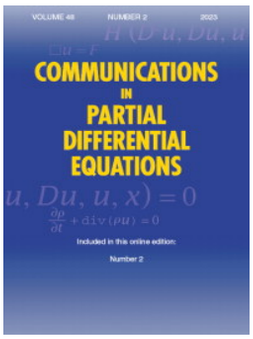

Taylor & Francis
Taylor & Francis Group

Communications in Partial Differential Equations

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/lpde20

On the growth of generalized Fourier coefficients of restricted eigenfunctions

Madelyne M. Brown

To cite this article: Madelyne M. Brown (2023) On the growth of generalized Fourier coefficients of restricted eigenfunctions, Communications in Partial Differential Equations, 48:2, 252-285, DOI: 10.1080/03605302.2023.2169939

To link to this article: https://doi.org/10.1080/03605302.2023.2169939

Published online: 31 Mar 2023.

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2023, VOL. 48, NO. 2, 252–285
https://doi.org/10.1080/03605302.2023.2169939

On the growth of generalized Fourier coefficients of restricted eigenfunctions

Madelyne M. Brown

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC, USA

ABSTRACT
Let (M, g) be a smooth, compact, Riemannian manifold and $\{\phi_h\}$ a sequence of L^2-normalized Laplace eigenfunctions on M. For a smooth submanifold $H \subset M$, we consider the growth of the restricted eigenfunctions $\phi_h|_H$ by testing them against a sequence of functions $\{\psi_h\}$ on H whose wavefront set avoids S^*H. That is, we study what we call the generalized Fourier coefficients: $\langle \phi_h, \psi_h \rangle_{L^2(H)}$. We give an explicit bound on these coefficients depending on how the defect measures for the two collections of functions ϕ_h and ψ_h relate. This allows us to get a little– o improvement whenever the collection of recurrent directions over the wavefront set of ψ_h is small. To obtain our estimates, we utilize geodesic beam techniques.

ARTICLE HISTORY
Received 6 May 2022
Accepted 15 January 2023

KEYWORDS
Eigenfunctions; Laplacian; Restriction

1. Introduction and main results

On a smooth, compact, n-dimensional Riemannian manifold (M, g), we consider a sequence of L^2-normalized Laplace eigenfunctions $\{\phi_h\}$ satisfying

$$(-h^2\Delta_g - 1)\phi_h = 0 \qquad \text{and} \qquad \|\phi_h\|_{L^2(M)} = 1. \tag{1}$$

From a quantum mechanics perspective, we can think of $\phi_h(x)$ as the wave function for a free quantum particle with fixed energy h^{-2}. Thus $|\phi_h(x)|^2$ gives the probability density for finding the quantum particle at $x \in M$. Understanding how these high-energy particles behave, corresponding to sending $h \to 0^+$, is a well-studied problem in mathematical physics. We are particularly interested in exploring how ϕ_h, on average, concentrates and grows on our manifold.

In this article, we lay the foundations for the defect measure approach to generalized Fourier coefficients of ϕ_h when restricted to a smooth, closed submanifold H. The Fourier expansion allows one to express $\phi_h|_H$ in terms of any complete orthonormal basis of $L^2(H)$. It is well known Laplace eigenfunctions on H can be used to build such a basis of $L^2(H)$. Particularly, there exists such an orthonormal basis consisting of eigenfunctions on H, $\{\psi_{h_j}\}_{j\in\mathbb{N}}$, which satisfy

$$-h_j^2\Delta_{g_H}\psi_{h_j} = E(h_j)\psi_{h_j}$$

CONTACT Madelyne M. Brown madelyne@live.unc.edu Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC, 27599, USA.

where g_H is the Riemannian metric on H induced by g. Thus we can express

$$\phi_h|_H = \sum_{j\in\mathbb{N}} \langle \phi_h|_H, \psi_{h_j}\rangle_{L^2(H)} \psi_{h_j} = \sum_{j\in\mathbb{N}} \left(\int_H \phi_h \bar{\psi}_{h_j} d\sigma_H \right) \psi_{h_j} \tag{2}$$

where $d\sigma_H$ is the volume measure on H induced by the metric g_H. We study the Fourier coefficients in (2), $\langle \phi_h, \psi_{h_j}\rangle_{L^2(H)}$ to gain an understanding of the restricted eigenfunctions $\phi_h|_H$. To extract more information we instead study the growth of $\left|\langle \phi_h, \psi_h\rangle_{L^2(H)}\right|$ where $\{\psi_h\}$ is any collection of functions on H. We will call these the generalized Fourier coefficients.

1.1. Summary of existing results

The growth of averages and weighted averages of eigenfunctions over a submanifold H has been widely studied. Much work has been done in the case where H is a smooth, closed curve, γ, and (M, g) is a surface. Good [1] and Hejhal [2] showed for γ a periodic geodesic and (M, g) a hyperbolic surface that there is a $C > 0$ such that as $h \to 0^+$

$$\left| \int_\gamma \phi_h d\sigma_\gamma \right| \leq C. \tag{3}$$

The integral in (3) is typically called a period integral. Further, for γ a unit length geodesic, Chen and Sogge [3] showed that $\left|\int_\gamma \phi_h d\sigma_\gamma\right| \leq C\|\phi_h\|_{L^2(M)}$. Reznikov [4] showed that on a compact hyperbolic surface, and γ a periodc geodesic, the Fourier coefficients, also called generalized periods, satisfy

$$\left| \int_\gamma \phi_h(\gamma(t)) e^{-2\pi i n t} dt \right| \leq C_\gamma |\gamma|^{1/2}$$

for any $|n| \leq c_\gamma h^{-1}$. Without needing to make any global assumptions on the surface M or curve γ, this was generalized by Xi [5] who proved for $0 \leq \alpha h < c < 1$ that

$$\left| \int_\gamma \phi_h(\gamma(t)) e^{-i\alpha t} dt \right| \leq C|\gamma| \tag{4}$$

where $|\gamma|$ is the length of γ.

More generally, for M an n-dimensional manifold and H a submanifold of codimension k, Zelditch [6] proved the sharp bound

$$\left| \int_H \phi_h d\sigma_H \right| = O\left(h^{\frac{1-k}{2}}\right) \tag{5}$$

which generalizes (3). This bound has since been improved under various assumptions on M and H by Canzani, Galkowski, Sogge, Toth, Wyman, Xi, and Zhang [5, 7–15]. Particularly in [8], Canzani and Galkowski show for a weight $w \in C^\infty(H)$ that

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left| \int_H \phi_h w \; d\sigma_H \right| \leq C_{n,k} \int_{SN^*H} |w| \sqrt{f|H_p R_H|^{-1}} d\sigma_{SN^*H},$$

where SN^*H is the unit conormal bundle of H, H_pR_H is a function measuring how fast geodesics flow out of the submanifold, and f is related to the defect measure of ϕ_h. They actually prove a stronger result for $\{\phi_h\}$ quasimodes of a wide class of semiclassical operators. To obtain their estimates, they develop a new technique that involves localizing ϕ_h near a family of geodesics emanating from points in H. Using this framework, they improve many existing results without needing global geometric conditions on their manifold.

Under various assumptions the standard restriction bound (5) has been logarithmically improved. In [11], Sogge, Xi, and Zhang study weighted period integrals on geodesics and show that there is a $C > 0$ such that

$$\left|\int_\gamma \phi_h \ w \ d\sigma_\gamma\right| \leq C(\log(1/h))^{-1/2}, \qquad h \ll 1,$$

for M a hyperbolic surface, γ a geodesic, and $w \in C_0^\infty$. Wyman extends this to the case where M is a surface with nonpositive curvature in [14] and further extends this to $k-$codimensional submanifolds in [15]. There he shows for manifolds with negative sectional curvature that

$$\left|\int_H \phi_h d\sigma_H\right| = O\left(\frac{h^{\frac{1-k}{2}}}{\sqrt{\log(1/h)}}\right). \tag{6}$$

In [7], Canzani and Galkowski give conditions on (H, M) for which (6) holds. This has also been extended to the Fourier coefficient case on surfaces by Wyman and Xi [16]. For example, under the condition that M has nonpositive curvature, they show

$$\left|\int_\gamma \phi_h(\gamma(t))e^{-i\alpha t}dt\right| \leq \frac{C}{\sqrt{\log(1/h)}}$$

where α is an integer multiple of $2\pi|\gamma|^{-1}$ and α is roughly size h^{-1}.

In this article, we work in any dimension, and allow the "weight" w or ($e^{-i\alpha t}$ as in [5, 16]) to be any collection of functions on H depending on h. We will utilize Canzani and Galkowski's technique to obtain our results. However, we face new challenges here as we must take into account the semiclassical behavior of both sequences.

1.2. Statement of results

Let $H \subset M$ be a closed, embedded submanifold of codimension k. Let $\{\psi_h\}$ be a collection of L^2-normalized functions on H,

$$\|\psi_h\|_{L^2(H)} = 1, \tag{7}$$

and let $A := \mathrm{WF}_h(\psi_h) \subset T^*H$ (see [17, pg. 188] for definition of the semiclassical wavefront set, denoted WF_h). We will use the coordinates (x', ξ') in T^*H.

We assume $\{\psi_h\}$ has defect measure ν (see [17, pg. 100] for definition of a defect measure). Note that supp $\nu \subseteq A$. Further, assume

$$\mathrm{WF}_h(\psi_h) = A \Subset B^*H \tag{8}$$

where B^*H denotes the coball bundle in T^*H. Using the coordinates on T^*H we can also write this as $A \Subset \left\{(x', \xi') : |\xi'|_{g_H} < 1\right\}$ where g_H is the metric induced by g on H. We define

$$\Sigma^A := \left\{\rho \in S_H^*M : \pi_{T^*H}\rho \in A\right\} \subseteq T^*M \tag{9}$$

where S_H^*M denotes the cosphere bundle with footprints in H and π_{T^*H} is the projection from T^*M onto T^*H. We use the defect measure ν to define a measure ν^A on Σ^A. Essentially ν^A is an extension of the defect measure ν to Σ^A. We later define ν^A more explicitly in (15).

In what follows we denote the recurrent set of Σ^A by $\mathcal{R}_A$ (see Section 5 for explicit definition). Roughly, the recurrent set of Σ^A is the collection of points $\rho \in \Sigma^A$ which, under the geodesic flow, return to Σ^A infinitely often and eventually get arbitrarily close to the initial point ρ.

Theorem 1.1. *Let $\{\phi_h\}$ be a sequence of Laplace eigenfunctions on M satisfying (1). Let $H \subset M$ be a closed, embedded, smooth submanifold of codimention k, and let $\{\psi_h\} \subset L^2(H)$ be a sequence of L^2-normalized functions on H with defect measure ν, satisfying $\mathrm{WF}_h(\psi_h) =: A \Subset B^*H$. If $\nu^A(\mathcal{R}_A) = 0$, where ν^A is defined in (15), then*

$$\left|\langle \phi_h, \psi_h\rangle_{L^2(H)}\right| = o\left(h^{\frac{1-k}{2}}\right), \qquad h \to 0^+. \tag{10}$$

Wyman, Xi, and Zelditch studied sums of Fourier coefficients of restricted eigenfunctions using the dynamics of the geodesic flows on M and H [18,19]. Here, we impose additional conditions on the dynamics to obtain improvements. A comparable result, [8, Theorem 2] due to Canzani and Galkowski, gives conditions on the recurrent set of SN^*H for which the period integral $\int_H \phi_h d\sigma_H$ is $o\left(h^{\frac{1-k}{2}}\right)$ as $h \to 0^+$. If we take the collection $\psi_h \equiv 1$ we recover their result (see Example 1.9). In Examples 1.4 and 1.5 we demonstrate how Theorem 1.1 can be used in two different ways: to study the generalized Fourier coefficients and to understand the size of the recurrent set.

Next, instead of taking $\{\phi_h\}$ to be exact Laplace eigenfunctions, we further generalize by considering quasimodes of the form

$$(-h^2\Delta_g - 1)\phi_h = o_{L^2(M)}(h) \quad \text{as} \quad h \to 0^+ \qquad \text{and} \qquad \|\phi_h\|_{L^2(M)} = 1. \tag{11}$$

We also assume ϕ_h is compactly microlocalized. That is, there exists a cutoff $\chi \in C_c^\infty(T^*M)$ such that

$$(1 - \mathrm{Op}_h(\chi))\phi_h = O_{C^\infty}(h^\infty).$$

Further, let μ be a defect measure for ϕ_h. We note that μ is supported in S^*M. Similar to [10, Lemma 6 & Remark 3] we use μ to define a measure on Σ^A, μ^A, by

$$\mu^A(\Omega) := \lim_{T \to 0^+} \frac{1}{2T}\mu\left(\bigcup_{|t| \le T} \varphi_t(\Omega)\right) \qquad \text{for } \Omega \subseteq \Sigma^A \text{Borel}. \tag{12}$$

The following theorem gives our main estimate for controlling generalized Fourier coefficients of quasimodes. Theorem 1.1 then follows as a corollary.

Theorem 1.2. *Let $\{\phi_h\}$ be a sequence of compactly microlocalized quasimodes on M satisfying (11) with defect measure μ. Let $H \subset M$ be a closed, embedded, smooth submanifold of dimension k, and let $\{\psi_h\} \subset L^2(H)$ be a sequence of L^2-normalized functions on H with defect measure ν, satisfying $\mathrm{WF}_h(\psi_h) = A \Subset B^*H$. Further, suppose we have a Radon-Nikodym decomposition of the form*

$$\mu^A = f\nu^A + \lambda^A$$

where $\nu^A \perp \lambda^A$ and $f \in L^1(\Sigma^A, \nu^A)$. Then there exists a constant $C_{n,k} > 0$ depending only on n and k such that

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\langle \phi_h, \psi_h\rangle_{L^2(H)}\right| \leq C_{n,k}\left(\int_{\Sigma^A} (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f d\nu^A\right)^{1/2}. \tag{13}$$

This gives much more explicit control on the constant in the standard restriction bound (5) which gives us more insight into when (5) can be improved upon. For example, if $f = 0$ in (13) then we see that we have a little-o improvement. Showing that $f = 0$ under the assumptions of Theorem 1.1 is exactly how we obtain (10). We will see how this bound on the generalized Fourier coefficients can be used to understand the true Fourier coefficients in Example 1.6. In the special case where ν^A is a volume measure on Σ^A, we can refine the proof of the theorem to get a finer bound as follows.

Theorem 1.3. *Let $\{\phi_h\}$ and $\{\psi_h\}$ satisfy the hypothesis of Theorem 1.2. Suppose $\Sigma^A \subseteq N \subseteq T^*M$ where N is a smooth submanifold of dimension $d \in \mathbb{N}$. Further, let m be the volume measure on N induced from the Liouville measure on T^*M. Moreover, suppose we have*

$$\mu^A = f\nu^A + \lambda^A \qquad \text{and} \qquad \nu^A = um$$

where $\nu^A \perp \lambda^A$, $f \in L^1(\Sigma^A, \nu^A)$ and $u \in C(\Sigma^A; \mathbb{R})$. Then there exists a constant $C_{n,k,d} > 0$ depending only on n, k, and d such that

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\langle \phi_h, \psi_h\rangle_{L^2(H)}\right| \leq C_{n,k,d}\int_{\Sigma^A} \sqrt{(1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f}\,|u|\ dm. \tag{14}$$

When we take $\{\psi_h\}$ to be an orthonormal collection of eigenfunctions on H the estimate in Theorem 1.2 allows us to study the growth of the Fourier coefficients of restricted eigenfunctions as we do in Example 1.6. We note that the theorem holds in more generality than this, as the collection $\{\psi_h\}$ does not necessarily consist of eigenfunctions. To the best of our knowledge, the only existing results in this direction are due to Wyman, Xi, and Zelditch [18,19], where the authors obtain asymptotics for sums of the norm-squares of the generalized Fourier coefficients over the joint spectrum. If we take our collection $\{\psi_h\}$ independent of h, we recover the weighted averages result in [8, Theorem 6], which we demonstrate in Example 1.9. We also show in Examples 1.7, 1.8 that we are able to recover the results of [5, Theorem 1.3] and [5, Theorem 1.4]. A similar argument to [8, Remark 1] could be used to show that we can use (14) with H a single point to recover L^∞ bounds of ϕ_h. Using such L^∞ bounds, if in addition we take $\psi \equiv 1$, we could also recover the main result of [20].

1.3. Examples

We next consider some examples to illustrate the use of Theorems 1.1,1.2, and 1.3. In the first two examples, we make use of Theorem 1.1 in two different ways. In the first, we show that the recurrent set has measure zero with respect to ν^A, and hence we obtain a little-o improvement. In the second example, we pick specific collections of ϕ_h and ψ_h and explicitly compute the generalized Fourier coefficients. Then we use Theorem 1.1 to obtain information on the size of the recurrent set. Next in Example 1.6, we show that when we take $\psi_h = \psi_{h_j}$ to be an orthonormal basis of eigenfunctions on H, we can use Theorem 1.2 to find a bound on true Fourier coefficients in the case where ψ_{h_j} has approximately the same frequency as ϕ_h. In the last three examples we use Theorems 1.2 and 1.3 to obtain bounds on the generalized Fourier coefficients in a few different settings. First, we take an explicit collection of ψ_h, second, we assume the collection of ϕ_h's are themselves restricted eigenfunctions, and third, when the collection of ϕ_h does not depend on the semiclassical parameter h.

We will use the coordinates $(x', \bar{x})$ with respect to H such that $H = \{\bar{x} = 0\}$ and work with dual coordinates $(\xi', \bar{\xi})$. In these coordinates we can write

$$\Sigma^A = \left\{(x', \bar{x}, \xi', \bar{\xi}) : |\bar{x}| = 0,\ \ (x', \xi') \in A,\ \ |\xi|_g = 1\right\}.$$

Note that Σ^A is parametrized by $(x', \xi', \bar{\xi})$ and that once (x', ξ') are fixed, the remaining coordinate lives on the $k-1$ dimensional sphere of radius $\sqrt{1 - |\xi'|^2_{g_H}}$. We define the measure ν^A, by

$$\int_{(x',0,\xi',\bar{\xi})\in\Sigma^A} f(x', \xi', \bar{\xi}) d\nu^A(x', \xi', \bar{\xi}) := \int_{(x',\xi')\in A}\int_{\pi^{-1}(x',\xi')} \frac{f(x', \xi', \bar{\xi})}{c_k(1 - |\xi'|^2_{g_H(x')})^{\frac{k-1}{2}}} \\ d\mathrm{vol}\ S^{k-1}_{\sqrt{1-|\xi'|^2_{g_H}}}(\bar{\xi}) d\nu(x', \xi') \tag{15}$$

where c_k is such that $\nu^A(\Sigma^A) = \nu(A) = 1$, π is the projection of Σ^A onto A, and f is any integrable function on Σ^A. The process of constructing ν^A from ν is sometimes referred to as disintegration.

Example 1.4. (Extracting information from the dynamics). Consider the torus $\mathbb{T} = \left\{(x, y) \in \mathbb{R}^2 : (x, y) \sim (x + 1, y) \sim (x, y + 1)\right\}$ and a collection of L^2-normalized eigenfunctions $\{\phi_h\}$ on $\mathbb{T}$. Furthermore, let $H = \{y = 0\}$ and consider the collection

$$\psi_h = C(h)\ \exp\left(\frac{i}{2h}\left(x - \frac{1}{2}\right)\right)\ \exp\left(-\frac{1}{2h}\frac{|x - 1/2|^2}{x(1 - x)}\right)$$

on H where $C(h)$ is such that $\|\psi_h\|_{L^2(H)} = 1$. The wavefront set for $\{\psi_h\}$ is $A = \left\{(x, \xi) : x = 1/2, \xi = 1/2\right\}$, and the defect measure is $\nu = \delta_{\{x=1/2,\xi=1/2\}}$. Therefore

$$\Sigma^A = \left\{(x, y, \xi, \eta) : x = 1/2,\ \ y = 0,\ \ \xi = 1/2,\ \ \eta = \pm\sqrt{3}/2\right\}$$

and ν^A is a point mass at both $\left(1/2, 0, 1/2, \sqrt{3}/2\right)$ and $\left(1/2, 0, 1/2, -\sqrt{3}/2\right)$ with mass 1/2. Geodesics emanating from Σ^A never return back to Σ^A since their directions have

irrational slopes. Therefore the recurrent set of Σ^A is empty and hence $\nu^A(\mathcal{R}_A) = 0$. Thus, Theorem 1.1 implies

$$\left|\langle \phi_h, \psi_h \rangle_{L^2(H)}\right| = o(1) \quad \text{as} \quad h \to 0^+.$$

Example 1.5. (Obtaining information on the recurrent set). Consider the torus $\mathbb{T}$ and the collection of eigenfunctions on $\mathbb{T}$, $\phi_h = e^{\frac{i}{h}(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)}$ where $h = \frac{\sqrt{2}}{4\pi n}$ and $n \in \mathbb{N}$. Furthermore, let $H = \{y = 0\}$ and consider the collection of functions on H, $\psi_h = e^{\frac{i\sqrt{2}}{2h}x}$. Then observe

$$\left|\langle \phi_h, \psi_h \rangle_{L^2(H)}\right| = \int_0^1 \phi_h|_{\{y=0\}} \overline{\psi}_h \, dx = \int_0^1 e^{\frac{i\sqrt{2}}{2h}x} e^{-\frac{i\sqrt{2}}{2h}x} dx = 1. \tag{16}$$

One can check that $A = \mathrm{WF}_h\{\psi_h\} = \left\{(x, \xi) : \xi = \sqrt{2}/2\right\}$, $\nu = \delta_{\left\{\xi = \sqrt{2}/2\right\}} dx$,

$$\Sigma^A = \left\{(x, y, \xi, \eta) : y = 0, \xi = \sqrt{2}/2, \eta = \pm\sqrt{2}/2\right\},$$

and $\nu^A = dx$, where we use (ξ, η) to denote the dual coordinates to (x,y). It is clear from (16) that $\left|\langle \phi_h, \psi_h \rangle_{L^2(H)}\right| \neq o(1)$ as $h \to 0^+$ and thus Theorem 1.1 implies $\nu^A(\mathcal{R}_A) > 0$.

For this example we can actually compute the recurrent set since the geometry is quite simple. Note that geodesics emanating from Σ^A return to their starting point after time $n\sqrt{2}$, where $n \in \mathbb{Z}$. Therefore every point of Σ^A is recurrent and so $\nu^A(\mathcal{R}_A) = \nu^A(\Sigma^A) = 1$.

Example 1.6. (True Fourier coefficients). Take $\psi_h = \psi_{h_j}$ an orthonormal basis of eigenfunctions on H satisfying $-h_j{}^2 \Delta_{g_H} \psi_{h_j} = E(h_j)^2 \psi_{h_j}$. Assume $E(h_j) \to c \in (0,1)$ as $h_j \to 0^+$. Then $\mathrm{WF}_h(\psi_{h_j}) \subseteq S_c^* H = \left\{(x', \xi') \in T^*H : |\xi'|^2_{g_H(x')} = c^2\right\}$. Since $c \in (0,1)$ we see that $A = \mathrm{WF}_h(\psi_{h_j}) \Subset B^*H$ and hence we can apply Theorem 1.2 to obtain

$$\limsup_{h \to 0^+} h^{\frac{k-1}{2}} \left|\langle \phi_h, \psi_{h_j} \rangle_{L^2(H)}\right| \leq C_{n,k}(1 - c^2)^{\frac{k-2}{4}} \left(\int_{\Sigma^A} f d\nu^A\right)^{1/2} \leq C_{n,k}(1 - c^2)^{\frac{k-2}{4}}$$

where we used that $\|f\|_{L^1(\Sigma^A, \nu^A)} \leq 1$.

Example 1.7. (Reproducing [5, Theorem 1.4]). Consider the simple case where we have a surface containing a smooth closed curve γ parametrized by t. We consider

$$\limsup_{h \to 0^+} \left|\int_\gamma \phi_h(\gamma(t)) e^{-i\alpha(h)t} dt\right|$$

for ϕ_h eigenfunctions, and some function α satisfying $0 \leq \alpha(h)h < c < 1$, $\lim_{h \to 0^+} \alpha(h)h = \alpha_0$, and such that $2\pi\alpha(h)$ is an integer multiple of $|\gamma|$. We note that this is a semiclassical version of [5, Theorem 1.4]. To apply our estimate, we need to normalize the exponential, thus we instead consider

$$|\gamma|^{1/2}\limsup_{h\to 0^+}\left|\int_\gamma \phi_h(\gamma(t))\frac{e^{-i\alpha(h)t}}{|\gamma|^{1/2}}dt\right| =: |\gamma|^{1/2}\limsup_{h\to 0^+}|\langle \phi_h, \psi_h\rangle_{L^2(\gamma)}|.$$

We note that the collection $\{\psi_h\} = \left\{e^{i\alpha(h)t}|\gamma|^{-1/2}\right\}$ has a defect measure $\nu = |\gamma|^{-1}\delta_{\{\tau=\alpha_0\}}dt$ where τ is dual to t and dt denotes the Lebesgue measure on γ. Furthermore, the wavefront set $A = \mathrm{WF}_h(\psi_h) = \{\tau = \alpha_0\}$. Now, using s to denote the coordinate on M normal to γ and σ dual to s, we have

$$\Sigma^A = \left\{(t, s, \tau, \sigma) : s = 0, \tau = \alpha_0, \sigma = \pm\sqrt{1-\alpha_0^2}\right\}$$

which is one dimensional. Furthermore we compute $\nu^A = |\gamma|^{-1}dt$ where dt is Lebesgue on Σ^A.

Thus, applying Theorem 1.3, we have that there is a $C > 0$ such that

$$|\gamma|^{1/2}\limsup_{h\to 0^+}|\langle \phi_h, \psi_h\rangle_{L^2(\gamma)}| \le C|\gamma|^{1/2}\int_{\Sigma^A}\sqrt{f(1-|\tau|^2)^{-1/2}}|\gamma|^{-1}dt.$$

Next, using Hölder's inequality and that $\|f\|_{L^1(\Sigma^A,\nu^A)} \le \mu^A(\Sigma^A) \le 1$ we obtain

$$\begin{aligned}|\gamma|^{1/2}\limsup_{h\to 0^+}|\langle \phi_h, \psi_h\rangle_{L^2(\gamma)}| &\le C|\gamma|^{1/2}\left(\int_{\Sigma^A}|f||\gamma|^{-1}dt\right)^{1/2}\left(\int_{\Sigma^A}(1-\alpha_0^2)^{-1/2}|\gamma|^{-1}dt\right)^{1/2}\\ &= C|\gamma|^{1/2}(f(^{1/2}_{L^1(\Sigma^A,\nu^A)}(1-\alpha_0^2)^{-1/4} \le \frac{C|\gamma|^{1/2}}{(1-\alpha_0^2)^{1/4}}.\end{aligned}$$

Finally since $\alpha_0 \le c < 1$ we have

$$\limsup_{h\to 0^+}\left|\int_\gamma \phi_h(\gamma(t))e^{-i\alpha(h)t}dt\right| \le \frac{C|\gamma|^{1/2}}{(1-\alpha_0^2)^{1/4}} \le \frac{C|\gamma|^{1/2}}{(1-c^2)^{1/4}}. \tag{17}$$

We see from (17) that we are able to bound the Fourier coefficients by $C|\gamma|^{1/2}$ which differs from Xi's bound of $C|\gamma|$ stated in (4). This discrepancy is because our method uses L^2 norms, while Xi uses L^1 norms. Actually, we believe that the wavefront set condition in Theorem 1.2 can be weakened to the condition that $\psi_h = o(h^{1/4})$ near S^*H. Then we could take $\alpha(h)$ not necessarily an integer multiple of $|\gamma|/2\pi$ in this example. This would then be a more general version of what Xi considered in [5, Theorem 1.4].

Example 1.8. (Reproducing [5, Theorem 1.3]). As in [5, Theorem 1.3] we consider the case where ϕ_h are eigenfunctions on M and ψ_h are the restrictions of a eigenfunctions on M to a hypersurface H. Let

$$\psi_h = \frac{\Psi_h|_H}{\|\Psi_h\|_{L^2(H)}} \qquad \text{where } \Psi_h \text{ satisfies } (-h^2\Delta_g - \alpha(h)^2)\Psi_h = 0 \text{ on } M.$$

We also assume that $0 \le \alpha(h) < c < 1$ as in [5, Theorem 1.3] and suppose $\alpha(h) \to \alpha_0$, taking a subsequence if necessary. Since $\mathrm{WF}_h(\Psi_h) = \left\{|\xi|_g = \alpha_0\right\}$ one can see that

$\mathrm{WF}_h(\psi_h) \subseteq \left\{ |\xi'|_{g_H} \leq \alpha_0 \right\}$ where we use coordinates $x = (x', \bar{x})$ on M such that $H = \{\bar{x} = 0\}$, and dual coordinates $\xi = (\xi', \bar{\xi})$. Applying Theorem 1.2 we have

$$\limsup_{h \to 0^+} \left| \langle \phi_h, \psi_h \rangle_{L^2(H)} \right| \leq C_{n,1} \left(\int_{\Sigma^A} f (1 - |\xi'|^2_{g_H(x')})^{-1/2} d\nu^A \right)^{1/2}.$$

Furthermore, since $|\xi'|_{g_H} \leq \alpha_0$ on Σ^A, $\|f\|_{L^1(\Sigma^A, \nu^A)} \leq 1$, and $\alpha_0 \leq c < 1$, we obtain

$$\limsup_{h \to 0^+} \left| \langle \phi_h, \psi_h \rangle_{L^2(H)} \right| \leq C_{n,1} \|f\|^{1/2}_{L^1(\Sigma^A, \nu^A)} (1 - \alpha_0^2)^{-1/4} \leq \frac{C_{n,1}}{(1 - c^2)^{1/4}}.$$

Thus, we find that for h small

$$\left| \langle \phi_h, \Psi_h \rangle_{L^2(H)} \right| \leq \frac{C_{n,1} \|\Psi_h\|_{L^2(H)}}{(1 - c^2)^{1/4}} \leq \frac{C \left(1 + \frac{\alpha(h)}{h}\right)^{1/4}}{(1 - c^2)^{1/4}}$$

where we use [21, Theorem 3] to bound $\|\Psi_h\|_{L^2(H)}$. In this case we recover the bound in [5, Theorem 1.3 (1.23)].

Example 1.9 (Reproducing [8, Theorem 6]). We study the case where our collection $\{\psi_h\}$ does not depend on h. We consider

$$\limsup_{h \to 0^+} h^{\frac{k-1}{2}} \left| \langle \phi_h, w \rangle_{L^2(H)} \right|$$

where ϕ_h are compactly microlocalized quasimodes, and $w \in C^\infty(H)$ is independent of h. We must normalize w to apply the theorem. We instead consider $\mathbf{w} = w \|w\|^{-1}_{L^2(H)}$. A short calculation shows that $\nu = \|w\|^{-2}_{L^2(H)} |w(x')|^2 \delta_{\{\xi' = 0\}} dx'$ is the defect measure for $\mathbf{w}$ where we use coordinates $x = (x', \bar{x})$ on M such that $H = \{\bar{x} = 0\}$, and dual coordinates $\xi = (\xi', \bar{\xi})$. Furthermore we observe that $A = \mathrm{WF}_h(\mathbf{w}) = N^*H$. Therefore $\Sigma^A = SN^*H$, which is $n - 1$ dimensional. Next we note

$$\nu^A = \|w\|^{-2}_{L^2(H)} |w(x')|^2 d\sigma_{SN^*H}$$

where σ_{SN^*H} is the measure on SN^*H induced by the Sasaki metric on T^*M. Applying Theorem 1.3 we have

$$\begin{aligned} \limsup_{h \to 0^+} h^{\frac{k-1}{2}} \left| \langle \phi_h, \mathbf{w} \rangle_{L^2(H)} \right| &\leq \frac{C_{n,k,n-1}}{\|w\|^2_{L^2(H)}} \int_{SN^*H} \sqrt{f (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}} |w|^2 d\sigma_{SN^*H} \\ &= \frac{C_{n,k}}{\|w\|^2_{L^2(H)}} \int_{SN^*H} \sqrt{f} |w|^2 d\sigma_{SN^*H} \end{aligned}$$

since $\xi' = 0$ on SN^*H. Note that in the notation of Theorem 1.3 we have $u = \|w\|^2_{L^2(H)} |w|^2$. In addition, since the dimension of Σ^A is $n - 1$ we just have that our constant depends on n and k. Thus, for the inner product with w, we have

$$\limsup_{h \to 0^+} h^{\frac{k-1}{2}} \left| \langle \phi_h, w \rangle_{L^2(H)} \right| \leq \frac{C_{n,k}}{\|w\|_{L^2(H)}} \int_{SN^*H} \sqrt{f} |w|^2 d\sigma_{SN^*H} = C_{n,k} \int_{SN^*H} \sqrt{f \|w\|^{-2}_{L^2(H)} |w|^2} |w| d\sigma_{SN^*H}.$$

The last equality matches with the bound in [8, Theorem 6], since under the square root is the Radon-Nikodym derivative of μ^A with respect to σ_{SN^*H}, which in this case is $fu = f\|w\|^{-2}_{L^2(H)}|w|^2$.

1.4. Organization of the paper

The remaining sections of our paper are organized as follows: Section 2 contains the proofs of Theorems 1.2 and 1.3 assuming a key quantitative estimate given in Proposition 2.1. Section 3 contains a few of the more technical lemmas, which focus on localizing to Σ^A, needed to prove Proposition 2.1. Section 3 can be omitted on a first read. Section 4 is dedicated to the proof of Proposition 2.1 in which the key idea is to first localize the generalized Fourier coefficients to geodesic tubes emanating from Σ^A. In Section 5 we define the recurrent set of Σ^A and use Theorem 1.2 to prove Theorem 1.1.

2. Proof of Theorem 1.2 and Theorem 1.3

In this section we present the proofs of Theorems 1.2 and 1.3. We first introduce notation that will be used throughout the paper. Then we state the main estimate, Proposition 2.1, which is central to the proof of Theorem 1.2, but we save its proof for Section 4. Assuming the proposition, we prove Theorem 1.2 and then modify its proof to obtain Theorem 1.3.

Throughout this section we assume $\{\phi_h\}$ is a compactly microlocalized collection of quasimodes on M satisfying (11) with defect measure μ. We also assume that the sequence of functions $\{\psi_h\}$ on H have defect measure ν and satisfy (7) and (8).

2.1. Preliminaries

We let $P(h) := -h^2\Delta_g - 1$ with principal symbol $p(x,\xi) = |\xi|_g^2 - 1$. Then we can rewrite the quasimode equation for ϕ_h as, $P(h)\phi_h = o_{L^2(M)}(h)$. Using properties of defect measures, we know that

$$\text{supp } \mu \subseteq \{p = 0\} = \left\{|\xi|_g^2 = 1\right\} = S^*M \subseteq T^*M,$$

so $\{\phi_h\}$ is localized near S^*M. Also, since $A = \mathrm{WF}_h(\psi_h)$, we note that Σ^A, defined in (9), can be thought of points where ϕ_h are concentrated which project onto where ψ_h are concentrated. Therefore, it is reasonable to expect contributions from $\left|\langle\phi_h, \psi_h\rangle_{L^2(H)}\right|$ to be small away from Σ^A. We prove this in Lemma 3.2.

We use H_p to denote the Hamilton vector field associated to p and $\varphi_t := \exp(tH_p)$ to denote the geodesic flow. Let $\mathscr{L} \subset T^*M$ be a smooth, embedded hypersurface containing Σ^A which is transversal to the flow, so

$$H_p \notin T\mathscr{L} \qquad \text{and} \qquad \Sigma^A \subset \mathscr{L}$$

as depicted in Figure 1. For $\rho \in \mathscr{L}$ and $R > 0$ define

$$B_{\mathscr{L}}(\rho, R) := B(\rho, R) \cap \mathscr{L}.$$

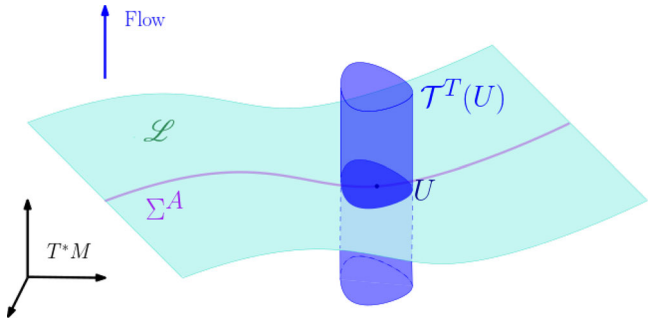

Figure 1. A schematic of the hypersurface $\mathscr{L}$ transverse to the flow containing Σ^A and a tube $\mathcal{T}^T(U)$, which is constructed by flowing $U \subseteq \mathscr{L}$ for time T.

We use the geodesic flow to form tubes in T^*M by flowing out of $\mathscr{L}$. For time $T > 0$ and $U \subset \mathscr{L}$ we define the tube

$$\mathcal{T}^T(U) := \bigcup_{|t| \leq T} \varphi_t(U). \tag{18}$$

Sometimes when U is a ball, we will write $\mathcal{T}^T_{\rho_0}(R) := \mathcal{T}^T(B_{\mathscr{L}}(\rho_0, R))$. For $U \subseteq \Sigma^A$ and $\varepsilon > 0$ we define

$$U(\varepsilon) := \bigcup_{\rho \in U} B_{\mathscr{L}}(\rho, \varepsilon) \subseteq \mathscr{L} \tag{19}$$

which is a version of U that has been thickened by ε into $\mathscr{L}$. We denote the "flowout" of Σ^A

$$\Lambda^T(\varepsilon) := \mathcal{T}^T(\Sigma^A(\varepsilon))$$

where $\Sigma^A(\varepsilon)$ denotes the fattened version of Σ^A defined in (19). Finally, define $\Gamma_H : C^\infty(M) \to C^\infty(H)$ which restricts functions on M to H.

To prove Theorem 1.2 we begin by using a cutoff χ to localize to the respective supports of our mutually singular measures, ν^A and λ^A. Thus we seek to understand how terms like $\left|\langle \mathrm{Op}_h(\chi)\phi_h, \psi_h \rangle_{L^2(H)}\right|$ grow as $h \to 0^+$. We control such terms in the following proposition.

Proposition 2.1. *There exist $T_0, R_0 > 0$ such that for all $0 < T \leq T_0$, $0 < \varepsilon \leq R_0$, and $\chi \in C_c^\infty(T^*M)$ with $H_p\chi \equiv 0$ on $\Lambda^{2T}(\varepsilon)$, there exists a constant $C_{n,k} > 0$ depending only on n and k such that*

$$\limsup_{h \to 0^+} h^{\frac{k-1}{2}} \left|\langle \mathrm{Op}_h(\chi)\phi_h, \psi_h \rangle_{L^2(H)}\right| \leq C_{n,k} \left(\nu^A(\mathrm{supp}\ \chi|_{\Sigma^A}) \int_{\Sigma^A} \left(1 - |\xi'|^2_{g_H(x')}\right)^{\frac{k-2}{2}} |\chi|^2 d\mu^A \right)^{1/2}.$$

To use Proposition 2.1, we need to work with cutoff functions $\chi \in C_c^\infty(T^*M)$ in which are flow invariant, meaning $H_p\chi \equiv 0$ on $\Lambda^{2T}(\varepsilon)$. In the following lemma, we show that a cutoff $\tilde{\chi} \in C_c^\infty(\Sigma^A(\varepsilon))$ can be extended to T^*M in this way.

Lemma 2.2. *For* $\varepsilon > 0$ *and* $\tilde{\chi} \in C_c^\infty(\Sigma^A(\varepsilon); [0,1])$ *there exists an extension* $\chi \in C_c^\infty(T^*M; [0,1])$ *such that* $\operatorname{supp} \chi \subseteq \Lambda^{3T}(\varepsilon)$ *and* $H_p\chi \equiv 0$ *on* $\Lambda^{2T}(\varepsilon)$.

Proof. Since $\Sigma^A(\varepsilon) \subseteq \mathscr{L}$ is transverse to the flow, for T small enough, we can use the map $X : (-4T, 4T) \times \mathscr{L} \to T^*M$ defined by

$$X(t, \rho) = \varphi_t(\rho)$$

as coordinates. Let $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset (-3T, 3T)$ and $f \equiv 1$ on $[-2T, 2T]$. Then take $\chi = (X^{-1})^*(f(t)\tilde{\chi})$. □

We first prove Theorem 1.2 assuming the proposition holds. The proof of Proposition 2.1 is saved for Section 4.

2.2. Proof of Theorem 1.2

Fix $\delta > 0$. Since ν^A and λ^A are mutually singular Radon measures on Σ^A there exists $K_\delta \subseteq \Sigma^A$ compact and $U_\delta \subseteq \Sigma^A$ open and containing K_δ such that

$$\nu^A(U_\delta) \leq \delta \qquad \text{and} \qquad \lambda^A(\Sigma^A \setminus K_\delta) \leq \delta.$$

Let $\tilde{\kappa}_\delta \in C_c^\infty(\Sigma^A; [0,1])$ such that

$$\tilde{\kappa}_\delta \equiv 1 \text{ on } K_\delta \qquad \text{and} \qquad \operatorname{supp} \tilde{\kappa}_\delta \subseteq U_\delta.$$

Furthermore, let $\kappa_\delta \in C_c^\infty(T^*M; [0,1])$ be a flow invariant extension of $\tilde{\kappa}_\delta$ as defined in Lemma 2.2. We split the inner product

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\langle \phi_h, \psi_h\rangle\right| \leq \limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left(\left|\langle \mathrm{Op}_h(\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right| + \left|\langle \mathrm{Op}_h(1-\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right|\right). \tag{20}$$

Next, we use Proposition 2.1 with $\chi = \kappa_\delta$ on the first term to obtain

$$\begin{aligned}\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\langle \mathrm{Op}_h(\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right| &\leq C_{n,k}\left(\nu^A(\operatorname{supp} \kappa_\delta|_{\Sigma^A}) \int_{\Sigma^A} (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} |\kappa_\delta|^2 d\mu^A\right)^{1/2} \\ &\leq C\delta^{1/2}.\end{aligned} \tag{21}$$

The last inequality follows from the fact that $\nu^A(\operatorname{supp} \kappa_\delta|_{\Sigma^A}) = \nu^A(\operatorname{supp} \widetilde{\kappa_\delta}) \leq \nu^A(U_\delta) \leq \delta$.

Next, to bound the second term in (20), we use Proposition 2.1 with $\chi = 1 - \kappa_\delta$ and the Radon-Nikodym decomposition of our measures, $\mu^A = f\nu^A + \lambda^A$. We have

$$\begin{aligned}&\limsup_{h\to 0^+} h^{\frac{k-1}{2}} \left|\langle \mathrm{Op}_h(1-\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right| \\ &\quad \leq C_{n,k}\nu^A(\operatorname{supp} (1-\kappa_\delta)|_{\Sigma^A})^{1/2}\left(\int_{\Sigma^A} (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} |1-\kappa_\delta|^2 (f d\nu^A + d\lambda^A)\right)^{1/2} \\ &\quad \leq C_{n,k}\nu^A(\Sigma^A)^{1/2}\left(\int_{\Sigma^A} (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f d\nu^A + C\delta\right)^{1/2},\end{aligned} \tag{22}$$

where, in the last line, we used that $\tilde{\kappa}_\delta \equiv 1$ on K_δ and so $(1-\kappa_\delta)|_{\Sigma^A}$ is supported on $\Sigma^A \setminus K_\delta$. Thus, since $\lambda^A(\Sigma^A \setminus K_\delta) \leq \delta$, the $d\lambda^A$ integral is bounded by $C\delta$. Since

$\nu^A(\Sigma^A) = 1$, and (21) and (22) hold for all $\delta > 0$, combining the above we have

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \phi_h, \psi_h\rangle\right| \le C_{n,k}\left(\int_{\Sigma^A}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f d\nu^A\right)^{1/2}$$

giving the bound in (13) as desired. □

2.3. Proof of Theorem 1.3

Let K_δ, U_δ, and κ_δ be as in the proof of Theorem 1.2. We similarly split the inner product:

$$\begin{aligned}\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \phi_h, \psi_h\rangle\right| &\le \limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left(\left|\langle \mathrm{Op}_h(\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right| + \left|\langle \mathrm{Op}_h(1-\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right|\right)\\ &=: I + II.\end{aligned}$$

Then applying Proposition 2.1 to I, we have

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \phi_h, \psi_h\rangle\right| \le C\delta^{1/2} + II.$$

By the Besicovitch-Federer Covering Lemma, there exists a constant $c_d > 0$ depending only on d, the dimension of Σ^A and R so that for all $0 < r < R$, there exist a cover of open balls $\{B(\rho_1, r), \ldots, B(\rho_{n(r)}, r)\} = \{B_1, \ldots, B_{n(r)}\} \subseteq \Sigma^A$ of radius r centered at $\{\rho_1, \ldots, \rho_{n(r)}\}$ with

$$n(r) \le c_d r^{-d} \qquad \text{and} \qquad m(B_j) \le c_d r^d$$

where m is Lebesgue on $N \supseteq \Sigma^A$. Furthermore $\Sigma^A \subseteq \bigcup_{j=1}^{n(r)} B_j$ and each point in Σ^A lies in at most c_d balls. Then we let $\tilde{\theta}_j$ be a partition of unity associated to $B_j(\varepsilon)$ and θ_j the flowed extensions into T^*M such that $\operatorname{supp} H_p\theta_j \subseteq \mathcal{T}^{3T}(B_j(\varepsilon)) \setminus \mathcal{T}^{2T}(B_j(\varepsilon))$ and $\sum_{j=1}^{n(r)}\theta_j \equiv 1$ on $\Lambda^{2T}(\varepsilon)$. Define $\Theta := \sum_{j=1}^{n(r)}\theta_j$. Next we split II :

$$\begin{aligned}&\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(1-\kappa_\delta)\phi_h, \psi_h\rangle_{L^2(H)}\right|\\ &\quad\le \limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left(\left|\langle \mathrm{Op}_h(\Theta(1-\kappa_\delta))\phi_h, \psi_h\rangle_{L^2(H)}\right| + \left|\langle \mathrm{Op}_h((1-\Theta)(1-\kappa_\delta))\phi_h, \psi_h\rangle_{L^2(H)}\right|\right)\\ &\quad\le \limsup_{h\to 0^+}\sum_{j=1}^{n(r)} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\theta_j(1-\kappa_\delta))\phi_h, \psi_h\rangle_{L^2(H)}\right|\\ &\qquad + \limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h((1-\Theta)(1-\kappa_\delta))\phi_h, \psi_h\rangle_{L^2(H)}\right|. \qquad (23)\end{aligned}$$

Taking $h \to 0^+$ we can apply Proposition 2.1 to both terms. Using the support properties of Θ, we find that the second term in (23) goes to 0. For the first term in (23), we have

$$\begin{aligned}
&\limsup_{h\to 0^+}\sum_{j=1}^{n(r)} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\theta_j(1-\kappa_\delta))\phi_h,\psi_h\rangle_{L^2(H)}\right| \\
&\quad\le C_{n,k}\sum_{j=1}^{n(r)}\nu^A(\mathrm{supp}\ (\theta_j(1-\kappa_\delta))|_{\Sigma^A})^{1/2}\left(\int_{\Sigma^A}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}|\theta_j(1-\kappa_\delta)|^2 d\mu^A\right)^{1/2} \\
&\quad\le C_{n,k}\sum_{j=1}^{n(r)}\left(\int_{B_j} u\ dm\right)^{1/2}\left(\int_{\Sigma^A}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}|\theta_j(1-\kappa_\delta)|^2(fu\ dm+d\lambda^A)\right)^{1/2}, \qquad (24)
\end{aligned}$$

where we used that supp $\theta_j|_{\Sigma^A}\subseteq B_j$ and $\nu^A = um$. As in the proof of Theorem 1.2, the $d\lambda^A$ integral can be bounded by $C\delta$, and we thus focus on the dm integral. Since u is uniformly continuous on Σ^A, we can find an $R>0$ such that if $\rho\in B(\rho_j,R)$ then $\left|u(\rho)-u(\rho_j)\right|\le\delta$. Therefore,

$$\int_{B_j} u\ dm \le \int_{B_j}(u(\rho_j)+\delta)\ dm = (u(\rho_j)+\delta)m(B_j)\le c_d r^d(u(\rho_j)+\delta)$$

for each B_j provided $r<R$ is small enough. Thus we can bound (24) by

$$C_{n,k,d}\ r^{d/2}\sum_{j=1}^{n(r)}\left(\int_{\Sigma^A}(u(\rho_j)+\delta)(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}|\theta_j(1-\kappa_\delta)|^2 fu\ dm\right)^{1/2}+C\delta^{1/2}.$$

Since supp $\theta_j|_{\Sigma^A}\subseteq B_j$ we can use the bound $u(\rho_j)+\delta\le u(\rho)+2\delta$. Continuing, we find

$$\begin{aligned}
&C_{n,k,d}\ r^{d/2}\sum_{j=1}^{n(r)}\left(\int_{\Sigma^A}(u(\rho_j)+\delta)(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}|\theta_j(1-\kappa_\delta)|^2 fu\ dm\right)^{1/2} \\
&\quad\le C_{n,k,d}\sum_{j=1}^{n(r)} m(B_j)^{1/2}\left(\int_{\Sigma^A}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}|\theta_j|^2 fu^2\ dm\right)^{1/2} \\
&\qquad + C_{n,k,d}\ r^{d/2}n(r)^{1/2}\left(\int_{\Sigma^A}\delta fu\ dm\right)^{1/2} \\
&\quad\le C_{n,k,d}\int_{\Sigma^A}\sum_{j=1}^{n(r)}\left(\frac{1}{m(B_j)}\int_{B_j}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} fu^2\ dm\right)^{1/2}\mathbb{1}_{B_j}\ dm + C\delta^{1/2}.
\end{aligned}$$

Therefore, combining the above steps we have

$$\begin{aligned}
&\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle\phi_h,\psi_h\rangle_{L^2(H)}\right| \qquad (25)\\
&\le C\delta^{1/2}+C_{n,k,d}\int_{\Sigma^A}\sum_{j=1}^{n(r)}\left(\frac{1}{m(B_j)}\int_{B_j}(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} fu^2\ dm\right)^{1/2}\mathbb{1}_{B_j}\ dm,
\end{aligned}$$

and since the left side does not depend on r, we may bound $\limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle\phi_h,\psi_h\rangle_{L^2(H)}\right|$ by the limit of the right side of (25) as $r\to 0$. We will use the Dominated Convergence Theorem to bring the limit inside the integral. To simplify our computations, we will write

$$F(\rho) := (1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f u^2.$$

First we calculate the limit of the integrand in (25). Using the Lebesgue Differentiation Theorem [22, Theorem 3.21] and that each point in Σ^A lies in finitely many balls of the cover, we see that

$$\limsup_{r\to 0} \sum_{j=1}^{n(r)} \left(\frac{1}{m(B_j)} \int_{B_j} F \ dm \right)^{1/2} \mathbb{1}_{B_j} \leq C_{n,k,d}\sqrt{F} = C_{n,k,d}|u|\sqrt{(1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f} \qquad m - a.e.$$

Lastly, to justify the use of the Dominated Convergence Theorem we need to show that the integrand in (25) is dominated by an L^1 function. We note that

$$\sum_{j=1}^{n(r)} \left(\frac{1}{m(B_j)} \int_{B_j} F \ dm \right)^{1/2} \mathbb{1}_{B_j} \leq \sum_{j=1}^{n(r)} \sqrt{HF(\rho_j)}\mathbb{1}_{B_j} \leq C\sqrt{HF(\rho)} \qquad m - a.e.$$

where H denotes the Hardy-Littlewood Maximal Functional. Furthermore, by the Maximal Theorem [22, Theorem 3.17] there exists a constant C so that for all $t > 0$

$$m\big(\{\rho \in \Sigma^A : HF(\rho) \geq t\}\big) \leq \frac{C}{t}$$

which implies that $\sqrt{HF} \in L^1(\Sigma^A, m)$. To see this we compute

$$\begin{aligned}
\int_{\Sigma^A} \sqrt{HF(\rho)} \ dm = \int_{\Sigma^A} \int_0^{\sqrt{HF(\rho)}} dt \ dm &= \int_{\Sigma^A} \int \mathbb{1}_{\{0 \leq t \leq 1\}} \ dt \ dm + \int_{\Sigma^A} \int \mathbb{1}_{\left\{1 \leq t \leq \sqrt{HF(\rho)}\right\}} \ dt \ dm \\
&\leq m(\Sigma^A) + \int_1^\infty \int_{\Sigma^A} \mathbb{1}_{\left\{\sqrt{HF(\rho)} \geq t\right\}} \ dm \ dt \\
&= C + \int_1^\infty m\left(\rho \in \Sigma^A : \left\{\sqrt{HF(\rho)} \geq t\right\}\right) \ dt \\
&\leq C + \int_1^\infty \frac{C}{t^2} \ dt < \infty,
\end{aligned}$$

where we use the Fubini-Tonelli Theorem to change the order of integration in the second line. Therefore, we are justified in applying the Dominated Convergence Theorem and we conclude that

$$\limsup_{h\to 0^+} h^{\frac{k-1}{2}} |\langle \phi_h, \psi_h \rangle| \leq C\delta^{1/2} + C_{n,k,d} \int_{\Sigma^A} |u| \sqrt{(1 - |\xi'|^2_{g_H(x')})^{\frac{k-2}{2}} f} \ dm$$

which holds for all $\delta > 0$ and hence we obtain (14). □

3. Localizing to Σ^A

We first present two technical results which will be needed in the proof of Proposition 2.1. First, Lemma 3.1 tells us how to construct a cutoff $\tilde{\chi} \in C_c^\infty(T^*H)$ such that $\mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi)\phi_h$ is $O(h^\infty)$. Next, Lemma 3.2 shows that the contributions of the inner product are negligible away from Σ^A. This section can be omitted on a first read. Once again, throughout this section we assume $\{\phi_h\}$ is a compactly microlocalized

collection of quasimodes on M satisfying (11) with defect measure μ. We also assume that the sequence of functions $\{\psi_h\}$ on H have defect measure ν and satisfy (7) and (8).

The following lemma gives a condition for which the composition $\mathrm{Op}_h(\tilde{\chi})\Gamma_H\mathrm{Op}_h(\chi)\phi_h$ is $O(h^\infty)$ where $\tilde{\chi} \in C_c^\infty(T^*H)$.

Lemma 3.1. *Let $\tilde{\chi} \in C^\infty(T^*H;[0,1])$ and $\chi \in C_c^\infty(T^*M;[0,1])$. Then*

$$\mathrm{Op}_h(\tilde{\chi})\Gamma_H\mathrm{Op}_h(\chi)\phi_h = O_{L^\infty(H)}(h^\infty)$$

provided $\{\rho \in T_H^*M : \rho \in \operatorname{supp}\chi, \pi_{T^*H}\rho \in \operatorname{supp}\tilde{\chi}\} = \emptyset$.

Proof. We write $\mathrm{Op}_h(\tilde{\chi})\Gamma_H\mathrm{Op}_h(\chi)\phi_h$ in coordinates:

$$\begin{aligned}&\mathrm{Op}_h(\tilde{\chi})\Gamma_H\mathrm{Op}_h(\chi)\phi_h\\ &\quad= (2\pi h)^{k-2n}\int\int\int e^{\frac{i}{h}\langle x',\eta'\rangle}e^{-\frac{i}{h}\langle y,\xi\rangle}\phi_h(y)\left(\int e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle}\tilde{\chi}(z',\eta')\chi(z',0,\xi',\bar{\xi})dz'\right)dy\ d\xi\ d\eta'.\end{aligned}$$

Consider the operator

$$\mathcal{L} := \frac{h\langle \xi'-\eta', D_{z'}\rangle}{|\xi'-\eta'|^2}$$

which satisfies

$$\mathcal{L}\ e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle} = e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle}.$$

We use $\mathcal{L}$ to repeatedly integrate by parts in the inner most integral. This is only possible provided $\xi' \neq \eta'$ on the support of $\tilde{\chi}\chi|_{\bar{x}=0}$. However, we assumed that there are no points such that $(z',0,\xi',\bar{\xi}) \in \operatorname{supp}\chi$ and $(z',\xi') \in \operatorname{supp}\tilde{\chi}$. Thus integrating by parts N times using $\mathcal{L}$ in the dz' integral we have

$$\begin{aligned}&\left|\int e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle}\tilde{\chi}(z',\eta')\chi(z',0,\xi',\bar{\xi})dz'\right|\\ &= \left|\int \mathcal{L}^N e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle}\tilde{\chi}(z',\eta')\chi(z',0,\xi',\bar{\xi})dz'\right|\\ &= \left(\frac{h}{|\xi'-\eta'|^2}\right)^N\left|\int e^{\frac{i}{h}\langle z',\xi'-\eta'\rangle}\sum_{i_1,\dots,i_N}(\xi'_{i_1}-\eta'_{i_1})\dots(\xi'_{i_N}-\eta'_{i_N})D_{z'_{i_1},\dots,z'_{i_N}}(\tilde{\chi}\chi)dz'\right|\\ &\le \left(\frac{h}{|\xi'-\eta'|^2}\right)^N\int C_N|\xi'-\eta'|^N\left|D_{z'}^N(\tilde{\chi}\chi)\right|dz' = C_N\left(\frac{h}{|\xi'-\eta'|}\right)^N\int|D_{z'}^N(\tilde{\chi}\chi)|dz'.\end{aligned}$$

Furthermore we have

$$\begin{aligned}&\left|\mathrm{Op}_h(\tilde{\chi})\Gamma_H\mathrm{Op}_h(\chi)\phi_h\right|\\ &\quad\le C_N h^{N+k-2n}\int\int\int\frac{|\phi_h(y)|}{|\xi'-\eta'|^N}\left(\int|D_{z'}^N(\tilde{\chi}(z',\eta')\chi(z',0,\xi',\bar{\xi}))|dz'\right)dy\ d\xi\ d\eta'\\ &\quad= C_N h^{N+k-2n}\|\phi_h\|_{L^1(M)}\int\int\int f_N(\xi,\eta')|\xi'-\eta'|^{-N}d\xi\ d\eta'\end{aligned}$$

where $f_N = \int|D_{z'}^N(\tilde{\chi}(z',\eta')\chi(z',0,\xi',\bar{\xi}))|dz'$ is smooth and compactly supported in ξ since $\chi \in C_c^\infty(T^*M)$. Furthermore, since $\tilde{\chi}\chi$ is supported away from $\xi' = \eta'$ so is f_N.

Also, since $\tilde{\chi}\chi$ is smooth and compactly supported in z', we know the dz' integral is finite. Moreover, for N large enough $|\xi' - \eta'|^{-N}$ is highly localized in $\left\{|\xi' - \eta'| \leq 1\right\}$. The compactness in ξ and this localization is enough to see that the last integral is finite and hence we have

$$\left|\mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi)\phi_h\right| \leq C_{N,M} h^{N+k-2n}$$

and hence $\mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi)\phi_h = O(h^\infty)$ as desired. □

Next, we show that away from Σ^A the contributions from the generalized Fourier coefficients are negligible.

Lemma 3.2. *Let $\chi_{S^*M} \in C_c^\infty(T^*M)$ such that $\chi_{S^*M} \equiv 1$ on a neighborhood of S^*M and supported in a neighborhood of S^*M. Similarly let $\chi_A \in C_c^\infty(T^*H)$ such that $\chi_A \equiv 1$ on a neighborhood of A and supported in a neighborhood of A. Then*

$$h^{\frac{k-1}{2}}\langle \phi_h, \psi_h\rangle_{L^2(H)} = h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \mathrm{Op}_h(\chi_A)\psi_h\rangle_{L^2(H)} + o(1) \quad \text{as} \quad h \to 0^+. \tag{26}$$

Proof. First we use $\mathrm{Op}_h(\chi_{S^*M}), \mathrm{Op}_h(1-\chi_{S^*M}), \mathrm{Op}_h(\chi_A)$ and $\mathrm{Op}_h(1-\chi_A)$ to split up the inner product:

$$\begin{aligned}
&h^{\frac{k-1}{2}}\langle \phi_h, \psi_h\rangle_{L^2(H)} \\
&\quad = h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \psi_h\rangle_{L^2(H)} + h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\phi_h, \psi_h\rangle_{L^2(H)} \\
&\quad = h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \mathrm{Op}_h(\chi_A)\psi_h\rangle_{L^2(H)} + h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \mathrm{Op}_h(1-\chi_A)\psi_h\rangle_{L^2(H)} \\
&\qquad + h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\phi_h, \psi_h\rangle_{L^2(H)} \\
&\quad =: I + II + III.
\end{aligned} \tag{27}$$

We just need to show that both II and III are $o(1)$ as $h \to 0^+$. We begin with III. First, since ϕ_h is compactly microlocalized, there exists a cutoff $\chi \in C_c^\infty(T^*M)$ such that $\mathrm{Op}_h(1-\chi)\phi_h = O_{C^\infty}(h^\infty)$. Using χ, we split III once more,

$$\begin{aligned}
III &= h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(\chi)\phi_h, \psi_h\rangle_{L^2(H)} + h^{\frac{k-1}{2}}\langle \Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(1-\chi)\phi_h, \psi_h\rangle_{L^2(H)} \\
&\leq h^{\frac{k-1}{2}}\|\Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(\chi)\phi_h\|_{L^2(H)} + h^{\frac{k-1}{2}}\|\Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(1-\chi)\phi_h\|_{L^2(H)},
\end{aligned}$$

where we also used that $\|\psi_h\|_{L^2(H)} = 1$. Using that ϕ_h is compactly microlocalized, we observe that the term with $\mathrm{Op}_h(1-\chi)\phi_h$ is $O(h^\infty)$. Next, for the other term, we use an elliptic parametrix to rewrite

$$\mathrm{Op}_h(1-\chi_{S^*M}) = R(h)\ P(h) + O(h^\infty)_{\Psi^{-\infty}}.$$

To do this, we need verify that $\mathrm{WF}_h(1-\chi_{S^*M}) \subseteq \mathrm{ell}_h(P(h))$. Since $1-\chi_{S^*M}$ does not depend on h, $\mathrm{WF}_h(1-\chi_{S^*M}) = \mathrm{esssupp}(1-\chi_{S^*M}) \subseteq (S^*M)^c$. Moreover $\mathrm{ell}_h(P(h)) = \{p \neq 0\} = (S^*M)^c$, and hence we have the inclusion necessary to use an elliptic parametrix. Therefore, we can write

$$
\begin{aligned}
&h^{\frac{k-1}{2}}\|\Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(\chi)\phi_h\|_{L^2(H)}\\
&\quad = h^{\frac{k-1}{2}}\|\Gamma_H R(h)P(h)\mathrm{Op}_h(\chi)\phi_h\|_{L^2(H)} + O(h^\infty)\\
&\quad \le h^{\frac{k-1}{2}}\|\Gamma_H R(h)\mathrm{Op}_h(\chi)P(h)\phi_h\|_{L^2(H)} + h^{\frac{k-1}{2}}\|\Gamma_H R(h)(h\mathrm{Op}_h(H_p\chi)+O(h^2))\phi_h\|_{L^2(H)} + O(h^\infty)\\
&\quad \le C_k h^{-1/2}\|P(h)\phi_h\|_{L^2(M)} + C_k h^{1/2}\|\phi_h\|_{L^2(M)} + O(h^\infty)
\end{aligned}
$$

where in the last line we used the standard restriction bound

$$
\|\Gamma_H \mathrm{Op}_h(\kappa)u\|_{L^2(H)} \le C_\gamma h^{-k/2}\|\mathrm{Op}_h(\kappa)u\|_{H_h^\gamma(M)} \le C_k h^{-k/2}\|u\|_{L^2(M)} \tag{28}
$$

for $\gamma > k/2$, and $\kappa \in C_c^\infty(T^*M)$. By (11) we know $h^{-1}\|P(h)\phi_h\|_{L^2(M)} \to 0$ as $h \to 0^+$ and $\|\phi_h\|_L^2(M) = 1$, and thus we obtain

$$
h^{\frac{k-1}{2}}\|\Gamma_H \mathrm{Op}_h(1-\chi_{S^*M})\mathrm{Op}_h(\chi)\phi_h\|_{L^2(H)} = o(1) \quad \text{as} \quad h \to 0^+
$$

as desired.

Next we show II is $O(h^\infty)$. To do so we first claim that there exists $\tilde{\chi} \in C_c^\infty(T^*H;[0,1])$ such that

$$
\Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h = \mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h + O(h^\infty). \tag{29}
$$

Using Lemma 3.1 we find that we get (29) if we take $\tilde{\chi}(z',\xi') \equiv 1$ on a small neighborhood, $\mathcal{U}$, of $\left\{|\xi'|_{g_H} \le 1\right\}$ and supported in a small neighborhood of $\mathcal{U}$. Using (29) we show II is $O(h^\infty)$. We rewrite

$$
II = \langle \mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \mathrm{Op}_h(1-\chi_A)\psi_h\rangle_{L^2(H)} + O(h^\infty) \tag{30}
$$

Next observe

$$
\begin{aligned}
&\left|\langle \mathrm{Op}_h(\tilde{\chi})\Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h, \mathrm{Op}_h(1-\chi_A)\psi_h\rangle_{L^2(H)}\right|\\
&\quad \le \|\Gamma_H \mathrm{Op}_h(\chi_{S^*M})\phi_h\|_{L^2(H)}\|\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A)\psi_h\|_{L^2(H)}\\
&\quad \le C_k h^{-\frac{k}{2}}\|\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A)\psi_h\|_{L^2(H)}
\end{aligned}
$$

where the last inequality follows from the standard restriction bound (28). Recall $A = \mathrm{WF}_h(\psi_h)$ and $\tilde{\chi}(x',\xi')$ is compactly supported in a neighborhood of $\left\{|\xi'|_{g_H} \le 1\right\}$. Let K denote the support of $\tilde{\chi}$. There exists $\rho_j \in \overline{A^c \cap K}$ for $j = 1,\ldots N$ and $\theta_j \in C_c^\infty(T^*H;[0,1])$ supported sufficiently close to ρ_j such that

$$
\|\mathrm{Op}_h(\theta_j)\psi_h\|_{L^2(H)} = O(h^\infty),
$$

and moreover

$$
\Theta := \sum_{j=1}^N \theta_j \equiv 1 \text{ on } \overline{A^c \cap K}.
$$

We use an elliptic parametrix to rewrite

$$
\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A) = R(h)\ \mathrm{Op}_h(\Theta) + O(h^\infty)_{\Psi^{-\infty}}
$$

which we are allowed to do since $\mathrm{WF}_h(\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A)) \subseteq \mathrm{ell}_h(\Theta)$. To see this, note by properties of wavefront sets

$$\mathrm{WF}_h(\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A)) = \mathrm{WF}_h(\mathrm{Op}_h(\tilde{\chi})) \cap \mathrm{WF}_h(\mathrm{Op}_h(1-\chi_A)) \subseteq K \cap A^c.$$

Furthermore, $\mathrm{ell}_h(\Theta) \supseteq \overline{A^c \cap K}$, and hence we have the inclusion needed to use the elliptic parametrix. Lastly, we have

$$\begin{aligned}\|\mathrm{Op}_h(\tilde{\chi})^*\mathrm{Op}_h(1-\chi_A)\psi_h\|_{L^2(H)} &= \|R(h)\ \mathrm{Op}_h(\Theta)\psi_h\|_{L^2(H)} + O(h^\infty)\\ &\le \|R(h)\|_{L^2\to L^2}\sum_{j=1}^{N}\|\mathrm{Op}_h(\theta_j)\psi_h\|_{L^2(H)} + O(h^\infty) = O(h^\infty).\end{aligned}$$

□

4. Localization to geodesic tubes: proof of Proposition 2.1

In this section we finally present the proof of Proposition 2.1. Once again, throughout this section we assume $\{\phi_h\}$ is a compactly microlocalized collection of quasimodes on M satisfying (11) with defect measure μ. We also assume that the sequence of functions $\{\psi_h\}$ on H have defect measure ν and satisfy (7) and (8). In the following we use coordinates $x = (x', \bar{x})$ such that $H = \{\bar{x} = 0\}$. Furthermore we write $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) = (\bar{x}_1, \tilde{x})$.

We will need a few lemmas before proving the Proposition.

4.1. A technical lemma

Lemma 4.1. *Fix $\rho_0 \in \Sigma^A$ and let $q \in C_c^\infty(\mathbb{R}_{\bar{x}_1} \times \mathbb{R}^{k-1}_{\tilde{\xi}})$. There exists $T_0, R_0 > 0$ such that for all $0 < T < T_0$ and $0 < R < R_0$, if $\chi \in C_c^\infty(T^*M)$ is such that* $\operatorname{supp}\chi \subseteq \mathcal{T}^{3T}(U)$ *and* $\operatorname{supp} H_p\chi \subset \mathcal{T}^{3T}(U) \setminus \mathcal{T}^{2T}(U)$, *where $U \subseteq B_{\mathscr{L}}(\rho_0, R)$, then we have,*

$$\begin{aligned}\|\mathrm{Op}_h(q)\mathrm{Op}_h(\chi)\phi_h(x', 0, \tilde{x})\|_{L^2_{x',\tilde{x}}} &\le C\left(\frac{1}{\sqrt{T\left|\partial_{\bar{\xi}_1}p(\rho_0)\right|}} + \sqrt{2T}\right)\|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h\|_{L^2_x}\\ &\quad + \frac{C\sqrt{2T}}{h}(\|P\phi_h\|_{L^2_x} + \|\mathrm{Op}_h(\chi)\left[P, \mathrm{Op}_h(q)\right]\phi_h\|_{L^2_x}) + C_T h^{1/2}\|\phi_h\|_{L^2_x}.\end{aligned}$$

The proof of Lemma 4.1 is very similar to [8, Lemma 13], but we include it for completeness.

Proof. Fix $\rho_0 \in \Sigma^A$. Then, as before, we have $\partial_{\bar{\xi}_1}p(\rho_0) > 0$. Let $\mathcal{O}$ be an open neighborhood of ρ_0 such that $\partial_{\bar{\xi}_1}p > 0$ on $\mathcal{O}$. Furthermore, let $\mathcal{T}^{3T}(U)$ be a tube contained in $\mathcal{O}$. Then we can write

$$p(x, \xi) = e(x, \xi)(\bar{\xi}_1 - a(x, \xi', \tilde{\xi})) \quad \text{for} \quad (x, \xi) \in \mathcal{O}$$

where e is elliptic on $\mathcal{O}$. Thus for $\tilde{\chi} \equiv 1$ on $\mathcal{T}^{3T}(U)$ and supported in $\mathcal{O}$ we have

$$p(x, \xi)\tilde{\chi}(x, \xi) = e(x, \xi)(\bar{\xi}_1 - a(x, \xi', \tilde{\xi}))\tilde{\chi}(x, \xi).$$

Using the notation $P = \mathrm{Op}_h(p)$, observe

$$\begin{aligned} P\mathrm{Op}_h(\chi) &= P\mathrm{Op}_h(\tilde{\chi})\mathrm{Op}_h(\chi) + O(h^\infty) \\ &= \big(\mathrm{Op}_h(p\tilde{\chi}) + h\mathrm{Op}_h(r_1)\big)\mathrm{Op}_h(\chi) + O(h^\infty) \\ &= \Big(\mathrm{Op}_h(e)\mathrm{Op}_h((\bar{\xi}_1 - a(x,\xi',\tilde{\xi}))\tilde{\chi}) + h\mathrm{Op}_h(r_2) + h\mathrm{Op}_h(r_1)\Big)\mathrm{Op}_h(\chi) + O(h^\infty) \\ &= \mathrm{Op}_h(e)(hD_{\bar{x}_1} - \mathrm{Op}_h(a(x,\xi',\tilde{\xi})))\mathrm{Op}_h(\chi) + h\mathrm{Op}_h(r)\mathrm{Op}_h(\chi) + O(h^\infty). \end{aligned}$$

Thus

$$\begin{aligned} (hD_{\bar{x}_1} - \mathrm{Op}_h(a(x,\xi',\tilde{\xi})))\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h = \mathrm{Op}_h(e)^{-1}(P\mathrm{Op}_h(\chi)\mathrm{Op}_h(q) \\ - h\mathrm{Op}_h(r)\mathrm{Op}_h(\chi)\mathrm{Op}_h(q))\phi_h \end{aligned}$$

where $\mathrm{Op}_h(e)^{-1}$ denotes a microlocal parametrix for $\mathrm{Op}_h(e)$ near supp χ. Since a is a real symbol, we know that $\mathrm{Op}_h(a(x,\xi',\tilde{\xi}))$ is an error of order h away from being self adjoint. Therefore we can replace $\mathrm{Op}_h(a)$ with $\tilde{A} + h\tilde{R}$ where $\tilde{A}$ is self adjoint. Therefore we have

$$\begin{aligned} (hD_{\bar{x}_1} - \tilde{A})\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h = \mathrm{Op}_h(e)^{-1}(P\mathrm{Op}_h(\chi)\mathrm{Op}_h(q) - h\mathrm{Op}_h(r)\mathrm{Op}_h(\chi)\mathrm{Op}_h(q))\phi_h \\ + h\tilde{R}\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h \end{aligned}$$

We set

$$\begin{aligned} u &:= \mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h \\ f &:= \mathrm{Op}_h(e)^{-1}(P\mathrm{Op}_h(\chi)\mathrm{Op}_h(q) - h(\mathrm{Op}_h(r) - \mathrm{Op}_h(e)\tilde{R})\mathrm{Op}_h(\chi)\mathrm{Op}_h(q))\phi_h. \end{aligned}$$

To later utilize the fact that $P\phi_h = o_{L^2(M)}(h)$ we rewrite f as

$$\begin{aligned} f = \mathrm{Op}_h(e)^{-1}(\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)P + \big[P, \mathrm{Op}_h(\chi)\big]\mathrm{Op}_h(q) + \mathrm{Op}_h(\chi)\big[P, \mathrm{Op}_h(q)\big] \\ - h(\mathrm{Op}_h(r) - \mathrm{Op}_h(e)\tilde{R})\mathrm{Op}_h(\chi)\mathrm{Op}_h(q))\phi_h. \end{aligned}$$

Thus we have a differential equation for u:

$$\left(\partial_{\bar{x}_1} - \frac{i}{h}\tilde{A}\right)u = \frac{i}{h}f.$$

To simplify notation, we write $\hat{x}$ to denote both x' and $\tilde{x}$ and similarly $\hat{\xi}$ for $\xi', \tilde{\xi}$. First we define

$$A(t,s,\hat{x}) := \int_t^s \tilde{A}(\bar{x}_1, \hat{x}, \hat{\xi})d\bar{x}_1.$$

We obtain

$$u(s,\hat{x}) = e^{\frac{i}{h}A(t,s,\hat{x})}u(t,\hat{x}) + \frac{i}{h}\int_t^s e^{-\frac{i}{h}A(s,\tau,\hat{x})}f(\tau,\hat{x})d\tau. \tag{31}$$

Next, define $\delta := T\left|\partial_{\bar{\xi}_1} p(\rho_0)\right|$ and note for $T > 0$

$$0 < \delta = T\left|\partial_{\bar{\xi}} p(\rho_0)\right| = 2T\sqrt{1 - |\xi_0'|^2_{g_H(x_0')}} < 2T \qquad \text{where} \quad \rho_0 = (x_0', 0, \xi_0', \bar{\xi}_0) \in \Sigma^A.$$

Further, let $\Phi(t) \in C_c^\infty(\mathbb{R}; [0, 2\delta^{-1}])$ with $\operatorname{supp} \Phi \subseteq [0, \delta]$ and $\|\Phi\|_{L^1_t} = 1$. Multiplying (31) through by $\Phi(t)$ and integrating in t we have

$$\begin{aligned} u(s, \hat{x}) &= \int_{\mathbb{R}} \Phi(t) u(s, \hat{x}) dt \\ &= \int_{\mathbb{R}} \Phi(t) e^{\frac{i}{h}A(t,s,\hat{x})} u(t, \hat{x}) dt + \frac{i}{h}\int_{\mathbb{R}} \Phi(t) \int_t^s e^{-\frac{i}{h}A(s,\tau,\hat{x})} f(\tau, \hat{x}) d\tau \ dt. \end{aligned}$$

Next, taking the $L^2_{\hat{x}}$ norm

$$\begin{aligned} \|u(s, \hat{x})\|_{L^2_{\hat{x}}} &\leq \int_{\mathbb{R}} \Phi(t) \|e^{\frac{i}{h}A(t,s,\hat{x})} u(t, \hat{x})\|_{L^2_{\hat{x}}} dt + \frac{1}{h}\int_{\mathbb{R}} \Phi(t) \int_t^s \|e^{-\frac{i}{h}A(s,\tau,\hat{x})} f(\tau, \hat{x})\|_{L^2_{\hat{x}}} d\tau \ dt \\ &= \int_{\mathbb{R}} \Phi(t) \|u(t, \hat{x})\|_{L^2_{\hat{x}}} dt + \frac{1}{h}\int_{\mathbb{R}} \Phi(t) \int_t^s \|f(\tau, \hat{x})\|_{L^2_{\hat{x}}} d\tau \ dt =: I + II \end{aligned}$$

where the last line follows from

$$\partial_s \|e^{\frac{i}{h}A(t,s,\hat{x})} u(t, \hat{x})\|^2_{L^2_{\hat{x}}} = 2\mathrm{Re}\left\langle \frac{i}{h}\tilde{A} e^{\frac{i}{h}A(t,s,\hat{x})} u(t, \hat{x}), e^{\frac{i}{h}A(t,s,\hat{x})} u(t, \hat{x}) \right\rangle_{L^2_{\hat{x}}} = 0$$

since $\tilde{A}$ is self adjoint. So $\|e^{\frac{i}{h}A(t,s,\hat{x})}\|_{L^2_{\hat{x}}} = \|e^{\frac{i}{h}A(t,t,\hat{x})}\|_{L^2_{\hat{x}}} = 1$. Using Hölder's inequality and properties of Φ we bound I:

$$I \leq \|\Phi\|_{L^2_t} \|u(t, \hat{x})\|_{L^2_{\hat{x},t}} \leq \frac{4}{\sqrt{\delta}} \|u(t, \hat{x})\|_{L^2_{\hat{x},t}}.$$

To find a bound for II, we first take the L^∞ norm in s and apply Hölder's inequality to get

$$II \leq \frac{1}{h}\int \|\mathbb{1}_{[0,\delta]}(t) \mathbb{1}_{[s,t]}(\tau)\|_{L^\infty_{t,s}} \|f(\tau, \hat{x})\|_{L^2_{\hat{x}}} d\tau. \tag{32}$$

Splitting f up into its components in (32) we see that the first term is

$$\frac{1}{h}\int \|\mathbb{1}_{[0,\delta]}(t) \mathbb{1}_{[s,t]}(\tau)\|_{L^\infty_{t,s}} \|\mathrm{Op}_h(e)^{-1} \mathrm{Op}_h(\chi) \mathrm{Op}_h(q) P\phi_h(\tau, \hat{x})\|_{L^2_{\hat{x}}} d\tau$$

which is bounded by $C\sqrt{\delta}h^{-1}\|P\phi_h\|_{L^2_x}$. We also have $\tau \leq t \leq \delta < 2T$, and recall that $\operatorname{supp} H_p\chi \equiv 0$ on $\{|\bar{x}_1| \leq 2T\}$. Thus we can bound the second term by,

$$\|\mathrm{Op}_h(e)^{-1}\left[P, \mathrm{Op}_h(\chi)\right] \mathrm{Op}_h(q) \phi_h(\tau, \hat{x})\|_{L^2(\tau \in [-2T, 2T], \hat{x})} \leq C_T h^2 \|\phi_h\|_{L^2_{\hat{x}}}.$$

Continuing we obtain

$$\begin{aligned} II &\le \frac{C\sqrt{\delta}}{h}\big(\|P\phi_h\|_{L^2_{\tilde{x}}} + C_T h^2\|\phi_h\|_{L^2_{\tilde{x}}} + \|\mathrm{Op}_h(e)^{-1}\mathrm{Op}_h(\chi)\big[P,\mathrm{Op}_h(q)\big]\phi_h\|_{L^2_{\tilde{x}}} \\ &\quad + h\|\mathrm{Op}_h(e)^{-1}(\mathrm{Op}_h(r) - \mathrm{Op}_h(e)\tilde{R})\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h\|_{L^2_{\tilde{x}}}\big) \\ &\le \frac{C\sqrt{\delta}}{h}\|P\phi_h\|_{L^2_{\tilde{x}}} + C_\delta\sqrt{\delta}h^{1/2}\|\phi_h\|_{L^2_{\tilde{x}}} \\ &\quad + \frac{C\sqrt{\delta}}{h}\|\mathrm{Op}_h(\chi)\big[P,\mathrm{Op}_h(q)\big]\phi_h\|_{L^2_{\tilde{x}}} + C\sqrt{\delta}\|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h\|_{L^2_{\tilde{x}}} \end{aligned}$$

where we used the standard estimate $\|\phi_h\|_{L^2_{\tilde{x}}} \le Ch^{-1/2}\|\phi_h\|_{L^2_{\tilde{x}}}$ in the last line. So finally, combining the bounds for I and II and rewriting u as $\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h$ we have

$$\begin{aligned} \|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} &\le C\left(\frac{1}{\sqrt{T\big|\partial_{\tilde{\xi}_1}p(\rho_0)\big|}} + \sqrt{2T}\right)\|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h\|_{L^2_{\tilde{x}}} \\ &\quad + \frac{C\sqrt{2T}}{h}\big(\|P\phi_h\|_{L^2_{\tilde{x}}} + \|\mathrm{Op}_h(\chi)\big[P,\mathrm{Op}_h(q)\big]\phi_h\|_{L^2_{\tilde{x}}}\big) + C_T h^{1/2}\|\phi_h\|_{L^2_{\tilde{x}}}. \end{aligned} \tag{33}$$

Therefore, using a commutator and the bound in (33) we have

$$\begin{aligned} \|\mathrm{Op}_h(q)\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} &\le \|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \\ &\quad + \|\big[\mathrm{Op}_h(q),\mathrm{Op}_h(\chi)\big]\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \\ &\le C\left(\frac{1}{\sqrt{T\big|\partial_{\tilde{\xi}_1}p(\rho_0)\big|}} + \sqrt{2T}\right)\|\mathrm{Op}_h(\chi)\mathrm{Op}_h(q)\phi_h\|_{L^2_{\tilde{x}}} \\ &\quad + \frac{C\sqrt{2T}}{h}\big(\|P\phi_h\|_{L^2_{\tilde{x}}} + \|\mathrm{Op}_h(\chi)\big[P,\mathrm{Op}_h(q)\big]\phi_h\|_{L^2_{\tilde{x}}}\big) + C_T h^{1/2}\|\phi_h\|_{L^2_{\tilde{x}}}, \end{aligned}$$

where the estimate on the commutator term comes from the Sobolev embedding estimate:

$$\begin{aligned} \|\big[\mathrm{Op}_h(q),\mathrm{Op}_h(\chi)\big]\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} &\le h\|\mathrm{Op}_h(H_q\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} + O(h^2)\|\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \\ &\le Ch\|\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \le Ch^{1/2}\|\phi_h\|_{L^2_{\tilde{x}}} \end{aligned}$$

which we regroup with the existing $O(h^{1/2})$ term. □

4.2. Further localizing to tubes

The proof of Proposition 2.1 relies on decomposing $\operatorname{supp}\chi|_{\Sigma^A}$ into many small "rectangles." Using the geodesic flow, we then extend the rectangles to create a collection of geodesic tubes covering $\operatorname{supp}\chi|_{\Sigma^A}$. We get a much finer estimate on these tubes, which is given in the lemma below (Figure 2).

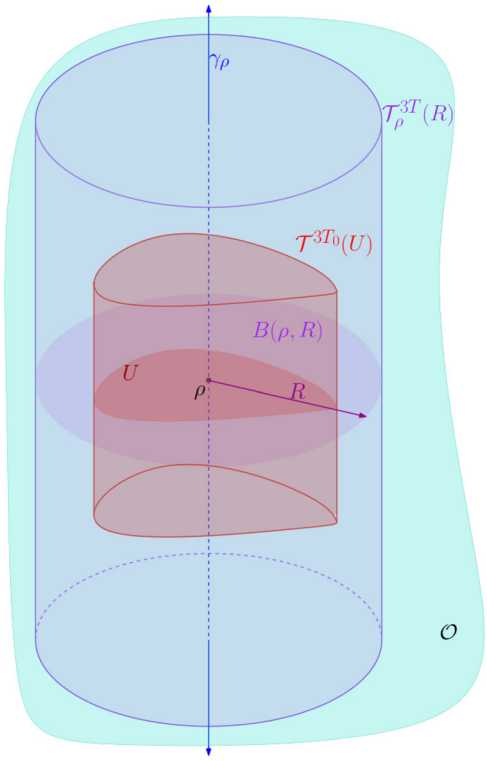

Figure 2. A diagram of the tube $\mathcal{T}^{3T_0}(U)$ sitting inside the tube $\mathcal{T}_\rho^{3T}(R)$.

Lemma 4.2. *Fix $\rho_0 = (x_0, 0, \xi_0', \bar{\xi}_0) \in \Sigma^A$. There exist $T_0, R_0 > 0$ such that for all $0 < T < T_0$ and $0 < R < R_0$, if $U \subseteq \mathscr{L}$ is a neighborhood of ρ_0 contained in $B_{\mathscr{L}}(\rho_0, R)$, and $\chi \in C_c^\infty(T^*M)$ is such that* supp $\chi \subseteq \mathcal{T}^{3T}(U)$ *and* supp $H_p\chi \subset \mathcal{T}^{3T}(U) \setminus \mathcal{T}^{2T}(U)$, *then there exists a constant C_k depending only on k for which*

$$\limsup_{h \to 0^+} h^{k-1} \|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(H)}^2 \leq \frac{C_k R^{k-1}}{2T\sqrt{1 - |\xi_0'|_{g_H(x_0')}^2}} \int_{T^*M} |\chi|^2 d\mu.$$

To prove Lemma 4.2 we will strategically pick the q's from Lemma 4.1 to be functions which vanish to order $k-1$ on the geodesic emanating from ρ_0. This will allow us to get a much better estimate on the geodesic tubes.

Proof. We choose local Fermi coordinates near $\rho_0 \in \Sigma^A$ with respect to H, $(x', \bar{x})$ such that $H = \{\bar{x} = 0\}$ and

$$|\xi|_g^2 = |\bar{\xi}|^2 + f(x', \bar{x})|\xi'|^2.$$

Thus note for ρ_0 we have $\left|\partial_{\bar{\xi}} p(\rho_0)\right| = 2\left|\bar{\xi}\right|(\rho_0) > 0$ since $\bar{\xi} \in S^{k-1}_{\sqrt{1-|\xi'|_{g_H}^2}}$. We note the importance of the assumption that $A \Subset \left\{(x', \xi') : |\xi'| < 1\right\}$ since otherwise we cannot assume $\left|\bar{\xi}\right| > 0$ on Σ^A. Next, since $\left|\partial_{\bar{\xi}} p(\rho_0)\right| > 0$ there exists a neighborhood $\mathcal{O}$ of ρ_0

such that $\left|\partial_{\bar{\xi}} p\right| > 0$ on $\mathcal{O}$. Without loss of generality we assume that $\partial_{\bar{\xi}_1} p(\rho_0) = \left|\partial_{\bar{\xi}} p(\rho_0)\right| > 0$ where $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2 ..., \bar{\xi}_k) = (\bar{\xi}_1, \tilde{\xi})$. Furthermore, in these coordinates we have $\|u\|_{L^2_{\bar{x}}} \leq 2\|u\|_{L^2(M)}$.

Let $T_0, R_0 > 0$ be such that $\mathcal{T}^{3T_0}_{\rho_0}(R_0) \subset \mathcal{O}$, where the tube $\mathcal{T}$ is as defined in (18). Note that for all $0 < R < R_0$ we still have $\mathcal{T}^{3T_0}_{\rho_0}(R) \subseteq \mathcal{O}$. Therefore, the "flowout" time T_0 is independent of the tube width R, for $R < R_0$ small enough. Let $\gamma_{\rho_0}(t) = (X(t), \Xi(t))$ denote a geodesic through ρ_0. From Hamilton's equations, we know the geodesic flow must satisfy

$$\dot{\bar{x}}_1 = p_{\bar{\xi}_1} \Rightarrow \frac{d}{dt}\bar{x}_1(t) = \partial_{\bar{\xi}_1} p(X(t), \Xi(t)) = \partial_{\bar{\xi}_1} p(\gamma_{\rho_0}(t)) > 0 \quad \text{for } |t| < 3T_0$$

as $\partial_{\bar{\xi}_1} p(\rho) > 0$ in $\mathcal{O}$. Thus for all $|t| \leq 3T_0$ we have $\frac{d}{dt}\bar{x}_1(t) > 0$. By the Inverse Function Theorem we can locally write $t = t(\bar{x}_1)$ and further we have

$$\bar{x}_1(\gamma_{\rho_o}(t(s))) = s.$$

We define

$$\begin{aligned} X'(\bar{x}_1) &:= x'(\gamma_{\rho_o}(t(\bar{x}_1))), & \bar{X}_1(\bar{x}_1) &:= \bar{x}_1(\gamma_{\rho_o}(t(\bar{x}_1))) = \bar{x}_1 \\ \tilde{X}(\bar{x}_1) &:= \tilde{x}(\gamma_{\rho_o}(t(\bar{x}_1))), & \Xi(\bar{x}_1) &:= \xi(\gamma_{\rho_o}(t(\bar{x}_1))). \end{aligned}$$

Therefore the geodesic through ρ_0 is parametrized by

$$\bar{x}_1 \mapsto (X'(\bar{x}_1), \bar{x}_1, \tilde{X}(\bar{x}_1), \Xi(\bar{x}_1)) = (x'(\gamma_{\rho_o}(t(\bar{x}_1))), \bar{x}_1, \tilde{x}(\gamma_{\rho_o}(t(\bar{x}_1))), \xi(\gamma_{\rho_o}(t(\bar{x}_1)))).$$

Moreover, we note that on the geodesic $\xi = \xi(\gamma_{\rho_0}(t(\bar{x}_1))) =: \xi_0(\bar{x}_1)$. This will be crucial in getting the improved estimate on the tubes.

In what follows, we write $\tilde{x}$ to denote the normal coordinates to H which are not $\bar{x}_1$, so $\bar{x} = (\bar{x}_1, \bar{x}_2, ...\bar{x}_k) = (\bar{x}_1, \tilde{x})$. We first use a version of the Sobolev Embedding Theorem (see [23, Lemma 6.1] or [24, Corollary 8]):

$$\begin{aligned} &\|\mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|_{L^\infty_{\tilde{x}}} \\ &\quad \leq C_k h^{\frac{1-k}{2}} \|\mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|^{1/2}_{L^2_{\tilde{x}}} \left(\sum_{i=2}^{k} \|(hD_{\bar{x}_i} - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1} \mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|^2_{L^2_{\tilde{x}}} \right)^{1/4}. \end{aligned}$$

Squaring both sides, integrating with respect to x' and applying Hölder's Inequality we have

$$\begin{aligned} &\|\mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|^2_{L^2_{x'}} \leq \int \|\mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|^2_{L^\infty_{\tilde{x}}} dx' \\ &\quad \leq C_k h^{1-k} \|\mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|_{L^2_{x',\tilde{x}}} \left(\sum_{i=2}^{k} \|(hD_{\bar{x}_i} - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1} \mathrm{Op}_h(\chi)\phi_h(x', \bar{x}_1, \tilde{x})\|^2_{L^2_{x',\tilde{x}}} \right)^{1/2}. \end{aligned}$$

Setting $\bar{x}_1 = 0$ and $\tilde{x} = 0$ on the left we have

$$h^{k-1}\|\mathrm{Op}_h(\chi)\phi_h\|^2_{L^2(H)} \leq C_k\|\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}}\left(\sum_{i=2}^{k}\|(hD_{\bar{x}_i}-\bar{\xi}_{0_i}(\bar{x}_1))^{k-1}\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|^2_{L^2_{x',\tilde{x}}}\right)^{1/2}. \tag{34}$$

Next, we will use Lemma 4.1 to bound the L^2 norms on the right side of (34). We denote $T_{\rho_0} := T\left|\partial_{\xi_1}p(\rho_0)\right| = 2T\sqrt{1-|\xi_0'|^2_{g_H(x_0')}}$. For $q \in C_c^\infty(\mathbb{R}_{\bar{x}_1}\times\mathbb{R}^{k-1}_{\tilde{\xi}})$ and $0 < T < T_0$ we have

$$\|q(\bar{x}_1,hD_{\tilde{x}})\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \leq C\left(\frac{1}{\sqrt{T_{\rho_0}}}+\sqrt{T}\right)\|\mathrm{Op}_h(\chi)q(\bar{x}_1,hD_{\tilde{x}})\phi_h\|_{L^2(M)} + \frac{C\sqrt{T}}{h}\left(\|P\phi_h\|_{L^2(M)} + \|\mathrm{Op}_h(\chi)\left[P,q(\bar{x}_1,hD_{\tilde{x}})\right]\phi_h\|_{L^2(M)}\right) + C_Th^{1/2}\|\phi_h\|_{L^2(M)}$$

where we have used that in our coordinates $\|u\|_{L^2_{\tilde{x}}} \leq 2\|u\|_{L^2(M)}$. Next, since $P\phi_h = o_{L^2(M)}(h)$ we know $h^{-1}\|P\phi_h\|_{L^2(M)} \to 0$ as $h \to 0^+$. We also have that $C_Th^{1/2}\|\phi_h\|_{L^2(M)} \to 0$ as $h \to 0^+$ since $\|\phi\|_{L^2(M)} = 1$. We regroup these two terms in a $o(1)$ error. Further, reordering the operators, we add an $O(h)$ error which we regroup with the $o(1)$ term to get

$$\begin{aligned}\|q(\bar{x}_1,hD_{\tilde{x}})\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} &\leq C\left(\frac{1}{\sqrt{T_{\rho_0}}}+\sqrt{T}\right)\|q(\bar{x}_1,hD_{\tilde{x}})\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} \\ &\quad + \frac{C\sqrt{T}}{h}\|\left[P,q(\bar{x}_1,hD_{\tilde{x}})\right]\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + o(1).\end{aligned} \tag{35}$$

First, taking $q = 1$ in (35) we get

$$\|\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \leq C\left(\frac{1}{\sqrt{T_{\rho_0}}}+\sqrt{T}\right)\|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + o(1). \tag{36}$$

Next, define $Q_i := (hD_{\bar{x}_i} - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1} = \mathrm{Op}_h(q_i)$ where $q_i = (\bar{\xi}_i - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1} + O(h)$. Then, using this q_i in (35) we have

$$\begin{aligned}&\|(hD_{\bar{x}_i} - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1}\mathrm{Op}_h(\chi)\phi_h(x',0,\tilde{x})\|_{L^2_{x',\tilde{x}}} \\ &\leq C\left(\frac{1}{\sqrt{T_{\rho_0}}}+\sqrt{T}\right)\|Q_i\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + \frac{C\sqrt{T}}{h}\|[P,Q_i]\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + o(1).\end{aligned} \tag{37}$$

Next define $\tilde{\chi} \in C_c^\infty(T^*M)$ such that $\operatorname{supp}\tilde{\chi} \subseteq \mathcal{T}^{3T}(U)$ and $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. We rewrite

$$Q_i\mathrm{Op}_h(\chi)\phi_h = Q_i\mathrm{Op}_h(\tilde{\chi})\mathrm{Op}_h(\chi)\phi_h + O(h^\infty).$$

Recall that on the geodesic γ_{ρ_0} we have $\xi = \xi_0(\bar{x}_1)$. Therefore, the principal symbol of Q_i, $\sigma(Q_i)$, vanishes to order $k-1$ on the geodesic. Furthermore, since $\tilde{\chi}$ is supported in

the tube $\mathcal{T}^{3T}(U)$ where $U \subseteq B_{\mathscr{L}}(\rho_0, R)$, the distance between any point in supp $\tilde{\chi}$ and the geodesic is approximately at most R. Thus we have

$$\sup\left|\sigma(Q_i \mathrm{Op}_h(\tilde{\chi}))\right| \leq 2R^{k-1}.$$

This implies that $\|Q_i \mathrm{Op}_h(\tilde{\chi})\|_{L^2 \to L^2(M)} \leq 2R^{k-1} + O(h)$ and in particular that

$$\begin{aligned}\|Q_i \mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} &= \|Q_i \mathrm{Op}_h(\tilde{\chi})\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + O(h^\infty)\|\phi_h\|_{L^2(M)} \\ &\leq (2R^{k-1} + O(h))\|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + O(h^\infty)\|\phi_h\|_{L^2(M)}.\end{aligned}$$

We also have that $H_p(\sigma(Q_i)) = H_p((\bar{\xi}_i - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1}) = (k-1)(\bar{\xi}_i - \bar{\xi}_{0_i}(\bar{x}_1))^{k-2}\, H_p(\bar{\xi}_i - \bar{\xi}_{0_i}(\bar{x}_1))$ vanishes to order $k-1$ on the geodesic γ_{ρ_0}. Since $\sigma([P, Q_i]) = \frac{h}{i} H_p((\bar{\xi}_i - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1})$, we similarly have

$$\|[P, Q_i]\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} \leq h(C_p R^{k-1} + O(h))\|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + O(h^\infty)\|\phi_h\|_{L^2(M)}$$

where C_p is a constant which depends on p. Using these estimates in (37) we have

$$\begin{aligned}&\|(hD_{\bar{x}_i} - \bar{\xi}_{0_i}(\bar{x}_1))^{k-1}\mathrm{Op}_h(\chi)\phi_h(x', 0, \tilde{x})\|_{L^2_{x',\tilde{x}}} \\ &\leq C\left(\frac{1}{\sqrt{T_{\rho_0}}} + \sqrt{T}\right)(2R^{k-1} + O(h))\|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} \\ &+ C\sqrt{T}(C_p R^{k-1} + O(h))\|\mathrm{Op}_h(\chi)\phi_h\|_{L^2(M)} + o(1).\end{aligned} \tag{38}$$

Finally, using (36) and (38) in (34) and taking h to zero, we have

$$\begin{aligned}&\limsup_{h \to 0^+} h^{k-1}\|\mathrm{Op}_h(\chi)\phi_h\|^2_{L^2(H)} \\ &\leq \limsup_{h \to 0^+} C_k R^{k-1}\left(\frac{1}{\sqrt{T_{\rho_0}}} + \sqrt{T}\right)\left(\frac{2}{\sqrt{T_{\rho_0}}} + C_p\sqrt{T}\right)\|\mathrm{Op}_h(\chi)\phi_h\|^2_{L^2(M)}.\end{aligned}$$

Using the defect measure μ associated to $\{\phi_h\}$ and that $T \ll 1$ we obtain the desired bound:

$$\limsup_{h \to 0^+} h^{k-1}\|\mathrm{Op}_h(\chi)\phi_h\|^2_{L^2(H)} \leq C_k \frac{R^{k-1}}{T_{\rho_0}} \int_{T^*M} |\chi|^2 d\mu.$$

□

4.3. Key quantitative estimate: proof of Proposition 2.1

The main estimate used in the proof of Theorem 1.2 lets us control terms of the form $\left|\langle \mathrm{Op}_h(\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right|$. To prove it we first cover supp $\chi|_{\Sigma^A}$ with tubes and apply Lemma 4.2. After localizing to the tubes, we will need to estimate $\langle \mathrm{Op}_h(\chi_j\chi)\phi_h, \psi_h\rangle_{L^2(H)}$ where χ_j is a cutoff localizing to a tube as in Lemma 4.2. If we use Cauchy-Schwarz to bound this by the L^2 norms,

$$\left|\langle \mathrm{Op}_h(\chi_j\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right| \leq \|\mathrm{Op}_h(\chi_j\chi)\phi_h\|_{L^2(H)}\|\psi_h\|_{L^2(H)} = \|\mathrm{Op}_h(\chi_j\chi)\phi_h\|_{L^2(H)}$$

we lose all the information from ψ_h since they are L^2-normalized on H. Thus we need to maintain the localization information of ψ_h too. To do this, we will use $\chi_j\chi \in$

$C_c^\infty(T^*M)$ and Lemma 3.1 to find a new cutoff $\theta_j \in C_c^\infty(T^*H)$ such that

$$\Gamma_H \mathrm{Op}_h(\chi_j\chi)\phi_h = \mathrm{Op}_h(\theta_j)\Gamma_H \mathrm{Op}_h(\chi_j\chi)\phi_h + O(h^\infty)$$

and thus

$$\begin{aligned}\langle \mathrm{Op}_h(\chi_j\chi)\phi_h, \psi_h\rangle_{L^2(H)} &= \langle \mathrm{Op}_h(\chi_j\chi)\phi_h, \mathrm{Op}_h(\theta_j)^*\psi_h\rangle_{L^2(H)} + O(h^\infty)\\ &\le \|\mathrm{Op}_h(\chi_j\chi)\phi_h\|_{L^2(H)}\|\mathrm{Op}_h(\theta_j)^*\psi_h\|_{L^2(H)}.\end{aligned}$$

Then we will be able to apply Lemma 4.2 to the first term and use the defect measure for ν in the second.

Proof of Proposition 2.1. Let $\chi \in C_c^\infty(T^*M)$ with $H_p\chi \equiv 0$ on $\Lambda^{2T}(\varepsilon)$. Consider sets of the form:

$$U_j := \left\{(x', 0, \xi', \bar{\xi}) \in \Sigma^A : (x', \xi') \in B_A(\rho_j', R_j), (x', 0, \xi', \bar{\xi}) \in B_{\Sigma^A}(\sigma_j(x', \xi'), R_j)\right\}$$

where $R_j > 0$, $\rho_j' = (x_j', \xi_j') \in A$ and σ_j is a smooth section, that is $\sigma_j : A \to \Sigma^A$ and $\pi(\sigma_j(\rho')) = \rho'$. These are essentially "rectangles" in Σ^A constructed by crossing a ball in A with balls in the spheres S^{k-1}. We note that the ν^A measure of these rectangles satisfy

$$\nu^A(U_j) \ge C_{n,k}\int_{(x',\xi')\in A} \frac{\mathbb{1}_{B_A(\rho_j', R_j)}}{\left(1 - |\xi'|^2_{g_H(x')}\right)^{\frac{k-1}{2}}} R_j^{k-1} d\nu(x', \xi')$$

provided R_j is small compared to $\sqrt{1 - |\xi'|_{g_H(x')}}$. Furthermore by uniform continuity of $\log(1 - |\xi'|^2_{g_H(x')})$ on $\left\{|\xi'|^2_{g_H(x')} < c < 1\right\}$, there exists an $R > 0$ independent of ρ_j' such that if $(x', \xi') \in B_A(\rho_j', R)$ then

$$\begin{aligned}\frac{1-k}{2}\log(1 - |\xi_j'|^2_{g_H(x_j')}) - \log(2) &\le \frac{1-k}{2}\log(1 - |\xi'|^2_{g_H(x')})\\ &\le \frac{1-k}{2}\log(1 - |\xi_j'|^2_{g_H(x_j')}) + \log(2).\end{aligned}$$

Thus for $R_j < R$, we have

$$\nu^A(U_j) \ge \frac{C_{n,k}\ \nu(B_A(\rho_j', R_j))R_j^{k-1}}{2(1 - |\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}}.$$

Fix $\delta > 0$. By outer regularity of ν^A there exist $\{U_j\}_{j=1}^{N(\delta)}$ covering supp $\chi|_{\Sigma^A}$ such that

$$\nu^A(\text{supp } \chi|_{\Sigma^A}) + \delta \ge \sum_{j=1}^{N(\delta)} \nu^A(U_j) \ge C_{n,k}\sum_{j=1}^{N(\delta)} \frac{\nu(B_A(\rho_j', R_j))R_j^{k-1}}{2(1 - |\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}}. \tag{39}$$

To construct the cover of tubes, we first "thicken" the U_j's into $U_j(\varepsilon) \subseteq \mathscr{L}$ as defined in (19). Finally, we flow U_j^ε's to form the collection of tubes $\left\{\mathcal{T}^{3T}(U_j(\varepsilon))\right\}$ where

$$T \leq \min_j \{T_{0_j}\} =: T_0 \qquad \text{and} \qquad 3R_j \leq \min_j(R_{0_j}) =: R_0 \tag{40}$$

where the T_{0_j}'s are the "T_0's" in the proof of lemma 4.2 and the R_{0_j}'s are the "R_0's" in the proof of lemma 4.2. We note that $U_j(\varepsilon) \subseteq B_{\mathscr{L}}(\rho_j, 3R_j)$ where $\rho_j = \sigma_j(\rho_j')$. By lemma 2.2 (or [24, Lemma 3.5]), for each j, we can take $\chi_j \in C_c^\infty(T^*M; [0,1])$ supported in $\mathcal{T}^{3T}(U_j(\varepsilon))$ such that supp $H_p\chi_j \subseteq \mathcal{T}^{3T}(U_j(\varepsilon)) \setminus \mathcal{T}^{2T}(U_j(\varepsilon))$ and furthermore that

$$\sum_{j=1}^{N(\delta)} \chi_j \equiv 1 \text{ on } \bigcup_{|t|\leq 2T} \varphi_t((\text{supp } \chi|_{\Sigma^A})(\varepsilon/2)).$$

Next we split the inner product into pieces localized to these tubes. We have

$$\begin{aligned} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right| &\leq h^{\frac{k-1}{2}}\left|\left\langle \mathrm{Op}_h\left(\sum_{j=1}^{N(\delta)}\chi_j\chi\right)\phi_h, \psi_h\right\rangle_{L^2(H)}\right| \\ &\quad + h^{\frac{k-1}{2}}\left|\left\langle \mathrm{Op}_h((1-\sum_{j=1}^{N(\delta)}\chi_j)\chi)\phi_h, \psi_h\right\rangle_{L^2(H)}\right| \\ &=: I + II. \end{aligned}$$

We claim $II = o(1)$ as $h \to 0^+$. We leave the proof of this to Lemma 4.3 at the end of this section. The rest of this proof is dedicated to controlling I. By Lemma 3.1 there exits $\theta_j \in C_c^\infty(T^*H)$ such that

$$\Gamma_H \mathrm{Op}_h(\chi_j\chi)\phi_h = \mathrm{Op}_h(\theta_j)\Gamma_H \mathrm{Op}_h(\chi_j\chi)\phi_h + O(h^\infty). \tag{41}$$

Particularly, we need to take θ_j equal to 1 on $B_{T^*H}(\rho_j', R_j + \varepsilon)$ and supp $\theta_j \subseteq B_{T^*H}(\rho_j', R_j + 2\varepsilon)$. Thus we have

$$\begin{aligned} I &\leq h^{\frac{k-1}{2}}\sum_{j=1}^{N(\delta)}\left|\langle \mathrm{Op}_h(\chi_j\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right| \leq h^{\frac{k-1}{2}}\sum_{j=1}^{N(\delta)}\left|\langle \mathrm{Op}_h(\theta_j)\Gamma_H \mathrm{Op}_h(\chi_j\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right| + O(h^\infty) \\ &\leq h^{\frac{k-1}{2}}\sum_{j=1}^{N(\delta)}\|\mathrm{Op}_h(\chi_j\chi)\phi_h\|_{L^2(H)}\|\mathrm{Op}_h(\theta_j)^*\psi_h\|_{L^2(H)} + O(h^\infty). \end{aligned} \tag{42}$$

We are now in position to apply Lemma 4.2 for "χ" $= \chi_j\chi$.

$$\begin{aligned} \limsup_{h\to 0^+} h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\chi)\phi_h, \psi_h\rangle_{L^2(H)}\right| &= \limsup_{h\to 0^+} h^{\frac{k-1}{2}}(I + II) \\ &\leq \limsup_{h\to 0^+} h^{\frac{k-1}{2}}\sum_{j=1}^{N(\delta)}\|\mathrm{Op}_h(\chi_j\chi)\phi_h\|_{L^2(H)}\|\mathrm{Op}_h(\theta_j)^*\psi_h\|_{L^2(H)} \\ &\leq \sum_{j=1}^{N(\delta)}\left(\frac{C_k R_j^{k-1}}{2T\sqrt{1-|\xi_j'|^2_{g_H(x_j')}}}\int_{T^*M}|\chi_j\chi|^2 d\mu\right)^{1/2}\left(\int_{T^*H}|\theta_j|^2 d\nu\right)^{1/2} \end{aligned} \tag{43}$$

where we used that ν is a defect measure associated to $\{\psi_h\}$. Next, to get $\nu^A(\text{supp } \chi|_{\Sigma^A})$ to appear, we work to make the second term in (43) look like the right side of (39).

Moving the R_j^{k-1} over, multiplying and dividing by $2(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}$ and applying Cauchy-Schwarz we find that (43) is bounded by

$$C_k\left(\frac{1}{T}\int_{T^*M}\sum_{j=1}^{N(\delta)}\frac{2(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}|\chi_j\chi|^2}{2\sqrt{1-|\xi_j'|^2_{g_H(x_j')}}}d\mu\right)^{1/2}\left(\sum_{j=1}^{N(\delta)}\frac{R_j^{k-1}}{2(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}}\int_{T^*H}\mathbb{1}_{B_{T^*H}(\rho_j',R_j+2\varepsilon)}d\nu\right)^{1/2}.$$

Next, since the χ_j's are supported in the tubes, the first integral can be rewritten as an integral over $\Lambda^{3T}(\varepsilon)$. Further, since the left side does not depend on ε we can take the limit as $\varepsilon\to 0$ on the right side. Using the dominated convergence theorem to bring the limit inside we have

$$\begin{aligned}&\limsup_{h\to 0^+}h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\chi)\phi_h,\psi_h\rangle_{L^2(H)}\right|\\&\quad\le C_k\left(\frac{1}{T}\int_{\Lambda^{3T}}|\chi|^2\sum_{j=1}^{N(\delta)}(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-2}{2}}|\chi_j|^2d\mu\right)^{1/2}\left(\sum_{j=1}^{N(\delta)}\frac{R_j^{k-1}\nu(B_A(\rho_j',R_j))}{2(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-1}{2}}}\right)^{1/2}\end{aligned}$$

where Λ^{3T} denotes $\mathcal{T}^{3T}(\Sigma^A)$. Next, since the second term is what we had in (39), we can replace it with $(\nu^A(\mathrm{supp}\ \chi|_{\Sigma^A})+\delta)^{1/2}$. Noticing that the left side does not depend on T, we take the limit as $T\to 0$ and use the definition of μ^A from (12) to get

$$\begin{aligned}&\limsup_{h\to 0^+}h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\chi)\phi_h,\psi_h\rangle_{L^2(H)}\right|\\&\quad\le C_{n,k}\left(\int_{\Sigma^A}|\chi|^2\sum_{j=1}^{N(\delta)}(1-|\xi_j'|^2_{g_H(x_j')})^{\frac{k-2}{2}}|\chi_j|^2d\mu^A\right)^{1/2}\left(\nu^A(\mathrm{supp}\ \chi|_{\Sigma^A})+\delta\right)^{1/2}.\end{aligned}$$

Finally, since $\{\chi_j|_{\Sigma^A}\}$ formed a partition of unity for $\mathrm{supp}\ \chi|_{\Sigma^A}$, $|\chi_j|\le 1$, $(1-|\xi'|^2_{g_H(x')})^{\frac{k-1}{2}}$ is continuous, and since $\delta>0$ was arbitrary, we have

$$\limsup_{h\to 0^+}h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h(\chi)\phi_h,\psi_h\rangle_{L^2(H)}\right|\le C_{n,k}\left(\int_{\Sigma^A}|\chi|^2(1-|\xi'|^2_{g_H(x')})^{\frac{k-2}{2}}d\mu^A\right)^{1/2}\left(\nu^A(\mathrm{supp}\ \chi|_{\Sigma^A})\right)^{1/2}$$

as desired. □

Finally, we show that term $II=h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h((1-\sum_j\chi_j)\chi)\phi_h,\psi_h\rangle_{L^2(H)}\right|$ in the proof of proposition 2.1 is $o(1)$ as $h\to 0^+$ as claimed above.

Lemma 4.3. *For χ,χ_j defined in the proof of proposition 2.1 we have*

$$h^{\frac{k-1}{2}}\left|\langle \mathrm{Op}_h\left(\left(1-\sum_j\chi_j\right)\chi\right)\phi_h,\psi_h\rangle_{L^2(H)}\right|\to 0,\quad as\quad h\to 0^+.$$

Proof. First, using Lemma 3.2 we obtain

$$h^{\frac{k-1}{2}}\left|\left\langle \mathrm{Op}_h\left(\left(1-\textstyle\sum_j \chi_j\right)\chi\right)\phi_h,\psi_h\right\rangle_{L^2(H)}\right|$$

$$\leq h^{\frac{k-1}{2}}|\langle \mathrm{Op}_h((1-\sum_{j=1}^{N(\delta)}\chi_j)\chi)\mathrm{Op}_h(\chi_{S^*M})\phi_h,\mathrm{Op}_h(\chi_A)\psi_h\rangle|+o(1)$$

$$\leq h^{\frac{k-1}{2}}\left\|\mathrm{Op}_h(\chi_A)^*\Gamma_H\mathrm{Op}_h\left(\left(1-\sum_{j=1}^{N(\delta)}\chi_j\right)\chi\right)\mathrm{Op}_h(\chi_{S^*M})\phi_h\right\|_{L^2(H)}\|\psi_h\|_{L^2(H)}+o(1)$$

where χ_{S^*M} and χ_A are defined in the statement of Lemma 3.2. We show that

$$\mathrm{Op}_h(\chi_A)^*\Gamma_H\mathrm{Op}_h\left(\left(1-\sum_{j=1}^{N(\delta)}\chi_j\right)\chi\right)\mathrm{Op}_h(\chi_{S^*M})\phi_h=O(h^\infty). \tag{44}$$

To do so, we employ Lemma 3.1. We just need to verify the hypothesis of the lemma. For contradiction, suppose there is a point $(z_0',\xi_0')\in\operatorname{supp}\chi_A$ and also $(z_0',0,\xi_0',\bar{\xi}_0)\in\operatorname{supp}((1-\sum_j\chi_j)\chi\chi_{S^*M})$. First, we note that $(z_0,0,\xi_0',\bar{\xi}_0)\notin(\operatorname{supp}\chi|_{\Sigma^A})(\varepsilon/2)$. However, since also $(z_0',0,\xi_0',\bar{\xi}_0)\in\operatorname{supp}\chi_{S^*M}$ and $(z_0',\xi_0')\in\operatorname{supp}\chi_A$ we know that $(z_0',0,\xi_0',\bar{\xi}_0)\in\Sigma^A(\alpha)=\bigcup_{\rho\in\Sigma^A}B_{\mathscr{L}}(\rho,\alpha)$ where $\alpha>0$ is small and depends on how tightly χ_{S^*M} and χ_A are localized. Furthermore $(z_0',0,\xi_0',\bar{\xi}_0)\in\operatorname{supp}\chi$ and so we have

$$(z_0',0,\xi_0',\bar{\xi}_0)\notin(\operatorname{supp}\chi|_{\Sigma^A})(\varepsilon/2)\qquad\text{and}\qquad(z_0',0,\xi_0',\bar{\xi}_0)\in\operatorname{supp}\chi|_{\Sigma^A(\alpha)} \tag{45}$$

but by taking χ_A and χ_{S^*M} supported sufficiently close to A and S^*M, we can find α such that $\operatorname{supp}\chi|_{\Sigma^A(\alpha)}\subseteq(\operatorname{supp}\chi|_{\Sigma^A})(\varepsilon/2)$ which contradicts (45). Thus use of Lemma 3.1 is justified and we have (44). □

5. Recurrence: proof of Theorem 1.1

In this section we prove Theorem 1.1 which gives the behavior of $\left|\langle\phi_h,\psi_h\rangle_{L^2(H)}\right|$ as $h\to 0^+$ when the recurrent set of Σ^A is ν^A-measure zero. First, we define the recurrent set and introduce some notation. Although the following can be defined more generally, we stick to defining loop set, recurrent set, etc., for Σ^A only. First for each point $\rho\in\Sigma^A$ we define the first return time $T_A:\Sigma^A\to\mathbb{R}\bigcup\{\infty\}$ by

$$T_A(\rho)=\inf\left\{t>0:\gamma_\rho(t)\in\Sigma^A\right\}$$

where $\gamma_\rho(t)$ is the geodesic emanating from ρ. This gives us the first time in which the geodesic $\gamma_\rho(t)$ returns to Σ^A. If the geodesic never returns to Σ^A, the return time will be infinite. We will primarily be interested in the points which return to Σ^A in finite time. We call the collection of such points the loop set, denoted

$$\mathcal{L}_A = \{\rho \in \Sigma^A : T_A(\rho) < \infty\}.$$

Since points in the loop set return to Σ^A in finite time, we denote the point in which $\rho \in \mathcal{L}_A$ returns to by $\eta(\rho)$ defined by $\eta : \mathcal{L}_A \to \Sigma^A$,

$$\eta(\rho) = \gamma_\rho(T_A(\rho)).$$

Next, define the infinite loop sets

$$\mathcal{L}_A^{+\infty} = \bigcap_{k\geq 0} \eta^{-k}(\mathcal{L}_A) \qquad \text{and} \qquad \mathcal{L}_A^{-\infty} = \bigcap_{k\geq 0} \eta^{k}(\mathcal{L}_A)$$

which are essentially the loop set points that return to Σ^A infinitely often forward and backward in time, respectively. Finally, the recurrent set $\mathcal{R}_A := \mathcal{R}_A^+ \cap \mathcal{R}_A^-$ where

$$\mathcal{R}_A^{\pm} := \left\{\rho \in \mathcal{L}_A^{\pm\infty} : \rho \in \bigcap_{N>0} \overline{\bigcup_{k\geq N} \eta^{\pm k}(\rho)}\right\},$$

which is essentially the collection of points $\rho \in \Sigma^A$ which return infinitely often and eventually get arbitrarily close to ρ.

Proof of Theorem 1.1. Suppose for contradiction that there is a sequence $h_j \to 0$ such that

$$\left|\langle \phi_{h_j}, \psi_{h_j}\rangle\right| \geq C h_j^{\frac{1-k}{2}}. \tag{46}$$

Taking a subsequence if necessary, there exists defect measure μ for $\{\phi_{h_j}\}$. Further note that ν is still a defect measure for $\{\psi_{h_j}\}$. Defining μ^A as in (12) we decompose $\mu^A = f\nu^A + \lambda^A$. Then applying Theorem 1.2 we have

$$\begin{aligned}
\lim_{j\to\infty} h_j^{\frac{k-1}{2}}\left|\langle \phi_{h_j}, \psi_{h_j}\rangle\right| &\leq C_{n,k}\left(\int_{\Sigma^A} f(1-|\xi'|^2)^{\frac{k-2}{2}} d\nu^A\right)^{1/2} \\
&= C_{n,k}\left(\int_{\Sigma^A\cap\mathcal{R}_A} f(1-|\xi'|^2)^{\frac{k-2}{2}} d\nu^A + \int_{\Sigma^A\setminus\mathcal{R}_A} f(1-|\xi'|^2)^{\frac{k-2}{2}} d\nu^A\right)^{1/2} \\
&= C_{n,k}\left(\int_{\Sigma^A\setminus\mathcal{R}_A} f(1-|\xi'|^2)^{\frac{k-2}{2}} d\nu^A\right)^{1/2}
\end{aligned}$$

where the last line follows from the fact that $\nu^A(\mathcal{R}_A) = 0$. Next, since ν^A and λ^A are mutually singular there exists V and W such that $\nu^A(V) = \lambda^A(W) = 0$ and $\Sigma^A = V \sqcup W$. Therefore we have

$$\begin{aligned}\lim_{j\to\infty} h_j^{\frac{k-1}{2}}\left|\langle\phi_{h_j},\psi_{h_j}\rangle\right| &\le C_{n,k}\left(\int_{\Sigma^A\cap\mathcal{R}_A^c} f(1-|\xi'|^2)^{\frac{k-2}{2}}d\nu^A\right)^{1/2}\\ &= C_{n,k}\left(\int_{(\Sigma^A\setminus\mathcal{R}_A)\cap V} f(1-|\xi'|^2)^{\frac{k-2}{2}}d\nu^A + \int_{(\Sigma^A\setminus\mathcal{R}_A)\cap W} f(1-|\xi'|^2)^{\frac{k-2}{2}}d\nu^A\right)^{1/2}\\ &= C_{n,k}\left(\int_{(\Sigma^A\setminus\mathcal{R}_A)\cap W} (1-|\xi'|^2)^{\frac{k-2}{2}}d\mu^A\right)^{1/2} \le C\mu^A(\Sigma^A\setminus\mathcal{R}_A)^{1/2}\end{aligned}$$

since $\nu^A(V)=0$ and since $\lambda^A(W)=0$ on W, so we can rewrite $\mu^A = f\nu^A$ on W. Next, we use that Lemma 5.1 below gives $\mu^A(\Sigma^A\setminus\mathcal{R}_A)=0$. Thus

$$\lim_{j\to\infty} h_j^{\frac{k-1}{2}}\left|\langle\phi_{h_j},\psi_{h_j}\rangle\right| = 0$$

which contradicts (46). □

Finally, we show that $\Sigma^A\setminus\mathcal{R}_A$ is μ^A-measure zero, which will complete the proof of Theorem 1.1.

Lemma 5.1. *Let $H\subseteq M$ and suppose that $\{\phi_h\}$ is a sequence of eigenfunctions with defect measure μ. Then*

$$\mu^A(\mathcal{R}_A) = \mu^A(\Sigma^A).$$

Proof. Let $B\subseteq\Sigma^A$ be an open set. For $\delta>0$ sufficiently small, define

$$B_\delta = \bigcup_{|t|<3\delta}\varphi_t(B).$$

Since (S^*M,μ,φ_t) forms a measure preserving system, the Poincaré Recurrence Theorem implies that for μ-a.e. $\rho\in B_\delta$ there exists $t_n^\pm\to\pm\infty$ such that $\varphi_{t_n^\pm}(\rho)\in B_\delta$. Moreover by definition of B_δ there exists $s_n^\pm$ such that $\left|s_n^\pm - t_n^\pm\right|<2\delta$ and $\varphi_{s_n^\pm}(\rho)\in B\subseteq\Sigma^A$. Therefore, for μ-a.e. $\rho\in B_\delta$ we have

$$\bigcap_{T>0}\overline{\bigcup_{t\ge T}\varphi_t(\rho)\cap B}\neq\emptyset \qquad\text{and}\qquad \bigcap_{T>0}\overline{\bigcup_{t\ge T}\varphi_{\bar{t}}(\rho)\cap B}\neq\emptyset \tag{47}$$

since the sets, $\overline{\bigcup_{t\ge T}\varphi_{\pm t}(\rho)\cap B}$ are non-empty, compact, and nested as T increases.

Next, we show that (47) also holds for μ^A-a.e. point in $B\subseteq\Sigma^A$. For contradiction, suppose that there is a set $B'\subseteq B$ with $\mu^A(B')>0$ and for each $\rho\in B'$, there exists a $T>0$ such that

$$\bigcup_{t\ge T}\varphi_t(\rho)\cap B=\emptyset \qquad\text{or}\qquad \bigcup_{t\ge T}\varphi_{-t}(\rho)\cap B=\emptyset.$$

Similarly to [10, Lemma 6], we have $\mu|_{B_\delta}=\mu^A dt$. Therefore, extending B' to $B'_{\delta/3}=\bigcup_{|t|\le\delta}\varphi_t(B')$ we have that

$$\mu(B'_{\delta/3}) = 2\delta \cdot \mu^A(B') > 0.$$

However, this implies (47) does not hold on $B'_{\delta/3} \subseteq B_\delta$ which is a set of positive μ measure, which is a contradiction.

Finally, let $\{B_k\}$ be a countable basis for topology on Σ^A. For all k there exists a $B'_k \subseteq B_k$ of full μ^A measure such that for all $\rho \in B'_k$ (47) holds (with B replaced with B_k). Let $X_k := B'_k \bigcup (\Sigma^A \setminus B_k)$. Following the same argument as in [8, Lemma 15] we find that $\cap_k X_k \subseteq \mathcal{R}_A$. However, we note $\mu^A(X_k) = \mu^A(B'_k) + \mu^A(\Sigma^A \setminus B_k) = \mu^A(B_k) + \mu^A(\Sigma^A \setminus B_k) = \mu^A(\Sigma^A)$. So each X_k has full measure and thus $\cap_k X_k$ has full measure too. Therefore $\mathcal{R}_A$ has full measure too, and we have

$$\mu^A(\mathcal{R}_A) = \mu^A(\Sigma^A)$$

as desired. □

Acknowledgements

The author would like to thank the referees for their constructive feedback. The author would like to thank Yaiza Canzani and Jeffrey Galkowski for many insightful conversations throughout the course of this project and for their feedback on multiple drafts of the article. The author would also like to thank Blake Keeler for the many helpful discussions, especially at the early stages of this project. The author is grateful to the National Science Foundation for their support through the Graduate Research Fellowship Program (DGE-1650116, DGE-2040435).

Funding

National Science Foundation (DGE-1650116, DGE-2040435, DMS-2045494).

ORCID

Madelyne M. Brown http://orcid.org/0000-0002-8249-0458

References

[1] Good, A. (1983). *Local analysis of Selberg's trace formula. volume 1040 of Lecture Notes in Mathematics*. Berlin: Springer-Verlag,

[2] Hejhal, D. A. (1982). Sur certaines séries de Dirichlet associées aux géodésiques fermées d'une surface Riemann compacte. *C.R. Acad. Sci. Paris Sér. I Math*. 294(8):273–276.

[3] Chen, X., Sogge, C. D. (2015). On integrals of eigenfunctions over geodesics. Proceedings of the American Mathematical Society, 143(1): 151–161.

[4] Reznikov, A. (2015). A uniform bound for geodesic periods of eigenfunctions on hyperbolic surfaces. *Forum Math*. 27(3):1569–1590. DOI: 10.1515/forum-2012-0185.

[5] Xi, Y. (2019). Inner product of eigenfunctions over curves and generalized periods for compact riemannian surfaces. *J. Geom. Anal*. 29:2674–2701.

[6] Zelditch, S. (1992). Kuznecov sum formulae and Szegö limit formulae on manifolds. *Commun Partial Differen Equat*. 17(1-2):221–260.

[7] Canzani, Y., Galkowski, J. (2019). Improvements for eigenfunction averages: an application of geodesic beams. *arXiv*. 1809.06296,

[8] Canzani, Y., Galkowski, J. (2019). On the growth of eigenfunction averages: microlocalization and geometry. *Duke J. Math*. 168(16):2991–3055.

[9] Canzani, Y., Galkowski, J. (2021). Eigenfunction concentration via geodesic beams. *J. für Die Reine Und Angew. Math.* 2021(775):197–257.
[10] Canzani, Y., Galkowski, J., Toth, J. A. (2018). Averages of eigenfunctions over hypersurfaces. *Commun. Math. Phys.* 360(2):619–637. DOI: 10.1007/s00220-017-3081-9.
[11] Sogge, C. D., Xi, Y., Zhang, C. (2017). Geodesic period integrals of eigenfunctions on Riemannian surfaces and the Gauss-Bonnet theorem. *Cambridge J. Math.* 5(1):123–151. DOI: 10.4310/CJM.2017.v5.n1.a2.
[12] Wyman, E. L. (2017). Integrals of eigenfunctions over curves in surfaces of nonpositive curvature. *arXiv.* 1702.03552,
[13] Wyman, E. L. (2019). Looping directions and integrals of eigenfunctions over submanifolds. *J Geom. Anal.* 29(2):1302–1319. DOI: 10.1007/s12220-018-0039-x.
[14] Wyman, E. L. (2020). Explicit bounds on integrals of eigenfunctions over curves in surfaces of nonpositive curvature. *J. Geom. Anal.* 30(3):3204–3232. DOI: 10.1007/s12220-019-00198-z.
[15] Wyman, E. L. (2020). Period integrals in nonpositively curved manifolds. *Math. Res. Lett.* 27(5):1513–1563. DOI: 10.4310/MRL.2020.v27.n5.a10.
[16] Wyman, E. L., Xi, Y. (2021). Improved generalized periods estimates over curves on Riemannian surfaces with nonpositive curvature. *Forum Math.* 33(3):789–807. DOI: 10.1515/forum-2020-0264.
[17] Zworski, M. (2012). *Semiclassical analysis, volume 138 of graduate studies in mathematics* Providence, R: American Mathematical Society.
[18] Wyman, E. L., Xi, Y., Zelditch, S. (2020). Fourier coefficients of restrictions of eigenfunctions. *arXiv.* 2011.11571,
[19] Wyman, E. L., Xi, Y., Zelditch, S. (2021). Geodesic bi-angles and Fourier coefficients of restrictions of eigenfunctions. *arXiv.* 2104.09470,
[20] Sogge, C. D., Toth, J. A., Zelditch, S. (2011). About the blowup of quasimodes on Riemannian manifolds. *J. Geom. Anal.* 21(1):150–173. DOI: 10.1007/s12220-010-9168-6.
[21] Burq, N., Gérard, P., Tzvetkov, N. (2007). Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. *Duke J. Math.* 138(3):445–486.
[22] Folland, G. B. (1999). *Real Analysis: Modern Techniques and Their Applications*, 2nd ed. New York: Wiley.
[23] Galkowski, J. (2019). Defect measures of eigenfunctions with maximal l^∞ growth. *Ann. L'inst. Fourier.* 69(4):1757–1798.
[24] Galkowski, J. (2019). Minicourse on eigenfunctions-outline.