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Voluntary environmental effort under (*s*,*S*) inventory policy*

Alain Bensoussan[a] and Fouad El Ouardighi[b]

[a]International Center for Decision and Risk Analysis, Jindal School of Management, University of Texas at Dallas, Richardson, TX, USA;
[b]Operations management, ESSEC Business School, Cergy-Pontoise, France

ABSTRACT

Prior research on inventory control has been wide ranging, yet the environmental implications of an (s, S) inventory policy remain uninvestigated. This paper seeks to bridge the gap by characterising a firm's voluntary environmental policy in the setup of an (s, S) inventory control policy. We suggest a mixed model structure wherein, due to the presence of fixed production costs, the inventory is determined *continuously* by sales and *impulsively* with ordering decisions obeying an optimal stopping process, while the uncertain sales process is controlled by continuous-time environmental goodwill-related decisions. We show that a firm should successively use voluntary environmental efforts to stimulate its sales when there is inventory and to increase backlogging to improve its production efficiency. Given the recurrent pattern of this policy, we conclude that voluntary environmental efforts under an (s, S) inventory control is not compatible with using these efforts as a means to generate ephemeral reputation insurance.

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1. Introduction

It has been shown that a voluntary approach is a valuable environmental policy instrument (e.g. Arimura, Hibiki, and Katayama 2008). Examples of such efforts are compensating for carbon emissions by planting trees or purchasing offsets, investing in clean energy, cleaning rivers, reducing the quantity of water or energy used for production, increasing the proportion of recycled waste used, and improving water quality. A voluntary approach is rewarded by relaxed regulatory scrutiny (e.g. Innes and Sam 2008), therefore firms facing higher regulatory pressure are more likely to participate in voluntary environmental programmes (e.g. Potoski and Prakash 2005; Wu 2009).

The question of how to design a voluntary environmental policy to be as effective as possible is crucial (Borck and Coglianese 2009; Koehler 2007). This paper seeks to contribute to the environmental management literature by characterising a firm's optimal voluntary environmental policy. In our setup, environmentally protective initiatives include voluntary efforts that provide goodwill due to customers' environmental awareness (Heydari, Govindan, and Basiri 2021; Hong et al. 2020) and therefore promote a firm's sales. In contrast with regulatory environmental initiatives that seek to avoid a penalty associated with the environmental impact of production, these efforts are disconnected from production cost considerations. Examples of prominent profit-oriented firms which voluntary invest in environmental processes are Shell,[1] Land Rover,[2] Unilever,[3] and Cemex.[4]

A main novelty in this paper lies in the fact that such a policy is characterised in the setup of an (s, S) inventory control policy. Though the concept of (s, S) policy is well-established both in theory and in practice (see the comprehensive presentation proposed by Dolgui and Proth 2010), and was extended in several directions (see the recent exhaustive survey by Perera and Sethi 2022), little is known about its implications in terms of environmentally protective management. This paper seeks to bridge the gap by developing an (s, S) inventory control model that extends towards environmental protection. A primary issue here is thus to induce an optimal voluntary environmental policy that is contingent on an (s, S) inventory policy.

Clearly, the response to this issue involves both operational production decisions and a marketing approach to satisfy a market demand.[5] The tradeoff thus requires the design of an operations-marketing interface that delivers both efficiency in production and environmental

CONTACT Fouad El Ouardighi elouardighi@essec.edu Operations Management Department, ESSEC Business School, Cergy-Pontoise 95021, France
*This work is dedicated to the memory of Jean Marie Proth. One of the greatest scholars in the domain of production management, Jean Marie became very concerned by environmental considerations. He was convinced that environment could not be absent in operations management modelling. He remains, in this domain, an inspirational figure.

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goodwill for sales promotion. In this regard, we consider a monopolist firm that faces a market demand that is sensitive to the firm's efforts to protect the ecological environment. In this setup, the firm can leverage a decision variable, i.e. a voluntary environmental effort, as a means to stimulate demand. That is, a greener (i.e. more environmentally protective) production will boost demand, though at additional cost. This assumption is consistent with a recent conclusion from a Mc Kinsey & Co. report that suggests that 66% of all respondents of a US cohort survey (and 75% of millennial respondents) say that they consider sustainability when they make a purchase (Mc Kinsey 2019).

In order to focus on the interactions between inventory control and environmental protection, we assume that the price is set exogenously once and for all at the beginning of the planning horizon. This means there is no pricing management, which simplifies the mathematical analysis. A diffusion model is suggested that accounts for the marketing impact of voluntary environmental efforts on an uncertain market demand.

The novelty of our approach lies in the fact that our model relies on an interface between operational and marketing instruments combining production and environmental goodwill decisions. More precisely, we suggest a mixed model structure wherein, due to the presence of fixed production costs, the inventory is determined *continuously* by sales and *impulsively* with ordering decisions obeying an optimal stopping process, while the sales process is controlled by continuous-time environmental goodwill-related decisions. The methodological resolution of the model proceeds in two steps: the inventory control is first characterised in terms of an s, S policy, and the environmental goodwill policy is then determined as an optimal feedback on the inventory.

Our results show that, in the context of an (s, S) inventory policy, a firm should leverage voluntary environmental efforts on a quasi-cyclical basis as a marketing tool when there is inventory and as a production efficiency tool in the case of backlogging. That is, voluntary environmental management should stimulate demand when the stock is replenished to maximise current profit, and subsequently generate backlogs when the stock is depleted to ensure that the fixed costs of future production can be covered. These results differ from Barcos et al. (2013) in that we find a U-shaped rather than an inverted U-shaped relationship between firms' voluntary environmental initiatives and their inventory levels under an (s, S) inventory policy.

The paper is organised as follows. We review the relevant literature in the next section, and develop our model in Section 3. In the subsequent section, we analyse the model and derive our results. Section 5 provides numerical illustrations. Section 6 concludes the paper.

2. Literature review

Our research lies at the intersection of two literatures, those concerning (s, S) inventory policy and analysis of related production decisions, and corporate environmental management and performance.

Regarding the (s, S) periodic-review inventory model and analysis of related production decisions, an important stream of literature has emerged since the model's introduction by Arrow, Harris, and Marschak (1951) and its consolidation by Scarf (1960). Consistent with many real-life scenarios, this model assumes fixed costs of ordering items or setting up a process. The (s, S) policy, where s is the ordering point and S is the order-up-to level, can be described as follows: if, at a certain period, the initial inventory level, x, is lower than the ordering point, s, then an order equal to the difference between the order-up-to level, S, and the initial inventory level, x, should be placed, and otherwise no order should be placed. The optimal expressions of s and S are derived by minimising a loss function representing the present value of the total expected lost sales incurred over a given time horizon for any initial inventory level.

Perera and Sethi (2022) provide an exhaustive survey of (s, S) policies in various settings, that is, discrete- and continuous-time reviews, finite- and infinite-time horizons, discounted- and average-cost objectives, backlogging and lost-sales settings, standard and generalised demand and cost structures, deterministic and stochastic delivery lead times, single- and multi-product settings, and coordinated pricing-inventory control.[6] Among the most representative extensions are those relating to Markovian demand (Sethi and Cheng 1997), demand Bayesian learning (Larson, Olson, and Sharma 2001), generalised loss functions (Benkherouf and Sethi 2010), lost sales (Bensoussan, Crouhy, and Proth 1983; Zipkin 2008), multiple items (Silver 1974), cutoff transaction size (Hollier, Makj, and Lam 1995), information delay (Bensoussan et al. 2009), deteriorating items (Ravichandran 1995), supply chain management (Kelle and Milne 1999), and competing suppliers (Fox, Metters, and Semple 2006). As in our study, some important papers assume a Wiener process demand where the Bellman equation of dynamic programming for the considered inventory problem reduces to a quasi-variational inequality (QVI) (Bensoussan and Lions 1987) that is solved to obtain an optimal impulse control policy and hence an optimal inventory policy. Among these papers are those of Bensoussan and Tapiero (1982), Bensoussan, Liu, and Sethi (2005), and Benkherouf and Bensoussan (2009).

Though the problem of joint pricing and inventory policy (s, S, p), where p stands for price, has been extensively investigated in a dynamic setting (Chen and Simchi-Levi 2004a; 2004b; Chen, Ray, and Song 2006; Song, Ray, and Boyaci 2009), the problem of joint goodwill advertising and inventory policy remains largely unexamined. Huh and Janakiraman (2008) consider a broad class of decisions, such as pricing and advertising, that influence the demand. In their study, a stationary, single-stage inventory system is assumed, so the dynamic impact of these decisions is not considered. To characterise the most protective voluntary environmental effort pattern associated with an (s, S) inventory policy, we follow the approach that consists of reducing the (s, S) inventory problem to a QVI (Benkherouf and Bensoussan 2009; Bensoussan and Tapiero 1982; Bensoussan, Liu, and Sethi 2005), with the innovation that the state of the system is determined by a continuous control of voluntary environmental efforts. The optimal voluntary environmental efforts are then determined as a feedback policy on the inventory level which depends on an impulse control, given as an increasing sequence of stopping times at which a certain quantity of products is produced.

As for corporate environmental management and performance, King and Lenox (2002) suggest that waste prevention often provides unanticipated innovation offsets while onsite waste treatment often engenders unexpected cost. They find evidence that the benefits of waste prevention alone are responsible for the observed association between lower emissions and profitability. Jacobs, Singhal, and Subramanian (2010) analyse the effects of environmental performance on the shareholder value by quantifying the stock market response associated with declarations of environmental performance. They find that, while the market does not react significantly to self-reported corporate efforts to avoid, mitigate, or offset the environmental impacts of the firm's products, services, or processes, announcements of philanthropic gifts for environmental causes are associated with significant positive market reaction. Kroes, Subramanian, and Subramanyam (2012) show a negative relationship between environmental performance and firm market performance over at least a three-year period. Barrage, Chyn, and Hastings (2020) suggest that firms may have incentives to engage in green advertising without investments in environmental stewardship in order to get reputation insurance (Minor and Morgan 2011).

Barcos et al. (2013) assume that there is an inverted U-shaped relationship between firms' corporate social responsibility and their inventory levels. The reason is that while customers put pressure on firms to increase inventories, environmental activists force firms to reduce inventories. Their empirical findings support their contention that for low levels of corporate social responsibility (CSR), customers are more relevant; and for higher levels of CSR, the natural environment gains importance.

An important stream of the corporate environmental management literature relates to process improvement with the objective of compliance to regulatory norms (see the reviews in Corbett and Klassen 2006; Kleindorfer, Singhal, and Van Wassenhove 2005; Sarkis and Zhu 2018). Recent contributions to the literature on operational compliance to regulatory norms include Drake, Kleindorfer, and Van Wassenhove (2016), Jaber, Glock, and El Saadany (2013), Krass, Nedorezov, and Ovchinnikov (2013), Porteous, Rammohan, and Lee (2015), and Jabbour et al. (2016). More recently, Xiao et al. (2019) use a dynamic, determinist setting to analyse a firm's investment in environmental process improvement to reduce the environmental impact of its manufacturing processes by taking into account various internal firm characteristics and different external regulatory drivers. An article by Chen and Monahan (2010) is the only paper that investigates the relationship between environmental management and production planning and inventory control policies. Based on a static model with both stochastic demand and environmental uncertainties, they analyse the optimal policies of production planning and inventory control under both regulatory and voluntary pollution control approaches, and investigate their operational and environmental effects. They show that a regulatory environmental standard that limits the total amount of waste may induce the firm to raise its planned stock level, which would lead to a higher expected amount of environmental waste before the standard is enforced. However, the additional planned stock level, termed 'environmental safety stock', can be reversed by using the voluntary control approach that provides the firm with the flexibility to occasionally exceed the environmental standard. Our paper adopts a different perspective from Barcos et al. (2013) and Chen and Monahan (2010) in that we investigate the effect of a firm's inventory level on its environmental management policy rather than the converse. Our approach also differs from Chen and Monahan (2010) and Xiao et al. (2019) in that we consider a product's uncertain diffusion process over time, which depends on voluntary environmental efforts with no impact on the unit production cost and no required compliance to an environmental regulation.

A growing stream of the corporate environmental management literature, to which we contribute, seeks to maximise the effectiveness of voluntary initiatives out of any regulatory pressure. Recent publications on voluntary environmental initiatives include topics such as polluting emissions abatement in a supply chain's manufacturing and sales processes. For instance, El Ouardighi, Sim, and Kim (2016), Sim, El Ouardighi, and Kim (2019), and El Ouardighi, Sim, and Kim (2021) investigate how polluting emissions and abatement are affected by competition and integration in an industry, and eventually how firms' strategy types modify the relative impact of horizontal and vertical competition on pollution. In this paper, we consider a single firm whose voluntary initiatives seek to carry out a broad spectrum of external environmental actions, such as the restoration of natural carbon sinks, which have the effect of promoting sales.

3. Model formulation

3.1. Background and general comments

We develop a novel decision aid model for inventory and production control. The standard inventory control theory leading in particular to the famous (s, S) policy introduced by Arrow, Harris, and Marschak (1951) and Scarf (1960) considers the demand as an external stochastic process with independent demands over time. The costs are inventory holding and shortage costs, namely costs were purely limited to physical aspects of production and inventories. The popularity of (s, S) policy lies in the fact that it is very intuitive and convenient to practitioners. After this remarkable progress in management science and operations research, a lot of effort has been devoted to meaningful extensions (see Perera and Sethi 2022). The natural extensions have notably dealt with modelling the demand process. A very natural question is: how sensitive is the (s, S) policy to factors influencing the demand process? In this setup, the assumption of independence of successive demands over time is indeed not anymore relevant. As an alternative, the demand process can be modelled as a Markov chain (Kalymon 1971; Sethi and Cheng 1997). Another type of extension is not to consider the demand process as external, but linked to the inventory, for instance a mean reverting connection (Cadenillas, Lakner, and Pinedo 2010). But there are also economic factors which influence the demand, many of them are not fully external but may depend on corporation decisions, for instance marketing decisions. Generally, the marketing and production decisions emanate from different departments of a company, but the issues are very connected. A famous case, which has generated huge research efforts, is pricing management. In pricing management, the issue is to decide the selling price of the product in connection with the production and inventory decisions. This problem is quite complicated because the price will affect the demand and therefore the production and inventory policy. In this regard, the price should not be decided independently from the inventory policy, as it is often the case.

In this paper, we introduce a different connection, related to environmental considerations. While the concern of environment was not so present when inventory control was developed, it has now become unthinkable to consider manufacturing and retailing activities without taking account of their environmental impact. Indeed, this problem has primarily to do with engineering issues, e.g. developing less polluting production processes, using alternative energy resources, etc. However, beyond compliance with restrictive regulations, the search for environmental improvements also involves an economic tradeoff because it entails additional expenses, on the one hand, and it stimulates demand, on the other hand. This comes from the fact that customers decide more and more their expenses with environmental issues in mind. Our objective is to analyse with a stylised model the cost–benefit problem of how much to spend on environmental aspects, in relation with the production and inventory control issues.

3.2. Description of the model

We build a model which gets inspiration on the way pricing management has been studied. In particular, we rely on the work of the first author on pricing management (Bensoussan, Skaaning, and Turi 2018) but with the difference that we take price as a fixed parameter and we introduce a different decision variable that is the effort of the manufacturer toward the environment. As in pricing management, a major issue here is how an economic factor such as environmental effort affect the demand and thus the inventory policy? It is obvious that this problem is not identical to the pricing management problem because the respective impacts of price and environmental effort are of different nature. Here, the decision variable considered, i.e. the manufacturer's effort toward the environment affects the demand in an opposite way to the price in pricing management. Also, environmental effort has a cost, which needs to be taken into account as part of the economic tradeoff. Regarding the inventory control part, the analysis is similar to the standard theory, and we shall obtain an (s, S) policy. The interconnection of the effort towards the environment and the (s, S) policy is the key mathematical challenge. This interaction is different from what occurs in pricing management.

3.2.1. A short presentation of the mathematical apparatus

We develop a model in continuous time (though discrete time is also possible) with infinite time horizon to get stationary problem. There are advantages with continuous time. We use stochastic control instead of Markov decision processes and get closed-form solutions. However, we have to accept that the inventory control is an impulse control and not a continuous control. In discrete time, there is no difference. A continuous control is a rate, and the state (here, the inventory level) evolves continuously. In contrast, an impulse control is a jump and the inventory is discontinuous (it changes instantaneously with the size of the jump). The reason why we have to consider impulses is because there are fixed costs. Each time a production is decided, whatever its amount, there is a fixed cost to be paid. So deciding continuously will entail an infinite cost, which is impossible. Therefore, an impulse control is a sequence of decision times at which we make an order or produce some quantity. Associated with these times are the quantities produced or ordered, which are also decision variables. They correspond to the jumps of the inventory at the decision times. We can neglect the production times in the sense that what accounts is when the production modifies the inventory. Note that these decision times and quantities produced are random. They cannot be decided *ex ante*, since we have to take into account the information that will be available in the future. Random times in probability theory are called stopping times. Information is characterised by σ -algebras. A time t, there is a σ -algebra W^t which is the collection of events observable at time t. The effort towards the environment is a function of time, which affects the rate of demand. An (s, S) policy is an example of impulse control (which turns out to be optimal). When the inventory, denoted hereafter by x, attains the threshold s, then the manufacturer replenishes its inventory, and the amount ordered is $S - x$.

3.2.2. Model

Given the need for the operations function to deliver efficiency and the necessity for environmental marketing to impact sales effectively, the trade-off between inventory level and sales-promoting environmental efforts deserves thorough examination. In this regard, we assume a company that manufactures a product and sells it at a fixed price. While the manufacturing process has a negative impact on the environment, we can assume that the firm complies with the applicable regulatory environmental norms. However, to gain a better reputation, the company is willing to make further environmental efforts that are voluntary. This may come in the form of planting trees and more generally restoring natural carbon sinks, purchasing offsets, reducing the quantity of water or energy used for production, increasing the proportion of recycled waste used, improving water quality, etc. These voluntary environmental efforts increase costs for the manufacturer but they also boost demand due to the marketing impact on customers who exhibit environmental consciousness (Heydari, Govindan, and Basiri 2021; Hong et al. 2020). We thus introduce a control variable called voluntary environmental efforts, denoted by $Q(t)$. The uncertainties on the demand are captured by a standard Wiener process $w(t)$, which is external. This process is built on a probability space Ω, A, P, and we denote by W^t the filtration generated by the Wiener process (i.e. a sequence of σ -algebras of information W^t). The control $Q(t)$ depends on W^t. This leads to the following control model for the cumulated demand $D(t)$ on $(0,t)$, that is:

$$dD(t) = \nu(Q(t))dt + \sigma dw(t) \tag{3.1}$$

where ν is the rate of demand by unit of time. This rate depends on the voluntary environmental effort.

The volatility σ is a constant. It may depend on $Q(t)$. The function $Q \to \nu(Q(t))$ is positive and monotone increasing. For mathematical convenience, we assume that it is a linear function. The ordering (or production) policy is not defined by a rate, which means that the inventory is not replenished continuously, but by an impulse control. This is because of fixed costs. In this case, we neglect the time of production of each impulse. So, a production by impulses is an increasing sequence of stopping times θ_i and positive random variables ξ_i, such that θ_i is a stopping time with respect to W^t and ξ_i is W^{θ_i} measurable. The variable ξ_i is the amount of product produced at θ_i (the time to produce ξ_i is neglected[7]). If $x(t)$ represents the inventory at time t, it is expressed by the formula:

$$x(t) = x - \int_0^t \nu(Q(s))ds + \sum_{\{i|\theta_i \leq t\}} \xi_i - \sigma w(t) \tag{3.2}$$

The impulse control $(\theta_i, \xi_i, i = 1, \dots)$ is denoted by V. So the state of the system $x(.)$ depends on two controls: a continuous control $Q(\cdot)$ and an impulse control V. The initial value of the inventory is x, which is a parameter. We then associate to the pair $(Q(.), V)$ a payoff functional given by the formula:

$$\begin{aligned} &J_x(Q(.), V) \\ &\quad = E\left\{\int_0^{+\infty} [\varpi \nu(Q(t)) - hx^+(t) - px^-(t) \right. \\ &\qquad \left. -\beta Q^2(t)]dt - \sum_{i=1}^{+\infty} e^{-\alpha\theta_i}(k + c\xi_i)\right\} \end{aligned} \tag{3.3}$$

The scalar ϖ is the selling price of the product. Our model accepts backlogs, so the inventory $x(t)$ can be positive or negative. When the inventory is positive, the firm pays a holding cost $hx^+(t)$ per unit of time. When the inventory is negative, the firm pays a shortage cost $px^-(t)$ per unit of time. The payment $\beta Q^2(t)$ per unit of time represents the cost of the effort towards the environment at time t. Finally, at each time θ_i, the firm pays a fixed cost k to produce and a variable cost $c\xi_i$ proportional to the quantity produced, ξ_i. The parameter α is the discount factor.

The model (3.1)–(3.3) differs from models based on regulatory environmental effort (e.g. Xiao et al. 2019) in two ways: first, in our model, there is no penalty associated with the environmental impact of production, and secondly, the production cost is not positively affected by the cumulative regulatory environmental efforts. However, one point in common with such models is the positive impact of either effort on the revenue drawn from production.

The objective is to maximise $J_x(Q(\cdot), V)$ on the pair $Q(.), V$. We set:

$$u(x) = \sup_{Q(\cdot)V} J_x(Q(.), V) \tag{3.4}$$

which is the value function. Using Dynamic Programming, we will define an analytic problem of which $u(x)$ is a solution.

4. Analysis

To simplify, we will take the following function of $\nu(Q(t))$:

$$\nu(Q(t)) = \nu_0 + \nu_1 Q(t) \tag{4.1}$$

We see from (3.3) that:

$$\begin{aligned} J_x(Q(.), V) &\leq E\int_0^{+\infty} e^{-\alpha t}[\varpi\nu(Q(t)) - \beta Q^2(t)]dt \\ &\leq \frac{sup_Q[\varpi\nu(Q(t)) - \beta Q^2(t)]}{\alpha} \end{aligned} \tag{4.2}$$

Therefore, we get:

$$u(x) \leq \frac{1}{\alpha}\left(\varpi\nu_0 + \frac{\varpi^2{\nu_1}^2}{4\beta}\right) \tag{4.3}$$

If we take $Q(.) = 0$ and $\theta_i = +\infty$, the state is reduced to:

$$x_0(t) = x - \nu_0 t - \sigma w(t) \tag{4.4}$$

Therefore, we can state:

$$u(x) \geq -E\int_0^{+\infty} e^{-\alpha t}[hx^+(t) + px^-(t)]dt \tag{4.5}$$

which provides a lower bound for the value function (3.4). It is possible to compute this lower bound explicitly.

Remark 1: Result (4.3) justifies the choice of Q^2 in the cost of effort. In this way, we obtain a finite upper bound for the value function. The lower bound, which is negative tends to $-\infty$ as x tends to $-\infty$. This is normal: if the initial inventory is $-\infty$, it is impossible to get rid of it and the holding cost will lead to an infinite penalty. This means also that the optimisation does not guarantee that the business is profitable. We intend to perform a sensitivity analysis in a further work.

4.1. Dynamic programming

4.1.1. Quasi-variational inequality (QVI)

We first introduce the function $\lambda \to \Phi(\lambda)$ defined by:

$$\Phi(\lambda) = \inf_{Q\geq 0}(\beta Q^2 + (\lambda - \varpi)\nu(Q)) \tag{4.6}$$

and with the specific function $\nu(Q)$ in (4.1), we obtain:

$$\Phi(\lambda) = \nu_0(\lambda - \varpi) - \frac{{\nu_1}^2}{4\beta}((\varpi - \lambda)^+)^2 \tag{4.7}$$

and:

$$\Phi'(\lambda) = \nu_0 + \frac{{\nu_1}^2}{2\beta}(\varpi - \lambda)^+ \tag{4.8}$$

The function $\Phi(\lambda)$ is monotone increasing and concave. The optimum $\hat{Q}(\lambda)$ in (4.6) is given by:

$$\hat{Q}(\lambda) = \frac{\nu_1}{2\beta}(\varpi - \lambda)^+ \tag{4.9}$$

In the case of impulse control, the Bellman equation of dynamic programming is replaced with a QVI, which can be written as follows:

$$\min\left[-\frac{1}{2}\sigma^2 u''(x) + \Phi(u'(x)) + \alpha u(x) + hx^+ \right. \\ \left. + px^-, u(x) - \sup_{\xi\geq 0}(u(x+\xi) - c\xi) + k\right] = 0, \text{a.e.} x \tag{4.10}$$

Note that the formulation in (4.10) requires some smoothness of the function $u(x)$, namely $u(x)$ is C^1, with second derivative $u''(x)$ defined a.e.

Heuristic derivation of equation (4.10)

Equation (4.5) can be written as a set of inequalities and complementarity slackness condition. Indeed, (4.5) is equivalent to:

$$u(x) \geq \sup_{\xi\geq 0}(u(x+\xi) - c\xi) - k \tag{4.10'}$$

$$-\frac{1}{2}\sigma^2 u''(x) + \Phi(u'(x)) + \alpha u(x) + hx^+ + px^- \geq 0 \tag{4.10''}$$

and for any x one of the two inequalities is an equality. The reason why the value function $u(x)$ satisfies the two inequalities can be seen intuitively as follows. At time 0, the manufacturer may decide to order the quantity ξ, then the inventory jumps from x to $x+\xi$. There is an immediate cost $c\xi + k$. From the optimality principle, the best profit starting with an inventory $x+\xi$ is $u(x+\xi)$. The right hand side of the first inequality (4.10') is the value function after the decision of ordering immediately at 0. The sup in ξ captures the fact that one can choose ξ. Since it may not be optimal to order immediately, the value function $u(x)$ must be larger than the right hand side. The second inequality (4.10") can be obtained by considering the alternative of putting an immediate order, namely not ordering for a small interval of time ϵ. At time ϵ, the inventory is approximately (since ϵ is small).$x - \epsilon\nu(Q) - \sigma w(\epsilon)$ Using the optimality principle, we can write:

$$u(x) \geq \epsilon(\varpi\nu(Q) - hx^{+} - px^{-} - \beta Q^{2}) + (1-\alpha\epsilon)Eu(x - \epsilon\nu(Q) - \sigma\omega(\epsilon))$$

Expressing the mathematical expectation with Ito's formula, rearranging, and optimising in Q, we obtain the second inequality (4.10"), using the definition of function Φ. The fact that at time 0, only two decisions are possible, putting an order immediately or postponing for at least a small amount of time implies that one of the inequalities must be an equality. So this is valid for any value of x.

4.1.2. Preliminaries

We begin with the simple transformation:

$$G(x) = u(x) - cx \tag{4.11}$$

We also define the nonlinear operator:

$$M(G)(x) = -k + \sup_{y \geq x} G(y) \tag{4.12}$$

M is an operator. It associates to the function $G \equiv G(x)$ a function $M(G)(x)$.

Then problem (4.6) becomes:

$$\min\left[-\frac{1}{2}\sigma^{2}G''(x) + \Phi(G'(x)+c) + \alpha G(x) + (h+\alpha c)x^{+} + (p-\alpha c)x^{-}, G(x) - M(G)(x)\right] = 0, \text{ a.e.}x \tag{4.13}$$

Conjectured solution:

We look for a solution of (4.13) as follows: Find s and a function $G_s(x)$, $x \geq s$, satisfying:

$$-\frac{1}{2}\sigma^{2}G_{s}''(x) + \Phi(G_{s}'(x)+c) + \alpha G(s) + (h+\alpha c)x^{+} + (p-\alpha c)x^{-} = 0, x > s \tag{4.14}$$

$G_s'(s) = 0$, $G_s'(x)$ bounded as $x \to +\infty$

$$G_s(s) = M(G_s)(s) \tag{4.15}$$

The logic is as follows: for s fixed, we solve the boundary value differential equations (4.14) and (4.15) is an algebraic equation to obtain the number s.

We shall need the fundamental assumption:

$$\frac{\sigma}{\sqrt{2\alpha}}(p-\alpha c) + \Phi(c) - \Phi\left(\frac{p}{\alpha}\right) > 0 \tag{4.16}$$

This condition means that $p - \alpha c$ must be sufficiently large. Since p is the unit cost of shortage, the higher p the more we produce to avoid shortage. On the other hand, the more we stimulate demand by efforts towards the environment the more we increase the risk of shortage, if production cannot follow.

The important understanding is that in (4.14) we identify an (s,S). The s is explicit, the S comes from the definition of the operator M (see (4.12)). S attains the maximum of $G_s(y)$ for $y > s$. The optimisation of efforts will be driven by another number, Σ, with $s < \Sigma < S$. The result is described in the next section, with details in the Appendix.

4.2. Optimal control

4.2.1. Main result

The main result is the following:

Theorem 1: *Assume* (4.16). *Set* $\gamma = (\sigma/\sqrt{2\alpha})$. *There exists a unique* $s^{*} < 0$ *such that:*

$$\frac{1}{\gamma}[p - \alpha c - (p+h)e^{\gamma s^{*}}] + \Phi(c) - \Phi\left(\frac{p}{c}\right) = 0$$

Then, there exists a set $(s, \Sigma, S, H(s))$ *satisfying:*

$$s < \Sigma < 0,\ S > max(\Sigma, s^{*}),\ H(s), \text{ is Lipschitz continuous,} \tag{4.17}$$

$$-\frac{1}{2}\sigma^{2}H''(x) + \alpha H(x) + \frac{d}{dx}\Phi(H(x)+c) + (h+\alpha c)1_{x>0} - (p-\alpha c)1_{x<0} = 0,\ x > s \tag{4.18}$$

$$H(s) = 0,\ H(+\infty) = -c - \frac{h}{\alpha},\ H'(+\infty) = 0,$$

$$H(S) = 0,\ k = \int_{s}^{S} H(x)dx,$$

$$H(x) > 0, \forall x \in (s,S),\ H(x) < 0,\ \forall x > S,$$

$$H'(x) > 0,\ s \leq x < \Sigma < S,\ H'(x) > 0,\ x > \Sigma,$$

and also:

$$-k + \int_x^y H(\xi)d\xi \leq 0, \forall s \leq x \leq y, \tag{4.19}$$

with the estimates:

$$-c - \frac{h}{\alpha} \leq H(x) \leq -c + \frac{p}{\alpha}$$

$$|H'(x)| \leq \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} max\left(\sqrt{2}(p - \alpha c), p + h\right).$$

The meaning of $s < 0$ is intuitive: a certain number of customers' orders are to be ensured before the launching of a production series. That is, $s < 0$ is the maximum backlogging rate required for production.

We next define:

$$G(x) = G(s) + \int_s^x H(\xi)d\xi, \; x > s$$

$$G(x) = G(s), \text{ if } x < s \tag{4.20}$$

$$G(s) = \frac{1}{\alpha}\left[\frac{1}{2}\sigma^2 H'(s) - \Phi(c) + s(p - \alpha c)\right]$$

We have also:

$$G(s) = -k + G(S) \tag{4.21}$$

$$\alpha G(s) + \Phi(c) - s(p - \alpha c) \geq 0 \tag{4.22}$$

and the function $G(x)$ is the solution of (4.13).

Remark 2: Since $G(x)$ Bellman equation (4.13), the s, S policy obtained by Theorem 1 is an optimal impulse control.

4.2.2. Optimal feedback

The inventory control is governed by an s, S policy, where the pair (s, S) is that defined in Theorem 2. The values of the pair (s, S) are different from those without effort towards environmental considerations. The optimal voluntary environmental effort is given by a feedback expression of the inventory level, defined by the formula:

$$\hat{Q}(x) = \hat{Q}(H(x) + c) \tag{4.23}$$

$$= \frac{\nu_1}{2\beta}(\varpi - c), \; \forall x \leq s \tag{4.24}$$

Assume $\varpi > c$, we get:

$$\hat{Q}(x) = \frac{\nu_1}{2\beta}(\varpi - c), \; \forall x \leq s \tag{4.25}$$

and:

$$\hat{Q}'(x) = \frac{\nu_1}{2\beta} H'(x) 1_{H(x)+c<\varpi} \tag{4.26}$$

Therefore, from Theorem 2, we have:

$$\hat{Q}'(x) \leq 0, \text{ if } x \leq \Sigma$$

$$\hat{Q}'(x) > 0, \text{ if } x > \Sigma \tag{4.27}$$

4.2.3. Study of condition (4.16)

Condition (4.16) reads:

$$\Phi\left(\frac{p}{\alpha}\right) - \Phi(c) = \int_c^{p/\alpha} \Phi'(\lambda)d\lambda \leq \frac{1}{\beta}(p - \alpha c)$$

Therefore, from (4.8) it follows:

$$\nu_0\left(\frac{p}{\alpha} - c\right) + \frac{\nu_1^2}{2\beta}\int_c^{p/\alpha} (\varpi + \lambda)^+ d\lambda \leq \frac{1}{\beta}(p - \alpha c) \tag{4.28}$$

This condition reads:

$$\frac{\nu_1^2}{4\beta}(\varpi - c)^2$$

$$\leq \left(\frac{\alpha}{\beta} - \nu_0\right)\left(\frac{p}{\alpha} - c\right) + \frac{\nu_1^2}{4\beta}\left(\left(\varpi - \frac{p}{\alpha}\right)^+\right)^2 \tag{4.29}$$

5. Discussion and managerial implications

From (4.25)–(4.27), it follows that for $+\infty < x < \Sigma$, $\hat{Q}'(x)$ decreases from $\nu_1/2\beta(\varpi + h/\alpha)$ to $\nu_1/2\beta(\varpi - H(\Sigma) - c)^+$, while from $\Sigma \leq x \leq s$, it increases from $\nu_1/2\beta(\varpi - H(\Sigma) - c)^+$ to $\nu_1/2\beta(\varpi - c)$. These results suggest the existence of a non-monotonic relationship between the voluntary environmental effort and the inventory level. From Theorem 2, the minimum inventory level s is negative, which means that the reaching of a maximum backlogging boundary should precede production. This is justified by the fact that it is less costly to incur a cumulative shortage cost over $s \leq x \leq 0$ rather than a fixed cost of production. On the other hand, the threshold $\Sigma > s$, for which $\hat{Q}(x)$ reaches a minimum –though positive –value, is also negative from Theorem 2. Regarding the order-up-to level S, we assume that it is positive, which requires that the fixed cost of production is sufficiently large, i.e. $k > \int_s^0 H(x)dx$, to justify a large enough production scale, ξ_i. Below the backlogging threshold Σ, i.e. $S \geq x \geq \Sigma$, the voluntary environmental effort should decrease as the inventory decreases and then as the backlogging increases. Between the backlogging threshold Σ and the maximum backlogging boundary s, the voluntary environmental effort should have an increasing pattern. Along the decreasing portion of the curve, there exists $\chi > 0$ such that $x = \Sigma + \chi (> / <) 0$, for which the voluntary environmental effort should have the same value as for $x = s$, that is, $\hat{Q}(x = \Sigma + \chi) = \hat{Q}(x = s) = \nu_1/2\beta(\varpi - c)$. Assuming that the fixed cost of production is sufficiently large, i.e. $k > \int_s^{\Sigma+\chi} H(x)dx$, so that $S > \Sigma + \chi$, the voluntary environmental effort should exhibit a U-shaped curve with respect to the inventory level as determined by a (s, S) policy over the interval $x \in [\Sigma + \chi, s]$. For $S > x >$

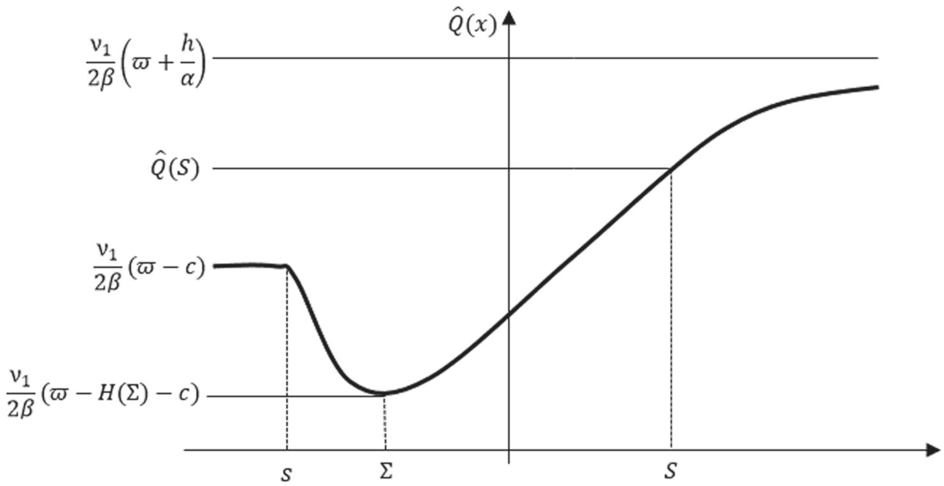

Figure 1. Optimal voluntary environmental effort policy.

$\Sigma + \chi$, the environmental effort should then decrease in a concave way, due to the convex cost function of voluntary environmental efforts. In any case, the environmental effort associated with a zero inventory level should be located on the decreasing portion of the curve. That is, even with zero inventory, the voluntary environmental effort remains worthwhile. Along the decreasing portion of the U-curve, the voluntary environmental effort is set at a higher level when the inventory is greater to boost the demand, which can then be instantaneously satisfied. Note that, from (4.1), the instantaneous sales also have a U-shaped relationship pattern with respect to the inventory level. As a consequence, the cumulative sales curve is concave–convex while the inventory level has a convex–concave pattern.

Figure 1 illustrates the optimal pattern of an voluntary environmental effort policy depending on the evolution from backlogging to inventory, that is, from $x = s$ to $x = S$, where the volatility has been neglected for convenience. Clearly, the voluntary environmental effort stimulates the sales not only to decrease the inventory at hand and thus the related holding cost, if any, but also to some extent to increase the backlogging when the inventory is exhausted. Below the backlogging threshold, the objective is clearly to generate economies of scale to improve production efficiency. Therefore, the backlogging threshold Σ is the turning point for the use of voluntary environmental effort as an effective sales policy to a cost-efficient production policy. To turn the voluntary environmental efforts toward an increasing pattern, it is therefore essential to identify the backlogging threshold Σ, which corresponds to when the cumulative sales reach a plateau with respect to inventory.

The managerial implications of the above results are summarised as follows: in the context of an (s, S) inventory policy, a firm should leverage voluntary

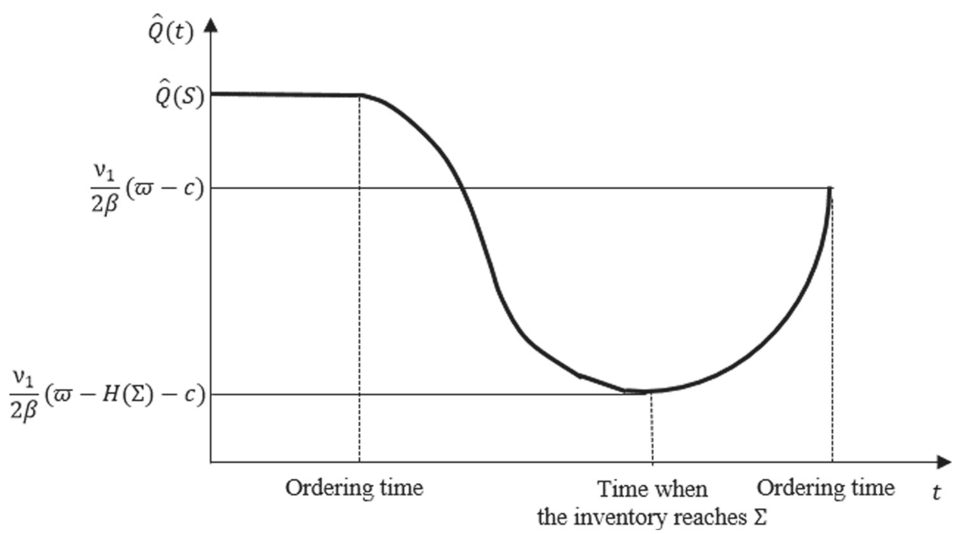

Figure 2. Evolution of voluntary environmental effort in a cycle inventory between two ordering times.

environmental efforts on a quasi-cyclical basis as a marketing tool when there is inventory, and as a production efficiency tool in the case of backlogging. This means that voluntary environmental efforts should be used to stimulate the demand when the stock is replenished to maximise the sales revenue and minimise the holding costs, and to generate backlogs when stock is depleted to ensure that the fixed costs of future production are covered (see Figure 2). Relatedly, the implementation of an (s, S) inventory policy should prevent a firm from using voluntary environmental policy as a simple means to get ephemeral reputation insurance. This conclusion is in sharp contrast with Minor and Morgan (2011) and Barrage, Chyn, and Hastings (2020). The reason lies with the intricacy among production and inventory operations and voluntary environmental policy. That is, the efficiency of production and inventory operations is contingent upon the meticulous deployment of the voluntary environmental policy while the effectiveness of the voluntary environmental policy depends upon the consistent execution of the operational planning.

Finally, a zero or negative inventory should not result in an absence of voluntary environmental efforts. Actually, current voluntary environmental efforts are necessary to prepare the next sales cycle. In this regard, the fact that increasing or decreasing the voluntary environmental effort is triggered by a threshold of the inventory, is a result of interest to practitioners.

6. Concluding remarks

In sharp contrast with Barcos et al. (2013), who assume an inverted U-shaped relationship between firms' environmental initiatives and their inventory levels, we show that a U-shaped relationship actually prevails when an (s, S) inventory policy is implemented. The justification for our result is clear: voluntary environmental efforts are successively deployed for marketing reasons when an inventory exists, and then after for production efficiency reasons when the inventory is exhausted. That is, a firm's voluntary environmental policy should not result from the search for balance between two opposing external pressures, but rather it should serve to reconcile marketing and production objectives.

A numerical study involving a sensitivity analysis of the model with respect to the way the demand is affected by the environmental effort and the parameters impact the profitability of the firm are needed to complement this study. An important research extension could be to look at price as another important driver of instantaneous sales and to characterise the nature of its interaction with voluntary environmental efforts under an (s, S) inventory policy. Another direction for future research would be to combine both voluntary and regulatory environmental efforts to determine whether both efforts act as mutual complements or substitutes in the context of an (s, S) inventory policy.

Notes

1. https://www.shell.com/shellenergy/othersolutions/welcome-to-shell-environmental-products/working-with-customers-to-compensate-for-their-emissions.html.
2. https://www.reliableplant.com/view/22448/land-rover-offsets-co2.
3. https://climatechampions.unfccc.int/unilever-and-the-race-to-halve-emissions-by-2030/.
4. https://cen.acs.org/materials/Chemex-goes-global-carbon-neutral/98/i42.
5. A similar approach can be found in, e.g., Khmelnitsky and Singer (2015) and Herbon (2021).
6. Earlier surveys are notably Aggarwal (1974) and Porteus (1990).
7. Note that the impulse times refer to the times when the inventory is modified. So if production takes some time, for example a fixed amount of time, the stopping time will be the time of delivery.

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Notes on contributors

Alain Bensoussan is Lars Magnus Ericsson Chair and the Director of ICDRiA (International Center for Decision and Risk Analysis) at the University of Texas at Dallas. He is also Chair Professor of Risk and Decision Analysis at the City University Hong Kong. He has been for 4 years World Class University Distinguished Professor at Ajou University, Korea. He is Professor Emeritus at the University Paris Dauphine. Professor Bensoussan served as President of National Institute for Research in Computer Science and Control (INRIA) from 1984 to 1996; President of the French Space Agency (CNES) from 1996 to 2003; and Chairman of the European Space Agency (ESA) Council from 1999 to 2002. He is a member of the

French Academy of Sciences, French Academy of Technology, Academia Europaea, and International Academy of Astronautics. His distinctions include AMS Fellow, IEEE Fellow, SIAM Fellow, Von Humboldt award, and the NASA public service medal. Professor Bensoussan is a decorated Officer of Legion d'Honneur, Commandeur Ordre National du Merite from France and Officer Bundes Verdienst Kreuz from Germany. He has received the W.T. and Idalia Reid Prize from SIAM in 2014. He has an extensive research background in stochastic control, risk analysis and decision-making. He has published 13 books and more than 400 papers and proceedings. He develops a comprehensive approach to Risk Analysis, to apprehend technical and socio-economic risks simultaneously. He has experience in aerospace and information technology industries. His main focus is presently in the energy sector.

Fouad El Ouardighi teaches operations and supply chain management as well as optimal control theory and continuous-time dynamic games at ESSEC Business School. His main research interests are in the interface of operations and marketing management, in R&D competition, environmental economics and management, and epidemic control. He visited many foreign academic institutions as University of Washington in Seattle, HEC-Montreal, McGill University, Bar-Ilan University, HEC-Paris, Technical University of Vienna, etc. Fouad was former academic director of various Advanced Masters programmes in Operations Management. He is currently the coordinator of the Operations Management and Operations Research concentration of the ESSEC PhD programme.

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Appendix

A1. Solution of (4.14)

We first solve the following problem, obtained by looking at the derivative $H_s(x) = G_s'(x)$, where s is fixed, that is:

$$\begin{aligned}&-\frac{1}{2}\sigma^2 H_s''(x) + \Phi'(H_s(x)+c)H_s'(x) + \alpha H_s(x)\\ &\quad + (h+\alpha c)1_{x>0} + (p-\alpha c)1_{x<0} = 0,\ x > s \qquad \text{(A1.1)}\\ &H_s(s) = 0,\ H_s(+\infty) = -c - \frac{h}{\alpha}\end{aligned}$$

We look for a solution $H_s(x)$ which is C^1, with bounded second derivative, satisfying the inequalities:

$$-c - \frac{h}{\alpha} \le H_s(x) \le -c + \frac{p}{\alpha} \qquad \text{(A1.2)}$$

We note that, in view of the boundary conditions, if a solution exists, then from the left inequality (A1.2):

$$0 \le \Phi'(H_s(x)+c) \le \nu_0 + \frac{{\nu_1}^2}{2\beta}\left(\varpi + \frac{h}{\alpha}\right) \qquad \text{(A1.3)}$$

Theorem 2: *We take $s < 0$. Then there exists one and only one solution of* (A1.1)*, which is $C^1(s, +\infty)$ with bounded second derivative. The second derivative has a discontinuity at* 0*. The inequalities* (A1.3) *are satisfied.*

The existence and uniqueness of $H_s(x)$ follows from standard results on second order two point boundary value problems with bounded coefficients.

We can immediately state that the problem:

$$\begin{aligned}&-\frac{1}{2}\sigma^2 G_s''(x) + \Phi(G_s'(x)+c) + \alpha G_s(x) + (h+\alpha c)x^+\\ &\quad + (p-\alpha c)x^- = 0,\ x > s \qquad \text{(A1.4)}\\ &G_s'(s) = 0,\ G_s'(+\infty) = -c - \frac{h}{\alpha}\end{aligned}$$

has one and only one solution given by:

$$\begin{aligned}G_s(s) &= \frac{1}{\alpha}\left[\frac{1}{2}\sigma^2 H_s'(s) - \Phi(c) + s(p-\alpha c)\right] \qquad \text{(A1.5)}\\ G_s(x) &= G_s(s) + \int_s^x H_s(\xi)d\xi\end{aligned}$$

The fact that $s < 0$ is not necessary to obtain a solution of (A1.1), but we shall need it later on. In fact, we shall restrict further the interval for s. Recalling $\gamma = \sqrt{2\alpha}/\sigma$, problem (A1.1) is equivalent to the integral equation:

$$\begin{aligned}H_s(x) &= \frac{2}{\sigma^2\gamma}e^{-\gamma(x-s)}\int_s^x e^{\gamma(\xi-s)}[p-\alpha c-(p+h)e^{-\gamma\xi^-}]d\xi\\ &\quad + \frac{1}{\sigma^2}\int_s^x e^{-\gamma(x-\xi)}[1+e^{-2\gamma(\xi-s)}]\Phi(H_s(\xi)+c)d\xi\\ &\quad - \frac{1-e^{-2\gamma(x-s)}}{\sigma^2}\int_x^{+\infty} e^{-\gamma(\xi-x)}\Phi(H_s(\xi)+c)d\xi\end{aligned} \qquad \text{(A1.6)}$$

We can then compute the derivative:

$$\begin{aligned}H_s'(x) &= \frac{2}{\sigma^2\gamma}\Big\{-\gamma e^{-\gamma(x-s)}\int_s^x e^{\gamma(\xi-s)}[p-\alpha c-(p+h)\\ &\quad e^{-\gamma\xi^-}]d\xi + p - \alpha c - (p+h)e^{-\gamma x^-}\Big\}\\ &\quad + \frac{2}{\sigma^2}\Phi(H_s(x)+c)\\ &\quad - \frac{\gamma}{\sigma^2}\int_s^x e^{-\gamma(x-\xi)}[1+e^{-2\gamma(\xi-s)}]\Phi(H_s(\xi)+c)d\xi\\ &\quad - \frac{\gamma}{\sigma^2}[1+e^{-2\gamma(x-s)}]\int_x^{+\infty} e^{-\gamma(\xi-x)}\Phi(H_s(\xi)+c)d\xi\end{aligned} \qquad \text{(A1.7)}$$

In particular, for $x = s$, we get, recalling that $s < 0$:

$$\begin{aligned}H_s'(s) &= \frac{2}{\sigma^2\gamma}[p-\alpha c-(p+h)e^{\gamma s}]\\ &\quad + \frac{2}{\sigma^2}\left[\Phi(c) - \gamma\int_s^{+\infty} e^{-\gamma(\xi-s)}\Phi(H_s(\xi)+c)d\xi\right]\end{aligned} \qquad \text{(A1.8)}$$

Since Φ is monotone increasing, we obtain from (A1.8) the estimate:

$$H_s'(s) \ge \frac{2}{\sigma^2\gamma}[p-\alpha c-(p+h)e^{\gamma s}] + \frac{2}{\sigma^2}\left(\Phi(c) - \Phi\left(\frac{p}{\alpha}\right)\right) \qquad \text{(A1.9)}$$

Recalling assumption (4.15) and the definition of s^*, we have:

$$\frac{1}{\gamma}[p-\alpha c-(p+h)e^{\gamma s}] + \Phi(c) - \Phi\left(\frac{p}{\alpha}\right) \ge 0,\ \forall s \le s^*, \qquad \text{(A1.10)}$$

therefore $H_s'(s) > 0, \forall s < s^*$.

Proposition 1: *Assume $H_s'(s) > 0$. This assumption is true as soon as $s \le s^*$. Then the function $H_s(x)$ (a solution of (A1.1)) has a unique zero, $S(s) > s$ as well as a unique maximum $\Sigma(s) < 0$. Moreover, $H_s'(x) > 0$, if $s \le x < \Sigma(s)$ and $H_s'(x) < 0$ if $x > \Sigma(s)$. Also $H_s(x) > 0$, if $s < x < S$ and $H_s(x) < 0$ if $x > S$. If $H_s'(s) \le 0$ then $H_s'(x) < 0, \forall x > s$. Also, $H_s(x) < 0, \forall x > s$. In this case, we define $S(s) = s$.*

The proof is technical and relies on maximum principle concepts. The method is almost identical to that of the paper of the first author (Bensoussan, Skaaning, and Turi 2018).

We next provide estimates on the derivative of $H_s(x)$.

Proposition 2: *Assume $H_s'(s) > 0$. We have the estimates:*

$$0 \le H_s'(x) \le \frac{2(p-\alpha c)}{\sigma\sqrt{\alpha}},\ if 0 \le x \le \Sigma(s), \tag{A1.11}$$

$$-\frac{\sqrt{2}}{\sigma\sqrt{\alpha}}(p+h) \le H_s'(x) \le 0,\ if\ \Sigma(s) \le x \le +\infty.$$

If $H_s'(s) \le 0$, then the second estimate holds for $s \le x \le +\infty$. So, in all cases:

$$|H_s'(x)| \le \frac{\sqrt{2}}{\sigma\sqrt{\alpha}} \max\left(\sqrt{2}(p-\alpha c), p+h\right). \tag{A1.12}$$

We also state the following:

Proposition 3: *The function $H_s(x)$ is continuous in s, and $H_s'(s)$ is also continuous.* ■

Consider next equation (A1.1) with $s = 0$. It writes:

$$-\frac{1}{2}\sigma^2 H_0''(x) + \frac{d}{dx}\Phi(H_0(x)+c) + \alpha H_0(x) + (h+\alpha c) = 0,\ x > 0 \tag{A1.13}$$

$$H_0(0) = 0,\ H_0(+\infty) = -c - \frac{h}{\alpha}.$$

We can check the properties:

$$H_0'(0) < 0,\ H_s'(s) > 0,\ \forall s \le s^*. \tag{A1.14}$$

From the continuity of the function $H_s'(s)$, there exists a point $\bar{s} \in (s^*, 0)$ such that $H_{\bar{s}}'(\bar{s}) = 0$. We can take the smallest one, so that $H_s'(s) > 0, \forall s < \bar{s}$. It follows that $S(s) > \max(s, s^*)$ for $s < \bar{s}$ and $S(\bar{s}) = \bar{s}$. Moreover, the function $S(s)$ is continuous on $(-\infty, \bar{s})$.

A2. Finding *s*

We obtain s by solving (4.15), which amounts to solving:

$$k = \int_s^{S(s)} H_s(x)dx \tag{A2.1}$$

where k corresponds to the fixed cost of production (see 3.3).

We then have:

Proposition 4: *There exists a solution of equation* (A2.1) *in the interval $(-\infty, \bar{s})$. We take the smallest value, in case there are several ones.*

If we introduce the function:

$$\zeta(s) = \int_s^{S(s)} H_s(x)dx$$

we can check that it is continuous. Moreover, $\zeta(\bar{s}) = 0$ and $\zeta(s) \to +\infty$ as $s \to -\infty$. It follows that equation (A2.1) has indeed a solution, and the solution is smaller than $\bar{s}$.