties that we will use.

Let Ω be a union of contours or a discrete set as described in Sect. 2.1, and let μ and $L^2(\Omega, \mu)$ be as in (2.1) and (2.2).

Let \mathbf{f} and \mathbf{g} be $(m+1)$ -dimensional column vector-valued functions (not necessarily of the form in Definition 2.1) in $L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ satisfying

$$\mathbf{f}(u)^T \mathbf{g}(u) = 0, \quad u \in \Omega, \quad \text{and set} \quad H(u, v) := \frac{\mathbf{f}(u)^T \mathbf{g}(v)}{u - v}, \quad u, v \in \Omega.$$

Observe that for \mathbf{f} and \mathbf{g} of the form in Definition 2.1, the assumption (2.8) implies this orthogonality. Assume that $1 - H$ is invertible. With I_k being the identity matrix of size k , define $(m+1)$ -dimensional column vectors

$$\mathbf{F} := (1 - H)^{-1} \mathbf{f}, \quad \mathbf{G} := (1 - H^T)^{-1} \mathbf{g}, \quad (4.1)$$

and the $(m+1) \times (m+1)$ matrix-valued function

$$\Phi(z) = I_{m+1} - \int_{\Omega} \frac{\mathbf{F}(u) \mathbf{g}(u)^T}{u - z} d\mu(u), \quad z \in \mathbb{C} \setminus \Omega. \quad (4.2)$$

An important property of integrable operators is that Φ is the unique solution to the following Riemann-Hilbert problem (RHP).

RHP (a) $\Phi : \mathbb{C} \setminus \Omega \rightarrow \mathbb{C}^{(m+1) \times (m+1)}$ is analytic.

RHP (b) In the case when Ω is a union of contours, Φ satisfies the jump condition

$$\Phi_+(u) = \Phi_-(u)J(u), \quad u \in \Omega, \quad \text{where } J(u) := I_{m+1} - 2\pi i f(u)g(u)^T. \quad (4.3)$$

In the case when Ω is a discrete set, the matrix Φ has simple poles at the points $u \in \Omega$, and its residues satisfy

$$\text{Res}_{z=u} \Phi(z) = \lim_{z \rightarrow u} \Phi(z)R(u) \quad \text{with} \quad R(u) := f(u)g(u)^T. \quad (4.4)$$

The convergence of the limit is a part of the condition.

RHP (c) Φ has an asymptotic series

$$\Phi(z) \sim I_{m+1} + \sum_{n=1}^{\infty} \frac{\Phi_n}{z^n} \quad (4.5)$$

as $z \rightarrow \infty$ uniformly away from Ω .

It can be shown that $\det \Phi(z) = 1$ for all z and

$$\Phi(z)^{-1} = I_{m+1} + \int_{\Omega} \frac{f(u)G(u)^T}{u-z} d\mu(u), \quad z \in \mathbb{C} \setminus \Omega. \quad (4.6)$$

Note that (4.2) and (4.6) imply that $\Phi(z) \sim I_{m+1} + \sum_{n=1}^{\infty} \frac{\Phi_n}{z^n}$ and $\Phi(z)^{-1} \sim I_{m+1} + \sum_{n=1}^{\infty} \frac{(\Phi^{-1})_n}{z^n}$ as $z \rightarrow \infty$ where

$$\Phi_n = \int_{\Omega} u^{n-1} F(u)g(u)^T d\mu(u), \quad (\Phi^{-1})_n = - \int_{\Omega} u^{n-1} f(u)G(u)^T d\mu(u), \quad n=1, 2, \dots \quad (4.7)$$

The notation Φ_1 is consistent with Definition 2.2.

4.2. Lax equations. The discussion in the last subsection is valid for general integrable operators. For the rest of Sect. 4, we further assume that H is a cubic integrable operator as introduced in Definition 2.1.

From (2.7), $f = c\Delta U$ and $g = c^{-1}\Delta^{-1}V$ for vector functions U and V that do not depend on x, y, t . Hence, the jump matrix and the residue matrix for the RHP in (4.3) and (4.4) take the form

$$J(u) = \Delta(u)J_0(u)\Delta(u)^{-1} \quad \text{and} \quad R(u) = \Delta(u)R_0(u)\Delta(u)^{-1}, \quad (4.8)$$

where the matrices $J_0(u) = I_{m+1} - 2\pi i U(u)V(u)^T$ and $R_0(u) = U(u)V(u)^T$ do not depend on x, y, t . We follow the standard dressing method of deriving Lax equations for RHPs when the jump or residue matrix is a conjugation of a parameter-independent matrix. We think of J and R as dressed-up versions of the matrices J_0 and R_0 which are constant in the parameters.

Recall that E_i denotes the $(m+1) \times (m+1)$ matrix whose (i, i) -entry is 1 and all other entries are 0. The linear ODEs (4.10) below are called Lax equations for Φ .

Lemma 4.1 (Lax equations). *Let $\Phi(z)$ be the solution, which is given by (4.2), of the Riemann-Hilbert problem associated to the cubic integrable operator \mathcal{H} . Then, the matrix*

$$\Psi(z) := \Phi(z)\Delta(z) \quad (4.9)$$

satisfies

$$\begin{aligned} \partial_{x_i} \Psi(z) &= D_1^{(i)}(z) \Psi(z), & \partial_{y_i} \Psi(z) &= D_2^{(i)}(z) \Psi(z), \\ \partial_{t_i} \Psi(z) &= D_3^{(i)}(z) \Psi(z), \end{aligned} \quad (4.10)$$

for $i \in \{1, \dots, m\}$, where

$$\begin{aligned} D_1^{(i)}(z) &= zE_i + C_1^{(i)}, & D_2^{(i)}(z) &= z^2E_i + zC_1^{(i)} + C_2^{(i)}, \\ D_3^{(i)}(z) &= z^3E_i + z^2C_1^{(i)} + zC_2^{(i)} + C_3^{(i)}, \end{aligned}$$

with

$$\begin{aligned} C_1^{(i)} &:= [\Phi_1, E_i], & C_2^{(i)} &:= [\Phi_2, E_i] - C_1^{(i)}\Phi_1, \\ C_3^{(i)} &:= [\Phi_3, E_i] - C_1^{(i)}\Phi_2 - C_2^{(i)}\Phi_1. \end{aligned} \quad (4.11)$$

Proof. From (4.8), the jump/residue condition for the new matrix becomes

$$\Psi_+(u) = \Psi_-(u)J_0(u) \quad \text{or} \quad \text{Res}_{z=u} \Psi(z) = \lim_{z \rightarrow u} \Psi(z)R_0(u).$$

Since $J_0(u)$ and $R_0(u)$ do not depend on x, y, t , we find that a partial derivative $\partial\Psi$ with respect any one of the parameters satisfies the same jump/residue condition as Ψ itself. Hence, $(\partial\Psi(z))\Psi(z)^{-1}$ is continuous across Ω , and thus it is an entire matrix function. From the definition (4.9),

$$(\partial\Psi)\Psi^{-1} = (\partial\Phi)\Phi^{-1} + \Phi(\partial\Delta)\Delta^{-1}\Phi^{-1}. \quad (4.12)$$

Also, from the formula of Δ in (2.4),

$$\partial_{x_i} \Delta(z) = zE_i \Delta(z), \quad \partial_{y_i} \Delta(z) = z^2E_i \Delta(z), \quad \partial_{t_i} \Delta(z) = z^3E_i \Delta(z).$$

Hence, using the fact that $\Phi \sim I_{m+1}$ as $z \rightarrow \infty$, Liouville's Theorem implies that the matrix functions $(\partial_{x_i} \Psi(z))\Psi(z)^{-1}$, $(\partial_{y_i} \Psi(z))\Psi(z)^{-1}$, and $(\partial_{t_i} \Psi(z))\Psi(z)^{-1}$ are polynomials in z of degrees 1, 2, and 3, respectively. Inserting (4.5) into (4.12), we obtain the coefficients of these polynomials in terms of Φ_1, Φ_2 and Φ_3 , and find (4.10). \square

4.3. NLS and mKdV: proof of Lemma 2.4 and Theorem 2.5. Recall that in Definition 2.2 $S \subset \{1, \dots, m\}$ is a given set, and the associated differential operators ∂_t, ∂_y , and ∂_x are given in (2.11) such as $\partial_x = \sum_{i \in S} \partial_{x_i}$. Set

$$E = E^S := \sum_{i \in S} E_i. \quad (4.13)$$

We first derive the following differential equations for Φ_1, Φ_2, Φ_3 (recall (4.7)).

Proposition 4.2. *The matrix Φ_1 satisfies the differential equations*

$$\begin{aligned}\partial_x^2 \Phi_1 + [\partial_y \Phi_1, E] + [\partial_x \Phi_1, [\Phi_1, E]] &= 0, \\ \partial_x \partial_y \Phi_1 + [\partial_t \Phi_1, E] + [\partial_y \Phi_1, [\Phi_1, E]] &= 0, \\ \partial_y^2 \Phi_1 - \partial_t \partial_x \Phi_1 - [\partial_y \Phi_1, \partial_x \Phi_1] &= 0.\end{aligned}\tag{4.14}$$

Furthermore, the three matrices Φ_1, Φ_2, Φ_3 are related by

$$\begin{aligned}\partial_x \Phi_1 &= -[\Phi_2, E] + [\Phi_1, E]\Phi_1, \\ \partial_y \Phi_1 &= -[\Phi_3, E] + [\Phi_1, E]\Phi_2 - (\partial_x \Phi_1)\Phi_1.\end{aligned}\tag{4.15}$$

Proof. The matrix $\Psi(z) := \Phi(z)\Delta(z)$ of (4.9) satisfies the Lax equations (4.10). Summing over $i \in S$,

$$\begin{aligned}\partial_x \Psi(z) &= (zE + C_1)\Psi(z), \\ \partial_y \Psi(z) &= (z^2E + zC_1 + C_2)\Psi(z), \\ \partial_t \Psi(z) &= (z^3E + z^2C_1 + zC_2 + C_3)\Psi(z),\end{aligned}\tag{4.16}$$

with

$$C_1 := [\Phi_1, E], \quad C_2 := [\Phi_2, E] - C_1\Phi_1, \quad C_3 := [\Phi_3, E] - C_1\Phi_2 - C_2\Phi_1.\tag{4.17}$$

Inserting $\Psi(z) = \Phi(z)\Delta(z)$, these equations become equations for $\Phi(z)$. We then insert the asymptotic series (4.5) and consider the coefficients of $\mathcal{O}(z^{-k})$. Collecting the coefficients, we obtain a sequence of equations relating the matrices Φ_n . In particular, the $\mathcal{O}(z^{-1})$ terms of the first two equations of (4.16) become

$$\partial_x \Phi_1 = -[\Phi_2, E] + C_1\Phi_1, \quad \partial_y \Phi_1 = -[\Phi_3, E] + C_1\Phi_2 + C_2\Phi_1.\tag{4.18}$$

Using these relations, we find that the three matrices in (4.17) can all be expressed in terms of Φ_1 as

$$C_1 = [\Phi_1, E], \quad C_2 = -\partial_x \Phi_1, \quad C_3 = -\partial_y \Phi_1.\tag{4.19}$$

Inserting the formulas of C_1 and C_2 into (4.18), we obtain (4.15).

To obtain the differential Eq. (4.14), we look at the so-called zero curvature equations for each pair of equations from (4.16). These conditions are obtained inserting (4.16) in the identities $\partial_y \partial_t \Psi = \partial_t \partial_y \Psi$, $\partial_x \partial_t \Psi = \partial_t \partial_x \Psi$, and $\partial_x \partial_y \Psi = \partial_y \partial_x \Psi$. These three equations become

$$\begin{aligned}\partial_y(z^2C_1 + zC_2 + C_3) - \partial_t(zC_1 + C_2) + z^2[C_3, E] + z[C_3, C_1] + [C_3, C_2] &= 0, \\ \partial_x(z^2C_1 + zC_2 + C_3) - \partial_t C_1 + z^2[C_2, E] + z[C_3, E] + z[C_2, C_1] + [C_3, C_1] &= 0, \\ \partial_x(zC_1 + C_2) - \partial_y C_1 + z[C_2, E] + [C_2, C_1] &= 0.\end{aligned}$$

These equations hold for all z . Considering the coefficients of z^i , we obtain 8 identities. The 2 identities from the last equation follow from the remaining equations. We also observe from (4.19) that $\partial_y C_2 = \partial_x C_3$. Using this, we see that the identity from the coefficients of z of the first equation and the identity from the constant terms of the second equation are equivalent. Thus, we are left with 5 identities. Two of them are

$\partial_x C_1 + [C_2, E] = 0$ and $\partial_y C_1 + [C_3, E] = 0$, but these follow from (4.19). Summarizing, the zero curvature conditions yield 3 new identities, which are

$$\begin{aligned}\partial_x C_2 - \partial_y C_1 + [C_2, C_1] &= 0, & \partial_x C_3 - \partial_t C_1 + [C_3, C_1] &= 0, \\ \partial_y C_3 - \partial_t C_2 + [C_3, C_2] &= 0.\end{aligned}\quad (4.20)$$

Inserting (4.19), these equations become (4.14). \square

We now show how the last result implies Lemma 2.4 and Theorem 2.5. Let Π^S be the $(m+1) \times (m+1)$ permutation matrix as introduced before Definition 2.2 and I_j the identity matrix of size j . Note that

$$\Pi^S E (\Pi^S)^T = \text{diag}(I_{|S|}, 0, \dots, 0)$$

with E as in (4.13).

Corollary 4.3. *Decompose $\Pi^S \Phi_1 (\Pi^S)^T$ as the sum of diagonal and off-diagonal block matrices,*

$$\Pi^S \Phi_1 (\Pi^S)^T = L + O, \quad L := \begin{pmatrix} q & 0 \\ 0 & s \end{pmatrix}, \quad O := \begin{pmatrix} 0 & p \\ r & 0 \end{pmatrix},$$

where we used the notations of (2.10). Define the $(m+1) \times (m+1)$ diagonal matrix

$$V = \text{diag}(I_{|S|}, -I_{m+1-|S|}).$$

Then,

$$\begin{aligned}\partial_x L &= -V O^2, & \partial_y L &= O(\partial_x O) - (\partial_x O)O, \\ \partial_y O &= V(\partial_x^2 O + 2O^3), & \partial_t O &= \partial_x^3 O + 3(\partial_x O)O^2 + 3O^2(\partial_x O).\end{aligned}\quad (4.21)$$

In terms of the matrices q, p, r, s from (2.10), the first two equations become

$$\partial_x q = -pr, \quad \partial_x s = rp, \quad \partial_y q = p(\partial_x r) - (\partial_x p)r, \quad \partial_y s = r(\partial_x p) - (\partial_x r)p,$$

the third equation becomes the coupled matrix NLS equation with complex time (2.12), and the fourth equation becomes the coupled matrix mKdV equations (2.13). Finally, the additional relation

$$\text{Tr}(s) = -\text{Tr}(q)$$

also holds.

Proof. For each $n = 1, 2, 3$, write in diagonal/off-diagonal block form,

$$\Pi^S \Phi_n (\Pi^S)^T = L_n + O_n, \quad L_n = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad O_n = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

In particular, $L_1 = L$ and $O_1 = O$. In the calculations that come, we make use of the identities

$$[\Pi^S \Phi_n (\Pi^S)^T, \Pi^S E (\Pi^S)^T] = O_n V = -V O_n, \quad L_n V = V L_n, \quad V^2 = I_{m+1},$$

which are straightforward to check.

Conjugate the Eqs. (4.14) and (4.15) by Π^S and consider the diagonal and off-diagonal blocks. The diagonal blocks of the first equation of (4.15) give the identity

$$\partial_x L_1 = -V O_1^2. \quad (4.22)$$

This is the first equation in (4.21).

The off-diagonal blocks of the first equation of (4.15) and the diagonal blocks of the second equation yield

$$\partial_x O_1 = V O_2 - V O_1 L_1, \quad \partial_y L_1 = -V O_1 O_2 - (\partial_x L_1) L_1 - (\partial_x O_1) O_1.$$

Using $V^2 = I_{m+1}$, the first equation implies $O_2 = V \partial_x O_1 + O_1 L_1$. Inserting this into the second equation and also using (4.22), we find $\partial_y L_1 = O_1 (\partial_x O_1) - (\partial_x O_1) O_1$. This is the second equation in (4.21).

The off-diagonal blocks of the first equation in (4.14) give us

$$-V \partial_y O_1 + \partial_x^2 O_1 - [\partial_x L_1, V O_1] = 0.$$

Inserting (4.22) and multiplying V on the left, we find $\partial_y O_1 = V \partial_x^2 O_1 + 2V O_1^3$, which is the third equation in (4.21).

The off-diagonal blocks of the second equation of (4.14) imply

$$-V \partial_t O_1 + \partial_x \partial_y O_1 - [\partial_y L_1, V O_1] = 0.$$

We remove $\partial_y L_1$ and $\partial_y O_1$ using the second and the third equations in (4.21), and find $\partial_t O_1 = \partial_x^3 O_1 + 3(\partial_x O_1) O_1^2 + 3O_1^2 (\partial_x O_1)$. This is the fourth equation in (4.21).

The equations for q, p, r, s follow by inserting the formulas of L and O .

Finally, since $1 = \det \Phi(z) = \det(I_{m+1} + \sum_{n=1}^{\infty} \frac{\Phi_n}{z^n})$ as $z \rightarrow \infty$, we have $\text{Tr } \Phi_1 = 0$, implying the identity $\text{Tr}(q) + \text{Tr}(s) = 0$. \square

Thus, we proved Lemma 2.4 and Theorem 2.5.

4.4. The matrix KP equation: proof of Theorem 2.6. We derive the matrix KP equation from the Eq. (4.21).

Proof of Theorem 2.6. We keep using the matrices O, L, V from Corollary 4.3, which satisfy $OV = -VO, LV = VL$, and $V^2 = I_{m+1}$. Let $u = pr$ and $v = rp$. Denote

$$U := O^2 = \begin{pmatrix} pr & 0 \\ 0 & rp \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

We first derive four equations for U, O , and L . The first claimed equation follows from the definition of U ,

$$\partial_x^3 U = (\partial_x^3 O) O + O (\partial_x^3 O) + 3(\partial_x^2 O)(\partial_x O) + 3(\partial_x O)(\partial_x^2 O). \quad (4.23)$$

To obtain the second equation, we use the second equation of (4.21) to get

$$\partial_y^2 L = (\partial_y O)(\partial_x O) + O(\partial_y \partial_x O) - (\partial_y \partial_x O) O - (\partial_x O)(\partial_y O).$$

Using the third equation of (4.21), this implies our second equation,

$$-V\partial_Y^2 L = (\partial_X^3 O)O + O(\partial_X^3 O) - (\partial_X^2 O)(\partial_X O) - (\partial_X O)(\partial_X^2 O) + 4O(\partial_X U)O. \quad (4.24)$$

Next, we multiply the fourth equation of (4.21) by O to the right, and separately to the left, and add both equations. The result is our third equation,

$$\partial_t U = (\partial_X^3 O)O + O(\partial_X^3 O) + 3\partial_X(U^2). \quad (4.25)$$

Finally, for the fourth and last one, we compute $[U, \partial_Y L]$ with the second equation of (4.21), obtaining

$$-[U, \partial_Y L] + \partial_X(U^2) = 2O(\partial_X U)O. \quad (4.26)$$

We combine the above four equations to eliminate O and obtain an equation of U and L as follows: Multiplying the Eq. (4.24) by 3 and adding it to the Eq. (4.23), and then subtracting the equation (4.25) multiplied by 4 and also the Eq. (4.26) multiplied by 6, we obtain

$$\partial_X^3 U - 3V\partial_Y^2 L - 4\partial_t U + 6[U, \partial_Y L] + 6\partial_X(U^2) = 0.$$

If we insert $U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $L = \begin{pmatrix} q & 0 \\ 0 & s \end{pmatrix}$, we obtain the first equations in (2.14) and (2.15). The second equations of them follow from $\partial_X L = -VU$, which is the first equation of (4.21). Thus the result is proved. \square

4.5. Multi-component KP hierarchy. In this subsection, we show that the three Lax equations (4.10) with respect to the parameters are related to multi-component KP hierarchy theory. This will give us another proof of Theorem 2.6. The article [20] gives a thorough exposition on the multi-component KP hierarchy theory, see also [28]. For the scalar KP hierarchy theory, a helpful overview is found in [14]. Multi-component KP hierarchy theory are already known to be connected with models related to the Airy₂ process and random matrix theory, see [2, 4] and references therein. We can also notice that cubic admissible determinant is multi-component KP tau function, but we don't use this fact in this paper.

The cubic integrable operator H contains $3m$ parameters $x_1, \dots, x_m, y_1, \dots, y_m, t_1, \dots, t_m$. We relabel them as

$$x_1^{(i)} = x_i, \quad x_2^{(i)} = y_i, \quad x_3^{(i)} = t_i, \quad i \in \{1, \dots, m\}.$$

Recall the matrices Φ_k for $k \geq 1$ from (4.7). Fix $n \in \{1, \dots, m\}$ and write in block form⁸

$$\Phi_k = \begin{pmatrix} q_k & p_k \\ r_k & s_k \end{pmatrix}, \quad (4.27)$$

where q_k is a matrix of size $n \times n$. For $k \geq 1$, let $E_j^{(k)}$ be the $k \times k$ matrix with 1 in the j -th diagonal entry and zero in the other entries.

⁸ We can also consider $\Pi^S \Phi_k (\Pi^S)^T$ similar to (2.10). Then the discussions below still hold if we define the parameters $x_\ell^{(i)}$, $i = 1, \dots, |S|$, $\ell = 1, 2, 3$ appropriately. For the convenience of presentation, we state results only when $\Pi^S = \{1, \dots, n\}$ here.

Now define the formal $n \times n$ matrix-valued pseudo-differential operators on the variables $x_1^{(1)}, \dots, x_1^{(n)}$,

$$\partial := \sum_{j=1}^n \partial_{x_1^{(j)}} \quad \text{and} \quad P := I_n + \sum_{k=1}^{\infty} q_k \partial^{-k}, \quad (4.28)$$

and for $j \in \{1, \dots, n\}$ and $\ell \in \{1, 2, 3\}$ also introduce

$$B_\ell^{(j)} := (PE_j^{(n)} \partial^\ell P^{-1})_+. \quad (4.29)$$

We stress that this convention for ∂ is only valid for this subsection. The subindex $+$ in (4.29) means that we take the part of the term between brackets that contains only non-negative powers of ∂ , and the inverse P^{-1} is in the sense of pseudo-differential operators, so that

$$P^{-1} = I_n - q_1 \partial^{-1} + (q_1^2 - q_2) \partial^{-2} + (-q_1^3 - q_1(\partial q_1) + q_1 q_2 + q_2 q_1 - q_3) \partial^{-3} + \dots$$

From the definition, we have⁹

$$\begin{aligned} B_1^{(j)} &= E_j^{(n)} \partial + [q_1, E_j^{(n)}], \\ B_2^{(j)} &= E_j^{(n)} \partial^2 + [q_1, E_j^{(n)}] \partial + [q_2, E_j^{(n)}] - [q_1, E_j^{(n)}] q_1 - 2E_j^{(n)} (\partial q_1) \\ B_3^{(j)} &= E_j^{(n)} \partial^3 + [q_1, E_j^{(n)}] \partial^2 + ([q_2, E_j^{(n)}] - [q_1, E_j^{(n)}] q_1 - 3E_j^{(n)} (\partial q_1)) \partial \\ &\quad + [q_3, E_j^{(n)}] + [q_1, E_j^{(n)}] q_1^2 - [q_1, E_j^{(n)}] q_2 - [q_2, E_j^{(n)}] q_1 \\ &\quad - 3E_j^{(n)} (\partial q_2) - 3E_j^{(n)} (\partial^2 q_1) - 2[q_1, E_j^{(n)}] (\partial q_1) + 3E_j^{(n)} (\partial q_1) q_1. \end{aligned} \quad (4.30)$$

These are particular combinations of q_1, q_2, q_3 and their derivatives.

There are several equivalent formulations for multi-component KP hierarchy. Remark 4.4 of [20] gives a good summary of their relations. Here, we prove the following version.

Proposition 4.4 (Sato equations for multi-component KP hierarchy). *Fix $1 \leq n \leq m$. The $n \times n$ matrix-valued pseudo-differential operators P and $B_\ell^{(j)}$ from (4.28) and (4.30) satisfy the following equations, known as the Sato equations¹⁰ for the n -component KP hierarchy:*

$$\frac{\partial P}{\partial x_\ell^{(i)}} = B_\ell^{(i)} P - P E_i^{(n)} \partial^\ell \quad \text{for } i \in \{1, \dots, n\} \text{ and } \ell \in \{1, 2, 3\}. \quad (4.31)$$

⁹ To compute it, note, for example, that $\partial^3 u \partial^{-2} = u \partial + 3(\partial u) + 3(\partial^2 u) \partial^{-1} + (\partial^3 u) \partial^{-2}$ and $\partial^{-1} u \partial^2 = u \partial - (\partial u) + (\partial^2 u) \partial^{-1} - \dots$ from the product rule of derivatives, and thus, $(\partial^3 u \partial^{-1})_+ = u \partial^2 + 3(\partial u) \partial$ and $(\partial^{-1} u \partial^2)_+ = u \partial - (\partial u)$ for any function u .

¹⁰ This equation is the special case when $\alpha = 0$ in equation (113) of [20]. The full Sato equations involve pseudo-differential operators parametrized by $\alpha \in \mathbb{Z}^{n-1}$. The multi-component KP hierarchy may also involve infinitely many parameters $x_\ell^{(i)}, i \in \{1, \dots, n\}, \ell \in \{1, 2, \dots\}$. Here we have the special case when $\ell \in \{1, 2, 3\}$.

Proof. Recall the Lax equations (4.10). Inserting $\Psi = \Phi\Delta$ into (4.10) and collecting coefficients of $\mathcal{O}(z^{-k})$ we obtain the identities

$$\begin{aligned}\partial_{x_1^{(i)}}\Phi_k &= -[\Phi_{k+1}, E_i^{(n+1)}] + C_1^{(i)}\Phi_k, \\ \partial_{x_2^{(i)}}\Phi_k &= -[\Phi_{k+2}, E_i^{(n+1)}] + C_1^{(i)}\Phi_{k+1} + C_2^{(i)}\Phi_k, \\ \partial_{x_3^{(i)}}\Phi_k &= -[\Phi_{k+3}, E_i^{(n+1)}] + C_1^{(i)}\Phi_{k+2} + C_2^{(i)}\Phi_{k+1} + C_3^{(i)}\Phi_k,\end{aligned}\quad (4.32)$$

for $k \in \{1, 2, \dots\}$ and $i \in \{1, \dots, n\}$, where $C_1^{(i)}$ are given by (4.11). Using the formula of $C_1^{(i)}$ and inserting the block form (4.27), the first equation of (4.32) yields

$$\partial_{x_1^{(i)}}\mathbf{q}_k = -[\mathbf{q}_{k+1}, E_i^{(n)}] + [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_k - E_i^{(n)}\mathbf{p}_1r_k \quad (4.33)$$

and

$$\begin{aligned}\partial_{x_1^{(i)}}\mathbf{p}_k &= E_i^{(n)}\mathbf{p}_{k+1} + [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_k - E_i^{(n)}\mathbf{p}_1s_k, \\ \partial_{x_1^{(i)}}r_k &= -r_{k+1}E_i^{(n)} + r_1E_i^{(n)}\mathbf{q}_k, \\ \partial_{x_1^{(i)}}s_k &= r_1E_i^{(n)}\mathbf{p}_k,\end{aligned}$$

Summing over $i = 1, \dots, n$ and noting $E_1^{(n)} + \dots + E_n^{(n)} = I_n$, we find

$$\partial\mathbf{q}_k = -\mathbf{p}_1r_k \quad (4.34)$$

and

$$\partial\mathbf{p}_k = \mathbf{p}_{k+1} - \mathbf{p}_1s_k, \quad \partial r_k = -r_{k+1} + r_1\mathbf{q}_k, \quad \partial s_k = r_1\mathbf{p}_k.$$

Using these four equations we can also compute higher derivatives of \mathbf{q}_k :

$$\begin{aligned}\partial^2\mathbf{q}_k &= -\mathbf{p}_2r_k + \mathbf{p}_1s_1r_k + \mathbf{p}_1r_{k+1} - \mathbf{p}_1r_1\mathbf{q}_k, \\ \partial^3\mathbf{q}_k &= 2\mathbf{p}_1s_1r_1\mathbf{q}_k + \mathbf{p}_1r_1\mathbf{q}_{k+1} + \mathbf{p}_1r_2\mathbf{q}_k - 2\mathbf{p}_2r_1\mathbf{q}_k - \mathbf{p}_1r_1\mathbf{q}_1\mathbf{q}_k - 2\mathbf{p}_1s_1r_{k+1} \\ &\quad + \mathbf{p}_1s_2r_k + \mathbf{p}_2s_1r_k - \mathbf{p}_1s_1^2r_k - \mathbf{p}_1r_{k+2} + 2\mathbf{p}_2r_{k+1} - \mathbf{p}_3r_k + 2\mathbf{p}_1r_1\mathbf{p}_1r_k.\end{aligned}\quad (4.35)$$

In addition, the top left blocks of the second and third equation of (4.32) are

$$\begin{aligned}\partial_{x_2^{(i)}}\mathbf{q}_k &= -[\mathbf{q}_{k+2}, E_i^{(n)}] + [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_{k+1} - E_i^{(n)}\mathbf{p}_1r_{k+1} + [\mathbf{q}_2, E_i^{(n)}]\mathbf{q}_k \\ &\quad - E_i^{(n)}\mathbf{p}_2r_k - [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_1\mathbf{q}_k + E_i^{(n)}\mathbf{p}_1r_1\mathbf{q}_k - [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_1r_k + E_i^{(n)}\mathbf{p}_1s_1r_k\end{aligned}\quad (4.36)$$

and

$$\begin{aligned}\partial_{x_3^{(i)}}\mathbf{q}_k &= -[\mathbf{q}_{k+3}, E_i^{(n)}] + [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_{k+2} - E_i^{(n)}\mathbf{p}_1r_{k+2} + [\mathbf{q}_2, E_i^{(n)}]\mathbf{q}_{k+1} \\ &\quad - E_i^{(n)}\mathbf{p}_2r_{k+1} - [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_1\mathbf{q}_{k+1} + E_i^{(n)}\mathbf{p}_1r_1\mathbf{q}_{k+1} - [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_1r_{k+1} \\ &\quad + E_i^{(n)}\mathbf{p}_1s_1r_{k+1} + [\mathbf{q}_3, E_i^{(n)}]\mathbf{q}_k - E_i^{(n)}\mathbf{p}_3r_k - [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_2\mathbf{q}_k + E_i^{(n)}\mathbf{p}_1r_2\mathbf{q}_k \\ &\quad - [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_2r_k + E_i^{(n)}\mathbf{p}_1s_2r_k - E_i^{(n)}\mathbf{p}_1s_1r_1\mathbf{q}_k + [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_1s_1r_k + E_i^{(n)}\mathbf{p}_2r_1\mathbf{q}_k \\ &\quad - [\mathbf{q}_2, E_i^{(n)}]\mathbf{p}_1r_k - E_i^{(n)}\mathbf{p}_1r_1\mathbf{q}_1\mathbf{q}_k + [\mathbf{q}_1, E_i^{(n)}]\mathbf{p}_1r_1\mathbf{q}_k + [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_1\mathbf{p}_1r_k + E_i^{(n)}\mathbf{p}_2s_1r_k \\ &\quad - E_i^{(n)}\mathbf{p}_1s_1^2r_k - E_i^{(n)}\mathbf{p}_1r_1\mathbf{p}_1r_k + [E_i^{(n)}, \mathbf{q}_2]\mathbf{q}_1\mathbf{q}_k + [\mathbf{q}_1, E_i^{(n)}]\mathbf{q}_1^2\mathbf{q}_k.\end{aligned}\quad (4.37)$$

Using (4.34), and (4.35), the right-hand side of (4.33), (4.36), and (4.37) can be expressed using only q_k 's. We obtain

$$\partial_{x_1^{(i)}} q_k = -[q_{k+1}, E_i^{(n)}] + [q_1, E_i^{(n)}] q_k + E_i^{(n)} \partial q_k, \quad (4.38)$$

$$\begin{aligned} \partial_{x_2^{(i)}} q_k &= -[q_{k+2}, E_i^{(n)}] + [q_1, E_i^{(n)}] q_{k+1} - [q_1, E_i^{(n)}] q_1 q_k + [q_2, E_i^{(n)}] q_k \\ &\quad + [q_1, E_i^{(n)}] \partial q_k + E_i^{(n)} \partial^2 q_k + 2E_i^{(n)} \partial q_{k+1} - 2E_i^{(n)} (\partial q_1) q_k, \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} \partial_{x_3^{(i)}} q_k &= -[q_{k+3}, E_i^{(n)}] + [q_1, E_i^{(n)}] q_{k+2} + [q_2, E_i^{(n)}] q_{k+1} - [q_1, E_i^{(n)}] q_1 q_{k+1} \\ &\quad - [q_1, E_i^{(n)}] q_2 q_k + [q_3, E_i^{(n)}] q_k - [q_2, E_i^{(n)}] q_1 q_k + [q_1, E_i^{(n)}] q_1^2 q_k \\ &\quad + 3E_i^{(n)} \partial q_{k+2} + 3E_i^{(n)} \partial^2 q_{k+1} + E_i^{(n)} \partial^3 q_k + 2[q_1, E_i^{(n)}] \partial q_{k+1} \\ &\quad + [q_1, E_i^{(n)}] \partial^2 q_k + [q_2, E_i^{(n)}] \partial q_k - 3E_i^{(n)} (\partial q_1) q_{k+1} \\ &\quad - 3E_i^{(n)} (\partial q_1) (\partial q_k) - 3E_i^{(n)} (\partial q_2) q_k \\ &\quad - 3E_i^{(n)} (\partial^2 q_1) q_k - [q_1, E_i^{(n)}] q_1 \partial q_k - 2[q_1, E_i^{(n)}] (\partial q_1) q_k + 3E_i^{(n)} (\partial q_1) q_1 q_k. \end{aligned} \quad (4.40)$$

Now consider again (4.31). In its the left-hand side, we use the definition $P := I_n + \sum_{k=1}^{\infty} q_k \partial^{-k}$ and take the derivative, $\frac{\partial P}{\partial x_\ell^{(i)}}$. We then insert the formulas (4.38), (4.39), and (4.40). For the right-hand side, we insert the definition of P and the formula (4.30). A tedious computation shows that the two sides are equal, and concluding the result. \square

Sato equations imply the following equations.

Corollary 4.5 (Zakharov-Shabat equations for multi-component KP hierarchy). *Fix $1 \leq n \leq m$. The $n \times n$ matrix-valued pseudo-differential operators $B_\ell^{(j)}$ from (4.30) satisfy the following equations, known as the Zakharov-Shabat equations for the n -component KP hierarchy:*

$$\frac{\partial B_\ell^{(i)}}{\partial x_{\ell'}^{(j)}} - \frac{\partial B_{\ell'}^{(j)}}{\partial x_\ell^{(i)}} = [B_{\ell'}^{(j)}, B_\ell^{(i)}], \quad i, j \in \{1, \dots, n\}, \quad \ell, \ell' \in \{1, 2, 3\}. \quad (4.41)$$

Proof. Proposition 4.4 of [20] proved that if P and $B_\ell^{(i)}$ satisfy the Sato equations (4.31), then $B_\ell^{(i)}$ satisfy (4.41). \square

Sato equations also give another proof of the matrix KP equation (2.14). The proof of the next result shows that Zakharov-Shabat equations (4.41) yield the matrix KP equation modulo an “integration constant” θ : see (4.45) below. We can then show that $\theta = 0$ using one of Sato equations.

Corollary 4.6. *Set*

$$u := -\partial_x q \quad \text{where} \quad q := q_1.$$

Then, the matrix KP equation (2.14) holds:

$$-4\partial_t u + \partial_x^3 u + 6\partial_x(u^2) - 3\partial_y^2 q + 6[u, \partial_y q] = 0.$$

Proof. We use the notations $x_1^{(i)} = x_i$, $x_2^{(i)} = y_i$, $x_3^{(i)} = t_i$, and

$$\partial_t = \sum_{i=1}^n \partial_{t_i}, \quad \partial_y = \sum_{i=1}^n \partial_{y_i}, \quad \partial_x = \sum_{i=1}^n \partial_{x_i} = \partial.$$

Set $B_\ell := \sum_{j=1}^n B_\ell^{(j)}$ for $\ell \in \{1, 2, 3\}$. From (4.30), we find that

$$B_1 = \partial, \quad B_2 = \partial^2 - 2(\partial q_1), \quad B_3 = \partial^3 - 3(\partial q_1)\partial - 3(\partial q_2) - 3(\partial^2 q_1) + 3(\partial q_1)q_1.$$

Setting

$$\alpha := -(\partial q_2) + (\partial q_1)q_1$$

and using $u := -\partial_x q_1$, these expressions become

$$B_1 = \partial, \quad B_2 = \partial^2 + 2u, \quad B_3 = \partial^3 + 3u\partial + 3(\partial_x u + \alpha). \quad (4.42)$$

Consider the $\ell' = 3$ and $\ell = 2$ case of the equations (4.41). Taking the sum over $i, j = 1, \dots, n$, it becomes

$$\partial_t B_2 - \partial_y B_3 = [B_3, B_2].$$

Inserting (4.42) and considering the terms of ∂^1 and ∂^0 we obtain two equations, namely

$$\partial_y u = \partial_x^2 u + 2\partial_x \alpha \quad (4.43)$$

and

$$2\partial_t u - 3\partial_y \partial_x u + \partial_x^3 u - 6u\partial_x u = 3\partial_y \alpha - 3\partial_x^2 \alpha + 6[\partial_x u + \alpha, u]. \quad (4.44)$$

Since $u := -\partial_x q$, (4.43) implies that there is a function θ such that

$$\alpha = -\frac{1}{2}(\partial_x u + \partial_y q_1 + \theta) \quad \text{and} \quad \partial_x \theta = 0.$$

Inserting it into (4.44), we obtain

$$-4\partial_t u + \partial_x^3 u + 6\partial_x(u^2) - 3(\partial_y^2 q + \partial_y \theta) + 6[u, \partial_y q + \theta] = 0. \quad (4.45)$$

Now, the case of $\ell = 2$ of Sato equations (4.31) is precisely (4.39). The sum of this equation over i implies that $-(\partial q_2) + (\partial q_1)q_1 = -\frac{1}{2}(\partial_x u + \partial_y q_1)$. This shows that $\theta = 0$ in (4.45), completing the proof. \square

4.6. Matrix ODE system à la Tracy and Widom: proof of Theorem 2.7 and Corollary 2.8. In this section, we assume that \mathbf{H} is a *strongly cubic integrable operator* as in Definition 2.1. Thus, Ω is a finite union of simple contours without endpoints and, furthermore, the matrix \mathbf{J}_0 in (4.8) is constant on each connected component of Ω . The last property allows us to derive an additional Lax equation for the derivative with respect to the variable z , which is often called the spectral variable. Because this equation will have coefficients which are polynomial in z , the corresponding Lax pairs are usually associated to isomonodromic deformations [13], but we will not explore this fact explicitly.

Proof of Theorem 2.7. Recall that

$$\partial = \sum_{j=1}^m \mathbf{t}_j \partial_{\mathbf{x}_j}$$

was defined in (2.17). Taking a weighted sum of the first equation of (4.10), we find that

$$\partial \Psi(z) = (z\mathbf{M}_t + \mathbf{D}_1)\Psi(z) \quad \text{with} \quad \mathbf{D}_1 := [\Phi_1, \mathbf{M}_t].$$

where recall $\mathbf{M}_t := \text{diag}(t_1, \dots, t_m, 0)$ from (2.16). Inserting $\Psi(z) = \Phi(z)\Delta(z)$, using the asymptotic series (4.5), and comparing the terms of $\mathcal{O}(z^{-1})$, we obtain

$$\partial \Phi_1 = -[\Phi_2, \mathbf{M}_t] + [\Phi_1, \mathbf{M}_t]\Phi_1. \quad (4.46)$$

The lower-right block of this identity is exactly (2.19).

On the other hand, consider the differentiation ∂_z with respect to the variable z . We have

$$\partial_z \Delta = \sum_{j=1}^m (3z^2 \mathbf{t}_j + 2zy_j + \mathbf{x}_j) \mathbf{E}_j \Delta = (3z^2 \mathbf{M}_t + 2z\mathbf{M}_y + \mathbf{M}_x) \Delta. \quad (4.47)$$

Since \mathbf{J}_0 is independent of z on each connected component of the contour, $\partial_z \Psi$ satisfies the same jump condition as Ψ , and hence, $(\partial_z \Psi)\Psi^{-1}$ is an entire function. Considering the large z asymptotic formula again, we obtain

$$\partial_z \Psi(z) = (3z^2 \mathbf{M}_t + z(3\mathbf{D}_1 + 2\mathbf{M}_y) + (3\mathbf{D}_2 + 2\mathbf{D}_3 + \mathbf{M}_x))\Psi(z) \quad (4.48)$$

with $\mathbf{D}_1 := [\Phi_1, \mathbf{M}_t]$ as before, and

$$\mathbf{D}_2 := [\Phi_2, \mathbf{M}_t] - [\Phi_1, \mathbf{M}_t]\Phi_1 = -\partial \Phi_1, \quad \mathbf{D}_3 := [\Phi_1, \mathbf{M}_y],$$

where we used (4.46) for the last equality.

Thus, Ψ is a common solution to the ∂ and ∂_z differential equations, and the compatibility of these two equations, namely $\partial \partial_z \Psi = \partial_z \partial \Psi$, implies the zero curvature equation

$$\begin{aligned} & z(3[\mathbf{M}_t, \mathbf{D}_2] + 2[\mathbf{M}_t, \mathbf{C}_3] + 2[\mathbf{D}_1, \mathbf{M}_y] - 3\partial \mathbf{D}_1) + (3[\mathbf{D}_1, \mathbf{D}_2] \\ & + 2[\mathbf{D}_1, \mathbf{D}_3] + [\mathbf{D}_1, \mathbf{M}_x] - 3\partial \mathbf{D}_2 - 2\partial \mathbf{D}_3) = 0. \end{aligned}$$

Using the formulas for \mathbf{D}_1 , \mathbf{D}_2 and \mathbf{D}_3 and the cyclicity of the commutator, we obtain that the coefficient of z is trivially identically zero. The constant term, on the other hand, is non-trivial and gives the equation

$$3[\mathbf{D}_1, \mathbf{D}_2] + 2[\mathbf{D}_1, \mathbf{D}_3] + [\mathbf{D}_1, \mathbf{M}_x] - 3\partial \mathbf{D}_2 - 2\partial \mathbf{D}_3 = 0.$$

Inserting the formula of \mathbf{D}_1 , \mathbf{D}_2 , \mathbf{D}_3 , we obtain (2.18). \square

Proof of Corollary 2.8. The equation (2.21) follows by inserting $\Phi_1 = \begin{pmatrix} q & p \\ r & s \end{pmatrix}$ into (2.18) after removing ∂S using (2.19). If $t_1 = \dots = t_m =: t$, the $m \times m$ upper left corner of (4.46) yields $\partial_x q = -pr$. This was also already proved in Lemma 2.4. Using this equation, the bottom two equations of (2.21) become the bottom two equations of (2.22). \square

4.7. Deformation formulas for cubic admissible determinants: proof of Proposition 2.10. Assume that H is cubic integrable and $1 - H$ is invertible, and consider the Fredholm determinant $\mathfrak{D} = \det(1 - H)$. We compute the derivatives of $\log \mathfrak{D}$ with respect to the variables x_i, y_i, t_i .

Proof of Proposition 2.10. Recall that E_i is the $(m+1) \times (m+1)$ matrix whose (i, i) entry is 1 and all other entries are 0. From (2.3) and (2.4), we have $\Delta(z) = \text{diag}(m_1(z), m_2(z), \dots, m_m(z), 1)$ with $m_j(z) = e^{t_j z^3 + y_j z^2 + x_j z}$. Thus,

$$\partial_{x_i} \Delta(z) = z E_i \Delta(z).$$

Hence, the vector functions f and g satisfy

$$\partial_{x_i} f(z) = z E_i f(z) + \alpha(z) f(z), \quad \partial_{x_i} g(z) = -z E_i g(z) - \alpha(z) g(z) \quad \text{where } \alpha(z) := \frac{\partial_{x_i} c(z)}{c(z)}.$$

Thus, for $z \neq w$,

$$\partial_{x_i} H(z, w) = \frac{(\partial_{x_i} f(z))^T g(w) + f(z)^T \partial_{x_i} g(w)}{z - w} = f(z)^T E_i g(w) + \alpha(z) H(z, w) - H(z, w) \alpha(w). \quad (4.49)$$

For $z = w$, since $H(z, z) = 0$ by definition, we have $\partial_{x_i} H(z, z) = 0$. On the other hand, noting that Δ is a diagonal matrix, the condition (2.8) implies that $f(z)^T E_i g(z) = 0$. Thus, the right-hand side of (4.49) is zero when $z = w$. Therefore, the equation (4.49) is valid for all $z, w \in \Omega$.

For vector functions u and v , let $u \otimes v$ be the integral operator given by the kernel $u(z)^T v(w)$. Let α be the operator of multiplication by $\alpha(z)$. Then, the equation (4.49) can be written in the operator form $\partial_{x_i} H = f \otimes (E_i g) + [H, \alpha]$ with $[\cdot, \cdot]$ being the commutator of operators. Thus,

$$\begin{aligned} \partial_{x_i} \log \det(1 - H) &= -\text{Tr}((1 - H)^{-1} \partial_{x_i} H) \\ &= -\text{Tr}((1 - H)^{-1} f \otimes (E_i g)) + \text{Tr}((1 - H)^{-1} [H, \alpha]). \end{aligned} \quad (4.50)$$

Since $(1 - H)^{-1} H = H(1 - H)^{-1}$, we use cyclicity of the trace and obtain

$$\text{Tr}((1 - H)^{-1} [H, \alpha]) = \text{Tr}((1 - H)^{-1} H \alpha) - \text{Tr}(\alpha H (1 - H)^{-1}) = 0. \quad (4.51)$$

Therefore, using (4.1),

$$\begin{aligned} \partial_{x_i} \log \det(1 - H) &= -\text{Tr}(F \otimes E_i g) = \int_{\Omega} F(z)^T E_i g(z) d\mu(z) \\ &= -\int_{\Omega} \text{Tr} \left[F(z) g(z)^T E_i \right] d\mu(z), \end{aligned} \quad (4.52)$$

where the last equality follows from the identity $v^T w = \text{Tr}(v^T w) = \text{Tr}(v w^T)$ which is valid for any two column vectors v, w . The proof is now complete recalling the definition of Φ_1 in (4.7). \square

In a similar spirit, we also obtain deformation formulas with respect to the other parameters. This result will be used in Sect. 8.

Proposition 4.7. *Under the same conditions as Proposition 2.10,*

$$\begin{aligned}\partial_{y_i} \log \det(1 - H) &= \text{Tr} \left[(\Phi_1^2 - 2\Phi_2) E_i \right], \\ \partial_{t_i} \log \det(1 - H) &= \text{Tr} \left[(-\Phi_1^3 + 2\Phi_1\Phi_2 + \Phi_2\Phi_1 - 3\Phi_3) E_i \right],\end{aligned}\quad (4.53)$$

where Φ_1 , Φ_2 , and Φ_3 are defined in (4.7).

Proof. Since $\partial_{y_i} \Delta(z) = z^2 E_i \Delta(z)$ and $\partial_{t_i} \Delta(z) = z^3 E_i \Delta(z)$, we find that

$$\begin{aligned}\partial_{y_i} f(z) &= z^2 E_i f(z) + \beta(z) f(z), & \partial_{y_i} g(z) &= -z^2 E_i g(z) - \beta(z) g(z), \\ \partial_{t_i} f(z) &= z^3 E_i f(z) + \gamma(z) f(z), & \partial_{t_i} g(z) &= -z^3 E_i g(z) - \gamma(z) g(z),\end{aligned}$$

where $\beta(z) := \frac{\partial_{y_i} c(z)}{c(z)}$ and $\gamma(z) := \frac{\partial_{t_i} c(z)}{c(z)}$. As in the proof of Proposition 2.10, we find that

$$\begin{aligned}\partial_{y_i} H(z, w) &= (z + w) f(z)^T E_i g(w) + \beta(z) H(z, w) - H(z, w) \beta(w), \\ \partial_{t_i} H(z, w) &= (z^2 + zw + w^2) f(z)^T E_i g(w) + \gamma(z) H(z, w) - H(z, w) \gamma(w),\end{aligned}$$

for $z \neq w$. As in the last proposition, this identity also extends to $z = w$ because both sides are equal to 0 in this case. Let M_z be the multiplication by z , $(M_z f)(z) = z f(z)$. In operator forms, the equations become

$$\begin{aligned}\partial_{y_i} H &= M_z E_i f \otimes g + f \otimes M_z E_i g + [\beta, H], \\ \partial_{t_i} H &= M_z^2 E_i f \otimes g + M_z f \otimes M_z E_i g + f \otimes M_z^2 E_i g + [\gamma, H].\end{aligned}$$

Calculating as in (4.50)–(4.51), we find

$$\begin{aligned}\partial_{y_i} \log \det(1 - H) &= -\text{Tr}((1 - H)^{-1} M_z E_i f \otimes g) - \text{Tr}((1 - H)^{-1} f \otimes M_z E_i g), \\ \partial_{t_i} \log \det(1 - H) &= -\text{Tr}((1 - H)^{-1} M_z^2 E_i f \otimes g) - \text{Tr}((1 - H)^{-1} M_z f \otimes M_z E_i g) \\ &\quad - \text{Tr}((1 - H)^{-1} f \otimes M_z^2 E_i g).\end{aligned}$$

We now relate each such term with the matrices Φ_n from (4.7). First, proceeding similarly to (4.52),

$$\text{Tr}((1 - H)^{-1} f \otimes M_z^n E_i g) = \text{Tr}(F \otimes M_z^n E_i g) = \int_{\Omega} \text{Tr} \left[z^n F(z) g(z)^T E_i \right] d\mu(z) = \text{Tr} [\Phi_{n+1} E_i]$$

for all $n \geq 1$. The identity $\text{Tr}(Ku \otimes v) = \text{Tr}(u \otimes (K^T v))$ is true for any vectors v and u and trace class operator K . Using this identity and (4.1) and (4.7), we find

$$\begin{aligned}\text{Tr}((1 - H)^{-1} M_z^n E_i f \otimes g) &= \text{Tr}(M_z^n E_i f \otimes (1 - H^T)^{-1} g) = \int_{\Omega} \text{Tr} \left[z^n E_i f(z) G(z)^T \right] d\mu(z) \\ &= \text{Tr} \left[E_i \int_{\Omega} z^n f(z) G(z)^T d\mu(z) \right] = -\text{Tr} [E_i (\Phi^{-1})_{n+1}].\end{aligned}$$

Finally, note that

$$\begin{aligned} [\mathbf{M}_z, (1 - \mathbf{H})^{-1}] &= (1 - \mathbf{H})^{-1} [1 - \mathbf{H}, \mathbf{M}_z] (1 - \mathbf{H})^{-1} = (1 - \mathbf{H})^{-1} [\mathbf{H}, \mathbf{M}_z] (1 - \mathbf{H})^{-1} \\ &= (1 - \mathbf{H})^{-1} \mathbf{f} \otimes \mathbf{g} (1 - \mathbf{H})^{-1} = (1 - \mathbf{H})^{-1} \mathbf{f} \otimes (1 - \mathbf{H}^T)^{-1} \mathbf{g} = \mathbf{F} \otimes \mathbf{G}. \end{aligned}$$

This identity implies that $(1 - \mathbf{H})^{-1} \mathbf{M}_z = \mathbf{M}_z (1 - \mathbf{H})^{-1} - \mathbf{F} \otimes \mathbf{G}$, and thus acting on \mathbf{f}^T , we find

$$(1 - \mathbf{H})^{-1} \mathbf{M}_z \mathbf{f}^T = \mathbf{M}_z \mathbf{F}^T - \mathbf{F}^T \left(\int_{\Omega} \mathbf{G}(z) \mathbf{f}(z)^T dz \right) = \mathbf{M}_z \mathbf{F}^T + \mathbf{F}^T (\Phi^{-1})_1^T.$$

Hence,

$$\begin{aligned} \text{Tr}((1 - \mathbf{H})^{-1} \mathbf{M}_z \mathbf{f} \otimes \mathbf{M}_z \mathbf{E}_i \mathbf{g}) &= \int_{\Omega} (z \mathbf{F}(z)^T + \mathbf{F}(z)^T (\Phi^{-1})_1^T) z \mathbf{E}_i \mathbf{g}(z) dz \\ &= \int_{\Omega} \text{Tr} \left[\left(z \mathbf{F}(z) + (\Phi^{-1})_1 \mathbf{F}(z) \right) z \mathbf{g}(z)^T \mathbf{E}_i \right] dz \\ &= \text{Tr} \left[\Phi_3 \mathbf{E}_i + (\Phi^{-1})_1 \Phi_2 \mathbf{E}_i \right]. \end{aligned}$$

Combining the above calculations, we obtain

$$\begin{aligned} \partial_{y_i} \log \det(1 - \mathbf{H}) &= \text{Tr} \left[((\Phi^{-1})_2 - \Phi_2) \mathbf{E}_i \right], \\ \partial_{t_i} \log \det(1 - \mathbf{H}) &= \text{Tr} \left[((\Phi^{-1})_3 - 2\Phi_3 - (\Phi^{-1})_1 \Phi_2) \mathbf{E}_i \right]. \end{aligned}$$

Since $\Phi(z) \sim 1 + \frac{\Phi_1}{z} + \frac{\Phi_2}{z^2} + \frac{\Phi_3}{z^3} + \dots$ as $z \rightarrow \infty$, we see that

$$\Phi(z)^{-1} = 1 - \frac{\Phi_1}{z} + \frac{\Phi_1^2 - \Phi_2}{z^2} - \frac{\Phi_1^3 - \Phi_1 \Phi_2 - \Phi_2 \Phi_1 + \Phi_3}{z^3} + \mathcal{O}(z^{-4}),$$

which determines $(\Phi^{-1})_j$ for $j = 1, 2, 3$ in terms of Φ_1, Φ_2, Φ_3 , and we conclude (4.53). \square

5. A Class of Fredholm Determinants

The formulas (2.23) and (2.24) of the multi-point distributions for the KPZ and the periodic KPZ fixed points involve certain Fredholm determinants. The multi-point distributions of other models such as continuous-time and discrete-time totally asymmetric simple exclusion processes (TASEPs) on the line and the ring also have similar Fredholm determinants [7, 22, 23]. The operators for these determinants all have a common structure. In this section, we consider a class of operators with this common structure and show that their Fredholm determinants are equal to the Fredholm determinants of integrable operators with more transparent structure. The main result is Theorem 5.3. In the special case of the KPZ and the periodic KPZ fixed point, the new integrable operators will be cubic integrable, and we prove this statement in Sect. 6. In a subsequent paper, we will consider the application of Theorem 2.11 to other models in the KPZ universality class, and derive differential equations for them.

5.1. A class of operators. We introduce a class of operators which arise in the study of the multi-point distributions of various models in the KPZ universality class.

Fix a positive integer m . Suppose that $\Omega_{1,1}, \dots, \Omega_{1,m}, \Omega_{2,1}, \dots, \Omega_{2,m}$ are $2m$ pairwise disjoint subsets of \mathbb{C} . We assume that they are either all finite unions of contours or all discrete sets. Set

$$\Omega_1 := \Omega_{1,1} \cup \dots \cup \Omega_{1,m} \quad \text{and} \quad \Omega_2 := \Omega_{2,1} \cup \dots \cup \Omega_{2,m}, \quad (5.1)$$

and

$$\begin{cases} \mu \text{ is the counting measure if } \Omega_{\ell,k} \text{ are discrete,} \\ d\mu = dz \text{ if } \Omega_{\ell,k} \text{ are contours.} \end{cases}$$

Let $A_1(z), \dots, A_m(z)$ and $B_1(z), \dots, B_m(z)$ be functions of $z \in \Omega_1 \cup \Omega_2$, and define the $m \times m$ diagonal matrices

$$A(z) = \text{diag}(A_j(z))_{j=1}^m \quad \text{and} \quad B(z) = \text{diag}(B_j(z))_{j=1}^m. \quad (5.2)$$

Finally, suppose that $m_1(z), \dots, m_m(z)$ are non-vanishing functions of $z \in \Omega_1 \cup \Omega_2$, and set

$$M(z) = \text{diag}(M_j(z))_{j=1}^m := \text{diag}\left(m_1(z), \frac{m_1(z)}{m_2(z)}, \frac{m_3(z)}{m_2(z)}, \frac{m_3(z)}{m_4(z)}, \frac{m_5(z)}{m_4(z)}, \frac{m_5(z)}{m_6(z)}, \dots\right). \quad (5.3)$$

Definition 5.1. With the notions (5.1)–(5.3), define integral operators

$$\widehat{K}_1 : L^2(\Omega_2, d\mu) \rightarrow L^2(\Omega_1, d\mu) \quad \text{and} \quad \widehat{K}_2 : L^2(\Omega_1, d\mu) \rightarrow L^2(\Omega_2, d\mu)$$

by their kernels

$$\widehat{K}_1(u, v) = \left(\delta_i(j) + \delta_i(j + (-1)^i)\right) \frac{M_i(u)A_i(u)B_j(v)}{u - v}, \quad \text{if } u \in \Omega_{1,i}, \quad v \in \Omega_{2,j},$$

and

$$\widehat{K}_2(u, v) = \left(\delta_i(j) + \delta_i(j - (-1)^i)\right) \frac{M_i(u)^{-1}A_i(u)B_j(v)}{u - v}, \quad \text{if } u \in \Omega_{2,i}, \quad v \in \Omega_{1,j},$$

for $i, j \in \{1, \dots, m\}$.

The m -point distributions of several models of the KPZ universality class are expressible in terms of the Fredholm determinant $\det(1 - \widehat{K}_1\widehat{K}_2)$ for some $\widehat{K}_1, \widehat{K}_2$ of the above form. The main differences among the models are the function m_i . For the KPZ and the periodic KPZ fixed points, m_i are cubic exponential functions $e^{\mathbf{t}jz^3 + \mathbf{y}jz^2 + \mathbf{x}jz}$ as in (2.3); see Sect. 6 for details. For other models, the cubic exponential functions are changed to

$$e^{t_i z} z^{k_i} (z + 1)^{n_i} \quad \text{for continuous-time (periodic) TASEP,}$$

$$(1 + pz)^{t_i - k_i} z^{k_i} (z + 1)^{n_i} \quad \text{for discrete-time (periodic) TASEP,}$$

$$z^{k_i} e^{-b_i z + a_i/z} \quad \text{for (periodic) PNG model,}$$

where a_i, b_i, k_i, n_i, t_i and p are parameters of the models. These results are found in [7, Section 3.4] and [23, Section 2.1.1] for the continuous-time TASEP, in [22, Sections 2.5 and 2.6] for the discrete-time TASEP [22, Sections 2.5 and 2.6], and in work in progress by Tejaswi Tripathi for the PNG model.

In the next subsection, we show that $\det(1 - \widehat{K}_1\widehat{K}_2)$ is equal to the Fredholm determinant of an integrable operator with a more convenient structure.

5.2. *A reformulation of a class of Fredholm determinants.* Set

$$\Omega := \Omega_1 \cup \Omega_2.$$

We introduce $m \times m$ diagonal matrices of indicator functions

$$\mathbf{X}_\ell = \text{diag}(\chi_{\Omega_{\ell,j}})_{j=1}^m \quad \text{for } \ell = 1, 2,$$

which, due to the pairwise disjointness of the sets $\Omega_{j,k}$, satisfy

$$\mathbf{X}_\ell(z) \mathbf{e} \mathbf{e}^T \mathbf{X}_{\ell'}(z) = \mathbf{X}_\ell(z) \delta_{\ell,\ell'} \quad (5.4)$$

for $\ell, \ell' = 1, 2$. Define the 2×2 matrix

$$\mathbf{U} := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and introduce the $m \times m$ matrices

$$\Lambda_1 := \begin{cases} \text{diag}(\underbrace{\mathbf{U}, \dots, \mathbf{U}}_{m/2}), & m \text{ even,} \\ \text{diag}(\underbrace{\mathbf{U}, \dots, \mathbf{U}, 1}_{(m-1)/2}), & m \text{ odd,} \end{cases} \quad \text{and} \quad \Lambda_2 := \begin{cases} \text{diag}(1, \underbrace{\mathbf{U}, \dots, \mathbf{U}}_{(m-2)/2}, 1), & m \text{ even,} \\ \text{diag}(1, \underbrace{\mathbf{U}, \dots, \mathbf{U}}_{(m-1)/2}), & m \text{ odd.} \end{cases} \quad (5.5)$$

Furthermore, define an $m \times m$ matrix, an $(m+1) \times (2m)$ projection matrix, and a column vector given by

$$\mathbf{E}_{11} := \text{diag}(1, 0, \dots, 0), \quad \mathbf{P} = (\mathbf{I}_{m+1} \ 0), \quad \mathbf{e} = (1, 1, \dots, 1)^T \in \mathbb{R}^m, \quad (5.6)$$

respectively. Recall the $m \times m$ diagonal matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ from (5.2), and define the $(m+1)$ -dimensional column vector-valued functions of $z \in \Omega$,

$$\begin{aligned} \mathbf{U}(z) &:= \mathbf{P} \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix} \begin{pmatrix} \mathbf{A}(z) \mathbf{X}_1(z) \mathbf{e} \\ \mathbf{A}(z) \mathbf{X}_2(z) \mathbf{e} \end{pmatrix}, \\ \mathbf{V}(z) &:= \mathbf{P} \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix} \begin{pmatrix} \mathbf{B}(z) \mathbf{X}_2(z) \mathbf{e} \\ \mathbf{B}(z) \mathbf{X}_1(z) \mathbf{e} \end{pmatrix}. \end{aligned} \quad (5.7)$$

Finally, introduce the $(m+1) \times (m+1)$ diagonal matrix-valued function

$$\Delta(z) := \text{diag}(\mathbf{m}_1(z), \mathbf{m}_2(z), \dots, \mathbf{m}_m(z), 1), \quad z \in \Omega,$$

where $\mathbf{m}_j(z)$ are the functions in (5.3).

Definition 5.2. Let $\mathbf{c}(z)$ be a non-vanishing scalar function of $z \in \Omega$. Define the $(m+1)$ -dimensional column vector-valued functions

$$\begin{aligned} \mathbf{f}(z) &= (\mathbf{f}_1(z), \dots, \mathbf{f}_{m+1}(z))^T := \mathbf{c}(z) \Delta(z) \mathbf{U}(z), \\ \mathbf{g}(z) &= (\mathbf{g}_1(z), \dots, \mathbf{g}_{m+1}(z))^T := \frac{1}{\mathbf{c}(z)} \Delta(z)^{-1} \mathbf{V}(z) \end{aligned} \quad (5.8)$$

for $z \in \Omega$, and the integral operator $\mathbf{H} : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$ by its kernel

$$\mathbf{H}(u, v) := \frac{\mathbf{f}(u)^T \mathbf{g}(v)}{u - v} \quad \text{for } u, v \in \Omega \text{ with } u \neq v, \quad \text{and} \quad \mathbf{H}(u, u) := 0, \quad u \in \Omega. \quad (5.9)$$

The following result is the main outcome of this section.

Theorem 5.3. *Let \widehat{K}_1 and \widehat{K}_2 be operators in Definition 5.1, and H in Definition 5.2. If the Fredholm determinant of $\det(1 - \widehat{K}_1 \widehat{K}_2)$ is well defined as a series expansion and H is a trace class operator, then*

$$\det(1 - \widehat{K}_1 \widehat{K}_2) = \det(1 - H). \quad (5.10)$$

Furthermore,

$$f_i(z)g_i(z) = 0 \quad \text{for every } z \in \Omega \text{ and } i \in \{1, \dots, m+1\}. \quad (5.11)$$

The proof of Theorem 5.3 is given in Sect. 5.3.

The scalar function $c(z)$ in (5.8) is arbitrary, and changing it does not affect the series expansion of $\det(1 - H)$. However, in Sect. 6, we choose it appropriately in applications so that H becomes a trace class operator.

The main difference of H from \widehat{K}_1 and \widehat{K}_2 is that the matrix $\Delta(z)$ does not involve the ratios of $m_i(z)$ unlike $M(z)$. Thus, the above identity “uncouples” $m_i(z)$ s in the kernel formula. This is useful when we derive differential equations as in Sect. 4.

The formulation (5.8) is useful in deriving differential equations. However, for other purposes, namely in proving that the operator is trace class and in obtaining asymptotic formulas, we find that alternative representations of f and g are more suitable, and we state them next.

Lemma 5.4. *Define the $m \times m$ matrix*

$$S(z) := \text{diag} \left(m_1(z)^{1/2}, \frac{m_1(z)^{1/2}}{m_2(z)^{1/2}}, \frac{m_3(z)^{1/2}}{m_2(z)^{1/2}}, \frac{m_3(z)^{1/2}}{m_4(z)^{1/2}}, \frac{m_5(z)^{1/2}}{m_4(z)^{1/2}}, \dots \right), \quad z \in \Omega, \quad (5.12)$$

for some fixed branch of the square-root. Choose the conjugating constant as

$$c(z) = e^T (X_1(z) + X_2(z)) \text{diag} \left(\frac{1}{\sqrt{m_1(z)}}, \frac{1}{\sqrt{m_1(z)m_2(z)}}, \dots, \frac{1}{\sqrt{m_{m-1}(z)m_m(z)}} \right) e. \quad (5.13)$$

Then the functions f and g in (5.8) can be written as

$$\begin{aligned} f(z) &= P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} \begin{pmatrix} A(z)S(z)X_1(z) & 0 \\ 0 & A(z)S(z)^{-1}X_2(z) \end{pmatrix} \begin{pmatrix} e \\ e \end{pmatrix}, \\ g(z) &= P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} \begin{pmatrix} B(z)S(z)^{-1}X_2(z) & 0 \\ 0 & B(z)S(z)X_1(z) \end{pmatrix} \begin{pmatrix} e \\ e \end{pmatrix}. \end{aligned} \quad (5.14)$$

Before proving the lemma, we first introduce the $m \times m$ diagonal matrices

$$\begin{aligned} M_O(z) &:= \text{diag}(m_1(z), m_1(z), m_3(z), m_3(z), m_5(z), \dots), \\ M_E(z) &:= \text{diag}(1, m_2(z), m_2(z), m_4(z), m_4(z), \dots), \end{aligned} \quad (5.15)$$

which we will also use later in this section.

Proof of Lemma 5.4. Define the $m \times m$ matrix $\tilde{\Delta} := \text{diag} (m_1, m_2, \dots, m_m)$. Note that

$$\Delta(z)^{\pm 1} P = (\Delta(z)^{\pm 1} \ 0) = P \begin{pmatrix} \Delta(z)^{\pm 1} & 0 \\ 0 & I_{m-1} \end{pmatrix} = P \begin{pmatrix} \tilde{\Delta}(z)^{\pm 1} & 0 \\ 0 & I_m \end{pmatrix}.$$

It is straightforward to check that

$$\tilde{\Delta}(z)^{\pm 1} \Lambda_1 = \Lambda_1 \mathbf{M}_0(z)^{\pm 1}, \quad \tilde{\Delta}(z)^{\pm 1} (\Lambda_2 - \mathbf{E}_{11}) = (\Lambda_2 - \mathbf{E}_{11}) \mathbf{M}_e(z)^{\pm 1},$$

and $\mathbf{E}_{11} = \mathbf{E}_{11} \mathbf{M}_e(z) = \mathbf{E}_{11} \mathbf{M}_e(z)^{-1}$. Applying these identities to (5.8), we find that

$$\begin{aligned} f(z) &= P \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix} \begin{pmatrix} \mathbf{M}_0(z) \mathbf{A}(z) \mathbf{X}_1(z) \mathbf{e} \\ \mathbf{M}_e(z) \mathbf{A}(z) \mathbf{X}_2(z) \mathbf{e} \end{pmatrix} c(z), \\ g(z) &= P \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix} \begin{pmatrix} \mathbf{M}_0(z)^{-1} \mathbf{B}(z) \mathbf{X}_2(z) \mathbf{e} \\ \mathbf{M}_e(z)^{-1} \mathbf{B}(z) \mathbf{X}_1(z) \mathbf{e} \end{pmatrix} \frac{1}{c(z)}. \end{aligned} \quad (5.16)$$

In terms of (5.15), the conjugating constant chosen in (5.13) can be written as

$$c(z) = \mathbf{e}^T (\mathbf{X}_1(z) + \mathbf{X}_2(z)) \mathbf{M}_0(z)^{-1/2} \mathbf{M}_e(z)^{-1/2} \mathbf{e},$$

and, due to the indicator functions, it satisfies

$$\frac{1}{c(z)} = \mathbf{e}^T (\mathbf{X}_1(z) + \mathbf{X}_2(z)) \mathbf{M}_0(z)^{1/2} \mathbf{M}_e(z)^{1/2} \mathbf{e}.$$

Now using (5.4), we find that

$$\begin{aligned} \mathbf{X}_\ell(z) \mathbf{e} c(z) &= \mathbf{X}_\ell(z) \mathbf{e} \mathbf{e}^T (\mathbf{X}_1(z) + \mathbf{X}_2(z)) \mathbf{M}_0(z)^{-1/2} \mathbf{M}_e(z)^{-1/2} \mathbf{e} \\ &= \mathbf{X}_\ell(z) \mathbf{M}_0(z)^{-1/2} \mathbf{M}_e(z)^{-1/2} \mathbf{e}, \\ \mathbf{X}_\ell(z) \mathbf{e} \frac{1}{c(z)} &= \mathbf{X}_\ell(z) \mathbf{e} \mathbf{e}^T (\mathbf{X}_1(z) + \mathbf{X}_2(z)) \mathbf{M}_0(z)^{1/2} \mathbf{M}_e(z)^{1/2} \mathbf{e} \\ &= \mathbf{X}_\ell(z) \mathbf{M}_0(z)^{1/2} \mathbf{M}_e(z)^{1/2} \mathbf{e} \end{aligned}$$

for $\ell = 1, 2$. Thus, using $\mathbf{S}(z) = \mathbf{M}_0(z)^{1/2} \mathbf{M}_e(z)^{-1/2}$, we find, for example,

$$\mathbf{M}_0(z) \mathbf{A}(z) \mathbf{X}_1(z) \mathbf{e} c(z) = \mathbf{A}(z) \mathbf{X}_1(z) \mathbf{M}_0(z)^{1/2} \mathbf{M}_e(z)^{-1/2} \mathbf{e} = \mathbf{A}(z) \mathbf{S}(z) \mathbf{X}_1(z) \mathbf{e}.$$

The other three vectors in (5.16) are computed in a similar way. \square

5.3. Proof of Theorem 5.3. In this Subsection we prove 5.3. Replacing the kernels in Definition 5.1 by

$$\chi_{\Omega_1}(u) \widehat{\mathbf{K}}_1(u, v) \chi_{\Omega_2}(v) \quad \text{and} \quad \chi_{\Omega_2}(u) \widehat{\mathbf{K}}_2(u, v) \chi_{\Omega_1}(v)$$

for $u, v \in \Omega$, we can extend $\widehat{\mathbf{K}}_1$ and $\widehat{\mathbf{K}}_2$ to operators from $L^2(\Omega, \mu)$ to $L^2(\Omega, \mu)$. We use the same notation $\widehat{\mathbf{K}}_1$ and $\widehat{\mathbf{K}}_2$ to denote the extended operators. The Fredholm determinant $\det(1 - \widehat{\mathbf{K}}_1 \widehat{\mathbf{K}}_2)$ is unchanged by this modification.

Lemma 5.5. For each $\ell = 1, 2$, the kernel of \widehat{K}_ℓ can be written as

$$\widehat{K}_\ell(u, v) = \frac{\widehat{a}_\ell(u)^T \widehat{b}_\ell(v)}{u - v}, \quad u, v \in \Omega,$$

with the $m \times 1$ column vectors

$$\begin{aligned} \widehat{a}_1(z) &= \Lambda_1 A(z) M(z) X_1(z) e, & \widehat{b}_1(z) &= \Lambda_1 B(z) X_2(z) e, \\ \widehat{a}_2(u) &= \Lambda_2 A(u) M(z)^{-1} X_2(u) e, & \widehat{b}_2(z) &= \Lambda_2 B(z) X_1(z) e. \end{aligned}$$

Proof. Set $\chi_{\Omega_\ell, 0} = \chi_{\Omega_\ell, m+1} = 0$ for $\ell = 1, 2$ and $B_0 = B_{m+1} = 0$. Due to indicator functions, the kernels in Definition 5.1 can be written as

$$\begin{aligned} (u - v) \widehat{K}_1(u, v) &= \sum_{\substack{k=1 \\ k \text{ odd}}}^m \chi_{\Omega_{1,k}}(u) M_k(u) A_k(u) (B_k(v) \chi_{\Omega_{2,k}}(v) + B_{k+1}(v) \chi_{\Omega_{2,k+1}}(v)) \\ &\quad + \sum_{\substack{k=1 \\ k \text{ even}}}^m \chi_{\Omega_{1,k}}(u) M_k(u) A_k(u) (B_k(v) \chi_{\Omega_{2,k}}(v) + B_{k-1}(v) \chi_{\Omega_{2,k-1}}(v)) \end{aligned}$$

and

$$\begin{aligned} (u - v) \widehat{K}_2(u, v) &= \sum_{\substack{k=1 \\ k \text{ odd}}}^m \chi_{\Omega_{2,k}}(u) M_k(u)^{-1} A_k(u) (B_k(v) \chi_{\Omega_{1,k}}(v) + B_{k-1}(v) \chi_{\Omega_{1,k-1}}(v)) \\ &\quad + \sum_{\substack{k=1 \\ k \text{ even}}}^m \chi_{\Omega_{2,k}}(u) M_k(u)^{-1} A_k(u) (B_k(v) \chi_{\Omega_{1,k}}(v) + B_{k+1}(v) \chi_{\Omega_{1,k+1}}(v)). \end{aligned}$$

It is direct to check that for any column vectors $\alpha = (\alpha_1, \dots, \alpha_m)^T \in \mathbb{R}^m$ and $\beta = (\beta_1, \dots, \beta_m)^T \in \mathbb{R}^m$,

$$\alpha^T \Lambda_1^T \Lambda_1 \beta = \sum_{\substack{k=1 \\ k \text{ odd}}}^m \alpha_k (\beta_k + \beta_{k+1}) + \sum_{\substack{k=1 \\ k \text{ even}}}^m \alpha_k (\beta_k + \beta_{k-1})$$

and

$$\alpha^T \Lambda_2^T \Lambda_2 \beta = \sum_{\substack{k=1 \\ k \text{ odd}}}^m \alpha_k (\beta_k + \beta_{k-1}) + \sum_{\substack{k=1 \\ k \text{ even}}}^m \alpha_k (\beta_k + \beta_{k+1}),$$

where we set $\beta_0 = \beta_{m+1} = 0$. The result then follows. \square

For the next lemma, recall the matrices M_0 and M_e defined in (5.15). Also introduce $m \times m$ diagonal matrices

$$\begin{aligned} D_1(z) &:= \text{diag}(m_1(z), 1, m_3(z), 1, m_5(z), \dots), \\ D_2(z) &:= \text{diag}(1, m_2(z), 1, m_4(z), 1, \dots). \end{aligned}$$

Lemma 5.6. Define the $m \times 1$ vector-valued functions

$$\begin{aligned} a_1(z) &:= \Lambda_1 A(z) M_o(z) X_1(z) e = D_1(z) \Lambda_1 A(z) X_1(z) e, \\ a_2(z) &:= \Lambda_2 A(z) M_e(z) X_2(z) e = D_2(z) \Lambda_2 A(z) X_2(z) e, \\ b_1(z) &:= \Lambda_1 B(z) M_o(z)^{-1} X_2(z) e = D_1(z)^{-1} \Lambda_1 B(z) X_2(z) e, \\ b_2(z) &:= \Lambda_2 B(z) M_e(z)^{-1} X_1(z) e = D_2(z)^{-1} \Lambda_2 B(z) X_1(z) e, \end{aligned} \quad (5.17)$$

for $z \in \Omega$. Then, there is a non-vanishing scalar function \widehat{c} on Ω such that for each $\ell = 1, 2$,

$$\widehat{a}_\ell(z) = \widehat{c}(z) a_\ell(z) \quad \text{and} \quad \widehat{b}_\ell(z) = \frac{1}{\widehat{c}(z)} b_\ell(z).$$

Proof. Define the scalar function

$$\widehat{c}(z) = e^T (M_e(z)^{-1} X_1(z) + M_o(z)^{-1} X_2(z)) e.$$

Due to the indicator functions, we find that

$$\frac{1}{\widehat{c}(z)} = e^T (M_e(z) X_1(z) + M_o(z) X_2(z)) e.$$

Observe that

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

holds for any complex numbers a and $*$. Using this identity, we find that

$$\Lambda_1 M_o(z)^{\pm 1} = D_1(z)^{\pm 1} \Lambda_1, \quad \Lambda_2 M_e(z)^{\pm 1} = D_2(z)^{\pm 1} \Lambda_2.$$

Since the matrices A, B, M_o, M_e , and X_ℓ are all diagonal, they commute. Since $M = M_o M_e^{-1}$, we find, using (5.4), that

$$\begin{aligned} \widehat{a}_1(z) \frac{1}{\widehat{c}(z)} &= \Lambda_1 A(u) M_o(z) M_e(z)^{-1} X_1(u) e e^T (M_e(z) X_1(z) + M_o(z) X_2(z)) e \\ &= \Lambda_1 A(z) M_o(z) X_1(z) e = D_1(z) \Lambda_1 A(z) X_1(z) e = a_1(z). \end{aligned}$$

The results for the other vector functions b_1, a_2 and b_2 follow similarly. \square

For the next result, observe that (5.17) and (5.4) imply that

$$b_1(z) a_1(z)^T = b_2(z) a_2(z)^T = 0. \quad (5.18)$$

Lemma 5.7. Let a_1, a_2, b_1, b_2 be the vectors defined in (5.17). Let $c(z)$ be an arbitrary non-vanishing scalar function on Ω . For $\ell = 1, 2$, define the operator $K_\ell : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$ by its kernel

$$K_\ell(u, v) := \frac{c(u) a_\ell(u)^T b_\ell(v) \frac{1}{c(v)}}{u - v} \quad \text{for } u, v \in \Omega \text{ with } u \neq v, \quad \text{and} \quad K_\ell(u, u) := 0, \quad u \in \Omega.$$

If the series definition of $\det(1 - \widehat{K}_1 \widehat{K}_2)$ is well-defined and K_1 and K_2 are trace class operators, then

$$\det(1 - \widehat{K}_1 \widehat{K}_2) = \det(1 - K_1 - K_2).$$

Proof. Lemma 5.5 and 5.6 show that for each $\ell = 1, 2$,

$$\widehat{K}_\ell(u, v) = \frac{\widehat{c}(u)}{c(u)} K_\ell(u, v) \frac{c(v)}{\widehat{c}(v)}.$$

Hence, from the series definition of Fredholm determinants, $\det(1 - \widehat{K}_1 \widehat{K}_2) = \det(1 - K_1 K_2)$.

Equation (5.18) implies that $K_\ell^2 = 0$ for $\ell = 1, 2$, and thus

$$1 - K_1 K_2 = (1 + K_1)(1 - K_1 - K_2)(1 + K_2).$$

Now for $\ell = 1, 2$, we have $\text{Tr } K_\ell = 0$ because $K_\ell(u, u) = 0$, and $\text{Tr } K_\ell^n = 0$ for all $n \geq 2$ since $K_\ell^2 = 0$. Thus, the Plemelj-Smithies formula for Fredholm determinants, which is valid for trace class operators, implies that $\det(1 + K_\ell) = 1$. Therefore, $\det(1 - K_1 K_2) = \det(1 - K_1 - K_2)$, and we obtain the result. \square

We are ready to prove Theorem 5.3. Introduce the $(2m) \times (2m)$ permutation matrix

$$\Pi := \begin{pmatrix} \Pi_+ & \Pi_- \\ \Pi_- & \Pi_+ \end{pmatrix} \quad \text{where} \quad \Pi_+ := \text{diag}(1, 0, 1, 0, \dots), \quad \Pi_- := \text{diag}(0, 1, 0, 1, \dots).$$

The diagonal matrices Ω_\pm are of size $m \times m$. Recall the matrices Λ_1 and Λ_2 from (5.5) and the matrix $E_{11} = \text{diag}(1, 0, \dots, 0)$ from (5.6). It is easy to check that

$$\Omega \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix}. \quad (5.19)$$

Proof of Theorem 5.3. Let K_1 and K_2 be the operators in Lemma 5.7. We first show that

$$K_1 + K_2 = H.$$

We need to show that

$$c(u)a(u)^T b(v) \frac{1}{c(v)} = f(u)^T g(v),$$

where we denote $a(z) := (a_1(z), a_2(z))^T$ and $b(z) := (b_1(z), b_2(z))^T$. Since $P^T P = \text{diag}(I_m, E_{11})$, we find from (5.19) that

$$\Pi \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = P^T P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} = P^T P \Pi \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}.$$

Hence, from the first formulas of b_1 and b_2 in (5.17), $P^T P \Pi b(z) = \Pi b(z)$. Using $\Pi^T \Pi = I_{2m}$, we find

$$a(u)^T b(v) = a(u)^T \Pi^T \Pi b(v) = a(u)^T \Pi^T P^T P \Pi b(v) = (P \Pi a(u))^T P \Pi b(v).$$

We now insert the second formulas of a_1 and a_2 in (5.17) to compute $P \Pi a(u)$. The relations

$$\Pi_+ D_1(z) = \Delta(z) \Pi_+, \quad \Pi_- D_1(z) = \Pi_-, \quad \Pi_+ D_2(z) = \Pi_+ \quad \text{and} \quad \Pi_- D_2(z) = \Delta(z) \Pi_-$$

are straightforward, and yield

$$P \sqcap \begin{pmatrix} D_1(z) & 0 \\ 0 & D_2(z) \end{pmatrix} \sqcap = \Delta(z) P \sqcap.$$

Thus, using (5.19) again, we obtain

$$\begin{aligned} P \sqcap a(z) &= \Delta(z) P \sqcap \begin{pmatrix} \Lambda_1 A(z) X_1(z) e \\ \Lambda_2 A(z) X_2(z) e \end{pmatrix} \\ &= \Delta(z) P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} \begin{pmatrix} A(z) X_1(z) e \\ A(z) X_2(z) e \end{pmatrix} = \frac{1}{c(z)} f(z). \end{aligned}$$

A similar computation yields

$$\begin{aligned} P \sqcap b(z) &= \Delta(z)^{-1} P \sqcap \begin{pmatrix} \Lambda_1 B(z) X_2(z) e \\ \Lambda_2 B(z) X_1(z) e \end{pmatrix} \\ &= \Delta(z)^{-1} P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} \begin{pmatrix} B(z) X_2(z) e \\ B(z) X_1(z) e \end{pmatrix} = c(z) g(z). \end{aligned}$$

Hence, $K_1 + K_2 = H$.

Since

$$\chi_{\Omega_1}(u) H(u, v) \chi_{\Omega_2}(v) = K_1(u, v), \quad \chi_{\Omega_2}(u) H(u, v) \chi_{\Omega_1}(v) = K_2(u, v),$$

if we assume that H is a trace class operator, then K_1 and K_2 are trace class operators. Thus, Lemma 5.7 applies and we obtain

$$\det(1 - \widehat{K}_1 \widehat{K}_2) = \det(1 - K_1 - K_2) = \det(1 - H),$$

and this proves (5.10).

Finally, we check that $f_i(z) g_i(z) = 0$ for $i \in \{1, \dots, m+1\}$, as claimed in (5.11). Since E_i is the $(m+1) \times (m+1)$ diagonal matrix which has 1 on the i -th diagonal entry and the other entries are all zero, we have $f_i(z) g_i(z) = (f(z)^T E_i g(z))_{ii}$. Inserting the formula (5.8) of f and g and noting that $\Delta(z) E_i \Delta(z)^{-1} = E_i$, we find that $f(z)^T E_i g(z)$ is equal to, with certain diagonal matrices α, β , determined from $P^T E_i P$,

$$\begin{pmatrix} X_1(z) \Lambda_1^T \alpha \Lambda_1 X_2(z) & X_1(z) \Lambda_1^T \alpha (\Lambda_2 - E_{11}) X_1(z) \\ X_2(z) (\Lambda_2^T - E_{11}) \alpha \Lambda_1 X_2(z) & X_2(z) (\Lambda_2^T - E_{11}) \alpha (\Lambda_2 - E_{11}) X_1(z) + X_2(z) E_{11}^T \beta E_{11} X_1(z) \end{pmatrix}$$

multiplied by the row vector $((A(z)e)^T, (A(z)e)^T)$ on the left and by the column vector $(B(z)e, B(z)e)^T$ on the right. It is direct to check that $\Lambda_1^T \alpha (\Lambda_2 - E_{11}) = 0$ and $(\Lambda_2^T - E_{11}) \alpha (\Lambda_2 - E_{11}) = 0$ for every diagonal matrix α . Thus, the off-diagonal blocks are zero. On the other hand, due to the characteristic functions, the diagonal blocks are also zero. Thus, $f_i(z)^T g_i(z) = 0$. \square

5.4. A lemma on trace class operators. In order to apply Theorem 5.3, we need to check that H is a trace class operator. We will use the following lemma in the next section. We use $|v|$ to denote the usual Euclidean norm of a vector v .

Lemma 5.8. *Let H be the operator in Definition 5.2. Suppose that each set Ω_ℓ can be split into a finite union of disjoint subsets*

$$\Omega_\ell = \bigcup_{j=1}^{N_\ell} \Sigma_{\ell,j} \quad (5.20)$$

satisfying the following property: For every pair of indices (j_1, j_2) with $j_\ell \in \{1, \dots, N_\ell\}$, $\ell \in \{1, 2\}$, there is a simple contour C_{j_1, j_2} such that (a) it is either closed or extends to infinity, (b) it separates Σ_{1, j_1} and Σ_{2, j_2} in the sense that they are in different regions of $\mathbb{C} \setminus C_{j_1, j_2}$, and (c) the estimates

$$\int_{\Sigma_{\ell, j_\ell}} \int_{C_{j_1, j_2}} \frac{|f(u)|^2}{|s-u|^2} |ds| |d\mu(u)| < \infty \quad \text{and} \quad \int_{\Sigma_{\ell, j_\ell}} \int_{C_{j_1, j_2}} \frac{|g(u)|^2}{|s-u|^2} |ds| |d\mu(u)| < \infty \quad (5.21)$$

hold for $\ell = 1, 2$. Then, H is a trace class operator.

Furthermore, there is a union C of contours that is disjoint from Ω such that H can be written as $H = T_2 T_1$ for Hilbert-Schmidt operators $T_1 : L^2(\Omega, \mu) \rightarrow L^2(C, dz)$, $T_2 : L^2(C, dz) \rightarrow L^2(\Omega, \mu)$ with Hilbert-Schmidt norms

$$\|T_1\|_2 \leq \left(\int_{\Omega} \int_C \frac{|f(u)|^2}{|s-u|^2} |ds| |d\mu(u)| \right)^{1/2} \quad \text{and} \quad \|T_2\|_2 \leq \left(\int_{\Omega} \int_C \frac{|g(u)|^2}{|s-u|^2} |ds| |d\mu(u)| \right)^{1/2}.$$

The decomposition $H = T_2 T_1$ will be used only in Lemma 7.1.

Proof. We express the kernel H in the form

$$H(u, v) = \sum_{j_1, j_2} \chi_{\Sigma_{1, j_1}}(u) H(u, v) \chi_{\Sigma_{2, j_2}}(v) + \sum_{j_1, j_2} \chi_{\Sigma_{2, j_2}}(u) H(u, v) \chi_{\Sigma_{1, j_1}}(v).$$

In this decomposition terms of the form $\chi_{\Sigma_{\ell, j_\ell}}(u) H(u, v) \chi_{\Sigma_{\ell', j'_\ell}}(v)$, $\ell \in \{1, 2\}$, do not appear due to the structure of the kernel and the indicator functions in (5.7) and (5.9). To show that H is trace class, it is enough to prove that each term in the sum defines a trace class operator. We only show it for the terms in the first sum since for the terms in the second sum the proof is analogous.

For $u \in \Sigma_{1, j_1}$ and $v \in \Sigma_{2, j_2}$, the separation assumption and the Cauchy integral formula imply that

$$\frac{1}{u-v} = \frac{\pm 1}{2\pi i} \int_{C_{j_1, j_2}} \frac{ds}{(s-u)(s-v)}$$

where the sign of the term ± 1 depends on the relative location of Σ_{1, j_1} to C_{j_1, j_2} . Thus,

$$\chi_{\Sigma_{1, j_1}}(u) H(u, v) \chi_{\Sigma_{2, j_2}}(v) = \frac{\pm 1}{2\pi i} \int_{C_{j_1, j_2}} \frac{\chi_{\Sigma_{1, j_1}}(u) f(u)^T g(v) \chi_{\Sigma_{2, j_2}}(v)}{(s-u)(s-v)} ds, \quad u, v \in \Omega.$$

This is the kernel of the product of two operators, one from $L^2(\Omega, d\mu)$ to $L^2(C_{j_1, j_2}, dz)$ and the other in the opposite way. Assumption (5.21) implies that these two operators are Hilbert-Schmidt, and thus H is a trace class operator.

Furthermore, taking the C to be the union of C_{j_1, j_2} , we find the decomposition $H = T_2 T_1$, with T_1, T_2 being Hilbert-Schmidt operators with the claimed Hilbert-Schmidt norm bounds. \square

We note that from the formula (5.14) and the definition of Ω , the condition (5.21) is satisfied if

- (i) $A, B \in L^\infty(\Omega, d\mu)$, and
- (ii) for every $\ell_1, \ell_2 \in \{1, 2\}$ with $\ell_1 \neq \ell_2$, and every j_1, j_2 ,

$$\int_{\Omega_{\ell_k, j_{\ell_k}}} \int_{C_{j_1, j_2}} \left| \frac{\sqrt{m_{j+1}(u)}}{\sqrt{m_j(u)}} \right| \frac{|ds| |d\mu(u)|}{|s-u|^2} < \infty \quad \text{when } \ell_k - j_{\ell_k} \text{ is even,} \quad (5.22)$$

and

$$\int_{\Omega_{\ell_k, j_{\ell_k}}} \int_{C_{j_1, j_2}} \left| \frac{\sqrt{m_j(u)}}{\sqrt{m_{j+1}(u)}} \right| \frac{|ds| |d\mu(u)|}{|s-u|^2} < \infty \quad \text{when } \ell_k - j_{\ell_k} \text{ is odd.} \quad (5.23)$$

6. Cubic Integrable Operators for the KPZ and the Periodic KPZ Fixed Points

We state the Fredholm determinants (2.23) and (2.24) obtained in [7, 23] explicitly, and then apply Theorem 5.3 of the last section to express them in terms of new Fredholm determinants which we show to be cubic admissible determinants. We also prove a symmetry for the case of the KPZ fixed point, thus proving Theorem 2.11.

6.1. KPZ fixed point - proof of Theorem 2.11 (i). The paper [23, Definition 2.23] showed that the function in (2.23) is given by

$$\mathfrak{D}^{(\text{KPZ})}(h, \gamma, \tau \mid \zeta) := \det(1 - \widehat{K}_1^{(\text{KPZ})} \widehat{K}_2^{(\text{KPZ})})$$

where the Fredholm determinant is defined through an absolutely convergent series definition and the operators $\widehat{K}_1^{(\text{KPZ})}$ and $\widehat{K}_2^{(\text{KPZ})}$ are of the form in Definition 5.1 which we now describe. The above formula is valid for the parameters satisfying

$$0 < \tau_1 \leq \dots \leq \tau_m \quad \text{where } \gamma_i < \gamma_{i+1} \text{ if } \tau_i = \tau_{i+1}, \quad (6.1)$$

which we can always assume by re-labeling the parameters. Note that the complex numbers $\zeta_1, \dots, \zeta_{m-1}$ are fixed complex numbers of different moduli satisfying $0 < |\zeta_i| < 1$ for all i .

Let $\Gamma_{m,L}^+, \dots, \Gamma_{2,L}^+, \Gamma_{1,L}, \Gamma_{2,L}^-, \dots, \Gamma_{m,L}^-$ be pairwise disjoint contours that are horizontal translates of each other in the left half of the complex plane extending from $\infty e^{i\theta_1}$ to $\infty e^{i\theta_2}$ for some angles $\theta_1 \in (7\pi/6, 5\pi/4)$ and $\theta_2 \in (3\pi/4, 5\pi/6)$ and are arranged as indicated in Fig. 1. In addition, let $\Gamma_{m,R}^+, \dots, \Gamma_{2,R}^+, \Gamma_{1,R}, \Gamma_{2,R}^-, \dots, \Gamma_{m,R}^-$ be the reflections of the above contours about the imaginary axis: see Fig. 1.

Set

$$\Gamma_{i,L} := \Gamma_{i,L}^+ \cup \Gamma_{i,L}^- \quad \text{and} \quad \Gamma_{i,R} := \Gamma_{i,R}^+ \cup \Gamma_{i,R}^-, \quad i = 2, \dots, m.$$

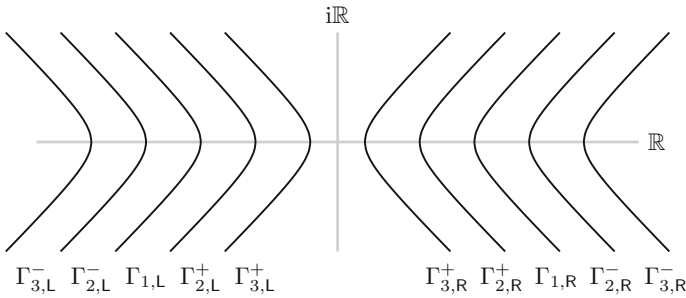


Fig. 1. Plots of contours when $m = 3$. The contours are oriented from the bottom to the top

In terms of the notations in Definition 5.1, define

$$\Omega_{1,j}^{(\text{KPZ})} := \begin{cases} \Gamma_{j,L}, & \text{if } j \text{ is odd,} \\ \Gamma_{j,R}, & \text{if } j \text{ is even,} \end{cases} \quad \Omega_{2,j}^{(\text{KPZ})} := \begin{cases} \Gamma_{j,R}, & \text{if } j \text{ is odd,} \\ \Gamma_{j,L}, & \text{if } j \text{ is even,} \end{cases}$$

and also $\Omega_\ell^{(\text{KPZ})} := \bigcup_{j=1}^m \Omega_{\ell,j}^{(\text{KPZ})}$ for $\ell = 1, 2$. Note that $\Omega_2^{(\text{KPZ})}$ is equal to $-\Omega_1^{(\text{KPZ})}$ if we reverse the orientation. The Hilbert spaces are $L^2(\Omega_\ell^{(\text{KPZ})}, dz)$ for $\ell = 1, 2$.

For each $j = 1, \dots, m$, let,¹¹

$$\mathbf{Q}_j^{(\text{KPZ})}(z) := \begin{cases} -(1 - \zeta_j)\chi_{\Omega_{1,j}}(z) + (1 - \frac{1}{\zeta_{j-1}})\chi_{\Omega_{2,j}}(z), & \text{if } j \text{ is odd,} \\ -(1 - \frac{1}{\zeta_{j-1}})\chi_{\Omega_{1,j}}(z) + (1 - \zeta_j)\chi_{\Omega_{2,j}}(z), & \text{if } j \text{ is even,} \end{cases}$$

with the convention that $\zeta_0 = \infty$, $\zeta_m = 0$, and.¹²

$$\mathbf{P}_j^{(\text{KPZ})}(z) := \begin{cases} \frac{1}{2\pi i}, & z \in \Gamma_1, \\ \frac{1}{2\pi i(1-\zeta_{j-1})}, & z \in \Gamma_{j,L}^+ \cup \Gamma_{j,R}^+, \quad j > 1, \\ \frac{-\zeta_{j-1}}{2\pi i(1-\zeta_{j-1})}, & z \in \Gamma_{j,L}^- \cup \Gamma_{j,R}^-, \quad j > 1. \end{cases}$$

Define the $m \times m$ diagonal matrix functions $\mathbf{A}^{(\text{KPZ})}$ and $\mathbf{B}^{(\text{KPZ})}$ in (5.2) with the functions

$$\mathbf{A}_j^{(\text{KPZ})}(z) := 1 \quad \text{and} \quad \mathbf{B}_j^{(\text{KPZ})}(z) := \mathbf{Q}_j^{(\text{KPZ})}(z)\mathbf{P}_j^{(\text{KPZ})}(z). \quad (6.2)$$

Finally, set

$$\mathbf{m}_j^{(\text{KPZ})}(z) := e^{\mathbf{t}_i z^3 + \mathbf{y}_i z^2 + \mathbf{x}_i z} \quad \text{where} \quad \mathbf{t}_i = -\tau_i/3, \quad \mathbf{y}_i = \gamma_i \quad \mathbf{x}_i = \mathbf{h}_i, \quad (6.3)$$

and define $\mathbf{M}^{(\text{KPZ})}$ as (5.3). Here, $\tau_j > 0$, $\gamma_j \in \mathbb{R}$, $\mathbf{h}_j \in \mathbb{R}$ are the time, position, and height variable of the KPZ fixed point, respectively. This notation is consistent with (2.25).

¹¹ In [23, Definition 2.25] the formulas $Q_1(j)$ and $Q_2(j)$ are used. Here, we use the function $\mathbf{Q}_j^{(\text{KPZ})}(z) = -Q_2(j)\chi_{\Omega_{1,j}}(z) + Q_1(j)\chi_{\Omega_{2,j}}(z)$.

¹² The factor $\mathbf{P}_j^{(\text{KPZ})}$ is not present in the kernels in [23, Equation (5) and Definition 2.4] This is because the L^2 space is weighted. Here we use $L^2(dz)$ space instead and include $\mathbf{P}_j^{(\text{KPZ})}$ as a part of the kernel.

The operators $\widehat{\mathcal{K}}_1^{(\text{KPZ})}$ and $\widehat{\mathcal{K}}_2^{(\text{KPZ})}$ are given by the form in Definition 5.1 with the above functions and matrices.

We now apply Theorem 5.3 to the above operators. Let $\mathcal{H}^{(\text{KPZ})}$ be the operator in Definition 5.2 with the specific choice of the conjugating constant $\mathfrak{C}(z)$ given by (5.13). The operator acts on $L^2(\Omega^{(\text{KPZ})}, dz)$ where $\Omega^{(\text{KPZ})} = \Omega_1^{(\text{KPZ})} \cup \Omega_2^{(\text{KPZ})}$. From the formula of the kernel, in particular due to the formula (6.3) of $\mathfrak{m}_j^{(\text{KPZ})}$, we see that $\mathcal{H}^{(\text{KPZ})}$ is a cubic integrable operator. Furthermore, since $\mathbf{A}_j^{(\text{KPZ})}$ and $\mathbf{B}_j^{(\text{KPZ})}$ do not depend on the parameters $\mathfrak{t}, \mathfrak{y}, \mathfrak{x}$, it is indeed strongly cubic integrable.

Corollary 6.1. *The operator $\mathcal{H}^{(\text{KPZ})}$ is trace class on $L^2(\Omega^{(\text{KPZ})}, dz)$ and*

$$\mathfrak{D}^{(\text{KPZ})}(h, \gamma, \tau \mid \zeta) = \det(1 - \mathcal{H}^{(\text{KPZ})}).$$

Proof. It only remains to check that $\mathcal{H}^{(\text{KPZ})}$ is trace class. We omit the superscript (KPZ) to lighten the notations. We use Lemma 5.8. Decompose Ω_ℓ so that each $\Sigma_{\ell,j}$ is one of the connected contours of Ω ; see Fig. 1. If Σ_{1,j_1} and Σ_{2,j_2} are in different half planes (such as $\Gamma_{1,L}$ and $\Gamma_{3,R}^-$), we take the separating contour to be $C_{j_1,j_2} = i\mathbb{R}$. On the other hand, if Σ_{1,j_1} and Σ_{2,j_2} are in the same half plane, then we take C_{j_1,j_2} to be a translation of them that sits between them. To check condition (c), it is enough to check (i) and (ii) in the discussion after Lemma 5.8. Since \mathbf{A} and \mathbf{B} are constants on each connected component, they are in $L^\infty(\Omega)$ and (i) holds. For (ii), due to the conditions (6.1) on the parameters, we see that

$$\frac{\sqrt{\mathfrak{m}_{i+1}(z)}}{\sqrt{\mathfrak{m}_i(z)}} = e^{\frac{1}{2}(\mathfrak{t}_i - \mathfrak{t}_{i+1})z^3 + \frac{1}{2}(\mathfrak{y}_i - \mathfrak{y}_{i+1})z^2 + \frac{1}{2}(\mathfrak{x}_i - \mathfrak{x}_{i+1})z}$$

decays super-exponentially fast as $|z| \rightarrow \infty$ for z in the sectors $\arg(z) \in (\pi/6, \pi/4) \cup (7\pi/4, 11\pi/6)$, and similarly, its reciprocal decays super-exponentially fast in the sectors $\arg(z) \in (3\pi/4, 5\pi/6) \cup (7\pi/6, 5\pi/4)$. From the geometry of the contours and this exponential decay, we find that the decay condition (ii) is satisfied, and we obtain the result. \square

From the above result and the next lemma, Theorem 2.11 (i) is proved.

Lemma 6.2. *The symmetry (2.26) holds true.*

Proof. For simplicity, we omit the superscript (KPZ) in the notations. From (4.5) and (4.7), we have $\Phi_1 = \lim_{z \rightarrow \infty} z(\Phi(z) - \mathbf{I})$. Thus, a symmetry of Φ_1 follows from a symmetry of $\Phi(z)$, the solution of the Riemann-Hilbert problem in Sect. 4.1. This further follows from a symmetry of the jump matrix.

From (4.3) and (5.8), the jump matrix is

$$\mathbf{J}(z) = \mathbf{I}_{m+1} - 2\pi i \mathfrak{f}(z) \mathbf{g}(z)^T = \mathbf{I}_{m+1} - 2\pi i \Delta(z) \mathbf{U}(z) \mathbf{V}(z)^T \Delta(z)^{-1} \quad (6.4)$$

where the $(m+1)$ -dimensional vector functions \mathbf{U} and \mathbf{V} are given in (5.7) with \mathbf{A} and \mathbf{B} given in (6.2). Since $\mathbf{A} = \mathbf{I}$, we have

$$\mathbf{U}(z) \mathbf{V}(z)^T = \mathbf{P} \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{B}(z) \mathbf{X}_1(z) \\ \mathbf{B}(z) \mathbf{X}_2(z) & 0 \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_2 - \mathbf{E}_{11} \\ 0 & \mathbf{E}_{11} \end{pmatrix}^T \mathbf{P}^T.$$

It is tedious but straightforward from the definition (6.2), also noting that $X_2(-z) = X_1(z)$, to check that

$$B_j(-z)\chi_{\Omega_{2,j}}(-z) = \begin{cases} -\frac{1-\frac{1}{\zeta_{j-1}}}{1-\zeta_j} B_j(z)\chi_{\Omega_{1,j}}(z) & \text{if } j \text{ is odd,} \\ -\frac{1-\zeta_j}{1-\frac{1}{\zeta_{j-1}}} B_j(z)\chi_{\Omega_{1,j}}(z) & \text{if } j \text{ is even,} \end{cases} \quad (6.5)$$

for $j = 1, \dots, m$ and $z \in \Omega$.

For arbitrary non-zero constants L_1, \dots, L_m , it is straightforward to check that

$$\text{diag}(L_1, \dots, L_m)^{\pm 1} \Lambda_1 = \Lambda_1 L_0^{\pm 1}, \quad \text{diag}(L_1, \dots, L_m)^{\pm 1} (\Lambda_2 - E_{11}) = (\Lambda_2 - E_{11}) L_e^{\pm 1},$$

with $m \times m$ matrices $L_0 := \text{diag}(L_1, L_1, L_3, L_3, \dots)$ and $L_e := \text{diag}(1, L_2, L_2, L_4, L_4, \dots)$. Thus, with the $(m+1) \times (m+1)$ matrix

$$L := \text{diag}(L_1, \dots, L_m, 1),$$

we find, noting $E_{11} L_e^{\pm} = E_{11}$, that

$$LU(z)V(z)^T L^{-1} = P \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix} \begin{pmatrix} 0 & L_0 L_e^{-1} B(z) X_1(z) \\ L_0^{-1} L_e B(z) X_2(z) & 0 \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_2 - E_{11} \\ 0 & E_{11} \end{pmatrix}^T P^T.$$

Now for L_j given in (2.27), we have

$$L_0 L_e^{-1} = \text{diag} \left(L_1, \frac{L_1}{L_2}, \frac{L_3}{L_2}, \frac{L_3}{L_4}, \dots \right) = \text{diag} \left(-(1 - \zeta_1), -\frac{1 - \frac{1}{\zeta_1}}{1 - \zeta_2}, -\frac{1 - \zeta_3}{1 - \frac{1}{\zeta_2}}, -\frac{1 - \frac{1}{\zeta_3}}{1 - \zeta_4}, \dots \right).$$

Thus, from (6.5) we find that

$$L_0^{-1} L_e B(-z) X_2(-z) = B(z) X_1(z), \quad L_0 L_e^{-1} B(-z) X_1(-z) = B(z) X_2(z)$$

and hence,

$$LU(-z)V(-z)^T L^{-1} = (U(z)V(z)^T)^T.$$

Noting that $m_j(-z \mid x, -y, t)^{-1} = m_j(z \mid x, y, t)$, we have $\Delta(-z \mid x, -y, t)^{-1} = \Delta(z \mid x, y, t)$. Thus, the jump matrix (6.4) satisfies

$$(LJ(-z \mid x, -y, t)L)^T = J(z \mid x, y, t).$$

A symmetry of the jump matrix implies a symmetry of the solution of the RHP, in our case here it reads

$$\left(L\Phi(-z \mid x, -y, t)L^{-1} \right)^{-T} = \Phi(z \mid x, y, t).$$

Finally, this implies the symmetry (2.26) for Φ_1 . □

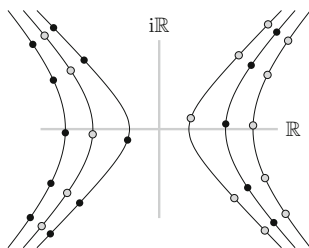


Fig. 2. The nodes are $S_{i,L}$, $S_{i,R}$ and the solid curves are the hyperbolas $\mathcal{P}_{i,L}$, $\mathcal{P}_{i,R}$ for $i = 1, 2, 3$ for certain choices of $\zeta_1, \zeta_2, \zeta_3$. The black-filled nodes represent $\Omega_1^{(\text{per})}$ while the gray-filled nodes represent $\Omega_2^{(\text{per})}$

6.2. Periodic KPZ fixed point - proof of Theorem 2.11 (ii). For the periodic KPZ fixed point, [7, Section 2.2.3] show that the formula in (2.24) is given by

$$\mathfrak{D}^{(\text{per})}(\mathbf{h}, \boldsymbol{\gamma}, \boldsymbol{\tau} \mid \boldsymbol{\zeta}) := \det(1 - \widehat{\mathbf{K}}_1^{(\text{per})} \widehat{\mathbf{K}}_2^{(\text{per})}),$$

for operators $\widehat{\mathbf{K}}_1^{(\text{per})}$ and $\widehat{\mathbf{K}}_2^{(\text{per})}$ of the form in Definition 5.1 which we now describe. The series formula of the Fredholm determinant was shown to be convergent and the above formula is valid for the parameters satisfying

$$0 < \tau_1 \leq \dots \leq \tau_m \quad \text{and} \quad \mathbf{h}_i < \mathbf{h}_{i+1} \text{ if } \tau_i = \tau_{i+1}. \quad (6.6)$$

The fixed constants ζ_1, \dots, ζ_m are arbitrary complex numbers satisfying $0 < |\zeta_1| < \dots < |\zeta_m| < 1$.

For each $i = 1, \dots, m$, consider the roots of the equation $e^{-s^2/2} = \zeta_i$, which are called *Bethe roots*, and define the discrete sets

$$\mathbf{S}_{i,L} := \{s \in \mathbb{C} \mid e^{-s^2/2} = \zeta_i, \operatorname{Re} s < 0\}, \quad \mathbf{S}_{i,R} := \{s \in \mathbb{C} \mid e^{-s^2/2} = \zeta_i, \operatorname{Re} s > 0\}.$$

Note that $\mathbf{S}_{i,L} = -\mathbf{S}_{i,R}$, and

$$\mathbf{S}_{i,R} \subset \mathcal{P}_{i,R} \quad \text{and} \quad \mathbf{S}_{i,L} \subset \mathcal{P}_{i,L},$$

where

$$\mathcal{P}_{i,R} := \{s \in \mathbb{C} \mid |e^{-s^2/2}| = |\zeta_i|, \operatorname{Re} s > 0\} = \{s \in \mathbb{C} \mid (\operatorname{Re} s)^2 - (\operatorname{Im} s)^2 = -2 \log |\zeta_i|, \operatorname{Re} s > 0\} \quad (6.7)$$

and $\mathcal{P}_{i,L} := -\mathcal{P}_{i,R}$ are hyperbolas. Thus, the sets $\mathbf{S}_{i,R}$ extend to ∞ along angles $7\pi/4$ and $\pi/4$ and the sets $\mathbf{S}_{i,L}$ extend to ∞ with angles $5\pi/4$ and $3\pi/4$. See Fig. 2.

In terms of the notation of (5.1), define the discrete sets

$$\Omega_{1,i}^{(\text{per})} := \begin{cases} \mathbf{S}_{i,L}, & \text{if } i \text{ is odd,} \\ \mathbf{S}_{i,R}, & \text{if } i \text{ is even,} \end{cases} \quad \Omega_{2,i}^{(\text{per})} := \begin{cases} \mathbf{S}_{i,R}, & \text{if } i \text{ is odd,} \\ \mathbf{S}_{i,L}, & \text{if } i \text{ is even,} \end{cases}$$

and

$$\Omega_\ell^{(\text{per})} := \bigcup_{i=1}^m \Omega_{\ell,i}^{(\text{per})}$$

for $\ell = 1, 2$. Note that $\Omega_1^{(\text{per})} = -\Omega_2^{(\text{per})}$. The Hilbert spaces are $\ell^2(\Omega_\ell^{(\text{per})})$ for $\ell = 1, 2$ with the counting measure.

For $i = 1, \dots, m$, set

$$\mathbf{Q}_{1,i}^{(\text{per})} := 1 - \frac{\zeta_i - (-1)^i}{\zeta_i}, \quad \mathbf{Q}_{2,i}^{(\text{per})} := 1 - \frac{\zeta_i + (-1)^i}{\zeta_i}$$

with the convention that $\zeta_0 = \zeta_{m+1} = 0$. Define

$$\mathbf{V}_i(z) := \begin{cases} \exp\left(-\int_{-i\infty}^{i\infty} \frac{\log(1-\zeta_i e^{w^2/2})}{w-z} dw\right) & \text{for } \text{Re}(z) < 0, \\ \exp\left(-\int_{-i\infty}^{i\infty} \frac{\log(1-\zeta_i e^{w^2/2})}{w+z} dw\right) & \text{for } \text{Re}(z) > 0. \end{cases}$$

The function $\mathbf{V}_i(z)$ does not depend on the parameters, it is analytic and non-vanishing on $\mathbb{C} \setminus i\mathbb{R}$, and satisfies $\mathbf{V}_i(-z) = \mathbf{V}_i(z)$ and $\mathbf{V}_i(z) \rightarrow 1$ as $z \rightarrow \infty$. Now, setting $\mathbf{V}_0 = \mathbf{V}_{m+1} \equiv 1$, define

$$\mathbf{A}_j^{(\text{per})}(z) := \frac{1}{z\mathbf{V}_j(z)} \times \begin{cases} \mathbf{V}_{j+1}(z) & \text{for } z \in \mathbf{S}_{j,\text{L}}, \\ \mathbf{V}_{j-1}(z) & \text{for } z \in \mathbf{S}_{j,\text{R}}, \end{cases}$$

and

$$\mathbf{B}_j^{(\text{per})}(z) := \left(\mathbf{Q}_{1,j}^{(\text{per})} \chi_{\mathbf{S}_1}(z) + \mathbf{Q}_{2,j}^{(\text{per})} \chi_{\mathbf{S}_2}(z) \right) \times \begin{cases} \mathbf{V}_{j+1}(z) & \text{for } z \in \mathbf{S}_{j,\text{L}}, \\ \mathbf{V}_{j-1}(z) & \text{for } z \in \mathbf{S}_{j,\text{R}}, \end{cases}$$

for $j = 1, \dots, m$, and define the matrices $\mathbf{A} = \mathbf{A}^{(\text{per})}$ and $\mathbf{B} = \mathbf{B}^{(\text{per})}$ by (5.2). Observe that $\mathbf{A}_j^{(\text{per})}, \mathbf{B}_j^{(\text{per})}$ are bounded on $\Omega^{(\text{per})}$.

Finally, define the functions

$$\mathbf{m}_j^{(\text{per})}(z) := e^{\mathbf{t}_j z^3 + \mathbf{y}_j z^2 + \mathbf{x}_j z} \quad \text{where} \quad \mathbf{t}_j = -\tau_j/3, \quad \mathbf{y}_j = \gamma_j/2 \quad \mathbf{x}_j = \mathbf{h}_j,$$

and define the matrix $\mathbf{M}^{(\text{per})}(z)$ as (5.3). Recall that $\tau_j > 0$, $\gamma_j \in [0, 1)$, $\mathbf{h}_j \in \mathbb{R}$ correspond to the time, position, and height variables, and which is consistent with (2.28). With these formulas, we define the operators $\widehat{\mathbf{K}}_1^{(\text{per})}$ and $\widehat{\mathbf{K}}_2^{(\text{per})}$ by the formula of Definition 5.1.

Let $\Omega^{(\text{per})} := \Omega_1^{(\text{per})} \cup \Omega_2^{(\text{per})}$ and define the operator $\mathbf{H}^{(\text{per})}$ on $\ell^2(\Omega_\ell^{(\text{per})})$ as Definition 5.2 with the specific choice of the conjugating constant $\mathbf{c}(z)$ given by (5.13). Then, from the formula of $\mathbf{m}_j^{(\text{per})}$, we find that $\mathbf{H}^{(\text{per})}$ is a cubic integrable operator. The next result proves Theorem 2.11 (ii).

Corollary 6.3. *The operator $\mathbf{H}^{(\text{per})}$ above is trace class on $\ell^2(\Omega_\ell^{(\text{per})})$ and*

$$\mathfrak{D}^{(\text{per})}(\mathbf{h}, \gamma, \tau \mid \zeta) = \det(1 - \mathbf{H}^{(\text{per})}).$$

Proof. It is enough to show that $\mathbf{H}^{(\text{per})}$ is trace class. We omit the superscript (per). We use Lemma 5.8 and the discussion that follows. We already observed that \mathbf{A} and \mathbf{B} are bounded on Ω .

We take the decomposition (5.20) of Ω_ℓ using $\Sigma_{\ell,j} = \Omega_{\ell,j}^{(\text{per})}$ for $\ell = 1, 2$ and $j = 1, \dots, m$. If the sets $\Omega_{1,j_1}^{(\text{per})}$ and $\Omega_{2,j_2}^{(\text{per})}$ are in different half planes, then we take

$C_{j_1, j_2} = i\mathbb{R}$ for the separating contour, and this case is easy, and we skip the detail. If the sets $\Omega_{1, j_1}^{(\text{per})}$ and $\Omega_{2, j_2}^{(\text{per})}$ are in the same half plane, then from the definition of the sets, we see that $j_1 \neq j_2$. Thus, $|\zeta_{j_1}| \neq |\zeta_{j_2}|$. We take the separating contour as

$$C_{j_1, j_2} = \left\{ s \in \mathbb{C} \mid |e^{-s^2/2}| = (|\zeta_{j_1}| + |\zeta_{j_2}|)/2 \right\}$$

It is enough to check that (5.22) and (5.23) in the discussion after Lemma 5.8 are satisfied. The hyperbola C_{j_1, j_2} does not intersect the hyperbolas $\{s \in \mathbb{C} \mid |e^{-s^2/2}| = |\zeta_{j_1}|\}$ and $\{s \in \mathbb{C} \mid |e^{-s^2/2}| = |\zeta_{j_2}|\}$ (see (6.7)) and all three hyperbolas have the same asymptotes. From the geometry we can check that

$$\int_{C_{j_1, j_2}} \frac{|ds|}{|s - u|^2} \quad (6.8)$$

is finite and grows at most like a polynomial as $u \rightarrow \infty$ on $\Omega_{\ell, j_1}^{(\text{per})}$ or $\Omega_{\ell, j_2}^{(\text{per})}$. We have

$$\frac{\sqrt{m_{j+1}(z)}}{\sqrt{m_j(z)}} = e^{\frac{1}{2}(t_j - t_{j+1})z^3 + \frac{1}{2}(y_j - y_{j+1})z^2 + \frac{1}{2}(x_j - x_{j+1})z}. \quad (6.9)$$

Note for $z \in \Omega^{(\text{per})}$, we have $e^{-z^2/2} = \zeta_i$ for some i , and thus,

$$|e^{\frac{1}{2}(y_j - y_{j+1})z^2}| = |\zeta_i|^{y_{j+1} - y_j}$$

does not depend on z . Therefore, since the set $\Omega^{(\text{per})}$ has asymptotes to angles $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, the conditions on (6.6) on parameters imply that (6.9) decays to zero exponentially fast on the set $\Omega_{\ell, j}^{(\text{per})}$ if it is on the right half plane. Similarly, the reciprocal of (6.9) decays exponentially on $\Omega_{\ell, j}^{(\text{per})}$ if it is on the left half plane. Due to exponential decay of $\frac{\sqrt{m_{j+1}(u)}}{\sqrt{m_j(u)}}$ or $\frac{\sqrt{m_j(u)}}{\sqrt{m_{j+1}(u)}}$ and the at most polynomial growth of (6.8), we find that (5.22) and (5.23) are satisfied, and we obtain the result. \square

7. Large Height Limits and the Proof of Proposition 2.12

It is interesting to consider the asymptotic properties of the solutions to the differential equations in Sect. 2.1 as some of the parameters tend to infinity. In this paper, we only consider one simple case and leave other cases for a future work. The next result shows the asymptotic behavior of the Fredholm determinant and Φ_1 for the (periodic) KPZ fixed points as the height parameters $x_i = h_i$ tend to positive infinity in a certain way. Since the result for both the KPZ and periodic KPZ fixed points are the same, we use the variables x, y, t instead of the physical variables h, γ, τ : see (2.25) and (2.28) for the correspondence.

Lemma 7.1. *Let either $\mathfrak{D} = \mathfrak{D}^{(\text{KPZ})}$ and $\Phi_1 = \Phi_1^{(\text{KPZ})}$, or $\mathfrak{D} = \mathfrak{D}^{(\text{per})}$ and $\Phi_1 = \Phi_1^{(\text{per})}$. Fix parameters x, y, t . Let $b_1, \dots, b_m > 0$ be constants and*

$$a := (a_1, a_2, \dots, a_m), \quad a_i := b_1 + \dots + b_i.$$

Set

$$\mathbf{x}_\xi := \mathbf{x} + \xi \mathbf{a} = (\mathbf{x}_1 + \xi a_1, \dots, \mathbf{x}_m + \xi a_m), \quad \xi \in \mathbb{R}.$$

Then, there exist constants $c > 0$ and $\xi_0 > 0$, independent of $\mathbf{x}, \mathbf{y}, \mathbf{t}$, for which the inequalities

$$|\mathfrak{D}(\mathbf{x}_\xi, \mathbf{y}, \mathbf{t}) - 1| \leq e^{-c\xi} \quad \text{and} \quad \max_{i,j=1,\dots,m} |(\Phi_1(\mathbf{x}_\xi, \mathbf{y}, \mathbf{t}))_{i,j}| \leq e^{-c\xi}$$

hold true for all $\xi \geq \xi_0$.

Proof. We omit the superscripts (KPZ) and (per). We also indicate the variable z and the parameter ξ , but omit other parameters. In this proof, we use the alternative formula (5.14) of the vectors \mathbf{f} and \mathbf{g} . This formula involves $\mathbf{S}(z)\mathbf{X}_1(z)$ and $\mathbf{S}(z)^{-1}\mathbf{X}_2(z)$, which we first consider.

From the definition in (2.3), $\mathbf{m}_i(z | \mathbf{x}_\xi) = \mathbf{m}_i(z | \mathbf{x})e^{\xi b_i z}$ holds. Thus, the matrix \mathbf{S} in (5.12) satisfies

$$\mathbf{S}(z | \mathbf{x}_\xi) = \mathbf{S}(z | \mathbf{x})\mathbf{R}(z | \xi) \quad \text{with} \quad \mathbf{R}(z | \xi) := \text{diag}(e^{\xi b_1 z}, e^{-\xi b_2 z}, e^{\xi b_3 z}, \dots). \quad (7.1)$$

Using the identity (5.4) for indicator functions and noting that \mathbf{R} and \mathbf{X}_ℓ are diagonal matrices, we find that

$$\begin{aligned} \mathbf{X}_1(z)\mathbf{e}\mathbf{e}^T(\mathbf{X}_1(z)\mathbf{R}(z | \xi) + \mathbf{X}_2(z)\mathbf{R}(z | \xi)^{-1})\mathbf{e} &= \mathbf{X}_1(z)\mathbf{R}(z | \xi)\mathbf{e} = \mathbf{R}(z | \xi)\mathbf{X}_1(z)\mathbf{e}, \\ \mathbf{X}_2(z)\mathbf{e}\mathbf{e}^T(\mathbf{X}_1(z)\mathbf{R}(z | \xi) + \mathbf{X}_2(z)\mathbf{R}(z | \xi)^{-1})\mathbf{e} &= \mathbf{X}_2(z)\mathbf{R}(z | \xi)^{-1}\mathbf{e} = \mathbf{R}(z | \xi)^{-1}\mathbf{X}_2(z)\mathbf{e}. \end{aligned} \quad (7.2)$$

The equations (7.1) and (7.2) imply that

$$\mathbf{S}(z | \mathbf{x}_\xi)\mathbf{X}_1(z)\mathbf{e} = \mathbf{S}(z | \mathbf{x})\mathbf{X}_1(z)\mathbf{e}\phi(z | \xi) \quad \text{and} \quad \mathbf{S}(z | \mathbf{x}_\xi)^{-1}\mathbf{X}_2(z)\mathbf{e} = \mathbf{S}(z | \mathbf{x})^{-1}\mathbf{X}_2(z)\mathbf{e}\phi(z | \xi),$$

where

$$\phi(z | \xi) := \mathbf{e}^T(\mathbf{X}_1(z)\mathbf{R}(z | \xi) + \mathbf{X}_2(z)\mathbf{R}(z | \xi)^{-1})\mathbf{e}$$

is a scalar function. Since \mathbf{A} and \mathbf{B} do not depend on \mathbf{x} , the formula (5.14) implies that

$$\mathbf{f}(z | \mathbf{x}_\xi) = \mathbf{f}(z | \mathbf{x})\phi(z | \xi) \quad \text{and} \quad \mathbf{g}(z | \mathbf{x}_\xi) = \mathbf{g}(z | \mathbf{x})\phi(z | \xi). \quad (7.3)$$

Now, $\phi(z | \xi)$ is a sum of $e^{\pm b_j \xi z} \chi_{\Omega_{\ell,j}}(z)$ for $\ell = 1, 2$ and $j = 1, \dots, m$ with the certain choice of the sign \pm . Being careful with the sign and noting that $\Omega_{\ell,j}$ is in either of the two halves of the complex plane, we find that

$$|\phi(z | \xi)| \leq e^{-b\xi|\text{Re } z|}, \quad b := \min\{b_1, \dots, b_m\}, \quad (7.4)$$

for every $z \in \Omega$ and every $\xi > 0$. Since the sets in Ω are away from the imaginary axis by a positive distance, say d , (see Fig. 1 and 2), this implies that $|\phi(z | \xi)| \leq e^{-bd\xi}$ for every $z \in \Omega$ and every $\xi > 0$.

Let $T_1 = T_1(x_\xi)$ and $T_2 = T_2(x_\xi)$ be the Hilbert-Schmidt operators from Lemma 5.8. From (7.3)–(7.4) we find

$$\|H(x_\xi)\| \leq \|H(x_\xi)\|_1 \leq e^{-bd\xi} \|T_1(x)\|_2 \|T_2(x)\|_2$$

where the norms are the operator norm, the trace class norm, and the Hilbert-Schmidt norm, respectively. The norms $\|T_1(x)\|_2$ and $\|T_2(x)\|_2$ are finite and independent of ξ . Thus, from the inequality $\|\det(1 - H) - 1\|_1 \leq \|H\|_1 e^{1+\|H\|_1}$ for trace class operators (see [27, Theorem 3.4]), the claim on $\mathfrak{D} = \det(1 - H)$ then follows. On the other hand, from (2.9) and (7.3)–(7.4), we find

$$\max_{i,j=1,\dots,m} |(\Phi_1(x_\xi))_{i,j}| \leq e^{-2bd\xi} \|(1 - H(x_\xi))^{-1}\| \|f(\cdot | x)\|_{L^2(\Omega,\mu)} \|g(\cdot | x)\|_{L^2(\Omega,\mu)}.$$

and the result for Φ_1 follows. \square

Thus, in particular, the matrix functions q, p, r, s for the (periodic) KPZ fixed points, which solve various differential equations, decay exponentially to zero as $\xi \rightarrow \infty$. Using the above lemma, we prove Proposition 2.12.

Proof of Proposition 2.12. Let \mathfrak{D} be a cubic admissible determinant and q_i the matrix-valued function in (2.10) corresponding to the choice of subset $S = \{1, 2, \dots, i\}$. Then, from Proposition 2.10,

$$(\partial_{x_1} + \dots + \partial_{x_i}) \log \mathfrak{D} = -\text{Tr}(q_i).$$

Thus, for setting $x_\xi := x + \xi a$ with $a = (1, \dots, m)$ as in (2.29) and $\xi \in \mathbb{R}$, we have

$$\frac{d}{d\xi} \log \mathfrak{D}(x_\xi) = \sum_{k=1}^m k \partial_{x_k} \log \mathfrak{D}(x_\xi) = \sum_{i=1}^m (\partial_{x_1} + \dots + \partial_{x_i}) \log \mathfrak{D}(x_\xi) = - \sum_{i=1}^m \text{Tr} q_i(x_\xi). \quad (7.5)$$

By Lemma 2.4, the above term is also equal to $\sum_{i=1}^m \text{Tr}(s_i(x_\xi))$. Now, when \mathfrak{D} is either $\mathfrak{D}^{(\text{KPZ})}$ or $\mathfrak{D}^{(\text{per})}$, the last lemma shows that $\mathfrak{D}(x_\xi) \rightarrow 1$ and $\text{Tr} q_i(x_\xi) \rightarrow 0$ exponentially fast as $\xi \rightarrow \infty$. Thus, integrating (7.5) we obtain Proposition 2.12. \square

8. Adler and van Moerbeke PDE and a Non-equal Time Extension

Let H be a strongly cubic integrable operator which is also trace class. Assume that $1 - H$ is invertible. Set $\bar{M} = \log \det(1 - H)$ for the case when $m = 2$ and with the change of variables given in (3.7),

$$t_2 = -\frac{t}{3}, \quad y_2 = y, \quad x_1 = \frac{E + W}{2}, \quad x_2 = \frac{E - W}{2} - \frac{y^2}{t} \quad (8.1)$$

with $t_1 = -\frac{1}{3}$ and $y_1 = 0$. Then, M is a function of 4 variables, t, E, W , and y . We now discuss how we can obtain partial differential Eqs. (3.8) and (3.9) for M . As we discussed in Sect. 3.4, the special cases of these equations when $t = 1$ yield the equal-time differential equations for the Airy₂ process obtained by Adler and van Moerbeke [3] and Quastel and Remenik [26].

In the derivation of the differential equations for cubic integrable operators, we used Lax equations obtained from a Riemann-Hilbert problem. For the NLS, mKdV, and KP derivations, we used the Lax equations with respect to the parameters t, x, y , which in terms of (8.1) become Lax equations with respect to t, E, W , and y . For the derivation of Tracy-Widom type ODE, we also considered the Lax equation with respect to the spectral variable z for the case of strongly cubic integrable operators, and used the compatibility of z -Lax equation and x -Lax equation. In this section, we use the compatibility of the z -Lax equation with each of the t -, y -, E -, and W -Lax equations.

We start with an analysis of the z -Lax equation. The solution $\Phi(z)$ of the RHP in Sect. 4.1 has the asymptotic series $\Phi(z) \sim I + \frac{\Phi_1}{z} + \frac{\Phi_2}{z^2} + \dots$ as $z \rightarrow \infty$. The asymptotic series can be written uniquely as

$$\Phi(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\Phi_k^{(o)}}{z^k} \right) \exp \left(\sum_{k=1}^{\infty} \frac{\Phi_k^{(d)}}{z^k} \right) \quad (8.2)$$

for diagonal matrices $\Phi_k^{(d)}$ and off-diagonal matrices $\Phi_k^{(o)}$. Considering the $O(z^{-1})$ and $O(z^{-2})$ terms, we see that

$$\Phi_1 = \Phi_1^{(o)} + \Phi_1^{(d)} \quad \text{and} \quad \Phi_2 = \Phi_2^{(o)} + \Phi_1^{(o)} \Phi_1^{(d)} + \Phi_2^{(d)} + \frac{1}{2} (\Phi_1^{(d)})^2. \quad (8.3)$$

Consider the Lax equation (4.48) with respect to the spectral parameter z , where recall (2.4) that $\Psi(z) := \Phi(z)\Delta(z)$. We can write it as

$$\partial_z \Psi(z) = (z^2 Q_2 + z Q_1 + Q_0) \Psi(z) \quad (8.4)$$

where, thanks to (8.3), the coefficient matrices are

$$Q_2 = 3M_t, \quad Q_1 = 2M_y + 3[\Phi_1^{(o)}, M_t], \quad Q_0 = M_x + 2[\Phi_1^{(o)}, M_y] + 3[\Phi_2^{(o)}, M_t] - 3[\Phi_1^{(o)}, M_t]\Phi_1^{(o)},$$

and where we also recall (2.16) for the definition of M_t, M_y , and M_x . Inserting $\Psi(z) = \Phi(z)\Delta(z)$, the equation becomes, using (4.47),

$$\partial_z \Phi(z) = (z^2 Q_2 + z Q_1 + Q_0) \Phi(z) - \Phi(z) (3z^2 M_t + 2zM_y + M_x).$$

We insert the asymptotic series (8.2) and for each $k = 1, 2, \dots$ collect the coefficients of z^{-k} to obtain an equation for $\Phi_i^{(o)}$ and $\Phi_i^{(d)}$. It is useful to note that $\Phi_i^{(d)}, M_t, M_y$, and M_x are diagonal matrices and commute. We find, after inserting the formula of Q_0, Q_1, Q_2 , that for $k \geq 1$,

$$\begin{aligned} & 3[\Phi_{k+2}^{(o)}, M_t] + 2[\Phi_{k+1}^{(o)}, M_y] - 3[\Phi_k^{(o)}, M_t]\Phi_{k+1}^{(o)} - 3[\Phi_{k+1}^{(o)}, M_t]\Phi_k^{(o)} + [\Phi_k^{(o)}, M_x] \\ & - (2[\Phi_k^{(o)}, M_y] - 3[\Phi_k^{(o)}, M_t]\Phi_k^{(o)})\Phi_k^{(o)} - (k-1)\Phi_{k-1}^{(o)} \\ & = (k-1)\Phi_{k-1}^{(d)} + \sum_{j=2}^{k-1} (j-1)\Phi_{k-j}^{(o)}\Phi_{j-1}^{(d)}. \end{aligned} \quad (8.5)$$

Note that all matrices are of size 3×3 since $m = 2$. These equations are recursive equations. Note that $\Phi_{k+2}^{(o)}$ is the matrix of the highest index for the off-diagonal matrices and $\Phi_{k-1}^{(o)}$ is the matrix of the highest index for the diagonal matrices. It turns out that the recursion determines all matrices if we know $\Phi_1^{(o)}$ and some entries of $\Phi_2^{(o)}$.

Lemma 8.1. Suppose that a_1, \dots, a_{10} , are 10 functions of t, E, W, y . Let

$$\Phi_1^{(o)} = \begin{pmatrix} 0 & a_1 & a_3 \\ a_2 & 0 & a_5 \\ a_4 & a_6 & 0 \end{pmatrix}, \quad \Phi_2^{(o)} = \begin{pmatrix} 0 & a_7 & c_1 \\ a_8 & 0 & c_2 \\ a_9 & a_{10} & 0 \end{pmatrix},$$

where c_1, c_2 are functions of a_1, \dots, a_{10} given by

$$\begin{aligned} c_1 &= -\frac{a_3 a_9}{a_4} - t a_1 a_5 + \frac{t a_1 a_8}{a_4} + \frac{t a_2 a_7}{a_4} - \frac{2 y a_1 a_2}{a_4} + a_1 a_5 + \frac{a_2 a_3 a_6}{a_4} - \frac{a_1 a_8}{a_4} - \frac{a_2 a_7}{a_4}, \\ c_2 &= \frac{a_1 a_4 a_5}{a_6} + a_2 a_3 - \frac{a_1 a_8}{a_6} - \frac{a_2 a_7}{a_6} - \frac{a_5 a_{10}}{a_6} + \frac{2 y a_1 a_2}{t a_6} + \frac{2 y a_5}{t} - \frac{a_2 a_3}{t} + \frac{a_1 a_8}{a_6 t} + \frac{a_2 a_7}{a_6 t}. \end{aligned} \quad (8.6)$$

Then, $\Phi_n^{(o)}$, $n \geq 3$, and $\Phi_n^{(d)}$, $n \geq 1$, are determined uniquely from the recursion (8.5). Furthermore, all entries of $\Phi_n^{(o)}$ and $\Phi_n^{(d)}$ are Laurent polynomials of a_1, \dots, a_{10} .

Proof. Since M_t is a diagonal matrix, the commutator of any matrix W with M_t is of the form

$$[W, M_t] = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \quad (8.7)$$

and each $*$ is a constant multiple of the entry of W at the same spot.

Consider the case $k = 1$ of (8.5). The right-hand side of the equation is zero, so it becomes an equation of $\Phi_1^{(o)}$, $\Phi_2^{(o)}$ and $\Phi_3^{(o)}$ only,

$$\begin{aligned} 3[\Phi_3^{(o)}, M_t] + 2[\Phi_2^{(o)}, M_y] - 3[\Phi_1^{(o)}, M_t]\Phi_2^{(o)} - 3[\Phi_2^{(o)}, M_t]\Phi_1^{(o)} \\ + [\Phi_1^{(o)}, M_x] - (2[\Phi_1^{(o)}, M_y] - 3[\Phi_1^{(o)}, M_t]\Phi_1^{(o)})\Phi_1^{(o)} = 0. \end{aligned}$$

Since $\Phi_3^{(o)}$ appears only in the term $[\Phi_3^{(o)}, M_t]$, the diagonal entries of this (matrix) equation yield 3 scalar equations involving only the entries of $\Phi_1^{(o)}$ and $\Phi_2^{(o)}$. Two of them are (8.6) and they imply the third equation. Now if we consider the off-diagonal entries of the equation, from (8.7) we see that all 6 off-diagonal entries of $\Phi_3^{(o)}$ are Laurent polynomials of a_1, \dots, a_{10} . Thus, $\Phi_3^{(o)}$, which is an off-diagonal matrix, is determined.

We now prove the lemma using an induction on k . For $k \geq 2$, suppose that $\Phi_1^{(o)}, \dots, \Phi_{k+1}^{(o)}$, and $\Phi_1^{(d)}, \dots, \Phi_{k-2}^{(d)}$ are known and their entries are Laurent polynomials of a_1, \dots, a_{10} . It is enough to show that we can determine $\Phi_{k+2}^{(o)}$ and $\Phi_{k-1}^{(d)}$ from the induction hypothesis, and their entries are Laurent polynomials of the quantities of the hypothesis. These two matrices appear in the Eq. (8.5) only in $3[\Phi_{k+2}^{(o)}, M_t]$ and $(k-1)\Phi_{k-1}^{(d)}$. Thus, considering the diagonal entries of the equation, we can solve $\Phi_{k+2}^{(o)}$ in terms of the terms of the induction hypothesis, and the off-diagonal entries of the equation allows us to solve $[\Phi_{k+2}^{(o)}, M_t]$, and thus $\Phi_{k+2}^{(o)}$ from (8.7), in terms of the induction hypothesis. From the solution, we find that they are polynomials of the terms of the induction hypothesis, and thus the lemma is proved. \square

Consider the Lax equations (4.16) with respect to t , y , and x . With the choice (8.1) of the parameters, they become four equations,

$$\begin{aligned}\partial_E \Psi(z) &= \left(\frac{1}{2} z E_1 + \frac{1}{2} z E_2 + \frac{1}{2} C_{1,1} + \frac{1}{2} C_{1,2} \right) \Psi(z), \\ \partial_W \Psi(z) &= \left(\frac{1}{2} z E_1 - \frac{1}{2} z E_2 + \frac{1}{2} C_{1,1} - \frac{1}{2} C_{1,2} \right) \Psi(z), \\ \partial_y \Psi(z) &= \left(z^2 E_2 + z C_{1,2} - z \frac{2y}{t} E_2 + C_{2,2} - \frac{2y}{t} C_{1,2} \right) \Psi(z), \\ \partial_t \Psi(z) &= \left(-\frac{1}{3} z^3 E_2 - \frac{1}{3} z^2 C_{1,2} - \frac{1}{3} z C_{2,2} + z \frac{y^2}{t^2} E_2 - \frac{1}{3} C_{3,2} + \frac{y^2}{t^2} C_{1,2} \right) \Psi(z),\end{aligned}$$

where $C_{i,1}$ and $C_{i,2}$ are C_i in (4.17) with E replaced by E_1 and E_2 , respectively. For each equation, consider the compatibility condition with the z -Lax equation (8.4). If we consider the entries of these 4 equations (called zero-curvature equations; cf. the derivation of (4.20)) yield 40 equations of the form

$$\partial_E a_j = P_j^{(1)}(a), \quad \partial_W a_j = P_j^{(2)}(a), \quad \partial_y a_j = P_j^{(3)}(a), \quad \partial_t a_j = P_j^{(4)}(a),$$

for $j = 1, \dots, 10$, where $P_j^{(k)}(a)$ are Laurent polynomials. On the other hand, from Propositions 2.10 and 4.7, the partial derivatives of $M = \log \det(1 - H)$ are polynomials of Φ_1 and Φ_2 , and hence, by Lemma 8.1, they and their higher partial derivatives are Laurent polynomials of a_1, \dots, a_{10} .

Motivated by the formula of the PDE of Adler and van Moerbeke, and of Quastel and Remenik, we consider the 27 partial derivatives

$$\begin{aligned}\partial_{X_a} M, \quad \partial_{X_b} \partial_{X_c} M, \quad \partial_{X_d} \partial_{X_e} \partial_{X_f} M, \quad \partial_{X_h} \partial_{X_i} \partial_{X_j} \partial_{X_k} M, \quad \partial_{X_k} \partial_{X_l} M \partial_{X_m} \partial_{X_n} M, \\ \partial_y M, \quad \partial_y \partial_{X_o} M, \quad \partial_y^2 M, \quad \partial_t M, \quad \partial_t \partial_{X_p} M\end{aligned}$$

with the indices a, b, c, \dots being either 1 and 2, and $X_1 = E$, $X_2 = W$. These are Laurent polynomials of a_1, \dots, a_{10} involving 163 Laurent monomials. We then asked if there are linear relations between these 27 Laurent polynomials. We regarded each of 163 Laurent monomials as independent vectors and considered the vector space of Laurent polynomials spanned by them. Symbolic computations using Maple showed that the above 27 Laurent polynomials are not linearly independent and furthermore, the rank of their coefficient matrix is 23. As a result we obtain 4 linear relations of these Laurent polynomials. Two of them are (3.8) and (3.9). The other two are more complicated and we do not present them here.

Acknowledgements. The work of Baik was supported in part by NSF grant DMS-1954790. Silva acknowledges his current support by São Paulo Research Foundation under grants # 2019/16062-1 and # 2020/02506-2, and by Brazilian National Council for Scientific and Technological Development (CNPq) under grant # 315256/2020-6. Prokhorov was supported by NSF MSPRF grant DMS-2103354, NSF grant DMS-1928930, and RSF grant 22-11-00070. The part of the work was done while Prokhorov was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2021 semester.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest in the subject matter or materials discussed in this manuscript.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

1. Ablowitz, M.J., Clarkson, P.A.: Solitons, nonlinear evolution equations and inverse scattering, London Mathematical Society Lecture Note Series, vol. 149, Cambridge University Press, Cambridge, (1991). (MR1149378)
2. Adler, M., Cafasso, M., van Moerbeke, P.: Non-linear PDEs for gap probabilities in random matrices and KP theory. *Phys. D* **241**(23–24), 2265–2284 (2012). (MR2998127)
3. Adler, M., van Moerbeke, P.: PDEs for the joint distributions of the Dyson, Airy and sine processes. *Ann. Probab.* **33**(4), 1326–1361 (2005). (MR2150191 (2006g:60118))
4. Adler, M., van Moerbeke, P., Vanhaecke, P.: Moment matrices and multi-component KP, with applications to random matrix theory. *Commun. Math. Phys.* **286**(1), 1–38 (2009). (MR2470922)
5. Baik, J.: KPZ limit theorems. [arXiv:2206.14086](https://arxiv.org/abs/2206.14086)
6. Baik, J., Liu, Z.: Fluctuations of TASEP on a ring in relaxation time scale. *Commun. Pure Appl. Math.* **71**(4), 747–813 (2018). (MR3772401)
7. Baik, J., Liu, Z.: Multipoint distribution of periodic TASEP. *J. Am. Math. Soc.* **32**(3), 609–674 (2019). (MR3981984)
8. Baik, J., Liu, Z., Silva, G.L.F.: Limiting one-point distribution of periodic TASEP. *Ann. Inst. Henri Poincaré Probab. Stat.* **58**(1), 248–302 (2022). (MR4374678)
9. Bertola, M., Cafasso, M.: Riemann–Hilbert approach to multi-time processes: the Airy and the Pearcey cases. *Phys. D* **241**(23–24), 2237–2245 (2012). (MR2998125)
10. Borodin, A.: Riemann–Hilbert problem and the discrete Bessel kernel. *Int. Math. Res. Notices* **9**, 467–494 (2000). (MR1756945)
11. Dauvergne, D., Ortman, J., Virág, B.: The directed landscape. [arXiv:1812.00309](https://arxiv.org/abs/1812.00309)
12. Deift, P.: Integrable operators, Differential operators and spectral theory, 1999, pp. 69–84. (MR1730504)
13. Fokas, A.S., Its, A.R., Kapaev, A. A., Yu, V.: Novokshenov, Painlevé transcendents, Mathematical Surveys and Monographs, vol. 128, American Mathematical Society, Providence, RI, 2006. The Riemann–Hilbert approach. (MR2264522)
14. Harnad, J., Balogh, F.: Tau functions and their applications, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge (2021). (MR4411375)
15. Its, A.R., Izergin, A.G., Korepin, V.E., Slavnov, N.A.: Differential equations for quantum correlation functions. In: Proceedings of the Conference on Yang–Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory, 1990, pp. 1003–1037. (MR1064758)
16. Its, A.R.: The Riemann–Hilbert problem and integrable systems. *Notices Am. Math. Soc.* **50**(11), 1389–1400 (2003). (MR2011605)
17. Johansson, K.: Two time distribution in Brownian directed percolation. *Commun. Math. Phys.* **351**(2), 441–492 (2017). (MR3613511)
18. Johansson, K.: The two-time distribution in geometric last-passage percolation. *Probab. Theory Relat. Fields* **175**(3–4), 849–895 (2019). (MR4026607)
19. Johansson, K., Rahman, M.: Multitime distribution in discrete polynuclear growth. *Commun. Pure Appl. Math.* **74**(12), 2561–2627 (2021). (MR4373163)
20. V. G. Kac, J. W. van de Leur, The n-component KP hierarchy and representation theory, 2003, pp. 3245–3293. Integrability, topological solitons and beyond. (MR2006751)
21. Krajenbrink, A., Le Doussal, P.: Inverse scattering of the Zakharov–Shabat system solves the weak noise theory of the Kardar–Parisi–Zhang equation. *Phys. Rev. Lett.* **127**(6), 8 (2021). (MR4312166)
22. Liao, Y.: Multi-point distribution of discrete time periodic TASEP. *Probab. Theory Relat. Fields* **182**(3–4), 1053–1131 (2022). (MR4408510)

23. Liu, Z.: Multipoint distribution of TASEP. *Ann. Probab.* **50**(4), 1255–1321 (2022). (**MR4420420**)
24. Matetski, K., Quastel, J., Remenik, D.: The KPZ fixed point. *Acta Math.* **227**(1), 115–203 (2021). (**MR4346267**)
25. Prähofer, M., Spohn, H.: Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* **108**(5–6), 1071–1106 (2002). (**MR1933446**)
26. Quastel, J., Remenik, D.: KP governs random growth off a 1-dimensional substrate, *Forum Math. Pi* 10 (2022), Paper No. e10, 26. (**MR4413350**)
27. Simon, B.: Trace ideals and their applications, Second, *Mathematical Surveys and Monographs*, vol. 120, American Mathematical Society, Providence, RI, 2005. (**MR2154153**) (2006f:47086)
28. Teo, L.P.: The multicomponent KP hierarchy: differential Fay identities and Lax equations. *J. Phys. A* **44**(22), 225201 (2011)
29. Tracy, C.A., Widom, H.: Level-spacing distributions and the Airy kernel. *Commun. Math. Phys.* **159**(1), 151–174 (1994). (**MR1257246 (95e:82003)**)
30. Tracy, C.A., Widom, H.: A system of differential equations for the Airy process. *Electron. Commun. Probab.* **8**, 93–98 (2003). (**MR1987098**)
31. Tracy, C.A., Widom, H.: Matrix kernels for the Gaussian orthogonal and symplectic ensembles. *Ann. Inst. Fourier (Grenoble)* **55**(6), 2197–2207 (2005). (**MR2187952 (2006j:82046)**)
32. Wang, D.: A PDE for the multi-time joint probability of the Airy process. *Phys. D* **238**(8), 819–833 (2009). (**MR2522977 (2010f:60020)**)

Communicated by K. Johansson