MEAN VIABILITY THEOREMS AND SECOND-ORDER HAMILTON–JACOBI EQUATIONS

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Abstract. We introduce the notion of mean viability for controlled stochastic differential equations and establish counterparts of Nagumo's classical viability theorems (necessary and sufficient conditions for mean viability). As an application, we provide a purely probabilistic proof of a comparison principle and of existence for contingent and viscosity solutions of second-order fully nonlinear path-dependent Hamilton–Jacobi–Bellman equations. We do not use compactness and optimal stopping arguments, which are usually employed in the literature on viscosity solutions for second-order path-dependent PDEs.

1. Introduction

We establish a comparison principle and existence for second-order Hamilton–Jacobi equations with viability-theoretical methods. To this end, we introduce the notion of mean viability in an appropriate setting. Roughly speaking, a pair (v, K) is mean viable if there is a control a such that $\mathbb{E}[v(t, X^a)] \in K$ for all $t \in \mathbb{R}_+$. Here, v is a function, K a set, and X^a solves a controlled stochastic differential equation. Mean viability theorems, which provide necessary and sufficient criteria for mean viability, are main contributions of this paper. Next, we consider (path-dependent) Hamilton–Jacobi–Bellman (HJB) equations of the form

$$\partial_t u + \inf_{a\in A}\left[b(t,\mathbf{x},a)\,\partial_{\mathbf{x}} u + \frac{1}{2}\sigma^2(t,\mathbf{x},a)\,\partial^2_{\mathbf{xx}} u\right] = 0 \quad \text{on } [0,T)\times C(\mathbb{R}_+,\mathbb{R}^d) \tag{1.1}$$

together with terminal conditions of the form $u(T, \cdot) = h$ on $C(\mathbb{R}_+, \mathbb{R}^d)$. Generalized sub- and supersolutions are defined via certain upper and lower directional derivatives. Using our mean viability theorems, we prove comparison and existence for generalized semisolutions. Finally, we introduce notions of smooth and viscosity solutions and provide connections to our previously defined generalized solutions. This yields a comparison principle for viscosity solutions.

1.1. **Background and motivation.** There is a large body of research that applies (deterministic) viability theory to first-order Hamilton–Jacobi equations and optimal control (see, e.g., [2, 3, 11, 15, 16, 36, 37, 52, 55]). In particular, existence and uniqueness of generalized nonsmooth solutions for Hamilton–Jacobi equations is established, including for HJB equations related to optimal control problems with state constraints and discontinuous data. There are additional applications

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in optimal control (e.g., optimal synthesis of feedback controls). Furthermore, viability theory allows for alternative proofs of the comparison principle for viscosity solutions (avoiding the doubling-the-variables approach).

While there are also many results on stochastic viability (see, e.g., [4, 7, 8, 13, 14, 39]), there have been less applications compared to the deterministic case. In particular, it is not clear if the existing results on stochastic viability can be used to show uniqueness for second-order HJB equations. Purpose of this work is to remedy this situation, i.e., to obtain counterparts for at least some of the mentioned deterministic results above by working with a different probabilistic notion of viability: Mean viability instead of stochastic viability.

1.2. **Stochastic and mean viability: A short overview and our contributions.** For simplicity, we consider the Markovian case in this subsection. Note that the rest of this paper considers the non-Markovian case.

We recall the standard notions of stochastic viability from [8, 13, 14]. Consider a controlled stochastic differential equation of the form

$$\begin{aligned} dX_s^{t,x,a} &= b(s, X_s^{t,x,a}, a_s)\,ds + \sigma(s, X_s^{t,x,a}, a_s)\,dW_s \text{ on } [t,\infty), \\ X_t^{t,x,a} &= x \in \mathbb{R}^d. \end{aligned} \tag{1.2}$$

A set $K \subset \mathbb{R}^d$ is *viable* for (1.2) if, whenever $x \in K$, there is a control a such that $X_s^{t,x,a}$ stays in K for all $s \geq t$. Similarly, K is *ε-viable* for (1.2) if, whenever $x \in K$, there is some constant $c > 0$ such that, for every $\varepsilon > 0$, there is a control a with

$$\mathbb{E}\left[\int_t^\infty e^{-cs} d_K^2(X_s^{t,x,a})\,ds\right] < \varepsilon.$$

Here, $d_K(x)$ denotes the distance from a point x to the set K. Note that for ε-viability, one still wants the *trajectories of* $X^{t,x,a}$ to be close to K in some sense.

Here, we consider another notion of viability, which formally subsumes stochastic viability. Given a function $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and a set $K \subset \mathbb{R}$, we call the pair (v, K) *mean viable* for (1.2) if, whenever $v(t, x) \in K$, there is a control a such that

$$\mathbb{E}\left[v(s, X_s^{t,x,a})\right] \in K \text{ for all } s \geq t. \tag{1.3}$$

To obtain applications for second-order Hamilton–Jacobi equations in the spirit of the first-oder case (e.g., as done in [37]), we focus on a suitable stochastic counterpart of the *deterministic viability* of the epigraph of v, i.e., if $y \geq v(t, x)$, then there is a trajectory $(x(\cdot), y(\cdot))$ starting at (x, y) at time t that satisfies $y(s) \geq v(s, x(s))$ for all $s \geq t$. Hence, we consider the following special case of (1.3), for which we establish sufficient and necessary tangential conditions:

$$\mathbb{E}\left[\widehat{v}(s, X_s^{t,x,a}, y)\right] \in K \text{ for all } s \geq t,$$

where $\widehat{v}$ is of the form $\widehat{v}(t, x, y) = v(t, x) - y$ and $K = (-\infty, 0]$. This corresponds, up to some technical details, to the notion of *stochastic u-stability* in [53, 54], where infinitesimal necessary and sufficient criteria are established as well. Note that the diffusion term in [53, 54] can only be time-dependent, i.e., of the form $\sigma = \sigma(t)$. Our diffusion term can also be state- and control-dependent, i.e., of the form $\sigma = \sigma(t, x, a)$. Moreover, (after this subsection) our setting is non-Markovian whereas the setting in [53, 54] is Markovian.

1.3. **Contributions to partial differential equations.** The probabilistic approach in this paper provides a third method of proving the comparison principle for a fairly general class of possibly degenerate parabolic HJB equations besides the standard viscosity approach (see [26]) and the approximation arguments used in [43] and many works in the path-dependent case (see below). Note that a probabilistic approach different from ours has been used in [49] albeit only in the semilinear case.

Our notion of generalized solutions for (1.1) in Section 5 is motivated by minimax solutions for Hamilton–Jacobi equations of the form $\partial_t v + \frac{1}{2}\sigma^2(t)\,\partial^2_{xx} v + H(t,x,\partial_x v) = 0$ on $[0,T)\times\mathbb{R}^d$ in [53, 54] (see also [9] for the path-dependent case). Our main contribution compared to [9, 53, 54] is the incorporation of the controlled volatility case.

Our notion of viscosity solutions in Subsection 6.3 is in the spirit of [21, 30, 31], i.e., test functions are required to be tangent in mean to the candidate solution. Thereby, the spaces of test functions are enlarged, which makes it, in principle, easier to prove uniqueness but more difficult to prove existence.

Note that our notion of viscosity solutions is relatively specific (only HJB equations of the form (1.1) are covered). However, our comparison principle (Theorem 6.12) is stronger than any other comparison principle for path-dependent fully nonlinear second-order HJB equations in the literature in the following sense: [57] requires continuity of viscosity sub- and supersolution (see, in particular, Remark 6.7 therein) whereas we allow for semicontinuity in time; [23] requires a stronger condition for the controlled diffusion term (the additional condition (C) therein); [32] requires first, strong uniform continuity assumptions (which means that $\sigma(\cdot,\cdot,\cdot)$ needs to be of the form $\sigma(t,\mathbf{x},a)=\sigma(a)$), and second, uniform ellipticity whereas we need just (A1) and (A2) from Section 3; [47,48] require a (stronger) $\|\cdot\|_{L^p}$-continuity compared to our $\|\cdot\|_\infty$-continuity; [33] operates in a "piece-wise Markovian" setting, which requires stronger conditions for existence.

Moreover, note that the comparison principle for fully nonlinear HJB equations in [21] was left as an open problem. Comparison was established therein for equations on $[0,T)\times C([0,T],H)$, H being a Hilbert space, that are *linear in the derivatives* and also involve an unbounded operator on H. Since our definition of viscosity solutions is substantially the same as in [21] and since we (like [21] in the linear case) do not rely on compactness arguments and also do not use the Crandall–Ishii lemma, our method gives new hope to attack the mentioned open problem in [21] (see also p. 364 in [35] for further related discussion).

1.4. **Organization of the rest of the paper.** In Section 2, we introduce general notation. Section 3 contains the data for our problem and the standing assumptions. Topic of Section 4 is mean viability: The notion itself and the mean viability theorems, i.e., necessary and sufficient conditions for mean viability. In Section 5, the mean viability theorems are applied to second-order HJB equations. We prove existence and a comparison principle for so-called quasi-contingent solutions. In Section 6, we introduce classical and viscosity solutions for our HJB equations and establish connections to quasi-contingent solutions, which provides us with a comparison principle for viscosity solutions. Section 7 contains technical material needed for the proof of one of the mean viability theorems.

2. Notation

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and $\mathbb{R}_+ = [0, \infty)$. Let $\mathcal{B}(E)$ be the Borel σ-field of a topological space E and $\mathcal{P}(E)$ the set of probability measures on $\mathcal{B}(E)$.

Fix $d \in \mathbb{N}$. Let $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathbb{P}$ be the Wiener measure on $\mathcal{B}(\Omega)$, $\mathbb{E} = \mathbb{E}^{\mathbb{P}}$ the corresponding expectation, $W = (W_t)_{t\geq 0}$ defined by $W_t(\omega) = \omega(t)$ the canonical process, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$ the canonical filtration on Ω. We also consider the shifted filtrations $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\geq t}$ defined by $\mathcal{F}^t_s = \sigma(W_r - W_t : t \leq r \leq s)$ with their progressive σ-fields $\mathrm{Prog}(\mathbb{F}^t)$.

Given a normed space E, a σ-field $\mathcal{G}$, and $t \in \mathbb{R}_+$, we use the spaces $\mathbb{L}^2(\mathcal{G}, \mathbb{P}, E)$ of all $\mathcal{G}$-measurable random variables $\xi : \Omega \to E$ with $\mathbb{E}[|\xi|^2] < \infty$ and

$$\mathbb{L}^0(\mathbb{F}^t, E) := \{\text{all } \mathbb{F}^t\text{-progressive processes from } \mathbb{R}_+ \times \Omega \text{ to } E\},$$

$$\mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, E) := \{X \in \mathbb{L}^0(\mathbb{F}^t, E) : \|X\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},E)} := \mathbb{E}\left[\int_t^\infty |X_s|^2 \, ds\right] < \infty\},$$

$$\mathbb{S}^2(\mathbb{F}^t, \mathbb{P}, E) := \{\text{all } \mathbb{P}\text{-a.s. continuous } X \in \mathbb{L}^0(\mathbb{F}^t, E) \text{ with } \mathbb{E}[\|X\|^2_\infty] < \infty\},$$

where $\mathcal{F}^t_s$ is implicitly understood to be equal to $\{\emptyset, \Omega\}$ whenever $s < t$ and $\|\cdot\|_\infty$ is the usual sup norm (cf. [56] for much of our notation).

The transpose of a matrix q is denoted by $q^\top$. Given real matrices q and $\tilde{q}$ of appropriate dimensions, we write $q : \tilde{q}$ for the trace of $q\tilde{q}^\top$. The identity matrix in $\mathbb{R}^{d\times d}$ is denoted by $I_{d\times d}$. We always write 0 for any zero vector.

We write $\mathbf{1}$ for indicator functions, i.e., given a set E, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise. Dirac measures are denoted by δ_x.

We use the usual notation for stochastic intervals (Definition 60 in [27]), e.g., for $\mathbb{F}$-stopping times τ_1 and τ_2, $[\![\tau_1, \tau_2]\!] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \tau_1(\omega) \leq t \leq \tau_2(\omega)\}$.

Given s, $t \in \mathbb{R}$, put $s \vee t := \max\{s, t\}$ and $s \wedge t := \min\{s, t\}$.

We write l.s.a. for lower semianalytic, l.s.c. for lower semicontinuous, and u.s.c. for upper semicontinuous.

Non-empty subsets of $\mathbb{R}_+ \times \Omega$ are always understood as pseudo-metric spaces equipped with the pseudo-metric $((t, \mathbf{x}), (t', \mathbf{x}')) \mapsto |t - t'| + \|\mathbf{x}_{\cdot\wedge t} - \mathbf{x}'_{\cdot\wedge t'}\|_\infty$.

3. Data and standing assumptions

Data are a Borel subset A of $[0, 1]$,[1] which is used as action space, and functions

$$b : \mathbb{R}_+ \times \Omega \times A \to \mathbb{R}^d,\ \sigma : \mathbb{R}_+ \times \Omega \times A \to \mathbb{R}^{d\times d},\ h : \Omega \to \mathbb{R}. \tag{3.1}$$

If there is danger of ambiguity, then we shall write $b(\cdot, \cdot, \cdot)$, etc. instead of b, etc. Moreover, we fix a triple

$$(a^\circ, p^\circ, q^\circ) \in A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}. \tag{3.2}$$

As set of admissible controls, we use

$$\mathcal{A} := \{a : \mathbb{R}_+ \times \Omega \to A \quad \mathbb{F}\text{-predictable}\}.$$

Note that $\mathcal{A}$ equipped with the metric given by $(\mathbb{E}[\int_0^\infty e^{-t} |a_t - \tilde{a}_t|^2 \, dt])^{1/2}$, a, $\tilde{a} \in \mathcal{A}$, is a Polish space (see p. 23 in [34] or p. 823 in [28]). We also use

$$\mathcal{A}^t := \{a \in \mathcal{A} : a \text{ is } \mathbb{F}^t\text{-predictable with } a|_{[\![0,t]\!]} = a^\circ\}, \quad t \in [0, T].$$

[1]This is not restrictive (see, e.g., p. 25 in [34]) but convenient for technical reasons.

Similarly, we assume that the restriction of any $(p,q) \in \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d})$ to $[\![0,t]\!]$ coincides with the constant process $(s,\omega) \mapsto (p^\circ, q^\circ)$.

We fix $T \in (0,\infty)$ and a universally measurable function $v : [0,T]\times\Omega \to \mathbb{R}$ that is non-anticipating, i.e., $v(t,\mathbf{x}) = v(t,\mathbf{x}_{\cdot\wedge t})$. Also set

$$\begin{aligned} \widehat{v}(t,\mathbf{x},y) &:= v(t,\mathbf{x}) - y, \quad (t,\mathbf{x},y) \in [0,T]\times\Omega\times\mathbb{R}, \text{ and} \\ K &:= (-\infty, 0]. \end{aligned} \tag{3.3}$$

The following standing assumptions are always in force:

(A1) The function h is $\mathcal{F}_T$-measurable and the functions $b(\cdot,\cdot,\cdot)$ and $\sigma(\cdot,\cdot,\cdot)$ are Borel measurable, they vanish after time T, and they are non-anticipating, i.e., for every $(t,\mathbf{x},a) \in \mathbb{R}_+ \times \Omega \times A$,

$$(b,\sigma)(t,\mathbf{x},a) = \mathbf{1}_{[0,T]}(t).(b,\sigma)(t,\mathbf{x}_{\cdot\wedge t},a) \quad \text{and} \quad h(\mathbf{x}) = h(\mathbf{x}_{\cdot\wedge T}).$$

(A2) There is a constant $L_b \geq 1$ such that, for all $(t,\mathbf{x},\tilde{\mathbf{x}},a) \in \mathbb{R}_+\times\Omega\times\Omega\times A$,

$$\begin{gathered} |b(t,\mathbf{x},a) - b(t,\tilde{\mathbf{x}},a)| + |\sigma(t,\mathbf{x},a) - \sigma(t,\tilde{\mathbf{x}},a)| \leq L_b \|\mathbf{x}_{\cdot\wedge t} - \tilde{\mathbf{x}}_{\cdot\wedge t}\|_\infty \quad \text{and} \\ |b(t,\mathbf{x},a)| + |\sigma(t,\mathbf{x},a)| + |h(\mathbf{x})| \leq L_b. \end{gathered}$$

The following hypothesis will also be used:

(H) The function v is bounded from below and there is a constant $L \geq 1$ such that, for all $\mathbf{x}$, $\tilde{\mathbf{x}} \in \Omega$ and s, $t \in \mathbb{R}_+$ with $s < t$,

$$\begin{gathered} |v(t,\mathbf{x}) - v(t,\tilde{\mathbf{x}})| \leq L\|\mathbf{x}_{\cdot\wedge t} - \tilde{\mathbf{x}}_{\cdot\wedge t}\|_\infty \text{ and} \\ |v(t,\mathbf{x}_{\cdot\wedge s}) - v(s,\mathbf{x})| \leq L(1 + \|\mathbf{x}_{\cdot\wedge s}\|_\infty)(t-s)^{1/2}. \end{gathered}$$

4. Mean viability

4.1. **The notions.** Given $a \in \mathcal{A}$ and $(t,\mathbf{x}) \in \mathbb{R}_+\times\Omega$, consider the controlled stochastic differential equation

$$\begin{aligned} dX_s^{t,\mathbf{x},a} &= b(s, X^{t,\mathbf{x},a}, a_s)\, ds + \sigma(s, X^{t,\mathbf{x},a}, a_s)\, dW_s \quad \text{on } (t,\infty), \ \mathbb{P}\text{-a.s.}, \\ X^{t,\mathbf{x},a}|_{[0,t]} &= \mathbf{x}|_{[0,t]}, \end{aligned} \tag{4.1}$$

which, under (A1) and (A2), has a unique (strong) solution in $\mathbb{S}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d)$ (this is a special case of Proposition 2.8 in [22]; alternatively one can slightly modify the proof of Theorem 3.3.1 in [56] to cover path-dependent coefficients).

We introduce now notions of mean viability for the pair $(\widehat{v}, K)$ defined by (3.3) (see [39] for corresponding stochastic viability notions).

Definition 4.1. The pair $(\widehat{v}, K)$ is *mean viable* for (4.1) if, for every $(t,\mathbf{x},y) \in [0,T]\times\Omega\times\mathbb{R}$ with $v(t,\mathbf{x}) \leq y$, there is an $a \in \mathcal{A}^t$ such that, for every $s \in [t,T]$,

$$\mathbb{E}\left[v(s, X^{t,\mathbf{x},a})\right] \leq y.$$

Definition 4.2. The pair $(\widehat{v}, K)$ is *approximately mean viable* for (4.1) if, for each $(t,\mathbf{x},y) \in [0,T]\times\Omega\times\mathbb{R}$ with $v(t,\mathbf{x}) \leq y$, we have

$$\inf_{a\in\mathcal{A}^t} \sup_{s\in[t,T]} \mathbb{E}\left[v(s, X^{t,\mathbf{x},a})\right] \leq y. \tag{4.2}$$

Remark 4.3. These notions can be formulated for more general pairs $(\widehat{v}, K)$, e.g., mean viability means that, if $\widehat{v}(t,\mathbf{x},y) \in K$, then there is an $a \in \mathcal{A}^t$ such that $\mathbb{E}\left[\widehat{v}(s, X^{t,\mathbf{x},a}, y)\right] \in K$ for all $s \in [t,T]$, but (3.3) is crucial for establishing our sufficient condition for mean viability (see Remark 4.7 for an explanation).

Remark 4.4. There is a natural connection between stochastic viability and stochastic target problems (see Section 7 in [50]). Similarly, mean viability is related to stochastic target problems with expected loss (cf. [12]), in particular, to superhedging of American contingent claims subject to expected loss constraints.

4.2. **Mean tangency.** The following definition is motivated by a deterministic counterpart in [17] and its stochastic version in [39].

Definition 4.5. Fix $(t, \mathbf{x}, y) \in [0,T] \times \Omega \times \mathbb{R}$ with $v(t,\mathbf{x}) \le y$. A non-empty set $\mathcal{E} \subset \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d})$ is *mean quasi-tangent* to $(\widehat{v}, K)$ at $(t, \mathbf{x}, y)$ if, for each $\varepsilon > 0$, there is a $(\delta, (b,\sigma), p, q) \in (0,\varepsilon] \times \mathcal{E} \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^d) \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^{d\times d})$ with

$$\|\mathbf{1}_{[\![t,t+\delta]\!]} . p)\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)} + \|\mathbf{1}_{[\![t,t+\delta]\!]} . q\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})} \le \varepsilon\delta \tag{4.3}$$

and $\mathbb{E}\left[v(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q})\right] \le y + \varepsilon\delta$ where $X = X^{t,\mathbf{x};b,\sigma;p,q}$ satisfies

$$\begin{aligned} X_s &= \mathbf{x}_t + \int_t^s [b_r + p_r]\, dr + \int_t^s [\sigma_r + q_r]\, dW_r \ \text{ on } (t, t+\delta],\ \mathbb{P}\text{-a.s.}, \\ X_s &= \mathbf{x}_s \quad \text{ on } [0,t]. \end{aligned} \tag{4.4}$$

We write $\mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y)$ for the *class of all mean quasi-tangent sets* to $(\widehat{v}, K)$ at $(t, \mathbf{x}, y)$.

4.3. **Necessary and sufficient conditions for mean viability.** Now, we can state appropriate versions of the classical viability theorems. To this end, consider

$$\begin{aligned} \mathcal{E}_+(t,\mathbf{x}) := \{ &(b,\sigma) \in \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d}) : \exists a \in \mathcal{A}^t : \\ &(b,\sigma)(s,\omega) = (b,\sigma)(s, \mathbf{x}_{\cdot\wedge t}, a(s,\omega)) \quad dt \times d\mathbb{P}\text{-a.e. on } [t,T]\times\Omega \}, \end{aligned} \tag{4.5}$$

where $(t, \mathbf{x}) \in [0,T] \times \Omega$ (we write $\mathcal{E}_+$ to indicate that this set depends on the values of $s \mapsto (b,\sigma)(s,\cdot,\cdot)$ from (3.1) in a right neighborhood of t).

Theorem 4.6. *Suppose that $(\widehat{v}, K)$ is mean viable or approximately mean viable for (4.1). Then, for each $(t,\mathbf{x},y) \in [0,T) \times \Omega \times \mathbb{R}$ with $v(t,\mathbf{x}) \le y$, we have*

$$\mathcal{E}_+(t,\mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y). \tag{4.6}$$

Proof. We consider the case that $(\widehat{v}, K)$ is approximately mean viable for (4.1). Fix $\varepsilon > 0$, $(t,\mathbf{x},y) \in [0,T) \times \Omega \times \mathbb{R}$ with $v(t,\mathbf{x}) \le y$, and $\delta \in (0, \varepsilon \wedge (T-t)]$. Then there is an $a \in \mathcal{A}^t$ such that $\mathbb{E}\left[v(t+\delta, X^{t,\mathbf{x},a})\right] \le y + \varepsilon\delta$. Note that, by (A2) and standard estimates (see, e.g., Section 3.2 in [56]),

$$\mathbb{E}\left[\|X^{t,\mathbf{x},a}_{\cdot\wedge(t+\delta)} - \mathbf{x}_{\cdot\wedge t}\|^2_\infty\right] \le C L_b^2\, \delta \tag{4.7}$$

for some $C > 0$ independent of δ and a. Next, we need to find $(b,\sigma) \in \mathcal{E}_+(t,\mathbf{x})$ and $(p,q) \in \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^d) \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^{d\times d})$ such that (4.3) holds and

$$b(s, X^{t,\mathbf{x},a}, a_s) = b_s + p_s \quad \text{and} \quad \sigma(s, X^{t,\mathbf{x},a}, a_s) = \sigma_s + q_s. \tag{4.8}$$

We take (b,σ) defined by $(b,\sigma)(s, \mathbf{x}_{\cdot\wedge t}, a_s)$ and (p,q) implicitly defined by (4.8). Then $\mathbb{E}\left[v(t+\delta, X)\right] \le y + \varepsilon\delta$ with X defined by (4.4) and, from (A2) and (4.7), we can deduce that (4.3) holds if $\delta \le (2CL_b^4)^{-1}\varepsilon$, i.e., we have (4.6). □

Remark 4.7. We need the pair $(\widehat{v}, K)$ to satisfy (3.3) for proving the following main result. The reason is that, in general, mean viability does not propagate in time, i.e., $X_t \in K \Rightarrow \mathbb{E}[X_{t+\delta}] \in K$ and $X_{t+\delta} \in K \Rightarrow \mathbb{E}[X_{t+2\delta}] \in K$ do not, in general, imply $X_t \in K \Rightarrow \mathbb{E}[X_{t+2\delta}] \in K$. This is in contrast to (stochastic) viability, where $X_t \in K \Rightarrow X_{t+\delta} \in K$ and $X_{t+\delta} \in K \Rightarrow X_{t+2\delta} \in K$ imply $X_t \in K \Rightarrow X_{t+2\delta} \in K$. The requirement (3.3) circumvents this issue explicitly in (7.20).

Theorem 4.8. *Let hypothesis (H) hold. If, for each* $(t, \mathbf{x}, y) \in [0, T) \times \Omega \times \mathbb{R}$ *with* $v(t, \mathbf{x}) \leq y$, *we have* $\mathcal{E}_+(t, \mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t, \mathbf{x}, y)$, *then* $(\widehat{v}, K)$ *is approximately mean viable for* (4.1).

The proof of this theorem is deferred to the end of Section 4.4.

Remark 4.9. In Theorem 4.8, one can drop the time regularity assumption in hypothesis (H). But then the conclusion becomes weaker. Instead of approximate mean viability we would have the property that, for each $(t, \mathbf{x}, y) \in [0, T] \times \Omega \times \mathbb{R}$ with $v(t, \mathbf{x}) \leq y$ and for each $s \in [t, T]$, we have $\inf_{a \in \mathcal{A}^t} \mathbb{E}\left[v(s, X^{t,\mathbf{x},a})\right] \leq y$.

4.4. **Approximate solutions.** We introduce now approximate solutions of (4.1). Existence of those solutions is crucial for the proof of Theorem 4.8. Our definition is an appropriate adaption of Definition 3 in [39] to our setting and the next proof follows the structure of the proof of Lemma 1 in [39]. However, details are different and relatively heavy technical arguments are needed (see especially Section 6).

Definition 4.10. Fix $\varepsilon > 0$ and $(t, \mathbf{x}, y) \in [0, T) \times \Omega \times \mathbb{R}$. An *$\varepsilon$-approximate solution of* (4.1) *for* $(\widehat{v}, K)$ *starting at* $(t, \mathbf{x}, y)$ is a sextuple $(\tau, \varrho, a, p, q, X)$ such that the following holds:

(i) τ is an $\mathbb{F}$-stopping time with $t < \tau \leq T$,

(ii) $\varrho = \varrho(s, \omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is non-decreasing and càdlàg in s, $\mathbb{F}$-adapted, and we have $s - \varepsilon \leq \varrho_s \leq s$ on $[\![0, \tau[\![$ and $\varrho_s = s$ on $[\![\tau, \infty[\![$,

(iii) $(a, p, q, X) \in \mathcal{A}^t \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^d) \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^{d \times d}) \times \mathbb{S}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d)$ such that

$$\|\mathbf{1}_{[\![t,\tau]\!]}.p)\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)} + \|\mathbf{1}_{[\![t,\tau]\!]}.q\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})} \leq \varepsilon \cdot \mathbb{E}[\tau - t], \tag{4.9}$$

(iv) $(\mathbf{1}_{[\![t,\tau]\!]}(s, W).(b, \sigma)(s, X_{\cdot \wedge \varrho_s}, a_s))_{s \geq 0} \in \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d \times d})$,

(v) $\mathbb{P}$-a.s. for every $s \geq t$,

$$X_{s\wedge\tau} = \mathbf{x}_t + \int_t^{s\wedge\tau} [b(r, X_{\cdot\wedge\varrho_r}, a_r) + p_r]\, dr + \int_t^{s\wedge\tau} [\sigma(r, X_{\cdot\wedge\varrho_r}, a_r) + q_r]\, dW_r,$$
$$X|_{[0,t]} = \mathbf{x}|_{[0,t]},$$

(vi) $\mathbb{E}\left[v(\varrho_s \wedge \tau, X)\right] \leq y + \varepsilon \cdot \mathbb{E}[s \wedge \tau - t]$ for all $s \in [t, T]$.

Proposition 4.11. *Let* v *be l.s.c. and bounded from below and* $T_1 \in (0, T]$. *If, for each* $(t, \mathbf{x}, y) \in [0, T_1) \times \Omega \times \mathbb{R}$ *with* $v(t, \mathbf{x}) \leq y$, *we have* $\mathcal{E}_+(t, \mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t, \mathbf{x}, y)$, *then, for each* $\varepsilon > 0$ *and* $(t, \mathbf{x}, y) \in [0, T_1) \times \Omega \times \mathbb{R}$ *with* $v(t, \mathbf{x}) \leq y$, *there is an* ε*-approximate solution* $(\tau, \varrho, a, p, q, X)$ *of* (4.1) *for* $(\widehat{v}, K)$ *starting at* $(t, \mathbf{x}, y)$ *such that* $\tau = T_1$, $\mathbb{P}$*-a.s.*

Proof. Fix $\varepsilon > 0$ and $(t, \mathbf{x}, y) \in [0, T) \times \Omega \times \mathbb{R}$ with $y \geq v(t, \mathbf{x})$. We consider only the case $T_1 = T$.

Step 1 (local existence for some τ). By Definition 4.5 and (4.5), the relation $\mathcal{E}_+(t, \mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t, \mathbf{x}, y)$ yields the existence of a quintuple

$$(\delta, X, a, p, q) \in (0, \varepsilon \wedge (T - t)] \times \mathbb{S}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathcal{A}^t \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^d) \times \mathbb{L}^0(\mathbb{F}^t, \mathbb{R}^{d\times d})$$

such that (4.3), (4.4) with $(b,\sigma)\in\mathcal{E}_+(t,\mathbf{x})$, and $\mathbb{E}[v(t+\delta,X)]\le\ y+\varepsilon\delta$ hold. Define $\varrho:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ by $\varrho_s:=s.\mathbf{1}_{[0,t)\cup[t+\delta,\infty)}(s)+t.\mathbf{1}_{[t,t+\delta)}(s)$ and $\tau:=t+\delta$. Then (τ,ϱ,a,p,q,X) is an ε-approximate solution of (4.1) for $(\widehat{v},K)$ starting at $(t,\mathbf{x},y)$.

Step 2 (existence for a maximal τ). Denote by $\mathcal{S}$ the set of all ε-approximate solutions of (4.1) for $(\widehat{v},K)$ starting at $(t,\mathbf{x},y)$. We equip $\mathcal{S}$ with a preorder $\preceq$ defined as follows: For two elements $\mathfrak{s}^{(i)}=(\tau^{(i)},\varrho^{(i)},a^{(i)},p^{(i)},q^{(i)},X^{(i)})$, $i=1$, 2, in $\mathcal{S}$, we have $\mathfrak{s}^{(1)}\preceq\mathfrak{s}^{(2)}$ if $\tau^{(1)}\le\tau^{(2)}$ and, outside of some $\mathbb{P}$-evanescent set, $(\varrho^{(2)},a^{(2)},p^{(2)},q^{(2)})(\cdot\wedge\tau^{(1)})=(\varrho^{(1)},a^{(1)},p^{(1)},q^{(1)})(\cdot\wedge\tau^{(1)})$. Also, define a function

$$\mathcal{N}:\mathcal{S}\to[t,T],\quad \mathfrak{s}=(\tau,\varrho,a,p,q,X)\mapsto\mathcal{N}(\mathfrak{s}):=\mathbb{E}[\tau].$$

We will invoke the Brezis-Browder ordering principle (see, e.g., Theorem 2.1.1 in [18]), which will provide us with an $\mathcal{N}$-maximal element $\mathfrak{s}_0$ of $\mathcal{S}$, i.e., $\mathfrak{s}\in\mathcal{S}$ and $\mathfrak{s}_0\preceq\mathfrak{s}$ imply $\mathcal{N}(\mathfrak{s}_0)=\mathcal{N}(\mathfrak{s})$. Since $\mathcal{N}$ is increasing, it suffices to verify that each increasing sequence in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. To this end, fix an increasing sequence $(\mathfrak{s}^{(n)})_{n\ge1}=(\tau^{(n)},\varrho^{(n)},a^{(n)},p^{(n)},q^{(n)},X^{(n)})_{n\ge1}$ in $\mathcal{S}$. Put $\bar{\tau}:=\sup_{n\ge1}\tau^{(n)}$. Recall $(a^\circ,p^\circ,q^\circ)$ from (3.2). Define $(\bar{a},\bar{p},\bar{q})\in\mathcal{A}^t\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^d)\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^{d\times d})$ by

$$(\bar{a},\bar{p},\bar{q})(s,\omega):=\begin{cases}\lim_{n\to\infty}(\tilde{a}^{(n)},\tilde{p}^{(n)},\tilde{q}^{(n)})(s,\omega) & \text{if this limit exists,}\\ (a^\circ,p^\circ,q^\circ) & \text{otherwise,}\end{cases}\tag{4.10}$$

where

$$(\tilde{a}^{(n)},\tilde{p}^{(n)},\tilde{q}^{(n)})(s,\omega):=\begin{cases}(a^{(n)},p^{(n)},q^{(n)})(s,\omega) & \text{if } s\in[0,\tau^{(n)}(\omega)],\\ (a^\circ,p^\circ,q^\circ) & \text{otherwise.}\end{cases}$$

Note that, outside of a $\mathbb{P}$-evanescent set, the limit in (4.10) always exists, as it is the limit of an eventually constant sequence (regarding $\mathrm{Prog}(\mathbb{F}^t)$–$\mathcal{B}(A)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^{d\times d})$ measurability, see, e.g., Lemma 4.29 in [1]). Further note that (4.9) holds for $(\bar{\tau},\bar{p},\bar{q})$. Next, we show that $(X^{(n)}_{\tau^{(n)}})_{n\ge1}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{F}_{\bar{\tau}},\mathbb{P},\mathbb{R}^d)$. To this end, let $m\ge n\ge1$. Then

$$\begin{aligned}\mathbb{E}\left[\left|X^{(m)}_{\tau^{(m)}}-X^{(n)}_{\tau^{(n)}}\right|^2\right]&\le C\mathbb{E}\left[\left(\int_{\tau^{(n)}}^{\tau^{(m)}}\left|b\left(s,X^{(m)}_{\cdot\wedge\varrho_s^{(m)}},\bar{a}_s\right)+\bar{p}_s\right|\,ds\right)^2\right]\\ &\quad+C\mathbb{E}\left[\int_{\tau^{(n)}}^{\tau^{(m)}}\left|\sigma\left(s,X^{(m)}_{\cdot\wedge\varrho_s^{(m)}},\bar{a}_s\right)\right|^2+|\bar{q}_s|^2\,ds\right]\end{aligned}$$

for some constant $C>0$ independent from m and n. Noting (A2), (4.9) for $(\bar{\tau},\bar{p},\bar{q})$, and $\tau^{(n)}\uparrow\bar{\tau}$, we can apply the dominated convergence theorem to deduce that the sequence $(X^{(n)}_{\tau^{(n)}})_{n\ge1}$ is a Cauchy sequence in $\mathbb{L}^2(\mathcal{F}_{\bar{\tau}},\mathbb{P},\mathbb{R}^d)$ and we denote its limit by ξ. Now, define $\bar{X}:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ by

$$\bar{X}(s,\omega):=\sum_{n=1}^{\infty}\mathbf{1}_{[\tau^{(n-1)}(\omega),\tau^{(n)}(\omega))}(s).X^{(n)}(s,\omega)+\mathbf{1}_{[\bar{\tau}(\omega),\infty)}(s).\xi(\omega),$$

where $\tau^{(0)}=0$. We have $\bar{X}\in\mathbb{S}^2(\mathbb{F},\mathbb{P},\mathbb{R}^d)$ because $(X^{(n)}_{\tau^{(n)}})_{n\ge1}$ has a subsequence $(X^{(n_k)}_{\tau^{(n_k)}})_{k\ge1}$ that converges to ξ, $\mathbb{P}$-a.s. This also yields that, $\mathbb{P}$-a.s., for every $s\ge t$,

$$\bar{X}_{s\wedge\bar{\tau}}=\mathbf{x}_t+\int_t^{s\wedge\bar{\tau}}[b(r,\bar{X}_{\cdot\wedge\bar{\varrho}_r},\bar{a}_r)+\bar{p}_r]\,dr+\int_t^{s\wedge\bar{\tau}}[\sigma(r,\bar{X}_{\cdot\wedge\bar{\varrho}_r},\bar{a}_r)+\bar{q}_r]dW_r,$$

where $\bar{\varrho} : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is defined by

$$\bar{\varrho}(s,\omega) := \sum_{n=1}^{\infty} \varrho^{(n)}(s,\omega).\mathbf{1}_{[\tau^{(n-1)}(\omega),\tau^{(n)}(\omega))}(s) + s.\mathbf{1}_{[\bar{\tau}(\omega),\infty)}(s).$$

Finally, by (3.3) and since v is l.s.c. and bounded from below, we have

$$\begin{aligned}\mathbb{E}\left[\widehat{v}(\bar{\varrho}_s \wedge \bar{\tau}, \bar{X}, y)\right] &\leq \varliminf_{n\to\infty} \mathbb{E}\left[\widehat{v}(\bar{\varrho}_s \wedge \tau^{(n)}, \bar{X}, y)\right] \\ &\leq \varliminf_{n\to\infty} \varepsilon \cdot \mathbb{E}[s \wedge \tau^{(n)} - t] = \varepsilon \cdot \mathbb{E}[s \wedge \bar{\tau} - t] \quad \text{for all } s \in [t,T].\end{aligned}$$

We can conclude that $(\bar{\varrho}, \bar{\tau}, \bar{a}, \bar{p}, \bar{q}, \bar{X}) \in \mathcal{S}$ and it is an upper bound of $(\mathfrak{s}^{(n)})_{n\geq 1}$.

Thus, by the Brezis-Browder ordering principle, $\mathcal{S}$ has an $\mathcal{N}$-maximal element.

Step 3 (existence for $\tau = T$, the "extension step"). Fix an $\mathcal{N}$-maximal element $\mathfrak{s}^\varepsilon = (\tau^\varepsilon, \varrho^\varepsilon, a^\varepsilon, p^\varepsilon, q^\varepsilon, X^\varepsilon)$ of $\mathcal{S}$ from Step 2. Assume $\mathbb{P}(\tau^\varepsilon < T) > 0$. By the extension lemma (Lemma 7.6), there is an $\mathfrak{s}^{\varepsilon,+} = (\tau^{\varepsilon,+}, \varrho^{\varepsilon,+}, a^{\varepsilon,+}, p^{\varepsilon,+}, q^{\varepsilon,+}, X^{\varepsilon,+}) \in \mathcal{S}$ such that $\mathfrak{s}^\varepsilon \preceq \mathfrak{s}^{\varepsilon,+}$ and $\mathbb{P}(\tau^\varepsilon < \tau^{\varepsilon,+}) > 0$. Hence $\mathcal{N}(\mathfrak{s}^\varepsilon) < \mathcal{N}(\mathfrak{s}^{\varepsilon,+})$, i.e., we have a contradiction to $\mathfrak{s}^\varepsilon$ being $\mathcal{N}$-maximal. Thus, $\tau^\varepsilon = T$, $\mathbb{P}$-a.s. □

Proof of Theorem 4.8. Fix $\varepsilon > 0$, $(t, \mathbf{x}, y) \in [0,T] \times \Omega \times \mathbb{R}$ with $v(t,\mathbf{x}) \leq y$. By Proposition 4.11, there exists an ε-approximate solution $(\tau, \varrho, a, p, q, X)$ of (4.1) for $(\widehat{v}, K)$ starting at $(t, \mathbf{x}, y)$ such that $\tau = T$, $\mathbb{P}$-a.s. Next, let $T_1 \in (t,T]$. Thanks to (A2), applying standard estimates (see, e.g., Section 3.2 in [56]) yields

$$\mathbb{E}[\|X^{t,\mathbf{x},a}_{\cdot\wedge(t+\delta)} - X_{\cdot\wedge(t+\delta)}\|^2_\infty] \leq C_0\,\varepsilon + \int_t^{t+\delta} C_0 \mathbb{E}[\|X^{t,\mathbf{x},a}_{\cdot\wedge s} - X_{\cdot\wedge\varrho_s}\|^2_\infty]\,ds \tag{4.11}$$

for all $\delta \in [0, T-t]$ and some constant $C_0 > 0$ independent of δ, ε, and T_1. Further note that, by condition (ii) of Definition 4.10 and by (A2), for every $s \in [0, T_1]$,

$$\begin{aligned}&\mathbb{E}\left[\|X_{\cdot\wedge s} - X_{\cdot\wedge\varrho_s}\|^2_\infty\right] \leq C_1\,\varepsilon \\ &\quad + C_1 \mathbb{E}\left[\int_{(s-\varepsilon)\vee 0}^{s} |b(r, X_{\cdot\wedge\varrho_r}, a_r)|^2 + |\sigma(r, X_{\cdot\wedge\varrho_r}, a_r)|^2\,dr\right] \leq C_1\,\varepsilon(1 + 2L_b^2)\end{aligned} \tag{4.12}$$

for some $C_1 > 0$ independent of s, ε, and T_1. Thus we can apply Gronwall's inequality to (4.11) combined with (4.12) to deduce $\mathbb{E}[\|X^{t,\mathbf{x},a}_{\cdot\wedge T_1} - X_{\cdot\wedge T_1}\|^2_\infty] \leq C\varepsilon$ for some $C > 0$ independent of ε and T_1. Finally, by (H) and (4.12), there is a constant $\tilde{C} > 0$ independent of ε and T_1 (cf. (4.7)) such that

$$\begin{aligned}&\mathbb{E}[v(T_1, X^{t,\mathbf{x},a})] \leq L(C\varepsilon)^{1/2} + \mathbb{E}[v(T_1, X)] \\ &\leq L(C\varepsilon)^{1/2} + L(C_1\,\varepsilon(1+2L_b^2))^{1/2} + \mathbb{E}[v(T_1, X_{\cdot\wedge\varrho_{T_1}})] \\ &\leq L(C\varepsilon)^{1/2} + L(C_1\,\varepsilon(1+2L_b^2))^{1/2} + L(1+\tilde{C})\varepsilon^{1/2} + \mathbb{E}[v(\varrho_{T_1}, X)] \\ &\leq y + L(C\varepsilon)^{1/2} + L(C_1\,\varepsilon(1+2L_b^2))^{1/2} + L(1+\tilde{C})\varepsilon^{1/2} + \varepsilon(T-t).\end{aligned}$$

The last inequality is due to Definition 4.10 (vi). We can see that (4.2) holds. □

5. Quasi-contingent solutions of HJB equations

In this section and the next, we consider the terminal value problem

$$\begin{aligned}\partial_t u + \inf_{a\in A}\left[b(t,\mathbf{x},a)\cdot\partial_{\mathbf{x}} u + \frac{1}{2}(\sigma\sigma^\top)(t,\mathbf{x},a) : \partial^2_{\mathbf{x}\mathbf{x}} u\right] &= 0 \qquad \text{on } [0,T)\times\Omega, \\ u(T,\mathbf{x}) &= h(\mathbf{x}) \qquad \text{on } \Omega,\end{aligned} \tag{5.1}$$

and the value function $V^S : [0,T] \times \Omega \to \mathbb{R}$ defined by[2]

$$V^S(t,\mathbf{x}) := \inf_{a\in\mathcal{A}} \mathbb{E}\left[h(X^{t,\mathbf{x},a})\right].$$

together with $\widehat{V}^S : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ defined by $\widehat{V}^S(t,\mathbf{x},y) := V^S(t,\mathbf{x}) - y$.

5.1. **Quasi-contingent supersolutions.** For our notion of supersolution, we use the following directional derivative (cf. [9,40,41,53,54] for similar stochastic derivatives and [16] for a related first-order derivative with function-valued direction).

Definition 5.1. Let the function v be bounded from below. The *contingent epiderivative* $D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E})$ *of* v *at* $(t,\mathbf{x}) \in [0,T)\times\Omega$ *in a multi-valued direction* $\mathcal{E} \subset \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})$ is defined by

$$\begin{aligned} D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}) := \sup_{\varepsilon>0} \inf \Bigg\{ &\mathbb{E}\left[\frac{v(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q}) - v(t,\mathbf{x})}{\delta}\right] : \,(b,\sigma)\in\mathcal{E}, \\ &\delta\in(0,\varepsilon\wedge(T-t)], \text{ and } (p,q)\in\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^d)\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^{d\times d}) \text{ with} \\ &\|\mathbf{1}_{[\![t,t+\delta]\!]}.p\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)} + \|\mathbf{1}_{[\![t,t+\delta]\!]}.q\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})} \le \varepsilon\delta \Bigg\}, \end{aligned}$$

where $X^{t,\mathbf{x};b,\sigma;p,q}$ satisfies (4.4).

Remark 5.2. It is perhaps more intuitive to write $D^{1,2}_{\uparrow}$ as a limit inferior:

$$D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}) = \varliminf_{\substack{\delta\downarrow 0,\\ (p,q)\to(0,0)}} \inf_{(b,\sigma)\in\mathcal{E}} \mathbb{E}\left[\frac{v(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q}) - v(t,\mathbf{x})}{\delta}\right].$$

Note that analogous directional derivatives with the same order of $\varliminf$ and inf already show up in [44,54]. Connections of $D^{1,2}_{\uparrow}$ with path derivatives of smooth functions can be found in Theorem 6.5 and its proof.

The next results link contingent epiderivatives with mean quasi-tangent sets from Definition 4.5. This allows the application of the mean viability theorems to (5.1).

Lemma 5.3. $D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}) \le 0 \iff \mathcal{E} \in \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},v(t,\mathbf{x}))$.

Proof. (i) If $D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}) \le 0$, then, for all $n\in\mathbb{N}$, there is a $(\delta_n,(b,\sigma)_n,p_n,q_n) \in (0,\frac{1}{n}]\times\mathcal{E}\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^d)\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^{d\times d})$ such that

$$\begin{aligned} &\mathbb{E}\left[v(t+\delta_n, X^{t,\mathbf{x};(b,\sigma)_n;p_n,q_n}) - v(t,\mathbf{x})\right]\delta_n^{-1} \le n^{-1} \quad \text{and} \\ &\|\mathbf{1}_{[\![t,t+\delta_n]\!]}.p_n\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)} + \|\mathbf{1}_{[\![t,t+\delta_n]\!]}.q_n\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})} \le \delta_n/n, \end{aligned} \tag{5.2}$$

i.e., $\mathcal{E}\in\mathcal{QTS}_{\widehat{v},(-\infty,0]}(t,\mathbf{x},v(t,\mathbf{x}))$.

(ii) Suppose that $\mathcal{E}\in\mathcal{QTS}_{\widehat{v},(-\infty,0]}(t,\mathbf{x},v(t,\mathbf{x}))$, i.e., there exists a sequence $(\delta_n,(b,\sigma)_n,p_n,q_n)_n$ in $(0,T-t]\times\mathcal{E}\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^d)\times\mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^{d\times d})$ with $\delta_n \le n^{-1}$ such that (5.2) holds. Then $D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}) \le 0$ follows. ☐

Lemma 5.4. $v(t,\mathbf{x}) \le y \Longrightarrow \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},v(t,\mathbf{x})) \subset \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y)$.

[2] Recall that $X^{t,\mathbf{x},a}$ solves (4.1).

Proof. Let $\mathcal{E} \in \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},v(t,\mathbf{x}))$. With ε, δ, and X as in Definition 4.5,

$$\varepsilon\delta \geq \mathbb{E}\left[\widehat{v}(t+\delta,X,v(t,\mathbf{x}))\right] = \mathbb{E}\left[v(t+\delta,X)-v(t,\mathbf{x})\right] \geq \mathbb{E}\left[v(t+\delta,X)-y\right]$$

because $v(t,\mathbf{x}) \leq y$. This immediately yields $\mathcal{E} \in \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y)$. □

Recall $\mathcal{E}_+$ defined by (4.5).

Definition 5.5. We say v is a *quasi-contingent supersolution* of (5.1) if v is l.s.c. and bounded from below, $v(T,\cdot) \geq h$, and, for each $(t,\mathbf{x}) \in [0,T) \times \Omega$,

$$D^{1,2}_{\uparrow} v(t,\mathbf{x})\,(\mathcal{E}_+(t,\mathbf{x})) \leq 0.$$

Theorem 5.6. *If V^S is l.s.c., then it is a quasi-contingent supersolution of* (5.1).

Proof. Fix $\varepsilon > 0$ and $(t,\mathbf{x},y) \in [0,T) \times \Omega$ with $y \geq V^S(t,\mathbf{x})$. Also let $s \in [t,T]$. By the dynamic programming principle (e.g., Theorem 3.5 in [34]), there is an $a \in \mathcal{A}$ with $\mathbb{E}[V^S(s,X^{t,\mathbf{x},a})] - V^S(t,\mathbf{x}) \leq \varepsilon$. Moreover, by Proposition 4 in [19] (cf. also Lemma 2.7 in [23]), we can assume that $a \in \mathcal{A}^t$. Thus (cf. (4.2) and Remark 4.9)

$$\inf_{\tilde{a}\in\mathcal{A}^t} \mathbb{E}\left[V^S(s,X^{t,\mathbf{x},\tilde{a}})\right] \leq y.$$

By the proof of Theorem 4.6, $\mathcal{E}_+(t,\mathbf{x}) \in \mathcal{QTS}_{\widehat{V^S},K}(t,\mathbf{x},y)$. Hence, by Lemma 5.3, $D^{1,2}_{\uparrow} V^S(t,\mathbf{x})\,(\mathcal{E}_+(t,\mathbf{x})) \leq 0$. □

Theorem 5.7. *Let $v = v(t,\mathbf{x})$ be a quasi-contingent supersolution of* (5.1) *that is Lipschitz in $\mathbf{x}$ uniformly in t. Then $V^S \leq v$.*

Proof. Note that, by Lemmata 5.3 and 5.4, the following holds:

$$\begin{aligned}&\forall (t,\mathbf{x},y) \in [0,T) \times \Omega \times \mathbb{R} \text{ with } v(t,\mathbf{x}) \leq y:\\ &\quad \mathcal{E}_+(t,\mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},v(t,\mathbf{x})) \subset \mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y).\end{aligned} \tag{5.3}$$

Now, fix $(t,\mathbf{x}) \in [0,T) \times \Omega$. For every $\varepsilon > 0$, there exists, by (5.3) and Remark 4.9, a control $a \in \mathcal{A}^t$ such that $\mathbb{E}[v(T,X^{t,\mathbf{x},a})] \leq v(t,\mathbf{x}) + \varepsilon$. Thus

$$V^S(t,\mathbf{x}) \leq \mathbb{E}[h(X^{t,\mathbf{x},a})] \leq \mathbb{E}[v(T,X^{t,\mathbf{x},a})] \leq v(t,\mathbf{x}) + \varepsilon.$$

This concludes the proof. □

5.2. **Quasi-contingent subsolutions.**

Definition 5.8. Let v be bounded from above. Then the *contingent hypoderivative $D^{1,2}_{\downarrow} v(t,\mathbf{x})\,(1,b,\sigma)$ of v at $(t,\mathbf{x}) \in [0,T) \times \Omega$ in a direction $(1,b,\sigma) \in \mathbb{R} \times \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})$* is defined by

$$\begin{aligned}D^{1,2}_{\downarrow} v(t,\mathbf{x})\,(1,b,\sigma) := \inf_{\varepsilon>0}\, \sup\Bigg\{ &\mathbb{E}\left[\frac{v(t+\delta,X^{t,\mathbf{x};b,\sigma;p,q}) - v(t,\mathbf{x})}{\delta}\right]:\\ &\delta \in (0,\varepsilon],\ \text{and } (p,q) \in \mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^d) \times \mathbb{L}^0(\mathbb{F}^t,\mathbb{R}^{d\times d}) \text{ with}\\ &\|\mathbf{1}_{[\![t,t+\delta]\!]}.p\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)} + \|\mathbf{1}_{[\![t,t+\delta]\!]}.q\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})} \leq \varepsilon\delta \Bigg\},\end{aligned} \tag{5.4}$$

where $X^{t,\mathbf{x};b,\sigma;p,q}$ satisfies (4.4).

Remark 5.9. We may write $D_{\downarrow}^{1,2}$ as a limit superior (cf. Remark 5.2):

$$D_{\downarrow}^{1,2}v(t,\mathbf{x})\,(1,b,\sigma) = \varlimsup_{\substack{\delta\downarrow 0,\\ (p,q)\to(0,0)}} \mathbb{E}\left[\frac{v(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q}) - v(t,\mathbf{x})}{\delta}\right].$$

Definition 5.10. We call v a *quasi-contingent subsolution* of (5.1) if v is u.s.c. and bounded from above, $v(T,\cdot)\le h$, and for all $(t,\mathbf{x})\in[0,T)\times\Omega$,

$$\inf_{a\in\mathcal{A}^t} D_{\downarrow}^{1,2}v(t,\mathbf{x})\,(1,(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a))\ge 0.$$

Given $a\in\mathcal{A}$ and $(t,\mathring{\mathbf{x}})\in\mathbb{R}_+\times\Omega$, define $a^{t,\mathring{\mathbf{x}}}\in\mathcal{A}^t$ by

$$a^{t,\mathring{\mathbf{x}}}(s,\omega) := a(s,\mathbf{1}_{[0,t)}.\mathring{\mathbf{x}} + \mathbf{1}_{[t,\infty)}.(\mathring{\mathbf{x}}_t+\omega-\omega_t)).$$

For fixed $a\in\mathcal{A}$, we consider here, instead of (4.1), the systems

$$\begin{aligned} &dX_s^{t,\mathbf{x},\mathring{\mathbf{x}},a} = b(s,X^{t,\mathbf{x},\mathring{\mathbf{x}},a},a_s^{t,\mathring{\mathbf{x}}})\,ds + \sigma(s,X^{t,\mathbf{x},\mathring{\mathbf{x}},a},a_s^{t,\mathring{\mathbf{x}}})\,dW_s \text{ on } (t,\infty),\ \mathbb{P}\text{-a.s.},\\ &X_s^{t,\mathbf{x},\mathring{\mathbf{x}},a} = \mathbf{x}_s \quad \text{on } [0,t], \end{aligned} \tag{5.5}$$

and, instead of (4.5), $\mathcal{E}_+^a(t,\mathbf{x},\mathring{\mathbf{x}}) := \{\mathbf{1}_{[0,T]}.(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a^{t,\mathring{\mathbf{x}}})\}$, where $(t,\mathbf{x},\mathring{\mathbf{x}})\in\mathbb{R}_+\times\Omega\times\Omega$.

Definition 5.11. Fix $a\in\mathcal{A}$. The pair $(\widehat{v},K)$ is *mean viable* for (5.5) if, for all $s\in[t,T]$ and $(t,\mathbf{x},\mathring{\mathbf{x}},y)\in\mathbb{R}_+\times\Omega\times\Omega\times\mathbb{R}$ with $v(t,\mathbf{x})\le y$, $\mathbb{E}\left[v(s,X^{t,\mathbf{x},\mathring{\mathbf{x}},a}\right]\le y$.

The next remark contains counterparts to Theorems 4.6 and 4.8, whose proofs have no essential differences and are omitted.

Remark 5.12. Let $u=u(t,\mathbf{x}):[0,T]\times\Omega\to\mathbb{R}$ be l.s.c., Lipschitz in $\mathbf{x}$ uniformly in t, and bounded from below. Define $\widehat{u}:[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ by $\widehat{u}(t,\mathbf{x},y):=u(t,\mathbf{x})-y$. Fix $a\in\mathcal{A}$. Then $(\widehat{u},K)$ is mean viable for (5.5) if and only if $u(t,\mathbf{x})\le y$ implies $\mathcal{E}_+^a(t,\mathbf{x},\mathring{\mathbf{x}})\in\mathcal{QTS}_{\widehat{u},K}(t,\mathbf{x},y)$ for every $(t,\mathbf{x},\mathring{\mathbf{x}},y)\in[0,T)\times\Omega\times\Omega\times\mathbb{R}$.

As in Theorem 4.6, we do not need u being Lipschitz in $\mathbf{x}$ and bounded from below for the necessary condition for mean viability.

Theorem 5.13. *If V^S is u.s.c, then it is a quasi-contingent subsolution of* (5.1).

Proof. Let $U:=-V^S$ and $\widehat{U}(t,\mathbf{x},y):=U(t,\mathbf{x})-y$. Fix $a\in\mathcal{A}$. By the dynamic programming principle (see, e.g., Theorem 3.5 in [34]), $V^S(t,\mathbf{x})\le\mathbb{E}[V^S(s,X^{t,\mathbf{x},\mathring{\mathbf{x}},a})]$. Thus $\mathbb{E}[U(s,X^{t,\mathbf{x},\mathring{\mathbf{x}},a}]\le y$ for every $(t,\mathbf{x},\mathring{\mathbf{x}})\in[0,T]\times\Omega\times\Omega$, $y\ge U(t,\mathbf{x})$, and $s\in[t,T]$, i.e., $(\widehat{U},K)$ is mean viable for (5.5). Hence, by Remark 5.12, we have $\mathcal{E}_+^a(t,\mathbf{x},\mathring{\mathbf{x}})\in\mathcal{QTS}_{\widehat{U},K}(t,\mathbf{x},y)$. Invoking Lemma 5.3 concludes the proof. □

Theorem 5.14. *Let $v=v(t,\mathbf{x})$ be a quasi-contingent subsolution of* (5.1) *that is Lipschitz in $\mathbf{x}$ uniformly in t. Then $v\le V^S$.*

Proof. Let $u:=-v$ and $\widehat{u}(t,\mathbf{x},y):=u(t,\mathbf{x})-y$. Fix $a\in\mathcal{A}$. Note that

$$D_{\downarrow}^{1,2}v(t,\mathbf{x})\,(1,(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a^{t,\mathring{\mathbf{x}}}))\ge 0 \Longleftrightarrow D_{\uparrow}^{1,2}u(t,\mathbf{x})\,(\mathcal{E}_+^a(t,\mathbf{x},\mathring{\mathbf{x}}))\le 0.$$

Thus, using Remark 5.12 instead of Remark 4.9, we can proceed as in the proof of Theorem 5.7 to deduce that $\mathbb{E}\left[\widehat{u}(T,X^{t,\mathbf{x},\mathring{\mathbf{x}}},u(t,\mathbf{x}))\right]\le 0$. Hence,

$$\mathbb{E}\left[h(X^{t,\mathbf{x},\mathring{\mathbf{x}},a})\right]\ge\mathbb{E}\left[v(T,X^{t,\mathbf{x},\mathring{\mathbf{x}},a})\right]\ge v(t,\mathbf{x}).$$

Finally, thanks to the proof of Proposition 4 (i) in [19], we obtain

$$V^S(t,\mathbf{x})= \inf_{a\in\mathcal{A}} \int_\Omega \mathbb{E}\left[h(X^{t,\mathbf{x},\mathring{\mathbf{x}},a})\right] \mathbb{P}(d\mathring{\mathbf{x}}) \geq v(t,\mathbf{x}).$$

This concludes the proof. □

5.3. **Comparison.** The next result follows directly from Theorems 5.7 and 5.14.

Theorem 5.15. *Let $v_- = v_-(t,\mathbf{x})$ be a quasi-contingent sub- and $v_+ = v_+(t,\mathbf{x})$ be a quasi-contingent supersolution of* (5.1)*. Suppose that v_- and v_+ are Lipschitz in $\mathbf{x}$ uniformly in t. Then $v_- \leq v_+$.*

5.4. **Quasi-contingent solutions.**

Definition 5.16. We call v a *quasi-contingent solution* of (5.1) if v is both, a quasi-contingent sub- and supersolution of (5.1).

Theorem 5.17. *Let h be Lipschitz. Then the value function $V^S = V^S(t,\mathbf{x})$ is the unique quasi-contingent solution of* (5.1) *that is Lipschitz in $\mathbf{x}$ uniformly in t.*

Proof. First note, that $V^S = V^S(t,\mathbf{x})$ is continuous and Lipschitz in $\mathbf{x}$ uniformly in t (see, e.g., Proposition 2.6 in [23]). Thus, by Theorems 5.6 and 5.13, V^S is a quasi-contingent solution of (5.1). Theorem 5.15 yields uniqueness. □

6. Classical and viscosity solutions of HJB equations

For the sake of notational simplicity, let $d = 1$ in this section. All results hold for higher dimensions subject to obvious modifications.

Given $(t,\mathbf{x}) \in [0,T)\times\Omega$ and $(b,\sigma) \in \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})\times\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})$, recall $X^{t,\mathbf{x};b,\sigma;p,q}$ from (4.4) and define $X^{t,\mathbf{x};b,\sigma} := X^{t,\mathbf{x};b,\sigma;0,0}$, i.e.,

$$\begin{aligned} dX_s^{t,\mathbf{x};b,\sigma} &= b_s\,ds + \sigma_s\,dW_s \quad \text{on } (t,T],\ \mathbb{P}\text{-a.s.}, \\ X_s^{t,\mathbf{x};b,\sigma} &= \mathbf{x}_s \quad \text{on } [0,t]. \end{aligned} \tag{6.1}$$

6.1. **Path derivatives and path differential operators.** A functional Itô calculus with first and second order path derivatives was introduced in [29]. In contrast to [29], we use an implicit definition for our path derivatives (adapted from [31]).

Definition 6.1. Let $t \in [0,T)$. The space $C_b^{1,2}([t,T]\times\Omega)$ consists of all bounded and continuous functions $\varphi : [t,T]\times\Omega \to \mathbb{R}$ for which there are bounded and continuous functions $\partial_t\varphi$, $\partial_\mathbf{x}\varphi$, $\partial^2_{\mathbf{x}\mathbf{x}}\varphi : [t,T]\times\Omega \to \mathbb{R}$ such that, for all $\mathbf{x}\in\Omega$, all $(b,\sigma) \in \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})\times\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})$ and all t_1, $t_2 \in [t,T]$ with $t_1 < t_2$,

$$\begin{aligned} \varphi(t_2,X) - \varphi(t_1,X) &= \int_{t_1}^{t_2} \partial_t\varphi(s,X) + b_s\,\partial_\mathbf{x}\varphi(s,X) + \frac{1}{2}\sigma_s^2\,\partial^2_{\mathbf{x}\mathbf{x}}\varphi(s,X)\,ds \\ &\quad + \int_{t_1}^{t_2} \sigma_s\,\partial_\mathbf{x}\varphi(s,X)\,dW_s,\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{6.2}$$

where $X = X^{t,\mathbf{x};b,\sigma}$ (see (6.1)).

Remark 6.2. The *path derivatives* $\partial_t\varphi$, $\partial_\mathbf{x}\varphi$, and $\partial^2_{\mathbf{x}\mathbf{x}}\varphi$ of any function φ in $C_b^{1,2}([t,T]\times\Omega)$ are uniquely determined (see [31] for details).

Definition 6.3. Let $\varphi \in C_b^{1,2}([0,T]\times\Omega)$ and $(t,\mathbf{x}) \in [0,T)\times\Omega$. For $a \in A$, define

$$\mathcal{L}^a\varphi(t,\mathbf{x}) := \partial_t\varphi(t,\mathbf{x}) + b(t,\mathbf{x},a)\,\partial_{\mathbf{x}}\varphi(t,\mathbf{x}) + \frac{1}{2}\sigma^2(t,\mathbf{x},a)\,\partial^2_{\mathbf{x}\mathbf{x}}\varphi(t,\mathbf{x}). \tag{6.3}$$

For $a \in \mathcal{A}$, define, with slight abuse of notation,[3]

$$\mathcal{L}^a\varphi(t,\mathbf{x}) := \overline{\lim_{\substack{\delta\downarrow 0,\\ (p,q)\to(0,0)}}}\ \frac{1}{\delta}\mathbb{E}\left[\int_t^{t+\delta}\mathcal{L}^{a(s)}\varphi(s,X^{p,q})\,ds\right], \tag{6.4}$$

where $dX_s^{p,q} = [b(s,\mathbf{x}_{\cdot\wedge t},a(s)) + p_s]\,ds + [\sigma(s,\mathbf{x}_{\cdot\wedge t},a(s)) + q_s]\,dW_s$ on $[t,T]$ with $X^{p,q}_{\cdot\wedge t} = \mathbf{x}_{\cdot\wedge t}$ and the limit superior in (6.4) is to be understood in the sense of the right-hand side of (5.4).

6.2. **Classical solutions.**

Definition 6.4. A function $u \in C_b^{1,2}([0,T]\times\Omega)$ is a *classical solution* of (5.1) if $u(T,\cdot) = h$ and $\inf_{a\in A}\mathcal{L}^a v(t,\mathbf{x}) = 0$ for all $(t,\mathbf{x}) \in [0,T)\times\Omega$.

Theorem 6.5. *Let the data from* (3.1) *be continuous. Then classical solutions of* (5.1) *are quasi-contingent solutions of* (5.1).

Proof. Let u be a classical solution of (5.1). Fix $(t,\mathbf{x}) \in [0,T)\times\Omega$. First, we establish the quasi-contingent supersolution property. To this end, we use

$$\begin{aligned}\mathcal{E}_+^\circ(t,\mathbf{x}) := \{(b,\sigma) \in \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})\times\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}) :\ \exists a \in A :\\ (b,\sigma)(s,\omega) = (b,\sigma)(s,\mathbf{x}_{\cdot\wedge t},a)\quad dt\times d\mathbb{P}\text{-a.e. on } [t,T]\times\Omega\}.\end{aligned} \tag{6.5}$$

Recall (6.1). Then, by (6.2),

$$\begin{aligned}D_\uparrow^{1,2}u(t,\mathbf{x})\,(\mathcal{E}_+(t,\mathbf{x})) &\le \sup_{\varepsilon>0}\ \inf_{\substack{(b,\sigma)\in\mathcal{E}_+^\circ(t,\mathbf{x}),\\ \delta\in(0,\varepsilon\wedge(T-t))}}\mathbb{E}\left[\frac{u(t+\delta,X^{t,\mathbf{x};b,\sigma}) - u(t,\mathbf{x})}{\delta}\right]\\ &\le \inf_{(b,\sigma)\in\mathcal{E}_+^\circ(t,\mathbf{x})}\ \underline{\lim}_{\delta\downarrow 0}\mathbb{E}\left[\frac{u(t+\delta,X^{t,\mathbf{x};b,\sigma}) - u(t,\mathbf{x})}{\delta}\right]\\ &= \inf_{a\in A}\mathcal{L}^a u(t,\mathbf{x}) = 0,\end{aligned}$$

i.e., u is a quasi-contingent supersolution of (5.1). Next, we establish the quasi-contingent subsolution property. Note that, for each $a \in \mathcal{A}^t$, (6.2) yields

$$\begin{aligned}D_\downarrow^{1,2}u(t,\mathbf{x})\,(1,(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)) &\ge \overline{\lim_{\delta\downarrow 0}}\,\mathbb{E}\left[\frac{u(t+\delta,X^{t,\mathbf{x};(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)}) - u(t,\mathbf{x})}{\delta}\right]\\ &= \overline{\lim_{\delta\downarrow 0}}\,\mathbb{E}\left[\frac{\int_t^{t+\delta}\mathcal{L}^{a(s)}u(s,X^{t,\mathbf{x};(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)})\,ds}{\delta}\right]\\ &\ge \overline{\lim_{\delta\downarrow 0}}\,\mathbb{E}\left[\frac{\int_t^{t+\delta}\inf_{\tilde a\in A}\mathcal{L}^{\tilde a}u(s,X^{t,\mathbf{x};(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)})\,ds}{\delta}\right]\\ &= \inf_{\tilde a\in A}\mathcal{L}^{\tilde a}u(t,\mathbf{x}) = 0,\end{aligned}$$

i.e., u is a quasi-contingent subsolution of (5.1), which concludes the proof. $\square$

[3]There should be no danger confusion because if $a \in \mathcal{A}$ is constant after time t, say $a(s,\omega) = \tilde a \in A$ for each $(s,\omega) \in [t,T]\times\Omega$, then $\mathcal{L}^a\varphi(t,\mathbf{x})$ in the sense of (6.4) coincides with $\mathcal{L}^{\tilde a}\varphi(t,\mathbf{x})$ in the sense of (6.3). Here, it is assumed that $b(\cdot,\cdot,\cdot)$ and $\sigma(\cdot,\cdot,\cdot)$ from (3.1) are continuous.

6.3. **Viscosity solutions.** The notions we employ here are in the framework of the notions of viscosity solutions in [21, 30, 31] for path-dependent PDEs that use tangency in mean as opposed to the usually used pointwise tangency.

First, we introduce test function spaces for our notion of viscosity solutions. They are very similar to the spaces in Remark 6.7 of [21]. For $u : [0,T] \times \Omega \to \mathbb{R}$, $(t,\mathbf{x}) \in [0,T) \times \Omega$, and $\mathcal{E} \subset \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}) \times \mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})$, put

$$\begin{aligned}\overline{\Phi}^{\mathcal{E}} u(t,\mathbf{x}) &:= \big\{\varphi \in C_b^{1,2}([t,T]\times\Omega) : \exists \varepsilon > 0 : \forall \delta \in (0,\varepsilon] : \forall (b,\sigma;p,q) \in \mathcal{E} \times \mathbf{O}^t_{\varepsilon,\delta} : \\ &\qquad 0 = (\varphi - u)(t,\mathbf{x}) \geq \mathbb{E}\left[(\varphi-u)(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q})\right]\big\}, \\ \underline{\Phi}^{\mathcal{E}} u(t,\mathbf{x}) &:= \big\{\varphi \in C_b^{1,2}([t,T]\times\Omega) : \exists \varepsilon > 0 : \forall \delta \in (0,\varepsilon] : \forall (b,\sigma;p,q) \in \mathcal{E} \times \mathbf{O}^t_{\varepsilon,\delta} : \\ &\qquad 0 = (\varphi - u)(t,\mathbf{x}) \leq \mathbb{E}\left[(\varphi-u)(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q})\right]\big\},\end{aligned}$$

where $X^{t,\mathbf{x};b,\sigma;p,q}$ has been defined in (4.4) and

$$\mathbf{O}^t_{\varepsilon,\delta} := \left\{(p,q) \in \mathbb{L}^0(\mathbb{F}^t,\mathbb{R})^2 : \, \|\mathbf{1}_{[\![t,t+\delta]\!]}.p\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})} + \|\mathbf{1}_{[\![t,t+\delta]\!]}.q\|^2_{\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R})} \leq \varepsilon\delta\right\}.$$

Next, recall $\mathcal{E}_+(t,\mathbf{x})$ from (4.5) and, given $a \in \mathcal{A}$, put $\mathcal{E}^a_+(t,\mathbf{x}) := \{(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)\}$.

Definition 6.6. Consider a function $u : [0,T] \times \Omega \to \mathbb{R}$.

(i) We call u a *viscosity supersolution* of (5.1) if u is l.s.c. and bounded from below, $u(T,\cdot) \geq h$, and, for all $(t,\mathbf{x}) \in [0,T) \times \Omega$ and $\varphi \in \overline{\Phi}^{\mathcal{E}_+(t,\mathbf{x})} u(t,\mathbf{x})$,

$$\inf_{a\in A} \mathcal{L}^a \varphi(t,\mathbf{x}) \leq 0.$$

(ii) We call u a *viscosity subsolution* of (5.1) if u is u.s.c. and bounded from above, $u(T,\cdot) \leq h$, and, for all $(t,\mathbf{x}) \in [0,T) \times \Omega$, $a \in \mathcal{A}^t$, and $\varphi \in \underline{\Phi}^{\mathcal{E}^a_+(t,\mathbf{x})} u(t,\mathbf{x})$,

$$\mathcal{L}^a \varphi(t,\mathbf{x}) \geq 0.$$

(iii) We call u a *viscosity solution* of (5.1) if u is both, a viscosity sub- and supersolution of (5.1).

Remark 6.7. Our definition of viscosity supersolution is nearly identical with the one in Definition 6.8 in [21]. The viscosity subsolution case is slightly different. A function u is a viscosity subsolution of $\inf_{a\in A} \mathcal{L}^a u(t,\mathbf{x}) = 0$ in our sense if it is a viscosity subsolution of $\mathcal{L}^a u(t,\mathbf{x}) = 0$ for every $a \in \mathcal{A}^t$ in the sense of Definition 4.6 in [21] (up to minor technical differences).

Theorem 6.8. *Let the data from* (3.1) *be continuous. Let $u : [0,T] \times \Omega \to \mathbb{R}$ be a classical solution of* (5.1)*. Then u is a viscosity solution of* (5.1)*.*

Proof. Fix $(t,\mathbf{x}) \in [0,T) \times \Omega$. First, we establish the viscosity supersolution property. Fix $(b,\sigma) \in \mathcal{E}^\circ_+(t,\mathbf{x})$ (see (6.5)) with corresponding $a \in A$, recall (6.1), and let $\varphi \in \overline{\Phi}^{\mathcal{E}_+(t,\mathbf{x})} u(t,\mathbf{x})$, i.e., there is an $\varepsilon > 0$ such that, for all $\delta \in (0,\varepsilon]$,

$$\mathbb{E}\left[\varphi(t+\delta, X^{t,\mathbf{x};b,\sigma}) - \varphi(t,\mathbf{x})\right] \leq \mathbb{E}\left[u(t+\delta, X^{t,\mathbf{x};b,\sigma}) - u(t,\mathbf{x})\right].$$

Dividing this inequality by δ and letting $\delta \downarrow 0$ yields $\mathcal{L}^a\varphi(t,\mathbf{x}) \leq \mathcal{L}^a u(t,\mathbf{x})$, i.e., $\inf_{\tilde{a}\in A} \mathcal{L}^{\tilde{a}}\varphi(t,\mathbf{x}) \leq \inf_{\tilde{a}\in A} \mathcal{L}^{\tilde{a}} u(t,\mathbf{x}) = 0$.

Next, we establish the viscosity subsolution property. To this end, fix $a \in \mathcal{A}^t$ and $\varphi \in \underline{\Phi}^{\mathcal{E}^a_+(t,\mathbf{x})} u(t,\mathbf{x})$. Then $(b,\sigma) \in \mathcal{E}^a_+(t,\mathbf{x})$ implies

$$\mathbb{E}\left[\varphi(t+\delta, X^{t,\mathbf{x};b,\sigma}) - \varphi(t,\mathbf{x})\right] \geq \mathbb{E}\left[u(t+\delta, X^{t,\mathbf{x};b,\sigma}) - u(t,\mathbf{x})\right].$$

Hence $\mathcal{L}^a\varphi(t,\mathbf{x}) \geq \overline{\lim}_{\delta\downarrow 0} \frac{1}{\delta}\mathbb{E}\left[\int_t^{t+\delta} \mathcal{L}^{a(s)} u(s, X^{t,\mathbf{x};b,\sigma})\, ds\right] \geq 0.$ □

Theorem 6.9. *If $u : [0,T] \times \Omega \to \mathbb{R}$ is a viscosity supersolution of* (5.1)*, then it is a quasi-contingent supersolution of* (5.1)*.*

Proof. Assume u is not a quasi-contingent supersolution of (5.1). Then there is a pair $(t,\mathbf{x}) \in [0,T) \times \Omega$ such that $D^{1,2}_{\uparrow} u(t,\mathbf{x})(\mathcal{E}_+(t,\mathbf{x})) > c$ for some constant $c > 0$, i.e., there is an $\varepsilon > 0$ such that, for all $\delta \in (0,\varepsilon]$, $(b,\sigma) \in \mathcal{E}_+(t,\mathbf{x})$, and $(p,q) \in \mathbf{O}^t_{\varepsilon,\delta}$,

$$\mathbb{E}[u(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q}) - u(t,\mathbf{x})] > c\delta.$$

Hence the function $\varphi : [t,T] \times \Omega \to \mathbb{R}$ defined by $\varphi(s,\omega) := u(t,\mathbf{x}) + c \cdot (s-t)$ belongs to $\overline{\Phi}^{\mathcal{E}_+(t,\mathbf{x})}\, u(t,\mathbf{x})$. However, $\inf_{a \in A} \mathcal{L}^a \varphi(t,\mathbf{x}) = \partial_t \varphi(t,\mathbf{x}) = c > 0$, which is a contradiction to u being a viscosity supersolution of (5.1). ☐

Theorem 6.10. *If $u : [0,T] \times \Omega \to \mathbb{R}$ is a viscosity subsolution of* (5.1)*, then it is a quasi-contingent subsolution of* (5.1)*.*

Proof. Assume u is not a quasi-contingent subsolution of (5.1). Then there are $(t,\mathbf{x}) \in [0,T) \times \Omega$ and $a \in \mathcal{A}^t$ such that $D^{1,2}_{\downarrow} u(t,\mathbf{x})(1,(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a)) < -c$ for some $c > 0$, i.e., there is an $\varepsilon > 0$ such that, for all $\delta \in (0,\varepsilon]$ and $(p,q) \in \mathbf{O}^t_{\varepsilon,\delta}$,

$$\mathbb{E}[u(t+\delta, X^{t,\mathbf{x};(b,\sigma)(\cdot,\mathbf{x}_{\cdot\wedge t},a);p,q}) - u(t,\mathbf{x})] < -c\delta.$$

Define $\varphi(s,\omega) := u(t,\mathbf{x}) - c\cdot(s-t)$, $(s,\omega) \in [t,T] \times \Omega$. Then $\varphi \in \underline{\Phi}^{\mathcal{E}^a_+(t,\mathbf{x})}\, u(t,\mathbf{x})$ but $\mathcal{L}^a \varphi(t,\mathbf{x}) = -c < 0$. This is a contradiction to u being a viscosity subsolution. ☐

Remark 6.11. It would be very interesting to obtain counterparts of Theorems 6.9 and 6.10 with the usual "Crandall–Lions" notion of viscosity solutions (as used in [23, 24, 57] for path-dependent PDEs). While equivalence results between viscosity solutions and contingent solutions (or similar nonsmooth solutions) have been established in the first-order case for standard PDEs on finite-dimensional spaces (see, e.g., [6, 37, 44, 52]), for PDEs on Hilbert space in [20], for PDEs on Wasserstein space in [5], and for path-dependent PDEs in [38], obtaining corresponding results in the second-order case remains a challenging largely open problem.

6.4. **Comparison for viscosity solutions.** The following result is a direct consequence of Theorems 5.15, 6.9, and 6.10.

Theorem 6.12. *Let $u_- = u_-(t,\mathbf{x})$ be a viscosity sub- and $u_+ = u_+(t,\mathbf{x})$ be a viscosity supersolution of* (5.1)*. If u_- and u_+ are Lipschitz in $\mathbf{x}$ uniformly in t, then $u_- \le u_+$.*

6.5. **Existence and uniqueness for viscosity solutions.**

Theorem 6.13. *Let the data from* (3.1) *be continuous. Moreover, assume that h is Lipschitz and $(t,\mathbf{x},a) \mapsto (b,\sigma)(t,\mathbf{x},a)$ is uniformly continuous in t uniformly in $(\mathbf{x},a)$. Then the value function V^S is the unique viscosity solution of* (5.1)*.*

Proof. We sketch parts of the proof for existence, as the arguments are essentially the same as in the proof of Theorem 3.4 in [23] (however in [23] a maximization problem is studied in contrast to our minimization problem). First note that $V^S = V^S(t,\mathbf{x})$ is continuous and Lipschitz in $\mathbf{x}$ uniformly in t by Proposition 2.6 in [23].

Next, fix $(t,\mathbf{x}) \in [0,T)$. We establish now the viscosity supersolution property. To this end, let $\varphi \in \overline{\Phi}^{\mathcal{E}_+(t,\mathbf{x})}\, V^S(t,\mathbf{x})$ with corresponding $\varepsilon > 0$. Note that, for every

$\delta > 0$, the dynamic programming principle provides us with an $a^\delta \in \mathcal{A}^t$ (cf. the proof of Theorem 5.6) such that

$$V^S(t,\mathbf{x}) + \delta^2 \geq \mathbb{E}\left[V^S(t+\delta, X^{t,\mathbf{x},a^\delta})\right] = \mathbb{E}\left[V^S(t+\delta, X^{t,\mathbf{x};b^\delta,\sigma^\delta;p^\delta,q^\delta})\right], \tag{6.6}$$

where

$$\begin{aligned}(b^\delta_s, \sigma^\delta_s) &:= (b,\sigma)(s, \mathbf{x}_{\cdot\wedge t}, a^\delta_s),\\ (p^\delta_s, q^\delta_s) &:= (b(s, X^{t,\mathbf{x},a^\delta}, a^\delta_s) - b^\delta_s,\ \sigma(s, X^{t,\mathbf{x},a^\delta}, a^\delta_s) - \sigma^\delta_s).\end{aligned}$$

By the proof of Theorem 4.6, there is a constant $C > 0$ independent from δ such that whenever $\delta < (2CL_b^4)^{-1}\varepsilon \wedge \varepsilon \wedge (T-t)$, we have $(p^\delta, q^\delta) \in \mathbf{O}^t_{\varepsilon,\delta}$. Thus we can assume from now on that δ is sufficiently small such that φ satisfies

$$\varphi(t,\mathbf{x}) + \delta^2 \geq \mathbb{E}\left[\varphi(t+\delta, X^{t,\mathbf{x},a^\delta})\right] = \mathbb{E}\left[\varphi(t+\delta, X^{t,\mathbf{x};b^\delta,\sigma^\delta;p^\delta,q^\delta})\right].$$

This is possible thanks to $\varphi \in \overline{\Phi}^{\mathcal{E}_+(t,\mathbf{x})}\, V^S(t,\mathbf{x})$ and (6.6). Now we can proceed as in the proof of Theorem 3.4 in [23] to deduce that $\inf_{a\in A} \mathcal{L}^a \varphi(t,\mathbf{x}) \leq 0$.

Next, we deal with the viscosity subsolution property. Let $a \in \mathcal{A}^t$ and $\varphi \in \underline{\Phi}^{\mathcal{E}^a_+(t,\mathbf{x})}\, V^S(t,\mathbf{x})$ with corresponding $\varepsilon > 0$. By the dynamic programming principle,

$$V^S(t,\mathbf{x}) \leq \mathbb{E}\left[V^S(t+\delta, X^{t,\mathbf{x},a})\right] = \mathbb{E}\left[V^S(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q})\right] \tag{6.7}$$

for all $\delta \in (0, T-t]$. Here, $(b,\sigma) \in \mathcal{E}^a_+(t,\mathbf{x})$ and (p,q) is defined as in the proof of Theorem 4.6, which also yields, for all sufficiently small $\delta > 0$, $(p,q) \in \mathbf{O}^t_{\varepsilon,\delta}$ and thus, by (6.7), we have $\varphi(t,\mathbf{x}) \leq \mathbb{E}\left[\varphi(t+\delta, X^{t,\mathbf{x};b,\sigma;p,q})\right]$. Hence $\mathcal{L}^a\varphi(t,\mathbf{x}) \geq 0$.

Finally, Theorem 6.12 yields uniqueness. This concludes the proof. □

7. The extension lemma

Purpose of this section is to prove the extension lemma (Lemma 7.6 below), which is needed in Step 3 of the proof of Proposition 4.11.

To deal with measurability issues and since we need to use regular conditional probability distributions (r.c.p.d.), we will work in a weak formulation, i.e., controls will be replaced by probability measures. An appropriate setting is introduced in Section 7.1 and the connection between controls and related probability measures is treated in Section 7.2. In Section 7.3, we obtain measurability of controls coming from mean quasi-tangential conditions with respect to initial data. This property is crucial. Finally, Section 7.4 contains statement and proof of the extension lemma.

7.1. **Setting.** The presentation is adapted from [34].

Given a topological space E, we define the spaces

$$\begin{aligned}\mathbb{M}(\mathbb{R}_+ \times E) &:= \{\text{all } \sigma\text{-finite measures on } \mathcal{B}(\mathbb{R}_+ \times E)\} \text{ and}\\ \mathbb{M}_E &:= \{m \in \mathbb{M}(\mathbb{R}_+ \times E) :\ m(dt, de) = m(t, de)\, dt\},\end{aligned}$$

i.e., the marginal distribution on $\mathbb{R}_+$ of any element of $\mathbb{M}_E$ equals the Lebesgue measure. Note that we use the same notation for an element $m(dt, de)$ of $\mathbb{M}_E$ and its kernel $m(t, de)$.

Our extended canonical space is

$$\tilde{\Omega} := \Omega \times \Omega \times \mathbb{M}_A \times \mathbb{M}_{\mathbb{R}^d} \times \mathbb{M}_{\mathbb{R}^{d\times d}}$$

with canonical process $(\tilde{W}, \tilde{X}, \tilde{M}_A, \tilde{M}_P, \tilde{M}_Q)$ defined by $\tilde{W}_t(\tilde{\omega}) := \mathring{\omega}_t$, $\tilde{X}_t(\tilde{\omega}) := \omega_t$, $\tilde{M}_A(\tilde{\omega}) := m_A$, $\tilde{M}_P(\tilde{\omega}) := m_P$, and $\tilde{M}_Q(\tilde{\omega}) := m_Q$ for every $t \geq 0$ and every $\tilde{\omega} = (\mathring{\omega}, \omega, m_A, m_P, m_Q) \in \tilde{\Omega}$. The (raw) filtration generated by this canonical process is denoted by $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$. Given $t \geq 0$, we also use the shifted canonical process $(\tilde{W}^t, \tilde{M}_A^t, \tilde{M}_P^t, \tilde{M}_Q^t)$ defined by $\tilde{W}_s^t := \tilde{W}_{s+t} - \tilde{W}_t$, $\tilde{M}_A^t(s, da)\, ds := \tilde{M}_A(t+s, da)\, ds$, $\tilde{M}_P^t(s, dp)\, ds := \tilde{M}_P(t+s, dp)\, ds$, and $\tilde{M}_Q^t(s, dq)\, ds := \tilde{M}_Q(t+s, dq)\, ds$, $s \geq 0$.

Solutions of controlled stochastic differential equations are expressed as solutions of martingale problems (quite similarly as in [42]). To this end, consider functions $\tilde{b} : \mathbb{R}_+ \times \Omega \times A \times \mathbb{R}^d \to \mathbb{R}^{2d}$ and $\tilde{\sigma} : \mathbb{R}_+ \times \Omega \times A \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ defined by

$$\tilde{b}(t, \mathbf{x}, a, p) := (0, b(t, \mathbf{x}, a) + p)^\top \text{ and } \tilde{\sigma}(t, \mathbf{x}, a, q) := \begin{pmatrix} I_{d\times d} \\ \sigma(t, \mathbf{x}, a) + q \end{pmatrix}.$$

Next, we define an infinitesimal generator $\tilde{\mathcal{L}}$ on $C_b^2(\mathbb{R}^{2d})$ as follows. For each $\varphi \in C_b^2(\mathbb{R}^{2d})$ and each $(t, \mathbf{x}, a, p, q, x) \in \mathbb{R}_+ \times \Omega \times A \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^{2d}$, put

$$(\tilde{\mathcal{L}}\varphi)(t, \mathbf{x}, a, p, q, x) := \tilde{b}(t, \mathbf{x}, a, p) \cdot D\varphi(x) + \frac{1}{2}(\tilde{\sigma}\tilde{\sigma}^\top)(t, \mathbf{x}, a, q) : D^2\varphi(x).$$

For each t, $t_0 \in \mathbb{R}_+$ and $\varphi \in C_b^2(\mathbb{R}^{2d})$, define a random variable $C_t^{t_0}(\varphi)$ on $\tilde{\Omega}$ by

$$C_t^{t_0}(\varphi)(\tilde{\omega}) := \Bigg[\varphi(\tilde{W}_t, \tilde{X}_t) - \int_0^t \int_A \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d\times d}} (\tilde{\mathcal{L}}\varphi)(s, \tilde{X}_{\cdot\wedge t_0}, a, p, q, \tilde{W}_s, \tilde{X}_s) \; \tilde{M}_Q(s, dq)\, \tilde{M}_P(s, dp)\, \tilde{M}_A(s, da)\, ds\Bigg](\tilde{\omega}) \tag{7.1}$$

if $\tilde{\omega} = (\mathring{\omega}, \omega, \delta_{\bar{a}(s)}(da)\, ds, \delta_{\bar{p}(s)}(dp)\, ds, \delta_{\bar{q}(s)}(dq)\, ds)$ for some Borel measurable function $(\bar{a}, \bar{p}, \bar{q}) : \mathbb{R}_+ \to A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ with $\int_0^s \left|\bar{p}^i(r)\right| dr$ and $\int_0^s \left|\bar{q}^{i,j}(r)\right| dr$ being finite for each i, $j \in \{1, \ldots, d\}$ and each $s \in \mathbb{R}_+$ (cf. (1.6) and Remark 1.4, both in [34]), and define $C_t^{t_0}(\varphi)$ by $C_t^{t_0}(\varphi)(\tilde{\omega}) := 0$ otherwise. We also use the map

$$\begin{aligned}\Upsilon : \mathcal{A} \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d}) &\to \mathcal{P}(\Omega \times \mathbb{M}_A \times \mathbb{M}_{\mathbb{R}^d} \times \mathbb{M}_{\mathbb{R}^{d\times d}}),\\ (\tilde{a}, \tilde{p}, \tilde{q}) &\mapsto \mathbb{P} \circ (W, \delta_{\tilde{a}_s(W)}(da)\, ds, \delta_{\tilde{p}_s(W)}(dp)\, ds, \delta_{\tilde{q}_s(W)}(dq)\, ds)^{-1},\end{aligned}$$

and its image $\Upsilon(\mathcal{A}\times\mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d}))$, which is a Borel set (Lemma 4.10 in [28] and p. 23 in [34]). Recall that $\mathcal{P}(\ldots)$ and $\delta_{(\cdot)}$ were defined in Section 2.

Definition 7.1. Let $(t, \mathbf{x}, \mathring{\mathbf{x}}) \in [0, T) \times \Omega \times \Omega$. Recall $(a^\circ, p^\circ, q^\circ)$ from (3.2). We call $\tilde{\mathbb{P}} \in \mathcal{P}(\tilde{\Omega})$ a *strong control rule starting at* $(t, \mathbf{x}, \mathring{\mathbf{x}})$ if, $\tilde{\mathbb{P}}$-a.s.,

$$(\tilde{X}, \tilde{W}, \tilde{M}_A(da), \tilde{M}_P(dp), \tilde{M}_Q(dq))|_{[0,t]} = (\mathbf{x}, \mathring{\mathbf{x}}, \delta_{a^\circ}(da), \delta_{p^\circ}(dp), \delta_{q^\circ}(dq))|_{[0,t]},$$

the processes $(C_s^t)(\varphi)_{t\leq s\leq T_1}$, $\varphi \in C_b^2(\mathbb{R}^{2d})$, $T_1 > t$, are $(\tilde{\mathbb{P}}, \tilde{\mathbb{F}})$-martingales, and

$$\begin{aligned}&\tilde{\mathbb{P}} \circ (\tilde{W}^t, \tilde{M}_A^t(s, da)\, ds, \tilde{M}_P^t(s, dp)\, ds, \tilde{M}_Q^t(s, dq)\, ds)^{-1}\\ &\qquad \in \Upsilon(\mathcal{A} \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d})).\end{aligned} \tag{7.2}$$

The *set of all strong control rules starting at* $(t, \mathbf{x}, \mathring{\mathbf{x}})$ is denoted by $\tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}}$.

7.2. **Connections between "strong" controls and strong control rules.** For every $(t, \mathbf{x}, \mathring{\mathbf{x}}) \in [0,T) \times \Omega \times \Omega$ and every $(\tilde{a}, \tilde{p}, \tilde{q}) \in \mathcal{A}^t \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d})$, consider the induced probability measure

$$\mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}} := \mathbb{P} \circ (W^{t,\mathring{\mathbf{x}}}, X^{t,\mathbf{x};\tilde{a},\tilde{p},\tilde{q}}, \delta_{\tilde{a}_s}(da)\, ds, \delta_{\tilde{p}_s}(dp)\, ds, \delta_{\tilde{q}_s}(dq)\, ds)^{-1} \tag{7.3}$$

in $\mathcal{P}(\tilde{\Omega})$, where $W^{t,\mathring{\mathbf{x}}} := \mathring{\mathbf{x}}.\mathbf{1}_{[0,t)} + (\mathring{\mathbf{x}}_t + W - W_t).\mathbf{1}_{[t,\infty)}$ and $X^{t,\mathbf{x};\tilde{a},\tilde{p},\tilde{q}}$ is the unique solution of

$$dX_s^{t,\mathbf{x};\tilde{a},\tilde{p},\tilde{q}} = [b(s, \mathbf{x}_{\cdot\wedge t}, \tilde{a}_s) + \tilde{p}_s]\, ds + [\sigma(s, \mathbf{x}_{\cdot\wedge t}, \tilde{a}_s) + \tilde{q}_s]\, dW_s, \quad \mathbb{P}\text{-a.s.}, \tag{7.4}$$

on $[t,\infty)$ with initial condition $X^{t,\mathbf{x};\tilde{a},\tilde{p},\tilde{q}}|_{[0,t]} = \mathbf{x}|_{[0,t]}$.

Lemma 7.2. $\mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}} \in \tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}}$.

Proof. We verify only (7.2). Showing the rest is more standard. First, we define a "path shifting map" $\iota_t : \Omega \to \Omega$ by

$$[\iota_t(\omega)](\theta) := \omega_0.\mathbf{1}_{[0,t)}(\theta) + \omega_{\theta-t}.\mathbf{1}_{[t,\infty)}(\theta), \quad \theta \geq 0. \tag{7.5}$$

Note that there is a Borel map $f : \mathbb{R}_+ \times \Omega \to A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that, for all $s \in \mathbb{R}_+$, we have $(\tilde{a}, \tilde{p}, \tilde{q})_s = f_s(W_{(\cdot\wedge s)\vee t} - W_t)$. Thus, $\mathbb{P}$-a.s., for all $s \in \mathbb{R}_+$,

$$(\tilde{a}, \tilde{p}, \tilde{q})_{t+s} = f_{t+s}((W_{\cdot\wedge(t+s)} - W_t).\mathbf{1}_{[t,\infty)}) = f_{t+s}([\iota_t(W_{t+\cdot} - W_t)](\cdot \wedge (t+s))).$$

With $\tilde{\mathbb{P}} = \mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}}$, we therefore have

$$\begin{aligned}
&\tilde{\mathbb{P}} \circ (\tilde{W}^t, \tilde{M}_A^t(s,da)\, ds, \tilde{M}_P^t(s,dp)\, ds, \tilde{M}_Q^t(s,dq)\, ds)^{-1} \\
&\quad = \mathbb{P} \circ (W_{t+\cdot}^{t,\mathring{\mathbf{x}}} - W_t^{t,\mathring{\mathbf{x}}}, \delta_{\tilde{a}_{t+s}}(da)\, ds, \delta_{\tilde{p}_{t+s}}(dp)\, ds, \delta_{\tilde{q}_{t+s}}(dq)\, ds)^{-1} \\
&\quad = \mathbb{P} \circ (W, \delta_{\tilde{a}'_s}(da)\, ds, \delta_{\tilde{p}'_s}(dp)\, ds, \delta_{\tilde{q}'_s}(dq)\, ds)^{-1},
\end{aligned}$$

where $(\tilde{a}', \tilde{p}', \tilde{q}')_s := f_{t+s}([\iota_t(W)](\cdot \wedge (t+s)))$, $s \in \mathbb{R}_+$. From (7.5), we can deduce that $\tilde{a}'$, $\tilde{p}'$, and $\tilde{q}'$ are $\mathbb{F}$-progressive and thus (7.2) holds. □

Lemma 7.3. $\forall \tilde{\mathbb{P}} \in \tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}} : \exists (\tilde{a}, \tilde{p}, \tilde{q}) : \tilde{\mathbb{P}} = \mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}}$.

Proof. Fix $\tilde{\mathbb{P}} \in \tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}}$. First, note that, by (7.2), there exists an $(\tilde{a}, \tilde{p}, \tilde{q}) \in \mathcal{A} \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d})$, such that

$$(\tilde{M}_A^t(s,da), \tilde{M}_P^t(s,dp), \tilde{M}_Q^t(s,dq)) = (\delta_{\tilde{a}_s(\tilde{W}^t)}(da), \delta_{\tilde{p}_s(\tilde{W}^t)}(dp), \delta_{\tilde{q}_s(\tilde{W}^t)}(dq)), \tilde{\mathbb{P}}\text{-a.s.}$$

Next, define $\tilde{\xi} : \mathbb{R}_+ \times \tilde{\Omega} \to \mathbb{R}^d$ by $\tilde{\xi}|_{[0,t]} := \mathbf{x}|_{[0,t]}$ and

$$\begin{aligned}
\tilde{\xi}_{t+s} := \mathbf{x}_t &+ \int_0^s \left[b_{t+r}(\mathbf{x}_{\cdot\wedge t}, \tilde{a}_r(\tilde{W}^t)) + \tilde{p}_r(\tilde{W}^t) \right] dr \\
&+ \int_0^s \left[\sigma_{t+r}(\mathbf{x}_{\cdot\wedge t}, \tilde{a}_r(\tilde{W}^t)) + \tilde{q}_r(\tilde{W}^t) \right] d\tilde{W}_r^t, \quad \tilde{\mathbb{P}}\text{-a.s.}, \quad s \geq 0.
\end{aligned}$$

Using $W^t := W_{t+\cdot} - W_t$, we define $(\tilde{a}', \tilde{p}', \tilde{q}') : \mathbb{R}_+ \times \Omega \to A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ by

$$(\tilde{a}', \tilde{p}', \tilde{q}')_s := (a^\circ, p^\circ, q^\circ).\mathbf{1}_{[0,t]}(s) + (\tilde{a}, \tilde{p}, \tilde{q})_{s-t}(W^t).\mathbf{1}_{(t,\infty)}(s),\ s \geq 0. \tag{7.6}$$

Then $(\tilde{a}', \tilde{p}', \tilde{q}') \in \mathcal{A}^t \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d})$ and

$$\tilde{\mathbb{P}} \circ (\tilde{W}, \tilde{\xi}, \tilde{M}_A(s,da)\, ds, \tilde{M}_P(s,dp)\, ds, \tilde{M}_Q(s,dq)\, ds)^{-1} = \mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a}',\tilde{p}',\tilde{q}'}.$$

It remains to verify that $\tilde{X} = \tilde{\xi}$, $\tilde{\mathbb{P}}$-a.s., which can be done by proceeding as in the proof of Lemma A.1 in [25], i.e., by applying Itô's formula to $(|\tilde{X}_s - \tilde{\xi}_s|^2)_{s\geq t}$

and utilizing the $(\tilde{\mathbb{P}}, \tilde{\mathbb{F}})$-martingale properties of the processes $(C^t_s)(\varphi)_{t\le s\le T_1}$, $\varphi \in C^2_b(\mathbb{R}^{2d})$, $T_1 > t$, from Definition 7.1. This concludes the proof. $\square$

7.3. **Existence of "extension" control rules.** Fix $\varepsilon > 0$. Suppose that $(t, \mathbf{x}, y) \in [0, T) \times \Omega$ with $\widehat{v}(t, \mathbf{x}, y) \in K$ implies $\mathcal{E}_+(t, \mathbf{x}) \in \mathcal{QTS}_{\widehat{v},K}(t, \mathbf{x}, y)$. Then there is a $(\delta, \tilde{a}, \tilde{p}, \tilde{q}) \in (0, \varepsilon] \times \mathcal{A}^t \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}^t, \mathbb{P}, \mathbb{R}^{d\times d})$ such that, for all $\mathring{\mathbf{x}} \in \Omega$, $\mathbb{E}^{\mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}}} \left[\widehat{v}(t + \delta, \tilde{X}, y)\right] = \mathbb{E}\left[\widehat{v}(t + \delta, X^{t,\mathbf{x};\tilde{a},\tilde{p},\tilde{q}}, y)\right] \le \varepsilon\delta$ and

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^{t+\delta}\left[\int_{\mathbb{R}^d} |p|^2\ \tilde{M}_P(s, dp) + \int_{\mathbb{R}^{d\times d}} |q|^2\ \tilde{M}_Q(s, dq)\right] ds\right] \le \varepsilon\delta, \tag{7.7}$$

where $\tilde{\mathbb{P}} = \mathbb{P}_{t,\mathbf{x},\mathring{\mathbf{x}};\tilde{a},\tilde{p},\tilde{q}}$. Put $\Lambda := \{(t, \mathbf{x}, y) \in [0, T) \times \Omega \times \mathbb{R} : \widehat{v}(t, \mathbf{x}, y) \in K\}$. Then, taking Lemma 7.2 into account, we can see that the following holds:

$$\forall (t, \mathbf{x}, y, \mathring{\mathbf{x}}) \in \Lambda \times \Omega : \exists (\delta, \tilde{\mathbb{P}}) \in (0, \varepsilon] \times \tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}} : \mathbb{E}^{\tilde{\mathbb{P}}}\left[\widehat{v}(t + \delta, \tilde{X}, y)\right] \le \varepsilon\delta. \tag{7.8}$$

Consider

$$\begin{aligned} \mathbf{M} := \{&(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}) \in \Lambda \times \Omega \times (0, \varepsilon] \times \mathcal{P}(\tilde{\Omega}) : \\ &\tilde{\mathbb{P}} \in \tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}},\ \mathbb{E}^{\tilde{\mathbb{P}}}[\widehat{v}(t + \delta, \tilde{X}, y)] \le \varepsilon\delta,\ \text{ and (7.7) holds}\}. \end{aligned} \tag{7.9}$$

Lemma 7.4. *Let $\widehat{v}$ be l.s.a. Then $\mathbf{M}$ is analytic.*

Proof. First, note that, by Lemma 7.30 (3) in [10], $(t, \delta, \mathbf{x}, y) \mapsto \widehat{v}(t+\delta, \mathbf{x}, y)$ is l.s.a. Hence, by Proposition 7.48 in [10], $(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}) \mapsto \mathbb{E}^{\tilde{\mathbb{P}}}[\widehat{v}(t+\delta, \tilde{X}, y)]$ is l.s.a. Thus, together with Lemma 7.30 (1) in [10], the set $\{(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}) : \mathbb{E}^{\tilde{\mathbb{P}}}[\widehat{v}(t + \delta, \tilde{X}, y)] \le \varepsilon\delta\}$ is analytic. We skip the rest of the proof, which is basically identical to the proof of Lemma 3.2 in [34] (cf. also Lemmata 4.5 and 4.10 in [28]). $\square$

Next, define a map $H : \Lambda \times \Omega \times (0, \varepsilon] \times \mathcal{P}(\tilde{\Omega}) \to \overline{\mathbb{R}}$ by

$$H(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}) := (+\infty).\mathbf{1}_{\mathbf{M}^c}(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}).$$

Since $\mathbf{M}$ is analytic (Lemma 7.4), H is l.s.a. Define $H^* : \Lambda \times \Omega \to \overline{\mathbb{R}}$ by

$$H^*(t, \mathbf{x}, y, \mathring{\mathbf{x}}) := \inf_{(\delta,\tilde{\mathbb{P}})\in(0,\varepsilon]\times\mathcal{P}(\tilde{\Omega})} H(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta, \tilde{\mathbb{P}}).$$

By (7.7) and (7.8), Proposition 7.50 in [10] yields the existence of a universally (in fact, even analytically) measurable minimizer $(\delta^*, \mathbb{Q}^*) : \Lambda \times \Omega \to (0, \varepsilon] \times \mathcal{P}(\tilde{\Omega})$, i.e.,

$$H^*(t, \mathbf{x}, y, \mathring{\mathbf{x}}) = H(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta^*_{t,\mathbf{x},y,\mathring{\mathbf{x}}}, \mathbb{Q}^*_{t,\mathbf{x},y,\mathring{\mathbf{x}}}) = 0.$$

Note that, since $(t, \mathbf{x}, y, \mathring{\mathbf{x}}, \delta^*_{t,\mathbf{x},y,\mathring{\mathbf{x}}}, \mathbb{Q}^*_{t,\mathbf{x},y,\mathring{\mathbf{x}}}) \in \mathbf{M}$ for every $(t, \mathbf{x}, y, \mathring{\mathbf{x}}) \in \Lambda \times \Omega$, we have, by the definitions of $\mathbf{M}$ and $\tilde{\mathcal{P}}_{t,\mathbf{x},\mathring{\mathbf{x}}}$ (see (7.9) and Definition 7.1),

$$\mathbb{Q}^*_{t,\mathbf{x},y,\mathring{\mathbf{x}}} = \mathbb{Q}^*_{t,\mathbf{x}_{\cdot\wedge t},y,\mathring{\mathbf{x}}_{\cdot\wedge t}}. \tag{7.10}$$

Remark 7.5. By (3.3), $\widehat{v}(t, \mathbf{x}, v(t, \mathbf{x})) \in K$ and thus

$$[0, T) \times \Omega \times \Omega \to (0, \varepsilon] \times \mathcal{P}(\tilde{\Omega}), (t, \mathbf{x}, \mathring{\mathbf{x}}) \mapsto (\delta^{**}_{t,\mathbf{x},\mathring{\mathbf{x}}}, \mathbb{Q}^{**}_{t,\mathbf{x},\mathring{\mathbf{x}}}) := (\delta^*_{\mathfrak{y}}, \mathbb{Q}^*_{\mathfrak{y}})|_{\mathfrak{y}=(t,\mathbf{x},v(t,\mathbf{x}),\mathring{\mathbf{x}})},$$

is universally measurable (Proposition 7.44 in [10]).

7.4. **The extension lemma with its proof.**

Lemma 7.6 (the extension lemma)**.** *Let v be l.s.a. and bounded from below. If, for each $(t,\mathbf{x},y)\in[0,T)\times\Omega\times\mathbb{R}$ with $\widehat{v}(t,\mathbf{x},y)\in K$, we have $\mathcal{E}_+(t,\mathbf{x})\in\mathcal{QTS}_{\widehat{v},K}(t,\mathbf{x},y)$, then, given $\varepsilon>0$ and $(t,\mathbf{x},y)\in[0,T)\times\Omega\times\mathbb{R}$ with $\widehat{v}(t,\mathbf{x},y)\in K$, every ε-approximate solution $(\tau^\varepsilon,\varrho^\varepsilon,a^\varepsilon,\bar{p},\bar{q},X^\varepsilon)$ of (4.1) for $(\widehat{v},K)$ starting at $(t,\mathbf{x},y)$ with $\mathbb{P}(\tau^\varepsilon<T)>0$ has an "extension" $\mathfrak{s}^{\varepsilon,+}=(\tau^{\varepsilon,+},\varrho^{\varepsilon,+},a^{\varepsilon,+},p^{\varepsilon,+},q^{\varepsilon,+},X^{\varepsilon,+})$ that satisfies the following:*

(i) $\mathfrak{s}^{\varepsilon,+}$ is an ε-approximate solution of (4.1) for $(\widehat{v},K)$ starting at $(t,\mathbf{x},y)$.
(ii) Outside of a $\mathbb{P}$-evanescent set, $(\varrho^{\varepsilon,+},a^{\varepsilon,+},p^{\varepsilon,+},q^{\varepsilon,+})_{\cdot\wedge\tau^\varepsilon}=(\varrho^\varepsilon,a^\varepsilon,\bar{p},\bar{q})_{\cdot\wedge\tau^\varepsilon}$.
(iii) $\tau^\varepsilon\le\tau^{\varepsilon,+}$.
(iv) $\mathbb{P}(\tau^\varepsilon<\tau^{\varepsilon,+})>0$.

Proof. Fix $\varepsilon>0$, $(t,\mathbf{x},y)\in[0,T)\times\Omega\times\mathbb{R}$ with $\widehat{v}(t,\mathbf{x},y)\in K$, and an ε-approximate solution $(\tau^\varepsilon,\varrho^\varepsilon,a^\varepsilon,\bar{p},\bar{q},X^\varepsilon)$ of (4.1) for $(\widehat{v},K)$ starting at $(t,\mathbf{x},y)$ and satisfying $\mathbb{P}(E_1)>0$, where

$$E_1:=\{\tau^\varepsilon<T\}.$$

The rest of the proof is structured as follows. In Step 1, we identify $(X^\varepsilon,a^\varepsilon,\bar{p},\bar{q})$ with the solution of a martingale problem specified by a probability measure $\mathbb{P}^\varepsilon$ on $\tilde{\Omega}$. Thanks to the mean quasi-tangency condition

$$\mathcal{E}_+(\tau^\varepsilon(\omega),X^\varepsilon(\omega))\in\mathcal{QTS}_{\widehat{v},K}(\tau^\varepsilon(\omega),X^\varepsilon(\omega),v(\tau^\varepsilon(\omega),X^\varepsilon(\omega))\text{ whenever }\tau^\varepsilon(\omega)<T,$$

we obtain from Section 7.3 extension control rules specified by probability measures. This leads to a kernel, whose measurability is shown in Step 2. In Step 3, we concatenate this kernel with $\mathbb{P}^\varepsilon$. This concatenation has appropriate martingale properties (Step 4) and corresponds to a "candidate" extended solution $\mathfrak{s}^+$ (Step 5). In Step 6, we verify that $\mathfrak{s}^+$ is indeed an ε-approximate solution.

Step 1. To relate X^ε to the solution of a martingale problem (cf. Section 7.1) define $(p^\varepsilon,q^\varepsilon)\in\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F}^t,\mathbb{P},\mathbb{R}^{d\times d})$ by

$$\begin{aligned}p^\varepsilon_s&:=\mathbf{1}_{[t,\tau^\varepsilon]}(s).[b(s,X^\varepsilon_{\cdot\wedge\varrho^\varepsilon_s},a^\varepsilon_s)+\bar{p}_s]-b(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},a^\varepsilon_s)\quad\text{and}\\ q^\varepsilon_s&:=\mathbf{1}_{[t,\tau^\varepsilon]}(s).[\sigma(s,X^\varepsilon_{\cdot\wedge\varrho^\varepsilon_s},a^\varepsilon_s)+\bar{q}_s]-\sigma(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},a^\varepsilon_s),\quad s\ge t,\end{aligned}$$

so that (cf. condition (v) of Definition 4.10), $\mathbb{P}$-a.s. on $[t,\infty)$, we have

$$dX^\varepsilon_{s\wedge\tau^\varepsilon}=[b(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},a^\varepsilon_s)+p^\varepsilon_s]\,ds+[\sigma(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},a^\varepsilon_s)+q^\varepsilon_s]\,dW_s.\tag{7.11}$$

Fix some $\mathring{\mathbf{x}}\in\Omega$. Similarly as in (7.3), consider the induced probability measure

$$\mathbb{P}^\varepsilon:=\mathbb{P}\circ(W^{t,\mathring{\mathbf{x}}},X^\varepsilon_{\cdot\wedge\tau^\varepsilon},\delta_{a^\varepsilon_s}(da)\,ds,\delta_{p^\varepsilon_s}(dp)\,ds,\delta_{q^\varepsilon_s}(dq)\,ds)^{-1}\in\mathcal{P}(\tilde{\Omega}).\tag{7.12}$$

For every $T_1>t$ and $\varphi\in C^2_b(\mathbb{R}^{2d})$, the process $C(\varphi):\mathbb{R}_+\times\tilde{\Omega}\to\mathbb{R}$ defined by[4]

$$C_s(\varphi):=C_s^{\tau^\varepsilon\circ\tilde{W}}(\varphi)$$

is, due to (7.11), a $(\mathbb{P}^\varepsilon,\tilde{\mathbb{F}})$-martingale on $[t,T_1]$. Also note that $\mathbb{P}^\varepsilon(\tilde{E}_1)>0$, where

$$\tilde{E}_1:=\{\tau^\varepsilon\circ\tilde{W}<T\}\subset\tilde{\Omega}.$$

Step 2. Using (7.10) and Remark 7.5, we can apply the same argument as in the paragraph before (2.7) in [46] (see, in particular, Lemma 2.5 therein) to deduce the

[4]Recall the definition of C^{t_0} for deterministic times t_0 in (7.1). The extension for stopping times is done in the usual way.

existence of an $\tilde{\mathcal{F}}_{\tau^\varepsilon\circ\tilde{W}}$-measurable function $\tilde{\delta}^*:\tilde{\Omega}\to\mathbb{R}_+$ and an $\tilde{\mathcal{F}}_{\tau^\varepsilon\circ\tilde{W}}$-measurable kernel $\tilde{\mathbb{Q}}^*_\cdot:\tilde{\Omega}\to\mathcal{P}(\tilde{\Omega})$ such that, for $\mathbb{P}^\varepsilon$-a.e. $\tilde{\omega}\in\tilde{E}_1$,

$$(\tilde{\delta}^*_{\tilde{\omega}},\tilde{\mathbb{Q}}^*_{\tilde{\omega}})=(\delta^{**}_{\mathfrak{x}},\mathbb{Q}^{**}_{\mathfrak{x}})|_{\mathfrak{x}=(\tau^\varepsilon\circ\tilde{W}(\tilde{\omega}),\tilde{X}_{\cdot\wedge\tau^\varepsilon\circ\tilde{W}}(\tilde{\omega}),\tilde{W}_{\cdot\wedge\tau^\varepsilon\circ\tilde{W}}(\tilde{\omega}))} \tag{7.13}$$

and that, for $\mathbb{P}^\varepsilon$-a.e. $\tilde{\omega}\in\tilde{\Omega}\setminus\tilde{E}_1$,

$$(\tilde{\delta}^*_{\tilde{\omega}},\tilde{\mathbb{Q}}^*_{\tilde{\omega}})=(0,\mathbb{P}^\varepsilon_{\tilde{\omega}}), \tag{7.14}$$

where $\{\mathbb{P}^\varepsilon_{\tilde{\omega}}\}_{\tilde{\omega}\in\tilde{\Omega}}$ is an r.c.pd. of $\mathbb{P}^*$ given $\tilde{\mathcal{F}}_{\tau^\varepsilon\circ\tilde{W}}$ (cf. pp. 13 and 25 in [34]).

Step 3. Following [42] (see, in particular, pp. 880-881) and [34] (see, in particular, p. 11), we introduce now concatenated probability measures relevant in our context. First, given $\tilde{\omega}\in\tilde{\Omega}$, consider the σ-field

$$\begin{aligned}(\tilde{\mathcal{F}})^{\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}:=\sigma(&\tilde{W}_s,\tilde{X}_s,\tilde{M}_{A,s}(\varphi_A)-\tilde{M}_{A,\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}(\varphi_A),\tilde{M}_{P,s}(\varphi_P)\\&-\tilde{M}_{P,\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}(\varphi_P),\tilde{M}_{Q,s}(\varphi_Q)-\tilde{M}_{Q,\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}(\varphi_Q):\ s\geq\tau^\varepsilon\circ W^\varepsilon(\tilde{\omega}),\\&\varphi_A\in C_b(\mathbb{R}_+\times A),\ \varphi_B\in C_b(\mathbb{R}_+\times\mathbb{R}^d),\ \text{and}\ \varphi_Q\in C_b(\mathbb{R}_+\times\mathbb{R}^{d\times d})),\end{aligned}$$

where $M_{A,s}(\varphi)=\int_0^s\int_A\varphi(r,a)\,M_A(dr,da)$, etc. Then we denote by $\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}}$ the unique probability measure in $\mathcal{P}(\tilde{\Omega})$ such that

$$\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}}\,((\tilde{X},\tilde{W},\tilde{M}_A(da),\tilde{M}_P(dp),\tilde{M}_Q(dq))|_{[0,\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})]}=\tilde{\omega}|_{[0,\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})]})=1$$

and that $\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}}$ coincides with $\tilde{\mathbb{Q}}^*_{\tilde{\omega}}$ on $(\tilde{\mathcal{F}})^{\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}$. Second, denote by

$$\tilde{\mathbb{P}}^*=\mathbb{P}^\varepsilon\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_\cdot$$

the unique probability measure in $\mathcal{P}(\tilde{\Omega})$ such that $\{\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}}\}_{\tilde{\omega}\in\tilde{\Omega}}$ is an r.c.p.d. of $\tilde{\mathbb{P}}^*$ given $\tilde{\mathcal{F}}_{\tau^\varepsilon\circ\tilde{W}}$ and that $\tilde{\mathbb{P}}^*$ coincides with $\mathbb{P}^\varepsilon$ on $\tilde{\mathcal{F}}_{\tau^\varepsilon\circ\tilde{W}}$. In particular, $\tilde{\mathbb{P}}^*(\tilde{E}_1)>0$.

Step 4. Martingale properties of $\tilde{\mathbb{P}}^*$ (cf. Lemma 3.3 in [34]): For $\mathbb{P}^*$-a.e. $\tilde{\omega}\in\tilde{\Omega}$, every $\varphi\in C_b^2(\mathbb{R}^{2d})$, and every $T_1\geq T$, the process $(C_s^{\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})}(\varphi))_{\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})\leq s\leq T_1}$ is a $(\tilde{\mathbb{Q}}^*_{\tilde{\omega}},\tilde{\mathbb{F}})$- and thus also a $(\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}},\tilde{\mathbb{F}})$-martingale. By Galmarino's test (see, e.g., Theorem 100 in [27]), $\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}}(\tau^\varepsilon\circ\tilde{W}=\tau^\varepsilon\circ\tilde{W}(\tilde{\omega}))=1$. Hence $(C_s(\varphi))_{\tau^\varepsilon\circ\tilde{W}(\tilde{\omega})\leq s\leq T_1}$ is a $(\delta_{\tilde{\omega}}\otimes_{\tau^\varepsilon\circ\tilde{W}}\tilde{\mathbb{Q}}^*_{\tilde{\omega}},\tilde{\mathbb{F}})$-martingale. Thus, by Theorem 1.2.10 in [51], $(C_s(\varphi))_{t\leq s\leq T_1}$ is a $(\tilde{\mathbb{P}}^*,\tilde{\mathbb{F}})$-martingale.

Step 5. Gluing controls together in a measurable way as it has been done in the context of volatility control in Section 3 of [45], we show that there exists an $(X^{\varepsilon,+},\tilde{a}^*,\tilde{p}^*,\tilde{q}^*)\in\mathbb{S}^2(\mathbb{F},\mathbb{P},\mathbb{R}^d)\times\mathcal{A}\times\mathbb{L}^2(\mathbb{F},\mathbb{P},\mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F},\mathbb{P},\mathbb{R}^{d\times d})$ that coincides with $(X^\varepsilon,a^\varepsilon,p^\varepsilon,q^\varepsilon)$ on $[\![0,\tau^\varepsilon]\!]$ and that satisfies

$$\tilde{\mathbb{P}}^*=\mathbb{P}\circ(W^{t,\mathring{\mathbf{x}}},X^{\varepsilon,+},\delta_{\tilde{a}^*_s}(da)\,ds,\delta_{\tilde{p}^*_s}(dp)\,ds,\delta_{\tilde{q}^*_s}(dq)\,ds)^{-1}. \tag{7.15}$$

First, we establish the existence of a 'measurable' map

$$\begin{aligned}\upsilon=(\upsilon_A,\upsilon_P,\upsilon_Q):\tilde{E}_1\to\mathcal{A}\times\mathbb{L}^1(\mathbb{F},\mathbb{P},\mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F},\mathbb{P},\mathbb{R}^{d\times d}),\\ \tilde{\omega}\mapsto(\tilde{a},\tilde{p},\tilde{q}),\end{aligned}$$

such that, with $\tilde{\tau}:=\tau^\varepsilon\circ\tilde{W}$, the measures $\tilde{\mathbb{Q}}^*_{\tilde{\omega}}$ satisfy

$$\begin{aligned}&(\tilde{M}_A^{\tilde{\tau}(\tilde{\omega})}(s,da),\tilde{M}_P^{\tilde{\tau}(\tilde{\omega})}(s,dp),\tilde{M}_Q^{\tilde{\tau}(\tilde{\omega})}(s,dq))\\&\quad=(\delta_{\tilde{a}_s(\tilde{W}^{\tilde{\tau}(\tilde{\omega})})}(da),\delta_{\tilde{p}_s(\tilde{W}^{\tilde{\tau}(\tilde{\omega})})}(dp),\delta_{\tilde{q}_s(\tilde{W}^{\tilde{\tau}(\tilde{\omega})})}(dq)),\qquad\tilde{\mathbb{Q}}^*_{\tilde{\omega}}\text{-a.s.}\end{aligned}$$

To this end, consider[5] (cf. the set A in Step 2 on p. 9 in [45]) the Borel set

$$\mathbf{Y} := \{(\tilde{\omega}, \tilde{a}, \tilde{p}, \tilde{q}) \in \tilde{E}_1 \times \mathcal{A} \times \mathbb{L}^1(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d}) : \\ \tilde{\mathbb{Q}}^*_{\tilde{\omega}} \circ (\tilde{W}^{\tilde{\tau}(\tilde{\omega})}, \tilde{M}_A^{\tilde{\tau}(\tilde{\omega})}(s, da)\, ds, \tilde{M}_P^{\tilde{\tau}(\tilde{\omega})}(s, dp)\, ds, \tilde{M}_Q^{\tilde{\tau}(\tilde{\omega})}(s, dq)\, ds)^{-1} = \Upsilon(\tilde{a}, \tilde{p}, \tilde{q})\}.$$

Then, by Proposition 7.49 in [10], there exists an analytically measurable map

$$\upsilon = (\upsilon_A, \upsilon_P, \upsilon_Q) : \mathrm{proj}_{\tilde{E}_1}(\mathbf{Y}) = \{\tilde{\omega} \in \tilde{E}_1 : \exists(\tilde{a}, \tilde{p}, \tilde{q}) : (\tilde{\omega}, \tilde{a}, \tilde{p}, \tilde{q}) \in \mathbf{Y}\} \\ \to \mathcal{A} \times \mathbb{L}^1(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d})$$

whose graph $\{(\tilde{\omega}, \upsilon(\tilde{\omega})) : \tilde{\omega} \in \mathrm{proj}_{\tilde{E}_1}(\mathbf{Y})\}$ is a subset of $\mathbf{Y}$. Note that, by (7.13), $\tilde{\mathbb{P}}^\varepsilon(\tilde{E}_1 \setminus \mathrm{proj}_{\tilde{E}_1}(\mathbf{Y})) = 0$. Thus (cf. the discussion before (7.13)) there is a map $\tilde{\upsilon} = (\tilde{\upsilon}_A, \tilde{\upsilon}_P, \tilde{\upsilon}_Q)$ from $\tilde{E}_1$ to $\mathcal{A}\times\mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d)\times\mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d})$ that is $\tilde{\mathcal{F}}_{\tilde{\tau}}$-measurable[6] and that coincides with υ, $\tilde{\mathbb{P}}^\varepsilon$-a.e. on $\tilde{E}_1$.[7] Now, for each $m \in \mathbb{N}$, let $\{B^{m,n}\}_{n\in\mathbb{N}}$ be a countable Borel partition of $\mathcal{A} \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^d) \times \mathbb{L}^2(\mathbb{F}, \mathbb{P}, \mathbb{R}^{d\times d})$, which is understood to be equipped with a suitable metric (cf. [45]). Under said metric, the diameter of each set $B^{m,n}$ is not to exceed $1/m$. For each m, $n \in \mathbb{N}$, fix $\beta^{m,n} \in B^{m,n}$. For each $m \in \mathbb{N}$, define $(\tilde{a}^m, \tilde{p}^m, \tilde{q}^m) : \mathbb{R}_+ \times E_1 \to A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ by[8] (cf. (7.6))

$$(\tilde{a}^m, \tilde{p}^m, \tilde{q}^m)(s, \omega) := \mathbf{1}_{[\tau^\varepsilon(\omega),\infty)}(s) \sum_{n\in\mathbb{N}} \beta^{m,n}_{s-\tau^\varepsilon(\omega)}(W^{\tau^\varepsilon(\omega)}(\omega)) \\ \cdot \mathbf{1}_{B^{m,n}}(\tilde{\upsilon}(\tilde{\omega}))|_{\tilde{\omega}=(W^{t,\mathring{\mathbf{x}}}, X^\varepsilon, \delta_{a^\varepsilon_r}(da)\, dr, \delta_{p^\varepsilon_r}(dq)\, dr, \delta_{q^\varepsilon_r}(dq)\, dr)(\omega)}.$$

Exactly as on pp. 10 and 11 in [45], we obtain the existence of a limit $(\tilde{a}^\dagger, \tilde{p}^\dagger, \tilde{q}^\dagger)$ of the sequence $(\tilde{a}^m, \tilde{p}^m, \tilde{q}^m)_m$. We use now concatenations $\omega \otimes_\tau \omega' \in \Omega$ of paths ω, $\omega' \in \Omega$ at $\mathbb{F}$-stopping times τ as in [45], i.e.,

$$(\omega \otimes_\tau \omega')_s := \mathbf{1}_{[0,\tau(\omega))}(s).\omega_s + \mathbf{1}_{[\tau(\omega),\infty)}(s).\left[\omega_{\tau(\omega)} + \omega'_{s-\tau(\omega)}\right].$$

Thus, given $\omega \in E_1$, we can define

$$(\tilde{a}^\omega, \tilde{p}^\omega, \tilde{q}^\omega) := \lim_m (\tilde{a}^m, \tilde{p}^m, \tilde{q}^m)(\cdot + \tau^\varepsilon(\omega), \omega \otimes_{\tau^\varepsilon} \cdot) : \mathbb{R}_+ \times \Omega \to A \times \mathbb{R}^d \times \mathbb{R}^{d\times d}.$$

Note that $\tilde{a}^\omega = \tilde{a}^\dagger(\cdot + \tau^\varepsilon(\omega), \omega \otimes_{\tau^\varepsilon} \cdot)$.[9] Then[10], for $\mathbb{P}$-a.e. $\omega \in E_1$,

$$\tilde{\upsilon}((W^{t,\mathring{\mathbf{x}}}, X^\varepsilon, \delta_{a^\varepsilon_s}(da)\, ds, \delta_{p^\varepsilon_s}(dq)\, ds, \delta_{q^\varepsilon_s}(dq)\, ds)(\omega)) = (\tilde{a}^\omega, \tilde{p}^\omega, \tilde{q}^\omega)$$

---

[5] Recall that $\Upsilon(\tilde{a}, \ldots) = \mathbb{P} \circ (W, \delta_{\tilde{a}_s}(da)\, ds, \ldots)$ and also keep Lemma 7.3, (7.9), and (7.13) in mind regarding the non-emptiness of $\mathbf{Y}$.

[6] This is needed below for progressive measurability of the processes $\tilde{a}^m$, etc.

[7] I.e., $\tilde{\upsilon} : \tilde{E}_1 \to \mathcal{A} \times \cdots$, $\tilde{\omega} \mapsto \tilde{\upsilon}(\tilde{\omega}) = \tilde{\upsilon}(\tilde{\omega}_{\cdot\wedge\tilde{\tau}})$ with $\tilde{\mathbb{Q}}^* \circ (\tilde{W}^{\tilde{\tau}(\tilde{\omega})}, \tilde{M}_A^{\tilde{\tau}(\tilde{\omega})}(s, da)\, ds, \ldots)^{-1} = \mathbb{P} \circ (W, \delta_{[\tilde{\upsilon}_A(\tilde{\omega})]_s}(da)\, ds, \ldots)^{-1} = \Upsilon(\tilde{\upsilon}(\tilde{\omega}))$.

[8] With $\tilde{\upsilon}^m : \tilde{E}_1 \to \mathcal{A} \times \cdots$ defined by $\tilde{\upsilon}^m(\tilde{\omega}) := \sum_{n\in\mathbb{N}} \beta^{m,n}.\mathbf{1}_{B^{m,n}}(\tilde{\nu}(\tilde{\omega}))$, we have $(\tilde{a}^m, \cdots)(s, \omega) = \mathbf{1}_{[\tau^\varepsilon(\omega),\infty)}(s).[\tilde{\upsilon}^m((W^{t,\mathring{\mathbf{x}}}, \ldots)(\omega)]_{s-\tau^\varepsilon(\omega)}(W^{\tau^\varepsilon(\omega)}(\omega))$. Note that $\tilde{\upsilon}^m \to \tilde{\upsilon}$ and thus (see below) $(\tilde{a}^\dagger, \ldots)(s, \omega) = \mathbf{1}_{[\tau^\varepsilon(\omega),\infty)}(s).[\tilde{\upsilon}((W^{t,\mathring{\mathbf{x}}}, \ldots)(\omega)]_{s-\tau^\varepsilon(\omega)}(W^{\tau^\varepsilon(\omega)}(\omega))$.

[9] The left-hand side is used to show that certain laws are equal. The (measurable!) right-hand side is used in the construction of the glued control $\tilde{a}^*$ below.

[10] To see this, note that

$$\begin{aligned}\tilde{a}^\omega_r(\omega') &= \tilde{a}^\dagger_{r+\tau^\varepsilon(\omega)}(\omega \otimes_{\tau^\varepsilon} \omega') \\ &= \mathbf{1}_{[\tau^\varepsilon(\omega),\infty)}(r + \tau^\varepsilon(\omega)).[\tilde{\upsilon}_A((W^{t,\mathring{\mathbf{x}}}, \cdots)(\omega))]_{[r+\tau^\varepsilon(\omega)]-\tau^\varepsilon(\omega)}(W^{\tau^\varepsilon(\omega)}(\omega \otimes_{\tau^\varepsilon} \omega')) \\ &= [\tilde{\upsilon}_A((W^{t,\mathring{\mathbf{x}}}, \cdots)(\omega))]_r((\omega \otimes_{\tau^\varepsilon} \omega')_{\cdot+\tau^\varepsilon(\omega)} - (\omega \otimes_{\tau^\varepsilon} \omega')_{\tau^\varepsilon(\omega)}) \\ &= [\tilde{\upsilon}_A((W^{t,\mathring{\mathbf{x}}}, \cdots)(\omega))]_r(\omega').\end{aligned}$$

and thus

$$\Upsilon(\tilde{a}^\omega,\tilde{p}^\omega,\tilde{q}^\omega)=\tilde{\mathbb{Q}}^*_{\tilde{\omega}}\circ(\tilde{W}^{\tilde{\tau}(\tilde{\omega})},\tilde{M}_A^{\tilde{\tau}(\tilde{\omega})}(s,da)\,ds,\tilde{M}_P^{\tilde{\tau}(\tilde{\omega})}(s,dp)\,ds,$$
$$\tilde{M}_Q^{\tilde{\tau}(\tilde{\omega})}(s,dq)\,ds)^{-1}|_{\tilde{\omega}=(W^{t,\mathring{\mathbf{x}}},X^\varepsilon,\delta_{a^\varepsilon_s}(da)\,ds,\delta_{p^\varepsilon_s}(dq)\,ds,\delta_{q^\varepsilon_s}(dq)\,ds)(\omega)}.$$

Next, define $(\tilde{a}^*,\tilde{p}^*,\tilde{q}^*):\mathbb{R}_+\times\Omega\to A\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$ by

$$(\tilde{a}^*,\tilde{p}^*,\tilde{q}^*)(s,\omega):=(a^\varepsilon,p^\varepsilon,q^\varepsilon)(s,\omega).\mathbf{1}_{[0,\tau^\varepsilon(\omega))}(s)+(\tilde{a}^\dagger,\tilde{p}^\dagger,\tilde{q}^\dagger)(s,\omega).\mathbf{1}_{[\tau^\varepsilon(\omega),\infty)}(s)$$

whenever $\omega\in E_1$ and $(\tilde{a}^*,\tilde{p}^*,\tilde{q}^*):=(a^\varepsilon,p^\varepsilon,q^\varepsilon)$ otherwise. We denote by $X^{\varepsilon,+}$ the solution of

$$dX^{\varepsilon,+}_s=[b(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},\tilde{a}^*_s)+\tilde{p}^*_s]\,ds+[\sigma(s,X^\varepsilon_{\cdot\wedge\tau^\varepsilon},\tilde{a}^*_s)+\tilde{q}^*_s]\,dW_s,\quad \mathbb{P}\text{-a.s.},\tag{7.16}$$

on $[t,\infty)$ with initial condition $X^{\varepsilon,+}|_{[0,t]}=\mathbf{x}|_{[0,t]}$. It suffices now to show that (7.15) holds or, equivalently (recall (7.14)), that (7.17) below holds. To simplify the notation, we omit from now on the (p,q)-components from the "control term" (a,p,q). Related obvious modficiations are implicitly used. Let $n\in\mathbb{N}$, $t\le t_1<\cdots<t_n$, $R_1=(R_1^1,R_1^2,R_1^3)$, …, $R_n=(R_n^1,R_n^2,R_n^3)\in\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$, $\varphi\in C_b(A)$. We write $\tilde{M}$ instead of $\tilde{M}_A$. Also put $\tilde{\mathfrak{X}}_{t_i}:=(\tilde{W}_{t_i},\tilde{X}_{t_i},\tilde{M}_{t_i}(\varphi))$. We will show that

$$\begin{aligned}&\tilde{\mathbb{P}}^*(R^{(1)}\cap\tilde{E}_1)\\&\quad=\mathbb{P}\left(\bigcap_{i=1}^n\left\{W^{t,\mathring{\mathbf{x}}}_{t_i}\in R_i^1,X^{\varepsilon,+}_{t_i}\in R_i^2,\int_0^{t_i}\varphi(\tilde{a}^*_s)\,ds\in R_i^3\right\}\cap E_1\right),\end{aligned}\tag{7.17}$$

where $R^{(i_0)}:=\cap_{i=i_0}^n\{\tilde{\mathfrak{X}}_{t_i}\in R_i\}$. First, note that (cf. p. 140 in [51])

$$\begin{aligned}\tilde{\mathbb{P}}^*(R^{(1)}\cap\tilde{E}_1)=\int_{\tilde{E}_1}\Bigg[&\mathbf{1}_{[0,t_1)}(\tilde{\tau}(\tilde{\omega}))\,\tilde{\mathbb{Q}}^*_{\tilde{\omega}}(R^{(1)})\\&+\sum_{i=1}^{n-1}\mathbf{1}_{[t_i,t_{i+1})}(\tilde{\tau}(\tilde{\omega}))\prod_{j=1}^{i}\mathbf{1}_{\{\tilde{\mathfrak{X}}_{t_j}\in R_j\}}(\tilde{\omega})\cdot\tilde{\mathbb{Q}}^*_{\tilde{\omega}}\left(\bigcap_{j=i+1}^n\{\tilde{\mathfrak{X}}_{t_j}\in R_j\}\right)\\&+\mathbf{1}_{[t_n,\infty)}(\tilde{\tau}(\tilde{\omega}))\prod_{i=1}^n\mathbf{1}_{\{\tilde{\mathfrak{X}}_{t_i}\in R_i\}}(\tilde{\omega})\Bigg]\,\mathbb{P}^\varepsilon(d\tilde{\omega}).\end{aligned}\tag{7.18}$$

Next, until the end of this paragraph, let $\tilde{\omega}=(W^{t,\mathring{\mathbf{x}}},X^\varepsilon,\delta_{a^\varepsilon_s}(da)ds)(\omega)$ for every $\omega\in\Omega$. Consider Borel maps $[\xi(\omega)]_{t_j}:\Omega\to\mathbb{R}^d$ that satisfy

$$\begin{aligned}[\xi(\omega)]_{t_j}(\tilde{W}^{\tilde{\tau}(\tilde{\omega})})&=\tilde{X}_{t_j}=\tilde{X}_{\tilde{\tau}(\tilde{\omega})}(\tilde{\omega})\\&+\int_0^{t_j-\tilde{\tau}(\tilde{\omega})}\left[b_{\tilde{\tau}(\tilde{\omega})+r}(\tilde{X}_{\cdot\wedge\tilde{\tau}(\tilde{\omega})}(\tilde{\omega}),\tilde{a}^\omega_r(\tilde{W}^{\tilde{\tau}(\tilde{\omega})}))\right]dr\\&+\int_0^{t_j-\tilde{\tau}(\tilde{\omega})}\left[\sigma_{\tilde{\tau}(\tilde{\omega})+r}(\tilde{X}_{\cdot\wedge\tilde{\tau}(\tilde{\omega})}(\tilde{\omega}),\tilde{a}^\omega_r(\tilde{W}^{\tilde{\tau}(\tilde{\omega})}))\right]d\tilde{W}^{\tilde{\tau}(\tilde{\omega})}_r,\ \tilde{\mathbb{Q}}^*_{\tilde{\omega}}\text{-a.s.},\end{aligned}$$

and[11] $[\xi(\omega)]_{t_j}(W^{\tilde\tau(\tilde\omega)}) = X_{t_j}^{\tilde\tau(\tilde\omega),\tilde X(\tilde\omega);\tilde a^\dagger}$, $\mathbb{P}$-a.s. (the results in Step 4 in this proof guarantee[12] the existence of such maps). Let $i \in \{1,\dots,n-1\}$. Then

$$
\begin{aligned}
&\tilde{\mathbb{Q}}^*_{\tilde\omega}(R^{(i)}) = \tilde{\mathbb{Q}}^*_{\tilde\omega}\left(\cap_{j=i}^n\{\tilde{\mathfrak{X}}_{t_j} \in R_j\}\right)\\
&= \tilde{\mathbb{Q}}^*_{\tilde\omega}\left(\cap_{j=i}^n\{\ \tilde W^{\tilde\tau(\tilde\omega)}_{t_j-\tilde\tau(\tilde\omega)} + \tilde W_{\tilde\tau(\tilde\omega)}(\tilde\omega) \in R^1_j, \tilde X_{t_j} \in R^2_j, \tilde M_{t_j}(\varphi) \in R^3_j\}\right)\\
&= \tilde{\mathbb{Q}}^*_{\tilde\omega}\Bigg(\cap_{j=i}^n\Bigg\{\ \tilde W^{\tilde\tau(\tilde\omega)}_{t_j-\tilde\tau(\tilde\omega)} + \tilde W_{\tilde\tau(\tilde\omega)}(\tilde\omega) \in R^1_j, [\xi(\omega)]_{t_j}(\tilde W^{\tilde\tau(\tilde\omega)}_{\cdot\wedge(t_j-\tilde\tau(\tilde\omega))}) \in R^2_j,\\
&\qquad\qquad \tilde M^{\tilde\tau(\tilde\omega)}_{t_j}(\varphi) + \tilde M_{\tilde\tau(\tilde\omega)}(\varphi) \in R^3_j\Bigg\}\Bigg)\\
&= \mathbb{P}\Bigg(\cap_{j=i}^n\Bigg\{W_{t_j-\tilde\tau(\tilde\omega)} + \tilde W_{\tilde\tau(\tilde\omega)}(\tilde\omega) \in R^1_j, [\xi(\omega)]_{t_j}(W_{\cdot\wedge(t_j-\tilde\tau(\tilde\omega))}) \in R^2_j,\\
&\qquad\qquad \int_0^{t_j-\tilde\tau(\tilde\omega)} \varphi(\tilde\alpha^\omega_s)\,ds + \tilde M_{\tilde\tau(\tilde\omega)}(\varphi) \in R^3_j\Bigg\}\Bigg)\\
&= \mathbb{P}\left(\cap_{j=i}^n\left\{W^{t,\mathring{\mathbf{x}}}_{t_j} \in R^1_j, X^{\varepsilon,+}_{t_j} \in R^2_j, \int_0^{t_j}\varphi(\tilde a^*_s)\,ds \in R^3_j\right\}\middle|\mathcal{F}_{\tau^\varepsilon}\right)(\omega)
\end{aligned}
$$

for $\mathbb{P}$-a.e. $\omega \in E_1$. Plugging the right-hand side above into (7.18) and using (7.12)[13] yields (7.17). We can conclude that (7.15) holds.

Step 6. Consider

$$
\tau^{\varepsilon,+} := \tau^\varepsilon + \tilde\delta^*_{(W^{t,\mathring{\mathbf{x}}},X^\varepsilon,\delta_{a^\varepsilon_s}(da)ds,\delta_{p^\varepsilon_s}(da)ds,\delta_{q^\varepsilon_s}(da)ds)}, \tag{7.19}
$$

which is an $\mathbb{F}$-stopping time (Lemma V.3 in [53]). Define $\varrho^{\varepsilon,+} : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ by

$$
\varrho^{\varepsilon,+}_s := \varrho^\varepsilon_s.\mathbf{1}_{[0,\tau^\varepsilon)}(s) + \tau^\varepsilon.\mathbf{1}_{[\tau^\varepsilon,\tau^{\varepsilon,+})}(s) + s.\mathbf{1}_{[\tau^{\varepsilon,+},\infty)}(s).
$$

[11]Recall the notation $X^{t,\mathbf{x};a}$ in (4.1).

[12]For more details, one can proceed as in the the proof of Lemma 7.3.

[13]In the present simplified notation, (7.12) is just $\mathbb{P}^\varepsilon = \mathbb{P}\circ(W^{t,\mathring{\mathbf{x}}}, X^{t,\mathbf{x};a^\varepsilon}, \delta_{a^\varepsilon_s}\,ds)^{-1}$.

By (7.9) and (7.19), $s-\varepsilon \le \varrho_s^{\varepsilon,+} \le s$ for all $s \in \mathbb{R}_+$. Put $(\tilde{\varrho}_s, \tilde{\varrho}_s^+, \tilde{\tau}^+) := (\varrho_s^\varepsilon, \varrho_s^{\varepsilon,+}, \tau^{\varepsilon,+})(\tilde{W})$. Then, for all $y \ge v(t,\mathbf{x})$ and $s \in [t,T]$,

$$
\begin{aligned}
\mathbb{E}\left[v(\varrho_s^{\varepsilon,+} \wedge \tau^{\varepsilon,+}, X^{\varepsilon,+})\right] &= \mathbb{E}^{\tilde{\mathbb{P}}^*}\left[v(\tilde{\varrho}_s^+ \wedge (\tilde{\tau} + \tilde{\delta}^*), \tilde{X})\right] \\
&= \int_{\tilde{E}_1 \cap \{\tilde{\tau}^+ \le s\}} \mathbb{E}^{\tilde{\mathbb{Q}}^*_{\tilde{\omega}}}\left[v(\tilde{\tau}(\tilde{\omega}) + \tilde{\delta}^*_{\tilde{\omega}}, \tilde{X})\right] \mathbb{P}^\varepsilon(d\tilde{\omega}) \\
&\quad + \int_{\tilde{E}_1^c \cup (\tilde{E}_1 \cap \{\tilde{\tau}^+ > s\})} v(\tilde{\varrho}_s(\tilde{\omega}) \wedge \tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega}))\, \mathbb{P}^\varepsilon(d\tilde{\omega}) \\
&\le \int_{\tilde{E}_1 \cap \{\tilde{\tau}^+ \le s\}} \left[v(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega})) + \varepsilon \tilde{\delta}^*_{\tilde{\omega}}\right] \mathbb{P}^\varepsilon(d\tilde{\omega}) \\
&\quad + \int_{\tilde{E}_1^c \cup (\tilde{E}_1 \cap \{\tilde{\tau}^+ > s\})} v(\tilde{\varrho}_s(\tilde{\omega}) \wedge \tilde{\tau}(\tilde{\omega}, \tilde{X}(\tilde{\omega}))\, \mathbb{P}^\varepsilon(d\tilde{\omega}) \\
&= \mathbb{E}\left[\mathbf{1}_{E_1 \cap \{\tau^{\varepsilon,+} \le s\}}.(v(\tau^\varepsilon, X^\varepsilon) + \varepsilon \cdot (\tau^{\varepsilon,+} - \tau^\varepsilon))\right] \\
&\quad + \mathbb{E}\left[\mathbf{1}_{E_1^c \cup (E_1 \cap \{\tau^{\varepsilon,+} > s\})}.(v(\varrho_s^\varepsilon \wedge \tau^\varepsilon, X^\varepsilon)\right] \\
&= \mathbb{E}\left[v(\varrho_s^\varepsilon \wedge \tau^\varepsilon, X^\varepsilon) + \varepsilon \cdot (\tau^{\varepsilon,+} - \tau^\varepsilon).\mathbf{1}_{E_1 \cap \{\tau^{\varepsilon,+} \le s\}}\right] \\
&\le y + \varepsilon \cdot \mathbb{E}[s \wedge \tau^\varepsilon - t] + \varepsilon \cdot \mathbb{E}[(\tau^{\varepsilon,+} - \tau^\varepsilon).\mathbf{1}_{\{\tau^{\varepsilon,+} \le s\}}],
\end{aligned} \tag{7.20}
$$

i.e., $\mathbb{E}[\widehat{v}(\varrho_s^{\varepsilon,+} \wedge \tau^{\varepsilon,+}, X^{\varepsilon,+}, y)] \le \varepsilon \cdot \mathbb{E}[s \wedge \tau^{\varepsilon,+} - t]$. The first inequality in (7.20) follows from $\widehat{v}(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega}), v(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega}))) \in K$, which (as assumed in the statement of our lemma) implies

$$
\mathcal{E}_+(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega})) \in \mathcal{QTS}_{\widehat{v},K}(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega}), v(\tilde{\tau}(\tilde{\omega}), \tilde{X}(\tilde{\omega}))),
$$

together with the definitions of $\tilde{\mathbb{Q}}^*_{\tilde{\omega}}$, $\mathbb{Q}^{**}$, and $\mathbf{M}$ in (7.13), Remark 7.5, and (7.9).

Define $(p^{\varepsilon,+}, q^{\varepsilon,+}) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ by

$$
(p^{\varepsilon,+}, q^{\varepsilon,+})_s := \mathbf{1}_{[0,\tau^\varepsilon)}(s).(\bar{p}, \bar{q})_s + \mathbf{1}_{[\tau^\varepsilon, \tau^{\varepsilon,+})}(s).(\tilde{p}^*, \tilde{q}^*)_s.
$$

By (4.9) for $(\tau^\varepsilon, \bar{p}, \bar{q})$ and (7.7) with (7.9), we have (4.9) for $(\tau^{\varepsilon,+}, p^{\varepsilon,+}, q^{\varepsilon,+})$.

We can conclude that $(\tau^{\varepsilon,+}, \varrho^{\varepsilon,+}, \tilde{a}^*, p^{\varepsilon,+}, q^{\varepsilon,+}, X^{\varepsilon,+})$ is our desired "extended" ε-approximate solution. □

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