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The tenets of quantile-based inference in Bayesian models

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ABSTRACT

Bayesian inference can be extended to probability distributions defined in terms of their inverse distribution function, i.e. their quantile function. This applies to both prior and likelihood. *Quantile-based likelihood* is useful in models with sampling distributions which lack an explicit probability density function. *Quantile-based prior* allows for flexible distributions to express expert knowledge. The principle of *quantile-based* Bayesian inference is demonstrated in the univariate setting with a Govindarajulu likelihood, as well as in a *parametric quantile regression*, where the error term is described by a quantile function of a Flattened Skew-Logistic distribution.

1. Introduction

Most statistics courses and textbooks introduce continuous random variables via the (cumulative) *distribution function* (CDF) and the *probability density function* (PDF). "An equally adequate representation" (Tukey, 1965) of a random variable can be made using the *inverse CDF*, known as the *quantile function* (QF), and its derivative, the *quantile density function* (QDF), but the use of such *quantile distributions* is rare. Defining a distribution via its quantile function has several advantages, including that the distributions with explicit quantile functions are easy to sample from and more complex distributions can be crafted using the simpler quantile functions as the building blocks (Gilchrist, 2000; Parzen, 1980; Hadlock, 2017; Powley, 2013).

Some of the widely used probability distributions defined in terms of the CDF and PDF (*density-defined* distributions) are not easily invertible (e.g. normal or gamma) and, therefore, the numerical approximation of their QF is used. Similarly, there are other distributions defined in terms of their QF and QDF (*quantile* distributions), that are also not invertible, and thus, the numerical approximation of their CDF can be used.

Most of the knowledge and methods for Bayesian inference have been developed for the *density-defined* distributions. While there have been several published articles where *quantile* distributions were used in the context of the likelihood-free approximate Bayesian computation (Allingham et al., 2009; Drovandi and Pettitt, 2011; Karabatsos and Leisen, 2018; Fearnhead and Prangle, 2012; Bernton et al., 2019; McVinish, 2012), the likelihood-based application of the Bayesian infer-

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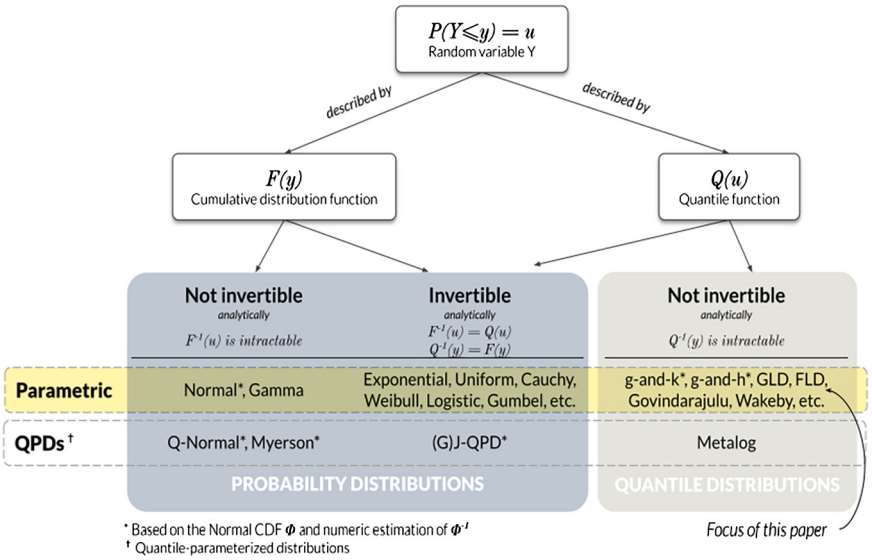

Fig. 1. Probability distributions, quantile distributions and parameterization.

ence for *quantile* distributions has been limited (Rayner and MacGillivray, 2002; Haynes and Mengersen, 2005; Nair et al., 2020).

This article builds on the ideas of Gilchrist (2000), Rayner and MacGillivray (2002), Nair et al. (2020) and systematically presents and illustrates the Bayesian inference using quantile functions. We refer to this method of inference as *quantile-based* because it deals with the inverse transformation of the *intermediate cumulative probabilities* (*depths*, indicating how "deep" an observation is into the distribution) corresponding to the observations given the parametrized model. We aim to show that the *quantile-based* Bayesian inference using the intermediate depths leads to the same posterior beliefs as the conventional density-based inference. We apply the principles of *quantile-based inference* to Bayesian updating of parameters in the univariate and regression settings using the flexible and extensible quantile sampling distributions.

Section 2 revisits the functions and identities for characterizing the distribution of a continuous random variable. Then in Section 3 we introduce the terms of *quantile-based likelihood* and *quantile-based prior* in Bayesian inference and show that the likelihood (and prior) can be expressed without the PDF. Section 4 discusses the computational aspects of the numerical inversion of quantile functions for approximating the intermediate depths in a quantile-based likelihood expressed by a quantile distribution. Section 5 discusses the applications of quantile-based inference in univariate and regression settings. We discuss the models and provide code examples implementing quantile-based likelihood in Stan (Gabry and Češnovar, 2022) and in R (R Core Team, 2021). The proposed models have been validated using the Simulation-Based Calibration (Modrák et al., 2022; Talts et al., 2020; Cook et al., 2006). The results of these simulation studies (provided in the Supplementary Materials) show successful recovery of model parameters for all widths of the posterior credible intervals. We conclude the paper with a discussion and summary of the results in Section 6.

Although the description of the *quantile-based likelihood* (Rayner and MacGillivray, 2002; King, 1999; Gilchrist, 2007; Nair et al., 2020) and prior (Nair et al., 2020) appeared in the literature before, they were presented in the context of specific distributions and not as a general principle of inference. In their recent work, Nair et al. (2020) presented Bayesian inference with quantile functions, but their presentation of what we describe here as *quantile-based prior* lacked the necessary adjustment due to the nonlinear transformation of the parameters involved (see Section 3.2 below).

In this paper, we apply the principles of *quantile-based inference* to implement the Bayesian version of the *parametric quantile regression* (Gilchrist, 2008) with the error term is described by a bespoke quantile function and estimate the regression parameters using MCMC.

2. Distribution specification

To set the scene for the discussion of density-based and quantile-based Bayesian inference we briefly review the different ways of specifying a probability distribution and discuss several examples of the distributions defined by a quantile function, found in the literature (Fig. 1).

2.1. Essential functions

Let Y be a continuous random variable. It can be expressed via the (cumulative) *distribution function* (CDF)

$$F_Y(y|\theta) = Pr(Y \leq y|\theta), \quad \theta \in \mathcal{A} \subset \mathbb{R}. \tag{1}$$

Table 1
Gilchrist's quantile function transformation rules.

Original QF	Rule	Resulting QF	Resulting variable
$Q_Y(u)$	Reflection rule	$-Q(1-u)$	QF of -Y
$Q_Y(u)$	Reciprocal rule	$1/Q(1-u)$	QF of $1/Y$
$Q_1(u)$, $Q_2(u)$	Addition rule	$Q_1(u)+Q_2(u)$	valid QF
$Q_1(u)$, $Q_2(u)$	Linear combination rule	$aQ_1(u)+bQ_2(u)$	valid QF for $a, b > 0$
$Q_1(u)$, $Q_2(u) > 0$	Multiplication rule	$Q_1(u)Q_2(u)$	valid QF
$Q_Y(u)$	Q-transformation	$T(Q_Y(u))$	QF of $T(Y)$, $T(Y)$ non-decreasing
$Q_Y(u)$	p-transformation	$Q_Y(H(u))$	p-transformation of $Q_Y(u)$, $H(u)$ non-decreasing

An alternative way of describing the random variable Y is via the *quantile function* (QF)

$$Q_Y(u|\theta) = \inf\{y : F_Y(y|\theta) \geq u\}, \quad 0 \leq u \leq 1. \tag{2}$$

The subscript $_Y$ is used to indicate the random variable that the depth u corresponds to.

New quantile functions can be easily created using Gilchrist's quantile function transformation rules (Gilchrist, 2000; Powley, 2013; Hadlock, 2017; Sharma and Chakrabarty, 2017) summarized in Table 1. We use these rules for crafting a bespoke quantile function for modeling the error term in a Bayesian parametric quantile regression in Section 5.

If $F_Y(y|\theta)$ is continuous and strictly monotonically increasing over the support of Y, then $Q_Y(u|\theta)$ is simply the inverse of $F_Y(y|\theta)$. Therefore, the quantile function is often referred to as the *inverse CDF*, i.e.

$$Q_Y(u|\theta) = F_Y^{-1}(y|\theta). \tag{3}$$

Not all QFs are analytically invertible (Fig. 1). A distribution whose quantile function $Q_Y(u|\theta)$ is not analytically invertible is called a *quantile distribution* (Gilchrist, 1997) or a *quantile-based distribution* (Sharma and Chakrabarty, 2020).

The derivative of the CDF is the *probability density function* (PDF) denoted by

$$f_Y(y|\theta) = \frac{dF_Y}{dy}. \tag{4}$$

Similarly, the derivative of the QF is the *quantile density function* (QDF) denoted by

$$q_Y(u|\theta) = \frac{dQ_Y}{du}, \quad 0 \leq u \leq 1. \tag{5}$$

The reciprocal of the QDF $[q_Y(u|\theta)]^{-1} = f(Q_Y(u|\theta))$ is referred to as the *density quantile function* (Parzen, 1980) or *p-pdf* (Gilchrist, 2000). Here and for the rest of the article, we will often omit the subscript $_Y$ and the conditioning on θ to simplify the notation.

$$f(Q(u)) = \frac{dF(Q(u))}{dQ(u)} = \frac{dF(Q(u))/du}{dQ(u)/du} = \frac{dF(F^{-1}(u))/du}{q(u)} = \frac{du/du}{q(u)} = [q(u)]^{-1}. \tag{6}$$

In Section 3 of this paper, we rely on the density quantile function (DQF) $[q(u)]^{-1}$, i.e. the density of a random variable expressed in terms of the cumulative distribution function (Perri and Tarsitano, 2007), to define the likelihood in a Bayesian model based on a quantile sampling distribution.

2.2. Derivatives of the inverses and the numerical approximation

Following the inverse function theorem (Price, 1984), for a function to be invertible in the neighborhood of a point, it should have a continuous non-zero derivative at that point. If the function is invertible, the derivative of the inverse is reciprocal to the function's derivative (Marsden et al., 1985). Formally, if dt/dy exists and $dt/dy \neq 0$, then dy/dt also exists and $dy/dt = [dt/dy]^{-1}$. Therefore, for a quantile function $Q(u) = y$, if a QDF $q(u)$ exists and $q(u) \neq 0$, then PDF $f(y)$ also exists and it is equal to $f(y) = [q(u)]^{-1}$.

Fig. 2 illustrates the relationship between the key probability functions. The distribution function (CDF) and the quantile function (QF) are depicted on the opposite sides of the Moebius strip. The derivatives of these functions (PDF and QDF, respectively) end up on the same side, due to the geometry of the Moebius surface. It means that the derivatives are no longer the inverses of each other, but rather the reciprocals, as stated in the equation at the bottom. The "do-it-yourself" copy of this probability function Moebius strip is included in the Supplementary Materials, along with the graphs of the five essential functions (CDF, PDF, QF, QDF, and DQF) for the common probability distributions (Normal, Logistic, Weibull, Exponential, and Kumaraswamy).

Even though the quantile distributions lack the closed-form CDF $F(y) = u$ in most cases the depths u can be approximated by numerically inverting the $Q(u)$. We denote the numerically inverted quantile function as $\widehat{Q^{-1}}(y)$ or $\widehat{F}(y)$. The

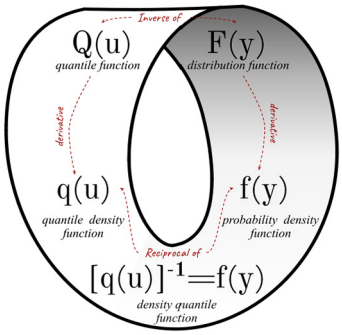

Fig. 2. Moebius strip of probability functions.

inverse of a quantile function $Q(u)$ at point u, corresponding to the observation y, is obtained by minimizing the difference between the actual observation y and $Q(u)$ by iteratively refining the depth u. The details of the numerical inversion algorithm are discussed in Section 4.

Figure provides examples of the CDF/PDF and QF/QDF for some common statistical distributions (normal, beta, lognormal, exponential and Weibull).

2.3. Quantile distributions

Statistical methods utilizing QF and QDF were pioneered by the seminal work of Parzen (1979). Some of the quantile distributions covered in the literature are generalized *g-and-h* and its sibling *g-and-k* distribution (Haynes and Mengersen, 2005; Jacob, 2017; Prangle, 2017; Rayner and MacGillivray, 2002), Tukey Lambda Distribution and its generalizations, known as GλD (Aldeni et al., 2017; Chalabi et al., 2012; Dedduwakumara et al., 2021; Ramberg and Schmeiser, 1974; Fournier et al., 2007; Freimer et al., 1988), Wakeby distribution (Rahman et al., 2015), flattened logistic distribution (Sharma and Chakrabarty, 2019) and Govindarajulu distribution (Nair et al., 2012, 2013).

The mathematical notation for describing probability distributions has been standardized and adopted across different domains. The first use of the tilde symbol $\sim$ to denote the CDF can be traced back to early 1960s. Olkin and Tate (1961) wrote: "$X \sim F(x)$ means that x is distributed according to the distribution function $F(x)$". Today this convention is adopted by the majority of Bayesian textbooks (Gelman et al., 2013; Johnson et al., 2022; O'Hagan et al., 2006; Lambert, 2018; Gelman et al., 2021; Koller and Friedman, 2009). For example, if a variable Y is normally distributed it is described as

$$Y \sim N(\mu, \sigma), \tag{7}$$

which means that the random variable Y has the distribution function $F_Y(y) = \Phi(y|\mu, \sigma)$, where Φ is the CDF of the normal distribution (Johnson et al., 1994).

When the distribution of a random variable Y is described by a non-invertible quantile function, such as, for example, the extensively researched Generalized Lambda Distribution (GLD) proposed by Ramberg and Schmeiser (1974)

$$Q_Y(u|\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left[u^{\lambda_3} + (1-u)^{\lambda_4}\right], \tag{8}$$

stating $Y \sim \text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is not strictly accurate, because the GLD quantile function is not invertible and its CDF can be computed only numerically as $F_Y(y) \approxeq \widehat{Q_Y^{-1}}(y)$.

Therefore, in this paper, we propose to denote this distribution as

$$u \overset{y}{\backsim} \text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{9}$$

where the *back-tilde* with the variable name overscript $\overset{y}{\backsim}$ should be read as "inversely distributed as" to indicate that the *depth* u is fully determined given the value of the random variable Y and the parameterized inverse distribution function indicated to the right of the *back-tilde* symbol (in this case, GLD). In situations where extra clarity is required, the *depth* variable name can also be mentioned in describing the density-defined distributions, e.g. $X \overset{p}{\sim} N(\mu, \sigma)$, where $\mu \overset{v}{\sim} N(\mu_0, \sigma_0)$.

Although in this paper we focus on the distributions with abstract parameters, the distributions parameterized by the quantile-probability pairs (*quantile-parameterized* distributions) are also worth a mention. The most prominent examples of the *quantile-parameterized density-defined* distributions are Myerson distribution (Myerson, 2005), Johnson QPD (J-QPD) with its generalizations (Hadlock and Bickel, 2017, 2019) and Simple Q-Normal distribution (Keelin and Powley, 2011). The group of the *quantile-parameterized quantile distributions* is represented by Metalog distribution (Keelin, 2016). Quantile-parameterized distributions play an important role in representing expert beliefs about variables, parameters, or quantities of interest (Dion et al., 2020; Gu et al., 2018; Larrain et al., 2021; Reinhardt et al., 2016; Baey et al., 2022), although they don't lend themselves easily as sampling distributions due to the special nature of their parameterization.

3. Bayesian inference with quantile functions

Gilchrist (2000), p. 209 used the term *quantile-based likelihood* while describing the method of maximum likelihood applied to a quantile distribution. Rayner and MacGillivray (2002) describe a three-step process of computing the log-likelihood for a quantile distribution and use it for the maximum likelihood estimation of parameters in *g-and-k* and generalized *g-and-h* distributions. Nair et al. (2020) performed quantile function substitution for both the observables $y_i = Q(u_i|\theta),\ i = \{1, 2, \dots n\}$, and parameters $\theta = Q_\Theta(v)$ and computed the Bayes estimator under the squared error loss for the Govindarajulu likelihood. In this section, we summarize this approach and use the terms *quantile-based* prior and *quantile-based* likelihood based on the identities and substitutions introduced in Section 2 to demonstrate the equivalence of the two ways of expressing the likelihood in Bayesian models.

3.1. Density-based and quantile-based likelihood

The traditional Bayesian inference formula can be restated using the substitutions involving quantile functions (Nair et al., 2020). Assume that the prior information about the scalar parameter θ can be summarized by the prior distribution over the parameter space Θ. Then, given a random sample of $\underline{y} = \{y_1, y_2, \dots y_n\}$, the posterior distribution of θ can be expressed as:

$$f(\theta|\underline{y}) \propto \mathcal{L}(\theta; \underline{y}) f(\theta), \tag{10}$$

where $f(\theta|\underline{y})$ is the posterior distribution of θ after having observed the sample $\underline{y}$, $f(\theta)$ is the prior distribution of θ, and $\mathcal{L}(\theta; \underline{y}) = \prod_{i=1}^{n} f(y_i|\theta)$ is the likelihood, which is a function of θ. We refer to this form of likelihood as *density-based*, because it is expressed using the density function (PDF) of the observable $\underline{y}$.

Given the random sample $\underline{y}$ and the value of the parameter θ, we can use the quantile function $Q_Y(u|\theta)$ to compute $\underline{Q} = \{Q(u_1), Q(u_2), \dots Q(u_n)\}$, such that $u_i = F(y_i|\theta),\ i = \{1, 2, \dots n\}$. The depths u_i are degenerate random variables because they are fully determined given the values of $\underline{y}$ and the parameter θ. Since $Q_Y(u_i|\theta) = y_i$ we can substitute $\underline{Q}$ for $\underline{y}$. Then the Bayesian inference formula (10) becomes

$$f(\theta|\underline{Q}) \propto \mathcal{L}(\theta; \underline{Q}) f(\theta). \tag{11}$$

We refer to the likelihood $\mathcal{L}(\theta; \underline{Q}) = \prod_{i=1}^{n} f(Q(u_i|\theta)) = \prod_{i=1}^{n} [q(u_i|\theta)]^{-1}$ as *quantile-based*, because it relies on computing the intermediate depths $u_i = F(y_i|\theta)$ corresponding to the observations $y_i,\ i = \{1, 2, \dots n\}$. The two forms of likelihood $\mathcal{L}(\theta; \underline{Q})$ and $\mathcal{L}(\theta; \underline{y})$ are equivalent to each other. Therefore, following the likelihood principle, the posterior beliefs about θ in both cases are identical.

Since the likelihood in the Equation (11) is expressed in terms of $\underline{Q}$, an additional transformation is required to arrive at $\underline{u} = F(\underline{y}|\theta)$. In case the CDF $F(\underline{y}|\theta)$ is not available, the numeric approximation of $\widehat{Q^{-1}}(\underline{y}|\theta)$ may be used. We discuss the details of the numerical approximation of the inverse quantile function in Section 4 of this paper.

3.2. Density-based and quantile-based prior

It is possible to extend the same logic of quantile function substitution to define *density-based* and *quantile-based* priors. In this section, we discuss the parameter transformation required for defining a quantile-based prior and show its connection to the inverse transform used for non-uniform sampling.

The Bayesian inference formula can be restated using the quantile form of the prior (Nair et al., 2020). Assume that the prior distribution of θ can be described using the invertible CDF $F_\Theta(\theta|\psi) = v$ with hyperparameter ψ, so that $Q_\Theta(v|\psi) = \theta$. Substituting the quantile values $Q_\Theta(v|\psi)$ for values of θ, prior beliefs about the parameter(s) of the distribution of θ can be expressed using the distribution of the *quantile values* corresponding to the random variate v, given hyperparameter ψ of the prior distribution as $f(Q_\Theta(v)|\psi) = [q_\Theta(v|\psi)]^{-1}$. We refer to the such formulation of the prior as *quantile-based* because it describes the prior distribution of the random variate v given hyperparameter ψ corresponding to the parameter $\theta = Q_\Theta(v|\psi)$ and not the distribution of the parameter θ itself.

Likewise, the likelihood $\mathcal{L}(Q_\Theta(v|\psi); \underline{y})$ will also rely on the parameter transformation $\theta = Q_\Theta(v|\psi)$. However, such non-linear parameter transformation requires a Jacobian adjustment (Andrilli and Hecker, 2010), which is equal to the absolute derivative of the transform, i.e. $J(Q_\Theta(v|\psi)) = |dQ_\Theta(v|\psi)/dv| = |q_\Theta(v|\psi)|$. Provided that the $Q_\Theta(v)$ is a valid (non-decreasing) quantile function, meaning that $q_\Theta(v|\psi)$ is non-negative on $v \in [0, 1]$, the density quantile term $[q_\Theta(v|\psi)]^{-1}$ representing the prior and the Jacobian adjustment $|q_\Theta(v|\psi)|$ can be dropped as they are reciprocal to each other. Therefore, the quantile-based posterior of the random variate v after observing the sample $\underline{y}$ can be expressed as

$$\begin{aligned} &[q_\Theta(v|\underline{y})]^{-1} \propto \mathcal{L}(Q_\Theta(v|\psi); \underline{y})[q_\Theta(v|\psi)]^{-1}|q_\Theta(v|\psi)| \implies \\ &[q_\Theta(v|\underline{y})]^{-1} \propto \mathcal{L}(Q_\Theta(v|\psi); \underline{y}), \end{aligned} \tag{12}$$

where $[q_\Theta(v|\underline{y})]^{-1}$ is the quantile form of the posterior, and the (quantile) prior density $[q_\Theta(v|\psi)]^{-1}$ is implied by QF transform $Q_\Theta(v|\psi) = \theta,\ v \in [0,1]$.

Quantile-based prior can also be used in combination with quantile-based likelihood, since, as we showed previously, regardless of the form of the likelihood used, the posterior beliefs about the parameter θ will be the same. In such a case, neither prior nor likelihood would require the PDF, and, therefore, both of them can be represented by quantile distributions.

4. Numerical inversion of quantile functions

4.1. Root-finding algorithms

The core element of the *quantile-based* likelihood method is the use of the intermediate depths $\underline{u}$, corresponding to the observables $\underline{y}$ given the parameter θ. These values can either be found analytically, as $F(y)$ for distributions with CDF, or numerically via root-finding algorithm, as $\widehat{F}_Y(y) \approxeq \widehat{Q_Y^{-1}}(y)$ e.g. in case of quantile distributions (Fig. 1).

The problem of inverting a quantile function is tantamount to finding the root of a target function

$$\Omega(u; y, \theta) = [y - Q_Y(u|\theta)], \tag{13}$$

where y is a known observation, θ is the parameter value, and u is the depth. Provided that the $Q_Y(u|\theta)$ is a non-decreasing function and y is a fixed observable value, the target function $\Omega(u; y, \theta)$ is non-increasing. The root-finding algorithm uses the target function to take an observable y and "pull in" its inverted equivalent $Q_Y(u|\theta)$ until the two values exactly meet by iteratively adjusting u.

Since the target function $\Omega(u; y, \theta)$ has a range $u \in [0,1]$, the *bracketing* root-finding algorithms, such as bisection or *regula falsi* (Atkinson, 2008; Burden and Faires, 2011) may be used, although depending on the shape of the quantile function their convergence can be slow. Modern bracketing methods, such as Chandrupatla (Chandrupatla, 1997), Ridders (Ridders, 1979), Zhang (Zhang, 2011; Stage, 2013) and TOMS748 (Alefeld et al., 1995), implemented in the Boost C++ library (Schäling, 2011), combine cubic, quadratic and linear (secant) interpolation to ensure robust and efficient convergence.

The convergence may be accelerated with the help of the *non-bracketing* root-finding algorithms e.g. Newton-Raphson, Halley and Schröder (Householder, 1970), which rely on computing the derivatives of the target function Ω. The first derivative of the target function is simply the negative QDF $\Omega'(u; y, \theta) = \frac{d[y - Q_Y(u|\theta)]}{du} = -q_Y(u|\theta)$. Unfortunately, the derivative-based algorithms do not guarantee that the root will be found and may end up in infinite loops and divergences. The bigger issue with trying to use a *non-bracketing* algorithm to find the root of the target function Ω is related to intermediate values of u falling outside of the [0,1] interval. In such a case $Q_Y(u|\theta)$ will return an error, which will cause the root-finding convergence check to fail. Modern derivative-based root-finding algorithms, such as NewtSafe (Acton, 1990; Press, 2007), perform the root search within a specified interval and fall back to bisection if the algorithm iteration leads the guess outside of the interval.

In this paper we used the Brent *bracketing* root-finding algorithm (Press, 2007) to invert the quantile functions. In R (R Core Team, 2021) the algorithm is available as the `uniroot()` function and in Stan (Gabry and Češnovar, 2022) we implemented it as a custom user-defined function. All source code is available in the Supplementary Materials.

4.2. Computational cost

Quantile-based method of inference comes at a computational cost associated with inverting a quantile function. In order to assess the cost of numerical inversion of quantile functions and to compare the integrated autocorrelation times (IAT) between the density-based and the quantile-based models, we performed simulation-based calibration (Modrák et al., 2022; Säilynoja et al., 2022; Talts et al., 2020; Cook et al., 2006) of the standard Exponential model, under the Gamma prior with $\alpha = 4$ and $\beta = 1$, which can be formulated both in the density-based and in the quantile-based form (since the exponential distribution is fully invertible). We refer to the Supplementary Materials for the details of the simulation-based calibration (SBC) algorithm.

We calculated IAT as the number of iterations of the sampler divided by the parameter's effective sample size (ESS) estimator (Betancourt, 2020). In addition to the standard rank-normalized ESS estimator, we calculated the minimum of the ESS for the 5% and 95% quantiles of the sample, known as the "tail ESS" (Vehtari et al., 2021).

We ran 200 replications of each of the models. Each SBC replication consisted of 2000 draws (of which half was used for burn-in) and 2 parallel chains (to assess the quality of chain mixing). The Stan code for running both density-based and the quantile-based Gamma-Exponential models is available in the Supplementary Materials.

We found that, on average, the numerical inversion of QF costs additional 6.43 sec/chain of 2000 samples (paired t-test 95% CI: [6.28, 6.59]). The MCMC proposals are also slightly more likely to be rejected (average increase in rejections is 0.345 based on the paired t-test with 95% CI of [0.216, 0.474]), as the quantile-based models are more dependent on the feasible initial values.

At the same time we found no significant difference in IAT for either the bulk of the samples (mean difference of 0.012 with 95% CI: [-0.0264, 0.0503], pair-wise t-test), nor the tail of the distribution (mean difference of -0.029 with 95% CI: [-0.0690, 0.0118], pair-wise t-test).

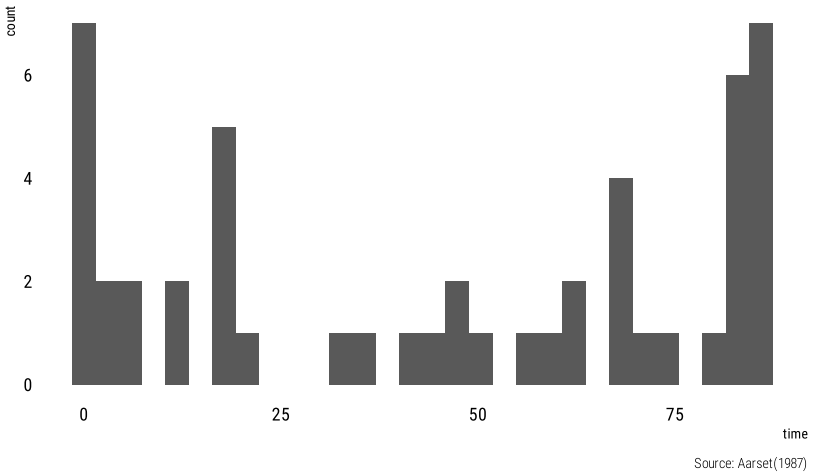

Fig. 3. Histogram of time-to-failure for 50 devices.

The cost may be more significant if the quantile function is expensive to compute, e.g. for distributions with a large number of parameters or involving computationally expensive transformations, or in the presence of covariates, as the case is for *parametric quantile regression*, discussed below.

5. Applications

In this section, we illustrate the application of the *quantile-based inference* to univariate and regression models and provide code examples for models based on the quantile sampling distributions in Stan (Gabry and Češnovar, 2022) and in R (R Core Team, 2021). For the univariate model, we update the shape parameter of a bathtub-shaped Govindarajulu distribution and for the regression model, we pick the flattened skew-logistic distribution to model the error term.

5.1. Univariate model

We take the dataset provided in Aarset (1987) on the time-to-failure of 50 devices (Fig. 3). Lifetime reliability data are often modeled using specialized distributions (Nadarajah, 2009) or 2(3)-component mixtures. Nair et al. (2020) obtained the Bayes estimator under the squared error loss function for the posterior mean of the parameter γ in the Govindarajulu distribution (Nair et al., 2012), under the generalized exponential prior (Gupta and Kundu, 2007). We reuse their example extending it to estimating the full posterior distribution by implementing the model in Stan (Gabry and Češnovar, 2022).

The QF and the QDF of the Govindarajulu distribution (Nair et al., 2012) are given by:

$$\begin{aligned} Q(u|\sigma,\gamma) &= \sigma[(\gamma+1)u^{\gamma} - \gamma u^{\gamma+1}] \\ q(u|\sigma,\gamma) &= \sigma\gamma(\gamma+1)u^{\gamma-1}(1-u), \end{aligned} \tag{14}$$

where $\sigma, \gamma > 0$. The distribution has support on $Q(u|\sigma,\gamma) \in [0,\sigma]$. Note, that the QDF in Nair et al. (2020) is slightly deviating from their original formula shown above. We refer to Nair et al. (2012) for the correct definition of the Govindarajulu distribution (including the same distribution with shifted support).

We adopt the generalized exponential prior with hyperparameters α = 0.59012 and λ = 1, used by Nair et al. (2020) for the parameter γ of the Govindarajulu distribution. The CDF and PDF of the generalized exponential distribution are given by

$$\begin{aligned} F(x|\lambda,\alpha) &= (1-e^{-\lambda x})^{\alpha} \\ f(x|\lambda,\alpha) &= \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}, \end{aligned} \tag{15}$$

where $\alpha, \lambda > 0$. The support of the distribution is $x \in [0,\infty]$. The quantile function and the quantile density of this distribution are:

$$Q(u|\lambda,\alpha) = \frac{1}{\lambda}[-\ln(1-u^{1/\alpha})] q(u|\lambda,\alpha) = \frac{u^{(1/\alpha)-1}}{\alpha\lambda(1-u^{1/\alpha})}, \tag{16}$$

where it is visible that Generalized Exponential distribution is a p-transformed exponential distribution (Gilchrist, 2000) with the scale parameter λ, and the shape parameter α.

Nair et al. (2020) estimated the σ parameter of the Govindarajulu distribution using L-moments and assumed it to be known and equal to 93.463 for this problem, which we adopt it as a fixed parameter, as well.

Table 2
Summary of the posterior samples from the GenExp-Govindarajulu model.

parameter	mean	median	q5	q95	rhat
gamma	2.132	2.1	1.638	2.73	1.001

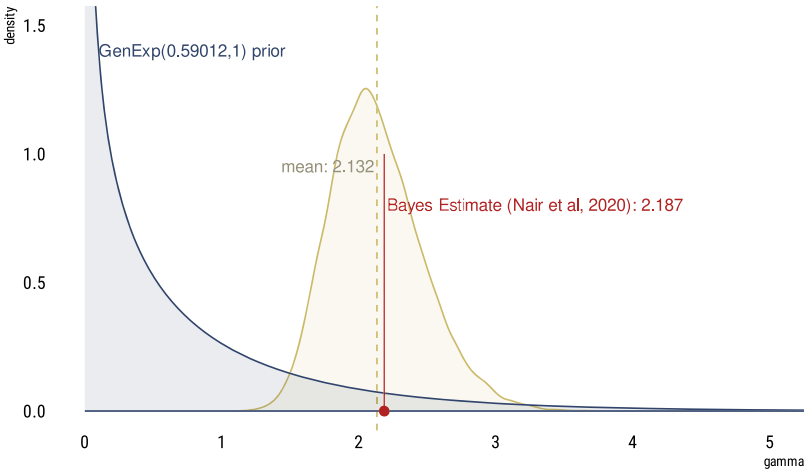

Fig. 4. Prior and posterior distributions of the parameter 'gamma' in the GenExp-Govindarajulu model.

As discussed Section 2.3, since Govindarajulu distribution does not have a closed-form CDF, it would be inappropriate to write $Y \sim \text{Govindarajulu}(\sigma, \gamma)$. Instead, our proposed notation highlights that the Govindarajulu distribution is defined via the QF, and it is, therefore, the degenerate random variate u that is inversely distributed according to this U-shaped distribution and not the observation y itself.

$$u \overset{y}{\backsim} \text{Govindarajulu}(\sigma, \gamma). \tag{17}$$

This notation also indicates the need to invert the QF to compute the random variate u corresponding to the observations y given the values of parameters σ, γ.

The resulting model takes the form

$$\begin{aligned} u &\overset{y}{\backsim} \text{Govindarajulu}(93.463, \gamma) \\ \gamma &\sim \text{GenExp}(1, 0.59012). \end{aligned} \tag{18}$$

The GenExp-Govindarajulu model has been validated using the Simulation-Based Calibration (Cook et al., 2006; Modrák et al., 2022; Talts et al., 2020). As evident from the diagnostic plots in the Supplementary Materials, the model parameters are successfully recovered for all widths of the posterior credible intervals.

We ran 2500 post-warmup iterations and 4 chains in Stan (Gabry and Češnovar, 2022). Table 2 summarizes the posterior distribution of the parameter γ of Govindarajulu distribution. We compare the prior and the posterior distribution in Fig. 4 and include the Bayes estimate by Nair et al. (2020), noting that the variation in the results could be attributed to the minor discrepancy in the quantile density formula between Nair et al. (2012) and Nair et al. (2020). The Stan code for this model can be found in the Supplementary Materials.

5.2. Parametric quantile regression

Quantile functions are useful not only for modeling the observables, but they can also be used to represent unobserved quantities of interest, such as the error term in a parametric quantile regression.

Using Gilchrist's Linear Combination rule in Table 1 any quantile function can be represented as

$$Q(u|\mu, \sigma, \theta) = \mu + \sigma\, Q_s(u|\theta), \tag{19}$$

where $Q_s(u)$ is a "basic" quantile function (Gilchrist, 2000), μ and σ are location and scale parameters, respectively, and θ is an optional shape parameter. Many quantile functions, such as logistic $Q(u) = \mu + \sigma \text{logit(u)}$ or normal $Q(u) = \mu + \sigma\, \Phi^{-1}(u)$, are already in this form. Others, such as the SLD discussed in Section 2, can have location and scale parameters added to them to enable shifted and scaled support, e.g. $Q_(u|\mu, \sigma, \delta) = \mu + \sigma\,[(1-\delta)\ln(u) - \delta \ln(1-u)]$. The basic quantile functions (i.e. $\text{logit}(u)$ for the logistic and $\Phi^{-1}(u)$ for the normal) can be useful as the building blocks for constructing more complex

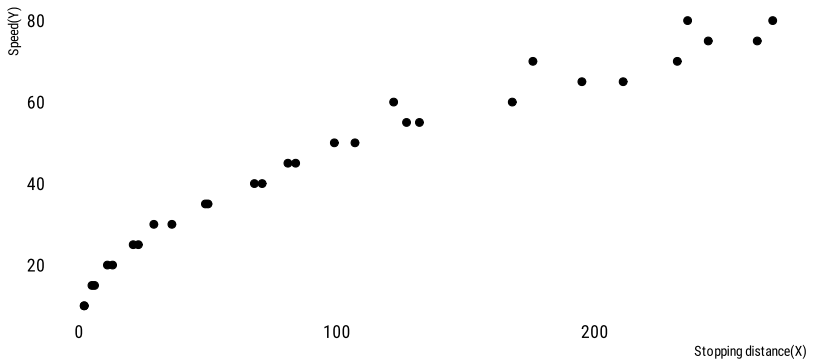

Fig. 5. Car stopping distance dataset.

distributions (Table 1). Basic quantile functions with the median centered at zero are called *standard* quantile functions (Gilchrist, 2000) denoted here as $S(u|\theta) = Q_s(u|\theta) - Q_s(0.5|\theta)$.

A simple linear regression of a random variable Y given the covariate X can be written as

$$y_i = \underbrace{\alpha + \beta x_i}_{\text{deterministic term}} + \underbrace{\varepsilon_i}_{\text{stochastic term}}, \tag{20}$$

where y_i is the i-th observation of Y, x_i is i-th observation of covariate X, α and β are unknown intercept and slope, respectively. The ε_i represents the error, which in ordinary least squares (OLS) regression is assumed (or forced through the link function) to be normally distributed with the mean of zero. An alternative way of representing the error term is through a *standard quantile function* $\varepsilon_i = S_\varepsilon(u_i|\theta)$, where u_i is the depth corresponding to the error ε_i in the regression model with the intercept α, slope β and the shape parameter θ (which are assumed to be independent).

$$y_i = \alpha + \beta x_i + S_\varepsilon(u_i|\theta), \tag{21}$$

We want to emphasize that the *traditional* quantile regression introduced by Koenker (2005) is in essence semi-parametric, because it does not require the user "to specify the distribution of the error term as it is allowed to take any form" (Yu and Moyeed, 2001). The regression Equation (21) represents the *parametric quantile regression* (PQR), because in this type of regression the error term is modeled explicitly (Gilchrist, 2008; Sharma and Chakrabarty, 2020; Su, 2015; Dean and King, 2009; Muraleedharan et al., 2016; Perri and Tarsitano, 2007, 2008).

Note that the deterministic term in (21) can be viewed as a location parameter in the quantile function

$$Q_Y(u_i|\mu_i, \theta) = \mu_i + S_\varepsilon(u_i|\theta), \tag{22}$$

where $\mu_i = \alpha + \beta x_i$. Likewise, if the stochastic component $S_\varepsilon(u_i|\theta)$ is made dependent on the covariate x_i, the resulting PQR QF can capture the heteroscedasticity of the error term.

The depth u_i can be found by inverting the quantile function $u_i \approxeq \widehat{Q_Y^{-1}}(y_i|\mu_i, \theta)$. In cases where inverting the PQR QF may be analytically difficult (e.g. when the $S_\varepsilon(u_i, \theta)$ is not invertible), the numerical approximation can be used (see Section 4 above). Once the depths $\underline{u} = \{u_1, u_2, \ldots, u_n\}$ are found, the likelihood of N observations $\underline{y} = \{y_1, y_2, \ldots, y_n\}$ given parameter θ can be calculated using the density quantile function corresponding to the PQR QF.

Because the deterministic term μ_i in PQR QF $Q_Y(u_i|\mu_i, \theta)$ is additive and does not depend on the depth u_i it can be dropped from the derivative.

$$[q_Y(u_i|\mu_i, \theta)]^{-1} = \left[\frac{dQ_Y(u_i|\mu_i, \theta)}{du}\right]^{-1} = \left[\frac{dS_\varepsilon(u_i|\theta)}{du}\right]^{-1} = [q_\varepsilon(u_i|\theta)]^{-1}, \tag{23}$$

where $[q_\varepsilon(u_i|\theta)]^{-1}$ is the *density quantile function* of the error term.

We illustrate the application of PQR using the car stopping distance data from Gilchrist (2000), sec. 12.4. The dataset (Fig. 5) contains 30 observations of the car speed and the corresponding stopping distances. As suggested by the physics' kinetic energy equation (Lutus, 2021) the speed of the car is proportional to the square root of the braking distance. We can draw a *mean regression line* through the observations, as shown in Fig. 5 relating the car to the square root of the stopping distance using ordinary least squares (OLS). In the rest of this section we will estimate the *quantile regression lines* for the *median*, the 5th, and the 95th quantile using the PQR.

One of the simplest quantile functions which could be used to model the error in PQR is the logistic quantile function $Q(u) = \ln(u) - \ln(1-u)$. The distribution of the errors in the stopping distance model might be less "peaked" than the standard logistic distribution due to various factors (about vehicles or the drivers) not included in the model. Therefore,

adding some effect of the standard uniform quantile function u (Lampasi, 2008) might be reasonable. Flattened Logistic Distribution (FLD) described by Sharma and Chakrabarty (2019) combines the standard QFs of logistic and uniform distributions by applying positive affine transformation for scale and shape parameters (ref. Addition and Linear Combination rules in Table 1).

$$Q_\varepsilon(u|\chi,\eta,\kappa) = \chi + \eta\left[\underbrace{\ln(u) - \ln(1-u)}_{\text{logistic}} + \kappa \times \underbrace{u}_{\text{uniform}}\right], \tag{24}$$

where χ is the location parameter, $\eta, \kappa > 0$ are scale and shape parameters, respectively. For the standard quantile function $S_\varepsilon(u|\kappa) = Q_\varepsilon(u|\kappa) - Q_\varepsilon(0.5|\kappa)$, the location should be set to 0 and the scale set to 1.

The Flattened Logistic Distribution is symmetrical. This assumption might be too restrictive for modeling the residuals in the car stopping distance model (e.g. because of inertia). Sharma and Chakrabarty (2020) replaced the logistic quantile function in the FLD with the skew-logistic quantile function; the resulting QF can be referred to as the flattened skew-logistic distribution (FSLD).

The FSLD QF and the DQF are

$$\begin{aligned} Q_\varepsilon(u|\chi,\eta,\delta,\kappa) &= \chi + \eta\left(\underbrace{(1-\delta)\ln(u) - \delta\ln(1-u)}_{\text{skew-logistic}} + \kappa \times \underbrace{u}_{\text{uniform}}\right) \\ [q_\varepsilon(u|\chi,\eta,\delta,\kappa)]^{-1} &= \left[\eta\left(\frac{1-\delta}{u} + \frac{\delta}{1-u} + \kappa\right)\right]^{-1}. \end{aligned} \tag{25}$$

Since the variance in the speed Y increases with the car stopping distance X, a heteroscedastic model can be used to describe the error term in the PQR for the stopping distances. The resulting PQR QF and the corresponding DQF can be expressed as

$$\begin{aligned} Q_Y(u|\alpha,\beta,\theta;x) &= \alpha + \beta\sqrt{x} + S_\varepsilon(u;\theta)\sqrt{x} \\ [q_Y(u|\theta;x)]^{-1} &= \left[\frac{dQ_Y(u|\alpha,\beta,\theta;x)}{du}\right]^{-1} = \frac{1}{\sqrt{x}}[q_\varepsilon(u;\theta)]^{-1}, \end{aligned} \tag{26}$$

where α, β are intercept and slope, $\theta = \{\eta, \delta, \kappa\}$ represent the parameters of the standard flattened logistic distribution $S_\varepsilon(u;\theta)$ with the density quantile function $[q_\varepsilon(u|\theta)]^{-1}$, u is the *depth* corresponding to the error in the model for the speed y given the stopping distance x and the regression parameters $\{\alpha,\beta,\theta\}$. The depth u can be computed by inverting the PQR QF $u \approxeq \widehat{Q_Y^{-1}}(y|\alpha,\beta,\theta;x)$ (26).

For each of the n observations of speed in the sample $\underline{Y} = \{y_1, y_2, \ldots, y_n\}$ we can compute $\underline{Q_Y} = \{Q_Y(u_1|\alpha,\beta,\theta,x_1), \ldots, Q_Y(u_n|\alpha,\beta,\theta,x_n)\}$, such that $u_i \approxeq \widehat{Q_Y^{-1}}(y_i|\alpha,\beta,\theta,x_i)$, $i = \{1,2,\ldots n\}$.

Let's further assume that the expert's prior belief about the intercept was elicited using a set of quantile-probability pairs and the best fit was achieved using the FLD quantile function with hyperparameters $\chi = 1, \eta = 1$, and $\kappa = 10$. Similarly, the expert belief about the slope is described by the FSLD with hyperparameters $\chi = 2, \eta = 2, \delta = 0.8$, and $\kappa = 2$.

Since FLD and FSLD are quantile distributions, the prior for the parameters α and β of PQR must be defined in the quantile form. This means that the density quantile functions $f(Q_\alpha(v))$, $f(Q_\beta(w))$ and the Jacobian adjustments $|q_\alpha(v)|$, $|q_\beta(w)|$ can be dropped, as explained in Section 3.2 above.

$$\begin{aligned} f(Q_\alpha(v), Q_\beta(w), \theta|\underline{Q_Y}, x) \propto \mathcal{L}(\theta;\underline{Q_Y}, x) f(Q_\alpha(v))|q_\alpha(v)| f(Q_\beta(w))|q_\beta(w)| f(\theta) \implies \\ f(Q_\alpha(v), Q_\beta(w), \theta|\underline{Q_Y}, x) \propto \mathcal{L}(\theta;\underline{Q_Y}, x) f(\theta). \end{aligned} \tag{27}$$

Therefore, the posterior distribution of the PQR parameters $\alpha = Q_\alpha(v), \beta = Q_\beta(w)$, and θ can be expressed using the *quantile-based likelihood* (and the *quantile-based prior* for parameters α and β).

Table 3
Summary of the posterior samples from the FSLD PQR model.

parameter	mean	median	q5	q95	rhat
v	0.3452	0.3458	0.3224	0.3632	1.012
Q(v)	3.8107	3.8201	3.4817	4.0702	1.012
w	0.4599	0.4598	0.4485	0.4715	1.013
Q(w)	4.5148	4.5133	4.4254	4.6059	1.013
eta	0.2691	0.2625	0.1937	0.3655	1.014
k	0.1142	0.0804	0.0063	0.3171	1.057
dlt	0.8065	0.8329	0.5766	0.9426	1.084

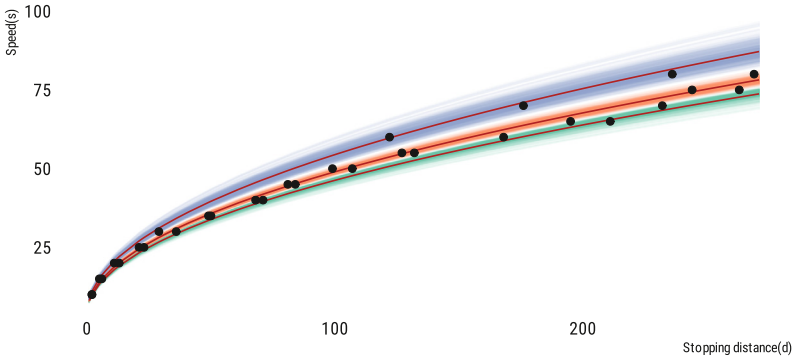

Fig. 6. Posterior predictive quantiles (0.05, 0.5, 0.95) for stopping distances. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$
\begin{aligned}
f(\alpha,\beta,\theta|\underline{Q_Y},\underline{x}) &\propto \mathcal{L}(\theta;\underline{Q_Y},\alpha,\beta,\theta,\underline{x})f(\theta)\\
\mathcal{L}(\theta;\underline{Q_Y},\alpha,\beta,\theta,\underline{x}) &= \prod_{i=1}^{n} f(Q_Y(u_i|\alpha,\beta,\theta;x_i)) = \prod_{i=1}^{n}\left[q_\varepsilon(u_i|\theta)\sqrt{x_i}\right]^{-1}\\
Q_Y(\underline{u}|\alpha,\beta,\theta;\underline{x}) &= \alpha+\beta\sqrt{\underline{x}}+S_\varepsilon(\underline{u};\theta)\sqrt{\underline{x}}\\
u &\overset{y}{\curvearrowleft} Q_Y(\alpha,\beta,\theta;\underline{x})\\
v &\overset{\alpha}{\curvearrowleft} \mathrm{FLD}(1,1,10)\\
w &\overset{\beta}{\curvearrowleft} \mathrm{FSLD}(2,2,0.8,2)\\
\eta &\sim \mathrm{Exp}(1/10)\\
\delta &\sim \mathrm{Beta}(2,1)\\
\kappa &\sim \mathrm{Exp}(1/0.1),
\end{aligned}
\tag{28}
$$

where $f(\theta) = f(\eta)f(\delta)f(\kappa)$. Note that, as we discussed in Section 2.3, the Parametric Regression Quantile Function $Q_Y(\underline{u}|\alpha,\beta,\theta;\underline{x})$ is not invertible and therefore it would be inappropriate to write $Y \sim Q_Y(\underline{u}|\alpha,\beta,\theta;\underline{x})$. Instead, we indicate $u \overset{y}{\curvearrowleft} Q_Y(\alpha,\beta,\theta;\underline{x})$, which means that the likelihood is defined via the QF and needs to be inverted to find the random variate u corresponding to observations y. This notation also helps distinguish between the random variate used for likelihood (u) and those used by the quantile-based priors (v and w for the parameters α and β, respectively).

The PQR model has been validated using the Simulation-Based Calibration (Cook et al., 2006; Modrák et al., 2022; Talts et al., 2020) in Stan. The diagnostic plots provided in the Supplementary Materials, show that the PQR model parameters are successfully recovered for all widths of the posterior credible intervals.

We ran 2500 post-warmup iterations and 4 chains using the Robust Adaptive Metropolis algorithm by Vihola (2012) implemented in `fmcmc` package (Vega Yon and Marjoram, 2019) in R (R Core Team, 2021). The code is provided in the Supplementary Materials.

Table 3 summarizes the posterior distribution of the parameters in the parametric quantile regression model for the car stopping distances.

Posterior predictive check (Gabry et al., 2019) can be done by generating a grid of values for the car stopping distances x and using randomly sampled parameters from the posterior distribution to compute the value of the response y using the PQR QF. Since in the PQR the regression equation is expressed in terms of the depth u we can extract the coherent (non-crossing) quantile regression lines for any set of fractiles. Fig. 6 illustrates hypothetical outcome plots for the 5th, 50th, and 95th quantile regression lines. The solid red lines are the conditional mean curves, representing the respective predictive quantiles.

In order to assess the empirical goodness of fit, we calculated the proportion of data points falling below the 5th, 50th, and 95th predictive quantile. Out of $n = 30$ observations, 93% of observations fell inside the conditional 95% posterior predictive interval (shown as the outer solid red lines on the plot), while 57% of observations turned out below the predictive median curve.

6. Discussion and conclusion

In the past 20 years, many examples of using quantile distributions for the approximate Bayesian computation (ABC) appeared in the literature (Allingham et al., 2009; Drovandi and Pettitt, 2011; Dunson and Taylor, 2005; McVinish, 2012; Smithson and Shou, 2017). ABC methods normally do not require computation of the likelihood, which, in case of the quantile distributions, is convenient, as these distributions lack an explicit CDF and PDF.

Regardless of whether the distribution is defined by the CDF of the QF, the defining function sometimes needs to be inverted. If the inverse does not exist in closed form, the function has to be inverted numerically. In the case of the *density-based likelihood*, the inverse distribution function may be needed for sampling from the posterior (e.g. for the posterior predictive check). In the case of the *quantile-based likelihood*, the inverse is needed for computing the intermediate depth values, corresponding to observations (conditional on covariates) for every draw of the parameters. No numerical inversion of the quantile function is needed for defining the *quantile-based prior*. A wide selection of efficient root-finding algorithm implementations in the popular statistical software makes the inversion of custom quantile functions accessible. We provide a generic wrapper for inverting arbitrary quantile functions using Brent method in the accompanying R package (Perepolkin, 2019). Further research of custom root-finding algorithms for non-decreasing functions on unit-interval can make inverting of quantile function even more computationally efficient.

The *quantile-based inference* opens up a wide set of new distributions to serve as likelihood and/or prior in Bayesian models. Although many flexible *density-defined* distributions have been proposed in recent decades (Jones, 2015; Steel and Rubio, 2015), *quantile* distributions play an important role in certain field applications (Nair et al., 2013; Chalabi et al., 2012), as well as in expert knowledge elicitation and decision analysis (Mikkola et al., 2021; Hadlock, 2017; Powley, 2013). Besides, the flexibility offered by the distributions defined in terms of the quantile function (Gilchrist, 2007), and in particular their easily extensible nature (Table 1), allows ultimate freedom in expressing the expert-informed priors. In this paper we showed the connection of quantile parameter transformation to inverse transform sampling and used quantile distribution as a prior for regression parameters.

Multivariate versions of quantile distributions have been explored in the past (Field and Genton, 2006; Vineshkumar and Nair, 2019), but their adoption in the scientific literature remains low. One possibility of utilizing the flexibility of the quantile distributions in a multivariate setting is to employ them as marginal distributions for bivariate copulas, which can be assembled into higher-dimensional structures using vines (Czado, 2019; Kurowicka and Joe, 2011). When used as priors (Wilson, 2018), the copula structure can be elicited from the experts (Elfadaly and Garthwaite, 2017) along with the marginal quantile-probability pairs for fitting the quantile distribution (O'Hagan et al., 2006; Mikkola et al., 2021). Versatile and user-friendly multivariate quantile distributions represent an opportunity for further research.

Gilchrist (2007) provides a review of the traditional approach to quantile regression, as proposed by Koenker and Bassett (1978) and contrasts it with the *fully parametric approach* taken by PQR (Gilchrist, 2000, 2008; Su, 2015). The parametric approach to regression provides coherent (non-crossing) estimates of posterior quantiles, allowing the scientists to model the distribution of the error term explicitly (instead of making assumptions). Note that the parametric quantile regression may also be used with invertible distributions (logistic, normal, etc), as long they have computable QF and QDF ($\Phi^{-1}(u)$ and $\Phi^{-1}(u)/du$, for normal distribution).

Traditionally, the fitting of parameters in quantile distributions was performed using the matching of moments or L-moments (Gilchrist, 2008; Asquith, 2007; Karvanen and Nuutinen, 2008), matching of percentiles (Karian and Dudewicz, 2011), location and scale-free shape functionals (King and MacGillivray, 2007), distributional least squares/absolutes (Gilchrist, 2007; Sharma and Chakrabarty, 2020), and maximum likelihood (Rayner and MacGillivray, 2002; Su, 2007; Tarsitano, 2005). The various methods of obtaining parameter estimates for the quantile distributions have been extensively studied and compared, primarily in application to GλD (King and MacGillivray, 1999; Karian and Dudewicz, 2011; Fournier et al., 2007; Tarsitano, 2010), but also to some other distributions (Rayner and MacGillivray, 2002; Jeong-Soo, 2005). This paper generalizes the approach to *quantile-based likelihood* (Gilchrist, 2000; Rayner and MacGillivray, 2002; Haynes and Mengersen, 2005; Nair et al., 2020) connecting the previous research on parametric quantile regression (Gilchrist, 2008; Su, 2015; Sharma and Chakrabarty, 2020) with more recently introduced work on *quantile-based priors* (Nair et al., 2020) and implementing both of these concepts in Stan (Gabry and Češnovar, 2022) and R (Vega Yon and Marjoram, 2019).

Since the definition of quantile function is usually mathematically simpler (and more easily extendable) than the respective CDF and PDF (Gilchrist, 2000), *quantile-based priors* represent an inexpensive and flexible way of incorporating prior knowledge in Bayesian models. The unit sampling space may offer some additional computational advantage for MCMC/HMC algorithms. The quantile-based formulation of the prior may not be appropriate if the sampler constraints need to be defined on the parameter level (e.g. prior truncation). In such a case, the traditional *density-based prior* may be more useful.

Quantile-based likelihoods open a wide range of possibilities for designing flexible data generative models for special areas of application (e.g. Govindarajulu for reliability problems, Wakeby for modeling of floods, GLD in other instances) where the *density-based* equivalent is not available. The alternatives usually involve falling back to the non-Bayesian estimation

methods (Karian and Dudewicz, 2011; Asquith, 2007; Karvanen and Nuutinen, 2008) or using the approximate computation algorithms (Drovandi et al., 2011; Dunson and Taylor, 2005; McVinish, 2012), both of which are outside of the scope for this paper. The availability of efficient MCMC samplers (Gabry and Češnovar, 2022) and modern root-finding algorithms (Schäling, 2011), make quantile-based likelihood computationally feasible. Gilchrist (2000) writes: "The lack of use of maximum likelihood is surprising as it is perfectly straightforward if one uses the general-purpose maximization software available rather than look for specific formulae for estimators". We share his sentiment.

Embracing and expanding the use of quantile distributions in Bayesian analysis can enable new solutions for old problems and enrich the toolkit available to scientists for performing hard inference tasks. We hope that the *quantile-based inference* methods presented in this paper can contribute to the expanding body of knowledge about the use of quantile functions in Bayesian statistics and fuel further research in the area of quantile distributions.

Data availability

The `qpd` R package used in this paper is available on Github at https://github.com/dmi3kno/qpd. Contact corresponding author Dmytro Perepolkin (Dmytro.Perepolkin@cec.lu.se) for requests for data.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.csda.2023.107795.

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