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CORRECTION

Correction to: Two weight Sobolev norm inequalities for smooth Calderón–Zygmund operators and doubling weights

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Abstract
We correct two errors in the paper "Two weight Sobolev norm inequalities for smooth Calderón–Zygmund operators and doubling weights" published in Math. Z. Here we give a precise description of where the errors occur in the original paper, followed by an overview of the corrections, pointing to details appearing on the arXiv. We include statements of the main corrected theorems.

1 Introduction

Two errors in our published article "Two weight Sobolev norm inequalities for smooth Calderón–Zygmund operators and doubling weights" have come to light recently. The first is a serious error, and requires weakening our main theorem by restricting the class of smooth Calderón–Zygmund operators in the statement of the theorem to just vector Riesz transforms (as this is the first $T1$ theorem for Sobolev spaces with doubling measures, this restriction does not seem unreasonable). The second error is easily fixed by introducing a new notion of full subgrids, and requires no further changes.

The bulk of the paper remains unchanged from its original form, with the only new material occurring in the first paragraph of Subsection 6.4 on the stopping form, and throughout the Appendix, which proves the necessity of the classical pivotal conditions for the testing

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conditions for vector Riesz transforms. A more detailed description of the errors and fixes appears below.

1.1 Description of the errors

In the second line on page 71 of the published article [1], the polynomial $Q_{I';\kappa}$ was incorrectly removed from an inner product. We could not overcome this serious error in the full generality of the main theorem, and this led to considering either vector Riesz transforms, or assuming pivotal side conditions.

Lemma 36 on page 35 is false as stated. Thus the strong hypothesis (2.15) used on page 27 of [1], cannot hold for general stopping time constructions with doubling weights. This led to the fix described below involving *full* grids.

1.2 Description of the changes

1.2.1 The serious error

The $T1$ theorem (Theorem 2 on page 4 of [1]) has been weakened in two different ways from its original form. First, the $T1$ theorem holds as stated, provided the operators are restricted to fractional vector Riesz transforms. Second, the $T1$ theorem holds as stated, provided a side condition of pivotal type holds. Here is a complete statement. For $0 \leq \alpha < n$ we define the α-fractional vector Riesz transform kernel $\mathbf{K}^\alpha(x, y)$ to be the standard function $\mathbf{K}^\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $c_{n,\alpha} \frac{z}{|z|^{n-\alpha}}$, and we denote by $\mathbf{R}^\alpha$ the associated α-fractional singular integral on $\mathbb{R}^n$.

Theorem 1 ($T1$ for doubling measures) *Let $0 \leq \alpha < n$, and let $\mathbf{R}^\alpha$ be the α-fractional vector Riesz transform on $\mathbb{R}^n$. Let σ and ω be doubling Borel measures on $\mathbb{R}^n$. Then there is a positive constant θ, depending only on the doubling constants of σ and ω, such that if $0 < s < \theta$, then $\mathbf{R}^\alpha_\sigma$, where $\mathbf{R}^\alpha_\sigma f \equiv \mathbf{R}^\alpha(f\sigma)$, is bounded from $W^s(\sigma)$ to $W^s(\omega)$, i.e.*

$$\left\|\mathbf{R}^\alpha_\sigma f\right\|_{W^s(\omega)} \leq \mathfrak{N}_{\mathbf{R}^\alpha} \|f\|_{W^s(\sigma)}, \tag{1.1}$$

provided the Sobolev $\mathbf{1}$-testing and $\mathbf{1}^*$-testing conditions for the operator T^α,

$$\left\|\mathbf{R}^\alpha_\sigma \mathbf{1}_I\right\|_{W^s(\omega)} \leq \mathfrak{T}_{\mathbf{R}^{\alpha,s}}(\sigma,\omega)\sqrt{|I|_\sigma}\,\ell(I)^{-s}, \qquad I \in \mathcal{Q}^n,$$
$$\left\|\mathbf{R}^{\alpha,*}_\omega \mathbf{1}_I\right\|_{W^{-s}(\sigma)} \leq \mathfrak{T}_{\mathbf{R}^{\alpha,-s,*}}(\omega,\sigma)\sqrt{|I|_\omega}\,\ell(I)^{s}, \qquad I \in \mathcal{Q}^n,$$

taken over the family of indicator test functions $\{\mathbf{1}_I\}_{I\in\mathcal{Q}^n}$.
If we assume in addition that the classical pivotal constants $\mathcal{V}_2^{\alpha,1}(\sigma,\omega)$ and $\mathcal{V}_2^{\alpha,1,*}(\omega,\sigma)$ hold, then for any smooth α -fractional singular integral T^α on $\mathbb{R}^n$, then T^α_σ, where $T^\alpha_\sigma f \equiv T^\alpha(f\sigma)$, is bounded from $W^s(\sigma)$ to $W^s(\omega)$, i.e.

$$\left\|T^\alpha_\sigma f\right\|_{W^s(\omega)} \leq \mathfrak{N}_{T^\alpha} \|f\|_{W^s(\sigma)}, \tag{1.2}$$

provided the classical fractional Muckenhoupt condition on the measure pair holds,

$$A_2^\alpha \equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\omega |Q|_\sigma}{|Q|^{2(1-\frac{\alpha}{n})}} < \infty,$$

as well as the Sobolev $\mathbf{1}$-testing and $\mathbf{1}^*$-testing conditions for the operator T^α,

$$\left\|T^\alpha_\sigma \mathbf{1}_I\right\|_{W^s(\omega)} \leq \mathfrak{T}_{T^{\alpha,s}}(\sigma,\omega)\sqrt{|I|_\sigma}\,\ell(I)^{-s}, \qquad I \in \mathcal{Q}^n,$$

$$\left\| T_{\omega}^{\alpha,*}\mathbf{1}_I \right\|_{W^{-s}(\sigma)} \leq \mathfrak{T}_{T^{\alpha,-s,*}}(\omega,\sigma)\sqrt{|I|_{\omega}}\ell(I)^s , \qquad I \in \mathcal{Q}^n ,$$

taken over the family of indicator test functions $\{\mathbf{1}_I\}_{I\in\mathcal{Q}^n}$.
Conversely, the testing conditions are necessary for (1.1)*, and if in addition* T^{α} *is a smooth convolution operator with homogeneous kernel that is nonvanishing in some coordinate direction, then* $A_2^{\alpha} < \infty$ *whenever the two weight norm inequality* (1.2) *holds for some* $s > 0$.

Consult the arXiv article [2, see page 3] for further details.

The start of the proof of the bound for the stopping form on page 70 of the published article [1] is altered to exploit an additional corona decomposition in which the classical 1-pivotal condition is controlled. The short sentence added for this is:

We apply the 1-pivotal stopping time construction to the Alpert projection $\mathsf{P}^{\sigma}_{\mathcal{C}_F;\kappa} f$ in order to obtain a further corona decomposition $\mathcal{C}_F = \bigcup_{H\in\mathcal{H}(F)}\mathcal{C}_H$ in which we obtain the stopping control bound

$$\mathrm{P}_1^{\lambda}(I,\mathbf{1}_H\sigma)^2 |I|_{\omega} \leq \Gamma |I|_{\sigma} \lesssim 2\mathcal{V}_2^{\lambda,1}(\sigma,\omega)|I|_{\sigma} , \qquad I \in \mathcal{C}_H \ , \ H \in \mathcal{H}(F) \ , \ F \in \mathcal{F}. \tag{1.3}$$

Consult [2, see page 59] for details.

The derivation of the pivotal condition from the testing conditions for a fractional vector Riesz transform, is carried out in the Appendix on pages 61 to 65 of [2]. Here is a summary of the main results in the appendix of the arXiv article [2].

Lemma 2 *For* $0 \leq \alpha < n$ *and* $|s|$ *sufficiently small, we have*

$$\mathcal{V}_{2,s}^{\alpha}(\sigma,\omega) \lesssim \mathfrak{T}_{\mathbf{R}^{\alpha,n,s}}(\sigma,\omega) + \sqrt{A_2^{\alpha}(\sigma,\omega)}.$$

The proof of Lemma 2 uses a new extreme reversal of energy, together with some familiar techniques from the two weight norm inequality literature. We introduce a *stronger* notion of energy reversal which we call extreme energy reversal. We say that a vector $\mathbf{T}^{\alpha} = \left\{T_{\ell}^{\alpha}\right\}_{\ell=1}^2$ of α-fractional transforms in the plane has *extreme* reversal of ω-energy on a cube J if there is a Haar function $h_J^{\omega}(x)$ and a positive constant C_0, such that for all $2 \leq \gamma \leq 2^{\mathbf{r}(1-\varepsilon)}$ and for all positive measures μ supported outside γJ, we have the inequality,

$$\begin{aligned}
&\mathbb{E}_J^{\omega}\left[\left(\mathbf{x}-\mathbb{E}_J^{\omega}\mathbf{x}\right)^2\right]\left(\frac{\mathrm{P}^{\alpha}(J,\mu)}{|J|^{\frac{1}{n}}}\right)^2 |J|_{\omega} = \mathsf{E}(J,\omega)^2\,\mathrm{P}^{\alpha}(J,\mu)^2\,|J|_{\omega}\\
&\leq C\left|\int_J\int_{\mathbb{R}^n\setminus\gamma J}\left[\mathbf{K}^{\alpha}(x,y)-\mathbf{K}^{\alpha}(c_J,y)\right]h_J^{\omega}(x)\,d\mu(y)\,d\omega(x)\right|^2 .
\end{aligned} \tag{1.4}$$

Lemma 3 *Let* $0 \leq \alpha < n$. *Then the* α*-fractional Riesz transform* $\mathbf{R}^{\alpha,n} = \left\{R_{\ell}^{n,\alpha}\right\}_{\ell=1}^n$ *has extreme reversal of* ω*-energy* (1.4) *on* all *cubes* J *provided* γ *is chosen large enough depending only on* n *and* α.

See the appendix of [2] for details.

1.2.2 The less problematic error

In Subsection 3.4 on pages 28–30 of [2], a strong new notion of a full grid is introduced in Definition 38, and is shown to arise in stopping time constructions using a κ-pivotal condition in Lemma 39. This new grid produces strong Carleson conditions for full grids. The strong Carleson conditions for full grids then yield the Quasiothogonality Lemma 30 on page 23 of [2] for full grids, which delivers a crucial control used throughout the paper—except for the stopping form, where quasiorthogonality is not used—only orthogonality. See [2] for details.

References

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