OXFORD

International Mathematics Research Notices, 2024, **2024**(12), 9936–9961

https://doi.org/10.1093/imrn/rnae082
Advance access publication date 29 April 2024
Article

A Geometric Approach to Polynomial and Rational Approximation

Christopher J. Bishop[1] and Kirill Lazebnik[2,*]

[1]Stony Brook University, 100 Nicolls Rd, Stony Brook, NY 11794, USA
[2]University of North Texas, Denton, TX, USA
[*]Correspondence to be sent to: e-mail: Kirill.Lazebnik@unt.edu
Communicated by Prof. Zeev Rudnick

We strengthen the classical approximation theorems of Weierstrass, Runge, and Mergelyan by showing the polynomial and rational approximants can be taken to have a simple geometric structure. In particular, when approximating a function f on a compact set K, the critical points of our approximants may be taken to lie in any given domain containing K, and all the critical values in any given neighborhood of the polynomially convex hull of $f(K)$.

1 Introduction

The study of approximating one class of more general functions by functions coming from a second more restricted class is termed *Approximation Theory*. Within complex analysis, the class of general functions is usually the class of functions analytic or continuous on a given set, whereas the restricted class of functions is the class of functions analytic on an extension of the given set. Approximation Theory has found a renewed interest within the field of complex dynamics, where approximation results have been used to prove the existence of various dynamical behaviors for functions in a number of recent works (see, e.g., [2, 8, 11, 12, 19, 20]). This approach, however, has the deficiency that the approximation theorems used do not provide a sufficient description of the approximant needed to describe the global dynamics of the approximant. The purpose of this paper is to give an improved, global description of the approximants in some of the fundamental results in approximation in one complex variable.

One such fundamental result is Runge's classical theorem on polynomial approximation.

Theorem 1.1. [21] Let f be a function analytic on a neighborhood of a compact set $K \subset \mathbb{C}$, and suppose $\mathbb{C} \setminus K$ is connected. For all $\varepsilon > 0$, there exists a polynomial p so that

$$||f - p||_K := \sup_{z \in K} |f(z) - p(z)| < \varepsilon.$$

This famous result does not say much about what the polynomial approximant p looks like off the compact set. For various applications (such as the dynamical applications mentioned above), it would be useful to understand the global behavior of p and, in particular, the location of the critical points and values of p. To this end, we state our first result (Theorem A below) after introducing the following notation.

Received: March 23, 2023. **Revised:** December 22, 2023. **Accepted:** April 2, 2024

Notation 1.2. For any compact set $K \subset \mathbb{C}$ we denote the ε-neighborhood of K by $N_\varepsilon K := \{z : \inf_{w\in K} |z-w| < \varepsilon\}$, and we denote by $\widehat{K}$ the union of K with all bounded components of $\mathbb{C}\setminus K$ (this is usually called the *polynomially convex hull* of K; see [1]). We say K is *full* if $\mathbb{C}\setminus K$ is connected. We let $\mathrm{CP}(f)$ denote the set of critical points of an analytic function f, and let $\mathrm{CV}(f) := f(\mathrm{CP}(f))$ denote its critical values. A *domain* in $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$.

Theorem A. (Polynomial Runge+) *Let $K \subset \mathbb{C}$ be compact and full, $\mathcal{D}$ a domain containing K, and suppose f is a function analytic in a neighborhood of K. Then for all $\varepsilon > 0$, there exists a polynomial p so that $||f-p||_K < \varepsilon$ and:*

1) $\mathrm{CP}(p) \subset \mathcal{D}$,
2) $\mathrm{CV}(p) \subset N_\varepsilon \widehat{f(K)}$.

We remark that no relation is assumed between the domain $\mathcal{D} \supset K$ and the neighborhood of K in which f is analytic. Analogous improvements of the polynomial approximation theorems of Mergelyan and Weierstrass will be stated and proved in Section 10 (see Theorem 10.7 and Corollary 10.8). When K is not full, uniform approximation by polynomials is not always possible, and so we turn to rational approximation. We denote the Hausdorff distance between two sets X, Y, by $\mathrm{d}_H(X,Y)$.

Theorem B. (Rational Runge+) *Let $K \subset \mathbb{C}$ be compact, $\mathcal{D}$ a domain containing K, f a function analytic in a neighborhood of K, and suppose $P \subset \widehat{\mathbb{C}} \setminus K$ contains exactly one point from each component of $\widehat{\mathbb{C}}\setminus K$. Then there exists $P' \subset P$ so that for all $\varepsilon > 0$, there is a rational function r so that $||f-r||_K < \varepsilon$ and:*

1) $\mathrm{d}_H(r^{-1}(\infty), P') < \varepsilon$ *and* $|r^{-1}(\infty)| = |P'|$,
2) $\mathrm{CP}(r) \subset \mathcal{D}$,
3) $\mathrm{CV}(r) \subset N_\varepsilon \widehat{f(K)}$.

We now briefly describe our approach. Let $\Omega \subset \mathbb{C}$ be a finitely connected domain with analytic boundary, and $F : \Omega \to \mathbb{D}$ analytic. By a theorem of Grunsky, F can be approximated on any compact subset of Ω by a *proper* holomorphic map $B : \Omega \to \mathbb{D}$ (recall proper means that the continuous extension of B to $\partial\Omega$ satisfies $B(\partial\Omega) = \mathbb{T} := \{z : |z| = 1\}$). Grunsky's proof (see Lemma 4.5.4 of [16]) uses a Riemann sum to approximate an integral representation of F involving the Green's function on Ω. Another approach can be found in [17]. When Ω is simply connected, B is a Blaschke product (up to a change of coordinates), and in this case the result is due to Carathéodory [9], with a much simpler proof based on power series.

Thus, the function f in Theorem A or B can be approximated on K by a holomorphic map B defined in a union $D := \cup_i D_i \supset K$ of pairwise disjoint domains $(D_i)_{i=1}^k$, so that B is proper in each D_i. Our approach in this manuscript is to extend B from D to a quasiregular mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with specified poles. The Measurable Riemann Mapping Theorem (MRMT for brevity) will then imply that there is a quasiconformal mapping ϕ so that $g \circ \phi^{-1}$ is rational. The bulk of the work in this paper will be to show that there exists a quasiregular extension g of B so that $\{z : g_{\overline{z}}(z) \neq 0\}$ is sufficiently small (in a suitable sense) so as to imply $\phi(z) \approx z$ and hence $g \circ \phi^{-1} = B \circ \phi^{-1} \approx B \approx f$ on K. We remark that our techniques build on the quasiconformal folding methods of the first author [4].

This approach yields not only information on the critical points and values of the approximants as in Theorems A and B, but also a detailed description of the geometric structure of these approximants. We end the introduction by describing this geometric structure in a few cases. First we introduce some more notation.

Notation 1.3. Let $V \subset \widehat{\mathbb{C}}$ be a simply connected domain so that $\infty \notin \partial V$. We let $\psi_V : \mathbf{E} \to V$ denote a Riemann mapping, where $\mathbf{E} = \mathbb{D}$ if V is bounded and $\mathbf{E} = \mathbb{D}_\infty := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ if V is unbounded, in which case we specify $\psi_V(\infty) = \infty$.

First consider the case when K is full and connected, and f is holomorphic in a neighborhood of K. Let Ω, Ω' be analytic Jordan domains containing $K, f(K)$, respectively, so that f is holomorphic in Ω. Then the mapping

$$F := \psi_{\Omega'}^{-1} \circ f \circ \psi_\Omega : \mathbb{D} \to \mathbb{D}$$

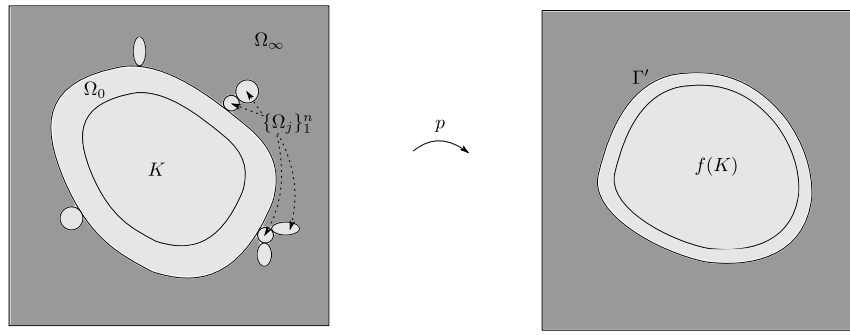

Fig. 1. This figure illustrates the geometry of the approximant p in Theorem A when K is connected. The notation is explained in the text. Both domain and co-domain are colored so that regions with the same color correspond to one another under p.

is holomorphic, and by the aforementioned theorem of Carathéodory, there is a finite Blaschke product $b : \mathbb{D} \to \mathbb{D}$ that approximates F on the compact set $\psi_\Omega^{-1}(K)$. Therefore,

$$B := \psi_{\Omega'} \circ b \circ \psi_\Omega^{-1} : \Omega \to \Omega'$$

is a holomorphic function that approximates f on K, and moreover B restricts to an analytic, finite-to-1 map of $\Gamma := \partial\Omega$ onto $\Gamma' := \partial\Omega'$.

In this paper, we will show that B can be approximated on Ω by a polynomial p so that $p^{-1}(\Gamma')$ is an approximation of Γ. More precisely, $p^{-1}(\Gamma')$ is connected, and consists of a finite union of Jordan curves $\{\gamma_j\}_0^n$ bounding pairwise disjoint Jordan domains $\{\Omega_j\}_0^n$ (see Figure 1): the $\{\Omega_j\}_0^n$ are precisely the connected components of $p^{-1}(\Omega')$. There is one "large" component Ω_0 that approximates Ω in the Hausdorff metric. The other components $\{\Omega_j\}_1^n$ can be made as small as we wish and to lie in any given neighborhood of $\partial\Omega$. Moreover, the collection $\{\Omega_j\}_0^n$ forms a tree structure with any two boundaries $\partial\Omega_j$, $\partial\Omega_k$ either disjoint or intersecting at a single point, and with Ω_0 as the "root" of the tree as in Figure 1. Let Ω_∞ denote the unbounded component of $\mathbb{C} \setminus p^{-1}(\Gamma')$, so that

$$\mathbb{C} \setminus p^{-1}(\Gamma') = \Omega_0 \sqcup \left(\sqcup_{j=1}^n \Omega_j\right) \sqcup \Omega_\infty. \tag{1.1}$$

Recalling Notation 1.3, the polynomial p has the following simple structure with respect to the domains in (1.1).

1) $p(\Omega_0) = \Omega'$ and $\psi_{\Omega'}^{-1} \circ p \circ \psi_{\Omega_0}$ is a finite Blaschke product.
2) $p(\Omega_j) = \Omega'$ and p is conformal on Ω_j for $1 \le j \le n$.
3) $p(\Omega_\infty) = \mathbb{C} \setminus \overline{\Omega'}$ and $p = \psi_{\mathbb{C}\setminus\overline{\Omega'}} \circ (z \mapsto z^m) \circ \psi_{\Omega_\infty}^{-1}$ on Ω_∞ for $m = \deg(p|_{\Omega_0}) + n$.

In other words, up to conformal changes of coordinates, p is simply a Blaschke product in Ω_0, a conformal map in each Ω_j, $1 \le j \le n$, and a power map $z \mapsto z^m$ in Ω_∞. The only finite critical points of p are either in Ω_0, or at a point where two of the curves $(\gamma_j)_{j=1}^n$ intersect, in which case the corresponding critical value lies on $\partial\Omega'$.

Next suppose K is connected, but $\mathbb{C} \setminus K$ has more than one component. In this case, in order to prove Theorem B, we will need to let Ω be a multiply connected analytic domain containing K, and Ω' an analytic Jordan domain containing $f(K)$. By Grunsky's Theorem, there exists a proper map b approximating $\psi_{\Omega'}^{-1} \circ f$ on K, so that $B := \psi_{\Omega'} \circ b$ is a holomorphic map approximating f on K, and B restricts to an analytic, finite-to-1 mapping of each component of $\Gamma := \partial\Omega$ onto Γ'. We will show B can be approximated on Ω by a rational map r so that each component of $\partial\Omega$ can be approximated by a component of $r^{-1}(\Gamma')$. These components of $r^{-1}(\Gamma')$ bound Jordan domains that form a decomposition of the plane as in the previously described polynomial setting, and in the interior of each such domain again r behaves either as a proper mapping, a conformal mapping, or a power mapping (up to conformal changes of coordinates).

Lastly, the case when K has more than one connected component is more intricate, and we will leave the precise description to later in the paper. (Briefly, quasiconformal folding is applied not just along

the boundary of a neighborhood of K, but also along specially chosen curves that connect different connected components of this neighborhood.)

We conclude the introduction by mentioning several related works. The location of $\mathrm{CP}(p)$ in relation to the zeros of a polynomial p is studied in the recent works [22, 23], and in [13] the problem of approximation in $\mathbb{C}^n$ with prescribed critical points was studied.

2 Approximation by Proper Mappings

Definition 2.1. We call a domain $D \subset \mathbb{C}$ an *analytic domain* if D is finitely connected, and each component of ∂D is an analytic Jordan curve.

We remark that a boundary component of an analytic domain D cannot be a single point.

Definition 2.2. Let D be an analytic domain. We will call a continuous mapping $f : D \to \mathbb{C}$ *proper* if $f(D) \subset \mathbb{D}$ and if for every compact $K \subset \mathbb{D}$, $f^{-1}(K)$ is a compact subset of D.

We remark that the definition of proper does not usually include the assumption of the function being $\mathbb{D}$-valued; however, this will be a useful convention to follow in this paper.

Remark 2.3. Since D in Definition 2.2 is assumed to be an analytic domain, the map $f : D \to \mathbb{D}$ extends continuously to a map $f : \overline{D} \to \overline{\mathbb{D}}$, and it is straightforward to check that the map $f : D \to \mathbb{D}$ is proper if and only if $f(\partial D) \subset \mathbb{T}$.

The following is Lemma 4.5.4 of [16]:

Theorem 2.4. Let $\varepsilon > 0$, D an analytic domain, $K \subset D$ compact, and $f : D \to \mathbb{D}$ holomorphic. Then there exists a proper map $B : D \to \mathbb{D}$ so that $||f - B||_K < \varepsilon$.

Remark 2.5. When $D = \mathbb{D}$, the function B in the conclusion of Theorem 2.4 is a Blaschke product (hence the notation B), and in this case Theorem 2.4 is due to Carathéodory (see, e.g., in Theorem I.2.1 of [14] or Theorem 5.1 of the survey [15]).

Notation 2.6. We denote $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. For a proper map B on an analytic domain D, we let $\mathcal{I}_B$ denote the connected components of $\partial D \setminus \{z : B(z) \in \mathbb{R}\}$. In other words, $\mathcal{I}_B$ are the preimages (under B) of the open upper and lower half-circles $\mathbb{T} \cap \mathbb{H}$, $\mathbb{T} \cap (-\mathbb{H})$. We will frequently be dealing with sequences of proper maps $(B_n)_{n=1}^{\infty}$ on D, in which case we abbreviate $\mathcal{I}_{B_n}$ by $\mathcal{I}_n$.

In order to prove Theorems A and B, we will need to approximate a given function by a sequence of proper approximants $(B_n)_{n=1}^{\infty}$ of increasing degree, so that $|B_n'|$ is uniformly comparable to n on the boundary (see (2.5) below). This will be done by post-composing the approximant coming from Theorem 2.4 with the following Blaschke products:

Definition 2.7. For $n \in \mathbb{N}$, $0 < \delta < 1$, and $0 < c < 1$, we define the Blaschke product:

$$\mathcal{B}_{n,\delta,c}(z) := z \cdot \prod_{j=0}^{n-1} \frac{e^{2\pi ij/n} z + r}{1 + r e^{2\pi ij/n} z}, \text{ where } r := 1 - \frac{\delta(1-c)}{n}. \tag{2.1}$$

Proposition 2.8. There exists a $\delta_0 > 0$ so that if $\delta < \delta_0$ and $0 < c < 1$, then:

$$\sup_{n \in \mathbb{N}} \sup_{|z| < c} \left| \mathcal{B}_{n,\delta,c}(z) - z \right| < 4\delta. \tag{2.2}$$

Proposition 2.9. There exists a constant C depending on δ and c, but not on n, so that

$$n/C < |\mathcal{B}_{n,\delta,c}'(z)| < nC \text{ for all } z \in \mathbb{T} \text{ and } n \in \mathbb{N}. \tag{2.3}$$

The proofs of Propositions 2.8 and 2.9 are straightforward but tedious calculations, and so we delay them until the end of the section. For now, we show how Theorem 2.4 together with Propositions 2.8 and 2.9 can be used to deduce the following result, Theorem 2.10. We remark that Theorem 2.10 is the only result from Section 2, which will be needed in the remainder of the paper.

Theorem 2.10. Let $\varepsilon > 0$, D an analytic domain, $K \subset D$ compact, and $f : D \to \mathbb{D}$ holomorphic. Then there exists $M < \infty$ and a sequence of proper maps $(B_n)_{n=1}^{\infty}$ on D satisfying:

$$\sup_{n\in\mathbb{N}} ||f - B_n||_K < \varepsilon, \text{ and} \tag{2.4}$$

$$n/M \le |B_n'(z)| \le nM \text{ for all } z \in \partial D \text{ and } n \in \mathbb{N}. \tag{2.5}$$

Proof. Fix ε, D, K, f as in the statement of the theorem. By Theorem 2.4, there exists a proper map $B : D \to \mathbb{D}$ satisfying $||f - B||_K < \varepsilon/2$. Fix c so that $\sup_{z\in K} |B(z)| < c < 1$, and fix $\delta < \varepsilon/8$ sufficiently small so that (2.2) holds. This defines the sequence $(\mathcal{B}_{n,\delta,c})_{n=1}^{\infty}$. We set

$$B_n := \mathcal{B}_{n,\delta,c} \circ B.$$

The relation (2.4) follows from (2.2) and the triangle inequality, and the relation (2.5) follows from (2.3) and the chain rule. ∎

We conclude this section with the proofs of Propositions 2.8 and 2.9.

Proof of Proposition 2.8. Let

$$\mathcal{B}(z) := \prod_{j=0}^{n-1} \frac{-\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } a_j := -r \exp(-2\pi ij/n), \tag{2.6}$$

so that $\mathcal{B}_{n,\delta,c}(z) = z \cdot \mathcal{B}(z)$. Recalling the definition of r from (2.1), we note that

$$\sup_{|z|\le c} \left| \frac{z + r}{1 + rz} - 1 \right| = \left| \frac{-c + r}{1 - cr} - 1 \right| = \frac{\delta(1 - c)(1 + c)/n}{1 - c\,(1 - \delta(1 - c)/n)} \le \frac{\delta(1 + c)}{n} \le \frac{2\delta}{n}.$$

Symmetry then gives us that

$$\sup_{|z|\le c} \left| \frac{-\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z} - 1 \right| \le \frac{2\delta}{n} \text{ for all } 0 \le j \le n.$$

Hence,

$$\left| \sum_{j=0}^{n-1} \log\left(\frac{-\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z} \right) \right| \le \sum_{j=0}^{n-1} \left| \log\left(\frac{-\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z} \right) \right| \le 3\delta$$

for sufficiently small δ and all $0 < c < 1$. Thus, we conclude that

$$|\mathcal{B}(z) - 1| = \left| \exp\left(\sum_{j=0}^{n-1} \log\left(\frac{-\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z} \right) \right) - 1 \right| \le 4\delta$$

for sufficiently small δ and all $0 < c < 1$, and this proves (2.2) since $\mathcal{B}_{n,\delta,c}(z) = z \cdot \mathcal{B}(z)$. ∎

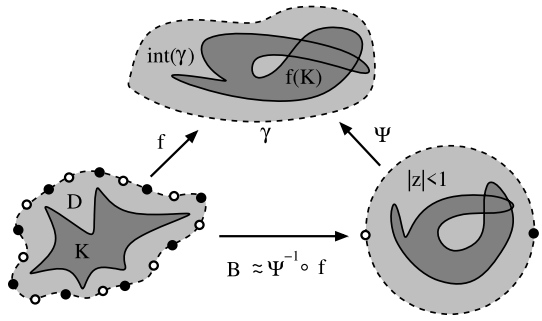

Fig. 2. This figure illustrates Notation 2.6, Definition 3.1, and Theorem 2.10. The vertices pictured on ∂D are $B^{-1}(\pm 1)$, and the components $\mathcal{I}_B$ of Notation 2.6 are the edges along ∂D connecting these vertices.

Proof of Proposition 2.9. We will use the notation $\lesssim, \simeq, \gtrsim$ to mean $\leq, =, \geq$ (respectively) up to a constant depending on δ and c, but not n. Let $\mathcal{B}(z)$ be as in (2.6). A calculation gives

$$\frac{\mathcal{B}'(z)}{\mathcal{B}(z)} = \sum_{j=0}^{n-1} \frac{1-|a_j|^2}{(1-\overline{a_j}z)^2} \cdot \left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{-1}$$

■

Thus, we have

$$|\mathcal{B}'(z)| \leq \sum_{j=0}^{n-1} P(z,a_j), \text{ where } P(z,\zeta) := \frac{1-|\zeta|^2}{|z-\zeta|^2} \text{ for } z \in \mathbb{T}, \zeta \in \mathbb{D} \tag{2.7}$$

is the Poisson kernel for the unit disc. The sum $\sum_{j=0}^{n-1} P(z,a_j)$ takes its maximum at $z=-1$ (as well as at any of the other n points $(\mathbf{exp}(\pi i + 2\pi ij/n))_{j=0}^{n-1}$), where we have

$$P(-1,r) \simeq n \text{ and } P(-1,a_j) \simeq n/j^2 \text{ for } j = 1, \ldots, \lfloor (n-1)/2 \rfloor. \tag{2.8}$$

Combining (2.7) and (2.8) together with the fact that $\sum_{j=1}^{\infty} 1/j^2 < \infty$, we have $|\mathcal{B}'(z)| \lesssim n$. Let us now prove $|\mathcal{B}'(z)| \gtrsim n$. The function $|\mathcal{B}'(z)|$ takes its minimum at $z_n := \mathbf{exp}(\pi i(1+1/n))$, where the triangle inequality yields

$$|\mathcal{B}'(z_n)| \geq \left| \sum_{j=0,1} \frac{1-|a_j|^2}{(1-\overline{a_j}z_n)^2} \cdot \left(\frac{z_n-a_j}{1-\overline{a_j}z_n}\right)^{-1} \right| - \sum_{j=2}^{n-1} P(z_n,a_j) \gtrsim n. \tag{2.9}$$

Since $\mathcal{B}_{n,\delta,c}(z) = z \cdot \mathcal{B}(z)$ and we have shown $|\mathcal{B}'(z)| \simeq n$ for all $z \in \mathbb{T}$. The relation (2.3) follows.

3 Applying Theorem 2.10

In this brief section, we apply Theorem 2.10 to the setting needed for the proofs of Theorem A and B. Given a Jordan curve $\gamma \subset \mathbb{C}$, we denote the bounded component of $\widehat{\mathbb{C}} \setminus \gamma$ by $\mathrm{int}(\gamma)$.

Notation 3.1. We refer to Figure 2 for a summary of the following. For the remainder of this section, we will fix a compact set K, an analytic domain D containing K, and a function f holomorphic in a neighborhood of $\overline{D}$ satisfying $||f||_D < 1$. Fix $\varepsilon > 0$. We assume that

$$d(z,K) < \varepsilon/2 \text{ and } d(f(z),f(K)) < \varepsilon/2 \text{ for every } z \in \partial D. \tag{3.1}$$

We remark that if f is a holomorphic function in a neighborhood of a compact set K, there does not necessarily exist an analytic domain D satisfying the assumptions of Notation 3.1 (in particular when K is disconnected); however it will be useful in the framework of our proof to work under the assumptions of Notation 3.1 for now.

Definition 3.2. We let γ be an analytic Jordan curve surrounding $f(D)$ such that

$$\operatorname{dist}(w, f(K)) < \varepsilon \text{ for every } w \in \gamma, \tag{3.2}$$

and let $\Psi : \mathbb{D} \to \operatorname{int}(\gamma)$ denote a Riemann mapping.

Recall Definition 2.2 of a proper mapping on a domain D.

Proposition 3.3. There exists $M < \infty$ and a sequence of proper mappings $(B_n)_{n=1}^{\infty}$ on D satisfying (2.5) so that

$$||f - \Psi \circ B_n||_K < \varepsilon \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Proof. Let $K' \subset D$ be compact so that $K \subset \operatorname{Int}(K')$ (the interior of K'). Fix $r < 1$ and $\delta > 0$ so that

$$\Psi^{-1} \circ f(K') \subset \{z : |z| < r\} \text{ and } \delta < \operatorname{dist}\left(\psi^{-1} \circ f(K), \partial(\psi^{-1} \circ f(K'))\right)$$

Apply Theorem 2.10 to:

$$\min\left(\frac{\varepsilon}{\sup_{|\zeta|\le r} |\Psi'(\zeta)|}, \varepsilon, \delta\right), D, K', \Psi^{-1} \circ f. \tag{3.4}$$

This produces a sequence of proper mappings $(B_n)_{n=1}^{\infty}$ on D satisfying (2.5) for some $M < \infty$. We claim $(B_n)_{n=1}^{\infty}$ also satisfies (3.3). Indeed, for any $z \in K$ we have by our choice of δ that $|B_n(z)| \le r$ and $|\Psi^{-1} \circ f(z)| \le r$. Thus, we deduce

$$|\Psi \circ B_n(z) - f(z)| = |\Psi \circ B_n(z) - \Psi \circ \Psi^{-1} \circ f(z)| \le \sup_{|\zeta|\le r} |\Psi'(\zeta)| \cdot |B_n(z) - \Psi^{-1} \circ f(z)|,$$

and so (3.3) follows from (3.4). ∎

Recall from the introduction that we plan to extend the definition of the approximant $\Psi \circ B_n \approx f$ from D to all of $\mathbb{C}$. To this end, it will be useful to define the following graph structure on ∂D.

Definition 3.4. For any $n \in \mathbb{N}$, we define a set of vertices on ∂D by $\mathcal{V}_n := (B_n|_{\partial D})^{-1}(\mathbb{R})$, where each vertex v is labeled black or white according to whether $B_n(v) > 0$ or $B_n(v) < 0$, respectively. The curves ∂D will be considered as a graph with edges defined by $\mathcal{I}_n$ (recall from Notation 2.6 that $\mathcal{I}_n$ is precisely the collection of components of $\partial D \setminus \mathcal{V}_n$). We will sometimes write D_n in place of D when we wish to emphasize the dependence of the graph ∂D on n.

Definition 3.5. We define a holomorphic mapping g_n in D by the formula

$$g_n(z) := \Psi \circ B_n(z). \tag{3.5}$$

In Sections 4–9 we will quasiregularly extend the definition of g_n to $\mathbb{C}$, and then in Section 10 we apply the MRMT to produce the rational approximant of Theorem B as described in the introduction.

Remark 3.6. Recall that in Notation 3.1, we fixed $\varepsilon > 0$, a compact set K contained in an analytic domain D, and a function f holomorphic in D (we note ε, K, D, f also satisfied extra conditions specified in Notation 3.1). The objects γ, Ψ, B_n, $\mathcal{V}_n$, g_n we then defined in this section were determined by our initial choice of ε, K, D, f. In future sections, it will be useful to think of γ, Ψ, B_n, $\mathcal{V}_n$, g_n as defining functions that take as input some quadruple (ε, K, D, f) (for any ε, K, D, f as in Notation 3.1), and output whatever object we defined in this section. For instance, $\mathcal{V}_n$ defines

a function that takes as input any (ε, K, D, f) as in Notation 3.1 and outputs (via Definition 3.4) a set of vertices $\mathcal{V}_n(\varepsilon, K, D, f)$ on ∂D. Similarly, B_n takes as input any (ε, K, D, f) as in Notation 3.1 and outputs (via Proposition 3.3) a proper mapping $B_n(\varepsilon, K, D, f)$ on D. Likewise for γ, Ψ, g_n.

4 Quasiconformal Folding

Given a compact set $K \subset \mathbb{C}$ and a function f holomorphic in a domain D containing K, we showed in Section 3 how to approximate f by a holomorphic function g_n defined in D (see Definition 3.5). If f is a function holomorphic in an arbitrary analytic neighborhood U (where U need not be connected) of a compact set K, then one can apply the results of Section 3 to each component of U, which intersects K (this is done precisely in Definition 5.1): this yields a holomorphic approximant of f defined in a finite union of domains. In Sections 4–8, we will build the apparatus necessary to extend this holomorphic approximant to a quasiregular function of $\mathbb{C}$, which is holomorphic outside a small set.

It was convenient to assume in Notation 3.1 that the compact set K was covered by a single domain D; however, we now begin to work more generally:

Remark 4.1. We refer to Figure 5 for a summary of the following. Throughout Sections 4–8, we will fix $\varepsilon > 0$, a compact set $K \subset \mathbb{C}$, a domain $\mathcal{D}$ containing K, a disjoint collection of analytic domains $(D_i)_{i=1}^k$ such that $K \subset U := \cup_i D_i \subset \mathcal{D}$, and a function f holomorphic in a neighborhood of $\overline{U}$ satisfying $||f||_{\overline{U}} < 1$. We assume that the following analog of Equation (3.1) holds:

$$d(z, K \cap D_i) < \varepsilon/2 \text{ and } d(f(z), f(K \cap D_i)) < \varepsilon/2 \text{ for all } z \in \partial D_i \text{ and } 1 \le i \le k.$$

Applying the methods of the previous section to each component D_i of U, we can define a sequence of proper mappings $(B_n)_{n=1}^\infty$ on each D_i (see Remark 3.6). We will let B_n denote the corresponding function defined on U. In particular, $(B_n)_{n=1}^\infty$ gives the following definition of vertices on the boundary of $U = \cup_i D_i$ (see Definition 3.4 and Remark 3.6).

Definition 4.2. For every $n \in \mathbb{N}$, we define a set of vertices $\mathcal{V}_n$ on ∂U by

$$\mathcal{V}_n := \bigcup_{i=1}^k \mathcal{V}_n(\varepsilon, K \cap D_i, D_i, f|_{D_i}) = \bigcup_{i=1}^k (B_n|_{\partial D_i})^{-1}(\mathbb{R}).$$

We now extend the graph structure on ∂U by connecting the different components of U by curves $\{\Gamma_i\}_{i=1}^{k-1}$ in Proposition 4.3 below, and defining vertices along these curves in Definition 4.4. We will need to prove a certain level of regularity for these curves and vertices in order to ensure that the dilatations of quasiconformal adjustments we will make later do not degenerate as $n \to \infty$. We will denote the curves by $\{\Gamma_i\}_{i=1}^{k-1}$, and we remark that the curves depend on n, although we suppress this from the notation.

Proposition 4.3. For each $n \in \mathbb{N}$, there exists a collection of disjoint, closed, analytic Jordan arcs $\{\Gamma_i\}_{i=1}^{k-1}$ in $(\widehat{\mathbb{C}} \setminus U) \cap \mathcal{D}$ satisfying the following properties:

1) Each endpoint of Γ_i is a vertex in $\mathcal{V}_n$;
2) Each Γ_i meets ∂U at right angles;
3) $U \cup (\cup_i \Gamma_i)$ is connected; and
4) For each $1 \le i \le k-1$, the sequence (in n) of curves Γ_i has an analytic limit.

Proof. The set $(\widehat{\mathbb{C}} \setminus U) \cap \mathcal{D}$ must contain at least one simply connected region V with the property that there are distinct i, j with both $\partial V \cap \partial D_i$ and $\partial V \cap \partial D_j$ containing non-trivial arcs (see Figure 3). By (2.5), for all sufficiently large n both $\partial V \cap \partial D_i$, $\partial V \cap \partial D_j$ contain vertices of $\mathcal{V}_n$, which we denote by $v_i \in \partial D_i$, $v_j \in \partial D_j$, respectively. Consider a conformal map $\phi : \mathbb{D} \to V$, and define Γ_1 to be the image under ϕ of the hyperbolic geodesic connecting $\phi^{-1}(v_i)$, $\phi_i^{-1}(v_j)$ in $\mathbb{D}$.

We now proceed recursively, making sure at step l we pick a V, which connects two components of U not already connected by a Γ_1,..., Γ_{l-1}, and so that V is disjoint from Γ_1,..., Γ_{l-1}. The curves Γ_i satisfy

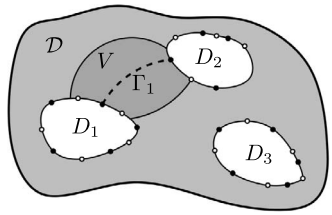

Fig. 3. Illustrated is the Definition of the domain V in the proof of Proposition 4.3

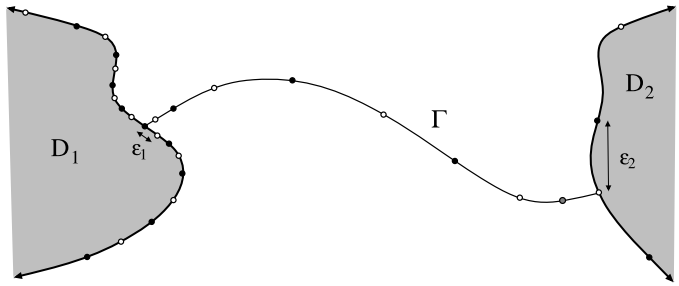

Fig. 4. Illustrated is Definition 4.4.

conclusions (1)-(3) of the proposition. We may ensure that for each $1 \le i \le k-1$, the sequence (in n) of curves Γ_i has an analytic limit by choosing v_i, v_j above to converge as $n \to \infty$. ■

Definition 4.4. Consider the vertices $\mathcal{V}_n \subset \partial U$ of Definition 4.2. We will augment $\mathcal{V}_n$ to include vertices on the curves $(\Gamma_i)_{i=1}^{k-1}$ as follows (see Figure 4). Let $\Gamma \in (\Gamma_i)_{i=1}^{k-1}$ denote both the curve as a subset of $\mathbb{C}$ and the arclength parameterization of the curve, and suppose Γ connects vertices

$$\Gamma(0) = v_i \in \partial D_i, \quad \Gamma(\text{length}(\Gamma)) = v_j \in \partial D_j.$$

For $k = i, j$, let ε_k denote the minimum length of the two edges with endpoint v_k in ∂D_k, and suppose without loss of generality $\varepsilon_j < \varepsilon_i$. Let $l \ge 0$ be so that

$$\varepsilon_j/2 \le \varepsilon_i/2^l \le 2\varepsilon_j.$$

We place vertices at $\Gamma(\varepsilon_i/2), \ldots, \Gamma(\varepsilon_i/2^l)$, and along $\Gamma([\varepsilon_i/2^l, \text{length}(\Gamma)])$ at equidistributed points. We can label the vertices black/white along Γ so that vertices connect only to vertices of the opposite color by adding one extra vertex at the midpoint of the segment having v_j as an endpoint, if need be.

We introduce the following notation.

Notation 4.5. Throughout Sections 4–8, we will let Ω denote a fixed (arbitrary) component of

$$\widehat{\mathbb{C}} \setminus \left(\overline{U} \cup \bigcup_{i=1}^{k-1} \Gamma_i \right), \tag{4.1}$$

and $p \in \Omega$. Note that Ω is simply connected by Proposition 4.3(3). Denote $\mathbb{D}_\infty := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and let σ denote any conformal mapping

$$\sigma : \mathbb{D}_\infty \to \Omega \tag{4.2}$$

satisfying $\sigma(\infty) = p$. For $z \in \Omega$, we define $\tau(z) := \sigma^{-1}(z)$. The map τ induces a partition of $\mathbb{T}$, which we denote by $V_n := \tau(\mathcal{V}_n)$.

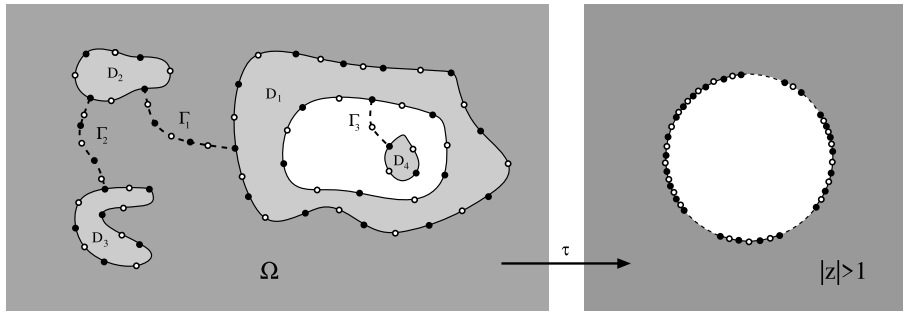

Fig. 5. This figure illustrates Remark 4.1 and Notation 4.5. As pictured, U has four components $(D_i)_{i=1}^4$ which are connected by curves $(\Gamma_i)_{i=1}^3$. Recall $K \subset U$ (the compact set K is not shown in the figure). The unbounded component Ω of (4.1) is pictured in dark grey. The map $\tau : \Omega \to \mathbb{D}_\infty$ is a conformal mapping, and sends the vertices on $\partial\Omega$ to (possibly unevenly spaced) vertices on the unit circle.

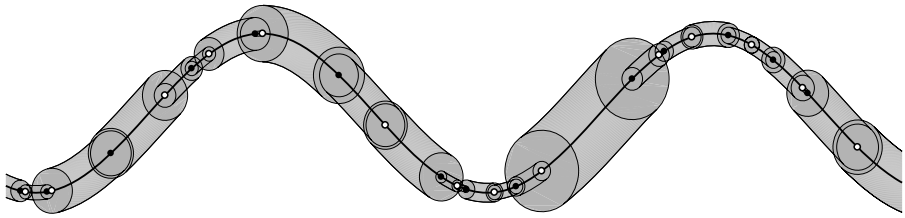

Fig. 6. Shown as a black curve is part of a graph G, and in light gray the neighborhood $\cup_{e\in G}\{z : \mathbf{dist}(z, e) < C \cdot \mathrm{diam}(e)\}$ of G.

Remark 4.6. We will sometimes write Ω_n, $\mathbb{D}^*_{\infty,n}$ in place of Ω, $\mathbb{D}_\infty$, respectively, when we wish to emphasize the dependence of the vertices $\mathcal{V}_n \subset \partial\Omega$, $V_n \subset \partial\mathbb{D}_\infty$ on the parameter n.

Proposition 4.7. For the graph $\partial\Omega_n$, we have $\mathbf{max}\{\mathrm{diam}(e) : e \text{is an edge of} \partial\Omega_n\} \xrightarrow{n\to\infty} 0$.

Proof. This follows from (2.5) and Definition 4.4. ∎

As explained in the introduction, in order to prove uniform approximation in Theorem B, we will need to prove that our quasiregular extension is holomorphic outside a region of small area. To this end it will be useful to introduce the following definition.

Definition 4.8. Suppose $V \subset \mathbb{C}$ is an analytic domain, and ∂V is a graph. Let $C > 0$. We say a quasiregular mapping $\phi : V \to \phi(V)$ is *C-vertex-supported* if

$$\mathrm{supp}(\phi_{\overline{z}}) \subset \bigcup_{e\in\partial V} \{z : \mathbf{dist}(z, e) < C \cdot \mathrm{diam}(e)\} \tag{4.3}$$

(see Figure 6), where the union in (4.3) is taken over all edges e on ∂V.

It will also be useful to have the following definition.

Definition 4.9. Suppose e, f are rectifiable Jordan arcs, and $h : e \to f$ is a homeomorphism. We say that h is *length-multiplying* on e if the push-forward (under h) of arc-length measure on e coincides with the arc-length measure on f multiplied by $\mathrm{length}(f)/\mathrm{length}(e)$.

First we will adjust the conformal map τ so as to be length-multiplying along edges of $\partial\Omega$. Recall the vertices $V_n \subset \mathbb{T}$ defined in Notation 4.5.

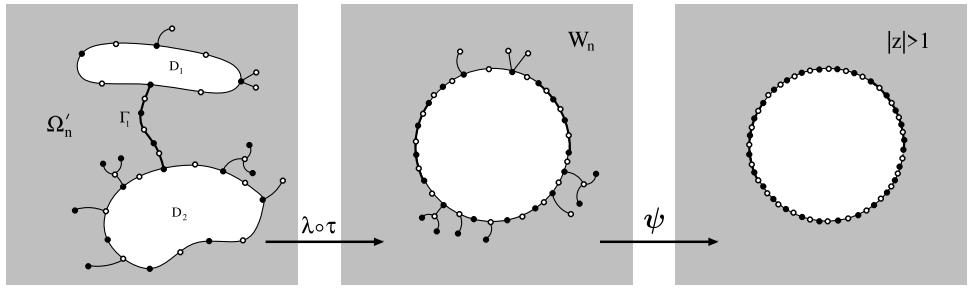

Fig. 7. This figure illustrates the Folding Theorem 4.13 and Notation 4.14. The simply connected domain Ω'_n is obtained by removing from Ω certain trees based at the vertices along $\partial\Omega$.

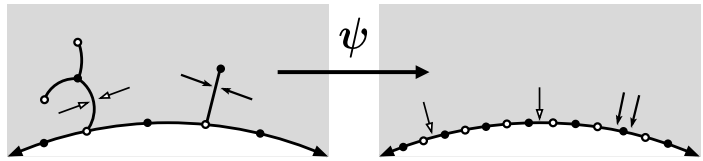

Fig. 8. For any $x \in \partial W_n \cap \mathbb{D}_\infty$, there are two limits $\lim_{W_n \ni z \to x} \psi(z) \in \mathbb{T}$ as illustrated in this figure. Theorem 4.13(3) says that these two limits are equidistant from the nearest black vertex, and are equidistant from the nearest white vertex.

Proposition 4.10. For every n, there is a K-quasiconformal mapping $\lambda : \mathbb{D}^*_{\infty,n} \to \mathbb{D}^*_n$ so that

1) λ is C-vertex-supported for some $C > 0$;
2) $\lambda(z) = z$ on V_n and off of $\mathrm{supp}(\lambda_{\bar{z}})$;
3) $\lambda \circ \tau$ is length-multiplying on every component of $\partial\Omega \setminus \mathcal{V}_n$; and
4) C, K do not depend on n.

Proof. This is a consequence of Theorem 4.3 of [4]. Indeed, let $\mathbb{H}_r := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and recall $\tau := \sigma^{-1}$ and consider the $2\pi i$-periodic covering map

$$\phi := \sigma \circ \exp : \mathbb{H}_r \mapsto \Omega. \tag{4.4}$$

The map ϕ induces a periodic partition $\phi^{-1}(\mathcal{V}_n)$ of $\partial\mathbb{H}_r$ which has *bounded geometry* (see the introduction of [4], or Section 2 of [5]) with constants independent of n by Proposition 4.3(2) and Definition 4.4. Thus, Theorem 4.3 of [4] applies to produce a $2\pi i$-periodic, C vertex-supported, and K-quasiconformal map $\beta : \mathbb{H}_r \to \mathbb{H}_r$ so that $\phi \circ \beta$ is length-multiplying on edges of $\mathbb{H}_r$, and C, K are independent of n. Thus, the inverse

$$\beta^{-1} \circ \log \circ \tau$$

is length-multiplying, and since $\exp$ is length-multiplying on vertical edges, the well-defined map

$$\lambda := \exp \circ \beta^{-1} \circ \log : \mathbb{D}_\infty \to \mathbb{D}_\infty$$

satisfies the conclusions of the Proposition. ∎

The main idea in defining the quasiregular extension in Ω is to send each edge of ∂D_i to the upper or lower half of the unit circle by following $\lambda \circ \tau$ with a power map $z \mapsto z^n$ of appropriate degree. The main difficulty in this approach, however, is that the images of different edges of ∂D_i under $\lambda \circ \tau$ may differ significantly in size, so that there is no single n with $z \mapsto z^n$ achieving the desired behavior. The solution is to modify the domain Ω by removing certain "decorations" from the domain Ω, so that each edge of ∂D_i is sent to an arc of roughly the same size under $\lambda \circ \tau$. This is formalized below in Theorem 4.13 (see also Figures 7 and 8), and is an application of the main technical result of [4] (see Lemma 5.1). The "decorations" are the trees in the following definition.

Definition 4.11. Let $V \subset \mathbb{T}$ be a discrete set. We call a domain $W \subset \mathbb{D}_\infty$ a *tree domain rooted at* V if W consists of the complement in $\mathbb{D}_\infty$ of a collection of disjoint trees, one rooted at each vertex of V (see the center of Figure 7).

Notation 4.12. For $m \in \mathbb{N}$, we let

$$\mathcal{Z}_m^{\pm} := \{z \in \mathbb{T} : z^m = \pm 1\},$$

$$\mathcal{Z}_m := \mathcal{Z}_m^+ \cup \mathcal{Z}_m^-.$$

In other words, $\mathcal{Z}_m^+$ denotes the m^{th} roots of unity, and $\mathcal{Z}_m^-$ the m^{th} roots of -1.

Theorem 4.13. For every n, there exists a tree domain W_n rooted at V_n, an integer $m = m(n)$, and a K-quasiconformal mapping $\psi : W_n \to \mathbb{D}_\infty$ so that

1) ψ is C-vertex-supported for some $C > 0$, and $\psi(z) = z$ off of $\operatorname{supp}(\psi_{\overline{z}})$;
2) on any edge e of $\partial W_n \cap \mathbb{T}$, ψ is length-multiplying and $\psi(e)$ is an edge in $\mathbb{T} \setminus \mathcal{Z}_m$;
3) for any edge e of $\partial W_n \cap \mathbb{D}_\infty$, $\psi(e)$ consists of two edges in $\mathbb{T} \setminus \mathcal{Z}_m$. Moreover, if $x \in e$, the two limits $\lim_{W_n \ni z \to x} \psi(z) \in \mathbb{T}$ are equidistant from $\mathcal{Z}_m^+$, and from $\mathcal{Z}_m^-$; and
4) C, K do not depend on n.

Proof. We consider the 2π-periodic covering map

$$\phi := \sigma \circ \lambda \circ \exp \circ (z \mapsto -iz) : \mathbb{H} \mapsto \Omega. \tag{4.5}$$

inducing a periodic partition $\phi^{-1}(\mathcal{V}_n)$ of $\partial\mathbb{H}$. By (2.5), Definition 4.4, and Proposition 4.10(2), any two edges of $\mathbb{H}$ have comparable lengths with constant independent of n. Therefore, Lemma 5.1 of [4] applies to yield a 2π-periodic K-quasiconformal map Ψ_n of $\mathbb{H}$ onto a subdomain $\Psi_n(\mathbb{H}) \subsetneq \mathbb{H}$, with K independent of n. We let

$$W_n := \exp(-i\Psi_n(\mathbb{H}))$$

and

$$\psi := \exp \circ -i\Psi_n^{-1} \circ i \log : W_n \to \mathbb{D}_\infty. \tag{4.6}$$

The map (4.6) is well defined, and the conclusions of the theorem follow from Lemma 5.1 of [4]. ∎

Notation 4.14. We will use the notation $\Omega_n' := (\lambda \circ \tau)^{-1}(W_n)$ (see Figure 7).

5 Annular Interpolation Between the Identity and a Conformal Mapping

Recall from Notation 4.1 that we have fixed $\varepsilon > 0$, a compact set K, disjoint analytic domains $(D_i)_{i=1}^k$ so that $U := \cup_i D_i$ contains K, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. In this section, we briefly define two useful interpolations in Lemmas 5.3 and 5.5, which we will need.

Since the domain D_i contains the compact set $K \cap D_i$, the definitions and results of Section 3 apply to $(\varepsilon, K \cap D_i, D_i, f|_{D_i})$ for each $1 \le i \le k$ (see Notation 3.1). Thus, Remark 3.6 applies to define (5.1), (5.2), and (5.3) in the following.

Definition 5.1. Let $1 \le i \le k$. Recalling Definition 3.2, we define the Jordan curve

$$\gamma_i := \gamma(\varepsilon, K \cap D_i, D_i, f|_{D_i}). \tag{5.1}$$

Recalling that $\operatorname{int}(\gamma_i)$ denotes the bounded component of $\widehat{\mathbb{C}} \setminus \gamma_i$, we define

$$\Psi_i := \Psi(\varepsilon, K \cap D_i, D_i, f|_{D_i}) \tag{5.2}$$

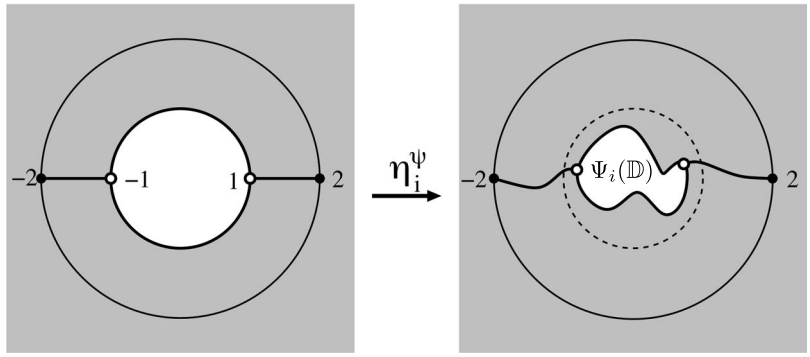

Fig. 9. Illustrated is the map $\eta_i^\Psi : \mathbb{D}_\infty \to \widehat{\mathbb{C}} \setminus \Psi_i(\mathbb{D})$ of Lemma 5.3. The dotted circle on the right depicts the unit circle.

to be a Riemann mapping $\Psi_i : \mathbb{D} \to \text{int}(\gamma_i)$. Lastly, we define the proper mappings

$$B_n := B_n(\varepsilon, K \cap D_i, D_i, f|_{D_i}) \text{ on } D_i, \tag{5.3}$$

where we suppress the dependence of $(B_n)_{n=1}^\infty$ on i from the notation.

Recall that in Section 4, we defined curves $\{\Gamma_i\}_{i=1}^{k-1}$ connecting the domains D_i, and in Notation 4.5 we fixed a component Ω of the complement of $\overline{U} \cup \cup_{i=1}^{k-1}\Gamma_i$.

Notation 5.2. After relabeling the $(D_i)_{i=1}^k$ if necessary, there exists $1 \le \ell \le k$ so that $\partial D_i \cap \partial\Omega \neq \emptyset$ if and only if $i \le \ell$ (see, e.g., Figure 5). For each $1 \le i \le \ell$, note that the intersection $\partial D_i \cap \partial\Omega$ consists of a single Jordan curve, which is mapped onto $\mathbb{T}$ by B_n.

The two interpolations we will need are given in Lemmas 5.3 and 5.5 below. In Lemma 5.3, we define an interpolation η_i^Ψ between $z \mapsto z$ on $|z| = 2$ with $z \mapsto \Psi_i(z)$ on $|z| = 1$ (see Figure 9), and in Lemma 5.5 we modify η_i^Ψ to define a map η_i so that $\eta_i(z) = \eta_i(\overline{z})$ for $|z| = 1$.

Lemma 5.3. For each $1 \le i \le \ell$, there is a quasiconformal mapping $\eta_i^\Psi : \mathbb{D}_\infty \to \widehat{\mathbb{C}} \setminus \Psi_i(\mathbb{D})$ satisfying the relations

$$\eta_i^\Psi(z) = z \text{ for } |z| \ge 2 \text{ and} \tag{5.4}$$

$$\eta_i^\Psi(z) = \Psi_i(z) \text{ for all } |z| = 1. \tag{5.5}$$

Moreover, if D_i, D_j for $1 \le i, j \le \ell$ are connected by one of the curves $(\Gamma_i)_{i=1}^{k-1}$, then

$$\eta_i^\Psi([-2,-1]) \cap \eta_j^\Psi([1,2]) = \eta_i^\Psi([1,2]) \cap \eta_j^\Psi([-2,-1]) = \emptyset. \tag{5.6}$$

Remark 5.4. The existence of a collection of quasiconformal mappings $(\eta_i^\Psi)_{i=1}^l$ satisfying (5.4) and (5.5) follows from a standard lemma on the extension of quasisymmetric maps between boundaries of quasiannuli (see, for instance, Proposition 2.30(b) of [3]). However, in order to ensure that condition (5.6) is also met for the collection $(\eta_i^\Psi)_{i=1}^l$, we will need the following different argument to define the maps $(\eta_i^\Psi)_{i=1}^l$.

Proof. First, observe that after renormalizing each conformal map Ψ_i (more specifically by pre-composing Ψ_i with an appropriate automorphism of $\mathbb{D}$), we may assume that:

$$\text{Re}(\Psi_i(-1)) = \inf\{\text{Re}(\Psi_i(\zeta)) : \zeta \in \mathbb{T}\}, \tag{5.7}$$

$$\text{Re}(\Psi_i(+1)) = \sup\{\text{Re}(\Psi_i(\zeta)) : \zeta \in \mathbb{T}\}, \tag{5.8}$$

for each i. We would like to find, for each i, a pair of smooth Jordan arcs $c_i^\pm$ satisfying

1) c_i^+ connects $+2$ to $\Psi_i(+1)$ (in other words c_i^+ has endpoints $+2, \Psi_i(+1)$), and c_i^- connects -2 to $\Psi_i(-1)$,
2) $c_i^+, c_i^- \subset (2\mathbb{D} \setminus \Psi_i(\mathbb{D}))$,
3) $c_i^+ \cap c_i^- = \emptyset$,

and so that the collection $(c_i^\pm)_{i=1}^l$ satisfies the following property analagous to (5.6):

($\star$) if D_i, D_j for $1 \le i, j \le \ell$ are connected by one of the curves $(\Gamma_i)_{i=1}^{k-1}$, then $c_i^- \cap c_j^+ = c_i^+ \cap c_j^- = \emptyset$.

We claim that in order to prove the Lemma, it suffices to show the existence of a collection $(c_i^\pm)_{i=1}^l$ satisfying the above properties; indeed one defines η_i^Ψ on the quadrilateral with four edges; $\mathbb{T} \cap \mathbb{H}$, $[-2,-1]$, $2\mathbb{T} \cap \mathbb{H}$, $[1,2]$ by setting $\eta_i^\Psi(z) := z$ on $2\mathbb{T} \cap \mathbb{H}$, $\eta_i^\Psi(z) := \Psi_i(z)$ on $\mathbb{T} \cap \mathbb{H}$, and as a smooth homeomorphism from $[-2,-1]$ (resp. $[1,2]$) onto the smooth Jordan curve c_i^- (resp. c_i^+). By a standard result on extension of quasisymmetric homeomorphisms of boundaries of quadrilaterals into their interiors (see, for instance, Lemma 2.24 of [3]), this definition of η_i^Ψ extends quasiconformally to the interior of this quadrilateral. An analagous definition of η_i^Ψ in the reflection of this quadrilateral in the real axis defines a quasiconformal map η_i^Ψ in $\{z : 1 \le |z| \le 2\}$, and the collection $(\eta_i^\Psi)_{i=1}^l$ satisfies (5.4) and (5.5) by definition of η_i^Ψ, and satisfies (5.6) since the collection $(c_i^\pm)_{i=1}^l$ satisfies ($\star$).

Thus it remains to argue the existence of Jordan curves $(c_i^\pm)_{i=1}^l$ satisfying 1.-3. and ($\star$). We proceed recursively, beginning by defining c_1^+ (resp. c_1^-) to be the straight line segment connecting $\psi_1(+1)$ (resp. $\psi_1(-1)$) to $+2$ (resp. -2); the normalizations (5.7) and (5.8) ensure $c_1^\pm$ satisfies 1.-3.. It will be important to note that the following property holds:

($\star\star$) $c_1^\pm$ intersects any vertical line $\{z : \mathrm{Re}(z) = x\}$ in at most one point.

Consider j so that D_1, D_j are connected by one of the curves $(\Gamma_i)_{i=1}^{k-1}$. In defining c_j^+, there are two cases to consider:

1) $\psi_j(+1) > \psi_1(-1)$, or
2) $\psi_j(+1) \le \psi_1(-1)$

In the first case, we define c_j^+ to be the straight line segment connecting $+2$ to $\psi_j(+1)$; then we have $c_1^- \cap c_j^+ = \emptyset$ by (5.7) and (5.8). In the second case we have that $\psi_j(\mathbb{D}) \cap \psi_1(\mathbb{D}) = \emptyset$, and so by ($\star\star$) there exists a Jordan curve $c_j^+ \subset \mathbb{D} \setminus (\psi_j(\mathbb{D}) \cup c_1^-)$ intersecting any vertical line $\{z : \mathrm{Re}(z) = x\}$ at most once, and connecting the point $\psi_j(+1)$ to $+2$. In particular, in either case, we have that $c_1^- \cap c_j^+ = \emptyset$, and c_j^+ satisfies ($\star\star$). An analagous procedure defines c_j^-. This defines the curves $c_j^\pm$ over all j so that the $c_j^\pm$ satisfy ($\star\star$), and ($\star$) holds for $i = 1$. Continuing in this way, one similarly defines $c_k^\pm$ for each D_k connected to one of the D_j considered in the previous step; the proof that $c_k^\pm$ satisfies the desired properties is the same as above (the only property of $c_1^\pm$ we used in defining $c_j^\pm$ was that $c_1^\pm$ satisfies ($\star\star$), which we have also ensured $c_j^\pm$ satisfies). ■

Lemma 5.5. For each $1 \le i \le \ell$, there is a quasiconformal mapping

$$\eta_i : \mathbb{D}_\infty \to \mathbb{C} \setminus \Psi_i([-1,1])$$

satisfying the relations

$$\eta_i(z) = z \text{ for } |z| \ge 2, \tag{5.9}$$

$$\eta_i(z) = \eta_i(\bar{z}) \text{ for } |z| = 1, \text{ and} \tag{5.10}$$

$$\eta_i(z) = \eta_i^\Psi(z) \text{ for } z \in \mathbb{R} \cap \mathbb{D}_\infty. \tag{5.11}$$

Proof. Define

$$\gamma_i^+ := \eta_i^\Psi(\partial(A(1,2) \cap \mathbb{H})).$$

Let η be a quasisymmetric mapping of $\mathbb{T} \cap \mathbb{H}$ onto $[-1,1]$ fixing ± 1 (one can take $\eta := M|_{\mathbb{T} \cap \mathbb{H}}$ where M is a Mobius transformation mapping $-1, 1, i$ to $-1, 1, 0$, respectively). Define a mapping g on γ_i^+ by:

$$g(z) := \begin{cases} \Psi_i \circ \eta \circ \Psi_i^{-1}(z) & z \in \Psi_i(\mathbb{T} \cap \mathbb{H}) \\ z & \text{otherwise} \end{cases} \tag{5.12}$$

Since g is a quasisymmetric mapping, a standard lemma on extension of quasisymmetric maps between boundaries of quasidisks (see, for instance, Proposition 2.30(a) of [3]) implies that g may be extended to a quasiconformal mapping of $\eta_i^\Psi(A(1,2)\cap\mathbb{H})$. Define g similarly in $\eta_i^\Psi(A(1,2)\cap(-\mathbb{H}))$. We let $\eta_i := g\circ\eta_i^\Psi$. It is then straightforward to check that η_i satisfies (5.9)–(5.11). ■

Remark 5.6. Lemmas 5.3 and 5.5 define 2ℓ many quasiconformal mappings: $\{\eta_i^\Psi\}_{i=1}^\ell$ and $\{\eta_i\}_{i=1}^\ell$. The definition of the mappings η_i^Ψ, η_i depend on the objects ε, K, $(D_i)_{i=1}^k$, f as fixed in Notation 4.1, but not on the parameter n in (5.3). Thus, we record the trivial but important observation that the mappings $\{\eta_i^\Psi\}_{i=1}^\ell$ and $\{\eta_i\}_{i=1}^\ell$ are quasiconformal with a constant independent of n.

6 Annular Interpolation Between a Proper Mapping and a Power Map

Recall that we have fixed $\varepsilon > 0$, a compact set K, disjoint analytic domains $(D_i)_{i=1}^k$ so that $U := \cup_i D_i$ contains K, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. The curves $\{\Gamma_i\}_{i=1}^{k-1}$ connect the domains $(D_i)_{i=1}^k$, and Ω is a component of the complement of $\overline{U}\cup\cup_{i=1}^{k-1}\Gamma_i$ with $\tau : \Omega \to \mathbb{D}_\infty$ conformal. Recall that the domain Ω_n' was defined in Theorem 4.13 and Notation 4.14 by removing from Ω a collection of trees rooted at the vertices along $\partial\Omega$, and the map $\psi\circ\lambda\circ\tau$ maps Ω_n' onto $\mathbb{D}_\infty$ (see Proposition 4.10 and Theorem 4.13).

Notation 6.1. Recall from Notation 5.2 that $\partial D_i \cap \partial\Omega \neq \emptyset$ if and only if $1 \le i \le \ell$. Hence exactly $\ell - 1$ of the curves $(\Gamma_i)_{i=1}^{k-1}$ intersect $\partial\Omega$. By relabelling the $(\Gamma_i)_{i=1}^{k-1}$ if necessary, we may assume Γ_j intersects $\partial\Omega$ if and only if $1 \le j \le \ell - 1$.

Let $m = m(n)$ be as in Theorem 4.13. To prove our main results, we will need to modify $z \mapsto z^m$ in $\mathbb{D}_\infty$ so that, roughly speaking, $(z \mapsto z^m)\circ\psi\circ\lambda\circ\tau(z)$ agrees with the proper mappings B_n (see Definition 5.1) along ∂D_i. This is done in Theorem 6.3 below (see [6] for a related result). Its proof uses the following.

Proposition 6.2. Suppose ϕ_1, ϕ_2 are C^1 homeomorphisms of a C^1 Jordan arc e such that

1) $\phi_1(e) = \phi_2(e)$;
2) ϕ_1, ϕ_2 agree on the two endpoints of e; and
3) $|\phi_1'(z)| = |\phi_2'(z)|$ for all $z \in e$.

Then $\phi_1 = \phi_2$ on e.

The proof of Proposition 6.2 is a consequence of the Fundamental Theorem of Calculus and is left to the reader.

Theorem 6.3. For every n, there exists a locally univalent K-quasiregular mapping $h_n : \mathbb{D}_\infty \to \mathbb{D}_\infty$ so that

1) $h_n(z) = z^m$ for $|z| \ge \sqrt[m]{2}$ where $m := m(n)$ is as in Theorem 4.13;
2) $h_n\circ\psi\circ\lambda\circ\tau(z) = B_n(z)$ for every $z \in \partial D_i$ and $1 \le i \le l$; and
3) K is independent of n.

Proof. Fix the standard branch of $\log$. Given an edge $e \in \partial D_i$, we have by Theorem 4.13 that

$$\log\circ\psi\circ\lambda\circ\tau(e) = \{0\}\times\left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right] \text{ for some } 0 \le j \le 2m-1. \tag{6.1}$$

Denote the vertical line segment in (6.1) by v_e. Let $f : v_e \mapsto e$ be a length-multiplying, C^1 homeomorphism so that f^{-1} agrees with $\log\circ\psi\circ\lambda\circ\tau$ on the two endpoints of e. Consider the maps:

$$z \mapsto mz \text{ for } z \in \left\{\frac{\log 2}{m}\right\}\times\left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right], \tag{6.2}$$

$$z \mapsto \log\circ B_n\circ f \text{ for } z \in \{0\}\times\left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right]. \tag{6.3}$$

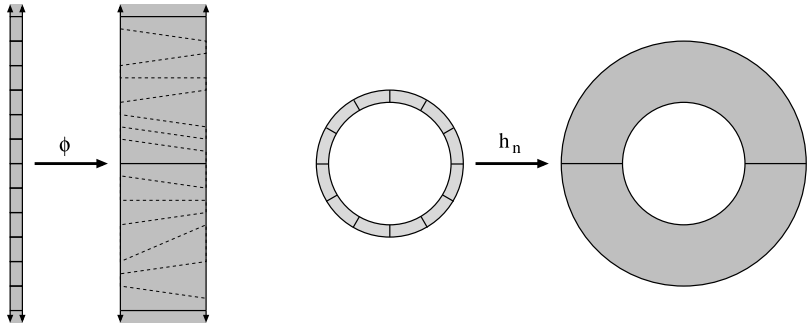

Fig. 10. Illustrated is the proof of Theorem 6.3. In logarithmic coordinates the desired interpolation is denoted ϕ, and h_n is then defined by (6.5) and (6.6).

For each $1 \leq i \leq l$, the proper mappings B_n are orientation-preserving on the unique outer boundary component of D_i, and orientation-reserving on all other boundary components of D_i. This implies that we may choose the branch of $\mathbf{log}$ in (6.3) so that the images of (6.2) and (6.3) are horizontal translates of one another (recall $B_n(e)$ is a circular arc of angle π), and the derivative of (6.3) is strictly positive for all $z \in v_e$. Since the derivative of (6.2) is also strictly positive, this means the linear interpolation between (6.2) and (6.3) is a homeomorphism.

By (2.5), we have that $|B_n'|$ is comparable at all points of e with constant independent of e and n. Thus, since f is length-multiplying and $\mathbf{log}$ is length-multiplying on Euclidean circles centered at 0, we conclude that the derivative of (6.3) is comparable to m at all points of v_e with constant independent of e and n. Thus, we conclude that the linear interpolation between (6.2) and (6.3) in the rectangle

$$\left[0, \frac{\log 2}{m}\right] \times \left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right] \tag{6.4}$$

is K-quasiconformal with K independent of e and n (see, for instance, Theorem A.1 of [18]). Denote the linear interpolation by ϕ (see Figure 10).

We define

$$h_n := \mathbf{exp} \circ \phi \circ \mathbf{log} \operatorname{in}\{z \in \mathbb{D}_\infty : z/|z| \in \psi \circ \lambda \circ \tau(e)\} \cap \{|z| \leq \sqrt[m]{2}\}. \tag{6.5}$$

The equation (6.5) defines $h_n(z)$ for z in $\{z : 1 \leq |z| \leq \sqrt[m]{2}\}$ and sharing a common angle with the image under $\psi \circ \lambda \circ \tau$ of an edge on some ∂D_i. We finish the definition of h_n by simply setting:

$$h_n(z) := z^m \operatorname{in}\{z \in \mathbb{D}_\infty : z/|z| \in \psi \circ \lambda \circ \tau(\partial \Omega_n' \setminus (\cup_i \partial D_i))\}. \tag{6.6}$$

The conclusion (1) now follows by definition of h_n, and (3) follows since h_n is a composition of holomorphic mappings and a K-quasiconformal interpolation where we have already noted that K is independent of n.

We now show that conclusion (2) follows from Proposition 6.2. Fix an edge e on ∂D_i. Recall $v_e := \mathbf{log} \circ \psi \circ \lambda \circ \tau(e)$. Thus, by (6.3) and (6.5) we have that:

$$h_n \circ \psi \circ \lambda \circ \tau = B_n \circ f \circ \mathbf{log} \circ \psi \circ \lambda \circ \tau \operatorname{on} e. \tag{6.7}$$

First note that (6.7) agrees set-wise with B_n on e and at the endpoints of e. The map $\psi \circ \lambda \circ \tau$ is length-multiplying (by Proposition 4.10(3) and Theorem 4.13(2)), $\mathbf{log}$ is length-multiplying on the circular segment $\psi \circ \lambda \circ \tau(e)$, and f is length-multiplying by definition. Thus, the modulus of the derivative of $f \circ \mathbf{log} \circ \psi \circ \lambda \circ \tau$ is constant on e, and so the derivatives of (6.7) and B_n have the same modulus at each point of e. Conclusion (2) now follows from Proposition 6.2. ■

7 Joining Different Types of Boundary Arcs: The Map E_n

Recall that in Section 5 we defined the maps η_i^Ψ, η_i where $1 \leq i \leq \ell$, and in Section 6 we defined the map h_n for all $n \in \mathbb{N}$. In this section we define a map E_n in $\mathbb{D}_\infty$, which is roughly given by either $z \mapsto \eta_i^\Psi \circ h_n(z)$ or $z \mapsto \eta_i(z) \circ h_n(z)$, where i is allowed to depend on $\arg(z)$ and which of η_i^Ψ, η_i we post-compose h_n with

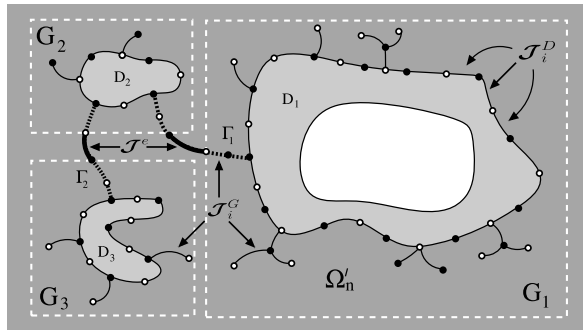

Fig. 11. Illustrated is Definition 7.1. The curves Γ_1, Γ_2 are depicted as black dotted lines, except for the edges $e_1 \subset \Gamma_1$, $e_2 \subset \Gamma_2$, which are in thick black.

is also allowed to depend on arg(z). Thus, we will need a way to interpolate between the definitions of η_i^Ψ, η_i, for different i. The interpolation regions are defined in Definition 7.1 below, and the map E_n in Proposition 7.2. It will be useful to keep Figure 11 in mind for the remainder of this section.

Definition 7.1. Mark one edge e_i on Γ_i for each $1 \le i \le \ell - 1$. Label the ℓ components of $\partial\Omega'_n \setminus \cup_i e_i$ as $(G_i)_{i=1}^{\ell}$, where $\partial D_i \subset G_i$. Let

1) $\mathcal{J}_i^D$ denote those edges in ∂D_i;
2) $\mathcal{J}_i^G$ denote those edges in $G_i \setminus \mathcal{J}_i^D$; and
3) $\mathcal{J}^e$ denote the edges $(e_i)_{i=1}^{\ell-1}$.

In other words, $\mathcal{J}_i^D$ are the edges shared by $\partial\Omega'_n$ and ∂D_i, $\mathcal{J}^e$ consists of $\ell - 1$ edges: one on each of the curves $(\Gamma_i)_{i=1}^{\ell}$, and $\mathcal{J}_i^G$ are the remaining edges on G_i. Thus, we have:

$$\partial\Omega'_n = \mathcal{J}^e \cup \bigcup_i \left(\mathcal{J}_i^D \cup \mathcal{J}_i^G\right).$$

For $z \in \mathbb{D}_\infty$, we define:

$$E_n(z) := \begin{cases} \eta_i^\psi \circ h_n(z) & \text{if } z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^D) \\ \eta_i \circ h_n(z) & \text{if } z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^G) \end{cases} \tag{7.1}$$

It remains to define $E_n(z)$ for $z \in \mathbb{D}_\infty$ satisfying $z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}^e)$. We do so in the following Proposition.

Proposition 7.2. The map E_n extends to a locally univalent K-quasiregular mapping $E_n : \mathbb{D}_\infty \to \mathbb{C}$ satisfying $E_n(z) = z^m$ for $|z| \ge \sqrt[m]{2}$, where $m = m(n)$ is as in Theorem 4.13. Moreover, K does not depend on n.

Proof. Consider (7.1). Note that if E_n is defined at z and $|z| \ge \sqrt[m]{2}$, then $E_n(z) = z^m$ by Theorem 6.3(1) and (5.4), (5.9). Thus, setting $E_n(z) := z^m$ for $|z| \ge \sqrt[m]{2}$ extends the definition of E_n.

It remains to extend the definition of E_n to:

$$\{z : 1 \le |z| \le \sqrt[m]{2} \text{ and } z/|z| \in \psi \circ \lambda \circ \tau(e_i)\}, \text{ for } 1 \le i \le \ell - 1. \tag{7.2}$$

Each of the $\ell - 1$ sets in (7.2) consists of 2 quadrilaterals which we denote by $\mathcal{Q}_i^{\pm}$. The curve Γ_i connects two distinct elements of $(D_i)_{i=1}^{\ell-1}$. In order to avoid complicating notation significantly, we will assume without loss of generality that Γ_i connects D_i to D_{i+1}. Let $\gamma_i \subset 2\mathbb{D}$ be a smooth Jordan arc connecting $\eta_i^\Psi(1)$ to $\eta_{i+1}^\Psi(-1)$ (see Figure 12). Moreover, by (5.6), we can choose γ_i so that the union of the arcs

$$\eta_i^\Psi([1,2]), 2\mathbb{T} \cap \mathbb{H}, \eta_{i+1}^\Psi([-2,-1]), \gamma_i \tag{7.3}$$

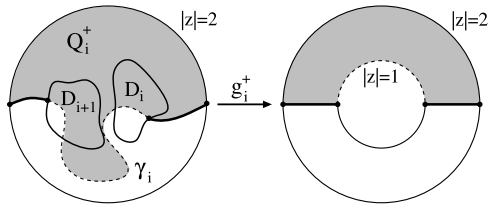

Fig. 12. Illustrated is the quadrilateral Q_i^+ and the map g_i^+ in the proof of Proposition 7.2.

forms a topological quadrilateral we denote by Q_i^+ (in particular none of the arcs in (7.3) intersect except at common endpoints).

Define a quasisymmetric homeomorphism $g_i^+ : \partial Q_i^+ \to \partial(A(1,2) \cap \mathbb{H})$ (see Figure 12) by

$$g_i^+(z) = z \text{ for } z \in 2\mathbb{T}$$

$$g_i^+(z) = (\eta_i^\Psi)^{-1}(z) \text{ for } z \in \eta_i^\Psi([1,2])$$

$$g_i^+(z) = (\eta_{i+1}^\Psi)^{-1}(z) \text{ for } z \in \eta_{i+1}^\Psi([-2,-1]),$$

and extending g_i^+ to a quasisymmetric homeomorphism of γ_i to $\mathbb{T} \cap \mathbb{H}$. The mapping g_i^+ extends to a quasiconformal homeomorphism $g_i^+ : Q_i^+ \to A(1,2) \cap \mathbb{H}$ (see Lemma 2.24 of [3]). We define $E_n(z) := (g_i^+)^{-1}(z^m)$ for $z \in \mathcal{Q}_i^+$. A similar definition of $g_i^- : Q_i^- \to A(1,2) \cap -\mathbb{H}$ is given (using the same curve γ_i) so that

$$g_i^+(z) = g_i^-(\overline{z}) \text{ for } z \in \mathbb{T} \cap \mathbb{H}.$$

We let $E_n(z) := (g_i^-)^{-1}(z^m)$ for $z \in \mathcal{Q}_i^-$.

To summarize, we have defined E_n in each of the three regions

$$\{z \in \mathbb{D}_\infty : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^D)\}, \tag{7.4}$$

$$\{z \in \mathbb{D}_\infty : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^G)\}, \tag{7.5}$$

$$\{z \in \mathbb{D}_\infty : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^e)\}. \tag{7.6}$$

Indeed, the definition of E_n in (7.4) and (7.5) was given already in (7.1), and in this proof we have defined E_n in (7.6). The definitions of E_n in each of (7.4), (7.5), (7.6) agree along any common boundary, and thus by removability of analytic arcs for quasiregular mappings, it follows that E_n is quasiregular on $\mathbb{D}_\infty$. Moreover, E_n has no branched points in $\mathbb{D}_\infty$, and hence E_n is locally quasiconformal. The dilatation of the map E_n depends only on the dilatation of h_n (which is independent of n by Theorem 6.3(3)) and the dilatations of the the finite collection of quasiconformal maps used in its definition: η_i^Ψ, η_i, g_i^+, g_i^-, and hence we may take K independent of n. ∎

8 Defining g_n in Ω_n'

First we recall our setup. We have fixed $\varepsilon > 0$, a compact set K, disjoint, analytic domains $(D_i)_{i=1}^k$ so that $K \subset U := \cup_i D_i$, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. We defined curves $\{\Gamma_i\}_{i=1}^{k-1}$ connecting the domains $(D_i)_{i=1}^k$, and we denoted by Ω a component of the complement of $\overline{U} \cup \cup_{i=1}^{k-1} \Gamma_i$ with $\tau : \Omega \to \mathbb{D}_\infty$ conformal. The domain Ω_n' is contained in Ω, and $\psi \circ \lambda \circ \tau$ maps Ω_n' onto $\mathbb{D}_\infty$. In Section 7 we defined the map E_n.

Definition 8.1. We define the mapping $g_n : \Omega_n' \to \widehat{\mathbb{C}}$ by

$$g_n := E_n \circ \psi \circ \lambda \circ \tau. \tag{8.1}$$

We will now record at which points the function $g_n|_{\Omega_n'}$ is locally $n:1$ for $n>1$.

Definition 8.2. Let g be a quasiregular function, defined in a neighborhood of a point $z\in\mathbb{C}$. We say that z is a *branched point* of g if for any sufficiently small neighborhood U of z, the map $g|_U$ is $n:1$ onto its image for $n>1$. We say $w\in\mathbb{C}$ is a *branched value* of g if $w=g(z)$ for a branched point z of g. We denote the branched points of a quasiregular mapping g by $\mathrm{BP}(g)$, and the branched values by $\mathrm{BV}(g)$.

Remark 8.3. Recall that in Notation 4.5 we fixed a point $p\in\Omega$ satisfying $\tau(p)=\infty$.

Proposition 8.4. The mapping $g_n:\Omega_n'\to\mathbb{C}$ of Definition 8.1 is K-quasiregular and C-vertex supported for K, C independent of n. Moreover, $g_n^{-1}(\infty)=\{p\}$,

$$\mathrm{BP}(g_n)\subset\bigcup_{e\in\partial\Omega_n}\{z:\mathbf{dist}(z,e)<C\cdot diam(e)\},\ and \tag{8.2}$$

$$\mathrm{BV}(g_n)\subset\bigcup_{i=1}^{k}\Psi_i(\mathbb{T}). \tag{8.3}$$

Proof. Since each of the mappings in the composition (8.1) are K-quasiregular and C-vertex supported for K, C independent of n, the same is true of g_n. The only points where the mapping g_n is locally $l:1$ for $l>1$ are a subset of the vertices of the graph $\partial\Omega_n'$. By Theorem 4.13, the vertices of $\partial\Omega_n'$ all lie in

$$\bigcup_{e\in\partial\Omega_n}\{z:\mathbf{dist}(z,e)<C\cdot\mathrm{diam}(e)\}.$$

Thus, (8.2) is proven. Moreover, any vertex of $\partial\Omega_n'$ is mapped to a point on one of the curves $\Psi_i(\mathbb{T})$ by g_n. Hence, (8.3) follows since $\mathrm{BV}(g_n)=g_n(\mathrm{BP}(g_n))$. It remains to show:

$$g_n^{-1}(\infty)=\{p\}. \tag{8.4}$$

Indeed, note that $E_n\circ\psi\circ\lambda$ fixes ∞ and has no finite poles. The map $\tau:\Omega\to\mathbb{D}_\infty$ is conformal and hence only one point p is mapped to ∞. The relation (8.4) now follows. ∎

It will be useful to record the following result.

Proposition 8.5. Let $r>1$. Then for all sufficiently large n, we have:

$$g_n(z)=\tau(z)^m \text{ for any } z\in\tau^{-1}(\{z:|z|>r\}). \tag{8.5}$$

Proof. Consider the functional equation (8.1) defining g_n. The maps λ, ψ are vertex-supported, and moreover λ (respectively, ψ) is the identity outside of the support of $\lambda_{\overline{z}}$, (respectively, $\psi_{\overline{z}}$). By Proposition 4.7, we therefore have that $\psi\circ\lambda(z)=z$ if $z\in\tau^{-1}(\{z:|z|>r\})$ and n is sufficiently large. The relation (8.5) now follows from (8.1) and Theorem 6.3(1) since $m\to\infty$ as $n\to\infty$. ∎

Remark 8.6. As in Remark 3.6, we note that our Definition 8.1 of g_n is determined by a choice of the objects K, U, $\mathcal{D}$, f, ε, Ω, p we fixed in Notations 4.1 and 4.5. When we wish to emphasize this dependence, we will write $g_n(K,U,\mathcal{D},f,\varepsilon,\Omega,p)$. In particular, it will be useful in the next section to think of g_n as a function taking as input any choice of K, U, $\mathcal{D}$, f, ε, Ω, p satisfying the conditions in Notations 4.1, 4.5, and outputting (via Definition 8.1) a quasiregular function $g_n(K,U,\mathcal{D},f,\varepsilon,\Omega,p)$ defined on Ω_n'.

9 Verifying g_n is Quasiregular on $\widehat{\mathbb{C}}$

In this section we combine our efforts in Sections 3-8 to define an approximant $g_n:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ of a given f. The approximant g_n will not be holomorphic as required in Theorems A and B, but we will solve this problem in the next section by applying the MRMT. We fix the following for Sections 9-10.

Notation 9.1. Fix $K, f, \mathcal{D}, \varepsilon, P$ as in the statement of Theorem b. Denote by U the neighborhood of K in which f is holomorphic. Define

$$P' := \{p \in P : p \text{ is contained in a component } V \text{ of } \widehat{\mathbb{C}} \setminus K \text{ such that } V \not\subseteq U\}. \tag{9.1}$$

Compactness of K implies that U contains all but finitely many components of $\widehat{\mathbb{C}} \setminus K$, and so the set P' is finite. Moreover, P' does not depend on ε. By shrinking U if necessary, we may assume that:

1) $U \cap P' = \emptyset$;
2) P' contains exactly one point in each component of $\widehat{\mathbb{C}} \setminus U$;
3) f is holomorphic in a neighborhood of $U \subset \mathcal{D}$; and
4) the components of U are a finite collection of analytic Jordan domains $(D_i)_{i=1}^k$ so that (3.1) holds for each D_i.

Let K' be a compact set such that $K \subset \mathrm{Int}(K') \subset K' \subset U$. We will assume for now that $||f||_{\overline{U}} < 1$.

We now define a quasiregular approximation g_n of f by applying the construction of Section 3 in each D_i, and by applying the folding construction of Sections 4-8 in each complementary component of $\cup_i(\overline{D_i} \cup \Gamma_i)$:

Definition 9.2. For every n, we define a quasiregular mapping g_n as follows. Recalling Remark 3.6, we first set

$$g_n := g_n(\varepsilon, K' \cap D_i, D_i, f|_{D_i}) \text{ in } D_i \text{ for } 1 \le i \le k \tag{9.2}$$

The equation (9.2) defines the curves $(\Gamma_i)_{i=1}^k$ by way of Proposition 4.3, and we enumerate the components of

$$\widehat{\mathbb{C}} \setminus \left(\overline{U} \cup \bigcup_{i=1}^{k-1} \Gamma_i \right)$$

by $(\Omega(i))_{i=1}^{\ell}$. Recalling Remark 8.6 and Notation 4.14, we extend the definition of g_n to the open set

$$\Omega := \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{\ell} \partial\Omega_n'(i) \right) \tag{9.3}$$

by the formula

$$g_n := g_n(K', U, \mathcal{D}, f, \varepsilon, \Omega(i), P' \cap \Omega(i)) \text{ in } \Omega_n'(i) \text{ for } 1 \le i \le \ell. \tag{9.4}$$

Proposition 9.3. The quasiregular function g_n is C-vertex supported and K-quasiregular for C, K independent of n.

Proof. For $g_n|_{\Omega_n'(i)}$ this is exactly Proposition 8.4, and so the conclusion follows since g_n is holomorphic in U. ∎

The function g_n is now defined on all of $\widehat{\mathbb{C}}$ except for the edges of each $\partial\Omega_n'(i)$. We show in Propositions 9.4 and 9.5 below that g_n in fact extends continuously across each edge of $\partial\Omega_n'(i)$, and deduce in Corollary 9.6 that g_n extends quasiregularly across $\partial\Omega$.

Proposition 9.4. The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a continuous function $g_n : \Omega \cup e \to \widehat{\mathbb{C}}$ for any edge $e \subset \partial\Omega \cap \partial U$.

Proof. Let i be so that $e \subset \partial D_i$ and denote the unique element of $(\Omega(i))_{i=1}^{\ell}$ that contains e on its boundary by $\Omega(j)$. Recall by Definitions 3.5 and 8.1 that

$$g_n|_{D_i} = \Psi_i \circ B_n, \tag{9.5}$$

$$g_n|_{\Omega_n'(j)} = E_n \circ \psi \circ \lambda \circ \tau. \tag{9.6}$$

By Theorem 6.3(2) we have

$$h_n \circ \psi \circ \lambda \circ \tau = B_n \text{one}.$$

By (5.5) and the definition (7.1) of E_n, it follows from (9.6) that

$$g_n|_{\Omega_n'(j)}(z) = \Psi_i \circ B_n(z) \text{ for } z \in e,$$

in other words $g_n|_{\Omega_n'(j)}$ and $g_n|_{D_i}$ agree pointwise on e. ∎

Proposition 9.5. The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a continuous function $g_n : \Omega \cup e \to \widehat{\mathbb{C}}$ for any edge $e \subset \partial\Omega \cap (\cup_{i=1}^{\ell} \Omega(i))$.

Proof. Let j be so that $e \subset \partial\Omega_n'(j)$, and as in the proof of Proposition 9.4, recall that

$$g_n|_{\Omega_n'(j)} = E_n \circ \psi \circ \lambda \circ \tau. \tag{9.7}$$

Let $x \in e$. There are two limits

$$\lim_{\Omega_n'(j) \ni z \to x} \psi \circ \lambda \circ \tau(z),$$

each lying on the unit circle. Denote them by $\zeta_{\pm}$. By Theorem 4.13(3), we have that $\zeta_+^m = \overline{\zeta_-^m}$. Thus, by (5.10) and (7.1), we conclude that there is a unique limit

$$\lim_{\Omega_n'(j) \ni z \to x} E_n \circ \psi \circ \lambda \circ \tau(z).$$

Hence, setting

$$g_n(x) := \lim_{\Omega_n'(j) \ni z \to x} E_n \circ \psi \circ \lambda \circ \tau(z)$$

defines a continuous extension of g_n across the edge e. ∎

Corollary 9.6. The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a K-quasiregular function $g_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Proof. The set $\widehat{\mathbb{C}} \setminus \Omega = \partial\Omega$ consists of a finite collection of analytic arcs: the edges of the graphs $\partial\Omega_n'(i)$ over $1 \le i \le \ell$. Thus, by removability of analytic arcs for quasiregular mappings, it suffices to show that $g_n : \Omega \to \widehat{\mathbb{C}}$ extends continuously across each such edge. There are two types of edges to check: those that lie on the boundary of a domain D_i, and those that lie in the interior of a domain $\Omega(i)$. We have already checked continuity across both types of edges in Propositions 9.4, 9.5, and so the proof is complete. ∎

10 Proof of the Main Theorems

In Section 10 we prove Theorems A and B. Recall that in Section 9 we fixed the objects $K, f, \mathcal{D}, \varepsilon, P$ as in Theorem B (see Notation 9.1), and we defined a quasiregular approximation g_n to f in Definition 9.2. We also showed in Section 9 that g_n in fact extends to a quasiregular function $g_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Now we apply the MRMT below in Definition 10.1 to obtain the rational maps $r_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which we will prove satisfy the conclusions of Theorems A and B for large n.

Definition 10.1. The mapping g_n induces a Beltrami coefficient $\mu_n := (g_n)_{\overline{z}}/(g_n)_z$, which, by way of the MRMT, defines a quasiconformal mapping $\phi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $r_n := g_n \circ \phi_n^{-1}$ is holomorphic. We normalize ϕ_n so that $\phi_n(\infty) = \infty$ and $\phi_n(z) = z + O(1/|z|)$ as $z \to \infty$.

We now begin deducing that for large n, the maps r_n satisfy the various conclusions in Theorems A and B.

Proposition 10.2. The function r_n of Definition 10.1 is rational, and $r_n^{-1}(\infty) = \phi_n(P')$. In particular, if K is full and $P = \{\infty\}$, then r_n is a polynomial.

Proof. The function r_n is holomorphic on $\widehat{\mathbb{C}}$ and takes values in $\widehat{\mathbb{C}}$: the only such functions are rational. Note that $g_n^{-1}(\infty) \cap \overline{U} = \emptyset$ since g_n is bounded on $\overline{U}$. Thus, by Proposition 8.4 and (9.4), we have that $g_n^{-1}(\infty) = P'$. Since $r_n := g_n \circ \phi_n^{-1}$, we conclude that $r_n^{-1}(\infty) = \phi_n(P')$. The last statement of the proposition follows since we normalized $\phi_n(\infty) = \infty$, and the only rational functions with a unique pole at ∞ are polynomials. ■

Proposition 10.3. For all $R < \infty$, the mapping ϕ_n satisfies:

$$||\phi_n(z) - z||_{R\cdot\mathbb{D}} \xrightarrow{n\to\infty} 0. \tag{10.1}$$

Proof. Since g_n is C-vertex supported by Proposition 9.3, we conclude from Proposition 4.7 that

$$\text{Area}(\text{supp}(\mu_n)) \xrightarrow{n\to\infty} 0. \tag{10.2}$$

The relation (10.1) now follows from (10.2) since $||\mu_n||_{L^\infty} \le K$ for all n by Proposition 9.3. ■

Theorem 10.4. For all sufficiently large n, the mapping r_n satisfies $CP(r_n) \subset \mathcal{D}$.

Proof. By Proposition 4.7, we have

$$\max\left\{\text{diam}(e) : e \text{ is an edge of } \bigcup_{i=1}^{\ell} \partial\Omega(i)\right\} \xrightarrow{n\to\infty} 0.$$

Thus, since $\mathcal{D}$ is a domain containing $\cup_{i=1}^{\ell}\partial\Omega(i)$, we have by (8.2) that $BP(g_n) \setminus U \subset \mathcal{D}$ for large n. Since $U \subset \mathcal{D}$ we conclude that $BP(g_n) \subset \mathcal{D}$ for large n. The result now follows from Proposition 10.3 since $\phi(BP(g_n)) = CP(r_n)$. ■

Theorem 10.5. For all sufficiently large n, we have

$$||f - r_n||_K < 2\varepsilon.$$

Proof. First we note that since f is uniformly continuous on K', there exists $\delta > 0$ so that if $z, w \in K'$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$. By Proposition 10.3, we can conclude that

$$||\phi_n(z) - z||_K < \min(\delta, \text{dist}(K, \partial K')) \tag{10.3}$$

for all sufficiently large n.

Let $z \in K$ and $w_n := \phi_n^{-1}(z)$. Then

$$|r_n(z) - f(z)| = |g_n(w_n) - f(z)| \le |g_n(w_n) - f(w_n)| + |f(w_n) - f(z)|. \tag{10.4}$$

It follows from (10.3) that $|f(w_n) - f(z)| < \varepsilon$ for sufficiently large n. Since $w_n \in K'$ for large n, we also have by (3.5) that $g_n(w_n) := \Psi_j \circ B_n(w_n)$, whence it follows from Proposition 3.3 that $|g_n(w_n) - f(w_n)| < \varepsilon$. ■

Theorem 10.6. For all sufficiently large n, we have

$$CV(r_n) \subset N_\varepsilon\widehat{f(K)}$$

Proof. Since

$$\mathrm{CV}(r_n) = \mathrm{BV}(g_n),$$

it suffices to show that for every $z \in \mathrm{BP}(g_n)$ and sufficiently large n, we have

$$g_n(z) \in N_\varepsilon \widehat{f(K)}. \tag{10.5}$$

For $z \in \mathrm{BP}(g_n) \setminus U$, it follows from (8.3) that $g_n(z) \in N_\varepsilon f(K) \subset N_\varepsilon \widehat{f(K)}$. For $z \in \mathrm{BP}(g_n) \cap D_i$, (10.5) follows from Definition 3.5 of $g_n|_{D_i}$. ■

Proof of Theorem B: In the special case that $||f||_K < 1$, we have already proven that the mappings r_n satisfy the conclusions of Theorem 1 for all sufficiently large n. Indeed, Theorem 10.5 says that $||f - r_n||_K < 2\varepsilon$, conclusion (2) in Theorem 1 is Theorem 10.4, and conclusion (3) is Theorem 10.6. Conclusion (1) follows from Propositions 10.2, 10.3. The general case follows by applying the above special case to an appropriately rescaled f. ■

Proof of Theorem A: When K is full, we may take $P = \{\infty\}$ and apply Theorem B, in which case Proposition 10.2 guarantees that the maps r_n are polynomials. ■

Theorem 10.7. (Mergelyan+) Let $K \subset \mathbb{C}$ be full, suppose $f \in C(K)$ is holomorphic in $\mathrm{Int}(K)$, and let $\mathcal{D}$ be a domain containing K. For every $\varepsilon > 0$, there exists a polynomial p so that $||f - p||_K < \varepsilon$ and:

1) $\mathrm{CP}(p) \subset \mathcal{D}$,
2) $\mathrm{CV}(p) \subset N_\varepsilon \widehat{f(K)}$.

Proof: By the usual version of Mergelyan's Theorem, there exists a polynomial q so that $||f - q||_K < \varepsilon/2$. Apply Theorem A to K, $\mathcal{D}$, q, $\varepsilon/2$ to obtain an approximant of q, which we denote by p. The polynomial p satisfies the conclusions of Theorem 10.7. ■

Corollary 10.8. (Weierstrass+) Suppose that $I \subset \mathbb{R}$ is a closed interval, $f : I \to \mathbb{R}$ is continuous, and U, $V \subset \mathbb{C}$ are planar domains containing I, $f(I)$, respectively. Then, for every $\varepsilon > 0$, there exists a polynomial p with real coefficients so that $\|f - p\|_I \le \varepsilon$, and

1) $\mathrm{CP}(p) \subset U$,
2) $\mathrm{CV}(p) \subset V$.

Proof. Let $I = [a, b]$, and f, U, V as in the statement of the corollary. By Theorem 10.7, there exists a complex polynomial q so that $||f - q||_{[a,b]} < \varepsilon/2$. The real polynomial

$$Q(z) := \frac{q(z) + \overline{q(\overline{z})}}{2}$$

satisfies $Q(x) = \mathrm{Re}(q(x))$ for $x \in \mathbb{R}$ and hence $||f - Q||_{[a,b]} < \varepsilon/2$. We will use the symbol $\Subset$ to mean compactly contained. Let $V_1 \Subset V$ be a sufficiently small, $\mathbb{R}$-symmetric domain containing $f(I)$ so that there is a component of $Q^{-1}(V_1)$ (which we denote by U_1) satisfying $U_1 \Subset U$. Let U_2 be a $\mathbb{R}$-symmetric, analytic domain satisfying $I \Subset U_1 \Subset U_2 \Subset U$. Recall Notation 9.1 and consider:

1) the compact set $\overline{U_1}$,
2) the analytic function Q,
3) the analytic domain U_2 containing $\overline{U_1}$,
4) $\min\{\varepsilon/2, \mathbf{dist}(\partial V_1, \partial V)\}$,
5) $P = \{\infty\}$.

Applying Definition 9.2 to (1)-(5) yields quasiregular mappings g_n with $\mathbb{R}$-symmetric Beltrami coefficient, so that

$$p_n := g_n \circ \phi_n^{-1} \tag{10.6}$$

is a real polynomial approximant of Q satisfying

1) $||f - p_n||_{[a,b]} < \varepsilon$,
2) $\mathrm{CP}(p_n) \subset U_2$, and
3) $\mathrm{CV}(p_n) \subset V$,

for large n. Thus, p_n satisfies the conclusion of Corollary 10.8 for large n. ■

Recall the notation $\Omega(\mathrm{i})$ from Definition 9.2, and let $\tau_{\mathrm{i}} : \Omega(\mathrm{i}) \to \mathbb{D}_\infty$ be the conformal mapping satisfying $\tau_{\mathrm{i}}^{-1}(\infty) = P' \cap \Omega(\mathrm{i})$ as in Notation 4.5. The following fact justifies part of our description in the introduction of the behavior of the rational approximants off K.

Proposition 10.9. Let $1 < r < R < \infty$. Then, for all sufficiently large n, we have

$$r_n \circ \phi_n(z) = \tau_{\mathrm{i}}(z)^m \text{ and} \tag{10.7}$$

$$|r_n(z)| > R \tag{10.8}$$

for all $z \in \tau_{\mathrm{i}}^{-1}(\{z : |z| > r\})$.

Proof. Fix r and R as in the statement. From (8.5) and the functional equation (9.4) defining g_n in $\Omega(\mathrm{i})$, it follows that

$$g_n(z) = \tau_{\mathrm{i}}(z)^m \text{ for all } z \in \tau_{\mathrm{i}}^{-1}(\{z : |z| > (r+1)/2\}) \tag{10.9}$$

for all large n. Since

$$r_n \circ \phi_n = g_n, \tag{10.10}$$

The relation (10.7) follows. Moreover, we have by Proposition 10.3 that

$$\phi_n \circ \tau_{\mathrm{i}}^{-1}(\{z : |z| > r\}) \subset \tau_{\mathrm{i}}^{-1}(\{z : |z| > (r+1)/2\}) \tag{10.11}$$

for all sufficiently large n. Since $((r+1)/2)^m > R$ for large n, the relation (10.8) also follows. ■

Remark 10.10. If we make further assumptions on f and K, the conclusion

$$\mathrm{CV}(r) \subset N_\varepsilon \widehat{f(K)}$$

of Theorems A and B can be improved to

$$\mathrm{CV}(r) \subset \widehat{f(K)}, \tag{10.12}$$

which is equivalent to $\mathrm{CV}(r) \subset f(K)$ if $f(K)$ is full. Indeed, if for instance the interiors of K, $f(K)$ are analytic domains and $f : \mathrm{Int}(K) \to \mathrm{Int}(f(K))$ is proper, then a similar strategy as in the proofs of Theorems A and B but replacing Ψ in (3.5) with a conformal map $\mathbb{D} \mapsto \mathrm{Int}(K)$ can be used to prove (10.12).

Remark 10.11. We remark that while Theorem A strictly improves on Runge's Theorem on polynomial approximation, the relationship between Theorem B and Runge's Theorem on rational approximation is more subtle. Both show existence of rational approximants, and

only Theorem B describes the critical point structure of the approximant, however the poles of the approximant in Theorem B are specified only up to a small perturbation, whereas in Runge's Theorem they are specified exactly. We do not know whether it is necessary to consider perturbations of P' in Theorem B, or if the improvement $r^{-1}(\infty) = P'$ is possible (a related problem appears in [5], [10], and [7]).

Funding

The first author is supported in part by NSF grant DMS-2303987, and the second author is supported in part by NSF grant DMS-2246876.

Acknowledgments

The authors would like to thank Dmitry Khavinson for pointing out the reference [16], which simplified several arguments in a previous version of this manuscript. The authors would also like to thank Jack Burkart, Franc Forstnerič, John Garnett, Oleg Ivrii, Xavier Jarque, and Malik Younsi for their comments on this manuscript.

References

1. Alexander, H. and J. Wermer. *Several Complex Variables and Banach Algebras*, Vol. 35 of *Graduate Texts in Mathematics*, 3rd ed. New York: Springer, 1998.
2. Benini, A. M., V. Evdoridou, N. Fagella, P. J. Rippon, and G. M. Stallard. "Classifying simply connected wandering domains." *Math. Ann.* **383**, no. 3-4 (2022): 1127–78. https://doi.org/10.1007/s00208-021-02252-0.
3. Branner, B. and N. Fagella. *Quasiconformal surgery in holomorphic dynamics*. Vol. 141 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, 2014. With contributions by Xavier Buff, Shaun Bullett, Adam L. Epstein, Peter Haïssinsky, Christian Henriksen, Carsten L. Petersen, Kevin M. Pilgrim, Tan Lei and Michael Yampolsky.
4. Bishop, C. J. "Constructing entire functions by quasiconformal folding." *Acta Math.* **214**, no. 1 (2015): 1–60. https://doi.org/10.1007/s11511-015-0122-0.
5. Bishop, C. J. and K. Lazebnik. "Prescribing the postsingular dynamics of meromorphic functions." *Math. Ann.* **375**, no. 3-4 (2019): 1761–82. https://doi.org/10.1007/s00208-019-01869-6.
6. Burkart, J. and K. Lazebnik. "Interpolation of power mappings." *Rev. Mat. Iberoamericana* **39**, no. 3 (2023): 1181–200. https://doi.org/10.4171/rmi/1359.
7. Bishop, C. J., K. Lazebnik, and M. Urbański. "Equilateral triangulations and the postcritical dynamics of meromorphic functions." *Math. Ann.* **387**, no. 3-4 (2023): 1777–818. https://doi.org/10.1007/s00208-022-02507-4.
8. Thaler, L. B. "On the geometry of simply connected wandering domains." *Bull. Lond. Math. Soc.* **53**, no. 6 (2021): 1663–73. https://doi.org/10.1112/blms.12518.
9. Caratheodory, C. *Theory of Functions of a Complex Variable*, Vol. 2. New York: Chelsea Publishing Co., 1954. Translated by F. Steinhardt.
10. DeMarco, L. G., S. C. Koch, and C. T. McMullen. "On the postcritical set of a rational map." *Math. Ann.* **377**, no. 1-2 (2020): 1–18. https://doi.org/10.1007/s00208-018-1732-6.
11. Erëmenko, A. È. and M. Y. Ljubich. "Examples of entire functions with pathological dynamics." *J. London Math. Soc. (2)* **36**, no. 3 (1987): 458–68.
12. Evdoridou, V., P. J. Rippon, and G. M. Stallard. "Oscillating simply connected wandering domains." arXiv e-prints, arXiv:2011.14736, November 2020.
13. Forstnerič, F. "Noncritical holomorphic functions on stein manifolds." *Acta Math.* **191**, no. 2 (2003): 143–89. https://doi.org/10.1007/BF02392963.
14. Garnett, J. B. *Bounded Analytic Functions*, Vol. 96 of *Pure and Applied Mathematics*. New York-London: Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], 1981.
15. Garcia, S. R., J. Mashreghi, and W. T. Ross. "Finite Blaschke products: a survey." *Harmonic Analysis, Function Theory, Operator Theory, and Their Applications*, Vol. 19 of *Theta Ser. Adv. Math.*,. 133–58. Theta, Bucharest, 2017.

16. Grunsky, H. *Lectures on Theory of Functions in Multiply Connected Domains*, Vol. 4 of *Studia Mathematica: Skript*. Göttingen: Vandenhoeck & Ruprecht, 1978.
17. Khavinson, S. Y. *Two Papers on Extremal Problems in Complex Analysis*, Vol. 129. American Mathematical Society Translations: Series 2, 1986. Translated by D. Khavinson.
18. Martí-Pete, D. and M. Shishikura. "Wandering domains for entire functions of finite order in the Eremenko-Lyubich class." *Proc. Lond. Math. Soc. (3)* **120**, no. 2 (2020): 155–91.
19. Martí-Pete, D., L. Rempe, and J. Waterman. "Eremenko's conjecture, wandering Lakes of Wada, and maverick points." *arXiv e-prints*, arXiv:2108.10256, August 2021.
20. Martí-Pete, D., L. Rempe, and J. Waterman. "Bounded Fatou and Julia components of meromorphic functions." *arXiv e-prints*, arXiv:2204.11781, April 2022.
21. Runge, C. "Zur Theorie der Eindeutigen Analytischen Functionen." *Acta Math.* **6**, no. 1 (1885): 229–44. https://doi.org/10.1007/BF02400416.
22. Totik, V. "The gauss-Lucas theorem in an asymptotic sense." *Bull. Lond. Math. Soc.* **48**, no. 5 (2016): 848–54. https://doi.org/10.1112/blms/bdw047.
23. Totik, V. "A quantitative gauss-Lucas theorem." *Ark. Mat.* **60**, no. 1 (2022): 195–212. https://doi.org/10.4310/ARKIV.2022.v60.n1.a9.