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Spectral stability of the Kohn Laplacian under perturbations of the boundary ☆

Siqi Fu [a], Howard Jacobowitz [a], Weixia Zhu [b,*]

[a] Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102, USA
[b] Faculty of Mathematics, University of Vienna, Vienna 1090, Austria

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ABSTRACT

In this paper, we study stability of the spectrum of the Kohn Laplacian $\Box_b$ on the boundary of a smoothly bounded domain in $\mathbb{C}^n$ as the boundary is perturbed in the C^2-topology. We obtain estimates for spectral stability of the Kohn Laplacian on smooth compact hypersurfaces that satisfy uniform subelliptic estimate, in particular for strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$ and pseudoconvex hypersurfaces of finite type in $\mathbb{C}^2$.

1. Introduction

In physical sciences, exact values are oftentimes impossible to obtain and approximations are used instead. It is then important to understand whether the observed quantity remains stable when other parameters are slightly perturbed. Stability of the spectrum for the classical Laplace operator with the Dirichlet or Neumann boundary condition on bounded domains in $\mathbb{R}^n$ has been studied extensively in the literature (see, e.g., [14,7,8,2] and references therein). In [13], the first and third authors initiated a systematic study of spectral stability of the $\bar{\partial}$-Neumann Laplacian on a bounded domain in $\mathbb{C}^n$ when the underlying domains are perturbed and established upper semi-continuity properties for the variational eigenvalues of the $\bar{\partial}$-Neumann Laplacian on bounded pseudoconvex domains in $\mathbb{C}^n$, lower semi-continuity properties on pseudoconvex domains that satisfy property (P), and quantitative estimates on smooth bounded pseudoconvex domains of finite D'Angelo type in $\mathbb{C}^n$.

In this paper, we study spectral stability of the Kohn Laplacian on the boundary of a smoothly bounded domain in $\mathbb{C}^n$ when the boundary is perturbed. Our main result can be stated as follows:

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* Corresponding author.
E-mail addresses: sfu@camden.rutgers.edu (S. Fu), jacobowi@camden.rutgers.edu (H. Jacobowitz), weixia.zhu@univie.ac.at (W. Zhu).

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Theorem 1.1. *Let Ω and Ω^t, $-1 \leq t \leq 1$, be bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries M and M^t respectively in $\mathbb{C}^n$, such that $M^0 = M$. Let ρ and ρ^t be the signed distance functions for M and M^t respectively. Assume that there exists a neighborhood U of M such that ρ and ρ^t are smooth with uniformly bounded C^∞-norms on U.*

(1) *Let $n \geq 3$. Suppose $1 \leq q \leq n-2$ and the Kohn Laplacian for $(0,q)$-forms on M^t satisfies a uniform subelliptic estimate. Let $\lambda_k^q(M)$ and $\lambda_k^q(M^t)$ be the k^{th} eigenvalues for the Kohn Laplacian on $(0,q)$-forms on M and M^t respectively. Then there exists a positive constant C_k independent of t such that*

$$|\lambda_k^q(M^t) - \lambda_k^q(M)| \leq C_k \delta^t, \tag{1.1}$$

provided that $\delta^t = \|\rho - \rho^t\|_{C^2,U}$ is sufficiently small. In particular, the above estimate holds if M is strictly pseudoconvex in $\mathbb{C}^n$.

(2) *Let $n \geq 3$. Suppose a uniform subelliptic estimate holds for the Kohn Laplacian on M^t for $(0,1)$-forms. Let $\lambda_k^0(M)$ and $\lambda_k^0(M^t)$ be the k^{th} non-zero eigenvalues for the Kohn Laplacian for functions on M and M^t respectively. Then (1.1) holds for $q = 0$. Analogously, suppose a uniform subelliptic estimate holds for the Kohn Laplacian on M^t for $(0,n-2)$-forms. Then (1.1) holds for $q = n-1$.*

(3) *Let $n = 2$. If M is pseudoconvex of finite type, then (1.1) holds for $q = 0,1$ provided $\delta^t = \|\rho^t - \rho\|_{C^\infty,U}$ is sufficiently small.*

Let $n \geq 3$ and $1 \leq q \leq n-2$. Recall that the Kohn Laplacian $\Box_b$ is said to satisfy a *subelliptic estimate* for $(0,q)$-forms on a smooth compact hypersurface M if there exist constants $0 < \varepsilon \leq 1/2$ and $C > 0$ such that

$$\|u\|_\varepsilon^2 \leq C(Q_b(u,u) + \|u\|^2) \tag{1.2}$$

for every $u \in \mathrm{Dom}(Q_b)$ such that $u \perp \mathrm{Ker}(Q_b)$, where $\|\cdot\|_\varepsilon$ denotes the L^2-Sobolev norm of order ε and $Q_b(u,u)$ the quadratic form associated with the Kohn Laplacian (see Section 2 for detail). We say that a *uniform subelliptic estimate* holds on M^t if estimate (1.2) holds on M^t and the constants ε and C in (1.2) can be chosen to be independent of the parameter t. Note that a subelliptic estimate (1.2) implies that the Kohn Laplacian $\Box_b$ has compact resolvent and its spectrum consists of discrete eigenvalues of finite multiplicity.

It follows from the works of D'Angelo ([6]) and Catlin ([3,4]) that the $\overline{\partial}$-Neumann Laplacian satisfies a subelliptic estimate for $(0,q)$-forms on a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ if and only if its boundary M is of finite D'Angelo q-type (*i.e.*, the order of contact of M with any q-dimensional complex analytic variety is finite). Furthermore, we know from the work of Kohn that the $\overline{\partial}$-Neumann Laplacian satisfies the subelliptic estimate on a smooth bounded pseudoconvex domain Ω if and only if the Kohn Laplacian satisfies the subelliptic estimate on its boundary $b\Omega$ ([15, Theorem 8.2]). Since strict pseudoconvexity of M is preserved under a sufficiently small perturbation in the C^2-topology, a uniform subelliptic estimate holds on M^t when M is strictly pseudoconvex. Similarly, for a smooth pseudoconvex hypersurface in $\mathbb{C}^2$, the finite type condition in the sense of D'Angelo is equivalent to the finite commutator type in the sense of Hörmander which is stable under a sufficiently small C^∞-perturbation. Thus a uniform subelliptic estimate holds on M^t when M is a pseudoconvex hypersurface of finite type in $\mathbb{C}^2$, provided M^t is a sufficiently small perturbation of M in the C^∞-topology.

Our paper is organized as follows. In Section 2, we recall the necessary definitions and set up the problem. In Section 3, we define the transition operator which plays an important role in the analysis. In Section 4, we establish an upper semicontinuity property for the eigenvalues under the assumption that the subelliptic estimate holds on M. In Section 5, we establish the lower semicontinuity property when a uniform subelliptic estimate holds. In Section 6, we study spectral stability of the Kohn Laplacian on functions and top degree $(0, n-1)$-forms. The last section contains further remarks.

2. Preliminaries

Let $(M, T^{1,0}M)$ be an orientable CR manifold of real dimensional $2n-1$, equipped with a Hermitian metric on $\mathbb{C}TM$ so that $T^{1,0}M$ is orthogonal to $T^{0,1}M := \overline{T^{0,1}M}$. Let T be the orthogonal complement of $T^{1,0}M \oplus T^{0,1}M$ in $\mathbb{C}TM$. Denote by $T^{*1,0}M$, $T^{*0,1}M$ and θ the dual bundles of $T^{1,0}M$, $T^{0,1}M$ and T, respectively. For $0 \le p, q \le n-1$, let $\Lambda^{p,q}M$ be the vector bundle defined by

$$\Lambda^{p,q}M = \Lambda^p T^{*1,0}M \otimes \Lambda^q T^{*0,1}M.$$

Denote by $\mathcal{E}^{p,q}(M)$ the space of smooth sections of $\Lambda^{p,q}M$ over M. Let $\overline{\partial}_b$ be the tangential Cauchy-Riemann operator defined intrinsically by

$$\overline{\partial}_b^{p,q} = P^{p,q+1} \circ d_M : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$

where d_M is the exterior differential operator on M and $P_M^{p,q+1} : \Lambda^{p+q+1}(M) \to \Lambda^{p,q+1}(M)$ the orthogonal projection. (We will drop the superscripts from $\overline{\partial}_b^{p,q}$ when they are clear from the contexts.) We also use $\overline{\partial}_b$ to denote the maximal extension of $\overline{\partial}_b$ on $L^2_{(p,q)}(M)$, the space of (p,q)-forms with L^2-coefficients. As such, the domain $\mathrm{Dom}(\overline{\partial}_b)$ of

$$\overline{\partial}_b^{p,q} : L^2_{(p,q)}(M) \to L^2_{(p,q+1)}(M)$$

consists of forms $u \in L^2_{(p,q)}(M)$ such that $\overline{\partial}_b u \in L^2_{(p,q+1)}(M)$ in the sense of distribution. Thus $\overline{\partial}_b$ is a linear, closed, and densely defined operator on $L^2_{(p,q)}(M)$. Let

$$\overline{\partial}_b^{p,q*} : L^2_{(p,q+1)}(M) \to L^2_{(p,q)}(M)$$

be the adjoint of $\overline{\partial}_b$ with

$$\mathrm{Dom}(\overline{\partial}_b^*) = \left\{ u \in L^2_{(p,q+1)}(M) \mid \exists C > 0 \text{ such that } |(u, \overline{\partial}_b \phi)| \le C\|\phi\|, \forall \phi \in \mathrm{Dom}(\overline{\partial}_b) \right\}.$$

For $0 \le p \le n$ and $1 \le q \le n-2$, the Kohn-Laplacian on $L^2_{(p,q)}(M)$ is given by

$$\Box_b^{p,q} = \overline{\partial}_b^{p,q-1} \overline{\partial}_b^{p,q-1*} + \overline{\partial}_b^{p,q*} \overline{\partial}_b^{p,q} : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$$

with

$$\mathrm{Dom}(\Box_b^{p,q}) = \left\{ u \in L^2_{(p,q)}(M) \mid u \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*), \right.$$
$$\left. \overline{\partial}_b u \in \mathrm{Dom}(\overline{\partial}_b^*), \overline{\partial}_b^* u \in \mathrm{Dom}(\overline{\partial}_b) \right\}.$$

It follows that $\Box_b$ is a linear, closed, and densely defined self-adjoint operator on $L^2_{(p,q)}(M)$ (see [12] for a spectral theoretic proof of this fact). Let

$$Q_b(u,v) = (\overline{\partial}_b u, \overline{\partial}_b v) + (\overline{\partial}_b^* u, \overline{\partial}_b^* v), \quad u, v \in \mathrm{Dom}(Q_b) = \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*)$$

be the sesquilinear form associated with the Kohn Laplacian $\Box_b$. Write

$$\mathrm{Ker}(Q_b) = \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*).$$

For $q = 0$, the Kohn Laplacian is given by

$$\Box_b^{p,0} = \overline{\partial}_b^{p,0*}\overline{\partial}_b^{p,0} : L^2_{(p,0)}(M) \to L^2_{(p,0)}(M)$$

with the associated sesquilinear form

$$Q_b(u,v) = (\overline{\partial}_b u, \overline{\partial}_b v) \quad u, v \in \mathrm{Dom}(Q_b) = \mathrm{Dom}(\overline{\partial}_b).$$

Similarly, for $q = n-1$, the Kohn Laplacian is

$$\Box_b^{p,n-1} = \overline{\partial}_b^{p,n-1}\overline{\partial}_b^{p,n-1*} : L^2_{(p,n-1)}(M) \to L^2_{(p,n-1)}(M)$$

with the associated sesquilinear form

$$Q_b(u,v) = (\overline{\partial}_b^* u, \overline{\partial}_b^* v), \quad u, v \in \mathrm{Dom}(Q_b) = \mathrm{Dom}(\overline{\partial}_b^*).$$

We now recall the extrinsic definition of the tangential Cauchy-Riemann operator when M is a smooth hypersurface in $\mathbb{C}^n$ with the inherited CR structure. Consider a neighborhood U of M in $\mathbb{C}^n$ and let $r \in C^\infty(U)$ be a defining function of M such that $r = 0$ and $|dr| = 1$ on M. Let $\mathcal{E}^{p,q}(U)$ be the space of smooth sections of $\Lambda^{p,q}U$ over U and let $\bar{\partial} : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p,q+1}(U)$ be the Cauchy-Riemann operator. Let $\Lambda^{p,q}(U)|_M$ be the restriction of the bundle $\Lambda^{p,q}(U)$ to M. More precisely, if

$$f = \sum_{I,J}{}' f_{I,J} dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q}(U),$$

then $f|_M \in \Lambda^{p,q}(U)|_M$ is obtained by restricting the coefficients $f_{I,J}$ to M. Let

$$I^{p,q}(U) = \{rf + \overline{\partial} r \wedge g \mid f \in \Lambda^{p,q}(U), g \in \Lambda^{p,q-1}(U)\} \tag{2.1}$$

be the ideal in $\Lambda^{p,q}(U)$ locally generated by r and $\overline{\partial} r$. Let $\Lambda^{p,q}(M)$ be the orthogonal complement of $I^{p,q}|_M$ in $\Lambda^{p,q}(U)|_M$ Let

$$\tau_M : \Lambda^{p,q}(U)|_M \to \Lambda^{p,q}(M)$$

be the orthogonal projection. For any $f \in \Lambda^{p,q}(U)|_M$, we refer to $\tau_M(f)$ as the tangential part of f. Since $|dr| = 1$ on M, it is easy to see that

$$\tau_M(f) = (\overline{\partial} r)^* \lrcorner (\overline{\partial} r \wedge f), \tag{2.2}$$

where

$$(\overline{\partial} r)^* = 4 \sum_{k=1}^n \frac{\partial r}{\partial z_k} \frac{\partial}{\partial \bar{z}_k}$$

is the dual vector to the form $\overline{\partial} r$ and $\lrcorner$ denotes the contraction operator of a vector with a form.

For an open set $W \subset M$, denote by $\mathcal{E}_M^{p,q}(W)$ the space of smooth sections of $\Lambda^{p,q}(M)$ over W and $\mathcal{D}_M^{p,q}(W)$ the space of compactly supported forms in $\mathcal{E}_M^{p,q}(W)$. The (extrinsic) tangential Cauchy-Riemann operator

$\overline{\partial}_M : \mathcal{E}^{p,q}_M(W) \to \mathcal{E}^{p,q+1}_M(W)$ is defined as follows: For $f \in \mathcal{E}^{p,q}_M(W)$, let $\tilde{f} \in \mathcal{E}^{p,q}_M(\widetilde{W})$ be an extension of f to some open set $\widetilde{W}$ in $\mathbb{C}^n$ such that $\widetilde{W} \cap M = W$ and $\tau_M(\tilde{f}|_M) = f$ on W. Then

$$\overline{\partial}_M f = \tau_M((\overline{\partial}\tilde{f})|_M).$$

Evidently, the definition of $\overline{\partial}_M$ is independent of the ambient extension. In the case when M is an embedded hypersurface in $\mathbb{C}^n$, the extrinsic and intrinsic approaches lead to different but isomorphic tangential Cauchy-Riemann complexes (see [1] for details). Since p plays no role in our analysis, hereafter we consider only the tangential Cauchy-Riemann operator on $(0, q)$-forms. We will also use $\overline{\partial}_b$ and $\overline{\partial}_M$ interchangeably to denote the tangential Cauchy-Riemann operator on M.

Let $\lambda^q_k(M)$ be the k^{th}-variational eigenvalue of $\Box_b$ on $L^2_{(0,q)}(M)$, given by the following min-max principle:

$$\lambda^q_k(M) = \inf_{\substack{L\subset \mathrm{Dom}(Q_b)\\ \dim L=k}} \sup_{u\in L\setminus\{0\}} Q_b(u,u)/\|u\|^2 \tag{2.3}$$

where the infimum takes over all linear k-dimension subspaces of $\mathrm{Dom}(Q_b)$. Recall that the spectrum of a non-negative self-adjoint operator S is purely discrete if and only if the variational eigenvalues $\lambda_k(S)$ defined as above go to ∞ as $k \to \infty$. In this case, $\lambda_k(S)$ is the k^{th}-eigenvalue of S when the eigenvalues are arranged in increasing order and repeated according to multiplicity (see [7, Chapter 4]). Note that in the cases when $q = 0$ and $q = n - 1$, the kernel $\mathrm{Ker}(Q_b)$ of Q_b is always infinite dimensional when M is embedded. In these cases, when we say k^{th}-variational eigenvalue of $\Box_b$, we refer to the k^{th}-variational eigenvalue of $\Box_b$ restricted to the orthogonal complement of $\mathrm{Ker}(Q_b)$. For example, in the case when $q = 0$,

$$\lambda^0_k(M) = \inf_{\substack{L\subset \mathrm{Dom}(\overline{\partial}_b)\cap \mathrm{Ker}(\overline{\partial}_b)^{\perp}\\ \dim L=k}} \sup_{u\in L\setminus\{0\}} \|\overline{\partial}_b u\|^2/\|u\|^2.$$

Let S_i, $i = 1, 2$, be non-negative self-adjoint operators on Hilbert space $\mathbb{H}$ with associated quadratic forms Q_i. One way to estimate the difference between variational eigenvalues $\lambda_k(S_1)$ of S_1 and $\lambda_k(S_2)$ of S_2 is to construct a transition operator $T\colon \mathrm{Dom}(Q_1) \to \mathrm{Dom}(Q_2)$ and estimate the difference between $\langle f,\ g\rangle_1$ and $\langle Tf,\ Tg\rangle_2$ and between $Q_1(f, g)$ and $Q_2(Tf, Tg)$ for f and g in any k-dimensional subspace of $\mathrm{Dom}(Q_1)$. The following lemma is a simple consequence of the min-max principle (2.3) (compare [13, Lemma 2.1] and the subsequent remark).

Lemma 2.1. *Let S_i, $i = 1, 2$, be non-negative self-adjoint operators on Hilbert spaces $\mathbb{H}_i$ with associated quadratic forms Q_i. Let $T\colon \mathrm{Dom}(Q_1) \to \mathrm{Dom}(Q_2)$ be a linear transformation from the domain of Q_1 to that of Q_2. Suppose there exist constants $0 < a < 1$ and $b > 0$ such that for any k-dimensional subspace $\mathcal{X}_k$ of $\mathrm{Dom}(Q_1)$ and any $u \in \mathcal{X}_k$,*

$$\|Tu\|^2_2 \geq (1-a)\|u\|^2_1 \quad \textit{and} \quad Q_2(Tu, Tu) \leq (1 + b)Q_1(u, u)$$

for any $u \in \mathrm{Dom}(Q_1)$. Then

$$\lambda_k(S_2) \leq \frac{1+b}{1-a}\lambda_k(S_1).$$

Proof. Let $\mathcal{X}_k$ be any k-dimensional subspace of $\mathrm{Dom}(Q_1)$. Since T is one-to-one, $T(\mathcal{X}_k)$ is a k-dimensional subspace of $\mathrm{Dom}(Q_2)$. Hence

$$\begin{aligned}\lambda_k(S_2) &\leq \sup\Big\{\frac{Q_2(Tu,Tu)}{\|Tu\|^2_2} \mid u \in \mathcal{X}_k, u \neq 0\Big\}\\ &\leq \sup\Big\{\frac{(1+b)Q_1(u,u)}{(1-a)\|u\|^2_1} \mid u \in \mathcal{X}_k, u \neq 0\Big\}.\end{aligned}$$

Taking infimum over all k-dimensional subspaces of $\mathrm{Dom}(Q_1)$, we then obtain the desired inequality. □

3. Transition operator

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $M = b\Omega$. For any $\sigma > 0$, set $U_\sigma(M) = \{z \in \mathbb{C}^n \mid \mathrm{dist}(z, M) < \sigma\}$, where $\mathrm{dist}(z, M)$ is the Euclidean distance from z to M. The hypersurface M is said to be of *positive reach* if there is a $\sigma > 0$ such that each $z \in U_\sigma(M)$ has a unique nearest point on M. Denote by $\mathrm{Reach}(M)$ the largest such σ. It follows from [9, Theorem 4.12] that when Ω is C^2-smooth, $\mathrm{Reach}(M) > 0$. Assume now that M is C^2-smooth. Let $\sigma_0 = \mathrm{Reach}(M)$ and $U_0 = U_{\sigma_0}(M)$. Let $\rho(z)$ be the signed distance from z to M such that $\rho(z) = -\mathrm{dist}(z, M)$ for $z \in \Omega$ and $\rho(z) = \mathrm{dist}(z, M)$ for $z \in \mathbb{C}^n \setminus \Omega$. Then $\rho \in C^2(U_0)$ and $|d\rho| \equiv 1$ on U_0. We will also use ρ to denote a C^2-extension of the signed distance function to $\mathbb{C}^n$. Let Ω^t be a family of bounded domains with C^2-smooth boundary M^t for $t \in (-1,\ 1)$ such that $\Omega^0 = \Omega$, $M^t \subset U_0$ and $d^t = d_H(M^t, M) \to 0$ as $t \to 0$, where

$$d_H(M^t, M) = \max_{z \in M} \mathrm{dist}(z, M^t) = \max_{w \in M^t} \mathrm{dist}(w, M)$$

is the Hausdorff distance between M and M^t. Let ρ^t be the signed distance defining function for Ω^t, extended to be C^2 on $\mathbb{C}^n$. Let

$$\delta^t = \|\rho^t - \rho\|_{C^2, U_0}$$

be the C^2-norm over U_0. Evidently, $d^t \le \delta^t$. Let

$$\pi \colon U_0 \to M$$

be the projection onto M along the real normal direction such that $\mathrm{dist}(z, M) = \mathrm{dist}(z, \pi(z))$. Then π is C^2 on U_0. Let

$$\pi^t \colon M^t \to M$$

be the restriction of π to M^t.

We can now define the transition operator. Let $P^{p,q}$ denote the natural orthogonal projection from $\Lambda^{p+q}(U_0)$ onto $\Lambda^{p,q}(U_0)$. The transition operator is then defined as follows:

$$T^t = \tau_{M^t} \circ P^{0,q}|_{M^t} \circ \pi^* \colon \Lambda^{0,q}(M) \to \Lambda^{0,q}(M^t), \tag{3.1}$$

where $\pi^* : \Lambda^{0,q}(M) \to \Lambda^q(U_0)$ is the pull-back operator and $P^{p,q}|_{M^t}(u)$ denotes the form obtained by restricting the coefficients of $P^{p,q}(u)$ to M^t. It is easy to see that T^t extends to a bounded linear transformation from $L^2_{(0,q)}(M)$ into $L^2_{(0,q)}(M^t)$ and it maps $\mathrm{Dom}(Q_b)$ into $\mathrm{Dom}(Q_{t,b})$.

In the remainder of this section, we will show that the L^2-norm of a $(0, q)$-form on M is stable under this transition operator. The following lemma is well known. We provide a proof for the reader's convenience.

Lemma 3.1. *π^t is a C^2-diffeomorphism between M^t and M, provided δ^t is sufficiently small.*

Proof. We first observe that π^t is surjective when δ^t is sufficiently small. In fact, for every $z \in M$, we have

$$\rho^t(z + s\vec{n}(z)) = \rho^t(z) + \nabla\rho^t(z) \cdot \vec{n}(z)s + O(s^2),$$

where $\vec{n}(z) = \nabla\rho(z)$ is the outward normal direction of M at z. Thus for any sufficiently small $s > 0$,

$$\rho^t(z - s\vec{n}(z)) \le \rho^t(z) - s/2 < 0 \quad \text{and} \quad \rho^t(z + s\vec{n}(z)) \ge \rho^t(z) + s/2 > 0,$$

provided δ^t is sufficiently small. Thus by the intermediate value theorem, there exists a $s_0 \in (-s,\ s)$ such that $\rho^t(z + s_0\vec{n}(z)) = 0$ and $\pi(z + s_0\vec{n}(z)) = z$.

We now show π is injective. Proving by contradiction, we assume that there are two distinct points z_1 and z_2 on M^t that project to the same point z on M. Write $z_i = z + s_i\vec{n}(z)$, $i = 1, 2$, where s_1 and s_2 are two distinct real numbers in $(-\sigma_0,\ \sigma_0)$. Set $g(s) = \rho^t(z + s\vec{n}(z))$. Then $g(s_1) = g(s_2)$. This contradicts the fact that

$$g'(s) = \nabla\rho^t(z + s\vec{n}(z)) \cdot \vec{n}(z) \ge 1/2,$$

when δ^t is sufficiently small. Then both π^t and its inverse are C^2-smooth is a consequence of the implicit function theorem. □

Proposition 3.2. *Let $\iota^t\colon M \to M^t$ be the inverse of π^t. Then there exists a constant $C > 0$ such that*

$$\left| |T^t u|^2(\iota^t(z)) |\, \mathrm{Jac}\ \iota^t_{*z}| - |u|^2(z) \right| \le C\delta^t |u|^2(z) \tag{3.2}$$

for any $u \in \Lambda^{0,q}(M)$, where $|\cdot|$ denotes the pointwise norm of a form and $\mathrm{Jac}\ \iota^t_{*z}$ the Jacobian determinant of ι^t at z. Furthermore,

$$\left| \|T^t u\|^2_{M^t} - \|u\|^2_M \right| \le C\delta^t \|u\|^2_M \tag{3.3}$$

for any $u \in L^2_{(0,q)}(M)$.

We do some preparations before proving this proposition. Let $p \in U_0$ and let $p_0 = \pi(p)$. After a unitary transformation, we might assume p_0 is the origin and the negative $\mathrm{Re}\ z_n$-direction is the outward normal direction at p_0. In this coordinate, $p = (0, \ldots, 0, d)$ and there exists a neighborhood U of the origin such that

$$M \cap U = \{(\tilde{x}, x_{2n}) \in U \mid x_{2n} = f(\tilde{x})\},$$

where $f(\tilde{x})$ is a C^3 function in the form $f(\tilde{x}) = \sum_{k,l=1}^{2n-1} a_{kl}x_kx_l + O(|\tilde{x}|^3)$. Here we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and use the notations $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$, $j = 1, \cdots, n$, and $\tilde{x} = (x_1, \ldots, x_{2n-1})$.

For $(\tilde{x}, x_{2n})$ near p_0, we write $\pi(\tilde{x}, x_{2n}) = (\tilde{y}, f(\tilde{y}))$ where $\tilde{y} = (y_1, \ldots, y_{2n-1})$ and $y_j = y_j(x_1, \ldots, x_{2n})$ are C^3-smooth near the origin. The following lemma is well known. We provide a proof, following the proof of Theorem 1.1 in [11] (compare also [16]) for the reader's convenience.

Lemma 3.3. *With the above notations, for sufficiently small d, we have:*

$$\frac{\partial\rho}{\partial x_k}(p) = 0, \quad \frac{\partial\rho}{\partial x_{2n}}(p) = -1; \tag{3.4}$$

$$\frac{\partial^2\rho}{\partial x_j\partial x_k}(p) = a_{jk} + O(d), \quad \frac{\partial^2\rho}{\partial x_j\partial x_{2n}}(p) = 0; \tag{3.5}$$

$$\frac{\partial y_j}{\partial x_k}(p) = \delta_{jk} + a_{jk}d + O(d^2), \quad \frac{\partial y_j}{\partial x_{2n}}(p) = 0; \tag{3.6}$$

and

$$\frac{\partial^2 y_j}{\partial x_k \partial x_h}(p) = O(d), \quad \frac{\partial^2 y_j}{\partial x_k \partial x_{2n}}(p) = a_{jk} + O(d), \quad \frac{\partial^2 y_j}{\partial x_{2n}^2}(p) = O(d), \tag{3.7}$$

for all $1 \le h, j, k \le 2n-1$*, where the constant depends on the* C^3*-norm of* ρ*.*

Proof. Estimates (3.4) and (3.5) were given in [11, Theorem 1.1]. For completeness, we provide a detailed proof for (3.6) and (3.7). Given $(\tilde{x}, x_{2n})$ near p_0, observe that for $\tilde{s} = (s_1, \dots, s_{2n-1})$ near the origin,

$$A(\tilde{s}) = \sum_{j=1}^{2n-1} (x_j - s_j)^2 + (x_{2n} - f(\tilde{s}))^2$$

attains a local minimum when $\tilde{s} = \tilde{y}$. Differentiating both sides with respect to s_j and then evaluating at $\tilde{s} = \tilde{y}$, we have

$$(x_j - y_j) + (x_{2n} - f(\tilde{y}))\frac{\partial f}{\partial y_j}(\tilde{y}) = 0, \quad 1 \le j \le 2n-1. \tag{3.8}$$

By taking $\partial/\partial x_k$, $1 \le k \le 2n-1$, to both sides of (3.8), we have

$$\delta_{jk} - \frac{\partial y_j}{\partial x_k} - \Big(\sum_{l=1}^{2n-1} \frac{\partial f}{\partial y_l}\frac{\partial y_l}{\partial x_k}\Big)\frac{\partial f}{\partial y_j}(\tilde{y}) + (x_{2n} - f(\tilde{y}))\sum_{l=1}^{2n-1}\frac{\partial^2 f(\tilde{y})}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_k} = 0. \tag{3.9}$$

Similarly, by taking $\partial/\partial x_{2n}$ to both sides of (3.8), we have

$$-\frac{\partial y_j}{\partial x_{2n}} + \Big(1 - \sum_{l=1}^{2n-1} \frac{\partial f}{\partial y_l}\frac{\partial y_l}{\partial x_{2n}}\Big)\frac{\partial f}{\partial y_j}(\tilde{y}) + (x_{2n} - f(\tilde{y}))\sum_{l=1}^{2n-1}\frac{\partial^2 f(\tilde{y})}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_{2n}} = 0. \tag{3.10}$$

Evaluating (3.9) and (3.10) at $p = (0, \dots, 0, d)$, we obtain

$$\delta_{jk} - \frac{\partial y_j}{\partial x_k} + d\sum_{l=1}^{2n-1} a_{jl}\frac{\partial y_l}{\partial x_k} = 0 \tag{3.11}$$

and

$$-\frac{\partial y_j}{\partial x_{2n}} + d\sum_{l=1}^{2n-1} a_{jl}\frac{\partial y_l}{\partial x_{2n}} = 0. \tag{3.12}$$

Applying Cramer's rule to the linear system (3.11), $1 \le j, k \le 2n-1$, we obtain

$$\frac{\partial y_j}{\partial x_k}(p) = \delta_{jk} + a_{jk}d + O(d^2).$$

Similarly, from (3.12), we obtain

$$\frac{\partial y_j}{\partial x_{2n}}(p) = 0.$$

We thus establish (3.6).

We now proceed to prove (3.7). Taking $\partial/\partial x_h$, $1 \le h \le 2n-1$ to (3.9) and then evaluating at p, we obtain

$$-\frac{\partial^2 y_j}{\partial x_k \partial x_h}(p) + d\frac{\partial}{\partial x_h}\sum_{l=1}^{2n-1}\frac{\partial^2 f(\tilde{y})}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_k} = 0. \tag{3.13}$$

Thus

$$\frac{\partial^2 y_j}{\partial x_k \partial x_h}(p) = O(d)$$

with the constant depending on the C^3-norm of f. Similarly, applying $\partial/\partial x_{2n}$ to (3.9), we obtain

$$-\frac{\partial^2 y_j}{\partial x_k \partial x_{2n}}(p) + \sum_{l=1}^{2n-1}\frac{\partial^2 f}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_k}(p) + d\frac{\partial}{\partial x_{2n}}\sum_{l=1}^{2n-1}\frac{\partial^2 f}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_k} = 0. \tag{3.14}$$

Together with (3.6), we then have

$$\frac{\partial^2 y_j}{\partial x_k \partial x_{2n}}(p) = a_{jk} + O(d). \tag{3.15}$$

Moreover, applying $\partial/\partial x_{2n}$ to (3.10) and then evaluating at p, we have

$$-\frac{\partial^2 y_j}{\partial x_{2n}^2}(p) + \sum_{l=1}^{2n-1}\frac{\partial^2 f}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_{2n}}(p) + d\frac{\partial}{\partial x_{2n}}\sum_{l=1}^{2n-1}\frac{\partial^2 f}{\partial y_j \partial y_l}\frac{\partial y_l}{\partial x_k} = 0. \tag{3.16}$$

It follows that

$$\frac{\partial^2 y_j}{\partial x_{2n}^2}(p) = O(d). \quad \square \tag{3.17}$$

As a consequence of this lemma, we have:

Lemma 3.4. *With the notation above, we have that*

$$\text{Jac } \pi_{*p} = 1 + O(d) \quad \textit{and} \quad \text{Jac } \iota^t_{*p_0} = 1 + O(d^t) \tag{3.18}$$

for any $p \in M^t$ and $p_0 \in M$ respectively.

For a given point $p_0 \in M$, since $|d\rho| = 1$ on M, we may assume without loss of generality that $\partial\rho/\partial z_n \neq 0$ on a neighborhood U of p_0. Let

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j}\left(\frac{\partial \rho}{\partial \bar{z}_n}\right)^{-1}\frac{\partial}{\partial \bar{z}_n}, \quad 1 \leq j \leq n-1. \tag{3.19}$$

Then $\{\bar{L}_1, \ldots, \bar{L}_{n-1}\}$ forms basis for $T^{(0,1)}(M)$ on U. Let

$$\bar{\omega}_\alpha = d\bar{z}_\alpha - 4\frac{\partial \rho}{\partial z_\alpha}\overline{\partial}\rho, \quad 1 \leq \alpha \leq n, \quad \text{and} \quad \theta = \overline{\partial}\rho. \tag{3.20}$$

Then $\{\bar{\omega}_1, \ldots, \bar{\omega}_{n-1}\}$ is a dual basis to $\{\bar{L}_1, \ldots, \bar{L}_{n-1}\}$ for $\Lambda^{0,1}(M)$ on U. Note that θ is orthogonal to $\bar{\omega}_j$, $1 \leq j \leq n-1$, and $\{\bar{\omega}_1, \ldots, \bar{\omega}_{n-1}, \theta\}$ is a local frame for $\Lambda^{0,1}(\mathbb{C}^n)$ on U. Furthermore, $\bar{\omega}_n$ is linearly dependent on $\bar{\omega}_1, \ldots, \bar{\omega}_{n-1}$ in $\Lambda^{0,1}(M)$. Indeed,

$$\bar{\omega}_n = -\rho_{\bar{n}}^{-1}\sum_{k=1}^{n-1}\rho_{\bar{k}}\bar{\omega}_k. \tag{3.21}$$

Hereafter, to simplify the notations, we write $\frac{\partial\rho}{\partial z_j}$ and $\frac{\partial\rho}{\partial\bar{z}_j}$ as ρ_j and $\rho_{\bar{j}}$ respectively. Also, lowercase roman indices will run from 1 to $n-1$, whereas lowercase Greek indices will run from 1 to n. Observe that

$$\tau_M(d\bar{z}_\alpha) = (\overline{\partial}\rho)^*\lrcorner(\overline{\partial}\rho\wedge d\bar{z}_\alpha) = \bar{\omega}_\alpha, \quad \alpha = 1,\cdots,n. \tag{3.22}$$

Moreover,

$$\tau_M(d\bar{z}_K) = \bar{\omega}_K, \tag{3.23}$$

where K is any tuple of integers from 1 to n. We define $\bar{L}_j^t$ and $\bar{\omega}_\alpha^t$ similarly by replacing ρ by ρ^t in (3.19) and (3.20) respectively. The identities (3.22) and (3.23) remain true when M is replaced by M^t, ρ by ρ^t, and $\bar{\omega}_\alpha$ by $\bar{\omega}_\alpha^t$.

Write $\pi(z) = (\pi_1(z),\dots,\pi_n(z))$, we have

$$\pi^*\bar{\omega}_j = \sum_{\alpha=1}^{n}\Big(\frac{\partial\bar{\pi}_j}{\partial\bar{z}_\alpha}d\bar{z}_\alpha + \frac{\partial\bar{\pi}_j}{\partial z_\alpha}dz_\alpha\Big) - 4\sum_{\alpha=1}^{n}(\rho_j\rho_{\bar{\alpha}}\circ\pi)\sum_{\beta=1}^{n}\Big(\frac{\partial\bar{\pi}_\alpha}{\partial\bar{z}_\beta}d\bar{z}_\beta + \frac{\partial\bar{\pi}_\alpha}{\partial z_\beta}dz_\beta\Big). \tag{3.24}$$

Thus

$$P^{0,1}\circ\pi^*\bar{\omega}_j = \sum_{\alpha=1}^{n}\frac{\partial\bar{\pi}_j}{\partial\bar{z}_\alpha}d\bar{z}_\alpha - 4\sum_{\alpha=1}^{n}(\rho_j\rho_{\bar{\alpha}}\circ\pi)\sum_{\beta=1}^{n}\frac{\partial\bar{\pi}_\alpha}{\partial\bar{z}_\beta}d\bar{z}_\beta. \tag{3.25}$$

Restricting the coefficients to M^t and applying τ_{M^t}, we then obtain

$$T^t(\bar{\omega}_j) = \sum_{\alpha=1}^{n}\frac{\partial\bar{\pi}_j}{\partial\bar{z}_\alpha}\bar{\omega}_\alpha^t - 4\sum_{\alpha=1}^{n}(\rho_j\rho_{\bar{\alpha}}\circ\pi)\sum_{\beta=1}^{n}\frac{\partial\bar{\pi}_\alpha}{\partial\bar{z}_\beta}\bar{\omega}_\beta^t. \tag{3.26}$$

Let p be any point on M^t and let $p_0 = \pi(p) \in M$. After a unitary transformation, we assume p_0 is the origin and the negative Re z_n-axis is the outward normal direction at p_0. It follows from (3.26) and Lemma 3.3 that

$$T^t(\bar{\omega}_j)(p) = \bar{\omega}_j^t(p) + O(d^t), \tag{3.27}$$

where $O(d^t)$ denotes a form whose pointwise norm is dominated by a constant multiple of d^t, the Hausdorff distance between M and M^t. More generally, when d^t is sufficiently small, we have

$$T^t(\bar{\omega}_J) = \bar{\omega}_J^t + O(d^t), \tag{3.28}$$

where $\bar{\omega}_J = \bar{\omega}_{j_1}\wedge\cdots\wedge\bar{\omega}_{j_q}$.

We are now in position to prove Proposition 3.2. Let $u \in \Lambda^{0,q}(M)$. Using the local frame defined by (3.20), we write

$$u = {\sum_J}' u_J\bar{\omega}_J.$$

Here the summation is taken over strictly increasing q-tuples of integers from $\{1,\dots,n-1\}$. By (3.28), we have

$$T^t(u)(z) = \sum_J{}' u(\pi(z))T^t(\bar{\omega}_J) = \sum_J{}' u(\pi(z))(\bar{\omega}^t_J + O(d^t)),$$

for $z \in M^t \cap U$. Hence

$$\left| |T^t(u)|^2(\iota^t(z)) - \sum_{J,K}{}' u_J(z)\bar{u}_K(z)\langle\bar{\omega}^t_J, \bar{\omega}^t_K\rangle(\iota^t(z))\right| \le Cd^t|u(z)|^2 \tag{3.29}$$

for any $z \in M \cap U$. (Throughout this paper, we will use C to denote a positive constant, independent of u and t, which might be different in different appearances.) Note that

$$|\rho_j(z) - \rho^t_j(\iota^t(z))| \le |\rho_j(z) - \rho_j(\iota^t(z))| + |\rho_j(\iota^t(z)) - \rho^t_j(\iota^t(z))| \le C\delta^t$$

for $z \in M$. It follows that for any q-tuples J and K,

$$\left|\langle\bar{\omega}_J, \bar{\omega}_K\rangle(z) - \langle\bar{\omega}^t_J, \bar{\omega}^t_K\rangle(\iota^t(z))\right| \le C\delta^t. \tag{3.30}$$

Combining (3.29) and (3.30) with (3.18), we then obtain (3.2). Moreover,

$$\left|\|T^t u\|^2_{M^t} - \|u\|^2_M\right| = \left|\int_M \left(|T^t u|^2(\iota^t(z))|\operatorname{Jac} \iota^t_{*z}| - |u|^2(z)\right)dS\right| \le C\delta^t\|u\|^2_M.$$

This concludes the proof of Proposition 3.2.

4. Upper semi-continuity

In this section, we establish an upper semi-continuity property for the variational eigenvalues of the Kohn Laplacian as the underlying boundaries vary in the C^2-topology. We have shown in the previous section that the difference between $\|T^t(u)\|^2_{M^t}$ and $\|u\|^2_M$ is under control (see Proposition 3.2). To obtain the desirable estimate for the variational eigenvalues, we need to show that both differences between $\|\overline{\partial}_{M^t}T^t(u)\|^2_{M^t}$ and $\|\overline{\partial}_M u\|^2_M$ and between $\|\overline{\partial}^*_{M^t}T^t(u)\|^2_{M^t}$ and $\|\overline{\partial}^*_M u\|^2_M$ are under control. This is how the subelliptic estimate comes into play. The following lemma is a direct consequence of the subelliptic estimate and the Sobolev embedding theorem.

Lemma 4.1. *Let M be the boundary of a smooth bounded domain in $\mathbb{C}^n$ such that a subelliptic estimate (1.2) holds. Let u be an eigenform of $\Box^q_b$ with associated eigenvalue $\lambda(M)$. Then for every $l \in \mathbb{N}$, there exist positive constant C_l such that*

$$\|u\|_{C^l} \le C_l(1 + \lambda(M))^{\frac{2(n+l)+1}{4\varepsilon}}\|u\|. \tag{4.1}$$

Proof. We provide a proof for completeness. Subelliptic estimate (1.2) implies that there exists a constant $C_s > 0$ such that

$$\|(\Box_b + I)^{-1}u\|_{s+2\varepsilon} \le C_s\|u\|_s. \tag{4.2}$$

(See the proof of Theorem 5.4.12 in [10].) Starting with $s = 0$ and repeatedly applying (4.2) to $(\Box_b + I)u = (\lambda(M) + 1)u$, we then have

$$\|u\|_{2m\varepsilon} \le C(1 + \lambda(M))^m\|u\|, \quad m \in \mathbb{N}. \tag{4.3}$$

The desired estimates (4.1) are then an immediate consequence of the Sobolev embedding theorem. $\Box$

Note that the constant C_l in (4.1) depends only on l, the constant C in (1.2), and the derivatives of the defining function. We are now in a position to state and prove the main result in this section:

Theorem 4.2. *Let M and M^t be boundaries of smooth bounded domains in $\mathbb{C}^n$, $n \geq 3$. Suppose a subelliptic estimate holds on M for the Kohn Laplacian on $(0, q)$-forms, $1 \leq q \leq n-2$. Then there exists a positive constant C_k independent of t such that*

$$\lambda_k^q(M^t) \leq \lambda_k^q(M) + C_k\delta^t, \tag{4.4}$$

provided that $\delta^t = \|\rho - \rho^t\|_{C^2}$ is sufficiently small.

We will keep the notations as in the previous section. Let $\bar{L}_j$, $1 \leq j \leq n-1$, be the local frame for $T^{0,1}(M)$ over a neighborhood U of a point p_0 on M defined by (3.19) and let $\bar{\omega}_\alpha$, $1 \leq \alpha \leq n$, be the $(0, 1)$-forms defined by (3.20). Note that

$$\overline{\partial}_M\bar{\omega}_j = \tau_M(\overline{\partial}\bar{\omega}_j) = \tau_M\Big(-\overline{\partial}\rho_j \wedge \overline{\partial}\rho\Big) = 0. \tag{4.5}$$

Let $u = \sum_J' u_J\bar{\omega}_J \in \Lambda^{0,q}(M \cap U)$. We have

$$\begin{aligned}\overline{\partial}_M u &= \sum_J{}' \big(\overline{\partial}_M u_J \wedge \bar{\omega}_J + u_J\overline{\partial}_M\bar{\omega}_J\big)\\ &= \sum_J{}' \sum_{j=1}^{n-1}(\bar{L}_j u_J)\,\bar{\omega}_j \wedge \bar{\omega}_J.\end{aligned} \tag{4.6}$$

We first compare the norms of $\overline{\partial}_{M^t}T^t(u)$ and $\overline{\partial}_M u$. From Proposition 3.2 applied to $\overline{\partial}_M u$, we have

$$\big|\|T^t(\overline{\partial}_M u)\|^2_{M^t} - \|\overline{\partial}_M u\|^2_M\big| \leq C\delta^t\|\overline{\partial}_M u\|^2_M. \tag{4.7}$$

It remains to estimate the difference of the norms of $\overline{\partial}_{M^t}T^t(u)$ and $T^t(\overline{\partial}_M u)$ on M^t. We have the following:

Lemma 4.3. *Let $u \in \mathrm{Dom}(Q_b)$. Then*

$$\big|\overline{\partial}_{M^t}T^t(u) - T^t(\overline{\partial}_M u)\big| \leq C\delta^t\big(|u| + |\nabla u|\big). \tag{4.8}$$

Proof. From (4.6), we have

$$\begin{aligned}T^t(\overline{\partial}_M u)(z) &= T^t\Big(\sum_J{}' \sum_{j=1}^{n-1}(\bar{L}_j u_J)\,\bar{\omega}_j \wedge \bar{\omega}_J\Big)(z)\\ &= \sum_J{}' \sum_{j=1}^{n-1}\big(\bar{L}_j u_J\big)(\pi(z))\big(\bar{\omega}_j^t \wedge \bar{\omega}_J^t + O(d^t)\big),\end{aligned} \tag{4.9}$$

for $z \in M^t \cap U$. It follows from (3.26) and (3.21) that

$$\begin{aligned}T^t(\bar{\omega}_j) &= \sum_{\alpha=1}^{n}\frac{\partial\bar{\pi}_j}{\partial\bar{z}_\alpha}\bar{\omega}_\alpha^t - 4\sum_{\alpha=1}^{n}(\rho_j\rho_{\bar{\alpha}} \circ \pi)\sum_{\beta=1}^{n}\frac{\partial\bar{\pi}_\alpha}{\partial\bar{z}_\beta}\bar{\omega}_\beta^t\\ &= \sum_{k=1}^{n-1} g_{jk}(z)\bar{\omega}_k^t,\end{aligned} \tag{4.10}$$

where

$$g_{jk}(z) = \frac{\partial \bar{\pi}_j}{\partial \bar{z}_k} - \frac{\partial \bar{\pi}_j}{\partial \bar{z}_n}\frac{\rho^t_{\bar{k}}}{\rho^t_{\bar{n}}} - 4\sum_{\alpha=1}^{n}(\rho_j\rho_{\bar{\alpha}} \circ \pi)\Big(\frac{\partial \bar{\pi}_\alpha}{\partial \bar{z}_k} - \frac{\partial \bar{\pi}_\alpha}{\partial \bar{z}_n}\frac{\rho^t_{\bar{k}}}{\rho^t_{\bar{n}}}\Big).$$

Hence

$$\overline{\partial}_{M^t} T^t(\bar{\omega}_j) = \sum_{l,k=1}^{n-1} \bar{L}^t_l g_{jk}(z)\, \bar{\omega}^t_l \wedge \bar{\omega}^t_k. \tag{4.11}$$

Let $p \in M^t$ and $p_0 = \pi(p) \in M$. After a unitary transformation, we assume as before that p_0 is the origin and the negative $\operatorname{Im} z_n$-axis is the outward normal direction at p_0. It follows from Lemma 3.3 that

$$\rho^t_{\bar{k}}(p),\ \rho_j(p_0),\ \frac{\partial \bar{\pi}_j}{\partial \bar{z}_n}(p),\ \frac{\partial \bar{\pi}_n}{\partial \bar{z}_k}(p),\ \text{and}\ \frac{\partial^2 \bar{\pi}_j}{\partial \bar{z}_l \partial \bar{z}_k}(p), \quad 1 \le j,k,l \le n-1,$$

are all dominated by a constant times δ^t. (Recall that $\delta^t = \|\rho^t - \rho\|_{C^2,U}$ where U is a neighborhood of M.) We then have

$$\bar{L}^t_l g_{jk}(p) = O(\delta^t),$$

and as a consequence

$$\overline{\partial}_{M^t} T^t(\bar{\omega}_j)(p) = O(\delta^t), \tag{4.12}$$

where as before $O(\delta^t)$ denotes a function or form whose pointwise norm is dominated by a constant multiple of δ^t. Therefore,

$$\begin{aligned}\overline{\partial}_{M^t} T^t(u)(z) &= \overline{\partial}_{M^t} {\sum_J}' u_J(\pi(z)) T^t(\bar{\omega}_J) \\ &= {\sum_J}' \sum_{j=1}^{n-1} \bar{L}^t_j\big(u_J(\pi(z))\big)\big(\bar{\omega}^t_j \wedge \bar{\omega}^t_J + O(\delta^t)\big) + {\sum_J}' u_J(\pi(z)) O(\delta^t)\end{aligned} \tag{4.13}$$

on $z \in M^t \cap U$. Combining (4.9) with (4.13), we then obtain (4.8). □

In comparing the norms of $\overline{\partial}^*_{M^t} T^t(u)$ and $\overline{\partial}^*_M u$, we likewise have

Lemma 4.4. *Let $u \in \operatorname{Dom}(Q^q_b)$. Then*

$$\big|\overline{\partial}^*_{M^t} T^t(u) - T^t(\overline{\partial}^*_M u)\big| \le C\delta^t(|u| + |\nabla u|). \tag{4.14}$$

Proof. Let $v = {\sum}'_{|K|=q-1} v_K \bar{\omega}_K$ be a smooth $(0, q-1)$-form, compactly supported on $M \cap U$. We have

$$(u, \overline{\partial}_M v)_M = {\sum_{J,K}}' \sum_{j=1}^{n-1} \int_M u_J \cdot \overline{\bar{L}_j v_K} \cdot \langle \bar{\omega}_J, \bar{\omega}_j \wedge \bar{\omega}_K \rangle dS. \tag{4.15}$$

Note that

$$\langle \bar{\omega}_J, \bar{\omega}_j \wedge \bar{\omega}_K \rangle = \delta^J_{jK} + R_1(\rho),$$

where

$$R_1(\rho) = P_1(\rho_j, \rho_{\bar{k}})$$

is a polynomial of first order partial derivative of ρ. Applying integration by parts to (4.15), we obtain

$$\overline{\partial}_M^* u = \sum_K{}' \sum_{j=1}^{n-1} (-L_j + R_2(\rho)) u_{jK}\, \bar{\omega}_K, \tag{4.16}$$

where $R_2(\rho)$ is a rational function where both the numerator and denominator are composed of up to second-order partial derivatives of ρ.

Likewise,

$$\langle \bar{\omega}_J^t, \bar{\omega}_j^t \wedge \bar{\omega}_K^t \rangle = \delta_{jK}^J + R_1(\rho^t).$$

Moreover, for $f \in \mathcal{E}^{0,q}(M^t \cap U)$, we have

$$\overline{\partial}_{M^t}^* f = \sum_K{}' \sum_{j=1}^{n-1} (-L_j^t + R_2(\rho^t)) f_{jK}\, \bar{\omega}_K^t.$$

Note that $R_k(\rho^t)$, $k = 1, 2$, can be obtained by replacing the derivatives of ρ in $R_k(\rho)$ by the corresponding derivatives of ρ^t. Hence

$$\|R_k(\rho) - R_k(\rho^t)\| \lesssim \delta^t, \quad k = 1, 2.$$

From (3.28) and (4.16), we see that

$$T^t(\overline{\partial}_M^* u)(z) = \sum_K{}' \sum_{j=1}^{n-1} \Big[\Big(\big(-L_j + R_2(\rho)\big) u_{jK}\Big)(\pi(z))\Big] \big(\bar{\omega}_K^t + O(\delta^t)\big). \tag{4.17}$$

Similar to the proof of (4.12), it follows from (4.10) and Lemma 3.3 that

$$\overline{\partial}_{M^t}^* T^t(\bar{\omega}_j) = \sum_{k=1}^{n-1} (-L_k^t + R_2(\rho^t)) g_{jk} = O(\delta^t). \tag{4.18}$$

Therefore,

$$\overline{\partial}_{M^t}^* T^t(u)(z) = \sum_K{}' \sum_{j=1}^{n-1} \big(-L_j^t + R_2(\rho^t)\big)(u_{jK}(\pi(z)))(1 + O(\delta^t))\, \bar{\omega}_K^t \tag{4.19}$$

for $z \in M^t \cap U$. Hence we have (4.14). □

Theorem 4.2 is then a consequence of Lemmas 2.1, 4.3, and 4.4. We sketch the proof as follows. Let $\mathcal{X}_k$ be the linear span of the normalized eigenforms u_j, $1 \le j \le k$, associated with the first k eigenvalues $\lambda_j^q(M)$ for the Kohn Laplacian for $(0, q)$-forms on M. It then follows from Lemma 4.1 that for any $u \in \mathcal{X}_k$,

$$\|u\|_{C^1} \le C_k \|u\|.$$

From Lemmas 4.3 and 4.4, we have

$$\left|Q_M(u,u) - Q_{M^t}(T^t u, T^t u)\right| \leq C_k \delta^t \|u\|^2.$$

Theorem 4.2 then follows from Lemma 2.1.

5. Lower semi-continuity

The proof of lower semi-continuity of the eigenvalues is similar. We provide details for the reader's convenience.

Theorem 5.1. *Let M be the boundary of a smooth pseudoconvex domain in $\mathbb{C}^n$ with normalized defining functions ρ. Let M^t be a family of boundaries of smooth pseudoconvex domains that satisfies the uniform subelliptic estimate. Let $1 \leq q \leq n-2$ and $k \in \mathbb{N}$. Then there exists a constant C_k which is independent of t, such that*

$$\lambda_k^q(M^t) \geq \lambda_k^q(M) - C_k \delta^t, \tag{5.1}$$

provided $\delta^t = \|\rho - \rho^t\|_{C^2}$ is sufficiently small.

Proof. The proof is similar in some respects to Theorem 4.2. The difference here is we use Lemma 4.1 to establish estimates that are uniform with regard to t.

We define $\hat{T}^t : \mathrm{Dom}(Q_b^t) \to \mathrm{Dom}(Q_b)$ by

$$\hat{T}^t = \tau_M \circ P^{0,q} \circ (\iota^t)^*.$$

For $u^t \in \mathrm{Dom}(Q_b^t)$, note that

$$\begin{aligned} \|\hat{T}^t(u^t)\|_M^2 - \|u^t\|_{M^t}^2 &= \int_M \langle \hat{T}^t(u^t), \hat{T}^t(u^t)\rangle dS - \int_{M^t} \langle u^t, u^t\rangle dS^t \\ &= \int_M \left(\left|\hat{T}^t(u^t)\right|^2(z) - |u^t|^2(\iota^t(z))|\mathrm{Jac}\,\iota^t_{*z}|\right) dS. \end{aligned} \tag{5.2}$$

As in the proof of (3.3), it is sufficient to estimate $\left|\hat{T}^t(u^t)\right|^2(z) - |u^t|^2(\iota^t(z))|\mathrm{Jac}\,\iota^t_{*z}|$ pointwise. We prove the case $q = 1$, and the general case follows from the same argument. Let $z \in M^t \cap U$, write $u^t(z) = \sum_{j=1}^{n-1} u_j^t(z)\bar{\omega}_j(z)$. From (3.24) and (3.27), we have

$$\hat{T}^t(u^t)(z) = \sum_{j=1}^{n-1} u_j^t \circ \iota^t(z)\big(\bar{\omega}_j(z) + O(\delta^t)\big) \tag{5.3}$$

and

$$|u^t|^2(\iota^t(z)) = \sum_{j,k=1}^{n-1} \langle u_j^t \circ \iota^t, u_k^t \circ \iota^t\rangle \big(\delta_{jk}(1 - 8|\rho_j^t|^2 \circ \iota^t) + 4(\rho_j^t \rho_{\bar{k}}^t) \circ \iota^t + O(\delta^t)\big). \tag{5.4}$$

From Lemma 3.3, we see that

$$\begin{aligned}\left|\hat{T}^t(u^t)\right|^2(z) - |u^t|^2(\iota^t(z))|\mathrm{Jac}\,\iota^t_{*z}| &= 8\sum_{j=1}^{n-1}|u^t_j\circ\iota^t|^2(-|\rho_j|^2+|\rho^t_j|^2\circ\iota^t)\\ &+4\sum_{j,k=1}^{n-1}\langle u^t_j\circ\iota^t, u^t_k\circ\iota^t\rangle\big(\rho_j\rho_{\bar{k}}-(\rho^t_j\rho^t_{\bar{k}})\circ\iota^t\big)+O(\delta^t)|u^t\circ\iota^t|^2\end{aligned} \tag{5.5}$$

on $z\in M\cap U$. Hence we obtain

$$\left|\|\hat{T}^t(u^t)\|^2_M - \|u^t\|^2_{M^t}\right| \lesssim \delta^t\|u^t\|^2. \tag{5.6}$$

We now assume that u^t is the normalized eigenform of $\Box^t_b$ associated with eigenvalue λ^t. As in the proof of Theorem 4.2, it suffices to prove estimates $\|\overline{\partial}_M\hat{T}^t(u^t)-\hat{T}^t(\overline{\partial}_{M^t}u^t)\|_M$ and $\|\overline{\partial}^*_M\hat{T}^t(u^t)-\hat{T}^t(\overline{\partial}^*_{M^t}u^t)\|_M$ that are uniform with respect to t. Suppose $u\in\mathcal{D}^{0,q}(U\cap M)$ and write $u=\sum'_J u_J\bar{\omega}_J$. It is not difficult to obtain that

$$\hat{T}^t(\overline{\partial}_{M^t}u^t) = {\sum_J}'\sum_{j=1}^{n-1}\big(\bar{L}^t_j u^t_J\circ\iota^t\big)(\bar{\omega}_j\wedge\bar{\omega}_J+O(\delta^t)) \tag{5.7}$$

and

$$\overline{\partial}_M\hat{T}^t(u^t) = {\sum_J}'\sum_{j=1}^{n-1}\bar{L}_j(u^t_J\circ\iota^t)\bar{\omega}_j\wedge(\bar{\omega}_J+O(\delta^t)) + {\sum_J}'(u^t_J\circ\iota^t)O(\delta^t). \tag{5.8}$$

Applying Lemma 4.1, we get

$$\begin{aligned}\|\overline{\partial}_M\hat{T}^t(u^t)-\hat{T}^t(\overline{\partial}_{M^t}u^t)\|^2_M &= \sum_{l=1}^m\int_M\psi_l^2\Big|{\sum_J}'\sum_{j=1}^{n-1}\big(\bar{L}_j(u^t_J\circ\iota^t)-\bar{L}^t_j u^t_J\circ\iota^t\big)\bar{\omega}^l_j\wedge\bar{\omega}^l_J\\ &\qquad+{\sum_J}'\sum_{j=1}^{n-1}\big(\bar{L}_j(u^t_J\circ\iota^t)-\bar{L}^t_j u^t_J\circ\iota^t+u^t_J\circ\iota^t\big)O(\delta^t)\Big|^2 dS\\ &\le C(\delta^t\|u^t\|_{C^1})^2 \le C(\delta^t(1+\lambda^t)^{\frac{n+2}{2\varepsilon}}\|u^t\|)^2,\end{aligned} \tag{5.9}$$

where $\{\psi_l\}^m_{l=1}$ denotes a partition of unity and here constant C is independent of t. The estimate $\|\overline{\partial}^*_M\hat{T}^t(u^t)-\hat{T}^t(\overline{\partial}_{M^t}u^t)\|_M\lesssim\delta^t$ can be obtained similarly. It follows that

$$|Q^t_b(u^t,u^t)-Q_b(\hat{T}^t(u^t),\hat{T}^t(u^t))|\le C\delta^t. \tag{5.10}$$

The desired inequality (5.1) then follows from Lemma 2.1 and the subsequent remark. □

6. Bottom and top degree cases

In this section, we prove Theorem 1.1, Parts (2) and (3). We first establish stability of eigenvalues for bottom $(0,0)$-degree and top $(0,n-1)$-degree forms.

Write $\lambda^0_k(M)$ as the k^{th}-positive eigenvalue of $\Box^0_b$. Since a subelliptic estimate for $(0,1)$-form holds on M, $\overline{\partial}^*_M$ has close range in $L^2_{(0,0)}(M)$. Hence $\mathcal{N}(\overline{\partial}_M)^\perp=\mathcal{R}(\overline{\partial}^*_M)$. Therefore

$$
\begin{aligned}
\lambda_k^0(M) &= \inf_{\substack{L\subset \mathrm{Dom}(\overline{\partial}_M)\cap \mathcal{N}(\overline{\partial}_M)^\perp \\ \dim L=k}} \sup_{u\in L\setminus\{0\}} \frac{\|\overline{\partial}_M u\|^2}{\|u\|^2} \\
&= \inf_{\substack{L\subset \mathcal{R}(\overline{\partial}_M^*) \\ \dim L=k}} \sup_{u\in L\setminus\{0\}} \frac{\|\overline{\partial}_M u\|^2}{\|u\|^2} \qquad (6.1)\\
&= \inf_{\substack{K\subset \mathrm{Dom}(\overline{\partial}_M^*)\cap \mathcal{N}(\overline{\partial}_M^*)^\perp \\ \dim K=k}} \sup_{f\in K\setminus\{0\}} \frac{\|\overline{\partial}_M \overline{\partial}_M^* f\|^2}{\|\overline{\partial}_M^* f\|^2}.
\end{aligned}
$$

A similar identity also holds for M^t.

Recall that

$$T^t = \tau_{M^t} \circ P^{0,1}|_{M^t} \circ \pi^* \colon \Lambda^{0,1}(M) \to \Lambda^{0,1}(M^t). \qquad (6.2)$$

We now use the same argument as in the cases $1 \le q \le n-2$ to demonstrate that $\lambda_k^0(M^t)$ satisfies upper-semicontinuity estimates. Letting $u = \overline{\partial}_M^* f$ in (3.3) and $u = f$ in (4.14), we have

$$\big|\|\overline{\partial}_{M^t}^* T^t(f)\|_{M^t}^2 - \|\overline{\partial}_M^* f\|_M^2\big| \le C\delta^t \|\overline{\partial}_M^* f\|_{C^1}^2. \qquad (6.3)$$

We claim that

$$\big|\|\overline{\partial}_{M^t}\overline{\partial}_{M^t}^* T^t(f)\|_{M^t}^2 - \|\overline{\partial}_M\overline{\partial}_M^* f\|_M^2\big| \le C\delta^t \|\overline{\partial}_M^* f\|_{C^2}^2 \qquad (6.4)$$

With $u = \overline{\partial}_M^* f$ in (4.8), we obtain the following pointwise estimate

$$\big|\overline{\partial}_{M^t} T^t(\overline{\partial}_M^* f) - T^t(\overline{\partial}_M\overline{\partial}_M^* f)\big| \le C\delta^t \big(|\overline{\partial}_M^* f| + |\nabla\overline{\partial}_M^* f|\big) \qquad (6.5)$$

on M^t. Substituting $u = \overline{\partial}_M\overline{\partial}_M^* f$ into (3.3), we get

$$\big|\|T^t(\overline{\partial}_M\overline{\partial}_M^* f)\|_{M^t}^2 - \|\overline{\partial}_M\overline{\partial}_M^* f\|_M^2\big| \le C\delta^t \|\overline{\partial}_M\overline{\partial}_M^* f\|_M^2, \qquad (6.6)$$

which gives

$$\big|\|\overline{\partial}_{M^t} T^t(\overline{\partial}_M^* f)\|_{M^t}^2 - \|\overline{\partial}_M\overline{\partial}_M^* f\|_M^2\big| \le C\delta^t \|\overline{\partial}_M\overline{\partial}_M^* f\|_M^2. \qquad (6.7)$$

In order to prove the claim, it is sufficient to show that

$$\big|\overline{\partial}_{M^t} T^t(\overline{\partial}_M^* f) - \overline{\partial}_{M^t}\overline{\partial}_{M^t}^* T^t(f)\big| \le C\delta^t \big(|\overline{\partial}_M^* f| + |\nabla\overline{\partial}_M^* f| + |\nabla^2\overline{\partial}_M^* f|\big). \qquad (6.8)$$

Indeed, this follows directly from (4.17) and (4.19). Since a uniform subelliptic estimate holds on M^t, we have

$$\|f\|_{\varepsilon/2}^2 \le C\|\overline{\partial}_{M^t}^* f\|^2, \qquad f \in \mathcal{N}(\overline{\partial}_{M^t}^*)^\perp.$$

(See the proofs of Theorem 8.4.10 and Theorem 8.4.14 in [5].) We then conclude the proof of the bottom degree case by applying the arguments of Lemma 4.1. The proof for the top degree case is similar and is left to the reader.

To prove Theorem 1.1 (3), we just need to apply Kohn's subelliptic estimate on the pseudoconvex hypersurface M of finite type in $\mathbb{C}^2$ and use the fact that these estimates are uniform on M^t provided $\|\rho^t - \rho\|_{C^\infty(U)} \to 0$.

7. Further remarks

(1) Our main result assumes that subelliptic estimate (1.2) with constants ε and C independent of t holds on M^t. It is of interest to know whether the subelliptic estimate remains stable under perturbations of the boundary in the C^∞-topology. More precisely, let M^t, $-1 \le t \le 1$, be a family of smooth compact pseudoconvex hypersurfaces in $\mathbb{C}^n$. Assume that the defining function ρ^t of M^t depends smoothly on t. Suppose subelliptic estimate (1.2) holds on M^0. Does it follow that a uniform subelliptic estimate holds on M^t (with possibly different constants ε and C that are independent of t) when t is sufficiently small? As we note in Section 1, this is the case if M^0 is strictly pseudoconvex in $\mathbb{C}^n$ or of finite type in $\mathbb{C}^2$. D'Angelo showed that if M^0 is of finite 1-type, then M^t is also of finite 1-type with a uniform bound on the type when t is sufficiently small ([6, Theorem 6.9]). Thus a subelliptic estimate holds on each M^t. However, it is not known to us whether the constants ε and C can be chosen to be both independent of t.

(2) Our method makes no use of the assumption that the defining function ρ^t of the hypersurface M^t depends smoothly on t. In fact, it is easy to see that the main results can be reformulated for a sequence of hypersurfaces M^{t_j} whose defining functions converge to that of M in C^2-norms.

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