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Critical thresholds in the Euler-Poisson-alignment system

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Abstract

This paper is concerned with the global wellposedness of the Euler-Poisson-alignment (EPA) system. This system arises from collective dynamics, and features two types of nonlocal interactions: the repulsive electric force and the alignment force. It is known that the repulsive electric force generates oscillatory solutions, which is difficult to be controlled by the nonlocal alignment force using conventional comparison principles. We construct *invariant regions* such that the solution trajectories cannot exit, and therefore obtain global wellposedness for subcritical initial data that lie in the invariant regions. Supercritical regions of initial data are also derived which leads to finite-time singularity formations. To handle the oscillation and the nonlocality, we introduce a new way to construct invariant regions piece by piece in the phase plane of a reformulation of the EPA system. Our result is extended to the case when the alignment force is weakly singular. The singularity leads to the loss of a priori bounds crucial in our analysis. With the help of improved estimates on the nonlocal quantities, we design non-trivial invariant regions that guarantee global wellposedness of the EPA system with weakly singular alignment interactions.

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1. Introduction

In this paper, the point of concern is the following one-dimensional Euler-Poisson-alignment (EPA) system

$$\rho_t + (\rho u)_x = 0, \quad (1.1a)$$

$$u_t + uu_x = -k\phi_x + \int_{\mathbb{R}} \psi(x-y)(u(y) - u(x))\rho(y)dy, \quad (1.1b)$$

$$-\phi_{xx} = \rho - c, \quad (1.1c)$$

subject to smooth initial density and velocity

$$(\rho(t, \cdot), u(t, \cdot))|_{t=0} = (\rho_0 \geq 0, u_0).$$

This system can be viewed as the pressureless Euler equations with two types of nonlocal interacting forces on the right-hand side of the momentum equation (1.1b): the *electric force* and the *alignment force*.

The electric force is modeled through an interacting potential ϕ , that is governed by the Poisson equation (1.1c), with a constant c representing the background charge that can be zero or a positive constant. The parameter k signifies the property of the underlying force: repulsive $k > 0$ or attractive $k < 0$. When only electric force is present, i.e. $\psi \equiv 0$, (1.1) reduces to the classical *Euler-Poisson system*. It has been an area of intensive study due to their vast relevance in modeling physical phenomena [5,12,14,20–22], including semiconductor and plasma dynamics.

The alignment force describes the collective motion of an interacting system, where the *influence function* ψ characterizes the strength of the pairwise velocity alignment interaction. Naturally, $\psi(x) = \psi(|x|)$ is assumed to be radial and decreasing in \mathbb{R}_+ . When only alignment force is present, i.e. $k = 0$, the system reduces to the *Euler-alignment system*, which serves as a macroscopic realization of the celebrated agent-based Cucker-Smale flocking model [7,8], cf. [11] for a derivation.

The EPA system lies in a framework of collective dynamics involving interactions among three zones: long-range attraction, short-range repulsion, and mid-range alignment [6]. The electric force within this system can be understood as an interactive force that arises from attractive or repulsive Newtonian potentials.

The purpose of this work is to study the global regularity of the EPA system (1.1) for general initial data. It is well-known that the finite-time breakdown of the pressureless Euler equations is generic, see e.g. [16]. In particular, for all smooth initial data such that u_0 is non-increasing, the solutions develop finite-time shock formations. On the other hand, the interacting forces intend to help avoiding the singularities.

For the 1D Euler-Poisson system with a repulsive force, a *critical threshold phenomenon* is shown in [10]: there exists a large class of *subcritical* initial data that lead to global smooth solutions, while a class of *supercritical* initial data lead to finite-time shock formations. See e.g. [17–19,27,28,30] on extensions to higher dimensions and with pressure.

For the Euler-alignment system, a similar critical threshold phenomenon is observed in [26] when the influence function ψ is bounded, cf. also [6,13]. Recently, there is a growing interest on singular influence function that are unbounded at the origin. When ψ is *strongly singular*, namely

ψ is non-integrable near the origin, it has been shown in [9,25] that all non-vacuous periodic initial data lead to global smooth solutions. When ψ is *weakly singular*, namely unbounded but integrable at the origin, critical thresholds are obtained in [29], also see [4] for improved bounds on density with any integrable ψ . For recent development on the Euler-alignment system, we refer readers to the book [24] and the references therein.

For the EPA system (1.1), we expect the critical threshold phenomenon when the influence function ψ in the alignment force is bounded. Such behavior has been first shown in [6], where the Poisson equation (1.1c) is assumed to have a zero background ($c = 0$). The result extends to any attractive or repulsive forces through a potential that is less singular than the Newtonian potential. The authors in [2] study the EPA system with attractive electric forces ($k < 0$) and nonzero, non-constant background ($c(x) > 0$). The dynamics are more subtle. They design highly non-trivial comparison principles to take care of the nonlocality that arises from the alignment force, and manage to obtain bounds on subcritical and supercritical regions of initial data, thus describing the critical threshold phenomenon.

Our main focus of this paper is on the EPA system (1.1) where the electric forces are repulsive ($k > 0$) and with non-zero background ($c > 0$). This type of electric forces is physically relevant. The solution to the corresponding Euler-Poisson system is known to generate solutions that oscillate, e.g. [10]. Such distinct feature makes it difficult to incorporate with the nonlocal alignment forces. In particular, the comparison principles used in [2] are no longer valid. New analytical tools are needed to capture the critical threshold phenomenon.

For convenience, we assume the spatial domain to be a torus $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$, namely we consider 1-periodic data. We shall comment that many of our results can be extended to the whole real line case with

$$\int_{-\infty}^{\infty} (\rho_0(x) - c) dx = 0.$$

We shall leave this case for future investigation.

Under the spatial domain \mathbb{T} , the Poisson equation (1.1c) requires the background charge to be the average density, that is conserved in time due to (1.1a). We have

$$c = \int_{\mathbb{T}} \rho_0(x) dx. \quad (1.2)$$

One useful parameter that plays an important role in quantifying the strength of the electric force is

$$\lambda = 2\sqrt{\frac{k}{c}}. \quad (1.3)$$

It is assumed to be a positive finite number throughout this paper. The alignment force can be equivalently expressed as

$$\int_{\mathbb{T}} \psi_{\text{per}}(y)(u(x+y) - u(x))\rho(x+y) dy,$$

with the periodic influence function

$$\psi_{\text{per}}(x) := \sum_{m \in \mathbb{Z}} \psi(x + m), \quad \forall x \in \mathbb{T},$$

which is symmetric with respect to zero. We will continually use ψ to represent the periodic influence function for simplicity.

Our first main result is on the global wellposedness of the EPA system (1.1) with repulsive electric force $k > 0$ and *bounded* alignment influence:

$$0 \leq \psi_{\min} \leq \psi(x) \leq \psi_{\max}, \quad \forall x \in \mathbb{T}. \quad (1.4)$$

We construct a class of subcritical initial data and show solutions are globally regular; on the other hand, we also find a class of supercritical initial data such that solutions experience finite-time singularity formations. The precise descriptions of such critical threshold phenomenon are stated in Theorems 2.2 and 2.5. Depending on the relative strength between the electric force and the alignment force, there are three different scenarios: (i). weak alignment ($\psi_{\max} < \lambda$), (ii). strong alignment ($\psi_{\min} \geq \lambda$), and (iii). medium alignment ($\psi_{\min} < \lambda \leq \psi_{\max}$). We construct subcritical regions Σ_i and supercritical regions Δ_i on initial data for each scenario, that leads to either global wellposedness or finite-time blowup, respectively.

In particular, when the alignment force is weak or medium, the solution is oscillatory. Instead of a direct comparison with an auxiliary system, we construct an *invariant region* in the phase plane of the solutions along each characteristic path. The novelty of our construction is that we use different auxiliary systems to build segments of the boundary of the invariant regions, and then glue them together. This allows us to handle the nonlocal alignment force while the underlying Euler-Poisson system is highly oscillatory.

We would like to point out a special case when ψ is a constant, known as *all-to-all* alignment interactions. In this case, the alignment force reduces to a local and linear damping, and (1.1) becomes the *damped Euler-Poisson system*. The invariant regions that we constructed are consistent with the sharp critical threshold conditions obtained in [1] on the damped Euler-Poisson system.

The next focus is on the singular alignment interactions. When ψ is strongly singular, the EPA system (1.1) was studied in [15]. The surprising result indicates that the alignment force dominates the electric force, regardless of whether the electric force is attractive or repulsive. Any smooth non-vacuous initial data lead to global smooth solutions. The argument holds even if we drop the assumption $\psi \geq 0$, namely misalignment is allowed, as discussed in [23].

Our second main result is on the EPA system (1.1) with repulsive electric force $k > 0$ and *weakly singular* alignment influence:

$$\psi(x) \geq 0, \quad \forall x \in \mathbb{T}, \quad \text{and} \quad \|\psi\|_{L^1(\mathbb{T})} < +\infty. \quad (1.5)$$

In particular, ψ can be unbounded at $x = 0$. Although the singularity is not strong enough to produce dominating dissipation like the strongly singular case, the global behavior is not expected to be the same as the case when ψ is bounded. Without the L^∞ bound on ψ , we do not have the following a priori bounds on the quantity $\psi * \rho$ (here $*$ denotes the spatial convolution)

$$\psi_m c \leq \int_{\mathbb{T}} \psi(y) \rho(t, x - y) dy \leq \psi_M c, \quad \forall t \geq 0, \quad (1.6)$$

which plays an essential role in the global regularity of the Euler-alignment system (see [29]), as well as our approach to the EPA system with bounded alignment interactions.

We construct a subcritical region on initial data such that the solution is globally regular. The main idea is to replace (1.6) by

$$\|\psi\|_{L^1} \rho_{\min} \leq \int_{\mathbb{T}} \psi(y) \rho(t, x - y) dy \leq \|\psi\|_{L^1} \rho_{\max}, \quad \forall t \geq 0, \quad (1.7)$$

where the bounds depend on the maximum and minimum of the solution ρ . Then choose appropriate constants ρ_{\min} and ρ_{\max} , and build an invariant region that is a subset of $\{\rho_0 : \rho_{\min} \leq \rho_0(x) \leq \rho_{\max}\}$. However, with the bound (1.7), we are not able to obtain a non-trivial invariant region using our analytical framework, with any choice of ρ_{\min} and ρ_{\max} . Indeed, for the Euler-alignment system, it is observed in [29] that, without the a priori bounds like (1.6), additional treatments are required to control ρ_{\max} , and the critical threshold is different from the scenario when ψ is bounded. The presence of the electric force adds another layer of complexity. To overcome such difficulty, we obtain refined bounds of (1.7), stated in Lemma 4.1, making use of the equation (1.2). With the refined bounds, we can obtain non-trivial invariant regions by the right choices of ρ_{\min} and ρ_{\max} , and show global regularity of the EPA system (1.1) if initial data lie in these subcritical regions. The precise statement is presented as Theorem 2.6.

This paper is arranged as follows. Section 2 contains the statements of the main results in this paper. Section 3 entails the constructions of the subcritical and supercritical regions for (1.1) with bounded alignment influence, proving Theorems 2.2 and 2.5. The first three subsections focus on the subcritical regions to the three different scenarios respectively. The fourth subsection is on the construction of the supercritical regions. Section 4 is devoted to the construction of the invariant region for (1.1) with weakly singular alignment influence, proving Theorem 2.6.

2. Main results

Let us start with a reformulation for the EPA system (1.1) through an auxiliary variable

$$G = u_x + \psi * \rho,$$

introduced in [6]. System (1.1) can be expressed in the following equivalent form

$$G_t + (Gu)_x = k(\rho - c), \quad (2.1a)$$

$$\rho_t + (\rho u)_x = 0, \quad (2.1b)$$

$$u_x = G - \psi * \rho. \quad (2.1c)$$

The velocity u can be recovered from (2.1c). It is uniquely defined up to a constant shift. The constant can then be uniquely determined by the total momentum $\int_{\mathbb{T}} \rho u dx$, which is conserved in time.

We state a local wellposedness result for smooth solutions to (2.1). The proof can be done using energy estimates on the derivatives of (G, ρ) . See [29, Theorem 2.1] for a complete proof when $k = 0$. The result can be easily extended to the case when $k \neq 0$, cf. also [3, 15].

Theorem 2.1 (*Local wellposedness*). *Consider the system (2.1) with initial data*

$$G_0 \in H^s(\mathbb{T}), \quad s > \frac{1}{2}, \quad \rho_0 \in (L_+^1 \cap H^s)(\mathbb{T}), \quad (2.2)$$

and interactions with $k \in \mathbb{R}$, $\psi \in L^1(\mathbb{T})$. Then, there exists a time $T > 0$ such that the solution

$$G \in C([0, T]; H^s(\mathbb{T})), \quad \rho \in C([0, T]; (L_+^1 \cap H^s)(\mathbb{T})).$$

Consequently, the EPA system (1.1) has a smooth solution

$$\rho \in C([0, T]; (L_+^1 \cap H^s)(\mathbb{T})), \quad u \in C([0, T]; H^{s+1}(\mathbb{T})).$$

Moreover, T can be extended as long as

$$\int_0^T \left(\|G(t, \cdot)\|_{L^\infty} + \|\rho(t, \cdot)\|_{L^\infty} \right) dt < \infty. \quad (2.3)$$

The regularity criterion (2.3) indicates: the global-in-time bounds on G and ρ are sufficient to obtain global regularity.

Our first main result focuses on repulsive electric force $\lambda > 0$ and bounded influence functions ψ in the alignment force (1.4).

Theorem 2.2 (*Global solutions*). *Consider (2.1) with repulsive electric force $k > 0$ and bounded alignment influence ψ satisfying (1.4). Suppose the initial data (G_0, ρ_0) satisfies (2.2). Then*

1. Weak alignment ($\psi_{\max} < \lambda$): under the admissible condition

$$\psi_{\max} - \psi_{\min} < \frac{e^{\frac{\tan^{-1} \frac{\lambda}{\hat{z}}}{\hat{z}}} \left(1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}} \right) \lambda}{2 \left(1 + e^{-\frac{\pi}{\hat{z}}} \right)}, \quad (2.4)$$

if the initial data lie in the subcritical region Σ_1 , namely

$$(G_0(x), \rho_0(x)) \in \Sigma_1, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

2. Strong alignment ($\psi_{\min} \geq \lambda$): if the initial data lie in the subcritical region Σ_2 , namely

$$(G_0(x), \rho_0(x)) \in \Sigma_2, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

3. Medium alignment ($\psi_{\min} < \lambda \leq \psi_{\max}$): under the admissible condition

$$\psi_{\max} - \psi_{\min} < \frac{e^{\frac{\tan^{-1}\hat{z}}{\tilde{z}}}}{2\left(1 + e^{-\frac{\pi}{\tilde{z}}}\right)}\lambda, \quad (2.5)$$

if the initial data lie in the subcritical region Σ_3 , namely

$$(G_0(x), \rho_0(x)) \in \Sigma_3, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

Consequently, (2.1) has a global smooth solution. Here, the parameters \hat{z} and \tilde{z} are defined as

$$\hat{z} := \sqrt{\left(\frac{\lambda}{\psi_{\max}}\right)^2 - 1} \quad \text{and} \quad \tilde{z} := \sqrt{\left(\frac{\lambda}{\psi_{\min}}\right)^2 - 1}. \quad (2.6)$$

Note that \hat{z}, \tilde{z} could be real, purely imaginary, as well as infinity. The regions Σ_1 , Σ_2 and Σ_3 are subsets of $\mathbb{R} \times \mathbb{R}_+$, defined in (3.20), (3.28) and (3.31) respectively.

Remark 2.3. The subcritical regions Σ_1 , Σ_2 are illustrated in Fig. 1. The shape of Σ_3 is similar to Σ_1 . We would like to point out that the steady-state solution $(G, \rho) = (c\|\psi\|_{L^1}, c)$ to (2.1) is included in the subcritical regions Σ_1 , Σ_2 and Σ_3 . This corresponds to the steady-state solution $\rho(x) \equiv c$ and $u(x) \equiv \bar{u}$ to (1.1). Therefore, our subcritical regions are non-empty, and contain a large class of physically meaningful initial data, including the states around a steady state.

Remark 2.4. When $\psi(x) \equiv \psi$ is a constant, the alignment force becomes a local and linear damping. Our constructed invariant regions agree with the sharp subcritical threshold obtained in [1]. The admissible conditions (2.4) and (2.5) automatically hold. For general ψ , the admissible conditions ensure the nonlocality is not too strong, and the invariant regions are non-trivial.

Theorem 2.5 (Finite time breakdown). *Under the same assumptions as Theorem 2.2, we have*

1. Weak alignment ($\psi_{\max} < \lambda$): If there exists $x_0 \in \mathbb{T}$ that lie in the supercritical region Δ_1 , namely

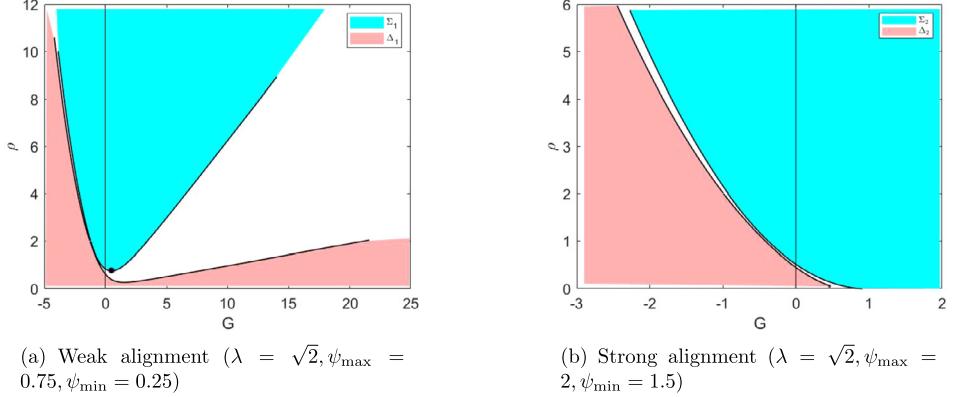
$$(G_0(x_0), \rho_0(x_0)) \in \Delta_1,$$

then (G, ρ) becomes unbounded at a finite time.

2. Strong and medium alignment ($\psi_{\max} \geq \lambda$): If there exists $x_0 \in \mathbb{T}$ that lie in the supercritical region Δ_2 , namely

$$(G_0(x_0), \rho_0(x_0)) \in \Delta_2,$$

then (G, ρ) becomes unbounded at a finite time.

Fig. 1. Shapes of $\Sigma_1, \Sigma_2, \Delta_1, \Delta_2$.

Moreover, at the blowup time t_c and location x_c , the solution generates a singular shock, with

$$\lim_{t \rightarrow t_c^-} \rho(t, x_c) = \infty \text{ or } 0, \quad \lim_{t \rightarrow t_c^-} G(t, x_c) = -\infty, \quad \lim_{t \rightarrow t_c^-} u_x(t, x_c) = -\infty.$$

The regions Δ_1, Δ_2 are defined in (3.32), (3.33) respectively.

Our second main result concerns the EPA system with weakly singular alignment influence (1.5). Although one would expect a similar critical threshold phenomenon for the global behaviors of the solutions, the lack of boundedness on ψ would yield a lack of apriori control on $\psi * \rho$, resulting a different subcritical region for global smooth solutions.

Theorem 2.6 (On weakly singular alignment force). *Consider (2.1) with repulsive electric force $k > 0$ and weakly singular alignment influence ψ satisfying (1.5). Suppose the initial data (G_0, ρ_0) satisfies (2.2). Then*

1. Weak alignment ($\|\psi\|_{L^1} - \gamma < \frac{\lambda}{2}$): under the admissible condition

$$4(\|\psi\|_{L^1} - 2\gamma) < \frac{e^{\frac{\tan^{-1}\hat{z}}{\hat{z}}}\left(1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}}\right)}{2\left(1 + e^{-\frac{\pi}{\hat{z}}}\right)}\lambda, \quad (2.7)$$

if the initial data lie in the subcritical region Σ_L^1 , namely

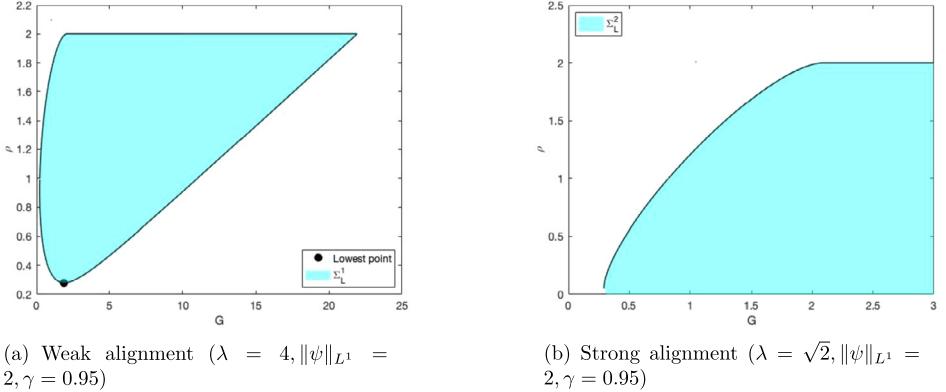
$$(G_0(x), \rho_0(x)) \in \Sigma_L^1, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

2. Strong alignment ($\gamma \geq \frac{\lambda}{2}$): if the initial data lie in the subcritical region Σ_L^2 , namely

$$(G_0(x), \rho_0(x)) \in \Sigma_L^2, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

Fig. 2. Shapes of Σ_L^1, Σ_L^2 .

3. Medium alignment ($\gamma < \frac{\lambda}{2} \leq \|\psi\|_1 - \gamma$): under the admissible condition

$$4(\|\psi\|_{L^1} - 2\gamma) < \frac{e^{\frac{\tan^{-1}\hat{z}}{\hat{z}}}}{2\left(1 + e^{-\frac{\pi}{\hat{z}}}\right)}\lambda, \quad (2.8)$$

if the initial data lie in the subcritical region Σ_L^3 , namely

$$(G_0(x), \rho_0(x)) \in \Sigma_L^3, \quad \forall x \in \mathbb{T},$$

then (G, ρ) remain bounded in all time.

Consequently, (2.1) has a global smooth solution. Here, $\gamma = \int_{1/2}^1 \psi^*(x) dx$, where $\psi^* : (0, 1] \rightarrow \mathbb{R}$ is the decreasing rearrangement of ψ on \mathbb{T} . The parameters \hat{z} and \tilde{z} are defined as

$$\hat{z} := \sqrt{\left(\frac{\lambda}{2(\|\psi\|_{L^1} - \gamma)}\right)^2 - 1} \quad \text{and} \quad \tilde{z} := \sqrt{\left(\frac{\lambda}{2\gamma}\right)^2 - 1}. \quad (2.9)$$

The regions Σ_L^1, Σ_L^2 and Σ_L^3 are subsets of $\mathbb{R} \times \mathbb{R}_+$ defined in (4.19), (4.21) and (4.22) respectively.

Remark 2.7. Unlike the case when ψ is bounded, the subcritical regions Σ_L^i 's are subsets of $\{(G_0, \rho_0) : \rho_{\min} \leq \rho_0 \leq \rho_{\max}\}$ for appropriate choices of $0 \leq \rho_{\min} < c < \rho_{\max} < \infty$. Fig. 2(a) illustrates the shape of Σ_L^1 and Σ_L^2 . The steady-state solution $(G, \rho) = (c\|\psi\|_{L^1}, c) \in \Sigma_L^i$. Hence, the region Σ_L^i contains initial data around the steady state.

Remark 2.8. The admissible conditions (2.7) and (2.8) are similar to (2.4) and (2.5) respectively. Since ψ is unbounded, $\psi_{\max} - \psi_{\min}$ is replaced by $4(\|\psi\|_{L^1} - 2\gamma)$. Note that $\|\psi\|_{L^1} - 2\gamma \geq 0$, and the equality holds if and only if $\psi(x) \equiv \psi$ is a constant. Hence, just like the comment in Remark 2.4, the admissible condition says that the nonlocality is not too strong. The parameters \hat{z} and \tilde{z} are also revised to adapt the unboundedness of ψ .

3. The EPA system with bounded alignment influence

Consider the characteristic path $x(t)$ originated at $\alpha \in \mathbb{T}$

$$\frac{dx}{dt} = u(t, x(t)), \quad x(0) = \alpha. \quad (3.1)$$

From (2.1b) and (2.1a), we obtain the system

$$G' = -G(G - \psi * \rho) + k(\rho - c), \quad (3.2a)$$

$$\rho' = -\rho(G - \psi * \rho), \quad (3.2b)$$

with initial data $G(0) = G_0(\alpha)$ and $\rho(0) = \rho_0(\alpha)$. Here ' denotes the derivative along the characteristic path

$$f'(t) = \frac{d}{dt} f(t, x(t)) = f_t(t, x(t)) + u(t, x(t)) f_x(t, x(t)).$$

In the proofs of Theorems 2.2 and 2.5, we will justify that the initial data when $\rho(0) = 0$ can be handled separately. For now, we assume that $\rho(0) > 0$. We can further apply the transformation

$$w := \frac{G}{\rho}, \quad s := \frac{1}{\rho} \quad (3.3)$$

to (3.2) and obtain the dynamics

$$w' = k - kcs, \quad (3.4a)$$

$$s' = w - s(\psi * \rho). \quad (3.4b)$$

This ODE system is not closed along each characteristic path due to the nonlocal nature of the term $\psi * \rho$. We shall analyze this nonlocal system by establishing a type of comparison argument. To this end, we introduce a family of auxiliary systems

$$p' = k - kcq, \quad (3.5a)$$

$$q' = p - \beta q, \quad (3.5b)$$

with $p = p(t; \beta)$, $q = q(t; \beta)$, where β is a parameter. For each given β , (3.5) is a linear system that can be solved explicitly. We can rewrite (3.5) as

$$\begin{bmatrix} p - \frac{\beta}{c} \\ q - \frac{1}{c} \end{bmatrix}' = \begin{bmatrix} 0 & -kc \\ 1 & -\beta \end{bmatrix} \begin{bmatrix} p - \frac{\beta}{c} \\ q - \frac{1}{c} \end{bmatrix},$$

where the coefficient matrix has two eigenvalues

$$\frac{-\beta \pm \sqrt{\beta^2 - 4kc}}{2}.$$

Note that $\psi * \rho$ has apriori bounds (1.4), which we recall here: $\beta_{\min} \leq \psi * \rho \leq \beta_{\max}$, where we denote

$$\beta_{\max} = c\psi_{\max}, \quad \beta_{\min} = c\psi_{\min}. \quad (3.6)$$

It is natural to consider the following two particular auxiliary systems with $\beta = \beta_{\max}$ and β_{\min} :

$$\begin{bmatrix} \hat{p} - \frac{\beta_{\max}}{c} \\ \hat{q} - \frac{1}{c} \end{bmatrix}' = \begin{bmatrix} 0 & -kc \\ 1 & -\beta_{\max} \end{bmatrix} \begin{bmatrix} \hat{p} - \frac{\beta_{\max}}{c} \\ \hat{q} - \frac{1}{c} \end{bmatrix}, \quad (3.7a)$$

$$\begin{bmatrix} \tilde{p} - \frac{\beta_{\min}}{c} \\ \tilde{q} - \frac{1}{c} \end{bmatrix}' = \begin{bmatrix} 0 & -kc \\ 1 & -\beta_{\min} \end{bmatrix} \begin{bmatrix} \tilde{p} - \frac{\beta_{\min}}{c} \\ \tilde{q} - \frac{1}{c} \end{bmatrix}. \quad (3.7b)$$

We would like to remark that there is no direct comparison principle between the solutions to the nonlocal system $(w(t), s(t))$ and the local auxiliary system $(\hat{p}(t), \hat{q}(t))$ or $(\tilde{p}(t), \tilde{q}(t))$, particularly when β is small, in which case the eigenvalues are not real, and the solutions are oscillatory. Instead, we shall obtain a comparison in the phase plane, and obtain an *invariant region* that the trajectory (w, s) cannot exit.

3.1. Weak alignment

We begin with the case where all admissible values of $\beta \in [\beta_{\min}, \beta_{\max}]$ are such that

$$\beta^2 < 4kc,$$

and in such case $(\beta/c, 1/c)$ is an asymptotically stable spiral point. Physically, this places a restriction on the upper bound of $\psi * \rho$. Hence, we call this scenario the weak alignment case. We will construct an invariant region using specific trajectories of the above auxiliary systems, see Fig. 3. At this point, we establish some notation to be used in this section,

$$\tilde{\theta} := \frac{1}{2}\sqrt{4kc - \beta_{\min}^2}, \quad \hat{\theta} := \frac{1}{2}\sqrt{4kc - \beta_{\max}^2}.$$

We will now construct the invariant region (Σ_1^*) as in Fig. 3. We divide this construction into three steps, each pertaining to one of the segments of the boundary of Σ_1^* . We will start from the origin and move backwards in time.

Step 1: The first segment of the curve is the trajectory to (3.7a) starting at the origin, going into the second quadrant, and ending when it hits the line $q = 1/c$ while going backwards in time. Hence, if we solve for \hat{p}, \hat{q} with $\hat{p}(0) = 0, \hat{q}(0) = 0$, then the other end point of the curve is $(\hat{p}(t_1^{we}), \hat{q}(t_1^{we}))$, where t_1^{we} is the first negative time for which $\hat{q}(t_1^{we}) = 1/c$. Let $p_1^{we} := \hat{p}(t_1^{we})$. Since (3.7a) is a simple linear system, we can explicitly solve for its solution with initial data $(\hat{p}(0), \hat{q}(0)) = (0, 0)$,

$$\hat{p}(t) = \frac{\beta_{\max}}{c} + e^{-\frac{\beta_{\max}t}{2}} \left(-\frac{\beta_{\max}}{c} \cos \hat{\theta}t + \frac{k - \frac{\beta_{\max}^2}{2c}}{\hat{\theta}} \sin \hat{\theta}t \right),$$

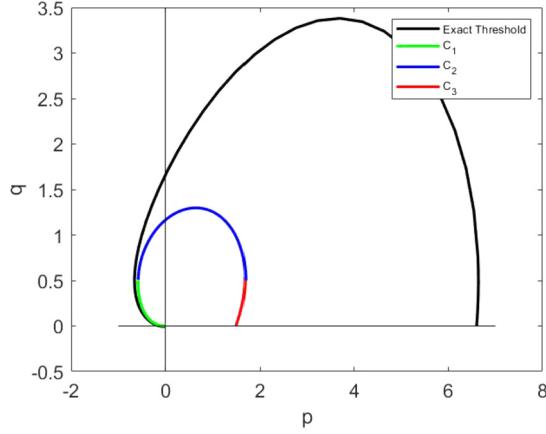


Fig. 3. Invariant region.

$$\hat{q}(t) = \frac{1}{c} - \frac{e^{-\frac{\beta_{\max} t}{2}}}{c} \left(\cos \hat{\theta} t + \frac{\beta_{\max}}{2\hat{\theta}} \sin \hat{\theta} t \right).$$

$$\hat{q}(t_1^{we}) = \frac{1}{c} \implies t_1^{we} = -\frac{1}{\hat{\theta}} \tan^{-1} \left(\frac{2\hat{\theta}}{\beta_{\max}} \right).$$

Hence,

$$p_1^{we} = \frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\beta_{\max}}{2\hat{\theta}} \tan^{-1} \left(\frac{2\hat{\theta}}{\beta_{\max}} \right)}$$

$$= \frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\tan^{-1}(\hat{z})}{\hat{z}}}, \quad (3.8)$$

with \hat{z} as defined at the end of Section 1.

Lemma 3.1. $p_1^{we} \in \sqrt{\frac{k}{c}} (-1, 2 - e)$.

Proof. We write the expression for p_1^{we} as,

$$f(\tau) = \sqrt{\frac{k}{c}} \left(\frac{1}{\tau} - e^{\frac{\tan^{-1} \sqrt{4\tau^2 - 1}}{\sqrt{4\tau^2 - 1}}} \right), \quad \tau = \frac{\sqrt{kc}}{\beta_{\max}} > \frac{1}{2}.$$

One can evaluate that f is a monotonically decreasing function with

$$\lim_{\tau \rightarrow \frac{1}{2}^+} f(\tau) = (2 - e) \sqrt{\frac{k}{c}} \quad \text{and} \quad \lim_{\tau \rightarrow \infty} f(\tau) = -\sqrt{\frac{k}{c}}.$$

Hence, the result holds. \square

Also note that $\hat{p}' > 0$ and $\hat{q}' < 0$ in this region. Hence, the first segment is given by,

$$C_1 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_1^{we}, 0]\}. \quad (3.9)$$

Step 2: The second segment is constructed using the trajectory of the system (3.7b). To have a closed region, the starting point of this segment should be the endpoint of the first segment. To this end, let \tilde{p}, \tilde{q} be solutions to IVP (3.7b) with $\tilde{p}(0) = p_1^{we}, \tilde{q}(0) = 1/c$. This segment starts at $(p_1^{we}, 1/c)$, traces the trajectory of (\tilde{p}, \tilde{q}) upwards and ends when it hits the $q = 1/c$ line again in the first quadrant. We denote the end point as $(p_2^{we}, 1/c)$. In particular, $p_2^{we} = \tilde{p}(t_2^{we})$ where t_2^{we} is the first negative time where $\tilde{q}(t_2^{we}) = 1/c$. We have

$$\begin{aligned} \tilde{p}(t) &= \frac{\beta_{\min}}{c} + e^{-\frac{\beta_{\min}t}{2}} \left[\left(p_1^{we} - \frac{\beta_{\min}}{c} \right) \cos \tilde{\theta}t + \left(\frac{p_1^{we}\beta_{\min}}{2\tilde{\theta}} - \frac{\beta_{\min}^2}{2c\tilde{\theta}} \right) \sin \tilde{\theta}t \right], \\ \tilde{q}(t) &= \frac{1}{c} + \frac{e^{-\frac{\beta_{\min}t}{2}}}{\tilde{\theta}} \left(p_1^{we} - \frac{\beta_{\min}}{c} \right) \sin \tilde{\theta}t. \\ \tilde{q}(t_2^{we}) &= \frac{1}{c} \implies t_2^{we} = -\frac{\pi}{\tilde{\theta}}. \end{aligned}$$

Consequently,

$$\begin{aligned} p_2^{we} &= \tilde{p}(t_2^{we}) = \frac{\beta_{\min}}{c} - e^{\frac{\beta_{\min}\pi}{2\tilde{\theta}}} \left(p_1^{we} - \frac{\beta_{\min}}{c} \right) \\ &= \frac{\beta_{\min}}{c} \left(1 + e^{\frac{\beta_{\min}\pi}{2\tilde{\theta}}} \right) - p_1^{we} e^{\frac{\beta_{\min}\pi}{2\tilde{\theta}}} \\ &= \frac{\beta_{\min}}{c} \left(1 + e^{\frac{\pi}{\tilde{z}}} \right) - p_1^{we} e^{\frac{\pi}{\tilde{z}}}. \end{aligned} \quad (3.10)$$

Here, we emphasize an important issue. We must have that $p_2^{we} > \beta_{\max}/c$, for otherwise we would not be able to obtain a closed invariant region. The following Lemma states a condition to ensure this.

Lemma 3.2. $p_2^{we} > \beta_{\max}/c$ if and only if

$$\sqrt{kc} e^{\frac{\tan^{-1} \hat{z}}{\hat{z}}} > (\beta_{\max} - \beta_{\min}) \left(1 + e^{-\frac{\pi}{\tilde{z}}} \right).$$

Proof. From (3.8) and (3.10), we have

$$\begin{aligned} p_2^{we} - \frac{\beta_{\max}}{c} &= \frac{\beta_{\min}}{c} \left(1 + e^{\frac{\pi}{\tilde{z}}} \right) - p_1^{we} e^{\frac{\pi}{\tilde{z}}} - \frac{\beta_{\max}}{c} \\ &= \frac{\beta_{\min}}{c} e^{\frac{\pi}{\tilde{z}}} - \left(\frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\tan^{-1}(\hat{z})}{\hat{z}}} \right) e^{\frac{\pi}{\tilde{z}}} - \frac{\psi_{\max}^* - \beta_{\min}}{c} \\ &= \sqrt{\frac{k}{c}} e^{\frac{\pi}{\tilde{z}} + \frac{\tan^{-1} \hat{z}}{\hat{z}}} - \left(\frac{\beta_{\max} - \beta_{\min}}{c} \right) \left(1 + e^{\frac{\pi}{\tilde{z}}} \right) > 0. \end{aligned} \quad (3.11)$$

We will use the inequality in the above form later in Lemma 3.4 to obtain a final condition. However, we can further rewrite this to obtain the inequality as in the statement of the Lemma. \square

Remark 3.3. The condition in Lemma 3.2 is sharp. A relaxed condition could be derived using Lemma 3.1,

$$\beta_{\max} - \beta_{\min} < e^{\frac{\pi}{\hat{z}}} \left(\beta_{\min} + (e-2)\sqrt{kc} \right).$$

However, we will make use of the sharp condition because it is evident from (3.11) that if $\psi \equiv \text{constant}$, then there is no need for such a condition.

The second segment of the boundary of invariant region is,

$$C_2 = \{(p, q) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_2^{we}, 0]\}. \quad (3.12)$$

Step 3: For the third segment, we again use (3.7a) but with different initial conditions than the ones for the first segment. The third segment should start from the ending point of the second segment, i.e., $\hat{p}(0) = p_2^{we}$ and $\hat{q}(0) = 1/c$. On solving, we obtain

$$\begin{aligned} \hat{p}(t) &= \frac{\beta_{\max}}{c} + e^{-\frac{\beta_{\max}t}{2}} \left[\left(p_2^{we} - \frac{\beta_{\max}}{c} \right) \cos \hat{\theta}t + \left(\frac{p_2^{we}\beta_{\max}}{2\hat{\theta}} - \frac{\beta_{\max}^2}{2c\hat{\theta}} \right) \sin \hat{\theta}t \right], \\ \hat{q}(t) &= \frac{1}{c} + \frac{e^{-\frac{\beta_{\max}t}{2}}}{\hat{\theta}} \left(p_2^{we} - \frac{\beta_{\max}}{c} \right) \sin \hat{\theta}t. \end{aligned}$$

Set

$$p_3^{we} := \hat{p}(t_3^{we}), \quad (3.13)$$

where t_3^{we} is the first negative time when $\hat{q}(t_3^{we}) = 0$. Hence, t_3^{we} is the largest negative root of the following,

$$e^{-\frac{\beta_{\max}t_3^{we}}{2}} \sin \hat{\theta}t_3^{we} = -\frac{\hat{\theta}}{cp_2^{we} - \beta_{\max}}.$$

To ensure the invariant region is closed, it should be that while traveling in the negative time direction, the trajectory hits the p -axis first before completing the outward spiral turn. The following Lemma ensures this.

Lemma 3.4. Let t_0 be the first negative time such that $\hat{q}'(t_0) = 0$. Then $\hat{q}(t_0) < 0$ if and only if

$$(\beta_{\max} - \beta_{\min}) \left(1 + e^{-\frac{\pi}{\hat{z}}} \right) e^{-\frac{\tan^{-1}(\hat{z})}{\hat{z}}} < \sqrt{kc} \left(1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}} \right). \quad (3.14)$$

Proof. Solving for $\hat{q}(t) = 0$, we obtain that

$$\hat{\theta}t_0 = -\pi + \tan^{-1}(\hat{z}).$$

Applying the condition $\hat{q}(t_0) < 0$, we obtain

$$\begin{aligned} \frac{1}{c} + \frac{e^{-\frac{\beta_{\max} t_0}{2}}}{\hat{\theta}} \left(p_2^{we} - \frac{\beta_{\max}}{c} \right) \sin \hat{\theta} t_0 &< 0 \\ e^{-\frac{\beta_{\max} t_0}{2}} \sin \hat{\theta} t_0 &< -\frac{\hat{\theta}}{c p_2^{we} - \beta_{\max}}. \end{aligned}$$

Plugging in the value of t_0 , we need that

$$e^{\frac{\pi}{\hat{z}} - \frac{\tan^{-1}(\hat{z})}{\hat{z}}} (c p_2^{we} - \beta_{\max}) - \sqrt{kc} > 0.$$

Note that (3.11) in Lemma 3.2 is indeed a necessary and sufficient condition for $p_2^{we} - \beta_{\max}/c > 0$ to hold. Hence, we can use (3.11) in the above expression to obtain a single final condition. To this end, we want

$$\begin{aligned} e^{\frac{\pi}{\hat{z}} - \frac{\tan^{-1}(\hat{z})}{\hat{z}}} (c p_2^{we} - \beta_{\max}) - \sqrt{kc} \\ = e^{\frac{\pi}{\hat{z}} - \frac{\tan^{-1}(\hat{z})}{\hat{z}}} \left(\sqrt{kc} e^{\frac{\pi}{\hat{z}} + \frac{\tan^{-1}(\hat{z})}{\hat{z}}} - (\beta_{\max} - \beta_{\min})(1 + e^{\frac{\pi}{\hat{z}}}) \right) - \sqrt{kc} \\ = \sqrt{kc} \left(e^{\frac{\pi}{\hat{z}} + \frac{\pi}{\hat{z}}} - 1 \right) - (\beta_{\max} - \beta_{\min})(1 + e^{\frac{\pi}{\hat{z}}}) e^{\frac{\pi}{\hat{z}} - \frac{\tan^{-1}(\hat{z})}{\hat{z}}} > 0. \end{aligned}$$

This finishes the proof to the Lemma. \square

Finally, we can define the last segment of the boundary of Σ_1^* ,

$$C_3 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_3^*, 0)\}. \quad (3.15)$$

We define the following set

$$\Sigma_1^* := \text{open set enclosed by } C_1, C_2, C_3 \text{ and } p\text{-axis}. \quad (3.16)$$

By our construction, we know that Σ_1^* is well-defined.

Next, we have the following Proposition.

Proposition 3.5. *Let $4kc > \beta_{\max}^2$. Let the initial conditions for (3.4) be such that $(w(0), s(0)) \in \Sigma_1^*$. Then $(w(t), s(t)) \in \Sigma_1^*$ for all $t > 0$.*

We will prove the Proposition by drawing comparison between the solution trajectory (w, s) and the boundary of Σ_1^* . Due to the presence of oscillations, a time based comparison between systems (3.4) and (3.7) cannot be derived. To circumvent this, we will draw comparisons in the (p, q) plane.

Proof. We will show that a solution trajectory to (3.4) with initial data in Σ_1^* can never touch its boundary as time increases. By getting rid of the time parameter t in the systems (3.7a) and (3.7b), we obtain the following two trajectory equations below. These will play a significant role in proving the invariance of Σ_1^* .

$$\frac{d\hat{q}}{dp} = \frac{p - \hat{q}\beta_{\max}}{k - kc\hat{q}}, \quad (3.17a)$$

$$\frac{d\tilde{q}}{dp} = \frac{p - \tilde{q}\beta_{\min}}{k - kc\tilde{q}}. \quad (3.17b)$$

We start by showing a contradiction if the trajectory touches C_1 . To this end, assume a point $(w_1, s_1) \in C_1$ where the trajectory meets C_1 . Therefore, $w_1 \leq 0$ and $s_1 < 1/c$. For a reminder, any portion of C_1 is $(p, \hat{q}(p))$ with appropriate initial conditions and values of p . We also get rid of the time parameter in (3.4) to write s as a function of another variable and satisfying,

$$\frac{ds}{dp} = \frac{p - s\psi * \rho}{k - kcs}. \quad (3.18)$$

We have $s_1 = \hat{q}(w_1) = s(w_1)$. Since $w'|_{(w_1, s_1)} > 0$, the trajectory $(p, s(p))$ was moving in the positive p direction before touching C_1 , see Fig. 4(a). Note that,

$$\begin{aligned} \frac{d(\hat{q} - s)}{dp} &= \frac{p - \hat{q}\beta_{\max}}{k - kc\hat{q}} - \frac{p - s(\psi * \rho)}{k - kcs} \\ &= \frac{-cp(s - \hat{q}) - (\beta_{\max}\hat{q} - (\psi * \rho)s) + cs\hat{q}(\beta_{\max} - \psi * \rho)}{k(1 - c\hat{q})(1 - cs)} \\ &= \frac{(cp - \psi * \rho)(\hat{q} - s) - \hat{q}(\beta_{\max} - \psi * \rho)(1 - cs)}{k(1 - c\hat{q})(1 - cs)} \\ &= \frac{(cp - \psi * \rho)}{k(1 - c\hat{q})(1 - cs)}(\hat{q} - s) - \frac{\hat{q}(\beta_{\max} - \psi * \rho)}{k(1 - c\hat{q})} \end{aligned}$$

In a neighborhood of $p = w_1$ (if $w_1 = 0$ consider left neighborhood),

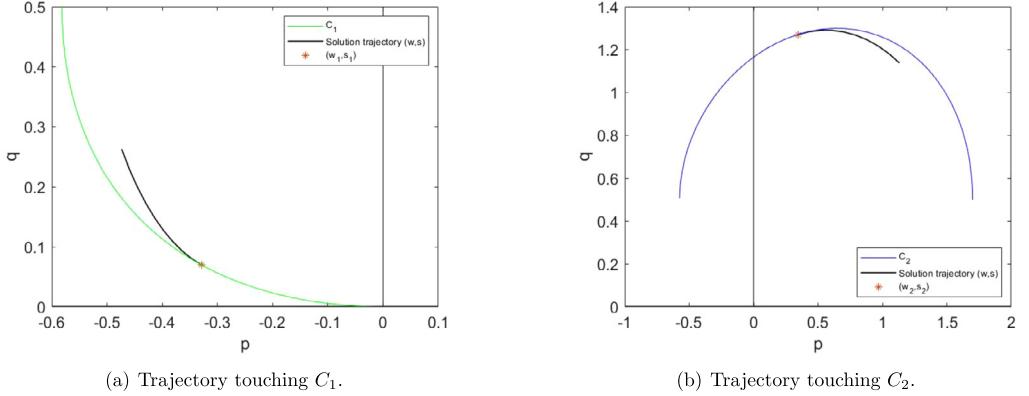
$$\frac{d(\hat{q} - s)}{dp} \leq \frac{(cp - \psi * \rho)}{k(1 - c\hat{q})(1 - cs)}(\hat{q} - s).$$

Upon integration in the interval $(w_1 - \epsilon, w_1)$, $\epsilon > 0$ being sufficiently small, we obtain

$$0 = \hat{q}(w_1) - s(w_1) \leq (\hat{q}(w_1 - \epsilon) - s(w_1 - \epsilon)) e^{\int_{w_1 - \epsilon}^{w_1} \frac{(cp - \psi * \rho)}{k(1 - c\hat{q}(p))(1 - cs(p))} dp} < 0.$$

This is a contradiction. Hence, a trajectory with initial point inside Σ_1^* can never touch C_1 . A very similar argument holds for C_3 .

Now we show for C_2 . For sake of contradiction, suppose there exists a point $(w_2, s_2) \in C_2$ where the trajectory, (w, s) , touches C_2 . For a reminder, any portion of C_2 is $(p, \tilde{q}(p))$ with appropriate initial conditions and range of p . Owing to our assumptions, we have $1/c < s_2 =$

Fig. 4. Trajectories touching boundary of Σ_1^* .

$s(w_2) = \tilde{q}(w_2)$. Since $w'|_{(w_2, s_2)} < 0$, the solution trajectory $(p, s(p))$ was traveling in the negative p direction when it touched C_2 , see Fig. 4(b). Similar to our previous calculations we obtain from (3.17b) and (3.18) that,

$$\frac{d(\tilde{q} - s)}{dp} = \frac{(cp - \psi * \rho)}{k(1 - cs)(1 - c\tilde{q})}(\tilde{q} - s) + \frac{\tilde{q}(\psi * \rho - \beta_{\min})}{k(1 - c\tilde{q})}.$$

In a neighborhood of $p = w_2$,

$$\frac{d(\tilde{q} - s)}{dp} \leq \frac{(cp - \psi * \rho)}{k(1 - cs)(1 - c\tilde{q})}(\tilde{q} - s).$$

Upon integration in the interval $(w_2, w_2 + \epsilon)$, for $\epsilon > 0$ sufficiently small, we obtain

$$0 < \tilde{q}(w_2 + \epsilon) - s(w_2 + \epsilon) \leq (\tilde{q}(w_2) - s(w_2)) e^{\int_{w_2}^{w_2 + \epsilon} \frac{cp - \psi * \rho}{k(1 - cs(p))(1 - c\tilde{q}(p))} dp} = 0.$$

Hence, the solution trajectory cannot cross C_2 .

Moreover, a trajectory (w, s) starting from any point $(p, 0)$ with $p > 0$ will go up into the region because at any such point,

$$s'|_{(p,0)} = w - (\psi * \rho)s|_{(p,0)} = p > 0.$$

This completes the proof to the proposition. \square

Now we will transform Σ_1^* to obtain an invariant region for (3.2). To this end, define a map by $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$F(p, q) = (p/q, 1/q). \quad (3.19)$$

F is invertible for $q > 0$. We define

$$\Sigma_1 := F(\Sigma_1^*), \quad (3.20)$$

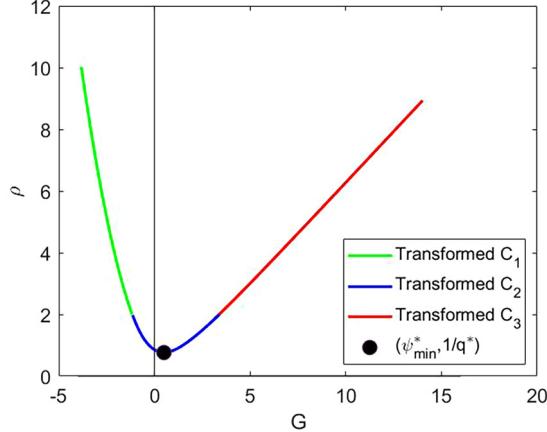


Fig. 5. C_1, C_2, C_3 as transformed to original coordinates (G, ρ) .

which is an invariant region for (G, ρ) . See Fig. 5 for the shape of the subcritical region, Σ_1 , in (G, ρ) coordinates.

Remark 3.6. In Fig. 5, the ρ coordinate of the tip of Σ_1 is $1/q^*$, where q^* is the q coordinate of the highest point of Σ_1^* , see Fig. 3. The expression of q^* can be explicitly written as,

$$q^* = \frac{1}{c} + \frac{e^{\frac{\pi}{\tilde{z}} - \frac{\tan^{-1}(\tilde{z})}{\tilde{z}}}}{\sqrt{kc}} \left(\frac{\beta_{\min}}{c} - p_1^{we} \right). \quad (3.21)$$

Since $p_1^{we} < 0$ from Lemma 3.1, we have that $q^* > 1/c$. Also, from Fig. 5, we see that there is no point (G, ρ) in Σ_1 such that $\rho < 1/q^*$.

As a direct result of Proposition 3.5 and transformation (3.3), we have the following corollary.

Corollary 3.7. *Let $4kc > \beta_{\max}^2$ and (3.14) holds. Let initial conditions for (3.2) be such that $(G(0), \rho(0)) \in \Sigma_1$. Then $(G(t), \rho(t)) \in \Sigma_1$ for all $t > 0$. In particular, G, ρ are bounded for any time.*

3.2. Strong alignment

Now, we handle the case where all admissible values of $\beta \in [\beta_{\min}, \beta_{\max}]$ in (3.5) are such that

$$\beta^2 \geq 4kc.$$

In such a case $(\beta/c, 1/c)$ is an asymptotically stable node and the solutions to (3.7a) and (3.7b) will not have any sinusoidal components. We call this scenario the strong alignment case. As before, we will construct an invariant region using specific trajectories. Unlike the invariant region constructed in Section 3.1, here we will have an unbounded subcritical region, Σ_2^* , see Fig. 6.

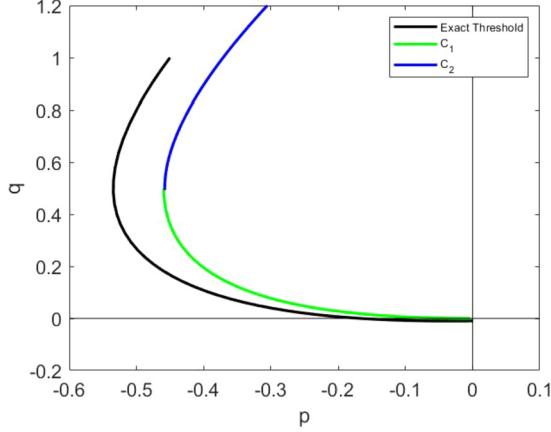


Fig. 6. Invariant region.

We first define some notation to be used in construction of Σ_2^* . Set

$$\begin{aligned}\hat{\gamma}_+ &:= \frac{\beta_{\max} + \sqrt{\beta_{\max}^2 - 4kc}}{2}, & \hat{\gamma}_- &:= \frac{\beta_{\max} - \sqrt{\beta_{\max}^2 - 4kc}}{2}, \\ \tilde{\gamma}_+ &:= \frac{\beta_{\min} + \sqrt{\beta_{\min}^2 - 4kc}}{2}, & \tilde{\gamma}_- &:= \frac{\beta_{\min} - \sqrt{\beta_{\min}^2 - 4kc}}{2}.\end{aligned}$$

Remark 3.8. In this Section as well as Section 3.3, we should point out that if $\beta_{\max}^2 = 4kc$, then the expressions of \hat{p}, \hat{q} have different form than the ones when $\beta_{\max}^2 > 4kc$, which is assumed for calculations below. However, the calculated expressions for $p_1^{se}, p_2^{se}, p_3^{me}$ always hold, although in the limit sense when $\beta_{\max}^2 = 4kc$. A more detailed note about this is mentioned right after the proof of Lemma 3.9.

Step 1: The first segment of the curve, C_1 , is the trajectory to (3.7a) with the starting point at origin and the ending point lying on the line $q = 1/c$ in the second quadrant. Set $p_1^{se} := \hat{p}(t_1^{se})$ so that $(p_1^{se}, 1/c)$ is the end point of C_1 lying in the second quadrant. Here, t_1^{se} is the negative time when $\hat{q}(t_1^{se}) = 1/c$ with $\hat{p}(0) = \hat{q}(0) = 0$. On solving, we obtain,

$$\begin{aligned}\hat{p}(t) &= \frac{\beta_{\max}}{c} - \frac{k}{\sqrt{\beta_{\max}^2 - 4kc}} \left(\frac{\hat{\gamma}_+}{\hat{\gamma}_-} e^{-\hat{\gamma}_- t} - \frac{\hat{\gamma}_-}{\hat{\gamma}_+} e^{-\hat{\gamma}_+ t} \right), \\ \hat{q}(t) &= \frac{1}{c} - \frac{1}{c\sqrt{\beta_{\max}^2 - 4kc}} \left(\hat{\gamma}_+ e^{-\hat{\gamma}_- t} - \hat{\gamma}_- e^{-\hat{\gamma}_+ t} \right).\end{aligned}$$

When $\hat{q}(t_1^{se}) = 1/c$,

$$e^{\left(\sqrt{\beta_{\max}^2 - 4kc}\right)t_1^{se}} = \frac{\hat{\gamma}_-}{\hat{\gamma}_+}.$$

Consequently, using the fact that $\hat{\gamma}_+ \hat{\gamma}_- = kc$, we have

$$\begin{aligned}
p_1^{se} &= \frac{\beta_{\max}}{c} - \frac{k}{\sqrt{\beta_{\max}^2 - 4kc}} e^{-\frac{\beta_{\max} t_1^{se}}{2}} \left(\frac{\hat{\gamma}_+}{\hat{\gamma}_-} e^{\frac{t_1^{se}}{2} \sqrt{\beta_{\max}^2 - 4kc}} - \frac{\hat{\gamma}_-}{\hat{\gamma}_+} e^{-\frac{t_1^{se}}{2} \sqrt{\beta_{\max}^2 - 4kc}} \right) \\
&= \frac{\beta_{\max}}{c} + \frac{k}{\sqrt{\beta_{\max}^2 - 4kc}} e^{-\frac{\beta_{\max} t_1^{se}}{2}} \left(-\sqrt{\frac{\hat{\gamma}_+}{\hat{\gamma}_-}} + \sqrt{\frac{\hat{\gamma}_-}{\hat{\gamma}_+}} \right) \\
&= \frac{\beta_{\max}}{c} - ke^{-\frac{\beta_{\max} t_1^{se}}{2}} \left(\frac{1}{\sqrt{\hat{\gamma}_- \hat{\gamma}_+}} \right) \\
&= \frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} \left(\frac{\hat{\gamma}_+}{\hat{\gamma}_-} \right)^{\frac{\beta_{\max}}{2\sqrt{\beta_{\max}^2 - 4kc}}}. \tag{3.22}
\end{aligned}$$

Lemma 3.9. $p_1^{se} \in \sqrt{\frac{k}{c}} [-(e-2), 0)$.

Proof. (3.22) can be rewritten as a function of essentially one variable,

$$g(\tau) = \sqrt{\frac{k}{c}} \left(\frac{1}{\tau} - \left(\frac{1 + \sqrt{1 - 4\tau^2}}{1 - \sqrt{1 - 4\tau^2}} \right)^{\frac{1}{2\sqrt{1-4\tau^2}}} \right), \quad \tau = \frac{\sqrt{kc}}{\beta_{\max}}, \quad \tau \in (0, 1/2].$$

One can check that the above function is decreasing with

$$\lim_{\tau \rightarrow (1/2)^-} g(\tau) = (2-e) \sqrt{\frac{k}{c}} \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} g(\tau) = 0.$$

Hence, the result holds. \square

Since the ODE system (3.7a) is well-posed, $g(1/2) = f(1/2) = -(e-2)\sqrt{k/c}$, where f is as defined in proof of Lemma 3.1. Moreover, if $\beta_{\max} = 2\sqrt{kc}$ (or equivalently $\tau = 1/2$), then the point $p_1^{se} = p_1^{we} = -(e-2)\sqrt{k/c}$. The relation between f and g is much more. In fact, they are equal if we extend each of their domains to \mathbb{R}^+ , see Remark 3.10.

Remark 3.10. We recall f here,

$$f(\tau) = \sqrt{\frac{k}{c}} \left(\frac{1}{\tau} - e^{\frac{\tan^{-1} \sqrt{4\tau^2 - 1}}{\sqrt{4\tau^2 - 1}}} \right), \quad \tau \geq \frac{1}{2}.$$

As a function into \mathbb{R} , f is defined only for $\tau \geq 1/2$. We aim to extend it to accommodate $\tau \in \mathbb{R}^+$. It turns out that

$$e^{\frac{\tan^{-1} \sqrt{4\tau^2 - 1}}{\sqrt{4\tau^2 - 1}}} = \left(\frac{1 + \sqrt{1 - 4\tau^2}}{1 - \sqrt{1 - 4\tau^2}} \right)^{\frac{1}{2\sqrt{1-4\tau^2}}}, \quad \tau \in (0, \infty).$$

To see this, let $z := \sqrt{4\tau^2 - 1}$, $y := \sqrt{1 - 4\tau^2}$. Consequently, $e^{\frac{\tan^{-1} z}{z}} = e^{\frac{\tan^{-1} iy}{iy}} =: h(y)$. We have,

$$\begin{aligned} \tan(\ln(h^{iy})) &= iy \\ \frac{e^{i \ln(h^{iy})} - e^{-i \ln(h^{iy})}}{i(e^{i \ln(h^{iy})} + e^{-i \ln(h^{iy})})} &= iy \\ \frac{h^{-y} - h^y}{h^{-y} + h^y} &= -y \\ h^{2y} &= \frac{1+y}{1-y}, \end{aligned}$$

and finally,

$$h(y) = \left(\frac{1+y}{1-y} \right)^{\frac{1}{2y}}.$$

Owing to Remark 3.10, the formula for p_1^{se} is the same as p_1^{we} , which is,

$$p_1^{se} = \frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\tan^{-1}(\hat{z})}{\hat{z}}},$$

where \hat{z} is purely imaginary and output of \tan^{-1} is the principal value.

We now define the first segment of boundary of Σ_2^* ,

$$C_1 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_1^{se}, 0]\}. \quad (3.23)$$

Step 2: Now, we move on to the second segment. For this part, we need the solutions to (3.7b) with initial condition $\tilde{p}(0) = p_1^{se}$ and $\tilde{q}(0) = 1/c$. Hence,

$$\begin{aligned} \tilde{p}(t) &= \frac{\beta_{\min}}{c} + \frac{(\beta_{\min} - cp_1^{se})}{c\sqrt{\beta_{\min}^2 - 4kc}} \left(\tilde{\gamma}_- e^{-\tilde{\gamma}_+ t} - \tilde{\gamma}_+ e^{-\tilde{\gamma}_- t} \right), \\ \tilde{q}(t) &= \frac{1}{c} + \frac{(\beta_{\min} - cp_1^{se})}{\sqrt{\beta_{\min}^2 - 4kc}} \left(e^{-\tilde{\gamma}_+ t} - e^{-\tilde{\gamma}_- t} \right). \end{aligned}$$

Note that \tilde{p}, \tilde{q} are strictly decreasing for $t < 0$ and $\lim_{t \rightarrow -\infty} \tilde{p}(t) = \lim_{t \rightarrow -\infty} \tilde{q}(t) = \infty$. We now define C_2 .

$$C_2 = \{(\tilde{p}, \tilde{q}) : (\tilde{p}(t), \tilde{q}(t)), t \in (-\infty, 0]\}. \quad (3.24)$$

This completes our construction and we are ready to define Σ_2^* .

$$\Sigma_2^* = \text{unbounded open set surrounded by } C_1, C_2, \{(p, 0) : p > 0\} \text{ on 3 sides.} \quad (3.25)$$

Our construction ensures Σ_2^* is well-defined. The following proposition states the invariance of Σ_2^* .

Proposition 3.11. *Let $4kc \leq \beta_{\min}^2$. Consider the ODE system (3.4). If $(w(0), s(0)) \in \Sigma_2^*$ then $(w(t), s(t)) \in \Sigma_2^*$ for all $t > 0$. In particular, w, s remain bounded and $s(t) > 0$ for all time.*

Proof. The proof for the part that (w, s) never crosses C_1 or C_2 is very similar to that in the proof of Proposition 3.5. So, we will omit it here. We prove that $s > 0$ and w, s remain bounded.

The only points where the trajectory (w, s) could cross the p -axis are of the form $(p, 0)$ where $p > 0$. However, at any such point, $s' > 0$ and therefore, the trajectory moves upwards. Consequently, $s(t) > 0$ for all $t > 0$. As a result,

$$w' = k - kcs < k.$$

Therefore, w is bounded from above. Moreover,

$$s' = w - s\psi * \rho \leq w,$$

and hence, s is bounded from above. \square

Similar to what we did in Section 3.1, we will now transform Σ_2^* to obtain an invariant region for (3.2). However, due to the fact that Σ_2^* is unbounded, through (3.3) we have that there are points in $F(\Sigma_2^*)$ with positive but arbitrarily small values of ρ . This indicates that the subcritical region might contain points where $\rho = 0$ which we miss in the above analysis due to working with the transformed variables, (3.3). Indeed if $\rho = 0$ in (3.2a), then

$$\begin{aligned} G' &= -G(G - \psi * \rho) - kc = -(G^2 - G\psi * \rho + kc) \\ &= -\left(G - \frac{\psi * \rho - \sqrt{(\psi * \rho)^2 - 4kc}}{2}\right)\left(G - \frac{\psi * \rho + \sqrt{(\psi * \rho)^2 - 4kc}}{2}\right). \end{aligned} \quad (3.26)$$

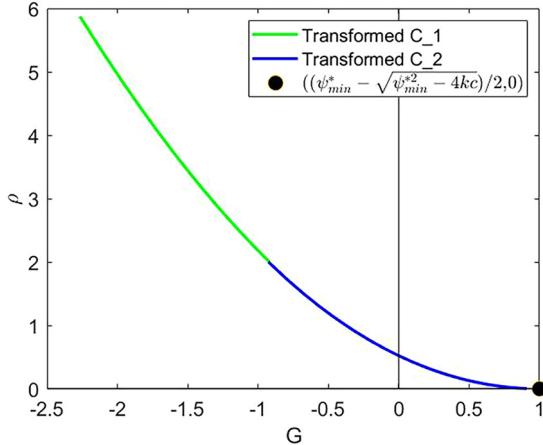
Noting that

$$\begin{aligned} \max\left(\psi * \rho - \sqrt{(\psi * \rho)^2 - 4kc}\right) &= \beta_{\min} - \sqrt{\beta_{\min}^2 - 4kc} \\ &< \beta_{\min} + \sqrt{\beta_{\min}^2 - 4kc} = \min\left(\psi * \rho + \sqrt{(\psi * \rho)^2 - 4kc}\right), \end{aligned}$$

therefore, if

$$G(0) > \frac{\beta_{\min} - \sqrt{\beta_{\min}^2 - 4kc}}{2}, \quad (3.27)$$

then $G(t)$ is bounded for all times. So, due to the balancing effect of the strong alignment, we have subcritical data for $\rho = 0$ as well, which was not the case for Σ_1 in Section 3.1. Owing to the above analysis and using F as in (3.19), we define

Fig. 7. C_1, C_2 as transformed to original coordinates (G, ρ) .

$$\Sigma_2 := F(\Sigma_2^*) \cup \left\{ (G, 0) : G > \frac{\beta_{\min} - \sqrt{\beta_{\min}^2 - 4kc}}{2} \right\}, \quad (3.28)$$

which is an invariant region for (G, ρ) . See Fig. 7 for the shape of the subcritical region in (G, ρ) coordinates.

Remark 3.12. We define the invariant region Σ_2 using the map (3.19). However, since C_1, C_2 are merely solution trajectories to a linear system, we can indeed denote Σ_2 through a function representing these solutions. In particular, there exists a Lipschitz continuous function ξ_a such that

$$\Sigma_2 = \{(G, \rho) : G > \xi_a(\rho), \rho \in [0, \infty)\}.$$

Proposition 3.13. Let $4kc < \beta_{\min}^2$. Let initial conditions for (3.2) be such that $(G(0), \rho(0)) \in \Sigma_2$. Then $(G(t), \rho(t)) \in \Sigma_2$ for all $t > 0$. In particular, $G(t), \rho(t)$ are bounded for any time.

Proof. Note that if $\rho(0) = 0$ in (3.2), then $\rho \equiv 0$. Also, if $(G(0), \rho(0)) \in F(\Sigma_2^*)$, then as a direct result of Proposition 3.11 and transformations (3.3), we conclude that $(G(t), \rho(t)) \in F(\Sigma_2^*)$ for all $t > 0$. Consequently, $\rho(0) > 0 \implies \rho(t) > 0$ for further times. In particular, this justifies that we can handle the $\rho(0) = 0 \equiv \rho$ case separately. From (3.27) above, we conclude the result for this case. This finishes the proof to the Proposition. \square

3.3. Medium alignment

This is the case where the range of $\beta \in [\beta_{\min}, \beta_{\max}]$ in (3.5) is such that $\beta_{\min}^2 < 4kc \leq \beta_{\max}^2$. We call this scenario the medium alignment case. Here, we will use analysis of both Sections 3.1 and 3.2. The invariant region here is closed as in Section 3.1 where $4kc > \beta_{\max}^2$. The procedure to calculate $p_1^{me}, p_2^{me}, p_3^{me}$ is very similar to what it is in Sections 3.1 and 3.2. So, we omit the calculations. We get

$$p_1^{me} = p_1^{se} = \frac{\beta_{\max}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\tan^{-1} \hat{z}}{\hat{z}}},$$

$$C_1 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_1^{se}, 0)\},$$

with \hat{p}, \hat{q} and t_1^{se} as in Step 1 of Section 3.2, and

$$p_2^{me} = \frac{\beta_{\min}}{c} \left(1 + e^{\frac{\pi}{\hat{z}}}\right) - p_1^{me} e^{\frac{\pi}{\hat{z}}}.$$

Similar to the condition (3.11) in Section 3.1, to have a closed invariant region, we need

$$p_2^{me} > \frac{\beta_{\max}}{c}.$$

We have the following Lemma.

Lemma 3.14. $p_2^{we} > \beta_{\max}/c$ if and only if

$$\sqrt{kce}^{\frac{\tan^{-1} \hat{z}}{\hat{z}}} > (\beta_{\max} - \beta_{\min}) \left(1 + e^{-\frac{\pi}{\hat{z}}}\right). \quad (3.29)$$

The proof is very similar to that of Lemma 3.2. We then have,

$$C_2 = \{(\tilde{p}, \tilde{q}) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_2^{we}, 0)\},$$

with \tilde{p}, \tilde{q} and t_2^{we} as in Step 2 of Section 3.1.

We complete our construction by finding the point p_3^{me} and the third segment of the boundary of the invariant region. The desired curve is the portion of the solution to (3.7a) with $\hat{p}(0) = p_2^{me}$ and $\hat{q}(0) = 1/c$. Using this, we obtain

$$\hat{p}(t) = \frac{\beta_{\max}}{c} + \frac{\left(p_2^{me} - \frac{\beta_{\max}}{c}\right)}{\sqrt{\beta_{\max}^2 - 4kc}} \left(\hat{\gamma}_+ e^{-\hat{\gamma}_- t} - \hat{\gamma}_- e^{-\hat{\gamma}_+ t}\right),$$

$$\hat{q}(t) = \frac{1}{c} + \frac{\left(p_2^{me} - \frac{\beta_{\max}}{c}\right)}{\sqrt{\beta_{\max}^2 - 4kc}} \left(e^{-\hat{\gamma}_- t} - e^{-\hat{\gamma}_+ t}\right).$$

\hat{q} is unbounded and strictly increasing for $t < 0$, hence, there exists a unique $t_3^{me} < 0$ such that $\hat{q}(t_3^{me}) = 0$. t_3^{me} is the unique solution of

$$e^{-\hat{\gamma}_+ t} - e^{-\hat{\gamma}_- t} = \frac{\hat{\gamma}}{cp_2^{me} - \beta_{\max}}.$$

And $p_3^{me} := \hat{p}(t_3^{me})$. Also,

$$C_3 = \{(\hat{p}, \hat{q}) : (\hat{p}(t), \hat{q}(t)), t \in [t_3^{me}, 0)\}.$$

And

$$\Sigma_3^* := \text{open set enclosed by } C_1, C_2, C_3 \text{ and } p\text{-axis.} \quad (3.30)$$

Finally, we obtain the invariant region for the (G, ρ) plane. Set

$$\Sigma_3 = F(\Sigma_3^*). \quad (3.31)$$

We can now have the following Proposition.

Proposition 3.15. *Let $\beta_{\min}^2 < 4kc \leq \beta_{\max}$ and (3.29) holds. Let initial conditions for (3.2) be such that $(G(0), \rho(0)) \in \Sigma_3$. Then $(G(t), \rho(t)) \in \Sigma_3$ for all $t > 0$.*

The proof is very similar to that of Proposition 3.7.

3.4. Global smooth solutions

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Assume the hypothesis of Assertion (1). $\psi_{\max} < \lambda$ implies $4kc > \beta_{\max}^2$ which means this case lies in the purview of Section 3.1. Also, the admissible condition (2.4) can be rewritten as (3.14). As a result, by Lemma 3.4, the invariant region Σ_1 is well defined.

Now suppose $(G_0(x), \rho_0(x)) \in \Sigma_1$ for all $x \in \mathbb{T}$. Then along any characteristic path (3.1), the initial data to (3.2), $(G(0), \rho(0)) \in \Sigma_1$. By Corollary 3.7, $(G(t), \rho(t)) \in \Sigma_1$ for all $t > 0$ and $G(t), \rho(t)$ are bounded along all characteristic paths. Finally, we can apply Theorem 2.1 and conclude that (ρ, u) are global-in-time smooth solutions to (1.1).

The proof to Assertions (2) and (3) in the Theorem is very similar only that in place of Corollary 3.7 used above, we use Propositions 3.13 and 3.15 respectively. \square

3.5. Finite time breakdown

This section is devoted to the proof of Theorem 2.5. The procedure of construction is very similar to that in Sections 3.1, 3.2 and 3.3. The only difference is that we use the system (3.7b) wherever we used (3.7a) and vice-versa. As a result, β_{\max} and β_{\min} interchange places in the relevant expressions. We only state the crucial steps and Propositions in obtaining the supercritical region.

Weak alignment ($4kc > \beta_{\max}^2$)

We have

$$B_1 = \{(p, q) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_1^{we}, 0]\},$$

where \tilde{p}, \tilde{q} are solutions to (3.7b) with initial conditions $\tilde{p}(0) = \tilde{q}(0) = 0$, and $t_1^{we} < 0$ is the first negative time when $\tilde{q}(t_1^{we}) = 1/c$. Also,

$$p_1^{we} := \tilde{p}(t_1^{we}) = \frac{\beta_{\min}}{c} - \sqrt{\frac{k}{c}} e^{\frac{\tan^{-1} \tilde{z}}{z}}.$$

We have the same bounds of p_1^{we} as in Lemma 3.1, $p_1^{we} \in \sqrt{k/c}(-1, -(2-e))$. Next, we have

$$B_2 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_2^{we}, 0]\},$$

where \hat{p}, \hat{q} are solutions to (3.7a) with initial conditions $\hat{p}(0) = p_1^{we}, \hat{q}(0) = 1/c$, and $t_2^{we} < 0$ is the first negative time when $\hat{q}(t_2^{we}) = 1/c$. Also,

$$p_2^{we} := \hat{p}(t_2^{we}) = \frac{\beta_{\max}}{c} \left(1 + e^{\frac{\pi}{z}}\right) - p_1^{we} e^{\frac{\pi}{z}}.$$

Since $p_1^{we} < 0$, we have that $p_2^{we} > \beta_{\min}/c$ and we do not need any extra condition (like (3.11)) to close the invariant region. Lastly,

$$B_3 = \{(\tilde{p}, \tilde{q}) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_3^{we}, 0]\},$$

where \tilde{p}, \tilde{q} are solutions to (3.7b) with initial conditions $\tilde{p}(0) = p_2^{we}, \tilde{q}(0) = 1/c$, and $t_3^{we} < 0$ is the first negative time when $\tilde{q}(t_3^{we}) = 0$. Here again, we do not need any extra condition (like (3.14)) for invariant region to be well-defined. To see this, just interchange β_{\max} and β_{\min} to see that the right hand side of (3.14) becomes negative. Therefore, the condition holds trivially. Finally, we define

$$\Delta_1^* = \text{unbounded open set outside } B_1 \cup B_2 \cup B_3 \text{ with } q > 0.$$

We then have the following Proposition.

Proposition 3.16. *Let $4kc > \beta_{\max}^2$. Let the initial conditions for (3.4) be such that $(w(0), s(0)) \in \Delta_1^*$. Then there exists $t_c > 0$ such that $s(t_c) = 0$. Also, $w(t_c) < 0$.*

Proof. The proof that the trajectory (w, s) does not touch Δ_1^* is very similar to that of Proposition 3.5. And from the signs of w', s' , it can be concluded that the trajectory hits $q = 0$ line for some time, $t_c > 0$, in the second quadrant. Hence, $w(t_c) < 0$. \square

Due to Δ_1^* being unbounded, we have supercritical region for points where $\rho = 0$ as well. Here, we have all such points in the supercritical region.

$$\Delta_1 = F(\Delta_1^*) \cup \{(G, 0) : G \in \mathbb{R}\}. \quad (3.32)$$

To see the inclusion of the points $(G, 0)$, we prove a Lemma.

Lemma 3.17. *Let a function h satisfy*

$$h' = -(h^2 - a(t)h + \omega),$$

where a is a bounded function and ω is a constant with $4\omega - \sup a^2 > 0$. Then for any initial data $h(0)$, there exists $t_c > 0$ such that $\lim_{t \rightarrow t_c^-} h(t) = -\infty$.

Proof. First, observe that

$$\begin{aligned} h' &= -h^2 + ah - \omega \\ &= -\left(h - \frac{a}{2}\right)^2 - \frac{4\omega - a^2}{4} \\ &\leq -\frac{4\omega - \sup a^2}{4} < 0. \end{aligned}$$

Therefore, h is strictly decreasing and can achieve any negative number. In particular, for some $t_0 \geq 0$, $h(t_0) < \min\{0, \inf a\}$. Consequently for $t > t_0$,

$$h' < -h(h - \inf a).$$

Since $h(t_0) < \min\{0, \inf a\}$, it admits a Riccati type blowup. Indeed on comparing above differential inequality with an equality, we obtain $\lim_{t \rightarrow t_c^-} h(t) = -\infty$ for some $t_c < t_0 + (-h(t_0))^{-1}$. \square

Owing to this Lemma, Proposition 3.16 and transformation (3.3), we have the following Corollary.

Corollary 3.18. *Let $4kc > \beta_{\max}^2$. Let initial conditions for (3.2) be such that $(G(0), \rho(0)) \in \Delta_1$. Then there exists $t_c > 0$ such that*

$$\lim_{t \rightarrow t_c^-} G(t, x_c) = -\infty, \quad \lim_{t \rightarrow t_c^-} \rho(t, x_c) = \infty \text{ or } 0,$$

for some $x_c \in \mathbb{T}$.

Proof. Note that if $\rho(0) = 0$ in (3.2), then $\rho \equiv 0$ and from (3.2a),

$$G' = -(G^2 - G\psi * \rho + kc).$$

We can have $h = G$, $a(t) = \psi * \rho$, $\omega = kc$ in Lemma 3.17 and the hypothesis is satisfied. Hence, for some $t_c > 0$, $\lim_{t \rightarrow t_c^-} G(t) = -\infty$ irrespective of $G(0)$. Also, if $\rho(0) > 0$ with $(G(0), \rho(0)) \in F(\Delta_1^*)$, then from Proposition 3.16 and transformations (3.3), we have the existence of $t_c > 0$ such that $\lim_{t \rightarrow t_c^-} \rho(t) = \infty$ and $\lim_{t \rightarrow t_c^-} G(t) = -\infty$. This finishes the proof to the Corollary. \square

3.6. Strong and medium alignment ($4kc \leq \beta_{\max}^2$)

These two cases are similar, so we state the construction together. We have

$$B_1 = \{(p, q) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_1^{se}, 0)\},$$

where \tilde{p}, \tilde{q} are solutions to (3.7b) with initial conditions $\tilde{p}(0) = \tilde{q}(0) = 0$, and $t_1^{se} < 0$ is the first negative time (for medium alignment) and the unique time (for strong alignment), when $\tilde{q}(t_1^{se}) = 1/c$. Next,

$$B_2 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in (-\infty, 0]\},$$

where \hat{p}, \hat{q} are solutions to (3.7a) with initial conditions $\hat{p}(0) = p_1^{we}, \hat{q}(0) = 1/c$. We can now define Δ_2^* .

$\Delta_2^* = \text{unbounded open set surrounded by } B_1, B_2, \{(p, 0) : p < 0\} \text{ on 3 sides.}$

Here, we have

$$\Delta_2 = F(\Delta_2^*) \cup \left\{ (G, 0) : G < \frac{\beta_{\max} - \sqrt{\beta_{\max}^2 - 4kc}}{2} \right\}. \quad (3.33)$$

Indeed when $\rho = 0$, from (3.26) we have that if

$$G(0) < \min \frac{\psi * \rho - \sqrt{(\psi * \rho)^2 - 4kc}}{2} = \frac{\beta_{\max} - \sqrt{\beta_{\max}^2 - 4kc}}{2},$$

then $G(t) \rightarrow -\infty$ in finite time.

We have the following Proposition.

Proposition 3.19. *Let $4kc \leq \beta_{\max}^2$. Let initial conditions for (3.2) be such that $(G(0), \rho(0)) \in \Delta_2$. Then there exists $t_c > 0$ such that*

$$\lim_{t \rightarrow t_c^-} G(t, x_c) = -\infty, \quad \lim_{t \rightarrow t_c^-} \rho(t, x_c) = \infty \text{ or } 0,$$

for some $x_c \in \mathbb{T}$.

We now give the proof to Theorem 2.5.

Proof of Theorem 2.5. Assume the hypothesis of Assertion (1). $\psi_{\max} < \lambda$ implies $4kc > \beta_{\max}^2$ which means this case lies in the purview of Section 3.1. Now suppose $(G_0(x_0), \rho_0(x_0)) \in \Delta_1$ for some $x_0 \in \mathbb{T}$. Then consider the dynamics (3.2) along the characteristic path (3.1) with $x(0) = x_0$, and apply Corollary 3.18. We have

$$\lim_{t \rightarrow t_c^-} G(t, x_c) = -\infty, \quad \lim_{t \rightarrow t_c^-} \rho(t, x_c) = \infty \text{ or } 0.$$

This proves Assertion (1). The proof to Assertion (2) in the Theorem is very similar only that in place of Corollary 3.18 used above, we use Proposition 3.19. \square

4. The EPA system with weakly singular alignment influence

In this section, we tackle the case when $\psi \in L_+^1(\mathbb{T})$. In particular, ψ need not be bounded as was assumed in Section 3. This type of alignment forces is known as *weakly singular*.

4.1. Improved bounds on $\psi * \rho$

The main difficulty of applying our theory in Section 3 to the EPA system with weakly singular alignment influence is that the bounds on $\psi * \rho$ in (1.4) no longer hold. A natural replacement of the bounds is (1.5), which we recall here:

$$\|\psi\|_{L^1} \rho_{\min} \leq \psi * \rho \leq \|\psi\|_{L^1} \rho_{\max}. \quad (4.1)$$

A major issue arises that the bounds depend on the unknown ρ . If we were to pick ρ_{\max} and ρ_{\min} and use the bounds (4.1) in place of $\beta_{\max}, \beta_{\min}$ as in Section 3, then the invariant region Σ need to satisfy

$$\inf\{\rho : (G, \rho) \in \Sigma\} \geq \rho_{\min}, \quad \sup\{\rho : (G, \rho) \in \Sigma\} \leq \rho_{\max}, \quad (4.2)$$

in order to keep (4.1) valid. However, after detailed analysis, it turns out that there are no values of ρ_{\max} and ρ_{\min} with which the constructed invariant region Σ satisfies (4.2).

To overcome this difficulty, we make improvements to the bounds (4.1) leveraging the additional property on ρ

$$\int_{\mathbb{T}} \rho(t, x) dx = c, \quad \forall t \geq 0.$$

In particular, we have the following key Lemma.

Lemma 4.1. *Let ρ be any nonnegative, periodic function satisfying*

$$\int_{\mathbb{T}} \rho(x) dx = c, \quad \text{and} \quad \rho_{\min} \leq \rho(x) \leq \rho_{\max}, \quad \forall x \in \mathbb{T}. \quad (4.3)$$

Let $\psi \in L^1_+(\mathbb{T})$. Then there exist two non-negative constants γ_1 and γ_2 such that

$$\rho_{\min} \|\psi\|_{L^1} + (\rho_{\max} - \rho_{\min}) \gamma_1 \leq \int_{\mathbb{T}} \psi(x - y) \rho(y) dy \leq \rho_{\max} \|\psi\|_{L^1} - (\rho_{\max} - \rho_{\min}) \gamma_2, \quad (4.4)$$

for any $x \in \mathbb{T}$. Moreover, γ_1 and γ_2 can be expressed by

$$\gamma_1 = \int_{\frac{\rho_{\max} - c}{\rho_{\max} - \rho_{\min}}}^1 \psi^*(y) dy, \quad \gamma_2 = \int_{\frac{c - \rho_{\min}}{\rho_{\max} - \rho_{\min}}}^1 \psi^*(y) dy,$$

where $\psi^ : (0, 1] \rightarrow \mathbb{R}$ is the decreasing rearrangement of ψ on \mathbb{T} .*

Proof. For the lower bound, fix an $x \in \mathbb{T}$. Consider the set

$$\mathcal{A} = \{y \in \mathbb{T} : \psi(x - y) > \psi^*(d)\}, \quad \text{with} \quad |\mathcal{A}| = d := \frac{\rho_{\max} - c}{\rho_{\max} - \rho_{\min}},$$

and define a function

$$\tilde{\rho}(y) = \rho_{\min} \mathcal{X}_{\mathcal{A}}(y) + \rho_{\max} \mathcal{X}_{\mathcal{A}^c}(y) = \begin{cases} \rho_{\min} & y \in \mathcal{A} \\ \rho_{\max} & y \in \mathcal{A}^c = \mathbb{T} \setminus \mathcal{A} \end{cases}, \quad (4.5)$$

where $\mathcal{X}_{\mathcal{A}}$ denotes the indicator function of set \mathcal{A} . Let us check

$$\begin{aligned} \int_{\mathbb{T}} \psi(x - y) \tilde{\rho}(y) dy &= \rho_{\min} \int_{\mathcal{A}} \psi(x - y) dy + \rho_{\max} \int_{\mathcal{A}^c} \psi(x - y) dy \\ &= \rho_{\min} \int_0^d \psi^*(y) dy + \rho_{\max} \int_d^1 \psi^*(y) dy = \rho_{\min} \|\psi\|_{L^1} + (\rho_{\max} - \rho_{\min}) \gamma_1. \end{aligned}$$

It remains to show

$$\int_{\mathbb{T}} \psi(x - y) (\rho(y) - \tilde{\rho}(y)) dx \geq 0.$$

Indeed, we have

$$\begin{aligned} \int_{\mathbb{T}} \psi(x - y) (\rho(y) - \tilde{\rho}(y)) dy &= \int_{\mathcal{A}} \psi(x - y) (\rho(y) - \rho_{\min}) dy + \int_{\mathcal{A}^c} \psi(x - y) (\rho(y) - \rho_{\max}) dy \\ &\geq \psi^*(d) \int_{\mathcal{A}} (\rho(y) - \rho_{\min}) dy + \psi^*(d) \int_{\mathcal{A}^c} (\rho(y) - \rho_{\max}) dy \\ &= \psi^*(d) \left(\int_{\mathbb{T}} \rho(y) dy - \rho_{\min} |\mathcal{A}| - \rho_{\max} |\mathcal{A}^c| \right) = \psi^*(d) (c - \rho_{\min} d - \rho_{\max} (1 - d)) = 0. \end{aligned}$$

The upper bound can be obtained similarly by considering the set

$$\hat{\mathcal{A}} = \{y \in \mathbb{T} : \psi(x - y) > \psi^*(d)\}, \quad \text{with} \quad |\hat{\mathcal{A}}| = \hat{d} := \frac{c - \rho_{\min}}{\rho_{\max} - \rho_{\min}},$$

and the function

$$\hat{\rho}(y) = \rho_{\max} \mathcal{X}_{\hat{\mathcal{A}}}(y) + \rho_{\min} \mathcal{X}_{\hat{\mathcal{A}}^c}(y).$$

We omit the details of the proof. \square

Remark 4.2. The function $\tilde{\rho}$ in (4.5) is the minimizer of the optimization problem

$$\min_{\rho} \int_{\mathbb{T}} \psi(x-y)\rho(y) dy,$$

subject to the constraints in (4.3). Indeed, the Lagrange function of the constraint minimization problem is

$$L(\rho, \mu_1, \mu_2, \kappa) = \int_{\mathbb{T}} \left(\psi\rho + \mu_1(\rho - \rho_{\max}) - \mu_2(\rho - \rho_{\min}) - \kappa\rho \right) dy + \kappa c,$$

where $\mu_1, \mu_2 \in L_+^\infty(\mathbb{T})$ and $\kappa \in \mathbb{R}$ are Lagrange multipliers. The Karush-Kuhn-Tucker (KKT) conditions read

$$\psi + \mu_1 - \mu_2 - \kappa = 0, \quad (\text{Stationarity}),$$

$$\rho_{\min} \leq \tilde{\rho} \leq \rho_{\max}, \quad \int_{\mathbb{T}} \tilde{\rho}(y) dy = c, \quad (\text{Primal feasibility}),$$

$$\mu_1, \mu_2 \geq 0, \quad (\text{Dual feasibility}),$$

$$(\tilde{\rho} - \rho_{\max})\mu_1 = (\tilde{\rho} - \rho_{\min})\mu_2 = 0, \quad (\text{Complementary slackness}).$$

We choose $\mu_1 = -\min\{0, \psi - \kappa\}$ and $\mu_2 = \max\{0, \psi - \kappa\}$. This ensures that the stationarity and dual feasibility conditions are satisfied. The complementary slackness conditions require $\tilde{\rho} \equiv \rho_{\max}$ when $\psi < \kappa$ and $\tilde{\rho} \equiv \rho_{\min}$ when $\psi > \kappa$. Finally, to ensure the primal feasibility, we obtain $\kappa = \psi^*(d)$. Altogether, we end up with (4.5).

As a special case of Lemma 4.1, if we choose ρ_{\max} and ρ_{\min} as

$$0 \leq \rho_{\min} < c < \rho_{\max} \leq 2c, \quad \rho_{\min} + \rho_{\max} = 2c, \quad (4.6)$$

then (4.4) holds with

$$\gamma_1 = \gamma_2 = \gamma := \int_{\frac{1}{2}}^1 \psi^*(y) dy.$$

It will dramatically simplify the analysis. We also observe

$$2\gamma < \int_0^1 \psi^*(y) dy = \|\psi\|_{L^1}.$$

In the following construction, we will choose

$$\rho_{\min} = 0, \quad \rho_{\max} = 2c. \quad (4.7)$$

We shall comment that (4.6) is not the only choice that leads to an invariant region. We will keep using the notations ρ_{\max} and ρ_{\min} throughout the construction for generality.

4.2. Construction of invariant region

We shall construct the invariant region in light of Σ^* as in Section 3. The main difference would be that the region Σ^* need to satisfy the additional restriction (4.2). We will make use of the improved bounds (4.4). Let us denote

$$\beta_{\max} := \rho_{\max} \|\psi\|_{L^1} - (\rho_{\max} - \rho_{\min})\gamma_2. \quad (4.8a)$$

$$\beta_{\min} := \rho_{\min} \|\psi\|_{L^1} + (\rho_{\max} - \rho_{\min})\gamma_1, \quad (4.8b)$$

Unlike definition (3.6), β_{\max} and β_{\min} depend on the density ρ_{\max} and ρ_{\min} .

We will carry over the same notations from Section 3 to avoid excess notations. However, it should be noted that the functions $\hat{z}, \tilde{z}, \hat{\theta}, \tilde{\theta}$ now depend on ρ_{\max}, ρ_{\min} . To avoid confusion we restate the expressions for \hat{z}, \tilde{z} ,

$$\hat{z} = \sqrt{\frac{4kc}{\beta_{\max}^2} - 1}, \quad \tilde{z} = \sqrt{\frac{4kc}{\beta_{\min}^2} - 1}. \quad (4.9)$$

Note that if we choose ρ_{\max} and ρ_{\min} as in (4.7), then \hat{z} and \tilde{z} have the explicit forms in (2.9).

Our construction of the invariant region will follow the procedure in Section 3. Here we focus on the construction of the weak alignment case Σ_1^* . The other two cases can be treated similarly. Let us assume $\beta_{\max} < 2\sqrt{kc}$.

Step 1: On the (p, q) plane, we construct the first segment of the boundary of the invariant region

$$C_1 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_1, 0]\}, \quad (4.10)$$

where (\hat{p}, \hat{q}) satisfy the dynamics

$$\hat{p}' = k - kc\hat{q}, \quad \hat{q}' = \hat{p} - \beta_{\max}\hat{q},$$

with initial data

$$\hat{p}(0) = \frac{\beta_{\max}}{\rho_{\max}}, \quad \hat{q}(0) = \frac{1}{\rho_{\max}}$$

and time $t_1 < 0$ such that $\hat{q}(t_1) = 1/c$. Here we choose a different initial point that Section 3.1. We take $q(0) = \frac{1}{\rho_{\max}}$ so that $\rho(0) \leq \rho_{\max}$. The choice of $p(0)$ ensures $\rho(t) \leq \rho_{\max}$, which is necessary for (4.2) to hold.

Using similar calculations as in Step 1 of Section 3.1, we have

$$\hat{p}(t) = \frac{\beta_{\max}}{c} - e^{-\frac{\beta_{\max}t}{2}} \left(\frac{1}{c} - \frac{1}{\rho_{\max}} \right) \left(\beta_{\max} \cos \hat{\theta}t - \frac{(kc - \frac{\beta_{\max}^2}{2})}{\hat{\theta}} \sin \hat{\theta}t \right),$$

$$\hat{q}(t) = \frac{1}{c} - e^{-\frac{\beta_{\max} t}{2}} \left(\frac{1}{c} - \frac{1}{\rho_{\max}} \right) \left(\cos \hat{\theta} t + \frac{\beta_{\max}}{2\hat{\theta}} \sin \hat{\theta} t \right).$$

The final point of C_1 is $(p_1, 1/c)$ where

$$p_1 := \hat{p}(t_1) = \frac{\beta_{\max}}{c} - \sqrt{kc} \left(\frac{1}{c} - \frac{1}{\rho_{\max}} \right) e^{\frac{\tan^{-1} \hat{z}}{\hat{z}}}. \quad (4.11)$$

The value of p_1 depends on the choices of ρ_{\max} and ρ_{\min} .

The point $(p_1, 1/c)$ should be the starting point of the next segment C_2 . To make sure C_2 continues to move upward as we trace time in the negative direction, we require that p_1 lies at the left hand-side of $\frac{\beta_{\min}}{c}$, which is the equilibrium state of the auxiliary system (3.7b). $p_1 < \frac{\beta_{\min}}{c}$ can be equivalently expressed as

$$\beta_{\max} - \beta_{\min} < \sqrt{kc} \left(1 - \frac{c}{\rho_{\max}} \right) e^{\frac{\tan^{-1} \hat{z}}{\hat{z}}}. \quad (4.12)$$

Step 2: Assume that (4.12) holds. We continue with the next segment of the boundary of the invariant region

$$C_2 = \{(p, q) : (\tilde{p}(t), \tilde{q}(t)), t \in [t_2, 0]\}, \quad (4.13)$$

where (\tilde{p}, \tilde{q}) satisfy the dynamics

$$\tilde{p}' = k - kc\tilde{q}, \quad \tilde{q}' = \tilde{p} - \beta_{\min}\tilde{q},$$

with initial data $\tilde{p}(0) = p_1$, $\tilde{q}(0) = \frac{1}{c}$, and t_2 is the first negative time such that $\tilde{q}(t_2) = 1/c$.

Using similar calculations as in Step 1 of Section 3.1, we have

$$\begin{aligned} \tilde{p}(t) &= \frac{\beta_{\min}}{c} + e^{-\frac{\beta_{\min} t}{2}} \left[\left(p_1 - \frac{\beta_{\min}}{c} \right) \cos \tilde{\theta} t + \left(\frac{p_1 \beta_{\min}}{2\tilde{\theta}} - \frac{\beta_{\min}^2}{2c\tilde{\theta}} \right) \sin \tilde{\theta} t \right], \\ \tilde{q}(t) &= \frac{1}{c} + \frac{e^{-\frac{\beta_{\min} t}{2}}}{\tilde{\theta}} \left(p_1 - \frac{\beta_{\min}}{c} \right) \sin \tilde{\theta} t. \end{aligned}$$

We find that the final point of C_2 is $(p_2, 1/c)$ where

$$p_2 := \tilde{p}(t_2) = \frac{\beta_{\min}}{c} \left(1 + e^{\frac{\pi}{\hat{z}}} \right) - p_1 e^{\frac{\pi}{\hat{z}}}, \quad (4.14)$$

which also depends on the choices of ρ_{\max} and ρ_{\min} .

The point $(p_2, 1/c)$ should be the starting point of the next segment C_3 . To make sure C_3 continues to move downward as we trace time in the negative direction, we require that p_2 lies at the right hand-side of $\frac{\beta_{\max}}{c}$, which is the equilibrium state of the auxiliary system (3.7a). $p_2 > \frac{\beta_{\max}}{c}$ can be equivalently expressed as

$$\beta_{\max} - \beta_{\min} < \sqrt{kc} \left(1 - \frac{c}{\rho_{\max}} \right) e^{\frac{\tan^{-1} \hat{z}}{\hat{z}}} \cdot \frac{e^{\frac{\pi}{\hat{z}}}}{1 + e^{\frac{\pi}{\hat{z}}}}, \quad (4.15)$$

where R is defined in (4.12). Note that condition (4.15) is stronger than (4.12).

Step 3: Assume that (4.15) holds. The next segment of the boundary of invariant region

$$C_3 = \{(p, q) : (\hat{p}(t), \hat{q}(t)), t \in [t_3, 0]\} \quad (4.16)$$

is constructed from the dynamics

$$\hat{p}' = k - kc\hat{q}, \quad \hat{q}' = \hat{p} - \beta_{\max}\hat{q},$$

with initial data $\hat{p}(0) = p_2$, $\hat{q}(0) = \frac{1}{c}$, and t_3 is the first negative time such that $\hat{q}(t_3) = 1/\rho_{\max}$. We have

$$\begin{aligned} \hat{p}(t) &= \frac{\beta_{\max}}{c} + e^{-\frac{\beta_{\max}t}{2}} \left[\left(p_2 - \frac{\beta_{\max}}{c} \right) \cos \hat{\theta}t + \left(\frac{p_2 \beta_{\max}}{2\hat{\theta}} - \frac{\beta_{\max}^2}{2c\hat{\theta}} \right) \sin \hat{\theta}t \right], \\ \hat{q}(t) &= \frac{1}{c} + \frac{e^{-\frac{\beta_{\max}t}{2}}}{\hat{\theta}} \left(p_2 - \frac{\beta_{\max}}{c} \right) \sin \hat{\theta}t. \end{aligned}$$

To ensure the existence of t_3 such that $\hat{q}(t_3) = 1/\rho_{\max}$, we state the following Lemma.

Lemma 4.3. *Let t_* be the first negative time such that $\hat{q}'(t_*) = 0$. Then $\hat{q}(t_*) < 1/\rho_{\max}$ if and only if*

$$(\beta_{\max} - \beta_{\min}) \left(1 + e^{-\frac{\pi}{\hat{z}}} \right) e^{-\frac{\tan^{-1}(\hat{z})}{\hat{z}}} < \sqrt{kc} \left(1 - \frac{c}{\rho_{\max}} \right) \left(1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}} \right),$$

or equivalently,

$$\beta_{\max} - \beta_{\min} < \sqrt{kc} \left(1 - \frac{c}{\rho_{\max}} \right) e^{\frac{\tan^{-1}(\hat{z})}{\hat{z}}} \cdot \frac{\left(1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}} \right)}{\left(1 + e^{-\frac{\pi}{\hat{z}}} \right)}. \quad (4.17)$$

The proof of the Lemma follows similar arguments as Lemma 3.4, which we will omit here. The admissible condition (4.17) is similar as (3.14), differed only by a factor, as the starting point of the construction is different. The Lemma ensures that the trajectory of C_3 hits the line $q = 1/\rho_{\max}$ first before completing the outward spiral turn. Moreover, at the intersection $p_3 = \hat{p}(t_3) > \frac{\beta_{\max}}{\rho_{\max}}$. It is easy to observe that condition (4.17) is stronger than (4.12) and (4.15).

Now we are ready to construct the invariant region

$$\Sigma_L^* = \text{open set enclosed by } C_1, C_2, C_3 \text{ and } C_4, \quad (4.18)$$

where C_4 is the line segment

$$C_4 = \left\{ (p, q) : p \in \left(\frac{\beta_{\max}}{\rho_{\max}}, p_3 \right), q = \frac{1}{\rho_{\max}} \right\}.$$

Fig. 8 gives an illustration of the invariant region. We can further make use of the transformation F as in (3.19) to obtain the invariant region Σ_L^1 in the (G, ρ) plane

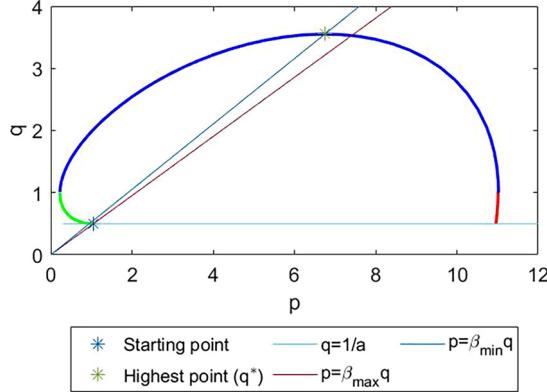


Fig. 8. Invariant region for $k = 4, c = 1, \rho_{\max} = 2, \rho_{\min} = 0$ and influence function with $\|\psi\|_{L^1} = 2, \gamma_1 = \gamma_2 = 0.95$.

$$\Sigma_L^1 := F(\Sigma_L^*). \quad (4.19)$$

See Fig. 2(a) for an illustration of Σ_L^1 .

Proposition 4.4 (Invariant region). *Let $4kc > \beta_{\max}^2$. Assume condition (4.17) holds. Consider the initial value problem of (3.4) with $(w(0), s(0)) \in \Sigma_L^*$. In addition, assume*

$$\beta_{\min} \leq \psi * \rho \leq \beta_{\max}. \quad (4.20)$$

Then the solution $(w(t), s(t)) \in \Sigma_L^$ for all $t > 0$.*

Proof. The arguments that the trajectory does not cross C_1, C_2, C_3 are entirely similar to the ones in the proof of Proposition 3.5. If $(w, s) \in C_4$, meaning $w > \frac{\beta_{\max}}{\rho_{\max}}$ and $s = \frac{1}{\rho_{\max}}$, we get from (3.4b) that

$$s' = w - s(\psi * \rho) > \frac{\beta_{\max}}{\rho_{\max}} - \frac{1}{\rho_{\max}} \cdot \beta_{\max} \geq 0.$$

Therefore, trajectories can not touch trajectories with initial point inside Σ_L^* never touch C_4 as well. By continuity of the trajectories, we conclude that $(w(t), s(t))$ stays in Σ_L^* all time. \square

For the other two cases, Σ_L^2, Σ_L^3 can be constructed very much alike as long the lines of Σ_2, Σ_3 respectively. The only difference is that the corresponding invariant regions on the (p, q) plane now start from a point in the first quadrant, namely $(\frac{\beta_{\max}}{\rho_{\max}}, \frac{1}{\rho_{\max}})$, instead of the origin. Since the respective calculations and consequent proof to the second and third assertions of Theorem 2.6 follows along the lines of the first assertion, we only prove the first assertion here and state the regions Σ_L^2, Σ_L^3 .

$$\Sigma_L^2 = F(\Sigma_2^*), \quad (4.21)$$

where,

$\Sigma_2^* = \text{unbounded open set surrounded by } C_1, C_2, \{(p, 0) : p > \frac{\beta_{\max}}{\rho_{\max}}\} \text{ on 3 sides,}$

with C_1 as in (3.23) and \hat{p}, \hat{q} with initial data $(\frac{\beta_{\max}}{\rho_{\max}}, \frac{1}{\rho_{\max}})$ and C_2 as in (3.24). Similarly,

$$\Sigma_L^3 = F(\Sigma_3^*), \quad (4.22)$$

where,

$$\Sigma_3^* = \text{open set enclosed by } C_1, C_2, C_3 \text{ and } q = \frac{1}{\rho_{\max}},$$

with C_1, C_2, C_3 as in Section 3.3 but C_1 obtained from \hat{p}, \hat{q} with initial data $(\frac{\beta_{\max}}{\rho_{\max}}, \frac{1}{\rho_{\max}})$.

4.3. Proof of Theorem 2.6

We are ready to apply Proposition 4.4 and prove Theorem 2.6. We will only prove the weak alignment case. The other two cases work similarly. We choose $\rho_{\max} = 2c$ and $\rho_{\min} = 0$ as in (4.7). It implies

$$\beta_{\max} = 2c(\|\psi\|_{L^1} - \gamma) \quad \text{and} \quad \beta_{\min} = 2c\gamma.$$

Let us validate all the assumptions in Proposition 4.4. First, the hypothesis of the Theorem $\|\psi\|_{L^1} - \gamma < \frac{\lambda}{2}$ implies

$$\beta_{\max}^2 = 4c^2(\|\psi\|_{L^1} - \gamma)^2 < 4kc.$$

Second, the admissible condition (2.7) implies (4.17). Indeed, we have

$$\beta_{\max} - \beta_{\min} = 2c(\|\psi\|_{L^1} - 2\gamma) < \frac{\sqrt{kc}}{2} \cdot \frac{e^{\frac{\tan^{-1}\hat{z}}{\hat{z}}} (1 - e^{-\frac{\pi}{\hat{z}} - \frac{\pi}{\hat{z}}})}{(1 + e^{-\frac{\pi}{\hat{z}}})}.$$

Finally, owing to Lemma 4.1, we conclude that (4.20) holds as long as ρ is uniformly bounded above by $2c$ (and below by 0).

Consider subcritical initial data $(G_0(x), \rho_0(x)) \in \Sigma_L^1$ for all $x \in \mathbb{T}$. Along each characteristic path (3.1), there is dynamics (3.4) with initial data $(w(0), s(0)) \in \Sigma_L^*$. We claim that $(w(t), s(t)) \in \Sigma_L^*$ for any $t \geq 0$ along any characteristic path.

Let us argue by contradiction. Suppose there exists a first time t_0 and a characteristic path such that $(w(t_0), s(t_0)) \notin \Sigma_L^*$. By continuity of the dynamics (3.4), we have that along every characteristic path $(w(t_0), s(t_0)) \in \overline{\Sigma_L^*}$. Since $\overline{\Sigma_L^*} \subset \{(p, q) : q \geq \frac{1}{\rho_{\max}}\}$, we obtain the uniform bound $s(t_0, x) \geq \frac{1}{2c}$ and hence $\rho(t_0, x) \in (0, 2c]$. Now, we can apply Proposition 4.4 and get $(w(t_0), s(t_0)) \in \Sigma_L^*$. This leads to a contradiction.

Collecting all characteristic paths, and applying the transformation F in (3.19), we conclude that $(G(t, x), \rho(t, x)) \in \Sigma_L^1$ for all $x \in \mathbb{T}$ and $t \geq 0$. Therefore, (G, ρ) remain bounded in all time. Consequently, by Theorem 2.1, we have that (ρ, u) is global-in-time smooth solution to (1.1).

Remark 4.5. We would like to remark the invariant region Σ_L^1 is a subset of

$$\{(G, \rho) : \frac{1}{q^*} \leq \rho \leq \rho_{\max}\},$$

where q^* is the highest tip of Σ_L^* . This leads to an improved bound on ρ_{\min} , and consequently better bounds on β_{\min} and β_{\max} . Repeating the procedure with the new bounds, we can obtain a larger invariant region. Finding the optimal (or largest) invariant region is beyond the scope of this paper. We shall leave this for future investigations.

Data availability

No data was used for the research described in the article.

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