# THE GENERALIZED DOUBLING METHOD: $(k, c)$ MODELS

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Abstract. One of the key ingredients in the recent construction of the generalized doubling method is a new class of models, called $(k, c)$ models, for local components of generalized Speh representations. We construct a family of $(k, c)$ representations, in a purely local setting, and discuss their realizations using inductive formulas. Our main result is a uniqueness theorem which is essential for the proof that the generalized doubling integral is Eulerian.

## Introduction

A model is a fundamental concept in representation theory and integral representations. Typically, a model for a representation arises from an equivariant functional, which allows one to realize the representation in a space of complex-valued functions with some natural geometric properties. The useful cases are when the functional is unique up to scaling, and the class of representations affording the model is broad. One important example is the Whittaker model, which has had a profound impact on the study of representations with a vast number of applications, perhaps most notably Shahidi's theory of local coefficients.

Let $F$ be a local field of characteristic 0. In this short note we discuss a new class of models, $(k, c)$ models, for representations of $\mathrm{GL}_{kc} = \mathrm{GL}_{kc}(F)$, which first appeared in the construction of the generalized doubling integral ([CFGK19]) in the context of generalized Speh representations. Our main result: Theorem 4, is that the local generalized Speh representation (defined in [Jac84]) of $\mathrm{GL}_{kc}$ corresponding to a unitary generic representation $\tau$ of $\mathrm{GL}_k$ admits a unique $(k, c)$ model. Our result is in fact stronger: We construct a map $\rho_c$ from irreducible generic representations $\tau$ of $\mathrm{GL}_k$ to $(k, c)$ representations $\rho_c(\tau)$. This general context is essential for the analysis of the local generalized doubling integrals ([CFGK19, GK]) when data are ramified or archimedean. We also discuss two realizations of the $(k, c)$ functional, which are also important for the study of such integrals.

The main application of Theorem 4 concerns the generalized doubling integrals. This theorem completes the proof that the global integral of [CFGK19] is Eulerian. See Corollary 5. In *loc. cit.* uniqueness was only proved when data are unramified, thus producing an "almost Eulerian" integral, i.e., only separating out the unramified places (cf. [CFGK19, (3.1)]). The existence of an Euler product is important for the development of the local theory, and Theorem 4 also plays a key role there, in the functional equation (see [GK]).

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## 1. Preliminaries

Let $F$ be a local field of characteristic 0. Identify linear algebraic $F$-groups with their $F$ points, i.e., $\mathrm{GL}_l = \mathrm{GL}_l(F)$. Fix the Borel subgroup $B_{\mathrm{GL}_l} = T_{\mathrm{GL}_l} \ltimes N_{\mathrm{GL}_l}$ of upper triangular invertible matrices in $\mathrm{GL}_l$, where $T_{\mathrm{GL}_l}$ is the diagonal torus. For a $d$ parts composition $\beta = (\beta_1, \ldots, \beta_d)$ of $l$, $P_\beta = M_\beta \ltimes V_\beta$ denotes the corresponding standard parabolic subgroup, where $V_\beta < N_{\mathrm{GL}_l}$. The unipotent subgroup opposite to $V_\beta$ is denoted $V_\beta^-$ and $\delta_{P_\beta}$ is the modulus character of $P_\beta$. For an integer $c \geq 0$, $\beta c = (\beta_1 c, \ldots, \beta_d c)$ is a composition of $lc$. Let $w_\beta$ be the permutation matrix consisting of blocks of identity matrices $I_{\beta_1}, \ldots, I_{\beta_d}$, with $I_{\beta_i} \in \mathrm{GL}_{\beta_i}$ on its anti-diagonal, beginning with $I_{\beta_1}$ on the top right, then $I_{\beta_2}$, etc. E.g., $w_{(a,b)} = \left(\begin{smallmatrix} & I_a \\ I_b & \end{smallmatrix}\right)$. We use $\tau_\beta$ to denote a representation of $M_\beta$, where $\tau_\beta = \otimes_{i=1}^d \tau_i$ ($\tau_i$ is then a representation of $\mathrm{GL}_{\beta_i}$). Let $\mathrm{Mat}_{a\times b}$ and $\mathrm{Mat}_a$ denote the spaces of $a\times b$ or $a\times a$ matrices. For $g \in \mathrm{Mat}_{a\times b}$, ${}^t g$ is the transpose of $g$. The trace map is denoted tr. For $x, y \in \mathrm{GL}_l$, ${}^x y = xyx^{-1}$ and if $Y < \mathrm{GL}_l$, ${}^x Y = \{{}^x y : y \in Y\}$.

All representations in this work are by definition complex and smooth. A generic representation of $\mathrm{GL}_l$ will be admissible, by definition. Over archimedean fields, by an admissible representation we mean admissible Fréchet of moderate growth. We use the smooth and normalized induction functor.

Let $U < R < \mathrm{GL}_l$ be closed subgroups such that $U$ is a unipotent subgroup, and fix a character $\psi$ of $U$. For a representation $\sigma$ of $R$ on a space $\mathcal{V}$, the Jacquet module $J_{U,\psi}(\pi)$ is the quotient of $\mathcal{V}$ by the subspace spanned by $\{\pi(u)\xi - \psi(u)\xi : \xi \in \mathcal{V}, u \in U\}$ over $p$-adic fields, and by the closure of this subspace for archimedean fields. Then $J_{U,\psi}(\pi)$ is a representation of $R$ and we normalize the action as in [BZ77, 1.8].

When the field is $p$-adic, an entire (resp., meromorphic) function $f(\zeta_1, \ldots, \zeta_m) : \mathbb{C}^m \to \mathbb{C}$ will always be an element of $\mathbb{C}[q^{\mp\zeta_1}, \ldots, q^{\mp\zeta_m}]$, (resp., $\mathbb{C}(q^{-\zeta_1}, \ldots, q^{-\zeta_m})$).

## 2. Representations of type $(k, c)$

2.1. **Definition.** Let $k, c \geq 1$ be integers. A partition $\sigma = (a_1, \ldots, a_l)$ of $kc$ such that $a_i > 0$ for all $i$ identifies a subgroup $V(\sigma) < N_{\mathrm{GL}_{kc}}$ as follows. Consider the multi-set of integers $\Lambda_\sigma = \{a_i - 2j + 1 : 1 \leq i \leq l, 1 \leq j \leq a_i\}$ and let $p_\sigma$ be the $kc$-tuple obtained by arranging $\Lambda_\sigma$ in decreasing order. For any $x \in F^*$, put $x^{p_\sigma} = \mathrm{diag}(x^{p_\sigma(1)}, \ldots, x^{p_\sigma(kc)}) \in T_{\mathrm{GL}_{kc}}$. The one-parameter subgroup $\{x^{p_\sigma} : x \in F^*\}$ acts on the Lie algebra of $N_{\mathrm{GL}_{kc}}$ by conjugation, and $V(\sigma)$ is the subgroup generated by the weight subspaces of weight at least 2. (This is not the subgroup $V_\beta$ defined for compositions). Let $G_\sigma < \mathrm{GL}_{kc}$ denote the centralizer of $\{x^{p_\sigma} : x \in F^*\}$, it acts on the set of characters of $V(\sigma)$. Under this action there is a unique open orbit $\mathcal{O}$ over $\mathbb{C}$, let $\psi_o \in \mathcal{O}$. A character $\psi'$ of $V(\sigma)$ is called generic if its stabilizer $G_{\sigma,\psi}$ in $G_\sigma$ is of the same type as $G_{\sigma,\psi_o}$ over $\mathbb{C}$. For further details see [Gin06], [CM93, § 5] and [Car93]. Let $\widehat{V}(\sigma)_{\mathrm{gen}}$ denote the set of generic characters of $V(\sigma)$. If $\sigma'$ is another partition of $kc$, write $\sigma' \gtrsim \sigma$ if $\sigma'$ is greater than or non-comparable with $\sigma$ under the natural partial ordering.

For $\sigma = (k^c)$, $V(\sigma) = V_{(c^k)}$ and $M_{(c^k)}$ acts transitively on $\widehat{V}(\sigma)_{\mathrm{gen}}$. We fix $\psi \in \widehat{V}(\sigma)_{\mathrm{gen}}$ by taking a nontrivial additive character $\psi$ of $F$ and extending it to a character of $V_{(c^k)}$ by

$$\psi(v) = \psi(\sum_{i=1}^{k-1} \mathrm{tr}(v_{i,i+1})), \qquad v = (v_{i,j})_{1\leq i,j\leq k}, \qquad v_{i,j} \in \mathrm{Mat}_c. \tag{2.1}$$

Then $G_{\sigma,\psi} = \mathrm{GL}_c^{\triangle}$, the diagonal embedding of $\mathrm{GL}_c$ in $M_{(c^k)}$.

**Definition 1.** *An admissible representation $\rho$ of $\mathrm{GL}_{kc}$ is of type $(k,c)$ (briefly, $\rho$ is $(k,c)$), if $\mathrm{Hom}_{V(\sigma)}(\rho,\psi')=0$ for all $\sigma \gtrsim (k^c)$ and $\psi' \in \widehat{V}(\sigma)_{\mathrm{gen}}$, and $\dim \mathrm{Hom}_{V_{(c^k)}}(\rho,\psi)=1$.*

We say that an admissible representation $\rho$ of $\mathrm{GL}_{kc}$ is $m$-weakly $(k,c)$, if it satisfies the vanishing condition of Definition 1 and $m=\dim \mathrm{Hom}_{V_{(c^k)}}(\rho,\psi)$ satisfies $1\le m<\infty$. Also note that $\rho$ is $(1,c)$ if and only if it is a character of $\mathrm{GL}_c$.

A $(k,c)$ functional on $\rho$ (with respect to $\psi$) is a nonzero element of $\mathrm{Hom}_{V_{(c^k)}}(\rho,\psi)$. If $\rho$ is of type $(k,c)$, the space of such functionals is one dimensional. The resulting model (which is unique by definition) is called a $(k,c)$ model, and denoted $W_\psi(\rho)$. If $\lambda$ is a fixed $(k,c)$ functional, $W_\psi(\rho)$ is the space of functions $g\mapsto \lambda(\rho(g)\xi)$ where $g\in \mathrm{GL}_{kc}$ and $\xi$ is a vector in the space of $\rho$. Then $W_\psi(\rho)$ is a quotient of $\rho$, and when $\rho$ is irreducible $W_\psi(\rho)\cong\rho$.

The following is an heredity-type result for $(k,c)$ representations.

**Proposition 2.** *For $1\le i\le d$, let $\rho_i$ be a $(k_i,c)$ representation. If $F$ is archimedean we further assume $k_1=\ldots=k_d=1$ and for each $i$, $\rho_i=\tau_i\circ\det$ for a quasi-character $\tau_i$ of $F^*$ and $\det$ defined on $\mathrm{GL}_c$. Then $\rho=\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d\rho_i)$ is of type $(k,c)$, where $\beta=(k_1,\ldots,k_d)$ and $k=k_1+\ldots+k_d$ (if $F$ is archimedean, $k=d$ and $\beta=(1^k)$). In the non-archimedean case if each $\rho_i$ is $m_i$-weakly $(k_i,c)$, $\rho$ is $\prod_i^d m_i$-weakly $(k,c)$.*

*Proof.* We need to prove $\mathrm{Hom}_{V(\beta')}(\rho,\psi')=0$ for any partition $\beta'\gtrsim(k^c)$ and character $\psi'\in\widehat{V}(\beta')_{\mathrm{gen}}$, and $\dim\mathrm{Hom}_{V_{(c^k)}}(\rho,\psi)=1$. We consider $m_i$-weakly representations at the end of the proof.

We will use the theory of detivatives of Bernstein and Zelevinsky [BZ76, BZ77] over $p$-adic fields, its partial extension to archimedean fields by Aizenbud *et. al.* [AGS15a, AGS15b], and the relation between derivatives and degenerate Whittaker models developed (over both fields) by Gomez *et. al.* [GGS17]. Let $P_l$ be the subgroup of matrices $g\in\mathrm{GL}_l$ with the last row $(0,\ldots,0,1)$ ($P_l<P_{(l-1,1)}$) and let $\psi_l$ be the character of $V_{(l-1,1)}$ given by $\psi_l(\left(\begin{smallmatrix} I_{l-1} & v\\ & 1\end{smallmatrix}\right))=\psi(v_{l-1})$. Then we have the functor $\Phi^-$ from (smooth) representations of $P_l$ to representations of $P_{l-1}$ given by $\Phi^-(\varrho)=J_{V_{(l-1,1)},\psi_l}(\varrho)$. For $0<r\le l$, the $r$-th derivative of a representation $\varrho$ of $\mathrm{GL}_l$ is defined over $p$-adic fields by $\varrho^{(r)}=J_{V_{(l-1,1)}}((\Phi^-)^{r-1}(\varrho|_{P_l}))$, and over archimedean fields by $\varrho^{(r)}=((\Phi^-)^{r-1}(\varrho|_{P_l}))|_{\mathrm{GL}_{n-r}}$ (more precisely this is the pre-derivative, we use the term derivative for uniformity). Also $\varrho^{(0)}=\varrho$. The highest derivative of $\varrho$ is the representation $\varrho^{(r_0)}$ such that $\varrho^{(r_0)}\ne0$ and $\varrho^{(r)}=0$ for all $r>r_0$.

According to the definition of $(k_i,c)$ representations and [GGS17, Theorems E, F] (which also apply over $p$-adic fields), the highest derivative of $\rho_i$ is $\rho_i^{(k_i)}$. Then the highest derivative of $\rho$ is $\rho^{(k)}$ and $\rho^{(k)}=\mathrm{Ind}_{P_{\beta(c-1)}}^{\mathrm{GL}_{k(c-1)}}(\otimes_{i=1}^d\rho_i^{(k_i)})$, where over archimedean fields $\rho_i^{(k_i)}=\rho_i^{(1)}=\tau_i\circ\det$ and $\det$ is the determinant of $\mathrm{GL}_{c-1}$. In the $p$-adic case this follows from [BZ77, Lemma 4.5]; in the archimedean case this follows from [AGS15a, Corollary 2.4.4] and [AGS15b, Theorem B] (see also [GGS17, § 4.4]). Now the highest derivative of $\rho^{(k)}$ is again its $k$-th derivative and we can repeat this process $c$ times to obtain a one dimensional vector space. We conclude from [GGS17, Theorems E, F] that $\rho$ admits a unique $(k,c)$ model.

If $\lambda\gtrsim(k^c)$, we can assume $\lambda_1>k$ then $\rho^{(k+1)}=0$, by [BZ77, Lemma 4.5] and [AGS15b, Theorem B]. By [GGS17, Theorems E, F], this proves the required vanishing properties.

The only difference regarding $m_i$-weakly representations, is that after taking the highest derivative of $\rho$ for $c$ times, we obtain a $\prod_i m_i$ dimensional space. Then again by [GGS17, Theorems E, F], $\dim\mathrm{Hom}_{V_{(c^k)}}(\rho,\psi)=\prod_i m_i$. □

**Remark 3.** *The result for p-adic fields is stronger, because we have a general "Leibniz rule" for derivatives (*[BZ77, Lemma 4.5]*).*

2.2. **The representation $\rho_c(\tau)$.** Let $\tau$ be an irreducible generic representation of $\mathrm{GL}_k$ (for $k=1$, this means $\tau$ is a quasi-character of $F^*$). It is of type $(k,1)$, by the uniqueness of Whittaker models and because $(k)$ is the maximal unipotent orbit for $\mathrm{GL}_k$. For any $c$, we construct a $(k,c)$ representation $\rho_c(\tau)$ as follows. First assume $\tau$ is unitary. For $\zeta\in\mathbb{C}^c$, consider the intertwining operator

$$M(\zeta,w_{(k^c)}):\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^c|\det|^{\zeta_i}\tau)\to\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^c|\det|^{\zeta_{c-i+1}}\tau).$$

Given a section $\xi$ of $\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^c|\det|^{\zeta_i}\tau)$ which is a holomorphic function of $\zeta$, $M(\zeta,w_{(k^c)})\xi$ is defined for $\mathrm{Re}(\zeta)$ in a suitable cone by the absolutely convergent integral

$$M(\zeta,w_{(k^c)})\xi(\zeta,g)=\int\limits_{V_{(k^c)}}\xi(\zeta,w_{(k^c)}^{-1}vg)\,dv,$$

then by meromorphic continuation to $\mathbb{C}^c$. Let $\rho_c(\tau)$ be the image of this operator at

$$\zeta=((c-1)/2,(c-3)/2,\ldots,(1-c)/2).$$

Since $\tau$ is unitary, this image is well defined and irreducible by Jacquet [Jac84, Proposition 2.2] (see also [MW89, § I.11]) and $\rho_c(\tau)$ is the unique irreducible quotient of

$$\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}((\tau\otimes\ldots\otimes\tau)\delta_{P_{(k^c)}}^{1/(2k)}) \tag{2.2}$$

and the unique irreducible subrepresentation of

$$\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}((\tau\otimes\ldots\otimes\tau)\delta_{P_{(k^c)}}^{-1/(2k)}). \tag{2.3}$$

Now assume $\tau$ is an arbitrary irreducible generic representation of $\mathrm{GL}_k$. Then $\tau=\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^d|\det|^{a_i}\tau_i)$ where $\tau_i$ are tempered and $a_1>\ldots>a_d$ (by Langlands' classification and [Vog78, Zel80, JS83]). Define

$$\rho_c(\tau)=\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d|\det|^{a_i}\rho_c(\tau_i)). \tag{2.4}$$

Clearly $\rho_c(|\det|^{s_0}\tau)=|\det|^{s_0}\rho_c(\tau)$ for any $s_0\in\mathbb{C}$.

Note that when $\tau$ is unitary, the definition as the image of an intertwining operator agrees with the definition (2.4). Indeed in this case by Tadić [Tad86] and Vogan [Vog86] we have $\tau\cong\mathrm{Ind}_{P_{\beta'}}^{\mathrm{GL}_k}(\otimes_{i=1}^{d'}|\det|^{r_i}\tau_i')$, where $\tau_i'$ are square-integrable and $1/2>r_1\geq\ldots\geq r_{d'}>-1/2$. Then by [MW89, § I.11] we can permute the blocks in $\rho_c(\tau)$ corresponding to different representations $\tau_i'$ hence

$$\rho_c(\tau)\cong\mathrm{Ind}_{P_{\beta'c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{d'}|\det|^{r_i}\rho_c(\tau_i')). \tag{2.5}$$

In particular, e.g.,

$$\rho_c(\mathrm{Ind}_{P_{(\beta_1',\beta_2')}}^{\mathrm{GL}_{\beta_1'+\beta_2'}}(\tau_1'\otimes\tau_2'))\cong\mathrm{Ind}_{P_{(\beta_1'c,\beta_2'c)}}^{\mathrm{GL}_{(\beta_1'+\beta_2')c}}(\rho_c(\tau_1')\otimes\rho_c(\tau_2')), \tag{2.6}$$

where the l.h.s. (left-hand side) is defined as the image of the intertwining operator. Hence if $a_1>\ldots>a_d$ are the $d\leq d'$ distinct numbers among $r_1,\ldots,r_{d'}$, $\tau_i$ is the tempered representation parabolically induced from $\otimes_{1\leq j\leq d':r_j=a_i}\tau_j'$ and $\beta=(a_1,\ldots,a_d)$,

$$\rho_c(\tau)\cong\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d|\det|^{a_i}\rho_c(\tau_i)),$$

which agrees with (2.4). Moreover, (2.6) implies that (2.4) also holds when $a_1 \geq \ldots \geq a_d$.

For example, if $\tau$ is irreducible unramified tempered, $\tau = \mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\otimes_{i=1}^k \tau_i)$ for unramified unitary characters $\tau_i$ of $F^*$. By definition $\rho_c(\tau)$ is the unique irreducible unramified quotient of (2.2), but by [MW89, § I.11], $\rho_c(\tau) = \mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k \tau_i \circ \det)$.

We extend the definition to certain unramified principal series, which are not necessarily irreducible. Assume $\tau = \mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\otimes_{i=1}^k |\,|^{a_i}\tau_i)$ $(k > 1)$ where $\tau_i$ are as above (e.g., unitary) but $a_1 \geq \ldots \geq a_k$ ($\tau$ is not a general unramified principal series because of the order). In this case $\tau$ still admits a unique Whittaker functional. We define $\rho_c(\tau) = \mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k |\,|^{a_i}\tau_i \circ \det)$, which is a $(k,c)$ representation by Proposition 2. Transitivity of induction and the example in the last paragraph imply that this definition coincides with (2.4), when $\tau$ is irreducible (in which case the order of $a_i$ does not matter).

**Theorem 4.** *Let $\tau$ be an irreducible generic representation of $\mathrm{GL}_k$. Then $\rho_c(\tau)$ is $(k,c)$.*

*Proof.* First assume $F$ is non-archimedean. We start by proving the result for square-integrable representations $\tau$. By Zelevinsky [Zel80], $\tau$ can be described as the unique irreducible subrepresentation of

$$\mathrm{Ind}_{P_{(r^d)}}^{\mathrm{GL}_k}((\tau_0 \otimes \ldots \otimes \tau_0)\delta_{P_{(r^d)}}^{1/(2r)}), \tag{2.7}$$

where $\tau_0$ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_r$. Then by Tadić ([Tad87, § 6.1]) the highest Bernstein–Zelevinsky derivative of $\rho_c(\tau)$ is its $k$-th derivative and equals $|\det|^{-1/2}\rho_{c-1}(\tau)$ (see also [Zel80, Tad86] and [LM14, Theorem 14]). Repeatedly taking highest derivatives, we see that $\rho_c(\tau)$ is supported on $(k^c)$. The uniqueness of the functional follows as in the proof of Proposition 2. The proof for an arbitrary (irreducible generic) $\tau$ now follows from (2.4) and Proposition 2.

Now consider an archimedean $F$, and an irreducible generic $\tau$. For a smooth representation $\vartheta$ of $\mathrm{GL}_l$ let $\mathcal{V}(\vartheta)$ denote its annihilator variety and $WF(\vartheta)$ be its wave-front set (see e.g., [GS13] for these notions). According to the results of Vogan [Vog91] and Schmid and Vilonen [SV00], $\mathcal{V}(\vartheta)$ is the Zariski closure of $WF(\vartheta)$ (see [GGS17, § 3.3.1]). Applying this to $\vartheta = \rho_c(\tau)$, by [GS13, Corollary 2.1.8] and [SS90, Theorem 3] (see [GS13, § 4.2]), $(k_1 + \ldots + k_d)^c = (k^c)$ is the maximal orbit in $WF(\rho_c(\tau))$. Note that this result holds although $\rho_c(\tau)$ may be reducible (associated cycles are additive, see e.g., [Vog91, p. 323]). Therefore $\mathrm{Hom}_{V_{(\beta')}}(\rho_c(\tau),\psi') = 0$ for all $\beta'$ greater than or not comparable with $(k^c)$ and generic character $\psi'$ of $V_{(\beta')}$, and also [GGS17, Theorem E] implies that $\rho_c(\tau)$ admits a $(k,c)$ functional. It remains to show $\dim \mathrm{Hom}_{V_{(\beta')}}(\rho_c(\tau),\psi') \leq 1$.

To this end, by [BSS90, SS90] (see also [GS13, Corollary 4.2.5]), each $\rho_c(\tau_i)$ with $\beta_i > 1$ is a quotient of $\mathrm{Ind}_{P_{(c^2)}}^{\mathrm{GL}_{2c}}(\rho_c(\chi_1) \otimes \rho_c(\chi_2))$ for suitable quasi-characters $\chi_1, \chi_2$ of $F^*$, hence $\rho_c(\tau)$ itself is also a quotient of a degenerate principal series $\mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k \rho_c(\chi_i))$. The latter is $(k,c)$ by Proposition 2, thus $\dim \mathrm{Hom}_{V_{(\beta')}}(\rho_c(\tau),\psi') \leq 1$. We deduce that $\rho_c(\tau)$ is $(k,c)$. □

**Corollary 5.** *The global generalized doubling integrals of [CFGK19, GK] are Eulerian, for decomposable data.*

*Proof.* The proof follows from Theorem 4 and the results of [CFGK19, GK]. In order to provide some details, we switch, in this proof alone, to a global setting. Let $F$ be a global number field with a ring of adeles $\mathbb{A}$ and fix a nontrivial additive character $\psi$ of $F\backslash\mathbb{A}$.

Let $\tau$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$, and denote the generalized Speh representation of $\mathrm{GL}_{kc}(\mathbb{A})$ corresponding to $\tau$ by $\rho_c(\tau)$ ([Jac84]). A global $(k,c)$ functional on $\rho_c(\tau)$ is given by $\Lambda(\xi) = \int_{V_{(c^k)}(F)\backslash V_{(c^k)}(\mathbb{A})} \xi(v)\,\psi^{-1}(v)\,dv$, where $\xi$ is an automorphic form in the space of $\rho_c(\tau)$ and $\psi$ is defined by (2.1). By Theorem 4 this functional is Eulerian: One can choose for each place $\nu$ of $F$ a local $(k,c)$ functional $\lambda_\nu$ on $\rho_c(\tau_\nu)$, such that for any decomposable vector $\xi = \otimes_\nu \xi_\nu$, $\Lambda(\xi) = \prod_\nu \lambda_\nu(\xi_\nu)$.

The generalized doubling integral was defined in [CFGK19, GK] for several reductive groups $G$. Since the details of the construction are similar, we take $G = \mathrm{SO}_c$. Define $H = \mathrm{SO}_{2kc}$. Fix a Siegel parabolic subgroup $P < H$ and a maximal compact subgroup $K < H$. Let $E(h;s,f)$ be the Eisenstein series attached to a standard $K$-finite section $f$ of $\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(|\det|^{s-1/2}\rho_c(\tau))$, $s \in \mathbb{C}$. One can choose a unipotent subgroup $U < H$ and a generic character $\psi_U$ of $U(F)\backslash U(\mathbb{A})$, such that the Fourier coefficient $E^{U,\psi_U}$ of the series along $(U,\psi_U)$ is an automorphic form on $G(\mathbb{A}) \times G(\mathbb{A})$.

Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, and let $\varphi_1$ and $\varphi_2$ be two cusp forms in the space of $\pi$. The global integral is defined by

$$Z(s,\varphi_1,\varphi_2,f) = \int\limits_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} \varphi_1(g_1)\,\overline{{}^{\iota}\varphi_2(g_2)}\,E^{U,\psi_U}((g_1,g_2);s,f)\,dg_1\,dg_2,$$

where $g \mapsto {}^{\iota}g$ is an involution of $G(\mathbb{A})$ and $(g_1,g_2)$ is the embedding of $G \times G$ in $H$. By [GK, § 3.2 and (3.8)] (see also [CFGK19, Theorem 1]), for $\mathrm{Re}(s) \gg 0$ we have

$$Z(s,\varphi_1,\varphi_2,f) = \int\limits_{G(\mathbb{A})}\int\limits_{U_0(\mathbb{A})} \langle \varphi_1, \pi(g)\varphi_2\rangle \Lambda \circ f(s, \delta u_0(I_c, {}^{\iota}g))\,\psi_U(u_0)\,du_0\,dg. \tag{2.8}$$

Here $U_0 < U$, $\langle , \rangle$ is the standard inner product and $\delta \in G(F)$. Consider decomposable vectors $f = \otimes_\nu f'_\nu$, $\varphi_1$ and $\varphi_2$. Then (by Theorem 4) $\Lambda \circ f = \prod_\nu f_\nu$ where for each $\nu$, $f_\nu = \lambda_\nu \circ f'_\nu$ belongs to the space of $\mathrm{Ind}_{P(F_\nu)}^{H(F_\nu)}(|\det|^{s-1/2}W_{\psi_\nu}(\rho_c(\tau_\nu)))$, and $\langle \varphi_1, \pi(g)\varphi_2\rangle = \prod_\nu \omega_\nu(g_\nu)$ where $\omega_\nu$ is a matrix coefficient of $\pi_\nu^\vee$. The local integral $Z(s,\omega_\nu,f_\nu)$ is of the form (2.8) but with local data and we integrate over $G(F_\nu)$ and $U_0(F_\nu)$. We obtain $Z(s,\varphi_1,\varphi_2,f) = \prod_\nu Z(s,\omega_\nu,f_\nu)$, as claimed. □

For an admissible representation $\varrho$ of $\mathrm{GL}_l$, let $\varrho^*(g) = \varrho(J_l{}^tg^{-1}J_l)$ where $J_l = w_{(1^l)}$. If $\varrho$ is irreducible, $\varrho^* \cong \varrho^\vee$.

**Claim 6.** *If $\tau$ is tempered, $\rho_c(\tau)^\vee = \rho_c(\tau^\vee)$. In general if $\tau$ is irreducible, $\rho_c(\tau)^* = \rho_c(\tau^\vee)$.*

*Proof.* The first assertion follows because $\rho_c(\tau)$ is a quotient of (2.2), hence both $\rho_c(\tau)^\vee$ and $\rho_c(\tau^\vee)$ are irreducible subrepresentations of $\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}((\tau^\vee \otimes \ldots \otimes \tau^\vee)\delta_{P_{(k^c)}}^{-1/(2k)})$, but there is a unique such. The general case follows from the definition, the tempered case and the fact that for any composition $\beta$ of $l$, $(\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_l}(\otimes_{i=1}^d \varrho_i))^* = \mathrm{Ind}_{P_{(\beta_d,\ldots,\beta_1)}}^{\mathrm{GL}_l}(\otimes_{i=1}^d \varrho_{d-i+1}^*)$. □

While (2.4) may be reducible, it is still of finite length and admits a central character. We mention that since the Jacquet functor is exact over non-archimedean fields, and the generalized Whittaker functor of [GGS17] is exact over archimedean fields ([GGS17, Corollary G]), $\rho_c(\tau)$ admits a unique irreducible subquotient which is a $(k,c)$ representation.

## 3. Realizations of $(k, c)$ functionals

3.1. **Explicit $(k, c)$ functionals from compositions of $k$.** Let $\tau$ be an irreducible generic representation of $\mathrm{GL}_k$, where $k > 1$. If $\tau$ is not supercuspidal, it is a quotient of $\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ for a nontrivial composition $\beta$ of $k$ and an irreducible generic representation $\tau_\beta$ (e.g., one can take $\tau_\beta$ to be supercuspidal). A standard technique for realizing the Whittaker model of $\tau$ is to write down the Jacquet integral on the induced representation (this integral stabilizes in the $p$-adic case). Since $\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ also affords a unique Whittaker model, the functional on the induced representation factors through $\tau$. One may also twist the inducing data $\tau_\beta$ using auxiliary complex parameters, to obtain an absolutely convergent integral which admits an analytic continuation in these parameters. See e.g., [Sha78, JPSS83, Sou93, Sou00].

We generalize this idea to some extent, for $(k, c)$ functionals.

**Lemma 7.** *If $\tau = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ with $\tau_\beta = \otimes_{i=1}^d \tau_i$, $\tau_i = |\det|^{a_i}\tau_{0,i}$, $a_1 \geq \ldots \geq a_d$ and each $\tau_{0,i}$ is square-integrable, or $\tau$ is the essentially square-integrable quotient of $\mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\tau_\beta)$ and $\tau_\beta$ is irreducible supercuspidal (this includes the case $\beta = (1^k)$, i.e., $\tau_\beta$ is a character of $T_{\mathrm{GL}_k}$, over any local field), then $\rho_c(\tau)$ is a quotient of $\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_i \rho_c(\tau_i))$.*

*Proof.* In the first case, this is true by (2.4) and (2.6). For the essentially square-integrable case, over an archimedean field the result follows from [GS13, Corollary 4.2.5] (see the proof of Theorem 4), and if $F$ is $p$-adic from [Tad86, Theorem 7.1]. □

Take $\beta$ as in the lemma. If $F$ is archimedean assume $\beta = (1^k)$, i.e., $\rho_c(\tau)$ is a quotient of a degenerate principal series, which is always possible (see the proof of Theorem 4). Denote $\beta = (\beta_1, \ldots, \beta_d)$, $\beta' = (\beta_d, \ldots, \beta_1)$ and consider the following Jacquet integral

$$\int\limits_{V_{\beta' c}} \xi(w_{\beta c} v)\psi^{-1}(v)\, dv, \tag{3.1}$$

where $\xi$ lies in the space of $\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d W_\psi(\rho_c(\tau_i)))$ and regarded as a complex-valued function, and $\psi$ is the restriction of (2.1) to $V_{\beta' c}$. The integral (3.1) is formally a $(k, c)$ functional on the full induced space. Twist the inducing data and induce from $\otimes_{i=1}^d |\det|^{\zeta_i}\rho_c(\tau_i)$, for fixed $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d$ with $\mathrm{Re}(\zeta_i - \zeta_{i+1}) \gg 0$ for all $i < d$. (Note that $W_\psi(|\det|^{\zeta_i}\rho_c(\tau_i)) = |\det|^{\zeta_i} W_\psi(\rho_c(\tau_i))$).

Both $\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d \rho_c(\tau_i))$ and $\mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^d |\det|^{\zeta_i}\rho_c(\tau_i))$ are $(k, c)$ representations, by Proposition 2. The condition on $\zeta$ implies that the integral is absolutely convergent for all $\xi$ (see e.g., [Sou00, Lemma 2.1]). If we let $\zeta$ vary and $\xi$ is analytic in $\zeta$, the integral admits analytic continuation, which over archimedean fields is continuous in the data $\xi$. Over $p$-adic fields this follows from Bernstein's continuation principle (the corollary in [Ban98, § 1]), since we have uniqueness and the integral can be made constant. Over archimedean fields this follows from Wallach [Wal88, Wal06] ($P_{(c^k)}$ is "very nice" and we induce from a degenerate principal series). For any $\zeta_0 \in \mathbb{C}^d$ one can choose data such that the continuation of (3.1) is nonzero at $\zeta = \zeta_0$. Taking $\zeta_0 = 0$ we obtain a $(k, c)$ functional, which is unique (up to scaling). Hence this functional factors through $\rho_c(\tau)$ and provides a realization of $W_\psi(\rho_c(\tau))$.

3.2. **Explicit $(k, c)$ functionals from compositions of $c$.** Let $\tau$ be an irreducible generic representation of $\mathrm{GL}_k$, and assume an unramified twist of $\tau$ is unitary. In this section we construct $(k,c)$ functionals on $\rho_c(\tau)$ using compositions of $c$. Fix $0<l<c$. Since now both $\rho_l(\tau)$ and $\rho_{c-l}(\tau)$ embed in the corresponding spaces (2.3), $\rho_c(\tau)$ is a subrepresentation of

$$\mathrm{Ind}_{P_{(kl,k(c-l))}}^{\mathrm{GL}_{kc}}((W_\psi(\rho_l(\tau))\otimes W_\psi(\rho_{c-l}(\tau)))\delta_{P_{(kl,k(c-l))}}^{-1/(2k)}). \tag{3.2}$$

Both $(k,l)$ and $(k,c-l)$ models exist by Theorem 4. We may regard vectors in the space of (3.2) as complex-valued functions. We construct a $(k,c)$ functional on (3.2), and prove it does not vanish on any of its subrepresentations, in particular on $\rho_c(\tau)$.

Let $v\in V_{(c^k)}$ and set $v_{i,j}=\begin{pmatrix} v_{i,j}^1 & v_{i,j}^2 \\ v_{i,j}^3 & v_{i,j}^4\end{pmatrix}$, where $v_{i,j}^1\in\mathrm{Mat}_l$ and $v_{i,j}^4\in\mathrm{Mat}_{c-l}$. For $t\in\{1,\ldots,4\}$, let $V^t<V_{(c^k)}$ be the subgroup obtained by deleting the blocks $v_{i,j}^{t'}$ for all $i<j$ and $t'\neq t$, and $V=V^3$. Also define

$$\kappa=\kappa_{l,c-l}=\begin{pmatrix} I_l \\ 0 & 0 & I_l \\ 0 & 0 & 0 & 0 & I_l & \ddots \\ & & & & & & I_l & 0 \\ 0 & I_{c-l} \\ 0 & 0 & 0 & I_{c-l} & & \ddots \\ & & & & & & & I_{c-l}\end{pmatrix}\in\mathrm{GL}_{kc}.$$

**Example 8.** *For $c=2$ (then $l=1$) and $k=3$,*

$$\kappa=\begin{pmatrix} 1 \\ & & 1 \\ & & & & 1 \\ & 1 \\ & & & 1 \\ & & & & & 1\end{pmatrix},\qquad V=\left\{\begin{pmatrix} 1 \\ & 1 & v_{1,2}^3 & & v_{1,3}^3 \\ & & 1 \\ & & & 1 & v_{2,3}^3 \\ & & & & 1 \\ & & & & & 1\end{pmatrix}\right\}.$$

Consider the functional on the space of (3.2),

$$\xi\mapsto\int\limits_V \xi(\kappa v)\,dv. \tag{3.3}$$

This is formally a $(k,c)$ functional. Indeed, the conjugation $v\mapsto {}^\kappa v$ of $v\in V_{(c^k)}$ takes the blocks $v_{i,j}^1$ onto the subgroup $V_{(l^k)}$ embedded in the top left $kl\times kl$ block of $M_{(kl,k(c-l))}$. Then these blocks transform by the $(k,l)$ functional realizing $W_\psi(\rho_l(\tau))$; $v_{i,j}^2$ is taken to $V_{(kl,k(c-l))}$ and $\xi$ is left-invariant on this group; and after the conjugation, the blocks $v_{i,j}^4$ form the subgroup $V_{((c-l)^k)}$ embedded in the bottom right $k(c-l)\times k(c-l)$ block, and transform by the $(k,c-l)$ functional realizing $W_\psi(\rho_{c-l}(\tau))$. Thus $V_{(c^k)}$ transforms under (2.1). Also note that this conjugation takes $v_{i,j}^3$ to $V_{(kl,k(c-l))}^-$ (in particular $V$ is abelian).

For $v\in V$, $y={}^\kappa v\in V_{(kl,k(c-l))}^-$ is such that its bottom left $kl\times k(c-l)$ block takes the form

$$\begin{pmatrix} 0 & v_{1,2}^3 & \cdots & v_{1,k}^3 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & v_{k-1,k}^3 \\ 0 & \cdots & \cdots & 0\end{pmatrix},\qquad v_{i,j}^3\in\mathrm{Mat}_{(c-l)\times l}. \tag{3.4}$$

Let $y_{i,j}\in V_{(kl,k(c-l))}^-$ be obtained from $y$ be zeroing out all the blocks in (3.4) except $v_{i,j}^3$. Also let $X<V_{(kl,k(c-l))}$ be the subgroup of matrices $x$ whose top right $kl\times k(c-l)$ block is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & x_{1,2}^3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & x_{1,k}^3 & \cdots & x_{k-1,k}^3\end{pmatrix},\qquad x_{i,j}^3\in\mathrm{Mat}_{l\times(c-l)}.$$

Define $x_{i,j}$ similarly to $y_{i,j}$. Let $X_{i,j}$ and $Y_{i,j}$ be the respective subgroups of elements. Then

$$\xi(y_{i,j}x_{i,j})=\psi(\mathrm{tr}(v_{i,j}^3x_{i,j}^3))\xi(y_{i,j}). \tag{3.5}$$

We show that (3.3) can be used to realize $W_\psi(\rho_c(\tau))$.

**Lemma 9.** *For $0<l<c$, realize $\rho_c(\tau)$ as a subrepresentation of (3.2). The integral (3.3) is absolutely convergent, and is a (nonzero) $(k,c)$ functional on $\rho_c(\tau)$.*

*Proof.* The proof technique is called "root elimination", see e.g., [Sou93, Proposition 6.1], [Sou93, § 7.2] and [Jac09, § 6.1] (also the proof of [LR05, Lemma 8]). We argue by eliminating each $y_{i,j}$ separately, handling the diagonals left to right, bottom to top: starting with $y_{k-1,k}$, next $y_{k-2,k-1}$, ..., up to $y_{1,2}$, then $y_{k-2,k}$, $y_{k-3,k-1}$, etc., with $y_{1,k}$ handled last. Let $\mathcal{W}$ be a subrepresentation of (3.2). For $\xi$ in the space of $\mathcal{W}$ and a Schwartz function $\phi$ on $\mathrm{Mat}_{l\times(c-l)}$ (over $p$-adic fields, Schwartz functions are in particular compactly supported), define

$$\xi_{i,j}(g)=\int_{X_{i,j}}\xi(gx_{i,j})\phi(x^3_{i,j})\,dx_{i,j},$$
$$\xi'_{i,j}(g)=\int_{Y_{i,j}}\xi(gy_{i,j})\widehat{\phi}(v^3_{i,j})\,dy_{i,j}.$$

Here $\widehat{\phi}$ is the Fourier transform of $\phi$ with respect to $\psi\circ\mathrm{tr}$. By smoothness over $p$-adic fields, or by the Dixmier–Malliavin Theorem [DM78] over archimedean fields, any $\xi$ is a linear combination of functions $\xi_{i,j}$, and also of functions $\xi'_{i,j}$. Using (3.5), the definition of the Fourier transform and the fact that $V$ is abelian we obtain, for $(i,j)=(k-1,k)$,

$$\int_{Y_{i,j}}\xi_{i,j}(y_{i,j}{}^{\kappa}v^\circ)\,dy_{i,j}=\xi'_{i,j}({}^{\kappa}v^\circ),$$

where $v^\circ\in V$ does not contain the block of $v^3_{i,j}$. We re-denote $\xi=\xi'_{i,j}$, then proceed similarly with $(i,j)=(k-2,k-1)$. This shows that the integrand is a Schwartz function on ${}^{\kappa}V$, thus the integral (3.3) is absolutely convergent. At the same time, the integral does not vanish on $\mathcal{W}$ because in this process we can obtain $\xi(I_{kc})$. Over archimedean fields the same argument also implies that (3.3) is continuous (see [Sou95, § 5, Lemma 2, p. 199]).

Over $p$-adic fields we provide a second argument for the nonvanishing part. Choose $\xi_0$ in the space of $\mathcal{W}$ with $\xi_0(I_{kc})\neq 0$. Define for a (large) compact subgroup $\mathcal{X}<X$, the function

$$\xi_1(g)=\int\limits_{\mathcal{X}}\xi(gx)\,dx,$$

which clearly also belongs to the space of $\mathcal{W}$. We show that for a sufficiently large $\mathcal{X}$, $\int\limits_V\xi_1({}^{\kappa}v)\,dv=\xi(I_{kc})$. Put $y={}^{\kappa}v$. We prove $\xi_1(y)=0$, unless $y$ belongs to a small compact neighborhood of the identity, and then $\xi_1(y)=\xi(I_{kc})$. We argue by eliminating each $y_{i,j}$ separately, in the order stated above. Assume we have zeroed out all blocks on the diagonals to the left of $y_{i,j}$, and below $y_{i,j}$ on its diagonal. Let $\mathcal{B}$ denote the set of indices $(i',j')$ of the remaining $y_{i',j'}$ and $\mathcal{B}^\circ=\mathcal{B}-(i,j)$. Denote $\mathcal{X}_{i,j}=\mathcal{X}\cap X_{i,j}$ and assume that if $(i',j')\notin\mathcal{B}$, $\mathcal{X}_{i',j'}$ is trivial. Write $\mathcal{X}=\mathcal{X}^\circ\times\mathcal{X}_{i,j}$. For $(i',j')\in\mathcal{B}^\circ$, ${}^{y_{i',j'}}x_{i,j}=ux_{i,j}$, where $u\in V_{(l^k)}\times V_{((c-l)^k)}$ and the $(k,l)$ and $(k,c-l)$ characters (2.1) are trivial on $u$. Therefore by (3.5),

$$\xi_1(y)=\int\limits_{\mathcal{X}^\circ}\xi(yx)\,dx\int\limits_{\mathcal{X}_{i,j}}\psi(\mathrm{tr}(v^3_{i,j}x^3_{i,j}))\,dx_{i,j}.$$

The second integral vanishes unless $v^3_{i,j}$ is sufficiently small, then this integral becomes a nonzero measure constant. Moreover, if $\mathcal{X}_{i,j}$ is sufficiently large with respect to $\xi$ and $\mathcal{X}_{i',j'}$ for all $(i',j')\in\mathcal{B}^\circ$, then for any $x\in\mathcal{X}^\circ$, ${}^{y_{i,j}}x=xz$ where $z$ belongs to a small neighborhood of the identity in $\mathrm{GL}_{kc}$, on which $\xi$ is invariant on the right. Therefore we may remove $y_{i,j}$

from $y$ in the first integral. Re-denote $\mathcal{X} = \mathcal{X}^\circ$. Repeating this process, the last step is for $y_{1,k}$ with an integral over $\mathcal{X}_{1,k}$, and we remain with $\xi(I_{kc})$. ☐

In contrast with the realization described in the previous section, here we regard $\rho_c(\tau)$ as a subrepresentation, therefore nonvanishing on $\rho_c(\tau)$ is not a consequence of uniqueness. In fact, the proof above implies $(k,c)$ functionals on the space of (3.2) are not necessarily unique. On the other hand (3.3) is already absolutely convergent (we do not need twists).

3.3. **Certain unramified principal series.** One may consider integral (3.1) (now with any $d \geq 2$) also when $\tau = \mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\otimes_{i=1}^k \tau_i)$ is an unramified principal series (possibly reducible), written with a weakly decreasing order of exponents (see before Theorem 4). Then $\rho_c(\tau) = \mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k \tau_i \circ \det)$. The integral admits analytic continuation in the twisting parameters $\zeta_1, \ldots, \zeta_k$, which is nonzero for all choices of $\zeta_i$. Hence it defines a $(k,c)$ functional on $\rho_c(\tau)$. In this setting we can prove a decomposition result similar to Lemma 9 but using (3.1). The results of this section are essentially an elaborative reformulation of [CFGK19, Lemma 22].

Put $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{C}^k$. Denote the representation $\mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i \circ \det)$ by $\rho_c(\tau_\zeta)$, with a minor abuse of notation (because we do not consider only $\zeta \in \mathbb{R}^k$ and $\zeta_1 \geq \ldots \geq \zeta_k$). Let $V(\zeta, \tau, c)$ be the space of $\rho_c(\tau_\zeta)$. A section $\xi$ on $V(\tau, c)$ is a function $\xi : \mathbb{C}^k \times \mathrm{GL}_{kc} \to \mathbb{C}$ such that for all $\zeta \in \mathbb{C}^k$, $\xi(\zeta, \cdot) \in V(\zeta, \tau, c)$, and we call it entire if $\zeta \mapsto \xi(\zeta, g) \in \mathbb{C}[q^{\pm\zeta_1}, \ldots, q^{\pm\zeta_k}]$, for all $g \in \mathrm{GL}_{kc}$. A meromorphic section is a function $\xi$ on $\mathbb{C}^k \times \mathrm{GL}_{kc}$ such that $\varphi(\zeta)\xi(\zeta, g)$ is an entire section, for some holomorphic and not identically zero $\varphi : \mathbb{C}^k \to \mathbb{C}$. The normalized unramified section $\xi^0$ is the section which is the normalized unramified vector for all $\zeta$.

Let $0 < l < c$. We use the notation $\kappa$, $v_{i,j}$, $v_{i,j}^t$, $t \in \{1, \ldots, 4\}$, $V^t$ and $V = V^3$ from the previous section. Denote $Z = {}^{(\mathrm{diag}(w_{(l^k)}, w_{((c-l)^k)})\kappa)}V^2$. Define an intertwining operator by the meromorphic continuation of the integral

$$m(\zeta, \kappa)\xi(\zeta, g) = \int\limits_Z \xi(\zeta, \kappa^{-1} z g)\, dz,$$

where $\xi$ is a meromorphic section of $V(\tau, c)$. When $\xi$ is entire, this integral is absolutely convergent for $\mathrm{Re}(\zeta)$ in a cone of the form $\mathrm{Re}(\zeta_1) \gg \ldots \gg \mathrm{Re}(\zeta_k)$, which depends only on the inducing characters.

**Lemma 10.** *Assume $1 - q^{-s}\tau_i(\varpi)\tau_j^{-1}(\varpi)$ is nonzero for all $i < j$ and $\mathrm{Re}(s) \geq 1$. Then for all $\zeta$ with $\mathrm{Re}(\zeta_1) \geq \ldots \geq \mathrm{Re}(\zeta_k)$, $m(\zeta, \kappa)\xi^0(\zeta, \cdot)$ is well defined, nonzero and belongs to the space of*

$$\mathrm{Ind}_{P_{(kl,k(c-l))}}^{\mathrm{GL}_{kc}}((\rho_l(\tau_\zeta) \otimes \rho_{c-l}(\tau_\zeta))\delta_{P_{(kl,k(c-l))}}^{-1/(2k)}). \tag{3.6}$$

*Proof.* The trivial representation of $\mathrm{GL}_c$ is an unramified subrepresentation of $\mathrm{Ind}_{B_{\mathrm{GL}_c}}^{\mathrm{GL}_c}(\delta_{B_{\mathrm{GL}_c}}^{-1/2})$, hence $\rho_c(\tau_\zeta)$ is an unramified subrepresentation of

$$\mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k \mathrm{Ind}_{B_{\mathrm{GL}_c}}^{\mathrm{GL}_c}(|\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_c}}^{-1/2})) = \mathrm{Ind}_{B_{\mathrm{GL}_{kc}}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_c}}^{-1/2}).$$

Looking at $\kappa$, we see that the image of $m(\zeta, \kappa)$ on the space of this representation is contained in

$$\begin{aligned} &\mathrm{Ind}_{B_{\mathrm{GL}_{kc}}}^{\mathrm{GL}_{kc}}(|\det|^{-(c-l)/2}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_l}}^{-1/2}) \otimes |\det|^{l/2}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_{c-l}}}^{-1/2})) \\ &= \mathrm{Ind}_{P_{(kl,k(c-l))}}^{\mathrm{GL}_{kc}}(\big(\mathrm{Ind}_{B_{\mathrm{GL}_{kl}}}^{\mathrm{GL}_{kl}}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_l}}^{-1/2}) \otimes \mathrm{Ind}_{B_{\mathrm{GL}_{k(c-l)}}}^{\mathrm{GL}_{k(c-l)}}(\otimes_{i=1}^k |\;|^{\zeta_i}\tau_i\delta_{B_{\mathrm{GL}_{c-l}}}^{-1/2})\big)\delta_{P_{(kl,k(c-l))}}^{-1/(2k)}). \end{aligned} \tag{3.7}$$

This representation contains (3.6) as an unramified subrepresentation. We show $m(\zeta,\kappa)\xi^0(\zeta,\cdot)$ satisfies the required properties for the prescribed $\zeta$. We may decompose $m(\zeta,\kappa)$ into rank-1 intertwining operators on spaces of the form

$$\mathrm{Ind}_{B_{\mathrm{GL}_2}}^{\mathrm{GL}_2}(|\,|^{\zeta_i-(c-2l+1)/2}\tau_i\otimes|\,|^{\zeta_j-(c-2l'+1)/2}\tau_j),\qquad i<j,\qquad l'\leq l-1.$$

According to the Gindikin–Karpelevich formula ([Cas80, Theorem 3.1]), each intertwining operator takes the normalized unramified vector in this space to a constant multiple of the normalized unramified vector in its image, and this constant is given by

$$\frac{1-q^{-1-\zeta_i+\zeta_j-l+l'}\tau_i(\varpi)\tau_j^{-1}(\varpi)}{1-q^{-\zeta_i+\zeta_j-l+l'}\tau_i(\varpi)\tau_j^{-1}(\varpi)}.$$

Since $\mathrm{Re}(-\zeta_i+\zeta_j)\leq 0$ and $-l+l'\leq -1$, if the quotient has a zero or pole, then $1-q^{-s}\tau_i(\varpi)\tau_j^{-1}(\varpi)=0$ for $\mathrm{Re}(s)\geq 1$, contradicting our assumption. Therefore $m(\zeta,\kappa)\xi^0(\zeta,\cdot)$ is well defined and nonzero, and because it is unramified, it also belongs to (3.6). □

Integral (3.1) is also absolutely convergent for $\mathrm{Re}(\zeta)$ in a cone $\mathrm{Re}(\zeta_1)\gg\ldots\gg\mathrm{Re}(\zeta_k)$, which depends only on the inducing characters. The proof is that of the known result for similar intertwining integrals.

**Lemma 11.** *In the domain of absolute convergence of (3.1) and in general by meromorphic continuation, for any meromorphic section $\xi$ on $V(\tau,c)$,*

$$\int\limits_{V_{(c^k)}}\xi(\zeta,w_{(c^k)}v)\psi^{-1}(v)\,dv=\int\limits_{V}m(\zeta,\kappa)\xi(\zeta,\kappa v)\,dv.$$

*Here $m(\zeta,\kappa)\xi$ belongs to the space obtained from (3.7) by applying the $(k,l)$ and $(k,c-l)$ functionals (3.1) on the respective factors of $P_{(kl,k(c-l))}$.*

*Proof.* Using matrix multiplication we see that $w_{(c^k)}=\kappa^{-1}\operatorname{diag}(w_{(l^k)},w_{((c-l)^k)})\kappa$. The character $\psi$ is trivial on $V^2$. Thus in its domain of absolute convergence integral (3.1) equals

$$\int\limits_{V}\int\limits_{V^4}\int\limits_{V^1}m(\zeta,\kappa)\xi(\zeta,\operatorname{diag}(w_{(l^k)}{}^{\kappa}v_1,w_{((c-l)^k)}{}^{\kappa}v_4)\kappa v)\psi^{-1}(v_1)\psi^{-1}(v_4)\,dv_1\,dv_4\,dv.$$

The integrals $dv_1dv_4$ constitute the applications of $(k,l)$ and $(k,c-l)$ functionals, e.g., ${}^{\kappa}V_1=\operatorname{diag}(V_{(l^k)},I_{k(c-l)})$ (see after (3.3)). Now combine this with the proof of Lemma 10. □

Combining this result for $\xi^0$ with Lemma 10, we obtain a result analogous to Lemma 9.

3.4. **Equivariance property under $\mathrm{GL}_c^{\triangle}$.** Let $g\mapsto g^{\triangle}$ be the diagonal embedding of $\mathrm{GL}_c$ in $\mathrm{GL}_{kc}$. Since $\dim\mathrm{Hom}_{V_{(c^k)}}(\rho_c(\tau),\psi)=1$, a $(k,c)$ functional on $\rho_c(\tau)$ translates under the action of $\mathrm{GL}_c^{\triangle}$ by a character. The following proposition explicates this character.

**Lemma 12.** *Let $\lambda$ be a $(k,c)$ functional on $\rho_c(\tau)$ and $\xi$ be a vector in the space of $\rho_c(\tau)$. For any $g\in\mathrm{GL}_c$, $\lambda(\rho_c(\tau)(g^{\triangle})\xi)=\tau(\det(g)I_k)\lambda(\xi)$.*

*Proof.* The claim clearly holds for $c=1$, since then $g^{\triangle}$ belongs to the center of $\mathrm{GL}_k$. Let $c>1$. We prove separately that $\lambda(\rho_c(\tau)(t^{\triangle})\xi)=\tau(\det(t))\lambda(\xi)$ for all $t\in T_{\mathrm{GL}_c}$ and $\lambda(\rho_c(\tau)(g^{\triangle})\xi)=\lambda(\xi)$ for all $g\in\mathrm{SL}_c$. Since all $(k,c)$ functionals are proportional, we may prove each equivariance property using a particular choice of functional.

First take $t=\operatorname{diag}(t_1,\ldots,t_c)$. Assume $\tau$ is irreducible essentially tempered. Consider the $(k,c)$ functional (3.3) with $l=1<c$. Conjugate $V$ by $t^{\triangle}$, then ${}^{\kappa}(t^{\triangle})=\operatorname{diag}(t_1I_k,t'^{\triangle})$ with

$t' = \mathrm{diag}(t_2, \ldots, t_c)$ and $t'^{\triangle} \in \mathrm{GL}_{k(c-1)}$. The change to the measure of $V$ is $\delta_{P_{(k,k(c-1))}}^{-1/2+1/(2k)}({}^{\kappa}t)$ and the result now follows using induction. The case of irreducible generic $\tau$ is reduced to the essentially tempered case using (3.1) and note that $\delta_{P_{(\beta c)}}(t^{\triangle}) = 1$ for any composition $\beta$ of $k$. If $\tau$ is a reducible unramified principal series we again compute using (3.1).

It remains to consider $g \in \mathrm{SL}_c$. By definition the Jacquet module of $\rho_c(\tau)$ with respect to $V_{(c^k)}$ and (2.1) is one dimensional (whether $\tau$ is irreducible or unramified principal series), hence $\mathrm{SL}_c^{\triangle}$ acts trivially on the Jacquet module, and the result follows. □

The above property is useful for the study of integrals involving $(k,c)$ models. For example, let $\pi_1$ and $\pi_2$ be irreducible admissible representations of $\mathrm{GL}_c$, $\tau_1$ and $\tau_2$ be irreducible generic representations of $\mathrm{GL}_k$, and $s \in \mathbb{C}$. Let $V(s, \tau_1 \times \tau_2)$ be the space of the representation $\mathrm{Ind}_{P_{(kc,kc)}}^{\mathrm{GL}_{2kc}}(|\det|^s W_\psi(\rho_c(\tau_1)) \otimes |\det|^{-s} W_\psi(\rho_c(\tau_2)))$. Denote $U = V_{(c^{k-1},2c,c^{k-1})}$ and fix a character $\psi_U$ of $U$ whose stabilizer in $M_{(c^{k-1},2c,c^{k-1})}$ is isomorphic to $\mathrm{GL}_c \times \mathrm{GL}_c$. Let $D = U \rtimes (\mathrm{GL}_c \times \mathrm{GL}_c)$. The study of the generalized doubling integral in this setup involves the space $\mathrm{Hom}_D(V(s, \tau_1 \times \tau_2), \psi_U \otimes \pi_1 \otimes \pi_2)$ (see [CFGK19, GK]). Lemma 12 can be used to determine the requirements on the central chatecters of $\pi_i$ and $\tau_j$, in order to ensure this space is nontrivial. This lemma is also important for the determination of the equivariance properties of the doubling integral for representations of $\mathrm{SO}_{2c+1} \times \mathrm{GL}_k$ with respect to varying the character $\psi$ of $F$. In that case, if $\psi$ is replaced by $\psi_b$ where $\psi_b(x) = \psi(bx)$, $b \in F^*$, Lemma 12 is applied with $g = \mathrm{diag}(I_c, b^{-1}, I_c)$.

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## References


[AGS15a] A. Aizenbud, D. Gourevitch, and S. Sahi. Derivatives for smooth representations of $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. *Israel J. Math.*, 206(1):1–38, 2015.

[AGS15b] A. Aizenbud, D. Gourevitch, and S. Sahi. Twisted homology for the mirabolic nilradical. *Israel J. Math.*, 206(1):39–88, 2015.

[Ban98] W. D. Banks. A corollary to Bernstein's theorem and Whittaker functionals on the metaplectic group. *Math. Res. Lett.*, 5(6):781–790, 1998.

[BSS90] D. Barbasch, S. Sahi, and B. Speh. Degenerate series representations for $\mathrm{GL}(2n, \mathbf{R})$ and Fourier analysis. In *Symposia Mathematica, Vol. XXXI (Rome, 1988)*, Sympos. Math., XXXI, pages 45–69. Academic Press, London, 1990.

[BZ76] I. N. Bernstein and A. V. Zelevinsky. Representations of the group $GL(n,F)$ where $F$ is a local non-Archimedean field. *Russian Math. Surveys*, 31(3):1–68, 1976.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $p$-adic groups I. *Ann. Scient. Éc. Norm. Sup.*, 10(4):441–472, 1977.

[CFGK19] Y. Cai, S. Friedberg, D. Ginzburg, and E. Kaplan. Doubling constructions and tensor product $L$-functions: the linear case. *Invent. Math.*, 217(3):985–1068, 2019.

[Car93] R. W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Cas80] W. Casselman. The unramified principal series of $p$-adic groups I: the spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[CM93] D. H. Collingwood and W. M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[DM78] J. Dixmier and P. Malliavin. Factorisations de fonctions et de vecteurs indéfiniment différentiables. *Bull. Sci. Math. (2)*, 102(4):307–330, 1978.

[Gin06] D. Ginzburg. Certain conjectures relating unipotent orbits to automorphic representations. *Israel J. Math.*, 151(1):323–355, 2006.

[GGS17] R. Gomez, D. Gourevitch, and S. Sahi. Generalized and degenerate Whittaker models. *Compos. Math.*, 153(2):223–256, 2017.

[GK] D. Gourevitch and E. Kaplan. Multiplicity one theorems for the generalized doubling method (with an appendix by A. Aizenbud and D. Gourevitch). *J. Eur. Math. Soc.*, DOI 10.4171/JEMS/1207, 2022.

[GS13] D. Gourevitch and S. Sahi. Annihilator varieties, adduced representations, Whittaker functionals, and rank for unitary representations of $\mathrm{GL}(n)$. *Selecta Math. (N.S.)*, 19(1):141–172, 2013.

[Jac84] H. Jacquet. On the residual spectrum of $\mathrm{GL}(n)$. In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 185–208. Springer, Berlin, 1984.

[Jac09] H. Jacquet. Archimedean Rankin-Selberg integrals. In *Automorphic forms and L-functions II. Local aspects*, volume 489 of *Contemp. Math.*, pages 57–172. Amer. Math. Soc., Providence, RI, 2009.

[JPSS83] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika. Rankin–Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.

[JS83] H. Jacquet and J. A. Shalika. The Whittaker models of induced representations. *Pacific. J. Math.*, 109(1):107–120, 1983.

[LM14] E. Lapid and A. Mínguez. On a determinantal formula of Tadić. *Amer. J. Math.*, 136(1):111–142, 2014.

[LR05] E. M. Lapid and S. Rallis. On the local factors of representations of classical groups. In J. W. Cogdell, D. Jiang, S. S. Kudla, D. Soudry, and R. Stanton, editors, *Automorphic representations, L-functions and applications: progress and prospects*, pages 309–359. Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[MW89] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989.

[SS90] S. Sahi and E. M. Stein. Analysis in matrix space and Speh's representations. *Invent. Math.*, 101(2):379–393, 1990.

[SV00] W. Schmid and K. Vilonen. Characteristic cycles and wave front cycles of representations of reductive Lie groups. *Ann. of Math. (2)*, 151(3):1071–1118, 2000.

[Sha78] F. Shahidi. Functional equation satisfied by certain $L$-functions. *Compositio Math.*, 37:171–208, 1978.

[Sou93] D. Soudry. Rankin–Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory. *Mem. Amer. Math. Soc.*, 105(500):vi+100, 1993.

[Sou95] D. Soudry. On the Archimedean theory of Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. *Ann. Sci. École Norm. Sup. (4)*, 28(2):161–224, 1995.

[Sou00] D. Soudry. Full multiplicativity of gamma factors for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. *Israel J. Math.*, 120(1):511–561, 2000.

[Tad86] M. Tadić. Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). *Ann. Sci. École Norm. Sup. (4)*, 19(3):335–382, 1986.

[Tad87] M. Tadić. Unitary representations of $\mathrm{GL}(n)$, derivatives in the non-Archimedean case. In *V. Mathematikertreffen Zagreb-Graz (Mariatrost/Graz, 1986)*, volume 274 of *Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum*, pages Ber. No. 281, 19. Forschungszentrum Graz, Graz, 1987.

[Vog78] D. A. Jr. Vogan. Gel′fand-Kirillov dimension for Harish-Chandra modules. *Invent. Math.*, 48(1):75–98, 1978.

[Vog86] D. A. Jr. Vogan. The unitary dual of $\mathrm{GL}(n)$ over an Archimedean field. *Invent. Math.*, 83(3):449–505, 1986.

[Vog91] D. A. Jr. Vogan. Associated varieties and unipotent representations. In *Harmonic analysis on reductive groups (Brunswick, ME, 1989)*, volume 101 of *Progr. Math.*, pages 315–388. Birkhäuser Boston, Boston, MA, 1991.

[Wal88] N. R. Wallach. Lie algebra cohomology and holomorphic continuation of generalized Jacquet integrals. In *Representations of Lie groups, Kyoto, Hiroshima, 1986*, volume 14 of *Adv. Stud. Pure Math.*, pages 123–151. Academic Press, Boston, MA, 1988.

[Wal06] N. R. Wallach. Holomorphic continuation of generalized Jacquet integrals for degenerate principal series. *Represent. Theory*, 10:380–398, 2006.

[Zel80] A. V. Zelevinsky. Induced representations of reductive $p$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980.

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