



On the local limit theorems for linear sequences of lower psi-mixing Markov chains

Magda Peligrad^a, Hailin Sang^b, Na Zhang^{c,*}

^a Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA

^b Department of Mathematics, University of Mississippi, University, MS 38677, USA

^c Department of Mathematics, Towson University, Towson, MD 21252, USA

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ABSTRACT

In this paper we investigate the local limit theorem for partial sums of linear sequences of the form $X_j = \sum_{i \in \mathbb{Z}} a_i \xi_{j-i}$. Here $(a_i)_{i \in \mathbb{Z}}$ is a sequence of constants satisfying $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$ and $(\xi_i)_{i \in \mathbb{Z}}$ are functions of a stationary Markov chain with mean zero and finite second moment. The Markov chain is assumed to satisfy one-sided lower psi-mixing condition.

1. Introduction and preliminaries

In this paper we obtain a local limit theorem for partial sums $(S_n)_{n \geq 1}$ of a stationary sequence of random variables, which are linear combinations of dependent innovations. This is a result about the rate of convergence of the probabilities of the type $\mathbb{P}(a \leq S_n \leq b)$.

We shall consider the case of non-lattice random variables. The lattice case means that there exists $v > 0$ and $a \in \mathbb{R}$ such that the values of all the variables in the sum S_n are concentrated on the lattice $\{a + kv : k \in \mathbb{Z}\}$, whereas the non-lattice case means that no such a and v exist.

In the non-lattice case, the local limit theorem has been well-studied for independent observations. We refer the books by Ibragimov and Linnik (1971), Petrov (1975), and Gnedenko (1962). See also the recent papers Dolgopyat (2016), Fortune et al. (2021), and the references therein. When the innovations are independent and have finite second moment the local limit theorem for linear random fields was studied in a paper by Fortune et al. (2021). In the case that the innovations are independent and have infinite second moment, we refer to the papers by Shukri (1976), Peligrad et al. (2022) and the references therein for the local limit theorems on linear processes or in general on linear random fields.

When the innovations are dependent, a first attempt to study the local limit theorem for linear processes can be found in Merlevède et al. (2021). In that paper, among other results, it was considered the problem of local limit theorem for linear processes with short memory (i.e. absolutely summable coefficients) and the innovations functions of ψ -mixing Markov chains. We are not aware of other results in this direction. We obtain here the local limit theorem for partial sums of linear processes with innovations functions of a lower-elliptic Markov chain, centered at expectations and with finite second moment. The general coefficients allow for both short and long memory.

* Corresponding author.

E-mail addresses: peligrm@ucmail.uc.edu (M. Peligrad), sang@olemiss.edu (H. Sang), nzhang@towson.edu (N. Zhang).

We assume now that $(v_k)_{k \in \mathbb{Z}}$ is a strictly stationary Markov chain defined on $(\Omega, \mathcal{K}, \mathbb{P})$ with values in a Polish space $(\mathcal{X}, B(\mathcal{X}))$, where $B(\mathcal{X})$ is a Borel σ -field on $\mathcal{X}$, with regular transition probabilities: for $k \in \mathbb{Z}$ and $A \in B(\mathcal{X})$

$$Q(x, A) = \mathbb{P}(v_k \in A | v_{k-1} = x) \quad (1)$$

and marginal distributions denoted by

$$m(A) = \mathbb{P}(v_0 \in A). \quad (2)$$

Throughout the paper we shall assume that a lower-elliptic condition holds, namely that there is a constant $a > 0$ and a $\mathcal{X}' \in B(\mathcal{X})$ with $\mathbb{P}(\mathcal{X}') = 1$ such that for all $A \in B(\mathcal{X})$ and $x \in \mathcal{X}'$ we have

$$Q(x, A) \geq a \cdot m(A). \quad (3)$$

Let h be a real-valued measurable function and define for all $j \in \mathbb{Z}$

$$\xi_j = h(v_j). \quad (4)$$

Assume that $\mathbb{E} \xi_0 = 0$, $\mathbb{E} \xi_0^2 < \infty$, and ξ_0 has a non-lattice distribution.

In this paper, we consider linear sequences of random variables of the form

$$X_j = \sum_{i \in \mathbb{Z}} a_i \xi_{j-i} \quad (5)$$

defined on $\mathbb{Z}$, where the collection $(a_i)_{i \in \mathbb{Z}}$ of real coefficients satisfies

$$\sum_{i \in \mathbb{Z}} a_i^2 < \infty. \quad (6)$$

As a matter of fact, the sequence $(X_j)_{j \in \mathbb{Z}}$ given in (5) is centered at expectation, stationary and exists in $L^2(\mathbb{R})$ if and only if (6) is satisfied. To see this, by relation (3.4) in Merlevède et al. (2021) or (21) in Merlevède et al. (2022), our condition (6) implies that, there exist $c_1, c_2 > 0$ such that

$$c_1 \sum_{|i| \leq n} a_i^2 \mathbb{E}(\xi_0^2) \leq \mathbb{E} \left(\sum_{|i| \leq n} a_i \xi_{j-i} \right)^2 \leq c_2 \sum_{|i| \leq n} a_i^2 \mathbb{E}(\xi_0^2), \quad (7)$$

which further implies that for any integers $m > n$, there is some constant $C > 0$

$$\mathbb{E} \left(\sum_{|i| \leq m} a_i \xi_{j-i} - \sum_{|i| \leq n} a_i \xi_{j-i} \right)^2 \leq C \sum_{n < |i| \leq m} a_i^2 \mathbb{E}(\xi_0^2).$$

Taking condition (6) into account, $\sum_{n < |i| \leq m} a_i^2 \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore $(\sum_{|i| \leq n} a_i \xi_{j-i})$ is Cauchy in L^2 . Therefore, $\sum_{|i| \leq n} a_i \xi_{j-i}$ has a unique limit, which we denote by

$$\sum_{i \in \mathbb{Z}} a_i \xi_{j-i} = \lim_{n \rightarrow \infty} \sum_{|i| \leq n} a_i \xi_{j-i} \text{ in } L^2.$$

We say that the process has *long memory (long range dependence)* if $\sum_{i \in \mathbb{Z}} |a_i| = \infty$.

Now define the partial sums of the stationary sequence (X_j) as

$$S_n = \sum_{j=1}^n X_j \quad (8)$$

with variance

$$B_n^2 = \text{Var}(S_n). \quad (9)$$

Since X_j defined in (5) can be written as $X_j = \sum_{i \in \mathbb{Z}} a_{j-i} \xi_i$, the sum $S_n = \sum_{j=1}^n X_j$ can be expressed as an infinite linear combination of the innovations $(\xi_i)_{i \in \mathbb{Z}}$, namely

$$S_n = \sum_{i \in \mathbb{Z}} b_{n,i} \xi_i, \text{ where } b_{n,i} = \sum_{j=1}^n a_{j-i}. \quad (10)$$

We also denote

$$\gamma_n = \sup_{i \in \mathbb{Z}} |b_{n,i}|.$$

We shall establish the following uniform local limit theorem:

Let (S_n) be a sequence of random variables, (B_n) be a sequence of positive numbers. We say that (S_n, B_n) satisfies the local limit theorem if for all continuous complex-valued functions $g(x)$ with $|g| \in L^1(\mathbb{R})$ and with Fourier transform $\hat{g}$ real and with compact support,

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left[\sqrt{2\pi} B_n \mathbb{E} g(S_n - u) - [\exp(-u^2/2B_n^2)] \int g(x) \lambda(dx) \right] = 0, \quad (11)$$

where λ is the Lebesgue measure. By arguments in Section 4 of [Hafouta and Kifer \(2016\)](#) this result implies that (11) also holds for the class of real continuous functions with compact support and by Theorem 10.7 in [Breiman \(1992\)](#) it follows that

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| \sqrt{2\pi} B_n \mathbb{P}(a + u \leq S_n \leq b + u) - [\exp(-u^2/2B_n^2)](b - a) \right| = 0,$$

for any $a < b$. In particular, since $B_n \rightarrow \infty$ as $n \rightarrow \infty$, then for fixed $A > 0$,

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq A} \left| \sqrt{2\pi} B_n \mathbb{P}(a + u \leq S_n \leq b + u) - (b - a) \right| = 0.$$

If we further take $u = 0$, then,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi} B_n \mathbb{P}(S_n \in [a, b]) = b - a. \quad (12)$$

In other words, the sequence of measures $\sqrt{2\pi} B_n \mathbb{P}(S_n \in [a, b])$ of the interval $[a, b]$ converges to Lebesgue measure.

It should be noted that the local limit theorem, as formulated in (11) is useful to the study of recurrence conditions for S_n , as explained in [Orey \(1966\)](#) and [Mineka and Silverman \(1970\)](#).

A few remarks about notation and terms used in the paper follow. To indicate relative growth rates at infinity, we use $a_n \propto b_n$ to indicate that $a_n/b_n \rightarrow C \in \mathbb{R}^+$, and the particular case when $C = 1$ is denoted $a_n \sim b_n$. By $a_n = o(b_n)$ we understand that $a_n/b_n \rightarrow 0$ and $a_n = O(b_n)$ means that $\limsup |a_n/b_n| < C$ for some positive number C . Throughout the paper, an indicator function will be denoted as I . A function $l : [0, \infty) \rightarrow \mathbb{R}$ is referred to as *slowly varying* (at ∞) if it is positive and measurable on $[A, \infty)$ for some $A \in \mathbb{R}^+$ such that $\lim_{x \rightarrow \infty} l(\lambda x)/l(x) = 1$ holds for each $\lambda \in \mathbb{R}^+$. The integer part of a real number x will be denoted by $\lfloor x \rfloor$.

2. Main results

In this work, we investigate the conditions under which the local limit theorem holds for the partial sums of the linear random sequences given by (5).

Denote the characteristic function of ξ_0 by $\varphi_{\xi_0}(t) := \mathbb{E} \exp(it\xi_0)$. It is well known that ξ_0 not having a lattice distribution is equivalent to $|\varphi_{\xi_0}(t)| < 1$ for all $t \neq 0$. On the other hand, the Cramér condition means that $\limsup_{|t| \rightarrow \infty} |\varphi_{\xi_0}(t)| < 1$. Thanks to the Riemann–Lebesgue lemma, the Cramér condition is automatically satisfied if the distribution function of ξ_0 is absolutely continuous with respect to the Lebesgue measure. It should be mentioned that ξ_0 has a non-lattice distribution whenever $\varphi_{\xi_0}(t)$ satisfies the Cramér condition. See Theorem 1 on page 10 in [Petrov \(1975\)](#) or Lemma 5.1 in [Fortune et al. \(2021\)](#).

Here is the local limit theorem, the main result.

Theorem 1. Assume that the innovations (ξ_i) of the linear process are defined by (4), have non-lattice distribution, mean 0, finite second moment, and satisfy (3). The coefficients satisfy $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$. Let S_n and B_n be defined as in (8) and (9). Assume $B_n \rightarrow \infty$. In the case the process has long range dependence, we assume that the innovations satisfy the Cramér condition. Under these conditions the local limit theorem in (11) holds.

Note that the conditions imposed on the coefficients (a_k) are minimal in the following sense. As we have seen in (7) and the comments following this relation, $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$ is necessary and sufficient for the existence of the linear process in $L^2(\mathbb{R})$. On the other hand condition $B_n \rightarrow \infty$ cannot be removed. To see this, we take $a_0 = 1, a_1 = -1$, and $a_i = 0$ for $i \neq 0, 1$ and (ξ_k) i.i.d. with mean zero and finite second moment. Then, $S_n = \xi_1 - \xi_{n+1}$, $B_n < \infty$, the distribution of S_n is not invariant, and (12) cannot hold.

In the short memory case for a smaller class of Markov chain, using an elliptic condition that implies ψ -mixing, [Merlevède et al. \(2021\)](#) in Corollary 2.6 obtained a local limit theorem under the additional assumption $\inf_j |a_1 + \dots + a_j| > 0$.

3. Examples

First we mention the example of a strictly stationary Markov chain satisfying (3), which is provided in [Bradley \(1997\)](#), Remark 1.5.

Example 1. Assume that in the Markov chain $(v_k)_{k \in \mathbb{Z}}$ the marginal distribution of v_0 is uniformly distributed on $[0, 1]$ and the one step transition probabilities are as follows:

For each $x \in [0, 1]$, let

$$\mathbb{P}(v_1 = x | v_0 = x) = 1/2$$

and for any Borel set $B \subset [0, 1] - x$,

$$\mathbb{P}(v_1 \in B | v_0 = x) = (1/2)\lambda(B),$$

where λ is the Lebesgue measure on $[0, 1]$. Then (3) holds with $a = 1/2$.

Next we give some particular examples of classes of coefficients for which the conclusion of Theorem 1 is satisfied, namely satisfying $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$ and $B_n^2 \rightarrow \infty$.

Example 2. Assume that $a_i = l(|i|)|i|^{-\alpha}$ with $\alpha \in (1/2, 1)$, where $l(x)$ is a slowly varying function at ∞ . In this case $B_n^2 \propto n^{3-2\alpha} l^2(n)$ since $B_n^2 \propto \tau_n^2 = \sum_{i \in \mathbb{Z}} b_{n,i}^2 \propto n^{3-2\alpha} l^2(n)$ (see, Peligrad and Utev (2006) or Theorem 2 later for the first part and Surgailis (1982), Theorem 2, or Wang et al. (2001) for the second part).

In particular, we mention the fractionally integrated processes defined by

$$X_k = (1 - B)^{-d} \xi_k = \sum_{i \geq 0} a_i \xi_{k-i} \text{ with } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)},$$

where $0 < d < 1/2$, B is the backward shift operator, $B\xi_k = \xi_{k-1}$. For this example, by the well-known fact that for any real x , $\lim_{n \rightarrow \infty} \Gamma(n+x)/n^x \Gamma(n) = 1$, we have $\lim_{n \rightarrow \infty} a_n/n^{d-1} = 1/\Gamma(d)$.

Example 3. Define $a_0 = 1$, $a_i = 0$ for $i < 0$ and $a_i = (i+1)^{-\alpha} - i^{-\alpha}$ for $i \geq 1$ with $\alpha \in [0, 1/2)$. By Example 2 of Dedecker et al. (2011), $B_n^2 \propto \tau_n^2 = \sum_{i \in \mathbb{Z}} b_{n,i}^2 \propto n^{1-2\alpha} \rightarrow \infty$.

Example 4. Define $a_0 = 1$, $a_i = 0$ for $i < 0$ and $a_i = i^{-1/2}(\log i)^{-\alpha}$ for $i \geq 1$ with $\alpha > 1/2$. Then $B_n^2 \propto \tau_n^2 = \sum_{i \in \mathbb{Z}} b_{n,i}^2 \propto n^2(\log n)^{1-2\alpha}$.

4. Proof

4.1. Facts about the lower-elliptic condition

Relation to lower ψ -mixing coefficient.

We introduce now a mixing condition which is comparable to condition (3).

Following Bradley (2007), for any two σ -algebras $\mathcal{A}$ and $\mathcal{B}$ define the lower ψ -mixing coefficient by

$$\psi'(\mathcal{A}, \mathcal{B}) = \inf \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)}; \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B}, \mathbb{P}(A)\mathbb{P}(B) > 0.$$

Obviously $0 \leq \psi'(\mathcal{A}, \mathcal{B}) \leq 1$.

Notice that, in terms of conditional probabilities defined by (1), we also have the following equivalent definition:

$$\psi'_1 = \text{ess inf}_x \inf_{A \in \mathcal{B}(\mathcal{X})} Q(x, A)/\mathbb{P}(v_0 \in A).$$

Note that if (3) holds, then $\psi'_1 \geq a > 0$ and if $\psi'_1 > 0$ then (3) holds with $a = \psi'_1$.

An important step in the proof of Theorem 1 is the following reformulation of Theorem 1 in Peligrad and Utev (2006):

Theorem 2 (Peligrad and Utev, 2006). Assume that the innovations (ξ_i) of the linear process are defined by (4), have mean 0, finite second moment, and satisfy (3). The coefficients satisfy $\sum_{i \in \mathbb{Z}} a_i^2 < \infty$. Let S_n and B_n be defined as in (8) and (9) and assume $B_n \rightarrow \infty$. Define

$$\tau_n^2 = \sum_{i \in \mathbb{Z}} b_{n,i}^2.$$

Then there is $\sigma > 0$, such that

$$\lim_{n \rightarrow \infty} \frac{B_n^2}{\tau_n^2} = \sigma^2, \quad (13)$$

and

$$\frac{S_n}{B_n} \Rightarrow N(0, 1).$$

The constant $\sigma^2 = 2\pi f(0)$, where $f(x)$ is the spectral density of $(\xi_i)_{i \in \mathbb{Z}}$.

Let us explain how to get this reformulation. By Theorem 7.4 (d) in Bradley (2007) and by condition (3), we have $1 - \psi'_n \leq (1 - a)^n \rightarrow 0$ (i.e. $(v_k)_{k \geq 0}$ is lower ψ -mixing).

Furthermore, if for $h \in L^2(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ we denote by

$$Qh(x) = \int h(y)Q(x, dy)$$

and if h is centered at expectation then, by Lemma 10 in Merlevède et al. (2021) (which is actually due to R. Bradley), we know that

$$\|Qh(x)\|_2 \leq (1 - \psi')\|h\|_2. \quad (14)$$

By stationarity and because $\sigma(\xi_n, n \leq -j) \subset \mathcal{F}_{-j} = \sigma(v_n, n \leq -j)$, and the sequence is Markov, we obtain

$$\begin{aligned} |\mathbb{E}(\xi_k \mathbb{E}(\xi_0 | \sigma(\xi_n, n \leq -j)))| &\leq \|\xi_k\|_2 \cdot \|\mathbb{E}(\xi_0 | \mathcal{F}_{-j})\|_2 \\ &= \|\xi_0\|_2 \cdot \|Q^j(\xi_0)\|_2 \leq (1 - a)^j \mathbb{E}(\xi_0^2). \end{aligned}$$

By this latter inequality we can see that condition (2) in Peligrad and Utev (2006) is satisfied and the conclusion of Theorem 1 there holds. Since a lower ν -mixing sequence is ergodic, the invariant sigma field is trivial. Furthermore, we have that the spectral density $f(x)$ of $(\xi_i)_{i \in \mathbb{Z}}$ exists, is continuous and $f(0) = \lim_{n \rightarrow \infty} \mathbb{E}(\sum_{i=1}^n \xi_i)^2 / n$. Furthermore $f(0) > 0$ because, as explained in Merlevède et al. (2022)

$$\frac{a}{2-a} \mathbb{E} \xi_0^2 \leq \frac{\mathbb{E}(\sum_{i=1}^n \xi_i)^2}{n} \leq \frac{2-a}{a} \mathbb{E} \xi_0^2. \quad (15)$$

By the Slutsky's theorem, an equivalent formulation of this result is

$$\frac{S_n}{\tau_n} \Rightarrow N(0, \sigma^2).$$

This theorem also follows from Theorem 12.8 in the book (Merlevède et al., 2019) and its proof.

In order to obtain our results we shall need a suitable bound on the characteristic function. For this purpose, we mention Proposition 10 in Merlevède et al. (2022):

Proposition 3 (Merlevède et al., 2022). Let $S_n = \sum_{j=1}^n Y_{j,n}$. Then, for all n , $Y_{j,n} = g_{n,j}(v_j)$ and (3) is satisfied. Then

$$|\mathbb{E}(\exp(iuS_n))| \leq \exp \left[-C \sum_{j=1}^n (1 - |\mathbb{E}(\exp(iuY_{j,n}))|^2) \right]$$

for all u and for some constant $C > 0$, which does not depend on u, n .

4.2. A sufficient condition for the local limit theorem

As a matter of fact, (11) is a special case (in which the following random variable L is the standard normal random variable) of the following local limit theorem. Let (S_n) be a sequence of random variables, (B_n) be a sequence of positive numbers and L be a random variable with characteristic function f_L . We assume the underlying convergence in distribution:

$$\frac{S_n}{B_n} \Rightarrow L, \text{ where } f_L \text{ is integrable and } B_n \rightarrow \infty. \quad (16)$$

In the case when the variables (S_n) do not have values in a fixed minimal lattice, the sequence is said to satisfy a local limit theorem if for any continuous function g on $\mathbb{R}$ with compact support,

$$\lim_{n \rightarrow \infty} \sup_{u \in \mathbb{R}} \left| B_n \mathbb{E} g(S_n + u) - h_L \left(-\frac{u}{B_n} \right) \int g(t) \lambda(dt) \right| = 0, \quad (17)$$

where λ is the Lebesgue measure.

Note that the integrability of f_L implies that L has a continuous density function which we denote by h_L (see pages 370–371 in Billingsley, 1995).

Here is a theorem in Peligrad et al. (2022) (Theorem 4.1), which is an important tool in the proof of Theorem 1.

Theorem 4 (Peligrad et al. 2022). Assume that the convergence in distribution in (16) holds. In addition, suppose that for each $D > 0$

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T < |t| \leq DB_n} \left| \mathbb{E} \exp \left(it \frac{S_n}{B_n} \right) \right| dt = 0. \quad (18)$$

Then (17) holds.

4.3. Proof of Theorem 1

To prove Theorem 1, we will first show that S_n defined by (10) can be reduced to a finite sum from $-N_n$ to N_n for N_n sufficiently large. Then we prove the local limit theorem for this finite sum by using an analytical and combinatorial approach.

Under the conditions of Theorem 1 we know that the CLT in Theorem 2 holds. Hence, to prove Theorem 1, we only need to verify (18) of Theorem 4.

Step 1. In this step, we show that we can reduce the problem to considering a finite sum. Let N_n be a positive integer such that

$$\sum_{i > N_n \text{ or } i < -N_n} a_i^2 \leq e^{-n}.$$

We use the decomposition

$$S_n := S'_n + S''_n \text{ where } S'_n = \sum_{|i| \leq N_n} b_{n,i} \xi_i, \quad S''_n = \sum_{|i| > N_n} b_{n,i} \xi_i.$$

But

$$\sum_{|i|>N_n} b_{n,i}^2 = \sum_{|i|>N_n} \left(\sum_{k=1}^n a_{k-i} \right)^2 \leq n \sum_{|i|>N_n} \sum_{k=1}^n a_{k-i}^2 \leq n^2 \sum_{i>N_n \text{ or } i<n-N_n} a_i^2 \leq n^2 e^{-n},$$

and so

$$\lim_{n \rightarrow \infty} \sum_{|i|>N_n} b_{n,i}^2 = 0.$$

Then, by relation (3.4) in Merlevède et al. (2021) or (21) in Merlevède et al. (2022)

$$\mathbb{E}(S_n'')^2 = \mathbb{E}(S_n - S_n')^2 \leq c_2 \mathbb{E}(\xi_0^2) \sum_{|i|>N_n} b_{n,i}^2 \leq c_2 \mathbb{E}(\xi_0^2) n^2 e^{-n}, \quad (19)$$

and so

$$\lim_{n \rightarrow \infty} \mathbb{E}(S_n'')^2 = 0.$$

We can see that, with the notation

$$(\tau_n')^2 = \sum_{|i| \leq N_n} b_{n,i}^2 \quad (20)$$

we easily obtain by the above arguments that

$$\lim_{n \rightarrow \infty} \frac{(\tau_n')^2}{\tau_n^2} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(S_n')^2}{\mathbb{E}S_n^2} = 1. \quad (21)$$

Now, by the mean value theorem and (19), for any $t \in \mathbb{R}$,

$$\begin{aligned} \left| \mathbb{E} \exp \left(it \frac{S_n}{B_n} \right) - \mathbb{E} \exp \left(it \frac{S_n'}{B_n} \right) \right| &\leq 2|t| \mathbb{E} \left| \frac{S_n - S_n'}{B_n} \right| \\ &\leq \frac{2|t|}{B_n} \sqrt{\mathbb{E}(S_n'')^2} = \frac{2|t|}{B_n} \sqrt{c_2 \mathbb{E}(\xi_0^2) n^2 e^{-n}}. \end{aligned}$$

Hence, for any $D > 0$,

$$\begin{aligned} &\left| \int_{T \leq |t| \leq DB_n} \mathbb{E} \exp \left(it \frac{S_n}{B_n} \right) dt - \int_{T \leq |t| \leq DB_n} \mathbb{E} \exp \left(it \frac{S_n'}{B_n} \right) dt \right| \\ &\leq \frac{2n}{B_n} \sqrt{c_2 \mathbb{E}(\xi_0^2) e^{-n}} \int_{T \leq |t| \leq DB_n} |t| dt \leq 2n D^2 B_n \sqrt{c_2 \mathbb{E}(\xi_0^2) e^{-n}}. \end{aligned}$$

Since X_n is stationary, $\text{Var}(S_n) = B_n^2 \leq n^2 \text{Var}(X)$, we have

$$n B_n \sqrt{e^{-n}} \leq n^2 \sqrt{e^{-n}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and we obtain

$$\begin{aligned} &\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T \leq |t| \leq DB_n} \left| \mathbb{E} \exp \left(it \frac{S_n}{B_n} \right) \right| dt \\ &= \lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T \leq |t| \leq DB_n} \left| \mathbb{E} \exp \left(it \frac{S_n'}{B_n} \right) \right| dt. \end{aligned}$$

Step 2. By the arguments in Step 1, we reduced the proof to showing that

$$\lim_{T \rightarrow \infty} \limsup_n \int_{T < |t| \leq DB_n} \left| \mathbb{E} \exp \left(it \frac{S_n'}{B_n} \right) \right| dt = 0.$$

By Proposition 3, there is $c > 0$ such that for any $t \in \mathbb{R}$

$$\left| \mathbb{E} \exp \left(it \frac{S_n'}{B_n} \right) \right| \leq \exp \left(-c \sum_{|j| \leq N_n} (1 - |\mathbb{E} \exp(it \frac{b_{n,j}}{B_n} \xi_0)|^2) \right).$$

Therefore the theorem is established if we are able to prove

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{T \leq |t| \leq DB_n} \exp \left(-c \sum_{|j| \leq N_n} (1 - |\mathbb{E} \exp(it \frac{b_{n,j}}{B_n} \xi_0)|^2) \right) dt = 0. \quad (22)$$

We order now the numbers $(|b_{n,i}|)_{-N_n \leq i \leq N_n}$ in a decreasing order and denote them as $(|\tilde{b}_{n,j}|)_{1 \leq j \leq 2N_n+1}$. Recall $(\tau_n')^2$ is defined by (20), i.e.

$$(\tau_n')^2 = \sum_{1 \leq k \leq 2N_n+1} \tilde{b}_{n,k}^2.$$

We shall construct a sequence $(k_n)_n$, with the property that for fixed n , k_n partitions $(\tau'_n)^2$ into two approximately equal parts, one containing the sum up to k_n , and the other from $k_n + 1$ to $2N_n + 1$. More exactly, k_n is defined in the following way:

$$k_n = \inf \left\{ \ell; 1 \leq \ell \leq 2N_n + 1 \text{ and } \sum_{1 \leq |k| \leq \ell} \tilde{b}_{n,k}^2 \geq 2^{-1} (\tau'_n)^2 \right\}.$$

Then

$$\frac{1}{2} \leq \frac{1}{(\tau'_n)^2} \sum_{1 \leq k \leq k_n} \tilde{b}_{n,k}^2 \leq \frac{1}{2} + \left(\frac{\gamma'_n}{\tau'_n} \right)^2, \quad (23)$$

where

$$\gamma'_n := \tilde{b}_{n,1} = \max_{1 \leq i \leq 2N_n+1} \tilde{b}_{n,i}.$$

Since $\gamma_n/B_n \rightarrow 0$ as $n \rightarrow \infty$, by [Lemma 8](#) in [Appendix](#) and the fact that $B_n \rightarrow \infty$, we also have $\gamma'_n/\tau'_n \rightarrow 0$, and therefore there exists a constant $C > 0$, such that for sufficiently large n ,

$$\frac{1}{2} \leq \frac{1}{(\tau'_n)^2} \sum_{1 \leq k \leq k_n} \tilde{b}_{n,k}^2 \leq \frac{3}{4}. \quad (24)$$

and also

$$\frac{1}{2} \leq \frac{1}{(\tau'_n)^2} \sum_{1 \leq k \leq k_n} \tilde{b}_{n,k}^2 \leq k_n \left(\frac{\gamma'_n}{\tau'_n} \right)^2.$$

Hence,

$$k_n \geq \frac{1}{2} \left(\frac{\tau'_n}{\gamma'_n} \right)^2. \quad (25)$$

It is convenient to denote

$$\begin{aligned} & \sum_{|j| \leq N_n} (1 - |\mathbb{E} \exp(it \frac{b_{n,j}}{B_n} \xi_0)|^2) \\ &= \sum_{1 \leq j \leq 2N_n+1} (1 - |\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2) = H_n(t). \end{aligned}$$

For a $d > 0$ fixed, to be selected later, we write

$$\begin{aligned} & \int_{T < |t| \leq dB_n} \exp(-cH_n(t)) dt \leq \\ &= \int_{T < |t| \leq dB_n/|\tilde{b}_{n,k_n}|} \exp(-cH_n(t)) dt + \int_{dB_n/|\tilde{b}_{n,k_n}| < |t| \leq dB_n} \exp(-cH_n(t)) dt \\ &= I_n(T) + II_n(T). \end{aligned}$$

For the first part $I_n(T)$, we use the fact that if $|t| \leq dB_n/|\tilde{b}_{n,k_n}|$, then $|t| \leq dB_n/|\tilde{b}_{n,j}|$ for all $k_n < j \leq 2N_n + 1$. Using [Lemma 8](#) in [Appendix](#) we easily get,

$$\sup_{1 \leq j \leq 2N_n+1} \frac{|\tilde{b}_{n,j}|}{B_n} \leq \frac{\gamma_n}{B_n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

by [Corollary 7](#) in the [Appendix](#), there exists $N(d)$ such that for $n > N(d)$ then

$$|\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2 \leq 1 - \frac{2}{3} t^2 \frac{\tilde{b}_{n,j}^2}{B_n^2} \mathbb{E} \xi_0^2.$$

So for all $j \geq k_n$ and $|t| \leq dB_n/|\tilde{b}_{n,k_n}|$

$$1 - |\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2 \geq \frac{2}{3} t^2 \frac{\tilde{b}_{n,j}^2}{B_n^2} \mathbb{E} \xi_0^2.$$

Therefore

$$\begin{aligned} H_n(t) &\geq \sum_{k_n < j \leq N_n} (1 - |\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2) \\ &\geq \frac{2}{3 B_n^2} t^2 \sum_{k_n < j \leq N_n} \tilde{b}_{n,j}^2 \mathbb{E} \xi_0^2, \end{aligned}$$

and by (13), (21), (24) we can find a constant $b > 0$ such that for n sufficiently large

$$H_n(t) \geq bt^2.$$

It follows that for n sufficiently large,

$$\begin{aligned} \int_{T < |t| \leq dB_n/|\tilde{b}_{n,k_n}|} \exp(-cH_n(t)) dt &\leq \int_{T < |t| \leq dB_n/|\tilde{b}_{n,k_n}|} \exp(-bt^2) dt \\ &\leq \int_{T < |t|} \exp(-bt^2) dt. \end{aligned}$$

We immediately obtain

$$\lim_{T \rightarrow \infty} \limsup_n I_n(T) = 0.$$

Now we analyze the part $II_n(T)$. We shall use separate arguments for the cases when the linear process has short memory or long memory. If the sequence (X_k) has short memory, i.e. $b_0 = \sum_{j \in \mathbb{Z}} |a_j| < \infty$, then the sequence $(|b_{n,i}|)$ is uniformly bounded by b_0 for $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Notice that $d \leq d|\tilde{b}_{n,j}|/|\tilde{b}_{n,k_n}| \leq t|\tilde{b}_{n,j}|/B_n \leq Db_0$ for $dB_n/|\tilde{b}_{n,k_n}| < |t| \leq DB_n$ and $1 \leq j \leq k_n$. Since the innovations have a non-lattice distribution and φ_{ξ_0} is continuous, there exist $0 < r_1 < 1$ and $d > 0$ such that

$$|\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)| = |\varphi_{\xi_0}(t\tilde{b}_{n,j}/B_n)| \leq r_1$$

for all $dB_n/|\tilde{b}_{n,k_n}| < |t| \leq DB_n$ and $1 \leq j \leq k_n$.

In the long memory case, by the Cramér condition we know that

$$\lim_{u \rightarrow \infty} |\mathbb{E}(e^{iu\xi_0})|^2 < r' < 1.$$

For $\varepsilon > 0$ small enough, we can find $d = d_\varepsilon > 0$ and have

$$|\mathbb{E}(e^{iu\xi_0})|^2 < r' + \varepsilon = r_2 < 1 \text{ for } |u| > d.$$

If $|t| > dB_n/|\tilde{b}_{n,k_n}|$ we have $|t\tilde{b}_{n,j}/B_n| > d$ for $1 \leq j \leq k_n$ and then

$$|\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2 < r_2 < 1.$$

We take $r = \max\{r_1, r_2\}$. Therefore, for $|t| > dB_n/|\tilde{b}_{n,k_n}|$ and $1 \leq j \leq k_n$ we have

$$1 - |\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2 \geq 1 - r,$$

and then,

$$H_n(t) \geq \sum_{1 \leq j \leq k_n} (1 - |\mathbb{E} \exp(it \frac{\tilde{b}_{n,j}}{B_n} \xi_0)|^2) \geq k_n(1 - r).$$

So we can find $c' > 0$ such that

$$\begin{aligned} \int_{dB_n/|\tilde{b}_{n,k_n}| < |t| \leq DB_n} \exp(-cH_n(t)) dt &\leq \exp(-ck_n(1 - r)) \int_{dB_n/|\tilde{b}_{n,k_n}| < |t| \leq DB_n} dt \leq \exp(-ck_n(1 - r)) DB_n \\ &\leq DB_n \exp(-c'k_n). \end{aligned}$$

Since by bound (25), (13) and (21) we can find $C > 0$ such that

$$k_n \geq \frac{1}{2} \left(\frac{\tau'_n}{\gamma'_n} \right)^2 \geq C \left(\frac{B_n}{\gamma_n} \right)^2.$$

we have

$$B_n \exp(-c'k_n) \leq B_n \exp(-c'C \left(\frac{B_n}{\gamma_n} \right)^2),$$

which converges to 0 as $n \rightarrow \infty$ by Lemma 8 in the Appendix and the fact that $B_n \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Data availability

No data was used for the research described in the article.

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Appendix

We gather here some preliminary results on the characteristic function of a random variable.

Lemma 5. *Let X be a random variable with real characteristic function. Then, for any $t \neq 0$*

$$\mathbb{E}(e^{itX}) \leq 1 - \frac{t^2}{3} \mathbb{E}(X^2 I(|X| \leq |t|^{-1})).$$

Proof. Clearly

$$\mathbb{E}(e^{itX}) = \mathbb{E}(\cos tX) = \mathbb{E}((\cos tX)I(|X| > |t|^{-1})) + \mathbb{E}((\cos tX)I(|X| \leq |t|^{-1})).$$

By Taylor expansion, for a certain ξ such that for $y > 0$, $0 < \xi < y$ and for $y < 0$, $y < \xi < 0$

$$\cos y = 1 - \frac{1}{2}y^2 + \frac{1}{6}y^3 \sin(\xi) \leq 1 - \frac{1}{2}y^2 + \frac{1}{6}|y|^3.$$

Replacing y by tX ,

$$\begin{aligned} \mathbb{E}(e^{itX}) &= \mathbb{E}(\cos(tX)I(|X| > |t|^{-1})) + \mathbb{E}(\cos(tX)I(|X| \leq |t|^{-1})) \\ &\leq \mathbb{P}(|X| > |t|^{-1}) + \mathbb{E}\left(\left(1 - \frac{1}{2}t^2X^2 + \frac{1}{6}|tX|^3\right)I(|X| \leq |t|^{-1})\right). \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(e^{itX}) &\leq \mathbb{P}(|X| > |t|^{-1}) + \mathbb{P}(|X| \leq |t|^{-1}) - \frac{1}{2}t^2 \mathbb{E}(X^2 I(|X| \leq |t|^{-1})) \\ &\quad + \frac{1}{6} \mathbb{E}(|tX|^3 I(|X| \leq |t|^{-1})) \\ &\leq 1 - \frac{1}{2}t^2 \mathbb{E}(X^2 I(|X| \leq |t|^{-1})) + \frac{1}{6}t^2 \mathbb{E}(X^2 I(|X| \leq |t|^{-1})) \\ &= 1 - \frac{1}{3}t^2 \mathbb{E}(X^2 I(|X| \leq |t|^{-1})). \quad \square \end{aligned}$$

Lemma 6. *Let X be a random variable with real characteristic function and let (d_n) be a sequence of constants with $d_n \rightarrow 0$ as $n \rightarrow \infty$. Then for any $A > 0$, there is $N(A)$ such that if $|td_n| < A$ and $n > N(A)$ then*

$$\mathbb{E}(e^{itd_nX}) \leq 1 - \frac{1}{3}t^2d_n^2 \mathbb{E}X^2.$$

Proof. Note first that, if $|t| < A$,

$$\mathbb{E}X^2 I(|X| \leq |A|^{-1}) \leq \mathbb{E}X^2 I(|X| \leq |t|^{-1}).$$

So, by Lemma 5

$$\mathbb{E}(e^{itX}) \leq 1 - \frac{1}{3}t^2 \mathbb{E}X^2 I(|X| \leq |t|^{-1}) \leq 1 - \frac{1}{3}t^2 \mathbb{E}X^2 I(|X| \leq |A|^{-1}). \quad (26)$$

Because d_n tends to 0,

$$\lim_{n \rightarrow \infty} \mathbb{E}X^2 I(|X| \leq \frac{1}{Ad_n}) = \mathbb{E}X^2.$$

So for any $A > 0$ there is $N(A)$ such that for $n > N(A)$,

$$\mathbb{E}X^2 I(|X| \leq \frac{1}{Ad_n}) \geq \frac{1}{2} \mathbb{E}X^2.$$

Then, clearly for any $n > N(A)$,

$$\mathbb{E}X^2 I(|X| \leq \frac{1}{Ad_n}) \geq \frac{1}{2} \mathbb{E}X^2.$$

Combining this with (26), for any A and n large enough, such that $|td_n| < A$,

$$\begin{aligned} \mathbb{E}(e^{itd_nX}) &\leq 1 - \frac{1}{3}t^2d_n^2 \mathbb{E}X^2 I(|d_nX| \leq |A|^{-1}) \\ &\leq 1 - \frac{1}{3}t^2d_n^2 \mathbb{E}X^2. \quad \square \end{aligned}$$

Now let X' be an independent copy of X (this means that the sequences are independent of another and have the same distribution), and denote their difference by $\tilde{X} = X - X'$. Then

$$|\mathbb{E} \exp(itX)|^2 = \mathbb{E}(\exp(it(X - X'))) = \mathbb{E}(\exp(it\tilde{X})).$$

Also $\mathbb{E}\tilde{X}^2 = 2\mathbb{E}X^2$, and we immediately obtain by Lemma 6 the following corollary:

Corollary 7. For any random variable X and any $A > 0$, there is $N(A)$ such that if $|td_n| < A$ and $n > N(A)$ then

$$|\mathbb{E}(e^{itd_n X})|^2 \leq 1 - \frac{2}{3}t^2 d_n^2 \mathbb{E}X^2. \quad (27)$$

Next, we give a Lemma concerning the behavior of γ_n/B_n .

Lemma 8. Under the conditions of Theorem 1

$$\frac{\gamma_n}{B_n} = O\left(\frac{1}{B_n^{1/2}}\right).$$

Proof. The main part of the proof is similar to that of Proposition 2 in Mallik and Woodroffe (2011), and we give it here for completeness. Let $\gamma_n = |b_{n,r_0}|$. By the triangle inequality, for any integer r

$$|b_{n,r_0}| \leq |b_{n,r_0+r}| + |b_{n,r_0+r} - b_{n,r_0}|.$$

Let us add these relations for r between 1 and m . We obtain

$$m|b_{n,r_0}| \leq \sum_{r=1}^m |b_{n,r_0+r}| + \sum_{r=1}^m |b_{n,r_0+r} - b_{n,r_0}|.$$

To estimate the first term in the right hand side, we apply first the Cauchy-Schwartz inequality. Then, by taking into account the definition of B_n and (13), we obtain

$$\sum_{r=1}^m |b_{n,r_0+r}| \leq \sqrt{m} \left(\sum_{k \in \mathbb{Z}} b_{n,k}^2 \right)^{1/2} = O(\sqrt{m} B_n).$$

For the second term, by the definition of $b_{n,k}$, by the Cauchy-Schwartz inequality and (6), we have

$$|b_{n,r_0+r} - b_{n,r_0}| \leq \left(2r \sum_{i \in \mathbb{Z}} a_i^2 \right)^{1/2} = O(r^{1/2}).$$

Overall,

$$m|b_{n,r_0}| = O(\sqrt{m} B_n + m^{3/2}).$$

Taking $m = B_n$, then

$$\frac{\gamma_n}{B_n} = \frac{|b_{n,r_0}|}{B_n} = O\left(\frac{1}{B_n^{1/2}}\right). \quad \square$$

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