Geometric triangulations and discrete Laplacians on manifolds: an update

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Abstract

This paper uses the technology of weighted triangulations to study discrete versions of the Laplacian on piecewise Euclidean manifolds. Given a collection of Euclidean simplices glued together along their boundary, a geometric structure on the Poincaré dual may be constructed by considering weights at the vertices. We show that this is equivalent to specifying sphere radii at vertices and generalized intersection angles at edges, or by specifying a certain way of dividing the edges. This geometric structure gives rise to a discrete Laplacian operator acting on functions on the vertices. We study these geometric structure in some detail, considering when dual volumes are nondegenerate, which corresponds to weighted Delaunay triangulations in dimension 2, and how one might find such nondegenerate weighted triangulations. Finally, we talk briefly about the possibilities of discrete Riemannian manifolds.

1. Introduction

In this paper we shall explore Euclidean structures on manifolds which lead to discrete Laplacians (sometimes called Laplace operators). Euclidean structures can be introduced on a triangulation of a manifold by giving each simplex the geometric structure of a Euclidean simplex. This structure gives the manifold a length space structure in the same way a Riemannian metric gives a manifold a length structure: the length between two points is the infimum of the lengths of paths between the two points. The length of a path is determined by the fact that each simplex it passes through has the structure of Euclidean space.

The purpose of this paper is to be able to do analysis on the piecewise Euclidean space. The Laplacian $\triangle$ is well defined on many geometric spaces, and is especially important as a natural operator on a Riemannian manifold and as a generator of Brownian motion. In this paper, we define a general Euclidean structure called a duality triangulation which not only allows one to measure distance between points and volumes of simplices, but also allows one

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to describe a geometric dual cell decomposition and the volume of dual cells. This allows one to define a Laplacian in a natural way. Similar Laplacians have found applications in fields such as image processing [95, 73] and physics [94].

The duality triangulation structure is very similar to other Euclidean structures used in both pure and applied math; specifically, we address the connection to weighted triangulations and Thurston triangulations. In addition, positivity of volumes of certain duals correspond to Delaunay or weighted Delaunay triangulations.

This paper is organized as follows. We begin in Section 3 with an introduction to Euclidean structures by recalling the definitions of weighted and Thurston triangulations, introducing dual triangulations, and relating the three types of triangulations. In Section 4 we introduce the Laplace operator $\triangle$ associated to a given duality triangulation and derive some of its properties. In Section 5 we discuss weighted Delaunay triangulations and in Section 6 we consider flip algorithms for constructing weighted Delaunay and Delaunay triangulations. Finally, in Section 7 we briefly discuss the status of piecewise linear Riemannian geometry.

Many of the results in this paper were motivated as generalizations of those described in [17].

Remark 1. *The original preprint version of this paper contained some of the results published in [58].*

2. Introduction to this update

The first version of this paper [59] appeared as a preprint. In this updated version, we have updated the terminology to adhere to current trends, especially the language regarding weighted Delaunay triangulations, and to give further comments and references to more recent work that uses or is closely related to the material herein. Other than these minor changes and additions, the paper is largely the same as the most recent preprint version. The main contribution of the original paper is to provide careful description of the types of duality structures that are found commonly in many applications of discrete and discrete differential geometry. The original paper also aimed to unite several different viewpoints that each utilize similar concepts. Some more recent treatments can be found as well, e.g. [1, 37].

Piecewise linear/piecewise Euclidean structures have been widely studied in the last 20 years as part of the body of literature devoted to discrete conformal structures and wider set of literature on discrete differential geometry. Moreover, graph and discrete Laplacians have been widely used in many contexts in order to describe both geometric and data generated questions. The work in this paper lays a foundation for a geometric Laplacian that at the same time restricts the class of weighted Laplacians to ones that are geometrically related to Laplace-Beltrami operators on Riemannian manifolds and generalizes weighted graph Laplacians to situations where weights are not always positive. In the latter case, the geometric character of the operators still can allow analysis of the operators even without the standard assumption of positive weights. Discussion on this issue can be found in Section 4.3.

While the application of (unweighted) Delaunay triangulations via 2D finite element is well-developed, a number of applications have found use for weighted Delaunay triangulations and their relatives. First among these is the use of finite volume methods where the control volumes can change, e.g., [47, 87]. Other applications involve modeling and simulation [93], mesh generation [99], spectral parametrization [131], discrete tensor fields [38], geometric processing [116], and surface representation [43].

3. Euclidean structures

3.1. Basic definitions

In this section we shall introduce three types of Euclidean structures: weighted triangulations, Thurston triangulations, and duality triangulations. All structures begin with a topological triangulation $\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n\}$ of an n-dimensional manifold (we shall usually use n to denote the dimension of the complex in this paper). The triangulation consists of lists of simplices σ^k, where the superscript denotes the dimension of the simplex, and $\mathcal{T}_k$ is a list of all k-dimensional simplices $\sigma^k = \{i_0, \ldots, i_k\}$. We shall often refer to 0-dimensional simplices as vertices, 1-dimensional simplices as edges, 2-dimensional simplices as faces or triangles, and 3-dimensional simplices as tetrahedra. We shall often denote vertices as j instead of $\{j\}$. Let $\mathcal{T}_1^+$ denote the directed edges, where we distinguish (i, j) from (j, i). When the order does not matter, we use $\{i, j\}$ to denote an edge. A triangulation is said to be an n-dimensional manifold if a neighborhood of every vertex is homeomorphic to a ball in $\mathbb{R}^n$. A two-dimensional manifold is often referred to as a surface. Throughout this paper we will be dealing exclusively with triangulations of manifolds or parts of manifolds.

In order to give the topological triangulation a geometric structure, each edge $\{i, j\}$ is assigned a length ℓ_{ij} such that for each simplex in the triangulation there exists a Euclidean simplex with those edge lengths. We call such an assignment a *Euclidean triangulation* $(\mathcal{T}, \ell)$, where we think of ℓ as a function

$$\ell : \mathcal{T}_1 \to (0, \infty).$$

The conditions on ℓ include the triangle inequality, but there are further restrictions in higher dimensions which ensure that the simplices can be realized as (non-degenerate) Euclidean simplices. The restrictions can be expressed in terms of the square of volume, which can be expressed as a polynomial in the squares of the edge lengths by the Cayley-Menger determinant formula [117]. Each pair of simplices σ_1^n and σ_2^n connected at a common boundary simplex σ^{n-1} is called a *hinge*. In a Euclidean triangulation every hinge can be embedded isometrically in $\mathbb{R}^n$.

Remark 2. *Throughout this paper, we will refer to simplices by their vertices, and hence use notation such as ℓ_{ij} for edge $\{i, j\}$. The work can be generalized to* *non-simplicial triangulations which may have multiple simplices that share the same vertices, but for notational convenience we do not consider these cases.*

Euclidean triangulations have the structure of a distance space with an intrinsically defined distance. Given any path γ whose length can be computed on each Euclidean simplex, we can compute the total length of the path $L(\gamma)$ as $L(\gamma) = \sum_\sigma L_\sigma(\gamma \cap \sigma)$ where $L_\sigma(\gamma \cap \sigma)$ is the length of the path in the simplex σ (if the path intersects the simplex many times, we simply add the contributions of each piece of the intersection) and the sum is over all simplices of highest dimension that intersect γ. In particular, we can consider paths that are differentiable when restricted to each simplex (these are called piecewise differentiable paths or curves). The intrinsic distance is defined as

$$d(P,Q) = \inf\{L(\gamma) : \gamma \text{ is a path from } P \text{ to } Q\}. \tag{1}$$

The class of paths can be either taken to be piecewise differentiable or piecewise linear since length is minimized on piecewise linear paths, as explained in [119, Section 2]. A path which locally minimizes length is called a *geodesic* and one which globally minimizes is called a *minimizing geodesic*.

We are now ready to introduce more structures on Euclidean triangulations.

3.2. Weighted triangulations

We begin with weighted triangulations.

Definition 3. *A* weighted triangulation *is a Euclidean triangulation* $(\mathcal{T}, \ell)$ *together with weights*

$$w : \mathcal{T}_0 \to \mathbb{R}.$$

We think of the weight w_i as the square of the radius of a circle centered at the vertex i, although we do not assume that weights are positive. These weighted triangulations are used in the literature on weighted Delaunay triangulations such as [45] and [6]. Thinking of the weights in this way, in each n-dimensional simplex there exists an $(n-1)$-dimensional sphere which is orthogonal to each of the spheres centered at the vertices (this means they are perpendicular if they intersect, or else orthogonal in the sense described in [103, Section 40]). In this way, each simplex σ has a corresponding center $C(\sigma)$, which is the center of this sphere, and the center has a weight $w_{C(\sigma)}$ which is the square of the radius of this sphere. See Figures 1 and 2.

An important particular case of weighted triangulations is that when $w_i = 0$ for all vertices i. This is the setting for Delaunay triangulations, but may not satisfy the Delaunay condition. We shall revisit this in Section 5.

3.3. Thurston triangulations

Definition 4. *A* Thurston triangulation *is a collection* $(\mathcal{T}, w, c)$, *where*

$$w : \mathcal{T}_0 \to \mathbb{R},$$
$$c : \mathcal{T}_1 \to \mathbb{R},$$

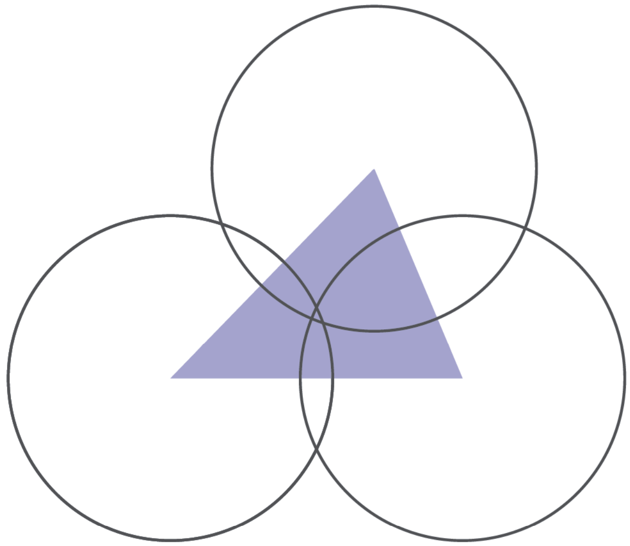

Figure 1: A weighted or Thurston triangulation with corresponding circles at the vertices.

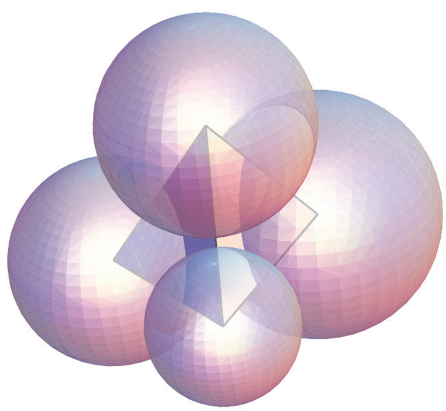

Figure 2: A weighted or Thurston triangulation with corresponding spheres at the vertices.

where $c_{ij} < w_i + w_j$ *and such that the induced lengths*

$$\ell_{ij} = \sqrt{w_i + w_j - c_{ij}}$$

make $(\mathcal{T}, \ell)$ *into a Euclidean triangulation.*

For a Thurston triangulation, one considers the weight w_i to be the square of the radius r_i of a sphere centered at vertex i, just as for weighted triangulations, and one considers $c_{ij} = 2r_i r_j \cos(\pi - \theta_{ij})$ where θ_{ij} is the angle between the spheres centered at vertices i and j. In this case, one derives the formula for ℓ_{ij} by the law of cosines. By considering c_{ij} instead of θ_{ij}, we have included some cases where the spheres do not intersect. These structures were studied by W. Thurston in the context of proving Andreev's theorem (see [120] and [92]). Alternatively, one could consider circles with inversive distance as in [64].

An important special case is that when $c_{ij} = -2r_i r_j$ (i.e. $\theta_{ij} = 0$). This is the case of a sphere packing on each simplex, since it corresponds to the spheres being mutually tangent (as in [36, 56, 57]).

Remark 5. *Note that weighted and Thurston triangulations are closely related to spaces of spheres, as described in classical references such as [103, 126, 25].*

3.4. Duality triangulations

Definition 6. *A* duality triangulation *is a collection* $(\mathcal{T}, d)$, *where*

$$d : \mathcal{T}_1^+ \to \mathbb{R}$$

which satisfies

$$d_{ij}^2 + d_{jk}^2 + d_{ki}^2 = d_{ji}^2 + d_{ik}^2 + d_{kj}^2 \tag{2}$$

for each $\{i, j, k\} \in \mathcal{T}_2$ *and such that the induced lengths*

$$\ell_{ij} = d_{ij} + d_{ji}$$

make $(\mathcal{T}, \ell)$ *into a Euclidean triangulation.*

We think of the weight d_{ij} as representing the portion of the length ℓ_{ij} of edge $\{i, j\}$ which has been assigned to vertex i while d_{ji} is the portion assigned to vertex j. We thus call them *local lengths.* The total length of $\{i, j\}$ is the sum of the contributions d_{ij} from vertex i and d_{ji} from vertex j. Hence each edge is assigned a center $C(\{i, j\})$ which is distance d_{ij} from vertex i and distance d_{ji} from vertex j. The condition (2) ensures that for each triangle $\{i, j, k\}$, the perpendiculars to the three edges through the edge centers meet at one point, which can be called the center of the triangle, $C(\{i, j, k\})$. We shall soon see that this condition on 2-dimensional simplices allows us to define a center for every simplex in the triangulation.

There are two canonical examples which automatically satisfy the condition (2). One is the case where d_{ij} depends only on i for all edges (i, j) (that is, $d_{ij} = d_{ik}$, etc.). We call this a *circle* or *sphere packing* as in [56], and the dual comes from the inscripted sphere, which is inscribed in the 1-skeleton. For instance, the center $C(\{i,j,k\})$ is the center of the circle inscribed in $\{i,j,k\}$ in 2D and the center $C(\{i,j,k,\ell\})$ is the center of the sphere tangent to each of the edges of the tetrahedron $\{i,j,k,\ell\}$ in 3D. Another important case is where $d_{ij}=d_{ji}$. This corresponds to the center $C(\{i,j,k\})$ coming from the circle circumscribed about the triangle $\{i,j,k\}$ and similar for all higher dimensions.

The structure is called a duality triangulation because the existence of a center $C(\sigma)$ for each σ puts a piecewise-Euclidean length structure on the dual of the triangulation in such a way that dual simplices are orthogonal to ordinary simplices. For example, in two dimensions, if an edge $\{i,j\}$ is incident on the two simplices $\{i,j,k\}$ and $\{i,j,\ell\}$, then we can define the length of the dual edge $\bigstar\{i,j\}$ to be equal to the distance between the center $C(\{i,j,k\})$ of the triangle $\{i,j,k\}$ and the center $C(\{i,j\})$ of the edge $\{i,j\}$ plus the distance between $C(\{i,j,\ell\})$ and $C(\{i,j\})$. When the hinge is isometrically embedded in $\mathbb{R}^2$, we see that $\bigstar\{i,j\}$ is a straight line which is perpendicular to the edge $\{i,j\}$. We shall now show that this can be done in all dimensions, and no additional restrictions must be made besides (2) for each triangle.

Proposition 7. *A duality triangulation in any dimension has unique centers $C(\sigma^m)$ for each simplex σ^m such that $C(\sigma^m)$ is at the intersection of the $(m-1)$-dimensional hyperplanes through $C(\{i,j\})$ and perpendicular to $\{i,j\}$ for each $\{i,j\}$ in σ^m.*

Proof. We construct the centers $C(\sigma^m)$ inductively for m-dimensional simplices. Each pair of m-dimensional simplices meeting at an $(m-1)$-dimensional simplex (a "hinge") can be embedded in $\mathbb{R}^m$ as two adjacent Euclidean simplices. To make the notation more readable, we shall not distinguish between the embedding of the hinge in $\mathbb{R}^m$ and the hinge as abstract simplices in the piecewise Euclidean manifold. A simplex σ^m is assumed to be Euclidean with the assigned edge lengths given by ℓ_{ij}. We now inductively construct the centers of each simplex. First, $C(\{i\})=i$ and $C(\{i,j\})$ is the point on $\{i,j\}$ which is a distance d_{ij} to $\{i\}$ and a distance d_{ji} to $\{j\}$. Now, given centers $C\left(\sigma^k\right)$ for $k\leq m-1$, we construct $C(\sigma^m)$ as follows. Label the vertices of σ^m to be $\{0,1,\ldots,m\}$.

Let $\Pi_{\{i,j\}}$ denote the plane in $\mathbb{R}^m$ through $C(\{i,j\})$ and perpendicular to $\{i,j\}$ (this is a hyperplane in $\mathbb{R}^m$). First we construct the center of a simplex $\{0,1,2\}$ $(m=2)$. One can embed the simplex in $\mathbb{R}^2$ as the three vertices $(0,0)$, $(\ell_{01},0)$, and $(\ell_{02}\cos\gamma_0,\ell_{02}\sin\gamma_0)$, where γ_0 is the angle at vertex 0. The centers of the three edges are realized as $C(\{0,1\})=(d_{01},0)$, $C(\{0,2\})=(d_{02}\cos\gamma_0,d_{02}\sin\gamma_0)$, and $C(\{1,2\})=(\ell_{01}-d_{12}\cos\gamma_1,d_{12}\sin\gamma_1)$. Hence

$$\begin{aligned}
\Pi_{\{0,1\}}&=\{(d_{01},t):t\in\mathbb{R}\}\,,\\
\Pi_{\{0,2\}}&=\{(d_{02}\cos\gamma_0+t\sin\gamma_0,d_{02}\sin\gamma_0-t\cos\gamma_0):t\in\mathbb{R}\}\,,\\
\Pi_{\{1,2\}}&=\{(\ell_{01}-d_{12}\cos\gamma_1+t\sin\gamma_1,d_{12}\sin\gamma_1+t\cos\gamma_1):t\in\mathbb{R}\}\,.
\end{aligned}$$

A quick calculation (using the law of cosines to compute $\cos\gamma_i$ and $\sin\gamma_i$ in terms of d_{ij}) shows that the three intersection points of these lines coincide if and only if (2) holds.

We now construct $C(\sigma^m)$ given $C(\sigma^{m-1})$ for all $(m-1)$-dimensional simplices. Since σ^m is a nondegenerate Euclidean simplex, the planes $\Pi_{\{0,1\}}, \ldots, \Pi_{\{0,m\}}$ intersect at one point, c. We need only show that the planes $\Pi_{\{i,j\}}$ also intersect c. This is true because inside $\{0,i,j\}$, the planes $\Pi_{\{0,i\}}$ and $\Pi_{\{0,j\}}$ meet each other and the plane $\Pi_{\{i,j\}}$ at $C(\{0,i,j\})$. Furthermore, since these planes are all perpendicular to $\{0,i,j\}$, the intersection $\Pi_{\{0,i\}} \cap \Pi_{\{i,j\}}$ is equal to the intersection $\Pi_{\{0,i\}} \cap \Pi_{\{0,j\}}$ and hence contains c. We call this point $C(\sigma^m) = c$. □

Remark 8. *Given the description in the previous proof using planes, it can be seen that the orthogonal planes intersect simplex planes to form extensions of dual edges. For this reason it is possible to find the simplex center by starting at the edge center, moving orthogonally to the triangle center, then orthogonally to the tetrahedron center, etc. to obtain simplex center. The plane description above ensures that the center gotten in this way does not depend on the choice of simplices along the way.*

As noted by Hirani [73], assignment of centers allows a geometric description of the Poincaré dual of the triangulation. Any triangulation of a manifold has a cell complex which is its Poincaré dual (see, for instance, [21] or [67]). See Figures 3 and 4 for two-dimensional and three-dimensional simplices with dual cells included. Duality structures determine centers, and thus allow one to define geometric duals (a realization of the Poincaré dual), each of which has a volume. Hirani restricted himself to “well-centered” triangulations, which means that the center of each simplex is inside the simplex. This is a very strong restriction, for even Delaunay triangulations may not be well-centered. More recent work replaces well-centered with Delaunay conditions, e.g. [74].

In general, some volumes may be considered to be negative. The k-dimensional volume of a simplex σ^k will be denoted $\left|\sigma^k\right|$ (for instance $|\{i,j\}| = \ell_{ij}$) and the $(n-k)$-dimensional (signed) volume of the dual of a simplex $\bigstar\sigma^k$ will be denoted $\left|\bigstar\sigma^k\right|$.

It is helpful to consider an example before considering the general definitions. Given a triangulation of a three-dimensional manifold, one defines the duals as follows (compare with Figure 4):

0. The dual of a 3-simplex $\{i,j,k,\ell\}$ is the center, $\bigstar\{i,j,k,\ell\} = C(\{i,j,k,\ell\})$, and its volume is one.
1. The dual of a 2-simplex $\{i,j,k\}$ contained in $\{i,j,k,\ell\}$ and $\{i,j,k,m\}$ is a 1-cell $\bigstar\{i,j,k\}$, which is the union of the line from $C(\{i,j,k,\ell\})$ to $C(\{i,j,k\})$ and the line from $C(\{i,j,k,m\})$ to $C(\{i,j,k\})$. Its volume is slightly tricky. We define the volume as

$$\begin{aligned}|\bigstar\{i,j,k\}| &= \pm d\left[C(\{i,j,k,\ell\}), C(\{i,j,k\})\right] \pm d\left[C(\{i,j,k,m\}), C(\{i,j,k\})\right] \\ &= \pm d\left[C(\{i,j,k,\ell\}), C(\{i,j,k,m\})\right]\end{aligned}$$

where d is the Euclidean distance in $\mathbb{R}^3$ (these are well defined because we can embed the hinge in $\mathbb{R}^3$) and the signs are defined appropriately. In the first line, the sign is positive if $C(\{i,j,k,\ell\})$ is on the same side

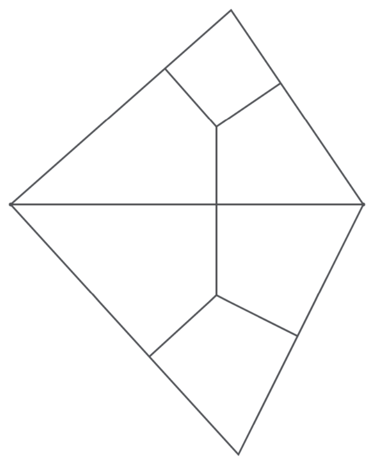

Figure 3: Two triangles with the pieces of dual edges intersecting the triangles included.

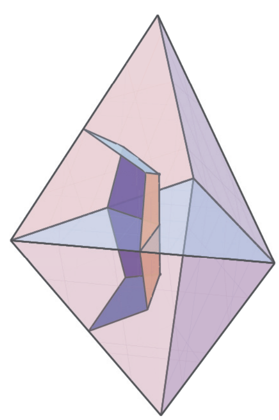

Figure 4: Two tetrahedra with the pieces of dual edges and faces intersecting the tetrahedra included.

of the plane containing the side $\{i,j,k\}$ as the simplex $\{i,j,k,\ell\}$ is, and negative if it is on the other side (similarly for $\{i,j,k,m\}$). The sign on the second line is defined to be compatible with the first line. Note that it is possible for $|\bigstar\{i,j,k\}|$ to be negative.

2. The dual of a 1-simplex $\{i,j\}$ is the union of triangles. For each k,ℓ such that $\{i,j,k,\ell\}$ is a simplex, the intersection of the simplex with the dual $\bigstar\{i,j\}$ is the union of the right triangle with vertices $C(\{i,j,k,\ell\})$, $C(\{i,j,k\})$, $C(\{i,j\})$ and the right triangle with vertices $C(\{i,j,k,\ell\})$, $C(\{i,j,\ell\})$, $C(\{i,j\})$. Each of these triangles has a signed area. The first is
$$\pm\frac{1}{2}d\left[C(\{i,j,k,\ell\}), C(\{i,j,k\})\right]\ d\left[C(\{i,j\}), C(\{i,j,k\})\right]$$
and the second is defined similarly. The sign is defined as the product of the appropriate signs in each of the two distances.

3. The dual of a vertex $\{i\}$ is a union of orthoschemes, which are tetrahedra with many right angles (see [115] or [18]). For each j,k,ℓ such that $\{i,j,k,\ell\}$ is a simplex, the intersection of $\bigstar\{i\}$ with $\{i,j,k,\ell\}$ is the union of the following six (orthoscheme) tetrahedra:
   (a) the tetrahedron defined by the vertices $C(\{i,j,k,\ell\})$, $C(\{i,j,k\})$, $C(\{i,j\})$, and i,
   (b) the tetrahedron defined by $C(\{i,j,k,\ell\})$, $C(\{i,j,k\})$, $C(\{i,k\})$, and i,
   (c) the tetrahedron defined by $C(\{i,j,k,\ell\})$, $C(\{i,j,\ell\})$, $C(\{i,j\})$, and i,
   (d) the tetrahedron defined by $C(\{i,j,k,\ell\})$, $C(\{i,j,\ell\})$, $C(\{i,\ell\})$, and i,
   (e) the tetrahedron defined by $C(\{i,j,k,\ell\})$, $C(\{i,k,\ell\})$, $C(\{i,k\})$, and i,
   (f) and the tetrahedron defined by $C(\{i,j,k,\ell\})$, $C(\{i,k,\ell\})$, $C(\{i,\ell\})$, and i.

   The volume of $\bigstar\{i\}$ is the sum of the volumes of these tetrahedra, namely
$$\pm\frac{1}{6}d\left[C(\{i,j,k,\ell\}), C(\{i,j,k\})\right]\ d\left[C(\{i,j\}), C(\{i,j,k\})\right]\ d\left[i, C(\{i,j\})\right]$$
   for the first and similarly for the others, where the signs are defined appropriately.

Remark 9. *Although we use terminology such as "intersection of $\bigstar\{i\}$ with $\{i,j,k,\ell\}$," if the region has negative volume, this part of $\bigstar\{i\}$ may physically lie outside of $\{i,j,k,\ell\}$ and have negative volume. It would be better termed the piece of $\bigstar\{i\}$* associated with $\{i,j,k,\ell\}$.

We can define the geometric duals in a triangulation of an n-dimensional manifold inductively as follows.

Definition 10. *Define the dual of $\{0,\ldots,n\}$ to be $\bigstar\{0,\ldots,n\} = C(\{0,\ldots,n\})$, and $|\bigstar\{0,\ldots,n\}| = 1$.*

Definition 11. *The signed distance*

$$d_{\pm}\left[C\left(\sigma^{n}\right), C\left(\sigma^{n-1}\right)\right]$$

for $\sigma^{n-1} \subset \sigma^{n}$ is equal to the distance between $C(\sigma^{n})$ and $C\left(\sigma^{n-1}\right)$ in any isometric embedding $\sigma^{n} \subset \mathbb{R}^{n}$ with the sign positive if $C(\sigma^{n})$ is on the same side of the hyperplane defined by $\sigma^{n-1} \subset \mathbb{R}^{n}$ as σ^{n} is, and negative if $C(\sigma^{n})$ is on the opposite side.

Remark 12. *This signed distance appears in different forms in other works, e.g., [74, 40].*

It will be useful to know the following formula for the distance between the center of a triangle and the center of a side. Consider a triangle $\{i, j, k\}$. Then some basic Euclidean geometry yields

$$d_{\pm}\left[C\left(\{i, j, k\}\right), C\left(\{i, j\}\right)\right] = \frac{d_{ik} - d_{ij}\cos\gamma_i}{\sin\gamma_i} \tag{3}$$

where γ_i is the angle at vertex i. The condition (2) ensures that

$$d_{\pm}\left[C\left(\{i, j, k\}\right), C\left(\{i, j\}\right)\right] = \frac{d_{jk} - d_{ji}\cos\gamma_j}{\sin\gamma_j}$$

as well, where we have switched the i and j.

Proposition 13. *For any $k \geq 1$, the volume of a simplex σ^{k} is*

$$\left|\sigma^{k}\right| = \frac{1}{k!}\sum_{\sigma^{0}\subset\cdots\subset\sigma^{k}}\prod_{j=0}^{k-1} d_{\pm}\left[C\left(\sigma^{j}\right), C\left(\sigma^{j+1}\right)\right] \tag{4}$$

where σ^{k} is fixed and the sum is over all strings of simplices contained in σ^{k}.

Proof. The proof is by induction on k. If $k = 1$, then $|\{i, j\}| = d_{ij} + d_{ji}$. Assume (4) is true and consider σ^{k+1}. Let the boundary of σ^{k+1} be made up of $\sigma_0^k, \ldots, \sigma_{k+1}^k$. The volume can be computed as

$$\left|\sigma^{k+1}\right| = \frac{1}{k+1}\sum_{i=0}^{k+1} d_{\pm}\left[C\left(\sigma_i^{k}\right), C\left(\sigma^{k+1}\right)\right]\left|\sigma_i^{k}\right|$$

where each term in the sum is the volume of the simplex consisting of the center $C\left(\sigma^{k+1}\right)$ union σ_i^k and the signs for $d_{\pm}$ tell us whether to add the area or subtract the area. It follows from the inductive hypothesis that

$$\left|\sigma^{k+1}\right| = \frac{1}{(k+1)!}\sum_{\sigma^{0}\subset\cdots\subset\sigma^{k+1}}\prod_{j=0}^{k} d_{\pm}\left[C\left(\sigma^{j}\right), C\left(\sigma^{j+1}\right)\right].$$

□

Note that the above argument works for any choice of center $C\left(\sigma^k\right) \in \mathbb{R}^k$ as long as $C\left(\sigma^\ell\right)$ are the orthogonal projections onto the subspaces spanned by σ^ℓ for each subsimplex. The volume of a dual simplex is defined as follows.

Definition 14. *The volume of a dual simplex $\bigstar\sigma^k$ is defined to be*

$$\left|\bigstar\sigma^k\right| = \frac{1}{(n-k)!} \sum_{\sigma^k \subset \cdots \subset \sigma^n} \prod_{j=k}^{n-1} d_\pm \left[C\left(\sigma^j\right), C\left(\sigma^{j+1}\right)\right] \tag{5}$$

where σ^k is fixed and the sum is over all strings of simplices containing σ^k.

Note that the volume is signed (it may be negative). The total volume is expressible in terms of volumes of the dual simplices.

Proposition 15. *Given a duality triangulation $\mathcal{T}$ of dimension n, the total volume is*

$$V = \sum_{\sigma^n \in \mathcal{T}_n} |\sigma^n| = \sum_{i \in \mathcal{T}_0} |\bigstar\{i\}|. \tag{6}$$

Proof. Using (5) and (4), we see that it is sufficient to show that

$$\sum_{i \in \mathcal{T}_0} \sum_{\{i\} \subset \cdots \subset \sigma^n}$$

is a reordering of

$$\sum_{\sigma^n \in \mathcal{T}_n} \sum_{\sigma^0 \subset \cdots \subset \sigma^n}.$$

Here is one way to see this. Make a graph whose vertices are all simplices of all dimensions and whose edges connect two simplices if one simplex is in the boundary of the other. An easy way to draw the graph in the plane is to put vertices corresponding to n-dimensional simplices in a horizontal line on top, then $(n-1)$-dimensional simplices in a horizontal line below those, and so on until at the bottom is a horizontal line containing all of the vertices corresponding to 0-dimensional simplices in the triangulation. Now draw the edges, which can only connect a vertex in a row to a vertex in the row above or below. Consider the sum over all paths between the top and bottom of this graph. We can count this in two ways: first start at the bottom with each path starting at a 0-dimensional simplex, or first start at the top with each path starting at an n-dimensional simplex. These are the two sums. □

3.5. Equivalence of metric triangulations

We shall now show that weighted triangulations are equivalent to Thurston triangulations, and that, up to a universal scaling of the weights, both are almost equivalent to the set of duality triangulations. This is motivated by the geometric interpretations of the lengths, weights, angles, etc.

First we show the equivalence of weighted triangulations and Thurston triangulations.

Theorem 16. *There is a bijection between weighted triangulations and Thurston triangulations.*

Proof. The definition of Thurston triangulation gives the map to weighted triangulations, keeping w_i the same and assigning

$$\ell_{ij} = \sqrt{w_i + w_j - c_{ij}}.$$

Since we assumed that $w_i + w_j - c_{ij} > 0$, it follows that ℓ_{ij} must be positive. Similarly, we can map the other way as

$$c_{ij} = w_i + w_j - \ell_{ij}^2.$$

Note that since $\ell_{ij} > 0$, we must have that $w_i + w_j - c_{ij} > 0$. □

Next we map weighted triangulations to duality triangulations. Notice that there is a one parameter family of deformations of a given weighted triangulation of a triangle $\{i, j, k\}$ which fix the center $C(\{i, j, k\})$. These deformations are given by

$$w_i \to w_i + t \tag{7}$$

for varying t. We call these *weight scaling deformations*, or just *weight scalings*.

Theorem 17. *Suppose the underlying manifold is connected. Then weighted triangulations modulo weight scalings can be mapped injectively into the set of duality triangulations. It is a bijection if the set of duality triangulations are required to satisfy*

$$\sum_{k=0}^{r} \left(d_{i_k i_{k-1}}^2 - d_{i_{k-1} i_k}^2 \right) = 0 \tag{8}$$

for all loops $j = i_0, i_1, \ldots, i_r = j$, where $\{i_k, i_{k+1}\} \in \mathcal{T}_1$.

Proof. Given spheres at the vertices of a simplex with radii $\sqrt{w_i}$, one can always construct a sphere which is orthogonal to each of these spheres. The center of that sphere will be the center of the simplex, and for that reason is often called the *orthogonal center* [45]. Then duals can be constructed for all dimensions; we get d_{ij} from the projections of the centers to the edges. One can do this very easily by embedding the circles in a vector space of signature $1, 1, 1, -1$ as in [103, 40.2]. Given a center, one can draw the lines perpendicular to the sides of the triangle through the center, and these determine d_{ij}. A careful calculation yields

$$d_{ij} = \frac{\ell_{ij}^2 + w_i - w_j}{2\ell_{ij}}. \tag{9}$$

This is the map to duality triangulations. Note that the condition (2) is automatically satisfied.

There appears to be more information in weighted triangulations, however, because the new circle centered at the orthogonal center has a radius, which can be calculated to be

$$r_{ijk}^2 = d_{ij}^2 + \left(\frac{d_{ik} - d_{ij}\cos\gamma_{ijk}}{\sin\gamma_{ijk}}\right)^2 - w_i \tag{10}$$
$$= \frac{d_{ij}^2 + d_{ik}^2 - 2d_{ij}d_{ik}\cos\gamma_{ijk}}{\sin^2\gamma_{ijk}} - w_i,$$

where γ_{ijk} is the angle at vertex i in triangle $\{i,j,k\}$. Note that $r_{ijk}^2 = w_{C(\{i,j,k\})}$, the weight assigned to the center of $\{i,j,k\}$. For any single triangle $\{i,j,k\}$, the weight scalings allow one to specify the value of r_{ijk}^2 while fixing the center $C(\{i,j,k\})$. Fixing the center means that each would map to the same duality triangulation. It is easy to see that the formula (9) is unchanged by scaling deformations like (7). If one chooses r_{ijk} then the map is unique. Once this scale is fixed in one triangle, however, the scale is determined on adjacent triangles, because weights on shared vertices have been fixed, and the deformation (7) must be done for all vertices i in the triangle. Thus there is one free scaling parameter for the whole triangulation.

The inverse map from duality triangulations to weighted triangulations must take $d_{ij} + d_{ji}$ to ℓ_{ij}. In order to get the weights, we must first fix w_0 for a given vertex (this is a free parameter since we are considering the weighted triangulation modulo scaling). Then each neighboring weight can be calculated using (9):

$$w_j = d_{ji}^2 - d_{ij}^2 + w_i. \tag{11}$$

We need only show that this is well defined. Suppose $\{i,j,k\} \in \mathcal{T}_2$ and consider a w_k which can be defined from w_j or w_i. Then we need that

$$d_{ki}^2 - d_{ik}^2 + w_i = d_{kj}^2 - d_{jk}^2 + w_j.$$

But since $w_j = d_{ji}^2 - d_{ij}^2 + w_i$, this follows from the fact that $d_{ki}^2 - d_{ik}^2 = d_{kj}^2 - d_{jk}^2 + d_{ji}^2 - d_{ij}^2$ from (2). It follows by a similar argument that any null-homotopic loop can be triangulated and property (8) holds automatically, showing that for any null-homotopic loop $j = i_0, i_1, \ldots, i_L = j$ of L vertices with $\{i_k, i_{k+1}\} \in \mathcal{T}_1$,

$$w_j = \sum_{k=1}^{L}\left(d_{i_k i_{k-1}}^2 - d_{i_{k-1} i_k}^2\right) + w_j.$$

Thus, in general, we need to assume property (8) is satisfied for the weights to be well-defined.

If we start with a weighted triangulation, property (8) is automatically satisfied and thus the map from weighted triangulations to duality triangulations is injective. □

The following triangulation of the torus does not satisfy (8) for all loops. Tile a torus with the two triangles $\{1,2,3\}, \{1,2,4\}$ where $d_{31} = d_{21} = d_{24} = 1 - \varepsilon$,

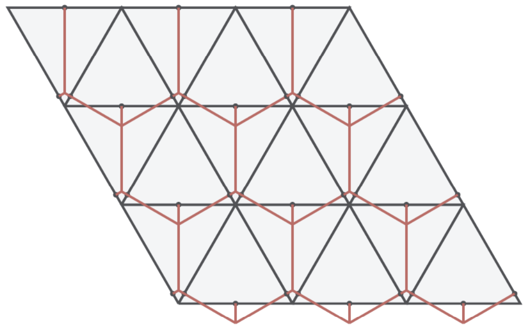

Figure 5: A triangulation of the torus together with dual edges.

$d_{13} = d_{12} = d_{42} = \varepsilon$, and $d_{32} = d_{23} = d_{14} = d_{41} = \frac{1}{2}$ for small ε, see Figure 5. Note that

$$d_{12}^2 + d_{23}^2 + d_{31}^2 = \varepsilon^2 + \frac{1}{4} + (1-\varepsilon)^2 = d_{21}^2 + d_{13}^2 + d_{32}^2$$
$$d_{12}^2 + d_{24}^2 + d_{41}^2 = \varepsilon^2 + \frac{1}{4} + (1-\varepsilon)^2 = d_{21}^2 + d_{14}^2 + d_{42}^2$$

and so on. The homotopy-nontrivial loop containing $\{1,2\}$ does not satisfy property (8).

The relationship between triangulations with duality structures and weighted triangulation on flat tori was considered in [46], where the term reciprocal triangulation is used to denote triangulations that admit duality structures and coherent triangulation to denote weighted Delaunay triangulations. The Maxwell-Cremona correspondence described there relates reciprocal triangulations with harmonic embeddings of graphs in a torus (see also [82]).

Corollary 18. *For a triangulation of a simply connected manifold, there is a bijection between weighted triangulations up to scaling and duality triangulations.*

Proof. Since the manifold is simply connected, any loop bounds a 2-dimensional disk, homeomorphic to $D^2 = \left\{ x \in \mathbb{R}^2 : |x|^2 \leq 1 \right\}$, which is triangulated. One can easily prove by induction on the number of triangles triangulating the disk that on the boundary of any such disk, (8) holds. $\square$

3.6. Discrete conformal structures

One of the motivation for the comparison of different metric structures is to relate the discrete conformal geometry of circle packing and others, encapsulated here in Thurston triangulations. Several types of discrete conformal structures were proposed by Thurston [120] as circle packing and independently by Rocek-Williams [113], Luo [89], and Bobenko-Pinkall-Springborn [19] as vertex scaling. Each of these forms of discrete conformal structures led to significant study. In

[60] the author gave an axiomatic viewpoint for discrete conformal structures as motivated by the fact that variations of curvature produce discrete Laplacians based on the duality structures described here. In [132] and [62] the possible discrete conformal structures of this type were completely classified, and shown to come from lengths assigned as

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}.$$

In this case, a discrete conformal variation arises from fixing all α_i and all η_{ij} and letting the functions f_i of the vertices vary. It turns out that this is related to the duality structure determined by

$$d_{ij} = \frac{\partial \ell_{ij}}{f_i} = \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i+f_j}}{\ell_{ij}}.$$

This is consistent with results for circle packing, where $\alpha_i = 1$ for all i and $\eta_{ij} = 1$ for all edges $\{i,j\}$, as well as with vertex scaling cases where $\alpha_i = 0$ for all i. It also encapsulates circles with fixed intersection angle and circles with fixed inversive distance, or more generally Thurston triangulations as described above with $w_i = \alpha_i e^{f_i}$ and $c_{ij} = 2\eta_{ij} e^{f_i+f_j}$. It was observed in [19] that vertex scaling conformal structures correspond to fixed distance cross ratio (or, equivalently, modulus of the cross ratio; see also [83]), and in [61] that other forms correspond to a version of cross ratio involving power distance. Other forms of discrete conformal structures that fix the argument of the cross ratio, instead of the modulus, can be found in [81].

4. Laplacians

Laplacians on graphs and on piecewise Euclidean manifolds have been studied in many different contexts, for instance [17, 28, 31, 56, 57, 68, 72, 73, 95, 104, 123, 125, 124]. The purpose of this section is to consider the comments from Bobenko and Springborn in [17], which suggests the use of Delaunay triangulations as a natural context in which to describe Laplace operators. We aim to generalize these comments to weighted Delaunay triangulations.

4.1. Laplacian defined

The suggested Laplacian on two-dimensional surfaces in [17] (also seen in [73, 95, 39]) is the following operator on functions $f : \mathcal{T}_0 \to \mathbb{R}$,

$$(\triangle f)_i = \sum_{j:\{i,j\}\in\mathcal{T}_1} w_{ij}\left(f_j - f_i\right) \tag{12}$$

where w_{ij} is defined by

$$w_{ij} = \frac{1}{2}\left(\cot\gamma_{kij} + \cot\gamma_{\ell ij}\right)$$

if γ_{kij} is the angle at vertex k in triangle $\{i,j,k\}$, and the hinge containing $\{i,j\}$ consists of the triangles $\{i,j,k\}$ and $\{i,j,\ell\}$. Note that if $w_{ij}>0$ then (i) this is a Laplacian with weights on the graph defined by the one-skeleton of the triangulation, and (ii) $\triangle f_i > 0$ if f_i is the minimal value of f and $\triangle f_i < 0$ if f_i is the maximal value of f. Bobenko and Springborn note that if the triangulation is Delaunay, then $w_{ij}>0$ and the Laplacian is a Laplacian on graphs in the classical sense (see [31]).

A simple calculation shows that if we take the weights at all vertices to be zero, then the signed distance

$$d_{\pm}\left[C\left(\{i,j,k\}\right), C\left(\{i,j\}\right)\right] = r_{ijk}\cos\gamma_{kij}$$

where r_{ijk} is the circumradius of triangle $\{i,j,k\}$. Since the circumradius can be computed to be

$$r_{ijk} = \frac{1}{2}\frac{\ell_{ij}}{\sin\gamma_{kij}}$$

we find that

$$d_{\pm}\left[C\left(\{i,j,k\}\right), C\left(\{i,j\}\right)\right] = \frac{1}{2}\ell_{ij}\cot\gamma_{kij},$$

which gives the well-known cotan formula for the finite element Laplacian.

In general, Hirani [73] suggests the following definition of Laplacian:

Definition 19. *The Laplacian of a discrete function f is the discrete function $\triangle f$ given by*

$$(\triangle f)_i = \frac{1}{|\bigstar\{i\}|}\sum_{j:\{i,j\}\in\mathcal{T}_1}\frac{|\bigstar\{i,j\}|}{|\{i,j\}|}\left(f_j - f_i\right). \tag{13}$$

This formula has roots in the divergence theorem for the smooth Laplacian:

$$\int_U \triangle f\, dV = \int_{\partial U}\nabla f\cdot n\, dS, \tag{14}$$

where n is the unit normal to ∂U. Taking $U=\bigstar\{i\}$ and slightly rearranging terms, we get the corresponding formula on piecewise Euclidean manifolds

$$(\triangle f)_i\ \ |\bigstar\{i\}| = \sum_{j:\{i,j\}\in\mathcal{T}_1}\frac{f_j - f_i}{|\{i,j\}|}\left|\bigstar\{i,j\}\right|,$$

where $\frac{f_j-f_i}{|\{i,j\}|}$ is the normal derivative and $|\bigstar\{i,j\}|$ is the surface area measure on the boundary of $\bigstar\{i\}$. This formula is well defined on any duality triangulation (which is the motivation for the definition) and coincides with (12) in the case of constant weights, except for the factor of $|\bigstar\{i\}|$. One can think of the difference between considering the induced measure $\triangle f\ dV$ instead of the pointwise Laplacian $\triangle f$. It is, in fact, natural to consider the measure instead since, if we consider the discrete Laplacian approximating a smooth one, the pointwise Laplacian is only accurate when considered on scales larger than the scale of the discretization.

It is notable that the Laplacian appears in Discrete Exterior Calculus as a special case of a Hodge Laplacian, also called Laplace-de Rham operator [39]. The Laplacian can be given as $d^T * d$, where d is the coboundary operator from vertices to edges considered as a matrix. Since we can also consider instead the boundary homomorphism ∂ from edges to vertices, which is the transpose of d, an equivalent expression is $\partial * \partial^T$. In finite elements literature, if piecewise linear continuous Lagrange finite elements are used then the Laplacian is called the stiffness matrix and the $*$ matrix is called the mass matrix (see, e.g., [37]).

We also note that the Laplacian given by (13) is the same as the Laplacian considered by Chow-Luo [28] in two dimensions as observed by Z. He [68], where the duality is defined by Thurston triangulations as described above. It also appears in [56, 57] in three dimensions, where Thurston triangulations are considered such that d_{ij} depend only on i. Also, the Laplacian described in [89] is actually the cotan Laplacian described above in (12) with the same weights w_{ij}. The interest in these Laplacians is that they are not derived from means such as (14) but instead as the induced time derivative of curvature quantities under geometric evolutions.

The Laplacian defined in (13) is a Laplacian with weights on graphs in the usual sense (see [31]) if the coefficients

$$\frac{|\bigstar\{i,j\}|}{|\bigstar\{i\}|}$$

are each positive. In two dimensions we see that this is implied by $d_{ij} > 0$ and $|\bigstar\{i,j\}| > 0$. Note that if we consider the Laplacian measure (the analogue of $\triangle f\ dV$), then the $|\bigstar\{i\}|$ term is not present and we need only look at the condition $|\bigstar\{i,j\}| > 0$, which is the condition that the triangulation is weighted Delaunay in two dimensions.

The Laplacian can be considered the gradient of a Dirichlet energy functional as described in [17], which is the analogue of the smooth functional

$$E(f) = \int_M |\nabla f|^2 \, dV.$$

The Dirichlet energy functional induced by the duality triangulation is

$$E(f) = \frac{1}{2} \sum_{\{i,j\} \in \mathcal{T}_1} \frac{|\bigstar\{i,j\}|}{|\{i,j\}|} (f_j - f_i)^2 . \tag{15}$$

This specializes in the case where the $w_i = 0$ for all $i \in \mathcal{T}_0$ (or, equivalently, $d_{ij} = d_{ji} = \ell_{ij}/2$ for all $\{i,j\} \in \mathcal{T}_1$) to the Dirichlet energy in [17]. This energy is positive if $|\bigstar\{i,j\}| > 0$.

4.2. Laplace, Poisson, and heat equations

Given a Laplacian, we can consider the standard elliptic and parabolic equations, namely the Laplace and Poisson equations

$$\triangle u = 0, \qquad \triangle u = f \tag{16}$$

and the heat equation

$$\frac{du}{dt} = \triangle u, \tag{17}$$

where the heat equation is an ordinary differential equation since $\triangle$ is a difference operator. A solution u to the Laplace equation is called a *harmonic function.* Two important properties we will study are the negative semidefiniteness of the operator $\triangle$ and the maximum principle for the heat equation.

It will sometimes be easier to consider $\triangle u = 0$ as a matrix equation. We think of $u : \mathcal{T}_0 \to \mathbb{R}$ as a vector and $\triangle$ corresponds to a matrix L whose off-diagonal pieces are

$$L_{ij} = \frac{|\bigstar \{i, j\}|}{|\{i, j\}|}$$

and whose diagonal pieces are

$$L_{ii} = -\sum_{j:\{i,j\}\in\mathcal{T}_1} \frac{|\bigstar \{i, j\}|}{|\{i, j\}|}.$$

Then one can write the Laplace equation as

$$Lu = 0.$$

Notice that Poisson's equation

$$\triangle u = f \tag{18}$$

is equivalent to to the vector equation

$$Lu = fV,$$

where $(fV)_i = f_i |\bigstar \{i\}|$. It is clear that L has the constant functions $f_i = a$ (or the vector $(a, a, \ldots, a)$) in the nullspace.

We first consider the definiteness of the $\triangle$, which is the same as definiteness for L. The first result concerns the case of $|\bigstar \{i, j\}| > 0$.

Theorem 20. *If $|\bigstar \{i, j\}| > 0$ for all edges $\{i, j\}$ then L is negative semidefinite with nullspace spanned by the constant vectors.*

Proof. In this case we have an $N\times N$ matrix L with diagonal entries negative and off-diagonal entries positive and with $\sum_{j=1}^{N} L_{ij} = 0$. We reiterate an argument from [36], though this is a standard result from linear algebra following from the fact that the matrix is an M-matrix [77]. Let $(v_1, \ldots, v_N)$ be an eigenvector corresponding to $\lambda \geq 0$. We may assume that $v_1 > 0$ is the maximum of v_i. We wish to show that $v_i = v_j$ for all i, j. Observe

$$\lambda v_1 = \sum_{i=1}^{N} L_{1i} v_i \leq \sum_{i=1}^{N} L_{1i} v_1 = 0.$$

Equality holds if and only if $v_i = v_1$ for all i. $\square$

Corollary 21. *If $|\bigstar\{i,j\}| > 0$ for all edges $\{i,j\}$ then Poisson's equation has a solution for any f such that*

$$\sum_{i\in\mathcal{T}_0} f_i V_i = 0.$$

This is the analogue of the smooth result that $\triangle u = f$ has a solution if $\int_M f dV = 0$. One may also consider boundary conditions such as Dirichlet and Neumann conditions. The condition $|\bigstar\{i,j\}| > 0$ is equivalent to a property called weighted Delaunay and will be studied in the next section. These cases for Delaunay triangulations in two dimensions were studied by Bobenko and Springborn [17]. This condition will also be important in our discussion of the maximum principle.

There are other conditions which guarantee that the Laplacian is negative semidefinite.

Theorem 22. *For any two-dimensional triangulation such that $d_{ij} > 0$ for all $(i,j) \in \mathcal{T}_1^+$, the Laplacian matrix L is negative semidefinite with nullspace spanned by the constant vectors.*

We shall prove this by a sequence of claims. For all of the claims it is assumed that the weights d_{ij} are all positive. We shall use $h_{ij} = d_{\pm}[C(\{1,2,3\}), C(\{i,j\})]$ and γ_i is the angle at vertex i. Consider only the 3×3 matrix M corresponding to $\{1,2,3\}$ with entries $M_{ij} = h_{ij}/\ell_{ij}$ if $i \neq j$ and $M_{ii} = -\sum_{j\neq i} M_{ij}$.

Claim 23. *If $h_{ij} < 0$ then $\gamma_i < \frac{\pi}{2}$ and $\gamma_j < \frac{\pi}{2}$.*

Proof. Let k be the third vertex so that $\{i,j,k\} = \{1,2,3\}$. We know that

$$h_{ij} = \frac{d_{ik} - d_{ij}\cos\gamma_i}{\sin\gamma_i}$$

by formula (3). If $h_{ij} < 0$ then $0 < d_{ik} < d_{ij}\cos\gamma_i$. Hence $\cos\gamma_i > 0$ and $\gamma_i < \pi/2$. We can also express h_{ij} as

$$h_{ij} = \frac{d_{jk} - d_{ji}\cos\gamma_j}{\sin\gamma_j}$$

and follow the same logic. $\square$

Thus only one M_{ij} may be negative. Suppose it is M_{12}.

Claim 24. $M_{12} + M_{13} = \frac{\ell_{23}(d_{12}\cos\gamma_2 + d_{13}\cos\gamma_3)}{2A_{123}}$.

Proof. We calculate

$$\begin{aligned} M_{12} + M_{13} &= \frac{d_{23} - d_{21}\cos\gamma_2}{\ell_{12}\sin\gamma_2} + \frac{d_{32} - d_{31}\cos\gamma_3}{\ell_{13}\sin\gamma_3} \\ &= \frac{\ell_{23}(\ell_{23} - d_{21}\cos\gamma_2 - d_{31}\cos\gamma_3)}{2A_{123}} \end{aligned}$$

and finally we use that $\ell_{23} = \ell_{12}\cos\gamma_2 + \ell_{13}\cos\gamma_2$. $\square$

Claim 25. $d_{12} \cos \gamma_2 + d_{13} \cos \gamma_3 > 0$.

Proof. If both γ_2 and γ_3 are less than or equal to $\pi/2$ then this is clear (since both may not be equal to $\pi/2$). Since $M_{12} < 0$, and hence $h_{12} < 0$, we can only have $\gamma_3 > \pi/2$. Since $h_{12} < 0$ and $h_{13} > 0$ we have that

$$\frac{d_{13}}{d_{12}} < \cos \gamma_1 < \frac{d_{12}}{d_{13}}$$

so $d_{12} > d_{13}$. Furthermore, since $\gamma_1 + \gamma_2 < \pi$ we have that

$$0 < -\cos \gamma_3 = \cos(\gamma_1 + \gamma_2) < \cos \gamma_2$$

so

$$-d_{13} \cos \gamma_3 < d_{12} \cos \gamma_2.$$

□

Lemma 26. $M_{ii} < 0$.

Proof. By the above argument, we know that $M_{11} = -M_{12} - M_{13} < 0$. Similar arguments hold for the other coefficients. □

Proof of Theorem 22. It is sufficient to prove that for any matrix M_{ij}, $1 \leq i, j \leq 3$, is negative semidefinite. We know that the vector $(1, 1, 1)$ is in the nullspace and we have already shown in Lemma 26 that the diagonal entries are negative. Hence it is sufficient to show that the determinant of the 2×2 submatrix M_{ij}, $1 \leq i, j \leq 2$, is positive. We find that the 2×2 determinant is equal to $M_{12}M_{13} + M_{12}M_{23} + M_{13}M_{23}$. We compute the determinant to be equal to

$$\frac{(d_{13}h_{23} + d_{23}h_{13}) \sin \gamma_2}{\ell_{12}\ell_{13}}$$

(to do this calculation, begin by writing the terms in the determinant using formula (3) choosing all of the denominators to contain $\sin \gamma_1 \sin \gamma_2$, then rearrange the terms using the facts that $\gamma_1 + \gamma_2 + \gamma_3 = \pi$, $d_{ij} + d_{ji} = \ell_{ij}$, and $\ell_{ij} = \ell_{ik} \cos \gamma_i + \ell_{jk} \cos \gamma_k$ several times and finally recollecting h_{23} and h_{13} again using formula (3)). Note that the determinant is symmetric in all permutations in $1, 2, 3$. We know by the claim above that two of the three h_{ij} must be positive, so choosing the two that are positive, we must have that the determinant is positive. Hence the matrix is negative semidefinite. □

We consider d_{ij} to be the length of a vector located at i and in the direction towards j. Thus the condition $d_{ij} > 0$ is like a positivity (or Riemannian) condition for a metric (which measures the length of vectors) and is thus a somewhat natural condition.

In the case where all the weights are the same and the triangulation is a triangulation of a domain in the plane, this is a well-known result that may be proven in a very different way. In this case, the Laplacian can be derived as the matrix $L_{ij} = \nabla \phi_i \cdot \nabla \phi_j$, where ϕ_i are the standard basis of piecewise linear finite elements, where $\phi_i(i) = 1$ and $\phi_i(j) = 0$ for vertices $j \neq i$. Thus L_{ij} comes from the restriction of a weak formulation of the differential Laplacian to the space of piecewise linear functions, and hence the negative semidefiniteness follows from the negative semidefiniteness of the differential Laplacian. This formulation goes back at least to Duffin [42], who derived the well-known cotan formula. Positive definiteness of discrete Hodge star matrices based on positivity of volumes of dual structures was also considered in [74, Theorems 4,5,6].

The following is a result on definiteness of the Laplacian in three-dimensions.

Theorem 27. *For a three-dimensional sphere packing triangulation, L is negative semidefinite with nullspace spanned by the constant vectors.*

Proof. It is proven in [57] (see also [111]) that the matrix $A_{\{1,2,3,4\}} = \left(\frac{\partial \alpha_i}{\partial r_j}\right)_{1\leq i,j\leq 4}$ is negative semidefinite with nullspace spanned by the vector $(r_1, \ldots, r_4)$. If we let $R_{\{1,2,3,4\}}$ be the diagonal matrix with r_i, $i = 1, \ldots, 4$ on the diagonal, we see that

$$L = \sum_{\sigma^3\in\mathcal{T}_3} \left(R_{\sigma^3} A_{\sigma^3} R_{\sigma^3}\right)_E .$$

where $\left(M_{\sigma^3}\right)_E$ is the matrix extended by zeros to a $|\mathcal{T}_0| \times |\mathcal{T}_0|$ matrix so that the $\left(M_{\sigma^3}\right)_E$ acts on a vector $\left(v_1, \ldots, v_{|\mathcal{T}_0|}\right)$ only on the coordinates corresponding to vertices in σ^3. Since $r_i > 0$ for all $i \in \mathcal{T}_0$, it follows that L is negative semidefinite with nullspace spanned by $(1, \ldots, 1)$. □

These results indicate that positivity of the dual area is a stronger assumption than the assumption that L is negative definite. If L is negative semidefinite with nullspace spanned by the constant vector $(1, \ldots, 1)$ then one can always solve the Poisson equation for f such that $\sum f_i A_i = 0$.

We now consider the maximum principle. The heat equation is the time-dependent, linear ordinary differential equation

$$\frac{du}{dt} = Lu,$$

whose short time existence is guaranteed by the existence theorem for ordinary differential equations. One of the key properties of the heat equation is the maximum principle, which says that the maximum decreases and the minimum increases. This is true if $|\bigstar\{i,j\}| > 0$.

Theorem 28. *If $|\bigstar\{i,j\}| > 0$ then for a solution $u_i(t)$ of the heat equation, $u_{\max}(t)$ decreases and $u_{\min}(t)$ increases, where $u_{\max} = \max\{u_i : i \in \mathcal{T}_0\}$ and $u_{\min} = \min\{u_i : i \in \mathcal{T}_0\}$.*

Proof. The proof is standard and is simply that for any operator Eu defined by

$$(Eu)_i = \sum_{j\neq i} e_{ij}\left(u_j - u_i\right)$$

for some weights $e_{ij} > 0$, then $(Eu)_i < 0$ if $u_i = u_{\max}$ and $(Eu)_i > 0$ if $u_i = u_{\min}$. □

Note that the proof uses that the off-diagonal entries of the matrix L are positive, which is why satisfying the maximum principle is more restrictive than being negative semidefinite. However, for certain functions (geometric ones which are related to the coefficients of the Laplacian), it may be possible to show that the maximum decreases and the minimum increases. We call this a maximum principle for the function f and we say that the operator is *parabolic-like* for the function f. In [57] it is proven that the sphere-packing case is parabolic-like for a curvature function K.

4.3. Remarks on Laplacian definiteness

We have seen that the maximum principal corresponds to triangulations being graph Laplacians with positive weights, which also corresponds to weighted Delaunay triangulations. This is the most important case in which we have both maximum principal and definiteness of the Laplacian operator. However, it is clear that the condition of positive weights is not necessary for definiteness. The most clear case is for that of the finite element Laplacian in two-dimensions, which yields the cotan Laplacian. Since it is the restriction of a definite operator to the subspace of piecewise linear functions, it is definite even if the triangulation is not Delaunay. Above we proved that in two dimensions, it is sufficient to have that the d_{ij} are positive. In [58] it is found that the two-dimensional Laplacian is definite if the centers of each triangle are inside the circumcircle of the triangle. Definiteness in situations where the coefficients may be negative occur in sphere packing [56] as well. Generalizations of these results in three and higher dimensions can be found in [130, 40].

5. Weighted Delaunay triangulations

5.1. Introduction to weighted Delaunay triangulations

The study of the Laplacian motivates, in two dimensions, the study of weighted Delaunay triangulations, also called coherent triangulations and regular triangulations. First, we recall the usual definition of a weighted Delaunay triangulation (see, for instance, [45] or [6]). Let $d(x,p)$ be the Euclidean distance between points p and x. Define the power distance

$$\pi_p : \mathbb{R}^n \to \mathbb{R}$$

by

$$\pi_p(x) = d(x,p)^2 - w_p \tag{19}$$

if p is a point weight w_p. The power is important as a function which is zero on the sphere centered at p with radius $\sqrt{w_p}$, positive outside the sphere, and negative inside the sphere. Notice that if p is a vertex of a simplex σ and $c = C(\sigma)$ then $\pi_c(p) = w_p$ and $\pi_p(c) = w_c$, where the weight w_c is defined as the square of the radius of the orthogonal sphere as described in Section 3.2.

Since we can embed any hinge in $\mathbb{R}^n$, the following local definition of weighted Delaunay makes sense on a piecewise Euclidean manifold.

Definition 29. *An $(n-1)$-dimensional simplex σ^{n-1} incident on two n-dimensional simplices $\sigma_1^n = \sigma^{n-1} \cup \{v_1\}$ and $\sigma_2^n = \sigma^{n-1} \cup \{v_2\}$ is* locally weighted Delaunay *if $\pi_{c_1}(v_2) > w_{v_2}$ and $\pi_{c_2}(v_1) > w_{v_1}$, where $c_i = C(\sigma_i^n)$ is the center of σ_i^n for $i = 1$ or 2. If the weights are all equal to zero, a locally weighted Delaunay simplex is said to be* locally Delaunay*.*

Sometimes we will instead say that the *hinge* is locally weighted Delaunay. A hinge is locally Delaunay if and only if it satisfies the local empty circumsphere property: the sphere circumscribing σ_1^n does not contain v_2. This is simply the interpretation of the definition when the weights are equal to zero. Note that the condition for being locally weighted Delaunay is unchanged by a weight scaling of the type (7) due to formula (10) for $w_{C(\{i,j,k\})}$.

There are actually global definitions of weighted Delaunay and Delaunay, since the definition of power (19) makes sense globally using the intrinsic distance (1) described in Section 3.1.

Definition 30. *An n-dimensional weighted triangulation is* weighted Delaunay *if for every $\sigma^n \in \mathcal{T}_n$, we have $\pi_{C(\sigma^n)}(v) > w_v$ for every vertex v in the complement of σ^n. In the case that the weights are all zero, we say the triangulation is* Delaunay*.*

It is a well known fact that for n-dimensional weighted Delaunay triangulations of points in $\mathbb{R}^n$ [6] and for 2-dimensional piecewise Euclidean surfaces with zero weights [17, 86] that every hinge being locally weighted Delaunay is equivalent to the triangulation being weighted Delaunay. It is likely that the proof in [86, Chapter 3] can be generalized to weighted Delaunay triangulations of any dimension, but we do not do that here.

The argument in [6] uses the fact that a geodesic must be a straight line, and along a geodesic line the power increases in the manner listed below. To generalize that argument, one needs the following assumption:

Criterion 31. *Suppose the hinge $\{\sigma_1^n, \sigma_2^n, \sigma^{n-1}\}$ is locally weighted Delaunay. Consider a minimizing geodesic ray γ starting at X_0 which intersects a hinge $\{\sigma_1^n, \sigma_2^n, \sigma^{n-1}\}$ by first entering σ_1^n and then σ_2^n. The simplex σ^{n-1} determines a plane which separates σ_1^n and σ_2^n and contains all points x such that $\pi_{C(\sigma_1^n)}(x) = \pi_{C(\sigma_2^n)}(x)$. Then $\pi_{C(\sigma_1^n)}(X_0) < \pi_{C(\sigma_2^n)}(X_0)$.*

This criterion is an attempt to give a condition under which arguments for triangulations in $\mathbb{R}^n$ can be extended to arguments on manifolds, as we will see later.

One might try to prove Criterion 31 by "developing the geodesic" in the plane in the following way (we consider two dimensions for simplicity). Start with a triangle and embed it in $\mathbb{R}^2$. For each new triangle which the geodesic goes through, embed a copy in $\mathbb{R}^2$ adjacent to the previous triangle so that it looks like we are unfolding the manifold. The geodesic must be a straight line if it does not go through a vertex and so we may try to make comparisons on this development. Note also that by the following theorem of Gluck, every two points have a minimizing geodesic between them.

Theorem 32 ([119, Prop. 2.1]). *If a piecewise Euclidean manifold is complete with respect to the intrinsic distance, in particular if M is a finite triangulation, then there is at least one minimizing geodesic between any two points of M.*

The problem with this is that geodesics do go through vertices and even by varying the endpoints slightly, a minimizing geodesic may still go through the vertex (see [96, Figure 14]). Hence it is not at all clear that Criterion 31 is always satisfied.

Note that Bobenko and Springborn [17] are able to prove that Delaunay is the same as all edges being locally Delaunay in general by developing the triangulation (not along a geodesic). Their argument appears to strongly use the fact that the edges are locally Delaunay (with all weights equal to zero), but does not use Criterion 31.

For completeness, we include the proof for weighted Delaunay triangulations of n-dimensional manifolds, assuming Criterion 31, which is proven using a similar method.

Theorem 33. *Under the assumption of Criterion 31, an n-dimensional weighted triangulation is weighted Delaunay if and only if all of its hinges are locally weighted Delaunay.*

Proof. This proof is essentially the one seen in [6] for Delaunay triangulations. Clearly if the triangulation is weighted Delaunay, then all hinges are locally weighted Delaunay. Now suppose all of the hinges of a weighted triangulation are locally weighted Delaunay. Given a vertex v and a simplex σ^n such that v is not in σ^n, we may consider the line L from v to a point in the simplex σ^n. Possibly by adjusting the line slightly, it must intersect, in order, a sequence of n-dimensional simplices $\sigma_1^n, \ldots \sigma_k^n = \sigma^n$ where v is in a simplex bordering σ_1^n. By Criterion 31 we know that

$$\pi_{C\left(\sigma_i^n\right)}(v) < \pi_{C\left(\sigma_{i+1}^n\right)}(v)$$

for $i = 1, \ldots, k-1$. Since the triangulation is locally weighted Delaunay,

$$w_v < \pi_{C\left(\sigma_1^n\right)}(v).$$

Stringing these together, we get that

$$w_v < \pi_{C(\sigma^n)}(v).$$

□

Although we have not proven that weighted Delaunay triangulations and locally weighted Delaunay triangulations are the same, we will often suppress the word "local" in the rest of this paper, always considering the local property.

5.2. Weighted Delaunay triangulations and duality structures

In order to have a definition of locally weighted Delaunay in terms of duality structures, we first look at the two-dimensional case. A weighted Delaunay hinge $\{\{i,j,k\},\{i,j,\ell\}\}$ must satisfy

$$\begin{aligned}
\pi_{C(\{i,j,k\})}(\ell) &= d\left(C\left(\{i,j,k\}\right),\{\ell\}\right)^2 - r_{ijk}^2 > w_\ell \\
\pi_{C(\{i,j,\ell\})}(k) &= d\left(C\left(\{i,j,\ell\}\right),\{k\}\right)^2 - r_{ij\ell}^2 > w_k.
\end{aligned}$$

Proposition 34. *The center $C(\{i,j,k\})$ and radius r_{ijk} are uniquely determined by the three equations*

$$\begin{aligned}
d\left(C\left(\{i,j,k\}\right),\{i\}\right)^2 - r_{ijk}^2 &= w_i \\
d\left(C\left(\{i,j,k\}\right),\{j\}\right)^2 - r_{ijk}^2 &= w_j \\
d\left(C\left(\{i,j,k\}\right),\{k\}\right)^2 - r_{ijk}^2 &= w_k.
\end{aligned}$$

Proof. Put the triangle in Euclidean space with vertices $v_i = \vec{0}, v_j, v_k$. We know that $C(\{i,j,k\}) = xv_j + yv_k$ for some x and y and let z be the unknown radius. Now we can write the first two equations as

$$\begin{aligned}
\left|xv_j + yv_k\right|^2 - z^2 &= w_i \\
\left|(xv_j + yv_k) - v_j\right|^2 - z^2 &= w_j
\end{aligned}$$

so

$$w_i - 2v_j \cdot (xv_j + yv_k) + \ell_{ij}^2 = w_j$$

which is linear in x, y. Similarly, we have

$$w_i - 2v_k \cdot (xv_j + yv_k) + \ell_{ik}^2 = w_k.$$

So the problem reduces to a linear system

$$\begin{aligned}
w_i + \ell_{ij}^2 - w_j &= 2\ell_{ij}^2 x + 2\ell_{ij}\ell_{ik}\left(\cos\gamma_i\right) y \\
w_i + \ell_{ik}^2 - w_k &= 2\ell_{ij}\ell_{ik}\left(\cos\gamma_i\right) x + 2\ell_{ik}^2 y,
\end{aligned}$$

where γ_i is the angle at vertex i, with solutions

$$\begin{aligned}
x &= \frac{\left(w_i + \ell_{ij}^2 - w_j\right)\ell_{ik} - \left(w_i + \ell_{ik}^2 - w_k\right)\ell_{ij}\cos\gamma_i}{2\left(\sin^2\gamma_i\right)\ell_{ij}^2\ell_{ik}} \\
y &= \frac{\left(w_i + \ell_{ik}^2 - w_k\right)\ell_{ij} - \left(w_i + \ell_{ij}^2 - w_j\right)\ell_{ik}\cos\gamma_i}{2\left(\sin^2\gamma_i\right)\ell_{ij}\ell_{ik}^2}
\end{aligned}$$

and

$$z^2 = x^2\ell_{ij}^2 + y^2\ell_{ik}^2 + 2xy\ell_{ij}\ell_{ik}\cos\gamma_i - w_i.$$

$\square$

Corollary 35. *If an edge is on the boundary of weighted Delaunay, i.e.*

$$\pi_{C(\{i,j,k\})}(\ell) = d\left(C(\{i,j,k\}), \{\ell\}\right)^2 - r_{ijk}^2 = w_\ell,$$

then $C(\{i,j,k\}) = C(\{i,j,\ell\})$ *and* $r_{ijk} = r_{ij\ell}$.

Proof. If $d\left(C(\{i,j,k\}), \ell\right)^2 - r_{ijk}^2 = w_\ell$ then $\left(C(\{i,j,k\}), r_{ijk}\right)$ satisfy the same three equations as $\left(C(\{i,j,\ell\}), r_{ij\ell}\right)$, which determine these uniquely. Hence they must be equal. □

Corollary 36. *An edge* $\{i,j\}$ *is weighted Delaunay if and only if* $|\bigstar\{i,j\}| > 0$.

Proof. Clearly $|\bigstar\{i,j\}| = 0$ on the boundary of weighted Delaunay as in Corollary 35 since the centers are the same. It is clear that $|\bigstar\{i,j\}| > 0$ if the edge is weighted Delaunay. □

One can now address the case of n dimensions. The corresponding proofs go through essentially untouched, and one has the following characterization of weighted Delaunay triangulations.

Proposition 37. *An* $(n-1)$*-dimensional simplex* σ^{n-1} *which forms a hinge with simplices* $\sigma_i^n = \sigma^{n-1} \cup \{i\}$ *and* $\sigma_j^n = \sigma^{n-1} \cup \{j\}$ *is weighted Delaunay if and only if* $\left|\bigstar\sigma^{n-1}\right| > 0$.

Note that $\bigstar\sigma^{n-1}$ is a one-dimensional simplex, so the property of being weighted Delaunay has to do with lengths dual to $(n-1)$-simplices being positive. The previous discussion motivates the following definitions which, in light of Theorem 17, are slight generalizations of those for weighted triangulations.

Definition 38. *An* n*-dimensional hinge at simplex* σ^{n-1} *is said to be* locally weighted Delaunay *if* $\left|\bigstar\sigma^{n-1}\right| > 0$. *An* n*-dimensional duality triangulation* $\mathcal{T}$ *is said to be* locally weighted Delaunay *if* $\left|\bigstar\sigma^{n-1}\right| > 0$ *for all* $\sigma^{n-1} \in \mathcal{T}_{n-1}$.

The duality structure is called a Voronoi diagram in the case the triangulation is Delaunay. Voronoi diagrams can be described in a more direct way. A point x is in the Voronoi cell $\bigstar\{i\}$ if it is closer to i than to any other vertex. The boundary of the Voronoi cells forms the $(n-1)$-dimensional complex called the Voronoi diagram. The analogue for weighted Delaunay triangulations is called a power diagram. A point x is in the power cell $\bigstar\{i\}$ if its power distance $\pi_i(x)$ is less than $\pi_j(x)$ for any $j \neq i$ (see [6] [45]).

Remark 39. *The dual cells described in Section 3.4 are slightly different than the usual usage of power cell. Power cells are described by inequalities and so the cells are either empty or have positive area. We have used equal power lines to give nonempty dual regions to each vertex. Instead of having empty dual cells, we have cells with negative area. From the perspective of duality triangulations, this is quite natural. In the case that all vertex duals have positive area, the two notions are the same.*

An interesting question is how to find a weighted Delaunay triangulation of a given manifold with given weights. A potential method of construction is via so called "flip algorithms."

6. Bistellar Flips

In this section we investigate the use of bistellar flips as a way to construct weighted Delaunay triangulations. While this method works well for two-dimensional Delaunay triangulations, there are some issues that arise for weighted Delaunay.

6.1. Flips in 2D

We first consider the case of two dimensions. One can imagine the following notion of a flip. Given a hinge consisting of two triangles $\{i, j, k\}$ and $\{i, j, \ell\}$ incident on one common edge $\{i, j\}$, there exists a flip which exchanges this hinge with a new hinge, namely $\{i, k, \ell\}$ and $\{j, k, \ell\}$. Note that the flip fixes the boundary quadrilateral which consists cyclically of the vertices i, k, j, ℓ. This exchange is called a $2 \to 2$ bistellar flip, or Pachner move ([102]). If the hinge is convex, then this can be done metrically. In fact, the flip can be made at the level of a duality structure. Given the hinge described above, to do the bistellar flip we need to construct $d_{k\ell}$ and $d_{\ell k}$ such that the condition (2) is satisfied in each of the new triangles. This is done by solving the following system of equations for $d_{k\ell}$ and $d_{\ell k}$,

$$\begin{aligned} d_{ik}^2 + d_{k\ell}^2 + d_{\ell i}^2 &= d_{ki}^2 + d_{i\ell}^2 + d_{\ell k}^2 \\ d_{k\ell} + d_{\ell k} &= d(k, \ell) \end{aligned}$$

where $d(k, \ell)$ is the distance between vertex k and vertex ℓ. This distance is the Euclidean distance because the entire hinge can be embedded in $\mathbb{R}^2$. Note that the first equation is equivalent to

$$d_{jk}^2 + d_{k\ell}^2 + d_{\ell j}^2 = d_{kj}^2 + d_{j\ell}^2 + d_{\ell k}^2$$

using (2) for triangles $\{i, j, k\}$ and $\{i, j, \ell\}$. The system can actually be written in a form easier to solve:

$$\begin{aligned} d_{k\ell} - d_{\ell k} &= \frac{d_{ki}^2 + d_{i\ell}^2 - d_{\ell i}^2 - d_{ik}^2}{d(k, \ell)} \\ d_{k\ell} + d_{\ell k} &= d(k, \ell) \end{aligned} \tag{20}$$

which is linear, although the dependence of $d(k, \ell)$ on the remaining d's is not obvious (although easy to find using trigonometry). Hence the $2 \to 2$ bistellar flip is well defined on duality triangulations, and the triangle inequality follows automatically. The two hinges which differ a bistellar flip are shown in Figure 6.

The flip requires that the quadrilateral is convex, otherwise the flip would require that one part is folded back, which complicates matters. This motivates the following definition:

Definition 40. *A hinge is* flippable *if the quadrilateral defined by the hinge is convex when embedded in* $\mathbb{R}^2$.

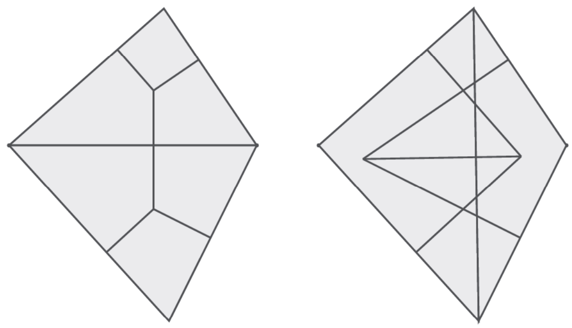

Figure 6: Two hinges differing by a bistellar flip, together with duals.

Now, given a convex quadrilateral, there exist two possible ways to make it into a hinge. The duals are uniquely determined by an assignment of centers to the edges on the quadrilateral. Let $L_{\{i,j\}}$ be the line perpendicular to $\{i,j\}$ and through $C(\{i,j\})$. Then $L_{\{i,k\}}$ and $L_{\{j,k\}}$ meet at a point which is the center $C(\{i,j,k\})$ and similarly $L_{\{i,\ell\}}$ and $L_{\{j,\ell\}}$ meet at a point which is the center $C(\{i,j,\ell\})$. However, also $L_{\{i,k\}}$ and $L_{\{i,\ell\}}$ meet at a point which becomes $C(\{i,k,\ell\})$ after the flip, and similarly with $L_{\{j,k\}}$ and $L_{\{j,\ell\}}$. Hence the centers in the hinge form another quadrilateral dual to the hinge (see the right side of Figure 6). One diagonal of the dual quadrilateral corresponds to $\bigstar\{i,j\}$ and the other corresponds to $\bigstar\{k,\ell\}$. One must have positive length and the other negative length (or both are zero if all dual lines meet at a single point), so either the hinge is weighted Delaunay, or it will become weighted Delaunay by a flip. One can also think of the flip of the hinge corresponding to a flip of the dual hinge. To make this argument rigorous, one simply uses the fact that $\bigstar\{i,j\}$ must be perpendicular to $\{i,j\}$, and considers the possible cases for $|\bigstar\{i,j\}|$ being positive, negative, or zero. If it is negative, then it must look like the right side Figure 6 and hence a flip makes $|\bigstar\{k,\ell\}|$ positive. If $|\bigstar\{i,j\}|$ is zero, then a flip maintains this.

6.2. Rippa's theorem and its generalization

Rippa [110] showed that if one considers the Dirichlet energy (15) on a triangulation of points in $\mathbb{R}^2$ where the weights are zero (or equivalently, $d_{ij} = d_{ji} = \ell_{ij}/2$ for all edges $\{i,j\}$), flipping to make an edge Delaunay increases the Dirichlet energy (see also [108]). Bobenko and Springborn [17] note that Rippa's Theorem extends trivially to piecewise Euclidean surfaces (2-dimensional manifolds). We shall express Rippa's theorem in a way closer to the exposition on [17], which is in line with the notation in this paper.

Theorem 41 ([110]). *Let $(\mathcal{T},\ell)$ be a piecewise Euclidean, triangulated surface with assigned edge lengths ℓ, which we think of as a weighted triangulation with all weights equal to zero. Let $\mathcal{T}_0$ be the vertices of the triangulation and let*

$f : \mathcal{T}_0 \to \mathbb{R}$ be a function. Suppose $\mathcal{T}'$ is another triangulation which is gotten from $\mathcal{T}$ by a $2 \to 2$ bistellar flip on edge e (in particular, $\mathcal{T}_0 = \mathcal{T}_0'$,) such that the hinge is locally Delaunay after the flip. Then

$$E_{\mathcal{T}'}(f) \leq E_{\mathcal{T}}(f),$$

where $E_{\mathcal{T}}$ and $E_{\mathcal{T}'}$ are the Dirichlet energies corresponding to $\mathcal{T}$ and $\mathcal{T}'$. As a consequence, the minimum is attained when all edges are Delaunay (and hence the triangulation is a Delaunay triangulation).

In [58], the following generalization of Rippa's theorem is proven for weighted Delaunay triangulations:

Theorem 42. *Let $(\mathcal{T}, d)$ be a duality triangulation of a surface with assigned local lengths d. Let $\mathcal{T}_0$ be the vertices of the triangulation and let $f : \mathcal{T}_0 \to \mathbb{R}$ be a function. Suppose $(\mathcal{T}', d')$ is another duality triangulation which is gotten from $(\mathcal{T}, d)$ by a $2 \to 2$ bistellar flip on edge e such that the hinge is locally weighted Delaunay after the flip. Then*

$$E_{\mathcal{T}'}(f) \leq E_{\mathcal{T}}(f),$$

where $E_{\mathcal{T}}$ and $E_{\mathcal{T}'}$ are the Dirichlet energies corresponding to $(\mathcal{T}, d)$ and $(\mathcal{T}', d')$.

In order to get the global statement, one needs to know that a weighted Delaunay triangulation can be found using flips. In the equal weight case, this is proved in [112] and [78]. This is not true in general (see [45]). However, we will investigate some conditions when a flip algorithm does work in Section 6.3.

As a corollary of Rippa's theorem, we get an entropy quantity that increases under the action of flipping to make a hinge weighted Delaunay.

Corollary 43. *Consider the entropy defined by*

$$\Lambda = \inf \left\{ E(f) : \sum_{i \in \mathcal{T}_0} f_i^2 = 1 \text{ and } \sum_{i \in \mathcal{T}_0} f_i = 0 \right\}.$$

Then Λ decreases when an edge is flipped to make the hinge weighted Delaunay.

Proof. Let Λ' denote the entropy after the flip and let f_0 be the f which realize Λ (since f is in a compact set, there must be an actual f which minimizes $E(f)$). Then

$$\Lambda' = \inf_f E_{\mathcal{T}'}(f) \leq E_{\mathcal{T}'}(f_0) \leq E_{\mathcal{T}}(f_0) = \Lambda.$$

□

We see that Λ is essentially the smallest nonzero eigenvalue of $-\triangle$. There are implications toward the eigenvalues of the Laplacian, e.g., [27].

In n dimensions, the weighted Delaunay condition corresponds to $\left|\bigstar\sigma^{n-1}\right| > 0$ while good Dirichlet energy corresponds to $\left|\bigstar\sigma^{1}\right| > 0$. Hence the correspondence between weighted Delaunay triangulations and the Dirichlet energy only occurs in dimension 2 because $1 = 2 - 1$, which is why the theorem is only described for dimension 2. Although we do not pursue it here, this may indicate that the Laplacian should instead be defined on functions on vertices of the dual complex, $f : \bigstar\mathcal{T}_n \to \mathbb{R}$, in which case the Laplacian would be

$$(\triangle f)_{\bigstar\sigma_0^n} = \frac{1}{|\sigma_0^n|} \sum_{\sigma^n \in \mathcal{T}_n} \frac{|\sigma^n \cap \sigma_0^n|}{|\bigstar(\sigma^n \cap \sigma_0^n)|} \left(f_{\bigstar\sigma^n} - f_{\bigstar\sigma_0^n}\right)$$

where the sum is over all n-simplices. In this case, positivity of the coefficients corresponds to being weighted Delaunay.

6.3. Flip algorithms

The most naive flip algorithm is to take a given weighted triangulation, look for a flippable edge which is not weighted Delaunay, and flip it. Continue until the triangulation is weighted Delaunay. This algorithm was first suggested by Lawson and shown to find Delaunay triangulations for points in $\mathbb{R}^2$ ([85], see also exposition in [44] and related result in [84]). It was later shown to work for any 2D piecewise Euclidean triangulation (where the weights are all equal) independently in [78] and [112]. This turns out not to work to find higher dimensional Delaunay triangulations or to find weighted Delaunay triangulations (if there are unequal weights) even in dimension 2. It was later found that points in $\mathbb{R}^n$ can be triangulated with weighted Delaunay triangulations (for any dimension) by incrementally adding one vertex at a time and doing all the flips before adding additional vertices. In this case one must pay close attention to the order of the flipping and the algorithm must either sort the hinges or dynamically decide which hinge to flip next [79] [45]. Unfortunately, it is not yet clear how to extend these algorithms to piecewise Euclidean manifolds, since their proofs rely on the fact that the triangulations are in $\mathbb{R}^n$. In this section we propose a subset of the space of all weighted triangulations for which the naive flip algorithm works, just as in the case of two-dimensional Delaunay triangulations.

Consider the following set.

Definition 44. *A 2-dimensional duality triangulation is said to be* edge positive *if $d_{ij} > 0$ for every directed edge (i, j) of the triangulation and for any possible flip, i.e. any solution of (20).*

Hence a triangulation is edge positive if the centers of each edge are inside the edge and if the center of the new edge after any flip is also inside that edge. This implies that any non-weighted Delaunay edge is flippable:

Lemma 45. *Given a 2D edge positive duality triangulation, if an edge is not weighted Delaunay, then it is flippable.*

Proof. We prove the contrapositive. Suppose a hinge consisting of $\{i, j, k\}$ and $\{i, j, \ell\}$ is not flippable, i.e. the quadrilateral is not convex. There can only be one interior angle larger than π, and it must be at vertex i or j. Say it is at i.

Let L_k be the line through vertex i which is perpendicular to $\{i,k\}$ and let L_ℓ be the line through vertex i which is perpendicular to $\{i,\ell\}$. Since $d_{ik} > 0$, the center $C(\{i,j,k\})$ must be on the side of L_k on which $\{i,k\}$ lies; call this open half-space H_k. Similarly, $C(\{i,j,\ell\})$ must lie on the side of L_ℓ on which $\{i,\ell\}$ lies; call this half space H_ℓ. Let H_j be the half-space containing $\{i,j\}$ whose boundary is the line L_j perpendicular to $\{i,j\}$ through i. Then $C(\{i,j,k\})$ must be in $H_k \cap H_j$ and $C(\{i,j,\ell\})$ must be in $H_k \cap H_\ell$. Since L_k, L_ℓ, and L_j intersect at i and since the angle at i is more than π, $H_k \cap H_j$ and $H_\ell \cap H_j$ are disjoint sectors in a half-space. Use Euclidean isometries to make put the hinge such that i is at the origin, $\{i,j\}$ is along the positive x-axis, and k has positive y-value (and hence ℓ must have negative y-value). Any possible segment $\bigstar\{i,j\}$ must be on a vertical line which intersects $\{i,j\}$. It is easy to see that any such line must intersect $H_k \cap H_j$ with a larger y-value than it intersects $H_\ell \cap H_j$, implying that $|\bigstar\{i,j\}| > 0$. □

Theorem 46. *The edge flip algorithm finds a weighted Delaunay triangulation given an edge positive duality triangulation.*

Proof. Since every flip maintains the edge positive property and every edge which is not weighted Delaunay is necessarily flippable, we can always do a flip if the triangulation is not weighted Delaunay. We now only need a monotone quantity which measures the progress of the algorithm to complete the proof in the same way as in [6, 45, 78, 112]. Since we are in two dimensions, we can use the Dirichlet energy for almost any function, since the energy increases if a flip makes the hinge weighted Delaunay (see Theorem 42). Since this function increases every time we perform a flip and there are finitely many possible configurations, the algorithm must terminate. □

Note that the edge flip algorithm to find Delaunay surfaces is a special case, since in that case, $d_{ij} = \ell_{ij}/2 > 0$. In the next section, we suggest the analogue of this proof for higher dimensions. However, the analogue of edge positive is possibly less natural in this setting.

6.4. Higher dimensional flips

First let's consider the analogue of the $2 \to 2$ bistellar move in higher dimensions. Recall that in any dimension, we can embed a hinge in $\mathbb{R}^n$, so the type of relevant flips must take place inside one or two simplices in $\mathbb{R}^n$. The relevant flip is the $2 \to n$ flip in $\mathbb{R}^n$ (see Figure 7 for the 3D version). The flip takes two simplices $\sigma_i^n = \sigma_0^{n-1} \cup \{i\}$ and $\sigma_j^n = \sigma_0^{n-1} \cup \{j\}$ meeting at a common face $\sigma_0^{n-1} = \{k_1, \ldots, k_n\}$ and replaces it with n simplices $\sigma_{k_p}^n = \left\{i, j, k_1, \ldots, \hat{k}_p, \ldots, k_n\right\}$, where $\hat{k}_p$ indicates that k_p is not present. The same argument as above shows that d_{ij} and d_{ji} can be chosen so that the duality conditions (2) hold for each face.

The duality structure gives each hinge a dual hinge. Figure 8 shows the 3D case. The boundary of σ_i^n consists of the faces $\sigma_0^n = \{k_1, \ldots, k_n\}$ and $\sigma_{ik_p}^{n-1} =$

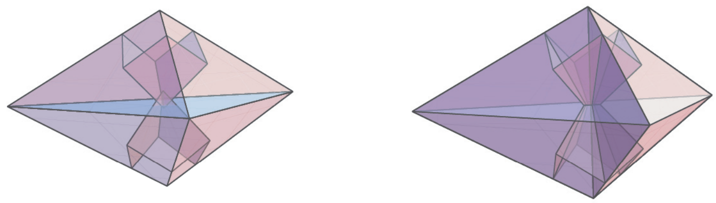

Figure 7: A $2 \to 3$ flip. There are two tetrahedra on the left and three tetrahedra on the right.

$\left\{i, k_1, \ldots, \hat{k}_p, \ldots, k_n\right\}$ for $p = 1, \ldots, n$ while the boundary of σ_j^n is similarly decomposed. Let $L_{\sigma^{n-1}}$ be the line through $C\left(\sigma^{n-1}\right)$ and perpendicular to σ^{n-1} for any $(n-1)$-dimensional simplex. We know that $L_{\sigma_{ik_p}}$ and $L_{\sigma_{ik_q}}$ intersect at the point $C\left(\sigma_i^n\right)$ for every $p, q = 1, \ldots, n$ by Proposition 7. After the $2 \to n$ flip, the boundary of $\sigma_{k_p}^n$ consists of $\sigma_{ik_p}^{n-1}$ and $\sigma_{jk_p}^{n-1}$ together with $\sigma_{k_pk_q}^{n-1} = \left\{i, j, k_1, \ldots, \hat{k}_p, \ldots, \hat{k}_q, \ldots, k_n\right\}$ for $q = 1, \ldots, n$ and $q \neq p$. Hence $L_{\sigma_{ik_p}}$ and $L_{\sigma_{jk_p}}$ intersect at the point $C\left(\sigma_{k_p}^n\right)$ for each $p = 1, \ldots, n$. We find that there is a polytope with vertices $C\left(\sigma_i^n\right)$, $C\left(\sigma_j^n\right)$, and $C\left(\sigma_{k_p}^n\right)$ for $p = 1, \ldots, n$. This is the dual hinge. The centers $C\left(\sigma_i^n\right)$ and $C\left(\sigma_j^n\right)$ are connected via the edge $\bigstar\sigma_0^{n-1}$. If $\left|\bigstar\sigma_0^{n-1}\right| < 0$ then the flip on the hinge does a $n \to 2$ flip on the dual hinge which results in removing $\bigstar\sigma_0^{n-1}$ and replaces it with $\left\{\bigstar\sigma_{k_pk_q}^{n-1}\right\}_{q \neq p}$, which are $\binom{n}{2}$ dual edges, each with positive length.

We see that this sort of flipping is exactly what is needed to make weighted Delaunay triangulations via some sort of flip algorithm. However, the condition of flippability is harder to guarantee. We now examine flippability.

Definition 47. *An n-dimensional triangulation is said to be m-central if $C\left(\sigma^k\right)$ is inside σ^k for all $k \leq m$.*

So edge positive is the same as 1-central. Furthermore, n-central is what is called well-centered in [73]. We now show that $(n-1)$-central assures that any hinge which is not weighted Delaunay is flippable.

Lemma 48. *Given an $(n-1)$-central triangulation of an n-dimensional manifold, if a hinge is not weighted Delaunay, then it is flippable.*

Proof. The proof is essentially the same as the proof of Lemma 46. Consider a hinge consisting of the simplices $\{i, k_1, \ldots, k_n\}$ and $\{j, k_1, \ldots, k_n\}$. The first claim is that if the hinge is unflippable, then at least one dihedral angle must be greater than π. This is clear because if every dihedral angle is less than

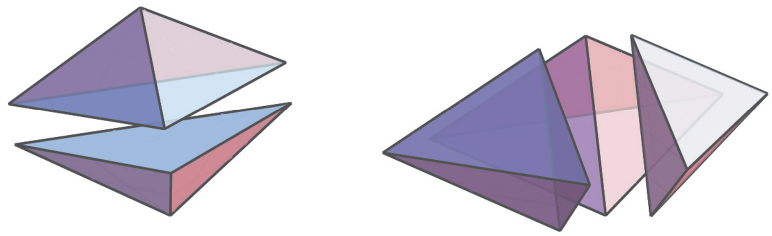

Figure 8: A flip in three dimensions together with dual cells.

or equal to π, then the hinge is the intersection of half-spaces defined by the $(n-1)$-simplices on the boundary and hence convex. Now consider the pairs of hyperplanes whose dihedral angle is greater than π. By relabeling we may assume that the hyperplanes are determined by faces $\sigma_{ik_n}^{n-1} = \{i, k_1, \ldots, k_{n-1}\}$ and $\sigma_{jk_n}^{n-1} = \{j, k_1, \ldots, k_{n-1}\}$ and intersect at $\sigma_0^{n-2} = \{k_1, \ldots, k_{n-1}\}$. Because $C\left(\sigma_{ik_n}^{n-1}\right) \subset \sigma_{ik_n}^{n-1}$, the $C\left(\sigma_i^n\right)$ must be inside the half-space defined by the plane Π_{ik_n}, the plane through σ_0^{n-2} and perpendicular to $\sigma_{ik_n}^{n-1}$, on the side containing $\sigma_{ik_n}^{n-1}$. We have the same for $C\left(\sigma_j^n\right)$ and since the angle is larger than π we must have that $\left|\bigstar\sigma_0^{n-1}\right| > 0$ by a similar argument to that in the proof of Lemma 45. $\square$

Weighted Delaunay triangulations of points in $\mathbb{R}^n$ can be produced via an incremental algorithm (see [45, 79]). The key observation is that if a new point is inserted into a weighted Delaunay triangulation, then there is at least one non-weighted Delaunay hinge which is flippable (or there are no non-weighted Delaunay hinges and it is weighted Delaunay). The generalization to the manifold setting is the following. Let $Star\,(v)$, the star of a vertex v, be defined as all simplices containing v.

Lemma 49. *Suppose Criterion 31 is true. If every hinge in a triangulation is weighted Delaunay except for hinges intersecting $Star\,(v)$ for some vertex v, then some if some hinge is not weighted Delaunay, there exists a flippable hinge that is not weighted Delaunay. Hence the triangulation can be made weighted Delaunay via a flipping algorithm.*

Proof (sketch). The proof in [45] (also with exposition in [44, Section 12]) can be applied to this situation. We are able to prove this lemma in the generality of manifolds because we have supposed Criterion 31 in that generality. $\square$

Using this lemma on subsets of $\mathbb{R}^n$, one is able to construct weighted Delaunay triangulations by: insert one vertex, make the triangulation weighted Delaunay, and then insert the next vertex, make the triangulation weighted Delaunay, etc. Unfortunately, on a manifold, the topology is defined by the initial triangulation, and so there is no obvious way to construct the triangulation incrementally. Also, if one starts with any triangulation, one may not have a weighted Delaunay triangulation which is reachable only by flips, as seen in the example [45, Fig. 5.1].

7. Toward discrete Riemannian manifolds

Much of this work arose out of an attempt to describe Riemannian manifolds using piecewise Euclidean methods. In this final section, we try to describe some of the work already done toward this end. There are two different philosophies. One is to find analogues of the Riemannian setting. The idea is to set up a framework on which variational-type arguments may be made analogously to those in the smooth setting. The other is to actually approximate smooth Riemannian geometry with discrete geometric structures. We shall briefly consider both of these.

7.1. Analogues of Riemannian geometry

In this paper we gave a discrete operator on duality triangulations which, it was argued, is an analogue of the Laplacian on a Riemannian manifold, often called the Riemannian Laplacian or Laplace-Beltrami operator. This gives rise to a discrete heat equation, which is an ordinary differential equation. It is not hard to imagine that similar arguments give rise to Laplace-Beltrami operators, or Hodge Laplacians, on forms with the proper definition of forms. A k-form can be defined to be an element of the dual space to the vector space spanned by the k-dimensional simplices. There are also dual k-forms which are elements of the space spanned by the duals of the $(n-k)$-dimensional simplices. Hirani [73] describes how to use duality information as we have described to define the Hodge star operation, and thereby the Hodge Laplacian on these forms. One may then ask about an analogue of the Hodge theorem. This has also been studied by Hiptmair [72]. Study of the Laplace-Beltrami operator on manifolds is also related to the study of the Laplacian and harmonic analysis on metrized graphs and electrical networks (see [41, 7, 8]).

Another important aspect of Riemannian geometry is the study of geodesics, which we recall are locally length-minimizing paths. In the setting of piecewise Euclidean manifolds, the geodesics are piecewise linear. One may then ask many questions about geodesics, such as the number of closed geodesics (see Pogorelov's work on quasi-geodesics on convex surfaces [105]) and the size of the cut locus to a basepoint, the locus of points with two or more geodesics connecting it to the basepoint (see Miller-Pak [96]). Many results on geodesics on piecewise Euclidean manifolds were found by Stone [119], which lead him to some possible definitions of curvature. The discrete geodesic problem for polytopes in $\mathbb{R}^3$ was studied extensively in [98]. Geodesics as "straightest" paths was considered by Polthier-Schmies in [106].

Much of modern Riemannian geometry is concerned with different notions of curvature, such as sectional, Ricci, and scalar. In the piecewise Euclidean setting, there are a number of definitions of curvatures, although it is still somewhat an open question which ones are the proper ones for classification purposes. Since the literature in this area is vast, we simply indicate some of the principle works. Stone [119] was successful in proving analogues of the Cartan-Hadamard theorem (that negatively curved manifolds have universal cover homeomorphic to $\mathbb{R}^n$) and Myer's theorem (that positively curved manifolds are compact with a bound on the diameter) on piecewise Euclidean manifolds using a quantity which he calls bounds on sectional curvature. Regge introduced a notion of scalar curvature which is described at each $(n-2)$-dimensional simplex as 2π minus the sum of the dihedral angles at that simplex [109]. This has been widely studied as the so-called "Regge calculus" (see, for instance, [50, 65, 66, 4]). There are even some convergence results, which we mention in the next section. Another potential curvature quantity in three dimensions is described by Cooper and Rivin in [36]. They consider the curvature at a vertex to be 4π minus the sum of the solid (or trihedral) angles at the vertex. This curvature is certainly weaker than the curvature introduced by Regge, but may be related to scalar curvature. Generalizations of this scalar curvature have recently been studied by the author [60]. It is possible that the right curvature quantity will lead to a geometric flow which simplifies geometry in a way similar to the way Ricci or Yamabe flow do in the smooth category. This has been studied a bit in [28, 89, 56, 57], and actually was the initial motivation for the definitions of Laplacian described in this paper. Banchoff [9] presented results related to total curvature of embedded polyhedra. Other applications of discrete analogues of Riemannian geometry or geometric operators can be found in [17, 78, 95, 94, 104, 122, 13]. In addition, techniques applying to metric spaces with sectional curvature bounded in the sense of Alexandrov may apply (see [24]).

Discrete forms of Ricci curvature and geometric flows have been explored through the lens of Regge calculus, e.g., [2, 97, 3, 55, 29]. Combinatorial forms of Ricci curvature for graphs has also been considered, e.g., [49, 100, 88, 107, 101].

The geometric Laplacian appears naturally in variation of discrete conformal structures. The first notice of the relationship of angle variation to discrete Laplacians of this form is found in [68], generalizing the observation of Thurston that the variation gives some weighted graph Laplacian [120]. This work was generalized to sphere packings in [56] and more generally to two- and three-dimensional discrete conformal structures in [60]. This work was generalized to piecewise hyperbolic and spherical structures in [62]. Other discrete conformal variations play key roles in arguments in [92, 35, 28, 89].

Laplacian spectrum results have been compared across the discrete and smooth settings, e.g., [34, 33, 32]. Many invariants, such as the Cheeger and Colin de Verdière invariants, were inspired by similar invariants on Riemannian manifolds.

There is also recent work on discrete vector bundles and discrete connections (including Riemannian connections) [30, 16].

7.2. Approximating Riemannian geometry

Another goal is to approximate Riemannian geometry by a discrete geometry such as piecewise Euclidean triangulations. One would hope to be able to find elements of Riemannian geometry such as Laplacian, Levi-Civita connection, sectional curvature, scalar curvature, and so forth and not only have analogous structures, but be able to show that as the triangulation gets finer and finer, the discrete versions converge to the smooth versions. We mention here some of the results which have been successful in this direction.

There has been some work on finite elements on manifolds, notably [75, 5, 76]. A general way of considering mappings of Euclidean simplices to manifolds with bounds on the first and second derivative depending on curvature bounds can be found in [121]. This allows for estimates of solutions to Poisson's equation using piecewise linear finite elements that map to geodesic simplices. Gawlik used mixed finite elements to consider approximation of the Gaussian curvature on surfaces [54].

One of the most influential works on curvature estimation is by Cheeger, Müller, and Schrader, who were able to relate discrete curvatures to Lipschitz-Killing curvatures [26] (see also [80]). The relevant discrete curvature is the sum certain angles and volumes of hinges. In particular, the scalar curvature measure (RdV) is concentrated on $(n-2)$-dimensional hinges in a triangulation, and under a condition that the triangulation does not degenerate, they find that the curvature quantity 2π minus the sum of the dihedral angles multiplied by the volume of the $(n-2)$-dimensional hinge converges to the scalar curvature measure. This version of scalar curvature is also the one suggested by Regge [109] and used extensively in the Regge calculus. The work in [26] proves convergence for each of the Lipschitz-Killing curvatures, and also gives a useful approximate cosine law for general Riemannian surfaces. Barrett and Parker [10] proved a pointwise convergence of piecewise-linear approximations of the Riemannian metric tensor and certain types of tensor fields. Berchenko-Kogan and Gawlik look at finite element approximation of the Levi-Civita connection and curvature on surfaces [14]. In addition, there has been work on the approximation of curvatures of submanifolds, notably by Brehm-Kühnel [22], Fu [51], and Borrelli-Cazals-Morvan [20],

G. Xu experimentally explored pointwise convergence of different discretized Laplace-Beltrami operators to the smooth ones [128, 129]. Some of the discretizations are the same or similar to those considered in this paper, while some are not. For surfaces embedded in $\mathbb{R}^3$, Hildebrandt-Polthier-Wardetzky [71] give results on convergence of discrete Laplacians. On graphs (one-dimensional manifolds and generalizations), it has been shown that the eigenvalues of the discrete Laplacians on metrized graphs converge to the eigenvalues of the smooth Laplacian on a metrized graph [52, 53, 48, 27]. There is also significant recent work on Laplacians on point cloud representation of submanifolds, often by a Laplacian on the k-nearest neighbor (KNN) graph, e.g., [11, 12, 15].

It was W. Thurston's idea to approximate the Riemann mapping between subsets of $\mathbb{C}$ by mappings of circle packings. Such a discretization has been shown to actually converge to the Riemann mapping [114, 69, 118]. Further work on convergence of discrete conformal mappings of various types can be found in [70, 90, 63, 133, 23, 127, 91, 127].

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