

# Distributed Control of Descriptor Networks: A Convex Procedure for Augmented Sparsity

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**Abstract**—For networks of systems, with possibly improper transfer function matrices, we present a design framework that enables  $\mathcal{H}_\infty$  control, while imposing sparsity constraints on the coprime factors of the controller. We propose a convex and iterative optimization procedure with guaranteed convergence to obtain distributed controllers. By exploiting the robustness-oriented nature of our proposed approach, we provide the means to obtain sparse representations of our control laws that may not be directly supported by the nominal model of the network.

**Index Terms**—Convex optimization, distributed processes, descriptor systems, sparse  $\mathcal{H}_\infty$  control.

## I. INTRODUCTION

### A. Motivation

When faced with a distributed control problem, one notices an acute lack of dedicated numerical tools, if compared with the classical centralized design context. Several computational methods, such as those proposed in [1], [2], [3], and [4], aim to exploit specialized techniques, in order to mitigate the numerical complexities inherent to distributed control.

Notably, previous efforts [5] have sought to enforce sparsity constraints directly upon a finite impulse response (FIR) approximation of the Youla parameter, under certain restrictive assumptions, such as quadratic invariance and strong stabilizability (see [6]). However, the technique proposed in [5, Sec. 5] cannot cope with enforcing sparsity patterns upon nonsparse *affine expressions* of the Youla parameter. These issues were tackled in [7], with the introduction of the framework dubbed *system-level synthesis* (SLS). Yet the focus on discrete-time systems meant that other architectures, such as the *network realization function* (NRF) representations discussed in [8] and [9], have been overshadowed by the FIR approximation methods from the SLS framework.

### B. Article Structure and Contributions

In this article, we propose tractable techniques and numerical procedures for the NRF-based framework formalized in [9], which offers distributed control laws in *both* continuous time and discrete time, without needing to communicate any internal states, i.e., plant or controller

states (see [9, Sec. IV] for a comparison with the SLS framework), thus promoting scalable control laws for large-scale networks. In Section II, we cover a set of preliminary notions, with our paper's problem statement forming Section III-A. Our contributions are structured via the subsequent sections and are listed as follows.

- 1) In Section III-B, we show how to impose sparsity in the NRF formalism and how it reduces to a model-matching problem that is solved via reliable procedures [10], [11], [12].
- 2) In Section III-C, we extend the robust stabilization approach from [13] to the distributed NRF-based setting.
- 3) In Section IV, we show how to particularize the convex and iterative procedure (with guaranteed convergence) from [14] to obtain robust NRF-based implementations.
- 4) In Section V, we consider a generalization of the network in [8], and we also show<sup>1</sup> how to employ our robustness-oriented approach, in order to retrieve the same sparse control architecture as in [8] for a more general case.

Finally, Section VI concludes this article.

## II. PRELIMINARIES

### A. Nomenclature and Definitions

Let  $\mathbb{C}$ ,  $\mathbb{C}^-$ ,  $j\mathbb{R}$ , and  $\mathbb{B}$  denote the complex plane, the open left-half plane, the imaginary axis, and the set  $\{0, 1\}$ , respectively. Let  $\mathcal{M}^{p \times m}$  stand for the set of all  $p \times m$  matrices having entries in a set denoted  $\mathcal{M}$ . We also denote by  $P \succ 0$  the fact that  $P \in \mathbb{R}^{q \times q}$  is positive definite and by  $\bar{\sigma}(Z)$  the maximum singular value of  $Z \in \mathbb{C}^{p \times m}$ . For any  $M \in \mathcal{M}^{p \times m}$ ,  $M^\top$  is its transpose. Let  $\text{Ker}(M)$  denote the null space of  $M \in \mathcal{M}^{p \times m}$ , and let  $\|Z\|_*$  denote the sum of the singular values belonging to  $Z \in \mathbb{C}^{p \times m}$ , which is termed the *nuclear norm*. The operator  $\otimes$  denotes the Kronecker product between any two matrices. We define the vectorization of  $M \in \mathcal{M}^{p \times m}$  as  $\text{vec}(M) := v \in \mathcal{M}^{pm \times 1}$ , where  $v_{i+(j-1)p} = M_{ij}$ , along with the diagonalization of  $M$  by  $\text{diag}(M) := V \in \mathcal{M}^{pm \times pm}$ , where  $V_{ii} = (\text{vec}(M))_i$ , for  $i \in 1 : pm$ , and  $V_{ij} = 0 \forall i \neq j$ .

For  $M_i \in \mathcal{M}^{p_i \times m_i}$ , with  $i \in 1 : \ell$ , and a natural number  $g$ , we define the block-diagonal concatenation operator by

$$\mathcal{D}(M_1, \dots, M_\ell) := \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_\ell \end{bmatrix} \in \mathcal{M}^{\sum_{i=1}^\ell p_i \times \sum_{j=1}^\ell m_j} \quad \text{and the}$$

block-diagonal repetition of  $M_i$  by  $\mathcal{D}_g(M_i) := \begin{bmatrix} M_i & & \\ & \ddots & \\ & & M_i \end{bmatrix} \in$

$\mathcal{M}^{gp_i \times gm_i}$ . For any  $R \in \mathcal{M}^{q \times q}$ , we denote its symmetric part by  $\text{sym}(R) := \frac{1}{2}(R + R^\top) = \text{sym}(R^\top)$  and its diagonal part by  $R_{ij}^{\text{diag}} := \begin{cases} R_{ij}, & i = j \\ 0, & i \neq j \end{cases}, \forall i, j \in 1 : q$ .

The matrix polynomial  $A - sE$  is called a pencil, with square ones that have  $\det(A - sE) \neq 0$  being termed *regular*. A regular

Manuscript received 7 December 2022; accepted 21 May 2023. Date of publication 4 August 2023; date of current version 5 December 2023. The work of Andrei Sperilă, Cristian Oară, and Bogdan D. Ciubotaru was supported by the Ministry of Research, Innovation and Digitization, CCCDI-UEFISCDI, under Grant PN-III-P2-2.1-PED-2021-1626, within PNCDI III. The work of Șerban Sabău was supported by the National Science Foundation Faculty Early Career Development Program under Award 1653756. Recommended by Associate Editor Z. Li. (*Corresponding author: Șerban Sabău.*)

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Digital Object Identifier 10.1109/TAC.2023.3301949

<sup>1</sup>All the implementations being compared in this article are available at the following link: <https://github.com/AndreiSperila/CONPRAS>

pencil without finite generalized eigenvalues in  $\mathbb{C} \setminus \mathbb{C}^-$  and without infinite generalized eigenvalues with partial multiplicities greater than 1 (see [15]) is called *admissible*. Let  $\Lambda(A - sE)$  be the collection of generalized eigenvalues (both finite and infinite) belonging to the regular pencil  $A - sE$ .

In this article, we will focus on systems described in the frequency domain by transfer function matrices (TFMs) of type  $(G(s))_{ij} = \frac{a_{ij}(s)}{b_{ij}(s)}$ , with  $a_{ij}(s)$  and  $b_{ij}(s)$  polynomials with coefficients in  $\mathbb{R}$ ,  $i \in 1:p$ ,  $j \in 1:m$ . We denote the set of all such TFMs with  $m$  inputs and  $p$  outputs by  $\mathcal{R}^{p \times m}$ , with  $\mathcal{R}_p^{p \times m}$  being the subset of proper TFMs ( $\deg a_{ij} \leq \deg b_{ij}$ ,  $\forall i \in 1:p$ ,  $j \in 1:m$ ). Let  $B \in \mathbb{R}^{p \times m}$ , which we use to express  $\mathcal{S}_B := \{G \in \mathcal{R}^{p \times m} | B_{ij} = 0 \Rightarrow G_{ij} \equiv 0, \forall i \in 1:p, \forall j \in 1:m\}$ . Note that  $G \in \mathcal{S}_B \iff (I - \text{diag}(B))\text{vec}(G) \equiv 0$ . We also define the restriction of  $\mathcal{S}_B$  to proper TFMs  $\tilde{\mathcal{S}}_B := \mathcal{S}_B \cap \mathcal{R}_p^{p \times m}$ .

A TFM without poles (see [16, Sec. 6.5.3]) located in  $\{\mathbb{C} \setminus \mathbb{C}^-\} \cup \{\infty\}$  is called *stable*. Let  $\mathcal{RH}_\infty$  denote the set of real-rational and stable TFMs, with the  $\mathcal{H}_\infty$  norm of any  $G \in \mathcal{RH}_\infty$  being given by  $\|G\|_\infty := \sup_{s \in j\mathbb{R}} \bar{\sigma}(G(s))$ . The systems considered in this article are usually represented in the time domain by differential and algebraic equations

$$E \frac{d}{dt} x(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) + Du(t). \quad (1b)$$

The dimension of the regular pencil  $A - sE$  and that of  $x$ , the vector that contains the realization's *descriptor variables*, is called the order of the realization (1a), (1b). If its order is the smallest out of all others of its kind, a realization is called *minimal* (see [17, Sec. 2.4]). Moreover, we have

$$G(s) = C(sE - A)^{-1}B + D =: \left[ \begin{array}{c|c} A - sE & B \\ \hline C & D \end{array} \right]. \quad (2)$$

Let the matrix  $S_\infty$  span  $\text{Ker} E$ . A pair  $(A - sE, B)$  or a realization (2) for which  $[A - sE \ B]$  has full row rank  $\forall s \in \mathbb{C} \setminus \mathbb{C}^-$  and  $[E \ AS_\infty \ B]$  has full row rank is called *strongly stabilizable*. By [18, Th. 1.1], strong stabilizability is equivalent to the existence of a matrix  $F$ , called an *admissible feedback*, such that the pencil  $A + BF - sE$  is admissible. By duality, a pair  $(C, A - sE)$  or realization (2) is deemed *strongly detectable* if  $(A^\top - sE^\top, C^\top)$  is strongly stabilizable.

Let both  $E_r$  and  $D_r^\top D_r$  be invertible and consider

$$\begin{aligned} E_r^\top X_r A_r + A_r^\top X_r E_r + C_r^\top C_r - (E_r^\top X_r B_r + C_r^\top D_r) \\ \times (D_r^\top D_r)^{-1} (B_r^\top X_r E_r + D_r^\top C_r) = 0 \end{aligned} \quad (3)$$

the generalized continuous-time algebraic Riccati equation (GCARE) (see [19]). A symmetric solution  $X_r$  of the GCARE is called *stabilizing* if  $F_r := -(D_r^\top D_r)^{-1} (B_r^\top X_r E_r + D_r^\top C_r)$  is a stabilizing feedback, i.e.,  $\Lambda(A_r + B_r F_r - sE_r) \subset \mathbb{C}^-$ .

## B. Parameterization of All Stabilizing Controllers

To obtain a tractable parameterization for NRF-based control laws, we employ the class of *all* controllers, which stabilize a network whose TFM  $G^n \in \mathcal{R}^{(p_u+p) \times (m_u+m)}$  is given by

$$G^n = \left[ \begin{array}{c|c} G_{11}^n & G_{12}^n \\ \hline G_{21}^n & G_{22}^n \end{array} \right] = \left[ \begin{array}{c|c|c} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array} \right] \quad (4)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $D_{11} \in \mathbb{R}^{p_u \times m_u}$ ,  $D_{22} \in \mathbb{R}^{p \times m}$ , and all other constant matrices have appropriate dimensions. Under certain assumptions of strong stabilizability and detectability, the aforementioned class coincides with that of the controllers, which render the closed-loop configuration from Fig. 1 well posed, i.e.,  $\det(I - G_{22}^n K) \neq 0$ , and internally stable, i.e., all the TFMs from  $w_1$  and  $w_2$  to  $u_1$ ,  $u_2$ ,  $y_1$ , and  $y_2$  are stable. We now state an extension of the Youla parameterization,

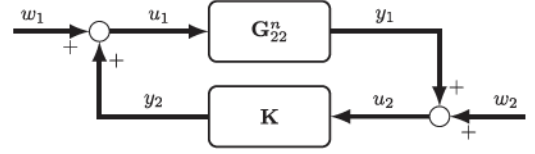


Fig. 1. Closed-loop configuration.

for a class of systems having possibly improper TFMs, by combining the notions from [20, Secs. 4.1 and 4.2].

**Theorem 2.1:** Let  $G^n \in \mathcal{R}^{(p_u+p) \times (m_u+m)}$  be given as in (4), with  $(A - sE, B_2)$  strongly stabilizable and  $(C_2, A - sE)$  strongly detectable. Let  $(N, \tilde{N}, M, \tilde{M}, X, \tilde{X}, Y, \tilde{Y})$  be a doubly coprime factorization (DCF) of  $G_{22}^n = NM^{-1} = \tilde{M}^{-1}\tilde{N}$  over  $\mathcal{RH}_\infty$ , with all eight TFMs being stable and satisfying

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (5)$$

Then, we have the following.

a) A DCF over  $\mathcal{RH}_\infty$  can be obtained, via (4), by

$$\left[ \begin{array}{c|c|c} \tilde{Y} & -\tilde{X} & \\ \hline F & I & 0 \\ \hline C_2 & -D_{22} & I \end{array} \right] := \left[ \begin{array}{c|c|c} A_H - sE & -B_2 - HD_{22} & H \\ \hline F & I & 0 \\ \hline C_2 & -D_{22} & I \end{array} \right] \quad (6a)$$

$$\left[ \begin{array}{c|c|c} M & X & \\ \hline F & I & 0 \\ \hline C_2 + D_{22}F & D_{22} & I \end{array} \right] := \left[ \begin{array}{c|c|c} A_F - sE & B_2 & -H \\ \hline F & I & 0 \\ \hline C_2 + D_{22}F & D_{22} & I \end{array} \right] \quad (6b)$$

with both the pencils  $A_H - sE := A + HC_2 - sE$  and  $A_F - sE := A + B_2F - sE$  being admissible.

b) The class of all stabilizing controllers is given by

$$K = (X + MQ)(Y + NQ)^{-1} = (\tilde{Y} + Q\tilde{N})^{-1}(\tilde{X} + Q\tilde{M}) \quad (7)$$

for all  $Q \in \mathcal{RH}_\infty^{m \times p}$ , which ensure that  $\det(Y + NQ) \neq 0$  and  $\det(\tilde{Y} + Q\tilde{N}) \neq 0$ .

c) For a stabilizing  $K$  given by a DCF over  $\mathcal{RH}_\infty$  of  $G_{22}^n$ ,

$$G_{CL} = \mathcal{F}_\ell(G^n, K) := G_{11}^n + G_{12}^n K(I - G_{22}^n K)^{-1} G_{21}^n$$

is expressed affinely in terms of  $Q$  from (7) by the identity  $G_{CL} = T_1 + T_2 QT_3$ . Given a realization of the employed DCF, as in (6a) and (6b), we have

$$\begin{aligned} T_1 &:= G_{11}^n + G_{12}^n X \tilde{M} G_{21}^n \\ &= \left[ \begin{array}{c|c|c} A_F - sE & -B_2 F & B_1 \\ \hline 0 & A_H - sE & B_1 + HD_{21} \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} \end{array} \right] \end{aligned} \quad (8a)$$

$$T_2 := G_{12}^n M = \left[ \begin{array}{c|c} A_F - sE & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array} \right] \quad (8b)$$

$$T_3 := \tilde{M} G_{21}^n = \left[ \begin{array}{c|c} A_H - sE & B_1 + HD_{21} \\ \hline C_2 & D_{21} \end{array} \right]. \quad (8c)$$

**Remark 2.1:** The two admissible feedback  $F$  and  $H$  can always be chosen via the two-step stabilization algorithm from [18]. Since  $A_H - sE$  and  $A_F - sE$  are admissible, the TFMs from (6a), (6b), and (8a)–(8c) are all stable and, thus, proper. State-space realizations for these TFMs can be obtained via the residualization procedure mentioned in [21, Sec. 3].

### III. THEORETICAL RESULTS

#### A. Problem Statement

The results presented in this article tackle the problem of obtaining *sparse* and *robustly stabilizing* control laws of type

$$\mathbf{u}_i = \sum_{j=1}^m \Phi_{ij} \mathbf{u}_j + \sum_{k=1}^p \Gamma_{ik} \mathbf{y}_k, \quad \Phi_{ii} \equiv 0 \quad \forall i \in 1:m \quad (9)$$

discussed in [9]. More specifically, we aim to impose  $\Gamma \in \hat{\mathcal{S}}_{\mathcal{X}}$  and  $\Phi \in \hat{\mathcal{S}}_{\mathcal{Y}}$ , for some binary matrices  $\mathcal{X} \in \mathbb{B}^{m \times p}$  and  $\mathcal{Y} \in \mathbb{B}^{m \times m}$ , with  $\mathcal{Y}^{\text{diag}} = 0$ , and to have the control laws from (9) stabilize all the network models  $\mathbf{G}_{\Delta} \in \mathcal{C}_{\mathcal{G}}^{\epsilon}$ , where the class  $\mathcal{C}_{\mathcal{G}}^{\epsilon}$  is of the type discussed in [22], due to its generality.

In the following, we show that this problem reduces to

$$\left\| \hat{\mathbf{T}}_1 + \sum_{i=1}^q \mathbf{x}_i \hat{\mathbf{T}}_{2i} \right\|_{\infty} < 1, \quad \mathbf{x}_i \in \mathcal{RH}_{\infty}^{1 \times 1}, \quad i \in 1:q \quad (10)$$

with  $\hat{\mathbf{T}}_1, \hat{\mathbf{T}}_{2i} \in \mathcal{RH}_{\infty}$ ,  $\forall i \in 1:q$ , being expressed in terms of (2) and (6a) and (6b). Finally, we particularize convex relaxation-based procedures [14] available in the literature to solve (10), and we compose  $(\Phi, \Gamma)$  from the obtained  $\mathbf{x}_i \in \mathcal{RH}_{\infty}^{1 \times 1}$ ,  $i \in 1:q$ .

#### B. Parameterization of NRF-Based Control Laws

Here, we show how the problem of obtaining the *sparse* and *stabilizing* distributed control laws of type (9) can be reduced to a readily solvable model-matching problem. As discussed in [9, Sec. III], this is primarily done by factorizing a stabilizing controller from the class expressed in Theorem 2.1 as  $\mathbf{K} = (\mathbf{I} - \Phi)^{-1} \Gamma$ , where  $(\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})^{\text{diag}}$  and  $(\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})$  have proper inverses and the NRF pair  $(\Phi, \Gamma)$  is obtained as

$$\Phi := \mathbf{I} - ((\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})^{\text{diag}})^{-1} (\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}}) \in \mathcal{R}_p^{m \times m} \quad (11a)$$

$$\Gamma := ((\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})^{\text{diag}})^{-1} (\tilde{\mathbf{X}} + \mathbf{Q}\tilde{\mathbf{M}}) \in \mathcal{R}_p^{m \times p}. \quad (11b)$$

**Remark 3.1:** When the realization of  $\mathbf{G}_{22}^{\epsilon} \in \mathcal{R}^{p \times m}$  (not necessarily proper) from (4) is strongly stabilizable and detectable, the guarantees of closed-loop internal stability and of scalability showcased in [9, Sec. III] for control laws of type (9) will also hold. Thus, since all the closed-loop transfers are stable and since the analogues of [23, Lemmas 5.2 and 5.3] (formulated for descriptor systems) are in effect, then the descriptor variables of both the plant and the controller's NRF-based implementation (along with their output signals in closed-loop interconnection) will be bounded and will tend to 0, when evolving freely from any finite initial conditions.

With the stability guarantees of (9) clarified in Remark 3.1, we now focus on imposing sparsity patterns on the  $(\Phi, \Gamma)$  pair. The following result offers a characterization of the stable Youla parameters, which, for a given DCF over  $\mathcal{RH}_{\infty}$ , produce the desired sparsity structure for the NRF pair in (11a) and (11b).

**Proposition 3.1:** Let  $\mathbf{G} \in \mathcal{R}^{p \times m}$  be given by a DCF over  $\mathcal{RH}_{\infty}$  (6a) and (6b); let  $\mathcal{X} \in \mathbb{B}^{m \times p}$  and  $\mathcal{Y} \in \mathbb{B}^{m \times m}$ , with  $\mathcal{Y}^{\text{diag}} = \mathbf{I}$ . Define  $F_{\mathcal{X}} := \mathbf{I} - \text{diag}(\mathcal{X})$  and  $F_{\mathcal{Y}} := \mathbf{I} - \text{diag}(\mathcal{Y})$ . If there exist  $\mathbf{Q}_0 \in \mathcal{RH}_{\infty}^{m \times p}$  and  $\hat{\mathbf{Q}} \in \mathcal{RH}_{\infty}^{m \times p}$  satisfying

$$\begin{bmatrix} F_{\mathcal{X}}(\tilde{\mathbf{M}}^{\top} \otimes \mathbf{I}) \\ F_{\mathcal{Y}}(\tilde{\mathbf{N}}^{\top} \otimes \mathbf{I}) \end{bmatrix} \text{vec}(\mathbf{Q}_0) + \begin{bmatrix} F_{\mathcal{X}} \text{vec}(\tilde{\mathbf{X}}) \\ F_{\mathcal{Y}} \text{vec}(\tilde{\mathbf{Y}}) \end{bmatrix} \equiv 0 \quad (12a)$$

$$\text{vec}(\hat{\mathbf{Q}}) \in \text{Ker} \begin{bmatrix} F_{\mathcal{X}}(\tilde{\mathbf{M}}^{\top} \otimes \mathbf{I}) \\ F_{\mathcal{Y}}(\tilde{\mathbf{N}}^{\top} \otimes \mathbf{I}) \end{bmatrix} \quad (12b)$$

$$\det((\tilde{\mathbf{Y}} + (\mathbf{Q}_0 + \hat{\mathbf{Q}})\tilde{\mathbf{N}})(\infty)) \neq 0 \quad (12c)$$

$$\det((\tilde{\mathbf{Y}} + (\mathbf{Q}_0 + \hat{\mathbf{Q}})\tilde{\mathbf{N}})^{\text{diag}}(\infty)) \neq 0 \quad (12d)$$

then the controller in (7), formed via the employed DCF over  $\mathcal{RH}_{\infty}$  of type (6a) and (6b) and via  $\mathbf{Q} := \mathbf{Q}_0 + \hat{\mathbf{Q}}$ , admits an NRF implementation of (11a) and (11b) with  $\Gamma \in \hat{\mathcal{S}}_{\mathcal{X}}$  and  $\Phi \in \hat{\mathcal{S}}_{(\hat{\mathcal{Y}} - \mathbf{I})}$ .

*Proof:* See the Appendix. ■

**Remark 3.2:** Equation (12a) can be solved for a stable  $\text{vec}(\mathbf{Q}_0)$ , as shown in [10]. Moreover, a least order solution can be obtained by employing the generalized minimum cover algorithm from [12]. A benefit of this approach is that it computes a (stable) basis for  $\text{Ker} \begin{bmatrix} F_{\mathcal{X}}(\tilde{\mathbf{M}}^{\top} \otimes \mathbf{I}) \\ F_{\mathcal{Y}}(\tilde{\mathbf{N}}^{\top} \otimes \mathbf{I}) \end{bmatrix}$ . Alternatively, a stable basis of least degree can be obtained as in [11].

**Remark 3.3:** Selecting a  $\mathbf{Q}$ , which ensures that  $\det((\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})(\infty)) \neq 0$ , thus guaranteeing that the TFM of the controller is well posed, can be done numerically by using the fact that

$$\begin{aligned} \det(\tilde{\mathbf{Y}}(\infty) + \mathbf{Q}(\infty)\tilde{\mathbf{N}}(\infty)) \neq 0 &\iff \\ \iff (\tilde{\mathbf{Y}}(\infty) + \mathbf{Q}(\infty)\tilde{\mathbf{N}}(\infty))^{\top} (\tilde{\mathbf{Y}}(\infty) + \mathbf{Q}(\infty)\tilde{\mathbf{N}}(\infty)) \succ 0. \end{aligned} \quad (13)$$

To ensure that  $\det((\tilde{\mathbf{Y}} + \mathbf{Q}\tilde{\mathbf{N}})^{\text{diag}}(\infty)) \neq 0$ , we first denote by  $\mathbf{e}_i$  the  $i$ th vector of the canonical basis of  $\mathbb{R}^{m \times 1}$  and impose that

$$\begin{aligned} \mathbf{e}_i^{\top} (\tilde{\mathbf{Y}}(\infty) + \mathbf{Q}(\infty)\tilde{\mathbf{N}}(\infty))^{\top} \mathbf{e}_i \\ \times \mathbf{e}_i^{\top} (\tilde{\mathbf{Y}}(\infty) + \mathbf{Q}(\infty)\tilde{\mathbf{N}}(\infty)) \mathbf{e}_i \succ 0 \quad \forall i \in 1:m. \end{aligned} \quad (14)$$

The bilinear matrix inequalities in (13) and (14) will be convexified and solved iteratively via the procedure given in Section IV.

#### C. Robust Stabilization and Augmented Sparsity

In this subsection, we show how to obtain a controller of type (7) whose NRF implementation (9) stabilizes all the network models  $\mathbf{G}_{\Delta}$  in a class  $\mathcal{C}_{\mathcal{G}}^{\epsilon}$  and how this technique can be used to obtain a sparse control architecture. However, before this, we begin by defining the aforementioned class of TFMs.

The class  $\mathcal{C}_{\mathcal{G}}^{\epsilon}$ , introduced in Section III-A, is expressed in terms of a stable right coprime factorization (RCF) of  $\mathbf{G} = \tilde{\mathbf{N}}\tilde{\mathbf{M}}^{-1} \in \mathcal{R}^{p \times m}$ , i.e.,  $\tilde{\mathbf{N}}, \tilde{\mathbf{M}} \in \mathcal{RH}_{\infty}$  and  $\exists \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathcal{RH}_{\infty}$  so that  $\tilde{\mathbf{Y}}\tilde{\mathbf{M}} - \tilde{\mathbf{X}}\tilde{\mathbf{N}} = \mathbf{I}$ , which is additionally normalized, i.e.,  $\tilde{\mathbf{N}}^{\top}(-s)\tilde{\mathbf{N}}(s) + \tilde{\mathbf{M}}^{\top}(-s)\tilde{\mathbf{M}}(s) = \mathbf{I}$ . With any (see [22]) such stable normalized right coprime factorization (NRCF) and  $\epsilon \in (0, 1]$ , we define

$$\begin{aligned} \mathcal{C}_{\mathcal{G}}^{\epsilon} := \left\{ (\hat{\mathbf{N}} + \Delta_{\hat{\mathbf{N}}}) (\hat{\mathbf{M}} + \Delta_{\hat{\mathbf{M}}})^{-1}, \Delta_{\hat{\mathbf{N}}}, \Delta_{\hat{\mathbf{M}}} \in \mathcal{RH}_{\infty} \right. \\ \left. \det(\hat{\mathbf{M}} + \Delta_{\hat{\mathbf{M}}}) \neq 0, \left\| \begin{bmatrix} \Delta_{\hat{\mathbf{N}}}^{\top} & \Delta_{\hat{\mathbf{M}}}^{\top} \end{bmatrix}^{\top} \right\|_{\infty} < \epsilon \right\}. \end{aligned} \quad (15)$$

Clearly, in order to manipulate  $\mathcal{C}_{\mathcal{G}}^{\epsilon}$ , we must first obtain a stable NRCF of  $\mathbf{G}$ . While (6b) readily provides a stable RCF of  $\mathbf{G}$ , a stable NRCF can be obtained via the following result.

**Lemma 3.1:** Let  $E_r$  be an invertible matrix, and let also  $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$ . Let the TFM

$$\begin{bmatrix} \mathbf{N}^{\top} & \mathbf{M}^{\top} \end{bmatrix}^{\top} = \left[ \begin{array}{c|c} A_r - sE_r & B_r \\ \hline C_r & D_r \end{array} \right] \in \mathcal{RH}_{\infty}^{(p+m) \times m} \quad (16)$$

designate a stable RCF of  $\mathbf{G} = \mathbf{N}\mathbf{M}^{-1} \in \mathcal{R}^{p \times m}$ , and let  $H_r \in \mathbb{R}^{m \times m}$  be invertible and satisfy  $H_r^{\top} H_r = D_r^{\top} D_r$ . Then, we have the following.

a) The GCARE from (3) has a symmetric stabilizing solution,  $X_r$ , along with a stabilizing feedback,  $F_r$ .

b) For  $\mathbf{G}_0 := \left[ \begin{array}{c|c} A_r - sE_r & B_r \\ \hline -H_r F_r & H_r \end{array} \right]$ , we get that  $[\hat{\mathbf{N}}^{\top} \quad \hat{\mathbf{M}}^{\top}]^{\top} := [\mathbf{N}^{\top} \quad \mathbf{M}^{\top}]^{\top} \mathbf{G}_0^{-1}$  designates a stable NRCF of  $\mathbf{G}$ .

*Proof:* For point (a), see the Appendix. Point (b) is precisely [21, Proposition 1]. ■

Having now the ability to express the TFMs that make up (15), we turn our attention to characterizing stabilizing controllers whose NRF

implementations of type (9) stabilize all the TFMs in  $\mathcal{C}_G^\epsilon$ , for a given  $\epsilon \in (0, 1]$ . The following result is central to this section and offers the means to do just so.

**Theorem 3.1:** Let  $G \in \mathcal{R}^{p \times m}$  be given by a strongly stabilizable and detectable realization (2), and let  $F$  ensure that  $A + BF - sE$  is admissible. Let also  $G = NM^{-1}$  be the stable RCF induced by  $F$  as in (6b), and for which a realization as in (16) is obtained (recall Remark 2.1), having  $E_r$  invertible and  $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$ . Let  $F_r$  be the stabilizing feedback of the GCARE from (3), and let  $\epsilon \in (0, 1]$  along with  $H_r \in \mathbb{R}^{m \times m}$  invertible, such that  $H_r^\top H_r = D_r^\top D_r$ . Then, we have the following.

- a) There exists a class of stabilizing controllers  $K \in \mathcal{R}^{m \times p}$ , based upon a DCF over  $\mathcal{RH}_\infty$  of  $T_{22}^\epsilon$ , for the system

$$\begin{aligned} T^\epsilon &:= \begin{bmatrix} 0 & -\epsilon \tilde{M}^{-1} & \epsilon \tilde{M}^{-1} \\ I & -G & G \end{bmatrix} = \begin{bmatrix} T_{11}^\epsilon & T_{12}^\epsilon \\ T_{21}^\epsilon & T_{22}^\epsilon \end{bmatrix} \\ &= \begin{bmatrix} A_r - sE_r & -B_r F & 0 & -B_r & B_r \\ 0 & A - sE & 0 & -B & B \\ -\epsilon H_r F_r & -\epsilon H_r F & 0 & -\epsilon H_r & \epsilon H_r \\ 0 & C & I & -D & D \end{bmatrix}. \end{aligned} \quad (17)$$

- b) Let  $K$  belong to the class from (a). If  $\|\mathcal{F}_\ell(T^\epsilon, K)\|_\infty \leq 1$  and  $K$  admits an NRF implementation as in (11a) and (11b), then the control laws from (9) stabilize all  $G_\Delta \in \mathcal{C}_G^\epsilon$ .

*Proof:* See the Appendix.  $\square$

**Remark 3.4:** The key to bypassing the feasibility of the model-matching problem tackled in Proposition 3.1 lies with judiciously employing Theorem 3.1. Let our network's TFM be  $\bar{G} \in \mathcal{R}^{p \times m}$  and assume that the chosen NRF architecture is either infeasible or difficult to satisfy for the available DCFs over  $\mathcal{RH}_\infty$  of  $\bar{G}$ . Then, we may resort to an approximation of  $\bar{G}$ , denoted  $G \in \mathcal{R}^{p \times m}$ , which satisfies  $\bar{G} \in \mathcal{C}_G^\epsilon$  and which is described by a DCF over  $\mathcal{RH}_\infty$  that supports the desired NRF architecture. By obtaining control laws of type (9) with the desired sparsity structure and which stabilize all  $G_\Delta \in \mathcal{C}_G^\epsilon$ , these sparse control laws will also stabilize  $\bar{G}$ . A concrete example of this design procedure will be shown in Section V.

Although we now possess the means to characterize robustly stabilizing NRF-based implementations of the controller, note that these are obtained by employing a DCF over  $\mathcal{RH}_\infty$  whose realization is of the same order as that in (17). The next result shows how to obtain descriptor representations for the DCF over  $\mathcal{RH}_\infty$  with the same order as that of the network's model.

**Proposition 3.2:** Let the same framework, hypotheses, and notation hold as in the statement of Theorem 3.1, and let  $T^\epsilon$  be defined as in (17). Then, we have the following.

- a) For any  $H$  so that the pencil  $A + HC - sE$  is admissible, a DCF over  $\mathcal{RH}_\infty$  of  $T_{22}^\epsilon$  is given by

$$\begin{bmatrix} \tilde{Y}^\epsilon & -\tilde{X}^\epsilon \\ -\tilde{N}^\epsilon & \tilde{M}^\epsilon \end{bmatrix} := \begin{bmatrix} A + HC - sE & -B - HD & H \\ F & I & 0 \\ C & -D & I \end{bmatrix} \quad (18a)$$

$$\begin{bmatrix} M^\epsilon & X^\epsilon \\ N^\epsilon & Y^\epsilon \end{bmatrix} := \begin{bmatrix} A + BF - sE & B & -H \\ F & I & 0 \\ C + DF & D & I \end{bmatrix}. \quad (18b)$$

- b) For any stabilizing controller obtained using (18a) and (18b) and an arbitrary  $Q \in \mathcal{RH}_\infty^{m \times p}$ , we may express  $\mathcal{F}_\ell(T^\epsilon, K) = T_1^\epsilon + T_2^\epsilon Q T_3^\epsilon$ , where we have

$$\begin{aligned} T_1^\epsilon &:= T_{11}^\epsilon + T_{12}^\epsilon X^\epsilon \tilde{M}^\epsilon T_{21}^\epsilon \\ &= \begin{bmatrix} A_r - sE_r & -B_r F & 0 & -B_r \\ 0 & A + HC - sE & H & -B - HD \\ -\epsilon H_r F_r & -\epsilon H_r F & 0 & -\epsilon H_r \end{bmatrix} \end{aligned} \quad (19a)$$

$$T_2^\epsilon := T_{12}^\epsilon M^\epsilon = \begin{bmatrix} A_r - sE_r & B_r \\ -\epsilon H_r F_r & \epsilon H_r \end{bmatrix} \quad (19b)$$

$$T_3^\epsilon := \tilde{M}^\epsilon T_{21}^\epsilon = \begin{bmatrix} A + HC - sE & H & -B - HD \\ C & I & -D \end{bmatrix}. \quad (19c)$$

*Proof:* See the Appendix.  $\blacksquare$

#### IV. CONVEX PROCEDURE FOR AUGMENTED SPARSITY

##### A. Procedure Setup and Norm Condition Reformulation

Recall that, in order to obtain sparse control laws of type (9), we aim to express controllers of type (7) for  $Q \in \mathcal{RH}_\infty^{m \times p}$  satisfying (12a) and (12d). For robust stability, point (b) of Theorem 3.1 argues that we need only satisfy  $\|T_1^\epsilon + T_2^\epsilon Q T_3^\epsilon\|_\infty \leq 1$ , where  $T_1^\epsilon$ ,  $T_2^\epsilon$ , and  $T_3^\epsilon$  are expressed as in (19a) and (19c).

The beginning of this section is dedicated to showing how this norm condition can be converted into (10). Owing to this being the setup of the iterative algorithm given in the following, this conversion will be given in an ordered sequence of steps.

**Step 1:** Solve (12a) for  $Q_0 \in \mathcal{RH}_\infty^{m \times p}$ , and obtain a basis  $B \in \mathcal{RH}_\infty^{m \times p}$  for  $\text{Ker} \begin{bmatrix} F_x(\tilde{M}^\top \otimes I) \\ F_y(\tilde{N}^\top \otimes I) \end{bmatrix}$  (recall Remark 3.2).

**Step 2:** Partition  $\tilde{B}$  via its columns, as follows:

$$B := [B_1 \mid \dots \mid B_i \mid \dots \mid B_p], \quad B_i \in \mathcal{RH}_\infty^{m \times 1}$$

to obtain minimal realizations  $B_i = \begin{bmatrix} A_i^B & B_i^B \\ C_i^B & D_i^B \end{bmatrix} \quad \forall i \in 1 : p$ .

**Step 3:** Using these realizations, write via (6b) a stable RCF of each  $B_i = N_{B_i} M_{B_i}^{-1}$ , which are given explicitly by

$$\begin{bmatrix} M_{B_i} \\ N_{B_i} \end{bmatrix} := \begin{bmatrix} A_i^B + B_i^B F_i^B & B_i^B \\ F_i^B & 1 \\ C_i^B + D_i^B F_i^B & D_i^B \end{bmatrix} \in \mathcal{RH}_\infty^{(mp+1) \times 1} \quad (20)$$

with  $F_i^B$  ensuring  $\Lambda(A_i^B + B_i^B F_i^B - sI) \subset \mathbb{C}^-$  to form

$$\hat{B} := [N_{B_1} \mid \dots \mid N_{B_i} \mid \dots \mid N_{B_p}] \in \mathcal{RH}_\infty^{m \times pq}. \quad (21)$$

**Step 4:** Partition  $\hat{B} = [\hat{B}_1^\top \mid \dots \mid \hat{B}_i^\top \mid \dots \mid \hat{B}_p^\top]^\top$ , noting that  $\hat{B}_i \in \mathcal{RH}_\infty^{m \times q}$ , in order to finally define

$$\bar{B} := [\hat{B}_1 \mid \dots \mid \hat{B}_p] = \begin{bmatrix} A_{\bar{B}} & B_{\bar{B}} \\ C_{\bar{B}} & D_{\bar{B}} \end{bmatrix} \in \mathcal{RH}_\infty^{m \times pq}. \quad (22)$$

**Remark 4.1:** Since  $B_i$  are the columns of stable basis of the null space in (12b), then so are  $N_{B_i} = B_i M_{B_i}$ , having realizations of the same order as those of  $B_i$ . Thus,  $\hat{B}$  is a stable basis for the same null space and may also be used to form  $\text{vec}(\hat{Q}) = \hat{B}x$ ,  $\forall x \in \mathcal{RH}_\infty^{q \times 1}$ , as in Proposition 3.1.

This concludes the setup of our procedure, and we now move on to converting  $\|T_1^\epsilon + T_2^\epsilon Q T_3^\epsilon\|_\infty \leq 1$  into (10), through the explicit use of  $\bar{B}$ . Recall that  $Q$  can be partitioned additively as  $Q = Q_0 + \hat{Q}$ , with  $Q_0$  having been obtained in Step 1 of the setup and with  $\text{vec}(\hat{Q})$  formed as in Remark 4.1. Thus, by (22) in Step 4 of the setup, it is straightforward to obtain

$$\hat{Q} = [\hat{B}_1 x \mid \dots \mid \hat{B}_p x] = \bar{B} D_p(x) \in \mathcal{RH}_\infty^{m \times p}.$$

Defining  $x \in \mathcal{RH}_\infty^{q \times 1}$  as  $x := \begin{bmatrix} A_x & b_x \\ C_x & d_x \end{bmatrix}$ , we may express

$$\hat{Q} = \begin{bmatrix} A_{\bar{B}} & B_{\bar{B}} D_p(C_x) & B_{\bar{B}} D_p(d_x) \\ 0 & D_p(A_x) & D_p(b_x) \\ C_{\bar{B}} & D_{\bar{B}} D_p(C_x) & D_{\bar{B}} D_p(d_x) \end{bmatrix} \quad (23)$$

whose realization is affine in terms of all variable matrices:  $A_x, b_x, C_x, d_x$ , and  $A_{\bar{B}}$  and  $C_{\bar{B}}$ , by way of  $F_i^B$ , for  $i \in 1 : q$ .

It now becomes clear, in terms of (10), that we have  $\hat{T}_1 = T_1^\epsilon + T_2^\epsilon Q_0 T_3^\epsilon$  and  $\hat{T}_{2i} = T_2^\epsilon \hat{Q}_i T_3^\epsilon$ , where we have defined

$$\hat{Q}_i := \bar{B} [\hat{e}_i \mid \dots \mid \hat{e}_{jq+i} \mid \dots \mid \hat{e}_{(p-1)q+i}] \quad (24)$$

with  $i \in 1 : q$ ,  $j \in 1 : p - 2$ , and  $\widehat{e}_i$  being the  $i$ th vector in the canonical basis of  $\mathbb{R}^{pq \times 1}$ . Moving on, the next section tackles the numerical details of satisfying the inequality from (10).

**Remark 4.2:** The free term of the Youla parameterization is now expressed as  $\mathbf{Q} = \mathbf{Q}_0 + \sum_{i=1}^q \mathbf{x}_i \widehat{\mathbf{Q}}_i$ , for some  $\mathbf{x}_i \in \mathcal{RH}_{\infty}^{1 \times 1}$ ,  $\forall i \in 1 : q$ . Thus, forming  $\widehat{\mathbf{B}}$  from only a subset of the  $q$  columns used in (21) may prove sufficient to solve (10), which has the benefit of cutting down on computational costs.

### B. Numerical Formulation and NRF Implementability

In order to formulate a numerical procedure meant to solve (10), we first require a state-space realization of  $\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon \mathbf{Q} \mathbf{T}_3^\epsilon$ . This can be obtained by first defining the following TFM:

$$\mathbf{T}^f := \left[ \begin{array}{c|c} \mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon \mathbf{Q}_0 \mathbf{T}_3^\epsilon & \mathbf{T}_2^\epsilon \\ \hline \mathbf{T}_3^\epsilon & 0 \end{array} \right] = \left[ \begin{array}{c|c|c} \mathbf{A}^f & \mathbf{B}_1^f & \mathbf{B}_2^f \\ \hline \mathbf{C}_1^f & \mathbf{D}_{11}^f & \mathbf{D}_{12}^f \\ \hline \mathbf{C}_2^f & \mathbf{D}_{21}^f & 0 \end{array} \right] \quad (25)$$

and obtaining a minimal state-space realization as in (25), with  $\Lambda(\mathbf{A}^f - s\mathbf{I}) \subset \mathbb{C}^-$  due to  $\mathbf{T}^f \in \mathcal{RH}_{\infty}$ , via one of  $\mathbf{Q}_0$  and via (19a) and (19c), as per Remark 2.1. Notice that  $\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon \mathbf{Q} \mathbf{T}_3^\epsilon = \mathcal{F}_\ell(\mathbf{T}^f, \widehat{\mathbf{Q}})$  to get, via (23) along with the formulas in [23, Sec. 10.4], the realization from (26), shown at the bottom of the this page. Crucially, notice that all the variable matrices that appear in the realization from (26) do so only via affine terms.

Before stating the numerical problem that will be tackled by our iterative procedure, we must ensure that the obtained controller is well defined and can be implemented as in (9) via its NRF pair. As indicated in Remark 3.3, this is ensured by satisfying (13) and (14), which can be written generically as

$$(\mathbf{Z}_1^k + \mathbf{Z}_2^k \mathbf{Q}(\infty) \mathbf{Z}_3^k)^\top (\mathbf{Z}_1^k + \mathbf{Z}_2^k \mathbf{Q}(\infty) \mathbf{Z}_3^k) \succ 0 \quad \forall k \in 1 : N_Z \quad (27)$$

where the various matrices  $\mathbf{Z}_1^k \in \mathbb{R}^{w_k \times w_k}$ ,  $\mathbf{Z}_2^k \in \mathbb{R}^{w_k \times m}$ , and  $\mathbf{Z}_3^k \in \mathbb{R}^{p \times w_k}$  are shown explicitly in (13) and (14). Finally, note that  $\mathbf{Q}(\infty) = \mathbf{Q}_0(\infty) + \sum_{i=1}^q d_{xi} \widehat{\mathbf{Q}}_i(\infty)$ , where we partition  $\mathbf{x}(\infty) = d_x = [d_{x1} \dots d_{xi} \dots d_{xq}]^\top$ ,  $i \in 2 : q - 1$ . Thus, we combine (10) and (27) into our numerical problem

$$\begin{cases} \|\mathcal{F}_\ell(\mathbf{T}^f, \widehat{\mathbf{Q}})\|_\infty < 1, \widehat{\mathbf{Q}} \text{ as in (23)} \\ (\widehat{\mathbf{Z}}_1^k)^\top \widehat{\mathbf{Z}}_1^k + \sum_{i=1}^q d_{xi} (\widehat{\mathbf{Z}}_1^k)^\top \widehat{\mathbf{Z}}_{2i}^k + \sum_{i=1}^q d_{xi} (\widehat{\mathbf{Z}}_{2i}^k)^\top \widehat{\mathbf{Z}}_1^k \\ + \sum_{i=1}^q \sum_{j=1}^q d_{xi} d_{xj} (\widehat{\mathbf{Z}}_{2i}^k)^\top \widehat{\mathbf{Z}}_{2j}^k \succ 0 \quad \forall k \in 1 : N_Z \\ \widehat{\mathbf{Z}}_1^k := \mathbf{Z}_1^k + \mathbf{Z}_2^k \mathbf{Q}_0(\infty) \mathbf{Z}_3^k, \widehat{\mathbf{Z}}_{2i}^k := \mathbf{Z}_2^k \widehat{\mathbf{Q}}_i(\infty) \mathbf{Z}_3^k \quad \forall i \in 1 : q \end{cases} \quad (28)$$

### C. Iterative Procedure With Guaranteed Convergence

We now introduce the most general form (recall Remark 4.2) of our convex and iterative procedure for solving (28), based upon the algorithm with guaranteed convergence in [14].

**Theorem 4.1:** Given the realization from (26) along with two tolerance values  $0 < \eta_1, 0 < \eta_2 \ll 1$ , define the following:

$$\overline{\mathbf{A}}^f := \begin{bmatrix} \mathbf{A}^f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{\mathbf{C}}_2^f := \begin{bmatrix} \mathbf{C}_2^f & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}$$

$$\mathbf{T}_1^\epsilon + \mathbf{T}_2^\epsilon \mathbf{Q} \mathbf{T}_3^\epsilon = \left[ \begin{array}{c|c} \overline{\mathbf{A}} & \overline{\mathbf{B}} \\ \hline \overline{\mathbf{C}} & \overline{\mathbf{D}} \end{array} \right] = \left[ \begin{array}{c|c|c} \mathbf{A}^f + \mathbf{B}_2^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{C}_2^f & \mathbf{B}_2^f \mathbf{C}_{\overline{\mathbf{B}}} & \mathbf{B}_2^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(C_x) & \mathbf{B}_1^f + \mathbf{B}_2^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{D}_{21}^f \\ \hline \mathbf{B}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{C}_2^f & \mathbf{A}_{\overline{\mathbf{B}}} & \mathbf{B}_{\overline{\mathbf{B}}} \mathcal{D}_p(C_x) & \mathbf{B}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{D}_{21}^f \\ \hline \mathcal{D}_p(b_x) \mathbf{C}_2^f & 0 & \mathcal{D}_p(A_x) & \mathcal{D}_p(b_x) \mathbf{D}_{21}^f \\ \hline \mathbf{C}_1^f + \mathbf{D}_{12}^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{C}_2^f & \mathbf{D}_{12}^f \mathbf{C}_{\overline{\mathbf{B}}} & \mathbf{D}_{12}^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(C_x) & \mathbf{D}_{11}^f + \mathbf{D}_{12}^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) \mathbf{D}_{21}^f \end{array} \right] \quad (26)$$

### Algorithm 1: Convex Approach to Solving (28)

**Initialization:** Solve the LMI system of (30), given on the next page, along with the equality constraint  $\overline{d}_x - d_x = 0$ , for  $(\mathbf{A}_x^0, \mathbf{b}_x^0, \mathbf{C}_x^0, \mathbf{d}_x^0, \overline{\mathbf{d}}_x^0, (\mathbf{F}_i^{\mathbf{B}})^0, \mathbf{P}^0, \overline{\mathbf{P}}^0, (\overline{\mathbf{P}}^D)^0, \mathbf{P}_x^0, \overline{\mathbf{P}}_x^0, (\mathbf{P}_i^{\mathbf{B}})^0, (\overline{\mathbf{P}}_i^{\mathbf{B}})^0)$ . Using these computed variables, form  $\mathbf{T}_A^0, \mathbf{T}_B^0, \mathbf{T}_C^0$  as in (29d)-(29f) and then set  $k = 0$  along with  $f^0 = \|\mathbf{T}_C^0 - \mathbf{T}_A^0 \mathbf{T}_B^0\|_* + \|\mathbf{I}_{n_T}\|_*$ ;

**repeat**

**if**  $k \bmod 2 < 1$  **then**

    Set  $k = k + 1$  followed by  $\Theta^k = \mathbf{T}_B - \mathbf{T}_B^{k-1}$ ;

**else**

    Set  $k = k + 1$  followed by  $\Theta^k = \mathbf{T}_A - \mathbf{T}_A^{k-1}$ ;

**end**

  Solve  $\mathcal{M}(\mathbf{T}_A^{k-1}, \mathbf{T}_B^{k-1}, \Theta^k)$  for  $(\mathbf{A}_x^k, \mathbf{b}_x^k, \mathbf{C}_x^k, \mathbf{d}_x^k, \overline{\mathbf{d}}_x^k, (\mathbf{F}_i^{\mathbf{B}})^k, \mathbf{P}^k, \overline{\mathbf{P}}^k, (\overline{\mathbf{P}}^D)^k, \mathbf{P}_x^k, \overline{\mathbf{P}}_x^k, (\mathbf{P}_i^{\mathbf{B}})^k, (\overline{\mathbf{P}}_i^{\mathbf{B}})^k)$

  and use them to form  $\mathbf{T}_A^k, \mathbf{T}_B^k, \mathbf{T}_C^k$  as in (29d)-(29f);

  Compute  $f^k := \|\mathbf{T}_C^k - \mathbf{T}_A^k \mathbf{T}_B^k\|_* + \|\mathbf{I}_{n_T}\|_*$ ;

**until**  $f^{k-1} - f^k < \eta_1$  or  $f^k - \|\mathbf{I}_{n_T}\|_* < \eta_2$ ;

$$\overline{\mathbf{B}}_1^f := \begin{bmatrix} (\mathbf{B}_1^f)^\top & 0 & 0 \end{bmatrix}^\top, \quad \overline{\mathbf{D}}_{21}^f := \begin{bmatrix} (\mathbf{D}_{21}^f)^\top & 0 & 0 \end{bmatrix}^\top \quad (29a)$$

$$\mathbf{G} := \begin{bmatrix} 2\text{sym}(\mathbf{P} \overline{\mathbf{A}}^f + \overline{\mathbf{P}} \mathbf{C}_2^f) & \mathbf{P} \overline{\mathbf{B}}_1^f + \overline{\mathbf{P}} \mathbf{D}_{21}^f & \overline{\mathbf{C}}^\top \\ (\overline{\mathbf{B}}_1^f)^\top \mathbf{P} + (\overline{\mathbf{P}} \mathbf{D}_{21}^f)^\top & -\mathbf{I} & \overline{\mathbf{D}}^\top \\ \overline{\mathbf{C}} & \overline{\mathbf{D}} & -\mathbf{I} \end{bmatrix} \quad (29b)$$

$$\mathbf{A}_S := \begin{bmatrix} \mathbf{B}_2^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) & \mathbf{B}_2^f \mathbf{C}_{\overline{\mathbf{B}}} & \mathbf{B}_2^f \mathbf{D}_{\overline{\mathbf{B}}} \mathcal{D}_p(C_x) \\ \mathbf{B}_{\overline{\mathbf{B}}} \mathcal{D}_p(d_x) & \mathbf{A}_{\overline{\mathbf{B}}} & \mathbf{B}_{\overline{\mathbf{B}}} \mathcal{D}_p(C_x) \\ \mathcal{D}_p(b_x) & 0 & \mathcal{D}_p(A_x) \end{bmatrix} \quad (29c)$$

$$\mathbf{T}_A := \mathcal{D}(\mathbf{P}_1^{\mathbf{B}}, \dots, \mathbf{P}_q^{\mathbf{B}}, \mathbf{P}_x, \overline{\mathbf{d}}_x, \mathbf{P}, 0) \in \mathbb{R}^{p_T \times n_T} \quad (29d)$$

$$\mathbf{T}_B := \mathcal{D}((\mathbf{F}_1^{\mathbf{B}})^\top, \dots, (\mathbf{F}_q^{\mathbf{B}})^\top, \mathbf{A}_x^\top, \mathbf{d}_x^\top, \mathbf{A}_S, 0) \in \mathbb{R}^{n_T \times m_T} \quad (29e)$$

$$\mathbf{T}_C := \mathcal{D}(\overline{\mathbf{P}}_1^{\mathbf{B}}, \dots, \overline{\mathbf{P}}_q^{\mathbf{B}}, \overline{\mathbf{P}}_x, \overline{\mathbf{P}}^D, \overline{\mathbf{P}}, \overline{\mathbf{d}}_x - d_x) \in \mathbb{R}^{p_T \times m_T}. \quad (29f)$$

Then, we have the following.

- If the problem from (28) is feasible, then a solution can be found by the iterative procedure with guaranteed convergence given in Algorithm 1, which involves the convex optimization problem from (30), shown at the bottom of the next page.
- If, at the proposed iteration's termination, we have that  $\|\mathbf{T}_C^k - \mathbf{T}_A^k \mathbf{T}_B^k\|_* < \eta_2$ , then  $\mathbf{A}_x^k, \mathbf{b}_x^k, \mathbf{C}_x^k, \mathbf{d}_x^k$ , and  $(\mathbf{F}_i^{\mathbf{B}})^k$  can be used to form  $\widehat{\mathbf{Q}}$  as in (20)-(23).

**Proof:** For point (a), see the Appendix. Point (b) follows directly from the fact that  $\|\mathbf{T}_C^k - \mathbf{T}_A^k \mathbf{T}_B^k\|_* < \eta_2 \ll 1$  indicates that the bilinear equality constraint belonging to the problem (given in the Appendix)

that is equivalent to (28) has been satisfied for a feasible tuple, which designates a solution. ■

## V. NUMERICAL EXAMPLE

### A. Design Procedure

Consider a set of  $\ell = 20$  subsystems, which are interconnected in a network with a ring topology, as depicted in Fig. 2. The input–output model of each subsystem can be written as

$$\begin{aligned} Y_{(i \bmod \ell)+1} &= \overline{G}_y Y_{((i-1) \bmod \ell)+1} + \overline{G}_u u_{(i \bmod \ell)+1} \quad \forall i \in 1 : \ell \\ \text{with } \overline{G}_y(s) &:= \left[ \begin{array}{c|c} A_y - sE_y & B_y \\ \hline C_y & 0 \end{array} \right] = \left[ \begin{array}{cc|c} -1-s & 0 & 1 \\ 0 & -1 & 1 \\ \hline 1 & 1 & 0 \end{array} \right] \quad \text{and} \\ \overline{G}_u(s) &:= \left[ \begin{array}{c|c} A_u - sE_u & B_u \\ \hline C_u & 0 \end{array} \right] = \left[ \begin{array}{cc|c} 1-s & 0 & 11 \\ 0 & -1 & 4 \\ \hline 1 & 1 & 0 \end{array} \right]. \quad \text{Define now} \\ \Xi : \mathbb{R} \rightarrow \mathbb{R}^{\ell \times \ell} \quad \text{and} \quad \Xi(\kappa) &:= \mathcal{D}_\ell(\kappa) \begin{bmatrix} O_{1,\ell-1} & 1 \\ I_{\ell-1} & O_{\ell-1,1} \end{bmatrix} \quad \text{to get that} \\ \overline{G}(s) &= \left[ \begin{array}{cc|c} \mathcal{D}_\ell(A_y - sE_y) + \mathcal{D}_\ell(B_y)\Xi(1)\mathcal{D}_\ell(C_y) & \mathcal{D}_\ell(B_y)\Xi(1)\mathcal{D}_\ell(C_u) & O_{2\ell,\ell} \\ \hline O_{2\ell} & \mathcal{D}_\ell(A_u - sE_u) & \mathcal{D}_\ell(B_u) \\ \hline \mathcal{D}_\ell(C_y) & \mathcal{D}_\ell(C_u) & O_\ell \end{array} \right] \quad (31) \end{aligned}$$

is the network's TFM, which is improper, having a strongly stabilizable and detectable realization and whose resulting descriptor vector is the concatenation of the descriptor vectors belonging to the realizations of all  $\overline{G}_y$  and  $\overline{G}_u$  subsystems. We aim to obtain a control law, for  $\Phi_K, \Gamma_K \in \mathcal{R}_p^{1 \times 1}$ , with

$$u_{(i \bmod \ell)+1} = \Phi_K u_{((i-1) \bmod \ell)+1} + \Gamma_K Y_{(i \bmod \ell)+1} \quad \forall i \in 1 : \ell. \quad (32)$$

Then, approximate  $\overline{G}(s)$  with  $G(s) := \mathcal{D}_\ell(\Psi)\Omega$ , where  $\Omega := \Xi(2) + I_\ell$  and  $\Psi(s) := \left[ \begin{array}{c|c} A_\Psi - sE_\Psi & B_\Psi \\ \hline C_\Psi & D_\Psi \end{array} \right] = \left[ \begin{array}{cc|c} 1 & -s & 0 \\ 0 & 1 & -1 \\ \hline 1 & 0 & 4 \end{array} \right]$ . Note that the latter realization is strongly stabilizable and detectable, such that  $F_\Psi = [1 \ 5]$  and  $H_\Psi^\top = [-5 \ -1]$  are admissible feedback for it. Thus, we are able to express

$$G(s) = \left[ \begin{array}{c|c} \mathcal{D}_\ell(A_\Psi - sE_\Psi) & \mathcal{D}_\ell(B_\Psi)\Omega \\ \hline \mathcal{D}_\ell(C_\Psi) & \mathcal{D}_\ell(D_\Psi)\Omega \end{array} \right] \quad (33)$$

with  $F = \Omega^{-1}\mathcal{D}_\ell(F_\Psi)$  and  $H = \mathcal{D}_\ell(H_\Psi)$  being admissible feedbacks. Obtaining stable NRCFs for (31) and (33) via Lemma 3.1, we use them to get the maximum stability radius  $b_{\text{opt}} > 0.9925$  of  $G$  (see [22]) and to compute an upper bound for some  $\overline{Q} \in \mathcal{RH}_\infty^{\ell \times \ell}$  (see [24, Ch. 8]) denoted  $\mu(\overline{Q}) < 0.5609$  of the directed gap metric between  $G$  and  $\overline{G}$ , as given in [22, Eq. (4)]. Then, we set  $\epsilon = 0.7 > \mu(\overline{Q})$ , and we get, by the same arguments as in [22, Proof of Lemma 2] applied for  $\overline{Q} \in \mathcal{RH}_\infty^{\ell \times \ell}$ , that  $\overline{G} \in \mathcal{C}_G^\epsilon$ . Note that  $\mu(\overline{Q}) < 1$  implies  $\det \overline{Q} \neq 0$ . Otherwise,  $\exists v \in \text{Ker } \overline{Q} \cap \mathcal{RH}_\infty^{\ell \times 1}$  with  $\mathcal{H}_2$  norm equal to 1, which can be used to obtain that  $\mu(\overline{Q}) \geq 1$ .

Now, use the realization from (33) and  $F$  to compute a stable RCF as in (16). With this stable RCF and  $H$ , employ Proposition 3.2 to compute  $(N^\epsilon, \tilde{N}^\epsilon, M^\epsilon, \tilde{M}^\epsilon, X^\epsilon, \tilde{X}^\epsilon, Y^\epsilon, \tilde{Y}^\epsilon)$  via (18a) and (18b) and  $T_1^\epsilon, T_2^\epsilon$ , and  $T_3^\epsilon$  as in (19a)–(19c). Then

$$K = (\Omega \tilde{Y}^\epsilon + \tilde{Q} \tilde{N}^\epsilon)^{-1} (\Omega \tilde{X}^\epsilon + \tilde{Q} \tilde{M}^\epsilon) \quad (34a)$$

$$\mathcal{F}_\ell(T^\epsilon, K) = T_1^\epsilon + (T_2^\epsilon \Omega^{-1}) \tilde{Q} T_3^\epsilon \quad (34b)$$

having defined  $\tilde{Q} := \Omega Q$ . Note that  $(N^\epsilon \Omega^{-1}, \tilde{N}^\epsilon, M^\epsilon \Omega^{-1}, \tilde{M}^\epsilon, X^\epsilon, \Omega \tilde{X}^\epsilon, Y^\epsilon, \Omega \tilde{Y}^\epsilon)$  is also a DCF over  $\mathcal{RH}_\infty$  of  $T_{22}^\epsilon = G$ , as all eight TFMs are stable, and they satisfy (5), with the added benefit of  $\Omega \tilde{Y}^\epsilon, \tilde{N}^\epsilon \in \hat{\mathcal{S}}_{(\Xi(1)+I_\ell)}$  and of  $\Omega \tilde{X}^\epsilon, \tilde{M}^\epsilon \in \hat{\mathcal{S}}_{I_\ell}$ .

We will employ this new DCF over  $\mathcal{RH}_\infty$  to form the controller as in (34a) and optimize the  $\mathcal{H}_\infty$  norm of (34b). The control laws in (32) can be obtained from an NRF pair of the controller with  $\Phi = \mathcal{D}_\ell(\Phi_K)\Xi(1) \in \hat{\mathcal{S}}_{\Xi(1)}$  and  $\Gamma = \mathcal{D}_\ell(\Gamma_K) \in \hat{\mathcal{S}}_{I_\ell}$ . By Proposition 3.1, a solution to (12a) is  $\tilde{Q}_0 = 0$ , and note that a stable basis for the null-space from (12b) is expressed as in (21) with  $q = \ell$  and  $B_i = N_{B_i} = \tilde{e}_{1+(i-1)\ell}$ ,  $\forall i \in 1 : \ell$ , where  $\tilde{e}_i$  is the  $i$ th vector of the canonical basis of  $\mathbb{R}^{\ell^2 \times 1}$ .

We now run Algorithm 1 with MOSEK [27], called through MATLAB via YALMIP [28]. A comparison with other techniques from the literature is given in Table I, located at the top of the next page, and their computational performance will be discussed in the next subsection. Taking  $\tilde{Q}(s) = \mathcal{D}_\ell(5.9844)$  produces,  $\forall i \in 1 : \ell$ , the distributed control laws of type (32)

$$u_{(i \bmod \ell)+1} = -2u_{((i-1) \bmod \ell)+1} + \frac{64.11s + 257.4}{s + 4} Y_{(i \bmod \ell)+1}.$$

*Remark 5.1:* Let  $\tilde{K} := (I - \tilde{\Phi})^{-1} \tilde{\Gamma}$ , where  $\tilde{\Phi} := -\Xi(2)$  and  $\tilde{\Gamma} := 64I_\ell$ , and notice that  $[\tilde{\Phi} \ \tilde{\Gamma}]$  internally stabilizes  $[I \ \overline{G}^\top]^\top$ . Then, the distributed implementation (9) of the approximated distributed controller  $\tilde{K}$  internally stabilizes  $\overline{G}$  even in the presence of communication disturbance (see [9] and recall  $b_{(i \bmod \ell)+1}$  from Fig. 2). Moreover, the control laws from (9) implemented with either  $(\Phi, \Gamma)$  or  $(\tilde{\Phi}, \tilde{\Gamma})$  stabilize all  $\overline{G}_\Delta \in \mathcal{C}_G^\delta$  and  $\delta = 0.8968$ , indicating satisfactory robustness.

### B. Computational Performance

We conclude this section by presenting a comparative discussion of the results showcased in Table I. With respect to our proposed procedure, inspired by Doelman and Verhaegen [14], we state the following.

- 1) [14, Alg. 2] is slightly more computationally demanding, due to optimizing over all decision variables during each iteration. However, this extra degree of freedom comes at the major cost of guaranteed convergence.
- 2) Although the individual iterations of [25, Alg. 1] are significantly less costly and convergence is initially quite rapid, the latter tapers off on later iterations, similarly to [25, Fig. 2]. Convergence can be sped up by the judicious choice of  $\rho_1$  and  $\rho_2$  from [25, Eq. (3.4)], yet our approach bypasses this empiric decision via the benefits of optimizing the trace heuristic (see [29]).
- 3) [26, Alg. 1] is based upon the same trace optimization heuristic proposed in [29] as our procedure, yet it requires an *explicit* eigenvalue decomposition and orthonormal eigenvector computation at every iteration. For large-scale problems (such as our numerical example), this may prove unreliable, with the accumulation of computational errors noticeably hampering convergence.

$$\mathcal{M}(X, Y, \Theta) := \begin{cases} \min_{A_x, b_x, C_x, d_x, \overline{a}_x, F_i^B, P, \overline{P}, \overline{P}^D, P_x, \overline{P}_x, P_i^B, \overline{P}_i^B} \left\| \begin{bmatrix} T_C + XY - T_A Y - X T_B & T_A - X \\ T_B - Y & I_{n_T} \end{bmatrix} \right\|_* \\ \text{s.t. } \begin{cases} (\hat{Z}_1^k)^\top \hat{Z}_1^k + \sum_{i=1}^q d_{xi} (\hat{Z}_1^k)^\top \hat{Z}_{2i}^k + \sum_{i=1}^q d_{xi} (\hat{Z}_{2i}^k)^\top \hat{Z}_1^k + \sum_{i=1}^q \sum_{j=1}^q \overline{P}_{ij}^D (\hat{Z}_{2i}^k)^\top \hat{Z}_{2j}^k \succ 0 \quad \forall k \in 1 : N_Z \\ \overline{P}^D = (\overline{P}^D)^\top, P = P^\top \succ 0, -G \succ 0, P_x = P_x^\top \succ 0, -2\text{sym}(\overline{P}_x) \succ 0 \\ P_i^B = (P_i^B)^\top \succ 0, -2\text{sym}(P_i^B (A_i^B)^\top + \overline{P}_i^B (B_i^B)^\top) \succ 0 \quad \forall i \in 1 : q, \Theta = 0 \end{cases} \end{cases} \quad (30)$$

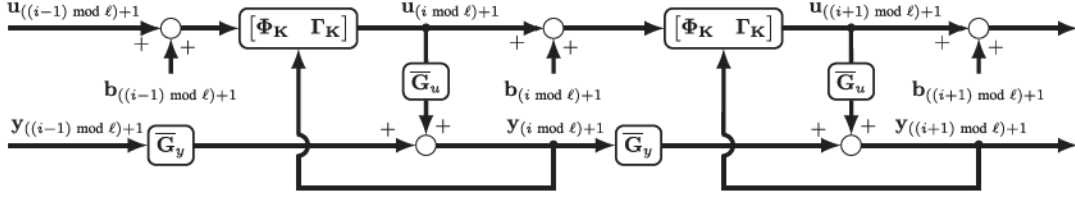


Fig. 2. Interconnection between the network's subsystems and the distributed subcontrollers.

TABLE I  
COMPARISON BETWEEN ALGORITHMS THAT SOLVE CONVEX RELAXATIONS OF (28)

| Employed procedure      | Guaranteed convergence | Runtime             | Solution                                             | $\ T_C - T_A T_B\ _*$ at convergence |
|-------------------------|------------------------|---------------------|------------------------------------------------------|--------------------------------------|
| Alg. 1 (Alg. 1 in [14]) | Yes                    | 18.41 s             | $\tilde{Q}(s) = \mathcal{D}_\ell(5.9844)$            | $2.8 \times 10^{-12}$                |
| Alg. 2 in [14]          | No                     | 20.49 s             | $\tilde{Q}(s) = \mathcal{D}_\ell(5.9844)$            | $2.1 \times 10^{-9}$                 |
| Alg. 1 in [25]          | Yes                    | Timeout after 900 s | $\tilde{Q}(s) = \mathcal{D}_\ell(6.0143)$ at timeout | $2.3 \times 10^2$ at timeout         |
| Alg. 1 in [26]          | Yes                    | Timeout after 900 s | $\tilde{Q}(s) = \mathcal{D}_\ell(5.7355)$ at timeout | $3.2 \times 10^1$ at timeout         |

## VI. CONCLUSION

In this article, we showed that the distributed control of a network (having a possibly improper TFM) can be tackled by imposing constraints upon affine expressions of the Youla parameter. A procedure was given on how to relax this problem, which reduced to solving a structurally constrained  $\mathcal{H}_\infty$  norm contraction. The latter was approached through a convex and iterative optimization algorithm with guaranteed convergence.

## APPENDIX

*Proof of Proposition 3.1:* Let there exist  $Q_0 \in \mathcal{RH}_\infty^{m \times p}$  so that  $\tilde{X} + Q_0 \tilde{M} \in \hat{S}_X$  and  $\tilde{Y} + Q_0 \tilde{N} \in \hat{S}_Y$ . They are equivalent to  $F_X \text{vec}(\tilde{X} + I Q_0 \tilde{M}) \equiv 0$  and  $F_Y \text{vec}(\tilde{Y} + I Q_0 \tilde{N}) \equiv 0$ . Using the properties of the vectorization operator (see [6, Lemma 1]), we retrieve (12a). Pick any  $\hat{Q} \in \mathcal{RH}_\infty^{m \times p}$  that satisfies (12b) and note that, when replacing  $Q_0$  with  $Q := Q_0 + \hat{Q}$  in (12a), the identity with 0 from (12a) will hold. In addition, ensuring (12c) is sufficient for the controller from (7) to be well posed, while ensuring (12d) is sufficient for  $\Gamma$  and  $\Phi$  to be both well posed and proper. Finally, the sparsity structures of  $\Gamma$  and  $\Phi$  follow from those of  $\tilde{X} + Q_0 \tilde{M}$  and  $\tilde{Y} + Q_0 \tilde{N}$ , respectively, by the way they are defined in (11a) and (11b). ■

*Proof of Lemma 3.1:* To prove point (a), define  $\hat{A}_r := E_r^{-1} A_r$ ,  $\hat{B}_r := E_r^{-1} B_r$ ,  $\hat{X}_r := E_r^\top X_r E_r$  to rewrite (3) as

$$\begin{aligned} & \hat{X}_r \hat{A}_r + \hat{A}_r^\top \hat{X}_r + C_r^\top C_r - (\hat{X}_r \hat{B}_r + C_r^\top D_r) \\ & \times (D_r^\top D_r)^{-1} (\hat{B}_r^\top \hat{X}_r + D_r^\top C_r) = 0 \end{aligned} \quad (35)$$

which is a standard continuous-time algebraic Riccati equation (see [23, Ch. 13]). Recall now that  $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$  and, thus,  $\Lambda(\hat{A}_r - sI) \subset \mathbb{C}^-$ . Then, both  $\begin{bmatrix} \hat{A}_r - sI & \hat{B}_r \end{bmatrix}$  and  $\begin{bmatrix} \hat{A}_r^\top - sI & C_r^\top \end{bmatrix}$  have full row rank  $\forall s \in \mathbb{C} \setminus \mathbb{C}^-$ , which, by [23, Sec. 3.2], means that  $(\hat{A}_r - sI, \hat{B}_r)$  is stabilizable and  $(C_r, \hat{A}_r - sI)$  is detectable. Note that  $\begin{bmatrix} N^\top(s) & M^\top(s) \end{bmatrix}^\top = C_r(sI - \hat{A}_r)^{-1} \hat{B}_r + D_r$  has full column rank  $\forall s \in j\mathbb{R} \cup \{\infty\}$ , or else there cannot exist  $\tilde{X}, \tilde{Y}$  stable so that  $\tilde{Y}M - \tilde{X}N = I$ . Then, by [23, Corollary 13.23, point (a)], (35) has a stabilizing solution,  $\hat{X}_r$ . Thus, (3) has a stabilizing solution,  $X_r = (E_r^\top)^{-1} \hat{X}_r E_r^{-1}$ , and its stabilizing feedback equals that of (35),  $\hat{F}_r := -(D_r^\top D_r)^{-1} (\hat{B}_r^\top \hat{X}_r + D_r^\top C_r) = F_r$ . ■

*Proof of Theorem 3.1:* To prove point (a), define first  $T := \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} 0 & -\tilde{M}^{-1} & \tilde{M}^{-1} \\ I & -\tilde{G} & \tilde{G} \end{bmatrix}$ , where  $\tilde{M}^{-1} = G_0 M^{-1}$  and

$G_0$  is expressed as in Lemma 3.1. Moreover, we have from (6b) in Theorem 2.1 that  $M^{-1} = \begin{bmatrix} A - sE & B \\ -F & I \end{bmatrix}$ . Expressing  $T = \begin{bmatrix} G_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -\tilde{M}^{-1} & \tilde{M}^{-1} \\ I & -\tilde{G} & \tilde{G} \end{bmatrix}$  and noticing that  $T^\epsilon = \begin{bmatrix} \epsilon I & 0 \\ 0 & I \end{bmatrix} T$ , for an  $\epsilon \in (0, 1]$ , we obtain the realization given in (17). Since  $G$  is given by a strongly stabilizable and detectable realization (2), then it is always possible to find  $\tilde{F}$  and  $\tilde{H}$  so that  $A + B\tilde{F} - sE$  and  $A + \tilde{H}C - sE$  are admissible. Thus, defining  $\hat{F} := [0 \quad \tilde{F}]$  and  $\hat{H} := [0 \quad \tilde{H}^\top]^\top$ , we extract the realization of  $T_{22}^\epsilon$  from (17),  $T_{22}^\epsilon = \begin{bmatrix} A_{22} - sE_{22} & B_{22} \\ C_{22} & D_{22} \end{bmatrix} := \begin{bmatrix} A_r - sE_r & -B_r F & B_r \\ 0 & A - sE & B \\ 0 & C & D \end{bmatrix}$ , to get that  $A_{22} + B_{22} \hat{F} - sE_{22}$  and  $A_{22} + \hat{H} C_{22} - sE_{22}$  are both admissible, since  $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$ . Therefore,  $\hat{F}$  and  $\hat{H}$  can be used, as in Theorem 2.1, in order to express the class of stabilizing controllers via a DCF over  $\mathcal{RH}_\infty$  of  $T_{22}^\epsilon$ .

To prove point (b), begin by defining the system

$$\bar{T} := \begin{bmatrix} A_r - sE_r & -B_r F & 0 & -B_r & B_r \\ 0 & A - sE & 0 & -B & B \\ -H_r F_r & -H_r F & 0 & -H_r & H_r \\ 0 & 0 & 0 & 0 & I \\ 0 & C & I & -D & D \end{bmatrix} = \begin{bmatrix} \bar{T}_{11} & \bar{T}_{12} \\ \bar{T}_{21} & \bar{T}_{22} \end{bmatrix} \quad (36)$$

and by considering the class of TFMs expressed through  $\mathcal{F}_u(\bar{T}, \begin{bmatrix} \Delta_{\hat{N}} \\ \Delta_{\hat{M}} \end{bmatrix}) := \bar{T}_{22} + \bar{T}_{21} \begin{bmatrix} \Delta_{\hat{N}} \\ \Delta_{\hat{M}} \end{bmatrix} (I - \bar{T}_{11} \begin{bmatrix} \Delta_{\hat{N}} \\ \Delta_{\hat{M}} \end{bmatrix})^{-1} \bar{T}_{12}$ , with  $\Delta_{\hat{N}}$  and  $\Delta_{\hat{M}}$  as in (15). Denoting now the class of TFMs  $G_\Delta := (\hat{N} + \Delta_{\hat{N}}) (\hat{M} + \Delta_{\hat{M}})^{-1}$ , it is straightforward to check that  $\mathcal{F}_u(\bar{T}, \begin{bmatrix} \Delta_{\hat{N}} \\ \Delta_{\hat{M}} \end{bmatrix}) = [I \quad G_\Delta]^\top$ . Thus, the proof of point (b) boils down to applying the small-gain theorem, as formulated in [30, Ch. 8], to confirm robust stability.

Note that the realization of  $T_{22}^\epsilon$  from (17) is strongly stabilizable and detectable, and so is the one belonging to  $\bar{T}_{22}$  in (36). Now, if  $(\Phi, \Gamma)$  is an NRF implementation of  $K$  as in (11a) and (11b) and  $K$  stabilizes  $G$ , then  $[\Phi \quad \Gamma]$  stabilizes  $\bar{T}_{22} = [I \quad G^\top]^\top$  (see [9]). Since the latter's realization in (36) is strongly stabilizable and detectable, then  $[\Phi \quad \Gamma]$  stabilizes  $\bar{T}$  (as in Theorem 2.1). Finally, it is straightforward to check that  $\mathcal{F}_\ell(T^\epsilon, K) = \epsilon \mathcal{F}_\ell(\bar{T}, [\Phi \quad \Gamma])$ . If  $\|\mathcal{F}_\ell(T^\epsilon, K)\|_\infty \leq 1$ , then  $\|\mathcal{F}_\ell(\bar{T}, [\Phi \quad \Gamma])\|_\infty \leq \frac{1}{\epsilon}$ , and by applying [30, Th. 8.1, point (a)], it follows that the closed-loop interconnection between  $[\Phi \quad \Gamma]$  and

$\mathcal{F}_u(\bar{T}, \Delta) = [I \quad G_\Delta^\top]^\top$  will be internally stable and well posed for any  $G_\Delta \in \mathcal{C}_G^e$ . As shown in the proof of the main result from [9], this ensures that the control laws from (9) will stabilize any  $G_\Delta \in \mathcal{C}_G^e$ . ■

*Proof of Proposition 3.2* Define first  $\hat{F} := [0 \quad F]$  and  $\hat{H} := [0 \quad H^\top]^\top$  and employ these two feedbacks to write, via (6a) and (6b), a DCF of  $T_{22}^e$  from (17). This factorization is indeed a DCF over  $\mathcal{RH}_\infty$  due to the fact that  $A + BF - sE$  and  $A + HC - sE$  are admissible and  $\Lambda(A_r - sE_r) \subset \mathbb{C}^-$ . The identities from (18a), (18b), and (19a)–(19c) follow by writing the realizations given by (6a) and (6b) and by (8a)–(8c) in Theorem 2.1 and then eliminating all the unobservable modes. ■

*Proof of Theorem 4.1* To prove point (a), we first ensure that the realization from (23) is stable by imposing that  $A_x^\top$  and  $(A_i^\top + B_i^\top F_i^\top)^\top$ ,  $\forall i \in 1:q$ , have eigenvalues only in  $\mathbb{C}^-$ , along with  $A_x^\top$  via (20)–(22). These conditions are equivalent to  $\exists P_x = P_x^\top \succ 0$  and  $P_i^\top = (P_i^\top)^\top \succ 0$ ,  $i \in 1:q$ , such that  $-2\text{sym}(A_x P_x) \succ 0$  and  $-2\text{sym}(A_i^\top P_i^\top + B_i^\top F_i^\top P_i^\top) \succ 0$ ,  $\forall i \in 1:q$ . To remedy the bilinearity induced by  $P_x A_x^\top$  and  $P_i^\top (F_i^\top)^\top$ , define  $\bar{P}_x := P_x A_x^\top$  and  $\bar{P}_i^\top := P_i^\top (F_i^\top)^\top$ , with  $i \in 1:q$ , and rewrite the inequalities as  $-2\text{sym}(\bar{P}_x) \succ 0$  and  $-2\text{sym}(P_i^\top (A_i^\top)^\top + \bar{P}_i^\top (B_i^\top)^\top) \succ 0$ ,  $\forall i \in 1:q$ .

If these new affine inequalities are satisfied, then due to  $\Lambda(A^f - sI) \subset \mathbb{C}^-$  and to  $C_2^f(sI - A^f)^{-1}B_2^f \equiv 0$ , it follows that  $\bar{A}$  from (26) has  $\Lambda(\bar{A} - sI) \subset \mathbb{C}^-$ . By the equivalence of [30, Corollary 12.3, points (i) and (vii)], we have that  $\|T_1^e + T_2^e Q T_3^e\|_\infty < 1$  if and only

if  $\exists P = P^\top \succ 0$  such that  $-\begin{bmatrix} 2\text{sym}(P\bar{A}) & P\bar{B} & \bar{C}^\top \\ \bar{B}^\top P & -I & \bar{D}^\top \\ \bar{C} & \bar{D} & -I \end{bmatrix} \succ 0$ , which

contains bilinear products of  $P$  with  $\bar{A}$  and  $\bar{B}$ , thus leading to nonconvex optimization. To obtain an affine expression, define  $A_S$  as in (29c), in order to introduce  $\bar{P} := P A_S$ . With this new matrix and the four matrices defined in (29a), notice that  $P\bar{A} = P\bar{A}^f + \bar{P}\bar{C}_2^f$  and that  $P\bar{B} = P\bar{B}_1^f + \bar{P}\bar{D}_{21}^f$ . The norm condition is equivalent to  $-G \succ 0$ , with  $G$  from (29b) being affine in all the variables.

Recall the inequalities from (28) that contain bilinear terms and denote  $\bar{P}^D := \bar{d}_x d_x^\top$ , to obtain the  $N_Z$  LMIs from (30), while additionally imposing that  $\bar{d}_x - d_x = 0$ . Form the matrices from (29c) to (29f) to notice that (28) is equivalent to

$$\begin{cases} (\hat{Z}_1^k)^\top \hat{Z}_1^k + \sum_{i=1}^q d_{xi} (\hat{Z}_{2i}^k)^\top \hat{Z}_{2i}^k + \sum_{i=1}^q d_{xi} (\hat{Z}_{2i}^k)^\top \hat{Z}_1^k + \\ + \sum_{i=1}^q \sum_{j=1}^q \bar{P}_{ij}^D (\hat{Z}_{2i}^k)^\top \hat{Z}_{2j}^k \succ 0 \quad \forall k \in 1:N_Z, \bar{P}^D = (\bar{P}^D)^\top, \\ P = P^\top \succ 0, -G \succ 0 \\ P_x = P_x^\top \succ 0, -2\text{sym}(\bar{P}_x) \succ 0, P_i^\top = (P_i^\top)^\top \succ 0 \\ -2\text{sym}(P_i^\top (A_i^\top)^\top + \bar{P}_i^\top (B_i^\top)^\top) \succ 0 \quad \forall i \in 1:q, T_A T_B = T_C \end{cases} \quad (37)$$

By selecting an artificial scalar  $\gamma > 0$  as the cost function and by applying [14, Th. 1], we get that (37) is equivalent to the problem in which  $T_A I_{n_T} T_B = T_C$  is replaced by  $\text{rank} \begin{bmatrix} T_C + XY - T_A Y - X T_B & T_A - X \\ T_B - Y & I_{n_T} \end{bmatrix} = \text{rank } I_{n_T}$ , for any matrices  $X$  and  $Y$ . Therefore, by applying [14, Th. 2] with a regularization parameter  $\lambda > 0$ , adapting [14, Alg. 1] for the resulting problem, scaling its cost function by  $1/\lambda$ , and then taking  $\lambda \rightarrow \infty$ , we obtain Algorithm 1, which solves (30) at each iteration. If the initialization is successful (the LMI system along with  $\bar{d}_x - d_x = 0$  must be feasible for the original BMI system to be feasible), we then set  $\Theta^1 = T_B - T_B^0$ , and we employ [14, Th. 3] for our algorithm (with the adapted cost function), which guarantees its convergence. ■

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