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Convergence of series of conditional expectations

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This paper deals with almost sure convergence for partial sums of Banach space valued random variables. The results are then applied to solve similar problems for weighted partial sums of conditional expectations. They are further used to treat partial sums of powers of a reversible Markov chain operator. The method of proof is based on martingale approximation. The conditions are expressed in terms of moments of individual summands.

1. Introduction

Let $(X_j)_{j\geq 1}$ be a sequence random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a separable Banach space, adapted to $(\mathcal{F}^j)_{j\geq 1}$, a decreasing sequence of sub-sigma algebras of $\mathcal{F}$. We are going to study the convergence of series $S_n = \sum_{j=1}^n X_j$. For integrable X, define the conditional expectation $E^j X = E(X|\mathcal{F}^j)$. Special attention will be given to the situation when $X_j = a_j E(X|\mathcal{F}^j)$ for $(a_j)_{j\geq 1}$ a sequence of real constants. Given a stationary and reversible Markov chain $(\xi_i)_{i\in Z}$ with values in a measurable space $(S, \mathcal{S})$, for an integrable function f defined on S with values in a separable Banach space, let $X_j = f(\xi_j)$. We denote $Q^k f(\xi_0) = E(X_k|\xi_0) = E_0(X_k)$, and also derive similar results for $\sum_{j=1}^k a_j Q^j f$. It should be noted that Cohen et al. (2017) studied the almost sure convergence of L_p contractions for the series in the form $\sum_{k=1}^n a_k Q^k f$ where a_k is a Kaluza sequence with divergent sum, Q a power bounded operator and $\sum_{k\geq 1} \beta_k z^k$ converges in the open unit disk. We consider a general sequence of constants and impose our conditions on the moments of $|E^k X|$.

The motivation for this study comes from a remarkable result, Theorem 3.11 in Derriennic and Lin (2001). In the context of additive functionals of stationary reversible Markov chains, for f centered at expectation and square integrable, if $E(S_n^2)/n \to \sigma^2$ then we have that $\sum_{k=1}^n k^{-1/2} Q^k f$ converges a.s. Theorem 3.9 in the same paper has a similar result for Harris recurrent Markov operators and some special functions f.

As in Dedecker and Merlevède (2008) or Cuny (2015), whenever possible, we shall work with variables in a separable Banach space B. For $x \in B$, for simplicity, we shall denote the norm by $|x| = |x|_B$.

We denote by L_p the set of measurable functions X defined on a probability space, with values in a separable Banach space such that $\|X\|_p^p = E|X|^p < \infty$. For random variables X in L_p, the notation $\|X\|_p \ll b$ means that there is a constant

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C_p such that $\|X\|_p \leq C_p b$. For two sequences of positive constants $(a_n)_n$, $(b_n)_n$, $a_n \ll b_n$ means that there is a constant C such that $a_n \leq C b_n$.

For integrable X, recall the notation $E^j X = E(X|\mathcal{F}^j)$, $j \geq 1$. For these definitions we direct the reader to the book of Ledoux and Talagrand (1991). Now denote the reverse martingale difference adapted to $(\mathcal{F}^i)_{i\geq 1}$ by

$$P^i(X) = E^i X - E^{i+1} X. \tag{1}$$

Sometimes we shall assume in addition that the Banach space is separable and r-smooth for an r such that $1 < r \leq 2$. We shall use this property or rather its consequence, for any sequence of B-valued martingale differences $(X_i)_{i\geq 1}$, if B is separable and r-smooth then for some $D > 0$,

$$E|X_1 + X_2 + \cdots + X_n|^r \leq D(E|X_1|^r + E|X_2|^r + \cdots + E|X_n|^r). \tag{2}$$

(see Assouad, 1975).

We should mention that, according to our knowledge, our results are also new for real-valued random variables.

The paper is divided into three parts. In Section 2, we obtain new maximal inequalities. Then, in Section 3, we apply these maximal inequalities to obtain convergence of series. Finally, in Section 4, we apply the results to reversible Markov chains.

2. Maximal inequalities

Proposition 1. *Let $p > 1$ and let $(X_j)_{j\geq 1}$ be a sequence of Banach space valued random variables in L_p, adapted to a sequence of decreasing sub-sigma fields of $\mathcal{F}$, $(\mathcal{F}^k)_{k\geq 1}$. Then*

$$E \max_{1\leq k\leq n} |S_k|^p \ll E \max_{1\leq k\leq n} |E^k S_k|^p + E\,|S_n|^p\,.$$

For $1 \leq p \leq 2$ and if B is separable and p-smooth, we also have

$$E \max_{1\leq k\leq n} |S_k|^p \ll E \max_{1\leq k\leq n} |E^k S_k|^p + \sum_{i=1}^{n-1} E|P^i(S_i)|^p\,.$$

Proof. Assume $S_0 = 0$. It is easy to see that

$$S_n = E^n S_n + \sum_{i=1}^{n-1} \left(E^i S_i - E^{i+1} S_i\right). \tag{3}$$

By taking the maximum

$$\max_{1\leq k\leq n} |S_k| \leq \max_{1\leq k\leq n} |E^k S_k| + \max_{2\leq k\leq n} \left|\sum_{i=1}^{k-1} P^i(S_i)\right|, \tag{4}$$

whence, by Minkowski's type inequality for norms in L_p we obtain

$$E \max_{1\leq k\leq n} |S_k|^p \ll E \max_{1\leq k\leq n} |E^k S_k|^p + E \max_{2\leq k\leq n} \left|\sum_{i=1}^{k-1} P^i(S_i)\right|^p.$$

By writing

$$\left|\sum_{i=1}^{k-1} P^i(S_i)\right|^p \ll \left|\sum_{i=1}^{n-1} P^i(S_i)\right|^p + \left|\sum_{i=k}^{n-1} P^i(S_i)\right|^p,$$

we deduce that

$$E \max_{2\leq k\leq n} \left|\sum_{i=1}^{k-1} P^i(S_i)\right|^p \ll E\left|\sum_{i=1}^{n-1} P^i(S_i)\right|^p + E \max_{1\leq k\leq n-1} \left|\sum_{i=k}^{n-1} P^i(S_i)\right|^p. \tag{5}$$

Since $P^k(S_k) = E^k S_k - E^{k+1} S_k$ is a reverse martingale difference adapted to the decreasing sequence of sigma algebras $(\mathcal{F}^k)_{k\geq 1}$, by Doob's maximal inequality for submartingales, for $p > 1$

$$E \max_{1\leq k\leq n-1} \left|\sum_{i=k}^{n-1} P^i(S_i)\right|^p \ll E\left|\sum_{i=1}^{n-1} P^i(S_i)\right|^p. \tag{6}$$

By relation (3),

$$E\left|\sum_{i=1}^{n-1} P^i(S_i)\right|^p \ll E|E^n S_n|^p + E\,|S_n|^p\,,$$

and the first inequality in this proposition follows.

The second inequality follows from (4) and (6) combined with (2). □

Remark 2. Note that if $p = 2$ and if the variables have values in a separable Hilbert space, then, for all $n \geq 1$,

$$E\left|\sum_{i=1}^{n} P^i(S_i)\right|^2 = \sum_{i=1}^{n}\left(E\left|E^i S_i\right|^2 - E\left|E^{i+1} S_i\right|^2\right).$$

We treat next linear combinations $\sum_{j=1}^{k} a_j E^j X$ with a_j a sequence of constants. We take a_j real, but complex valued constants can be treated in the same way. We shall assume that $\sum_{i=1}^{\infty}|a_i| = \infty$ since otherwise, by Doob's maximal inequality, we immediately get for $p > 1$

$$E\max_{1\leq k\leq n}\left|\textstyle\sum_{j=1}^{k} a_j E^j X\right|^p \ll E\max_{1\leq k\leq n}\left|E^k X\right|^p \leq E|X|^p,$$

which is finite as soon as $X \in L_p$.

We shall also use the following notations

$$s_j = \sum_{i=1}^{j} a_i,\;\; s_0 = 0,\;\; s_k^* = \max_{1\leq j\leq k}\Big|\sum_{i=1}^{j} a_i\Big|,$$

and

$$b_k = \max\left(k^{-1}\left(s_{4k}^*\right)^2, s_k^2 - s_{k-1}^2\right). \tag{7}$$

The next corollary follows from Proposition 1 applied to $X_i = a_i E^i X$.

Corollary 3. *Let X be a random variable with values in a separable Banach space and, for any $p > 1$, $E|X|^p < \infty$. Then*

$$E\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j E^j X\right|^p \ll E\max_{1\leq k\leq n}\left|s_k E^k X\right|^p + E\left|\sum_{j=1}^{n} a_j E^j X\right|^p. \tag{8}$$

For $1 \leq p \leq 2$ and if B is p-smooth, we also have

$$E\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j E^j X\right|^p \ll E\max_{1\leq k\leq n}\left|s_k E^k X\right|^p + \sum_{i=1}^{n-1} E\left|s_i P^i(X)\right|^p. \tag{9}$$

Remark 4. For $p \geq 1$, an estimate of $E\max_{1\leq k\leq n}\left|s_k E^k X\right|^p$ is

$$E\max_{1\leq k\leq n}\left|s_k E^k X\right|^p \ll \sum_{k=1}^{n} k^{-1}\left(s_{4k}^*\right)^p E\left|E^k X\right|^p.$$

Proof of Remark 4. By Doob's maximal inequality

$$E\max_{2^i\leq k\leq 2^{i+1}}\left|E^k X\right|^p \ll E\left|E^{2^i} X\right|^p.$$

Note that

$$E\max_{1\leq k\leq 2^r}\left|s_k E^k X\right|^p \leq \sum_{i=0}^{r-1} E\max_{2^i\leq k\leq 2^{i+1}}\left|s_k E^k X\right|^p$$

$$\leq \sum_{i=0}^{r-1}\left(s_{2^{i+1}}^*\right)^p E\max_{2^i\leq k\leq 2^{i+1}}\left|E^k X\right|^p \ll \sum_{i=0}^{r-1}\left(s_{2^{i+1}}^*\right)^p E\left|E^{2^i} X\right|^p.$$

Using the fact that $(s_n^*)_{n\geq 1}$ is increasing and $\left(E|E^n X|^p\right)_{n\geq 1}$ is decreasing, we easily obtain

$$E \max_{1\leq k\leq 2^r} \left|s_k E^k X\right|^p \ll \sum_{i=0}^{r-1} \left(s^*_{2^{i+1}}\right)^p E|E^{2^i} X|^p$$
$$\ll \sum_{i=1}^{2^{r-1}} i^{-1} \left(s^*_{4i}\right)^p E|E^i X|^p.$$

Now if $2^{r-1} < n \leq 2^r$, clearly

$$E \max_{1\leq k\leq n} \left|s_k E^k X\right|^p \leq E \max_{1\leq k\leq 2^r} \left|s_k E^k X\right|^p \ll \sum_{k=1}^{n} k^{-1} \left(s^*_{4k}\right)^p E|E^k X|^p,$$

and the result follows. □

For $p = 2$ we also have the following result:

Corollary 5. *Assume that X has values in a separable Hilbert space and $E|X|^2 < \infty$. Then*

$$E \max_{1\leq k\leq n} \left|\sum_{j=1}^{k} a_j E^j X\right|^2 \ll \sum_{k=1}^{n} b_k E|E^k X|^2.$$

Remark 6. In particular, under the conditions of Corollary 5, if $a_j = j^{-1/2}$ for all $j \geq 1$,

$$E \max_{1\leq k\leq n} \left|\sum_{j=1}^{k} j^{-1/2} E^j X\right|^2 \ll \sum_{j=1}^{n} E|E^j X|^2,$$

and if $a_j = 1$ for all $j \geq 1$, then

$$E \max_{1\leq k\leq n} \left|\sum_{j=1}^{k} E^j X\right|^2 \ll \sum_{j=1}^{n} jE|E^j X|^2.$$

Proof of Corollary 5. In a Hilbert space $P^i(X)$ and $P^j(X)$ are orthogonal for $i \neq j$. By the properties of the conditional expectations and Remark 2, ($s_0 = 0$, $n \geq 2$)

$$E\left|\sum_{i=1}^{n-1} s_i P^i(X)\right|^2 = \sum_{i=1}^{n-1} s_i^2 E|P^i(X)|^2$$
$$= \sum_{i=1}^{n-1} s_i^2 \left(E|E^i X|^2 - E|E^{i+1} X|^2\right) \leq \sum_{j=1}^{n-1} \left(s_j^2 - s_{j-1}^2\right) E|E^j X|^2.$$

Combining the latter inequality with Proposition 1 and Remark 4, we get

$$E \max_{1\leq k\leq n} \left|\sum_{j=1}^{k} a_j E^j X\right|^2 \ll \sum_{k=1}^{n} k^{-1} \left(s^*_{4k}\right)^2 E|E^k X|^2$$
$$+ \sum_{k=1}^{n-1} \left(s_k^2 - s_{k-1}^2\right) E|E^k X|^2$$
$$\leq \sum_{k=1}^{n} b_k E|E^k X|^2,$$

where b_k is given in (7). □

3. Convergence of series

We give some straightforward applications of the maximal inequalities established before.

Proposition 7. *Let $p > 1$ and let $(X_j)_{j\geq 1}$ be a sequence of Banach space valued random variables. Assume that $(S_n)_{n\geq 1}$ is bounded in L_p and $(E^n S_n)_{n\geq 1}$ converges in L_p. Then $(S_n)_{n\geq 1}$ converges in L_p. If in addition $(E^n X_n)_{n\geq 1}$ converges a.s. then $(S_n)_{n\geq 1}$ converges a.s.*

Proof of Proposition 7. We start from the representation given in (3).

Now note that $\sum_{i=1}^{n-1}\left(E^i S_i - E^{i+1} S_i\right)$ is a reverse martingale, which converges a.s. and in L_p provided

$$\sup_n E\left|\sum_{i=1}^{n-1}\left(E^i S_i - E^{i+1} S_i\right)\right|^p < \infty.$$

By (3) and the triangle inequality this condition is satisfied under the conditions of this proposition, and the result follows. □

If we apply the previous proposition to $X_i = a_i X$ we obtain the following corollary. We denote as before $s_n = \sum_{i=1}^n a_i$.

Corollary 8. *Let $p > 1$. Assume that $(\sum_{i=1}^n a_i E^i X)_{n\geq 1}$ is bounded in L_p and $(s_n E^n X)_{n\geq 1}$ converges in L_p. Then $\left(\sum_{i=1}^n a_i E^i X\right)_{n\geq 1}$ converges in L_p. If in addition $(s_n E^n X)_{n\geq 1}$ converges a.s. then $\left(\sum_{i=1}^n a_i E^i X\right)_{n\geq 1}$ converges a.s.*

Corollary 9. *Assume that X has values in a separable Hilbert space. Define (b_k) by (7), and assume that*

$$\sum_{k\geq 1} b_k E|E^k X|^2 < \infty.$$

Then $\left(\sum_{i=1}^n a_i E^i X\right)_{n\geq 1}$ converges in L_2 and a.s.

Proof of Corollary 9. We start from the representation (3), namely

$$\sum_{j=1}^k a_j E^j X = s_k E^k X + \sum_{i=1}^{k-1} s_i\left(E^i X - E^{i+1} X\right).$$

The reverse martingale $\sum_{i=1}^{k-1} s_i\left(E^i X - E^{i+1} X\right)$ converges a.s. and in L_2 provided it is bounded in L_2. Because the variables have values in a Hilbert space, by Remark 2

$$E\left|\sum_{i=1}^{k-1} s_i\left(E^i X - E^{i+1} X\right)\right|^2 = \sum_{i=1}^{k-1} s_i^2\left(E|E^i X|^2 - E|E^{i+1} X|^2\right),$$

which is bounded because $\sum_{k=1}^{n-1}\left(s_k^2 - s_{k-1}^2\right) E|E^k X|^2$ is positive and we assumed it is bounded.

By a similar proof as of Corollary 5, because $(s_k^*)_k$ is increasing and $(E^k X)_k$ is decreasing, we obtain

$$E(\max_{2^a\leq k\leq 2^b} |s_k E^k X|^2) \leq \sum_{j=a}^{b-1} E(\max_{2^i\leq k\leq 2^{i+1}} |s_k E^k X|^2)$$
$$\ll \sum_{j=a}^{b-1}\left(s_{2^{j+1}}^*\right)^2 E|E^{2^j} X|^2 \ll \sum_{j=2^{a-1}}^{2^b} j^{-1}\left(s_{4j}^*\right)^2 E|E^j X|^2.$$

Now if $2^a \leq m < 2^{a+1}$ and $2^{b-1} \leq n < 2^b$, with a, b integers $a \leq b$, then

$$E(\max_{m\leq k\leq n} |s_k E^k X|^2) \ll \sum_{j=2^a}^{2^b} j^{-1}\left(s_{4j}^*\right)^2 E|E^j X|^2.$$

This implies relation (22.11) in Billingsley (1999) and the proof continues as there. □

4. Reversible Markov chains

All the results presented so far, are useful to treat power series of operators associated to stationary reversible Markov chains. Let $(\xi_i)_{i\in Z}$ be defined on $(\Omega, \mathcal{F}, P)$ with values in a general measurable space $(S, \mathcal{S}, \pi)$ with stationary transition probabilities $Q(x, A)$. Reversible, means that the distribution of (ξ_i, ξ_{i+1}) is the same as the distribution of (ξ_{i+1}, ξ_i). For f defined on S with values in a separable Banach space B, and any $n \in N$ denote $Q^n f(\xi_0) = E(f(\xi_n)|\xi_0)$.

We also denote by Q the operator on an integrable function f defined by

$$Qf(x) = \int_S f(y)Q(x, dy).$$

We denote the invariant distribution by π, which is a measure on $\mathcal{S}$. The integral with respect to π is denoted by E_π.

We shall use below notations similar to those in previous sections,

$$s_k^e = a_2 + \cdots + a_{2k},\ \ s_k^{e*} = \max_{1\le j\le k} |s_j^e|$$

$$b_k^e = \max\left(k^{-1}\left(s_{4k}^{e*}\right)^2, \left(s_k^e\right)^2 - \left(s_{k-1}^e\right)^2\right)$$

$$s_k^0 = a_1 + \cdots + a_{2k-1}, s_k^{o*} = \max_{1\le j\le k} |s_j^o|$$

$$b_k^0 = \max\left(k^{-1}\left(s_{4k}^{o*}\right)^2, \left(s_k^o\right)^2 - \left(s_{k-1}^o\right)^2\right)$$

$$b_k^* = \max\left(b_k^e, b_k^o\right).$$

Theorem 10. *For f with values in a separable Hilbert space with $E_\pi|f|^2 < \infty$, we have*

$$E_\pi \max_{1\le k\le 2n}\left|\sum_{j=1}^k a_j Q^j f\right|^2 \ll \sum_{j=1}^n b_j^* E_\pi |Q^j f|^2.$$

Corollary 11. *In particular*

$$E_\pi \max_{1\le k\le n}\left|\sum_{j=1}^k Q^j f\right|^2 \ll \sum_{j=1}^n jE_\pi |Q^j f|^2$$

and

$$E_\pi \max_{1\le k\le n}\left|\sum_{j=1}^k j^{1/2} Q^j f\right|^2 \ll \sum_{j=1}^n E_\pi |Q^j f|^2.$$

To relate our result with Theorem 3.11 in Derriennic and Lin (2001), we give the following result similar to their theorem.

Corollary 12. *If f is real valued with mean 0 and has finite second moment, then*

$$E_\pi \sup_{k\ge 1}\left|\sum_{j=1}^k j^{1/2} Q^j f\right|^2 < \infty$$

provided one of the following equivalent conditions hold

$$(a)\quad \sum_{k=1}^n E_\pi(fQ^k f) \text{ is bounded in } L_2. \tag{10}$$

$$(b)\quad \sup_n ES_n^2/n < C.$$

$$(c)\quad \lim_{n\to\infty} ES_n^2/n = \sigma^2.$$

$$(d)\quad \int_{-1}^1 \frac{1}{1-t} d\rho_f < \infty.$$

$$(e)\quad f \in \sqrt{1-Q}L_2^0.$$

Above, ρ_f denotes the spectral measure of f associated with the self-adjoint operator Q and function f on $L_2(S, \pi)$. Also L_2^0 is the set of functions which are square integrable and have mean 0.

The proofs of the results in this section are based on the following identity, relating $Q^j f$ to $E^j f$.

Lemma 13. *For $n \geq 1$*

$$\sum_{j=1}^{2n} a_j Q^j f(\xi_0) = E_0 \left(\sum_{j=1}^{n} a_{2j} E^j f\ (\xi_0) + \sum_{j=0}^{n-1} a_{2j+1} E^{j+1} f(\xi_1) \right).$$

Proof of Lemma 13. We estimate $E(X_{2j}|\mathcal{F}_0) = Q^{2j} f(\xi_0)$. By the Markov property and reversibility

$$\begin{aligned} a_{2j} Q^{2j} f(\xi_0) &= a_{2j} E_0 E(f(\xi_{2j})|\xi_j) \\ &= a_{2j} E_0 E(f(\xi_0)|\mathcal{F}^j) = a_{2j} E_0 E^j f(\xi_0). \end{aligned}$$

By the above identity

$$\sum_{j=1}^{n} a_{2j} Q^{2j} f(\xi_0) = E_0 \sum_{j=1}^{n} a_{2j} E^j f(\xi_0). \tag{11}$$

Similarly

$$a_{2j+1}(Q^{2j+1} f)(\xi_0) = a_{2j+1} E(f(\xi_{2j+1})|\xi_0) =$$
$$a_{2j+1} E_0 E(f(\xi_{2j+1})|\xi_{j+1}) = a_{2j+1} E_0 E(f(\xi_1)|\xi_{j+1}) = a_{2j+1} E_0 E^{j+1} f(\xi_1),$$

and so

$$\sum_{j=0}^{n-1} a_{2j+1} Q^{2j+1} f(\xi_0) = E_0 \sum_{j=0}^{n-1} a_{2j+1} E^{j+1} f(\xi_1). \tag{12}$$

Overall, by (11) and (12), we have the result of this lemma. □

Remark 14. A similar result as in Lemma 13 holds for odd sums. We easily deduce

$$\begin{aligned} \max_{1\leq k\leq 2n} \left| \sum_{j=1}^{k} a_j Q^j f(\xi_0) \right| &\leq E_0 \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_j E^j f(\xi_0) \right| \\ &+ E_0 \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_{2j+1} E^j f(\xi_1) \right|, \end{aligned}$$

whence, for every $p \geq 1$,

$$\begin{aligned} E \max_{1\leq k\leq 2n} \left| \sum_{j=1}^{k} a_j Q^j f(\xi_0) \right|^p &\ll E \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_{2j} E^j f(\xi_0) \right|^p \\ &+ E \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_{2j+1} E^j f(\xi_0) \right|^p. \end{aligned}$$

It is straightforward now to combine Remark 14 with the results in the previous sections. It can be easily combined with Corollary 3.

In particular, for $p = 2$ we combine Remark 14 with Corollary 5 and we get the result in Theorem 10.

Proof of Corollary 12. First we find a more flexible maximal inequality, which has interest in itself. From Theorem 10 with f replaced by $f + Qf$, we get

$$E_\pi \max_{1\leq k\leq 2n} \left(\sum_{j=1}^{k} (Q^j f + Q^{j+1} f) \right)^2 \ll \sum_{j=1}^{n} j E_\pi \left(Q^j f + Q^{j+1} f \right)^2.$$

But, by the fact that on L_2 the operator Q is self-adjoint, with the notation $\langle f, g \rangle = E_\pi fg$,

$$E_\pi \left(Q^j f + Q^{j+1} f \right)^2 = \langle f, Q^{2j} f \rangle + 2 < f, Q^{2j+1} > + \langle f, Q^{2j+2} f \rangle.$$

So, after some algebraic computation, using that $\langle f, Q^{2j} f \rangle = \|Q^j f\|^2 \geq 0$ we obtain

$$\sum_{j=1}^{k} j E_\pi \left(Q^j f + Q^{j+1} f \right)^2 \ll \left| \sum_{j=1}^{2k} j E_\pi \left(f Q^j f \right) \right| + \langle f, Q^2 f \rangle,$$

and therefore

$$E_\pi \max_{1\le k\le 2n}\left(\sum_{j=1}^{k}(Q^j f+Q^{j+1}f)\right)^2 \ll \left|\sum_{j=1}^{2n} jE_\pi\left(fQ^j f\right)\right| + \langle f, Q^2 f\rangle. \qquad (13)$$

On the other hand, because

$$\sum_{j=1}^{k}(Q^j f+Q^{j+1}f) = 2\sum_{j=1}^{k} Q^j f - Qf + Q^{k+1}f,$$

we obtain

$$\max_{1\le k\le 2n}\left|\sum_{j=1}^{k} Q^j f\right| \ll \max_{1\le k\le 2n}\left|\sum_{j=1}^{k}(Q^j f+Q^{j+1}f)\right| + \max_{1\le k\le 2n}|Q^{k+1}f| + |Qf|.$$

To estimate the second moment of $\max_{1\le k\le n+1}|Q^k f|$ we use the Stein Theorem (see page 106 in Stein (1970) or Krengel (1985), page 190).

$$E_\pi \max_{1\le k\le 2n}|Q^{k+1}f|^2 \le E_\pi(Qf)^2 = E_\pi(fQ^2 f).$$

By combining these results with the estimate in (13) we get

$$E_\pi \max_{1\le k\le 2n}\left|\sum_{j=1}^{k} Q^j f\right|^2 \ll \left|\sum_{j=1}^{2n} jE_\pi\left(fQ^j f\right)\right| + E_\pi(fQ^2 f),$$

and the result follows.

The equivalences (a)–(e) are well-known results in the literature (see for instance pages 3 and 4 in Kipnis and Varadhan (1986) and Cuny (2009)). □

Data availability

No data was used for the research described in the article.

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