

STABILITY OF THE BULK GAP FOR FRUSTRATION-FREE TOPOLOGICALLY ORDERED QUANTUM LATTICE SYSTEMS

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ABSTRACT. We prove that uniformly small short-range perturbations do not close the bulk gap above the ground state of frustration-free quantum spin systems that satisfy a standard local topological quantum order condition. In contrast with earlier results, we do not require a positive lower bound for finite-system spectral gaps uniform in the system size. To obtain this result, we extend the Bravyi-Hastings-Michalakis strategy so it can be applied to perturbations of the GNS Hamiltonian of the infinite-system ground state.

1. INTRODUCTION

One of the characteristic properties of gapped topologically ordered ground state phases of quantum many-body systems is the stability of the spectral gap above the ground state with respect to small perturbations of the Hamiltonian. Stability results for the ground state gap have a long history. The first result, due to Yarotsky [64], included the stability of the gap of the AKLT chain, named after Affleck, Kennedy, Lieb, and Tasaki [1]. The approach we follow in this paper has a much broader range of applicability; it was introduced by Bravyi, Hastings, and Michalakis [12] and further developed in [13, 26, 41, 48]. Other approaches have been introduced in recent years [17, 20–22]. These new approaches can also treat some cases of models with unbounded on-site Hamiltonians, see [48, Section 1] for a more detailed discussion. The Bravyi-Hastings-Michalakis strategy, however, is the only approach that handles general cases with non-trivial topological order.

One obstacle to proving spectral gaps for topological insulators is the common occurrence of gapless edge states. Spectral analysis for interacting many-body systems is usually carried out for finite systems for which edge states typically imply that there is no spectral gap uniform in the system size. Nevertheless, there may be a bulk gap, meaning excitations away from the boundary of the system have energy bounded below uniformly in the system size. The goal of this work is to prove stability for the bulk gap in a way that does not require the assumption of a uniform positive lower bound in the spectrum of finite systems. Previously, it was shown how certain cases can be handled by considering sequences of finite systems with suitable boundary conditions. For example, such an approach may work if the edge states are absent in the model considered with periodic boundary conditions [41, 48]. In general, however, there may not be a suitable boundary condition that ‘gaps out’ the boundary modes or we may not know whether such a boundary condition exists. Systems defined on a quasicrystal structure, for example, may be an instance where no simple way of removing gapless edge modes is available [39]. In our approach here, we only assume that the infinite system described in the Gelfand-Naimark-Segal (GNS) representation of the ground state has a gap. Under natural assumptions consistent with the previous works cited above, we prove that sufficiently small but extensive perturbations do not close the gap.

We adapt the strategy of Bravyi, Hastings, and Michalakis [12, 13, 41] and use the techniques we developed in [47, 48] to handle the infinite system setting. From a certain perspective, and apart from the technical aspects to deal with unbounded Hamiltonians, the infinite system setting allows for a simplification in the statement of conditions and the main result. In particular, the local topological quantum order (LTQO) condition is simpler to state directly for the infinite system.

The LTQO property is well known to hold for one-dimensional systems with matrix product state (MPS) ground states [48, Appendix B]. It is also well-established for Kitaev's quantum double models [16,34] and the Levin-Wen string-net model [37,57]. LTQO was recently also shown to hold for the AKLT model on a decorated hexagonal lattice [40].

For concreteness, we work in the quantum spin system setting but, using the arguments of [46], our approach is applicable to frustration-free lattice fermion systems too. We will provide a detailed account of this in a forthcoming paper, thus providing an alternative and generalization of the gap stability results for quasi-free lattice fermions in the literature [17, 26, 35].

The assumption that the bulk Hamiltonian has a gap in the spectrum above the ground state appears in several important recent works. For example, the construction of an index for the classification of symmetry-protected topological phases in the works of Ogata and co-authors makes use of this assumption [43, 49–51]. Other examples are in the recent work on adiabatic theorems for infinite many-body systems [3, 4, 29]. All of these works use the same general setting as described here in Section 2.

In addition to the main stability result, we also prove Theorem B.1, which shows that a differentiability assumption introduced in [43] and also used in later work [49, 51, 52, 56] is always satisfied. A similar result appeared in the PhD dissertation of Moon [42, Appendix]. As a consequence, we establish a stability theorem for the bulk gap in the exact setting of the series of works of Ogata and others on the classification of Symmetry Protected Topological (SPT) phases cited above. This was the main motivation for this work.

Before closing this introduction, we comment on two specific applications of the stability theorem for infinite systems proved in this paper, and briefly elaborate on the relevance of the spectral analysis for infinite systems.

The first concerns spin liquids, specifically the ongoing search for two-dimensional spin models with topologically non-trivial spin liquid ground states characterized by a bulk gap and gapless edge modes. A plausible example of a rich phase diagram showing such a phase is found based on various numerical approaches for an $SU(3)$ -invariant antiferromagnetic in [63]. As the discussion in that work illustrates, the existence of such phases and the precise nature of the edge modes is a topic of ongoing intense study. Assuming one has established the existence of such a phase, even if the expected properties of the local gaps and LTQO can be demonstrated, the question of the nature of the edge states may remain a challenge. What we can say based on our work is that the putative chiral spin liquid phases, if predictions based on numerical results are confirmed, are stable gapped phases in the usual sense. Granted, if the relevant models can be shown to be uniformly gapped on finite volumes with a torus geometry, there would be no need work in the GNS representation to prove stability of the gap. Next, we discuss an application where periodic boundary conditions are not an option and a general strategy to exclude edge modes in finite volume is not known.

The second application is the case of models defined on quasi-crystal lattices we mentioned before. Gapless edge modes are also expected to be relevant there. So far, they have been studied only in the single-particle setting with the goal to numerically distinguish the behavior of edge modes from bulk spectrum [39] or to prove the existence of a bulk gap in the presence of gapless edge modes [28]. These approaches have yet to be generalized to the many-body setting. Our approach here assumes only a general regularity property of the lattice, which is easily satisfied by quasi-crystals.

As a final remark, we wish to emphasize that significant progress has been made over the years studying both infinite systems as well as properties of finite systems and studying their asymptotics. Valid points in favor of either approach have been made in the literature. For example, in [15, Section 2.5], the authors argue that the mathematics of the infinite system setting could be a distraction for the audience they have in mind. In [62], the authors are motivated by the common observation that ‘many features of physical systems, both qualitative and quantitative, become sharply defined or tractable only in some limiting situation’, and introduce a general approach to

formulate the dynamics of such limiting systems. The infinite system setting has recently seen increased interest and continues to serve as the setting of new interesting results, for example in [31, 53]. In our view, both approaches are fruitful in their particular contexts, and so proving stability directly in the infinite volume setting is interesting in its own right. Other works have similarly obtained infinite-system results for automorphic equivalence [43], the Lieb-Schultz-Mattis Theorem [54, 55], invertible states [5], and adiabatic theory [29].

In a previous paper on the topic, we discussed in some detail the literature on spectral gap stability, including historically important works and alternative methods [48]. Here, we just like to add to that discussion by mentioning two recent works based on adaptions of the iterative Lie-Schwinger block-diagonalization method introduced by Fröhlich and Pizzo [22]. The new work aims at a more detailed analysis of two well-known quantum spin chains, the XXZ chain [19] and the AKLT model [18], supplementing the basic spectral gap stability obtained in previous works, including [41, 48, 61, 64].

The content of this paper is structured as follows. Section 2 provides the setup for interactions with stretched exponential decay we use, and includes the statement of the main results. This decay class can be regarded as a minor variation of the notions of almost-local observables used in [5, 30] and the Banach spaces of g -local observables in [43]. The main technical advance in this work is the incorporation of the (unbounded) GNS Hamiltonian in the development of the Bravyi-Hastings-Michalakis (BHM) strategy for proving gap stability. Properties such as the definition of the transformed Hamiltonian and its decomposition require a new look. Some properties that are obvious in the finite-system setting become non-trivial because they involve super operators formally acting on densely defined unbounded operators on a domain for which an explicit description is not available. We lay the ground work for dealing with these issues in Section 3, and carry out the BHM strategy in Section 4. Section 5 contains the proof of the form bound for the transformed Hamiltonian, which is a mild adaptation of a result of Michalakis and Zwolak [41] to the GNS setting. The final arguments needed to prove the main results are also contained in this section. In two appendices we prove two results that may be of independent interest. The first is that the LTQO property implies the kernel of the GNS Hamiltonian is one dimensional (something we use and that is often introduced as a separate assumption on other works). The second is about the differentiability of the quasi-adiabatic dynamics and is a variant of a result that first appeared in the PhD dissertation of Moon [42].

2. SETUP AND STATEMENT OF THE MAIN RESULTS

2.1. Setup and notation. The models considered in this work are defined on a ν -regular discrete metric space (Γ, d) , for some $\nu > 0$. This means that there exists $\kappa > 0$ so that for all $x \in \Gamma, n \geq 1$, $|b_x(n)| \leq \kappa n^\nu$, where $b_x(n) = \{y \in \Gamma \mid d(x, y) \leq n\}$. For $\Lambda \in \mathcal{P}_0(\Gamma)$, the finite subsets of Γ , and $n \geq 0$, we also define the sets $\Lambda(n)$ by

$$(2.1) \quad \Lambda(n) = \bigcup_{x \in \Lambda} b_x(n).$$

The algebra of local observables of the system is the usual $\mathcal{A}^{\text{loc}} = \bigcup_{\Lambda \in \mathcal{P}_0(\Gamma)} \mathcal{A}_\Lambda$. Here, \mathcal{A}_Λ is the matrix algebra $\bigotimes_{x \in \Lambda} M_{d_x}$ with d_x the dimension of the spin at x . The C^* -algebra of quasi-local observables \mathcal{A} is the completion of \mathcal{A}^{loc} with respect to the operator norm. For $A \in \mathcal{A}^{\text{loc}}$, the *support* of A , denoted by $\text{supp } A$, is the smallest $X \subset \Gamma$ such that $A \in \mathcal{A}_X$. For any $X \subset \Gamma$, $\Pi_X : \mathcal{A} \rightarrow \mathcal{A}_X$ is the conditional expectation with respect to the tracial state ρ on \mathcal{A} :

$$(2.2) \quad \Pi_X = \rho \upharpoonright_{\Gamma \setminus X} \otimes \text{id}_{\mathcal{A}_X}.$$

In particular, for local A , $\Pi_X(A)$ is a normalized partial trace.

We are specifically interested in systems defined on *infinite* Γ and often want to consider approximations $A_n \in \mathcal{A}_{\Lambda_n}$ of $A \in \mathcal{A}$, where $\Lambda_n \in \mathcal{P}_0(\Gamma)$ is an increasing sequence of finite volumes

such that $\bigcup_n \Lambda_n = \Gamma$. We call such a sequence (Λ_n) an *increasing and absorbing sequence* (IAS). It will often be important to have an estimate for the speed of convergence of $A_n \rightarrow A$, in terms of a non-increasing function $g : [0, \infty) \rightarrow (0, \infty)$ that vanishes at infinity, which we call a *decay function*. In this paper we will only use decay functions that satisfy a moment condition of the form

$$(2.3) \quad \sum_{n \geq 0} (n+1)^\mu g(n) < \infty, \quad \text{for some } \mu \geq \nu.$$

In particular, we will often work with decay functions of the form

$$(2.4) \quad g(r) = \frac{1}{(1+r)^\xi} e^{-ar^\theta}, \quad \xi \geq 0, a \geq 0, \theta \in [0, 1].$$

For $a > 0$ and $\theta \in (0, 1)$, such functions g are said to have *stretched exponential decay*. One checks that these functions satisfy

$$(2.5) \quad g(n)g(m) \leq Cg(n+m) \quad \text{for all } n, m \geq 0,$$

for some constant C .

Consider an IAS (Λ_n) in (Γ, d) , and a decay function g . Define a norm $\|\cdot\|_{(\Lambda_n),g}$ on \mathcal{A}^{loc} and a Banach space $\mathcal{A}^{(\Lambda_n),g}$ by

$$(2.6) \quad \|A\|_{(\Lambda_n),g} = \|A\| + \sup_{n \in \mathbb{N}} \left(\frac{\|A - \Pi_{\Lambda_n}(A)\|}{g(n)} \right), \quad \mathcal{A}^{(\Lambda_n),g} = \overline{\mathcal{A}^{\text{loc}}}^{\|\cdot\|_{(\Lambda_n),g}}.$$

For a proof that $\mathcal{A}^{(\Lambda_n),g}$ is the Banach space of all $A \in \mathcal{A}$ for which $\|A\|_{(\Lambda_n),g} < \infty$, see [43]. In fact, $\mathcal{A}^{(\Lambda_n),g}$ is a Banach $*$ -algebra.

For each $x \in \Gamma$, $\Lambda_n := b_x(n)$ defines a IAS. In this case we set $\|\cdot\|_{(b_x(n)),g} = \|\cdot\|_{x,g}$. Define the set

$$(2.7) \quad \mathcal{A}^g := \bigcup_{x \in \Gamma} \mathcal{A}^{(b_x(n)),g}.$$

For any decay function g satisfying (2.5), any two norms from $\{\|\cdot\|_{x,g} \mid x \in \Gamma\}$ are equivalent. Hence, for all $x \in \Gamma$

$$(2.8) \quad \mathcal{A}^{(b_x(n)),g} = \mathcal{A}^g.$$

In this case, \mathcal{A}^g is a Banach $*$ -algebra. Elements $A \in \mathcal{A}^g$ are called *g-local*.

We will often also assume that a decay function g is uniformly summable over Γ , i.e.,

$$(2.9) \quad \|g\|_1 := \sup_{x \in \Gamma} \sum_{y \in \Gamma} g(d(x, y)) < \infty,$$

and additionally, that there is a constant $C > 0$ such that

$$(2.10) \quad \sum_{z \in \Gamma} g(d(x, z))g(d(z, y)) \leq Cg(d(x, y)), \quad \text{for all } x, y \in \Gamma.$$

Any decay function g satisfying (2.9) and (2.10) will be called an *F*-function. For any ν -regular Γ , the following are *F*-functions appearing in this work:

$$(2.11) \quad F(r) = \frac{1}{(1+r)^\xi} e^{-ar^\theta}, \quad \xi > \nu + 1, a \geq 0, \theta \in (0, 1].$$

In the case $\Gamma = \mathbb{Z}^\nu$, which is ν -regular, (2.11) defines an *F*-function for all $\xi > \nu$. For a discussion of these examples and some basic inequalities, see [47, Appendix].

Assumption 2.1 (Initial Interaction). We assume the initial model is defined by a finite-range, uniformly bounded, frustration-free interaction h given in terms of a family $h = \{h_x\}_{x \in \Gamma}$ which satisfies:

- i. There is a number $R \geq 0$, called the interaction radius, for which $h_x^* = h_x \in \mathcal{A}_{b_x(R)}$ for all $x \in \Gamma$.
- ii. These terms are uniformly bounded in the sense that

$$(2.12) \quad \|h\|_\infty = \sup_{x \in \Gamma} \|h_x\| < \infty.$$

- iii. The interaction is *frustration-free*, meaning that $h_x \geq 0$ for all $x \in \Gamma$ and for any $\Lambda \in \mathcal{P}_0(\Gamma)$,

$$(2.13) \quad \min \text{spec}(H_\Lambda) = 0 \quad \text{where} \quad H_\Lambda = \sum_{\substack{x \in \Lambda: \\ \text{supp}(h_x) \subset \Lambda}} h_x.$$

The frustration-free condition implies that the ground state space is $\ker(H_\Lambda)$ for any finite volume Λ . Moreover, $\psi \in \ker(H_\Lambda)$ if and only if $\psi \in \ker(h_x)$ for each $x \in \Lambda$ with $\text{supp}(h_x) \subset \Lambda$. Thus, denoting by P_Λ the orthogonal projection onto $\ker(H_\Lambda)$, for any $\Lambda_0 \subset \Lambda$, one has

$$(2.14) \quad P_\Lambda P_{\Lambda_0} = P_{\Lambda_0} P_\Lambda = P_\Lambda.$$

For such a model, the derivation δ_0 determining the infinite system dynamics is given by

$$(2.15) \quad \delta_0(A) = \sum_{x \in \Lambda(R)} [h_x, A] \quad \text{for any } A \in \mathcal{A}_\Lambda \text{ and } \Lambda \in \mathcal{P}_0(\Gamma).$$

It is a standard result that there is a closed derivation extending δ_0 , which we also denote by δ_0 , with domain $\text{dom}(\delta_0)$ for which \mathcal{A}^{loc} is a core [11, Theorem 6.2.4] (note that the factor i is absorbed in the definition of the derivation in this reference). The system dynamics is then the strongly continuous one-parameter group of C^* -automorphisms $\{\tau_t^{(0)} \mid t \in \mathbb{R}\}$ satisfying

$$(2.16) \quad \frac{d}{dt} \tau_t^{(0)}(A) = i \tau_t^{(0)}(\delta_0(A)) \quad \text{for all } A \in \mathcal{A}^{\text{loc}}.$$

In fact, this differential equation holds for all $A \in \text{dom}(\delta_0)$. Two other general properties are:

- i. $\tau_t^{(0)}(\text{dom}(\delta_0)) \subset \text{dom}(\delta_0)$ for all $t \in \mathbb{R}$;
- ii. $\tau_t^{(0)}(\delta_0(A)) = \delta_0(\tau_t^{(0)}(A))$ for all $A \in \text{dom}(\delta_0)$ and $t \in \mathbb{R}$.

More generally, quantum spin models can be defined by an interaction on Γ which, by definition, is a map $\Phi : \mathcal{P}_0(\Gamma) \rightarrow \mathcal{A}^{\text{loc}}$, with the property that $\Phi(X)^* = \Phi(X) \in \mathcal{A}_X$ for all $X \in \mathcal{P}_0(\Gamma)$. For any decay function g , an interaction norm is defined by

$$(2.17) \quad \|\Phi\|_g = \sup_{x,y \in \Gamma} g(d(x,y))^{-1} \sum_{\substack{X \in \mathcal{P}_0(\Gamma): \\ x,y \in X}} \|\Phi(X)\|.$$

When the above quantity is finite for some interaction Φ , the function g is said to measure the decay of Φ . If g is an F -function, the norm $\|\cdot\|_g$ is called an *F -norm*. If g is summable, in the sense of (2.9), and $\|\Phi\|_g < \infty$, then a closable derivation on \mathcal{A}^{loc} can be defined by setting

$$(2.18) \quad \delta(A) = \sum_{Y, Y \cap X \neq \emptyset} [\Phi(Y), A] \quad \text{for } A \in \mathcal{A}^{\text{loc}} \text{ with } \text{supp}(A) \subset X \in \mathcal{P}_0(\Gamma).$$

One can prove conditions that guarantee that the derivation δ defined on \mathcal{A}^{loc} is a generator of a strongly continuous dynamics given by automorphisms of \mathcal{A} [10, 11]. In practice, however, one usually directly proves the existence of the thermodynamic limit of the Heisenberg dynamics $\{\tau_t \mid t \in \mathbb{R}\}$. Standard results along these lines prove the existence of the dynamics for Φ in a suitable Banach space of interactions [11, 59, 60] starting from a convergent series for small $|t|$. An alternative approach, based on Lieb-Robinson bounds [38], was introduced by Robinson [58]. Lieb-Robinson bounds can be derived for any interaction Φ with a finite F -norm [44], and this allows one to extend the results for existence of the dynamics beyond the Banach spaces of interactions \mathcal{B}_λ introduced by Ruelle [59]. These ideas are important for the construction of the spectral flow

automorphisms [7]. This and some other generalizations relevant for the present work are discussed in detail in [47].

Recall that infinite-volume ground states associated to δ are those states ω on \mathcal{A} that satisfy

$$(2.19) \quad \omega(A^* \delta(A)) \geq 0 \quad \text{for all } A \in \mathcal{A}^{\text{loc}}.$$

In the case of a frustration-free model as in (2.13), a state ω is called a zero-energy ground state, or a frustration free ground state, if $\omega(h_x) = 0$ for all $x \in \Gamma$. It is easy to see that a zero-energy ground state satisfies (2.19).

Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of ω . This means that $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a representation of the C^* -algebra \mathcal{A} on a Hilbert space \mathcal{H} for which $\{\pi(A)\Omega | A \in \mathcal{A}^{\text{loc}}\}$ is dense in \mathcal{H} . Moreover, the normalized vector $\Omega \in \mathcal{H}$ is such that $\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathcal{A}$. For the GNS representation of a ground state, as in (2.19), there exists a unique, non-negative self-adjoint operator H on \mathcal{H} , with dense domain $\text{dom } H$, satisfying $H\Omega = 0$ and

$$(2.20) \quad \pi(\tau_t(A)) = e^{itH} \pi(A) e^{-itH} \quad \text{for all } A \in \mathcal{A} \text{ and } t \in \mathbb{R}.$$

The full domain of H is seldom described explicitly. However, for all systems we consider in this paper, $\pi(\mathcal{A}^{\text{loc}})\Omega$ is a core for H .

The (GNS) *gap* of the model in the state ω is defined as

$$(2.21) \quad \text{gap}(H) = \sup\{\gamma > 0 \mid (0, \gamma) \cap \text{spec}(H) = \emptyset\}.$$

If the set on the RHS is empty, one defines $\text{gap}(H) = 0$. We say that a ground state ω is gapped if $\text{gap}(H) > 0$.

The equivalence of the following two conditions is easy to verify:

i. For some $\gamma > 0$, ω satisfies

$$(2.22) \quad \omega(A^* \delta(A)) \geq \gamma \omega(A^* A) \quad \text{for all } A \in \mathcal{A}^{\text{loc}} \text{ with } \omega(A) = 0;$$

ii. The ground state of the GNS Hamiltonian H is unique and $\text{gap}(H) \geq \gamma$.

A case of special interest is when Γ is infinite and describes the bulk of a physical model while the same system on a subset of Γ with a boundary would describe an edge. In the first situation we will refer to the GNS gap as the *bulk gap* of the system. A model with the same interaction restricted to a subspace of Γ describing an edge, may have a vanishing gap while the bulk gap is positive. This is precisely the situation of interest here.

In this setting, the GNS representation π is an isometry. This follows from the fact that \mathcal{A} is simple [23, Theorem 5.1], which implies that $\ker \pi = \{0\}$. We find it convenient to use this fact, see e.g. the proof of Lemma 4.5, however in many arguments the contraction property of π suffices.

2.2. Main results.

We now state the assumptions for the main results.

Assumption 2.2 (Bulk gap). We assume $\gamma_0 := \text{gap}(H_0) > 0$, where H_0 is the GNS Hamiltonian of an infinite-volume, zero-energy ground state ω_0 of a finite-range, uniformly bounded, frustration free interaction $\{h_x\}$ as in Assumption 2.1.

We also need to impose a condition that the local gaps do not close too fast. There generally is some freedom in choosing the family of finite volumes on which to impose this condition. We will assume that there is a family $\mathcal{S} = \{\Lambda(x, n) \mid n \geq 0, x \in \Gamma\} \subset \mathcal{P}_0(\Gamma)$, with $b_x(n) \subset \Lambda(x, n)$ for all x and n , and an associated family of partitions of Γ which *separates \mathcal{S} and has at most polynomial growth*. Concretely, this means there is a family of sets $\mathcal{T} = \{\mathcal{T}_n \mid n \geq 0\}$ and positive numbers c and ζ , such that for each $n \geq 0$, $\mathcal{T}_n = \{\mathcal{T}_n^i : i \in \mathcal{I}_n\}$ is a partition of Γ satisfying $|\mathcal{I}_n| \leq cn^\zeta$ and

$$(2.23) \quad \Lambda(x, n) \cap \Lambda(y, n) = \emptyset \quad \text{for all } x, y \in \mathcal{T}_n^i \text{ with } x \neq y.$$

In such cases, we say that \mathcal{T} is of (c, ζ) -polynomial growth.

As an example, in the case of $\Gamma = \mathbb{Z}^\nu$, we may take for $\Lambda(x, n)$ the ℓ^∞ -ball of radius n centered at x , define $\mathcal{I}_n = \Lambda(0, n)$ and, for each $i \in \mathcal{I}_n$, set

$$(2.24) \quad \mathcal{T}_n^i = \{x \in \mathbb{Z}^\nu \mid x_j = i_j \pmod{2n+1}, 1 \leq j \leq \nu\}.$$

Assumption 2.3 (Local gaps). For an interaction $\{h_x\}$ of range R , we assume there exist families \mathcal{S} and \mathcal{T} , such that \mathcal{T} separates \mathcal{S} and is of ζ -polynomial growth, and an exponent $\alpha \geq 0$ and constant $\gamma_1 > 0$, and such that the finite-volume Hamiltonians satisfy:

$$(2.25) \quad \text{spec}(H_{\Lambda(x, n)}) \subset \{0\} \cup [n^{-\alpha}\gamma_1, \infty) \quad \text{for all } x \in \Gamma \text{ and } n \geq R.$$

It is important here that the local gaps are allowed to vanish in the limit of infinite system size. For example, certain types of topologically ordered two-dimensional systems are expected to have chiral edge modes with an energy of order L^{-1} on a finite volume of diameter L . Whether or not such edge modes occur in frustration-free systems, however, is not clear. For the class of systems studied in [36], the authors find that finite-volume gaps of a system with gapless edge modes in the thermodynamic limit would have to decay at least as fast as $L^{-3/2}$. Other results of this type are in [2, 24, 32]. This is consistent with the gapless boundary modes found in a class of toy models called Product Vacua with Boundary States which are of order L^{-2} [6, 8]. In any case, regardless of the possible values of the exponent α , we will prove stability of the *bulk* gap.

The next assumption was introduced in the form we use here in [41] where it is called *Local Topological Quantum Order* (LTQO).

Assumption 2.4 (LTQO). There is a decay function $G_0 : [0, \infty) \rightarrow [0, \infty)$, with

$$(2.26) \quad \sum_{n \geq 0} n^q G_0(n) < \infty \text{ for some } q > 2(\nu + \zeta + \alpha),$$

and such that for all $m \geq k \geq 0$, $x \in \Gamma$, and $A \in \mathcal{A}_{b_x(k)}$, the ground state projections satisfy

$$(2.27) \quad \|P_{b_x(m)} A P_{b_x(m)} - \omega_0(A) P_{b_x(m)}\| \leq \|A\| (1+k)^\nu G_0(m-k).$$

As explained in detail in [48, Section 8], if both the initial Hamiltonian and the perturbation (see below) have a local gauge symmetry, only observables A that commute with this symmetry need to satisfy (2.27). Other discrete symmetries can be treated similarly (see [48, Section 8]). Therefore, the stability results proved here (Theorems 2.7 and 2.8) will also hold for symmetry-protected topological phases.

An interesting observation is that the GNS Hamiltonians associated to frustration free models which satisfy Assumption 2.4 automatically have a unique ground state. Since we also use this fact, see e.g. Section 3.2.3, we include a short proof in Appendix A.

Next, we turn to the perturbations of the Hamiltonian H_0 . We consider $\Phi(x, n)^* = \Phi(x, n) \in \mathcal{A}_{b_x(n)}$ for all $x \in \Gamma$ and $n \geq 0$. These define what we call an *anchored* interaction Φ . By regrouping, we need only consider those terms with $n \geq R$.

Assumption 2.5 (Short-range perturbation). There is a constant $\|\Phi\| \geq 0$, $a > 0$, and $\theta \in (0, 1]$ such that for all $x \in \Gamma$

$$(2.28) \quad \|\Phi(x, n)\| \leq \|\Phi\| e^{-an^\theta} \quad \text{for all } n \geq R.$$

Remark 2.6. For a particular perturbation Φ , Assumption 2.5 is typically straight forward to check. Moreover, any such Φ also has a finite F -norm for some F as in (2.11). This implies the general locality results found, e.g., in [47] necessarily hold. In fact, let Φ satisfy Assumption 2.5. For any $0 < a' < a$ and $\xi > \nu$, the function $F : [0, \infty) \rightarrow (0, \infty)$ given by

$$(2.29) \quad F(r) = \frac{e^{-a'r^\theta}}{(1+r)^\xi} \quad \text{for all } r \geq 0,$$

is an F -function on Γ . Let $\delta = a - a' > 0$ and note that for any $x, y \in \Gamma$ with $d(x, y) \geq R$, we have

$$(2.30) \quad \begin{aligned} \sum_{z \in \Gamma} \sum_{\substack{n \geq R: \\ x, y \in b_z(n)}} \|\Phi(z, n)\| &= \sum_{n \geq d(x, y)} \sum_{z \in b_y(n) \cap b_x(n)} \|\Phi(z, n)\| \\ &\leq \kappa \|\Phi\| \sum_{n \geq d(x, y)} n^\nu e^{-an^\theta} \leq CF(d(x, y)) \end{aligned}$$

with $C := \kappa \|\Phi\| \sum_{n \geq 0} (1+n)^{\nu+\xi} e^{-\delta n^\theta} < \infty$. Thus, $\|\Phi\|_F \leq C$, with F as in (2.29).

The focus of this work is to analyze the stability of the bulk gap under the presence of perturbations given by an anchored interaction Φ satisfying Assumption 2.5. We will prove two different stability results. The first, Theorem 2.7 below, establishes that finite-volume perturbations of the GNS Hamiltonian H_0 (defined in (2.31-2.32) below) have a stable spectral gap uniform in the support of the perturbation. In this case, the initial GNS Hamiltonian and the perturbed Hamiltonians are all defined on the same GNS Hilbert space.

To describe the second result, Theorem 2.8 below, let δ_Φ denote the derivation defined by Φ as in (2.18). This result shows that, for sufficiently small s , there exists a ground state ω_s of the perturbed derivation $\delta_0 + s\delta_\Phi$, whose GNS Hamiltonian has a positive gap. Although the GNS representations of ω_s are, in general, inequivalent for different values of s , Theorem 2.8 will follow from Theorem 2.7 by a simple argument given in Section 5.2.

We consider perturbed Hamiltonians of the form

$$(2.31) \quad H(\Lambda, s) = H_0 + s\pi_0(V_\Lambda), \quad s \in \mathbb{R}$$

where, for any finite volume $\Lambda \in \mathcal{P}_0(\Gamma)$,

$$(2.32) \quad V_\Lambda = \sum_{x \in \Lambda} \sum_{\substack{n \geq R: \\ b_x(n) \subset \Lambda}} \Phi(x, n).$$

Clearly, $V_\Lambda \in \mathcal{A}_\Lambda$ is bounded and self-adjoint, and so $H(\Lambda, s)$ defines for all $s \in \mathbb{R}$ a self-adjoint Hamiltonian on \mathcal{H} with the same dense domain as H_0 .

In the next several sections we will prove the following theorem, which establishes that the spectral gap of $H(\Lambda, s)$ remains open for small $|s|$ uniformly in the finite volume Λ .

Theorem 2.7 (Stability of the gap uniformly in the perturbation region). *Suppose that $\{h_x\}$ and ω_0 satisfy Assumptions 2.2 – 2.4, and Φ is an anchored interaction satisfying Assumption 2.5. Then, for all $\gamma \in (0, \gamma_0)$, there exists $s_0(\gamma) > 0$, such that for all real s , $|s| < s_0(\gamma)$, and $\Lambda \in \mathcal{P}_0(\Gamma)$, we have*

$$(2.33) \quad \text{spec } H(\Lambda, s) \subset \{E(\Lambda, s)\} \cup [E(\Lambda, s) + \gamma, \infty)$$

with $H(\Lambda, s)$ as in (2.31) and

$$(2.34) \quad E(\Lambda, s) = \inf \text{spec } H(\Lambda, s).$$

We remark that the quantity $s_0(\gamma)$ only depends on the values of κ and ν of the lattice, $\|h\|$, the gap γ_0 , the parameters in Assumption 2.3, the decay function in Assumption 2.4, and a suitable F -norm of the perturbation Φ . In particular, $s_0(\gamma)$ is independent of the finite volume Λ . From the arguments in this paper, one can derive an explicit lower bound for $s_0(\gamma)$ in terms of these quantities, see Section 5.2.

We also investigate the situation where the perturbation region Λ tends to all of Γ . Consider any IAS (Λ_n) . We will denote by $\tau_t^{(\Lambda_n, s)}$ the dynamics on \mathcal{A} generated by the derivation

$$(2.35) \quad \delta_s^{\Lambda_n}(A) = \delta_0(A) + [sV_{\Lambda_n}, A] \quad \text{for } A \in \mathcal{A}^{\text{loc}}.$$

As discussed in [47, Definition 3.7], the sequence of interactions $h + s\Phi \restriction_{\Lambda_n}$ converges locally in F -norm to the interaction $h + s\Phi$. Using [47, Theorem 3.8], we conclude local convergence in the sense that

$$(2.36) \quad \lim_{n \rightarrow \infty} \tau_t^{(\Lambda_n, s)}(A) = \tau_t^{(s)}(A) \quad \text{for all } A \in \mathcal{A}^{\text{loc}}$$

as well as

$$(2.37) \quad \lim_{n \rightarrow \infty} \delta_s^{\Lambda_n}(A) = \delta_s(A) \quad \text{for all } A \in \mathcal{A}^{\text{loc}}$$

with $\tau_t^{(s)}$ (respectively, δ_s) being the *a priori* well-defined strongly continuous dynamics on \mathcal{A} (respectively, the closure of the derivation restricted to \mathcal{A}^{loc}) generated by the interaction $h + s\Phi$. Neither of these limits depend on the choice of IAS sequence Λ_n .

Our second result is then concerned with the ground state and its gap for a family of extensive perturbations. In particular, the uniformity of the stability result in Theorem 2.7 allows one to prove, almost as a corollary, that for all $|s| \leq s_0(\gamma)$ there is a gapped ground state ω_s of δ_s in the sense of (2.22). To make this precise, we introduce the limiting spectral flow. For any $\gamma > 0$ and IAS (Λ_n) , take

$$(2.38) \quad \alpha_s(A) = \lim_n \alpha_s^{\Lambda_n}(A) \quad \text{for all } A \in \mathcal{A},$$

where the spectral flows $\alpha_s^{\Lambda_n}$ will be introduced in more detail in the next section, see (3.23). For now, it suffices to observe that this limit exists and is independent of the choice of IAS. In fact, the interactions defining the spectral flows $\alpha_s^{\Lambda_n}$ converge locally in F -norm by arguments as in [47, Section VI.E.2]. This limiting spectral flow α_s defines a strongly continuous co-cycle of automorphisms of \mathcal{A} , and moreover, under the assumptions we have made, for $A \in \mathcal{A}^{\text{loc}}$, $s \mapsto \alpha_s(A)$ is differentiable to all orders. We prove bounded differentiability for $A \in \mathcal{A}^g$, for suitable g in Theorem B.1.

Theorem 2.8 (Stability of the bulk gap). *Under the assumptions of Theorem 2.7, let $\gamma \in (0, \gamma_0)$ and take s with $|s| < s_0(\gamma)$. The state $\omega_s = \omega_0 \circ \alpha_s$ is a gapped ground state of the perturbed infinite dynamics δ_s , i.e.*

$$(2.39) \quad \omega_s(A^* \delta_s(A)) \geq \gamma \omega_s(A^* A) \quad \text{for all } A \in \mathcal{A}^{\text{loc}} \text{ with } \omega_s(A) = 0.$$

In particular, the GNS Hamiltonian H_s of ω_s has a one-dimensional kernel and $\text{spec } H_s$ has a gap above its ground state bounded below by γ .

3. QUASI-LOCALITY, DOMAINS AND LOCAL DECOMPOSITIONS

The strategy used here for proving spectral gap stability of infinite systems relies in an essential way on quasi-locality properties of the observables, the dynamics, and several transformations defined in terms of the dynamics. To this end, we first review these locality properties of the algebra and then, importantly, record how this local structure is mapped into the GNS space. Quasi-locality of observables is the topic of Section 3.1. In Section 3.1.1, we recall general methods for making strictly local approximations of both quasi-local observables and maps. The specific quasi-local maps and estimates used in the stability proof are discussed in Sections 3.1.2 - 3.1.4. Our stability results will follow from spectral perturbation theory applied in the GNS space. Section 3.2 succinctly makes clear the distinction between the relevant objects in the algebra and their counterparts in the GNS representation. Finally, in Section 3.3 we prove how the action of certain unbounded operators on a dense domain can be expressed as limits of sequences of bounded operators with finite support.

3.1. Quasi-LocalitY. We first recall some general features of quasi-locality estimates and then turn to some important examples relevant for this work.

3.1.1. *Quasi-locality estimates.* Let $A \in \mathcal{A}$, $X \in \mathcal{P}_0(\Gamma)$, and $\epsilon > 0$. In [14, 45] it was shown that if

$$(3.1) \quad \| [A, B] \| \leq \epsilon \| B \|, \quad \text{whenever } B \in \mathcal{A}_{\Gamma \setminus X}^{\text{loc}},$$

then

$$(3.2) \quad \| A - \Pi_X(A) \| \leq \epsilon.$$

A linear map $\mathcal{K} : \mathcal{A} \rightarrow \mathcal{A}$ is said to be *quasi-local* with constant $C \geq 0$, power $p \geq 0$, and decay function G if

$$(3.3) \quad \| [\mathcal{K}(A), B] \| \leq C |X|^p \|A\| \|B\| G(d(X, Y)) \quad \text{for all } A \in \mathcal{A}_X \text{ and } B \in \mathcal{A}_Y.$$

Using (3.2), for such a map \mathcal{K} and $A \in \mathcal{A}_{b_x(k)}$, we have

$$(3.4) \quad \| \mathcal{K}(A) - \Pi_{b_x(k+n)}(\mathcal{K}(A)) \| \leq C |b_x(k)|^p \|A\| G(n).$$

When the corresponding decay function G is summable, this estimate guarantees the absolute convergence of telescopic sums, i.e. for any $n_0 \geq 0$,

$$(3.5) \quad \mathcal{K}(A) = \Pi_{b_x(k+n_0)}(\mathcal{K}(A)) + \sum_{n=n_0+1}^{\infty} (\Pi_{b_x(k+n)} - \Pi_{b_x(k+n-1)}) (\mathcal{K}(A))$$

since the terms satisfy

$$(3.6) \quad \| (\Pi_{b_x(k+n)} - \Pi_{b_x(k+n-1)}) (\mathcal{K}(A)) \| \leq 2C |b_x(k)|^p \|A\| G(n-1) \quad \text{for } n \geq 1.$$

A common choice is $n_0 = 0$ and we adopt the notation

$$(3.7) \quad \mathcal{K}(A) = \sum_{n \geq k} \Delta_{b_x(k)}^n(\mathcal{K}(A)) \quad \text{where} \quad \Delta_{b_x(k)}^n = \begin{cases} \Pi_{b_x(k)}, & n = k \\ \Pi_{b_x(n)} - \Pi_{b_x(n-1)}, & n \geq k+1. \end{cases}$$

We now review a few examples of quasi-local maps and indicate some of their important properties which will be used to prove stability of the gap. For more details of these maps see [47]. Throughout Sections 3.1.2–3.2 we work under the assumptions of Theorem 2.7.

3.1.2. *Dynamics.* It is well-known that the unperturbed dynamics $\tau_t^{(0)}$ defined as in (2.16) satisfies an exponential Lieb-Robinson bound [38]. Namely, for every $\mu > 0$ there exists $C_\mu > 0$ and $v_\mu > 0$ such that the bound

$$(3.8) \quad \| [\tau_t^{(0)}(A), B] \| \leq C_\mu \min(|X|, |Y|) \|A\| \|B\| e^{-\mu(d(X, Y) - v_\mu|t|)}$$

holds for any $X, Y \in \mathcal{P}_0(\Gamma)$, all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$.

It is easy to check that the perturbed interaction $h + s\Phi|_\Lambda$ has a finite F -norm for the same F as Φ , and that this F -norm is uniformly bounded in $|s| \leq 1$ and Λ . As a consequence, there are $C_F > 0$ and $v_F > 0$, independent of s and Λ , such that for any choice of $X, Y \in \mathcal{P}_0(\Gamma)$,

$$(3.9) \quad \| [\tau_t^{(\Lambda, s)}(A), B] \| \leq C_F \|A\| \|B\| e^{v_F|t|} \sum_{x \in X} \sum_{y \in Y} F(d(x, y))$$

for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$.

Since each sV_Λ is bounded and self-adjoint, [11, Proposition 5.4.1] implies that

$$(3.10) \quad \tau_t^{(\Lambda, s)}(A) = (K_t^{(\Lambda, s)})^* \tau_t^{(0)}(A) K_t^{(\Lambda, s)} \quad \text{for all } A \in \mathcal{A} \text{ and } t \in \mathbb{R}$$

where $\{K_t^{(\Lambda, s)} \mid t \in \mathbb{R}\}$ is a one-parameter family of unitaries on \mathcal{A} which are uniquely defined as the \mathcal{A} -valued solution of

$$(3.11) \quad \frac{d}{dt} K_t^{(\Lambda, s)} = -i \tau_t^{(0)}(sV_\Lambda) K_t^{(\Lambda, s)} \quad \text{with} \quad K_0^{(\Lambda, s)} = \mathbb{1}.$$

These unitaries are quasi-local as, for any $A \in \mathcal{A}^{\text{loc}}$ and $t > 0$,

$$(3.12) \quad \begin{aligned} \|[K_t^{(\Lambda,s)}, A]\| &= \|(K_t^{(\Lambda,s)})^* A K_t^{(\Lambda,s)} - A\| = \left\| \int_0^t \frac{d}{du} \left((K_u^{(\Lambda,s)})^* A K_u^{(\Lambda,s)} \right) du \right\| \\ &\leq \int_0^t \|\tau_u^{(0)}(sV_\Lambda), A\| du. \end{aligned}$$

An application of (3.8) then shows that for any $\mu > 0$ and $A \in \mathcal{A}_X$ with $X \in \mathcal{P}_0(\Gamma \setminus \Lambda)$,

$$(3.13) \quad \|[K_t^{(\Lambda,s)}, A]\| \leq C_\mu |s| |\Lambda| \|V_\Lambda\| \|A\| |t| e^{\mu v_\mu |t|} e^{-\mu d(\Lambda, X)}$$

for any $s, t \in \mathbb{R}$. Thus, $K_t^{(\Lambda,s)} \in \mathcal{A}^g$ for any exponential g , by (3.2).

3.1.3. Weighted Integral Operators. Fix $\gamma > 0$. For each $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in \mathbb{R}$, we define two weighted integral operators $\mathcal{F}_s^\Lambda : \mathcal{A} \rightarrow \mathcal{A}$ and $\mathcal{G}_s^\Lambda : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(3.14) \quad \mathcal{F}_s^\Lambda(A) = \int_{-\infty}^{\infty} \tau_t^{(\Lambda,s)}(A) w_\gamma(t) dt \quad \text{and} \quad \mathcal{G}_s^\Lambda(A) = \int_{-\infty}^{\infty} \tau_t^{(\Lambda,s)}(A) W_\gamma(t) dt,$$

where the real-valued functions $w_\gamma, W_\gamma \in L^1(\mathbb{R})$, are defined in [47, Section VI.B]. In particular, they decay faster than any stretched exponential. Both of these maps depend on the choice of γ through their weight functions, w_γ and W_γ respectively, but we suppress this in the notation. Arguing as in [47, Section VI.E.1], see also [48, Section 4.3.2], we find that for all $A \in \mathcal{A}$

$$(3.15) \quad \|\mathcal{F}_s^\Lambda(A)\| \leq \|A\| \quad \text{and} \quad \|\mathcal{G}_s^\Lambda(A)\| \leq \|W_\gamma\|_1 \|A\|,$$

where we have used that, by our choice of normalization, $\|w_\gamma\|_1 = 1$. As a result, these maps are bounded uniformly with respect to $s \in \mathbb{R}$ and $\Lambda \in \mathcal{P}_0(\Gamma)$. Moreover, they are uniformly quasi-local in the sense that for each $\mathcal{K} \in \{\mathcal{F}, \mathcal{G}\}$ there is a decay function $G_{\mathcal{K}}$ such that: for any choice of $X, Y \in \mathcal{P}_0(\Gamma)$, we have

$$(3.16) \quad \sup_{s \in [-1, 1]} \|[\mathcal{K}_s^\Lambda(A), B]\| \leq 2\|A\| \|B\| |X| G_{\mathcal{K}}(d(X, Y))$$

for all $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. As shown in [47, Lemma 6.10–6.11], the decay functions $G_{\mathcal{K}}$ can be made explicit. For our purposes here, we need only stress that they can be taken independent of $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in [-1, 1]$, and with decay faster than any power. Thus, for any $\mu \geq 0$,

$$(3.17) \quad \sum_{n=1}^{\infty} (n+1)^\mu G_{\mathcal{K}}(n) < \infty.$$

3.1.4. The Spectral Flow. Fix $\gamma > 0$. For each $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in \mathbb{R}$, denote by

$$(3.18) \quad D(\Lambda, s) = \mathcal{G}_s^\Lambda(V_\Lambda) = \int_{-\infty}^{\infty} \tau_t^{(\Lambda,s)}(V_\Lambda) W_\gamma(t) dt$$

with \mathcal{G}_s^Λ as defined in (3.14). Clearly, $D(\Lambda, s)$ is self-adjoint and $s \mapsto D(\Lambda, s)$ is uniformly bounded by (3.15).

For $t \in \mathbb{R}$ fixed, the strong derivative of $s \mapsto \tau_t^{(\Lambda,s)}$ is given by the Duhamel formula [47, Proposition 2.7]:

$$(3.19) \quad \frac{d}{ds} \tau_t^{(\Lambda,s)}(A) = i \int_0^t \tau_r^{(\Lambda,s)}([V_\Lambda, \tau_{t-r}^{(\Lambda,s)}(A)]) dr.$$

Using (3.19), one obtains the norm continuity of $s \mapsto D(\Lambda, s)$ from the following estimate:

$$\begin{aligned} \|D(\Lambda, s) - D(\Lambda, s_0)\| &\leq \int_{-\infty}^{\infty} \|\tau_t^{(\Lambda, s)}(V_{\Lambda}) - \tau_t^{(\Lambda, s_0)}(V_{\Lambda})\| |W_{\gamma}(t)| dt \\ (3.20) \quad &\leq 2\|V_{\Lambda}\|^2 |s - s_0| \int_{-\infty}^{\infty} |t| |W_{\gamma}(t)| dt. \end{aligned}$$

Given these properties of $D(\Lambda, s)$, there is a unique solution of

$$(3.21) \quad \frac{d}{ds} U(\Lambda, s) = -i D(\Lambda, s) U(\Lambda, s) \quad \text{with} \quad U(\Lambda, 0) = \mathbb{1},$$

which is given by unitaries in \mathcal{A} . Using similar arguments as in (3.13) with (3.16) and (3.18), one can show that for $s > 0$

$$(3.22) \quad \| [U(\Lambda, s), A] \| \leq \int_0^s \| [\mathcal{G}_r^{\Lambda}(V_{\Lambda}), A] \| dr \leq 2s \|A\| |\Lambda| \|V_{\Lambda}\| G_{\mathcal{G}}(d(X, \Lambda))$$

for any $A \in \mathcal{A}_X$ with $X \in \mathcal{P}_0(\Gamma \setminus \Lambda)$. Thus, $U(\Lambda, s) \in \mathcal{A}^g$ for some g with finite moments of all orders by (3.17).

The *spectral flow* is then the family of inner automorphisms on \mathcal{A} induced by $U(\Lambda, s)$:

$$(3.23) \quad \alpha_s^{\Lambda}(A) = U(\Lambda, s)^* A U(\Lambda, s) \quad \text{for all } A \in \mathcal{A}.$$

This is Hastings' quasi-adiabatic evolution [25, 27]. Quasi-locality of this map is then a consequence of a Lieb-Robinson bound. To this end, first rewrite the generator as

$$(3.24) \quad D(\Lambda, s) = \mathcal{G}_s^{\Lambda}(V_{\Lambda}) = \sum_{x \in \Lambda} \sum_{\substack{n \geq R: \\ b_x(n) \subset \Lambda}} \mathcal{G}_s^{\Lambda}(\Phi(x, n))$$

using (2.32). Applying the conditional expectations and telescopic sum from (3.7), we further write

$$(3.25) \quad D(\Lambda, s) = \sum_{x \in \Lambda} \sum_{m \geq R} \Psi_{\Lambda}(x, m, s) \quad \text{where} \quad \Psi_{\Lambda}(x, m, s) = \sum_{\substack{R \leq n \leq m: \\ b_x(n) \subset \Lambda}} \Delta_{b_x(n)}^m(\mathcal{G}_s^{\Lambda}(\Phi(x, n))).$$

Arguing as in [48, Appendix A], there is a decay function G_{Ψ} and a positive number $\|\Psi\|_{G_{\Psi}}$ such that for all $\Lambda \in \mathcal{P}_0(\Gamma)$, $s \in [-1, 1]$, $x \in \Lambda$ and $k \geq R$,

$$(3.26) \quad \sum_{m \geq k} \|\Psi_{\Lambda}(x, m, s)\| \leq \|\Psi\|_{G_{\Psi}} G_{\Psi}(k).$$

One can be explicit about estimates for G_{Ψ} , see [48, Corollary A.3], but for our purposes, we only need that it has finite moments of all orders. Given (3.25) and (3.26), well-known Lieb-Robinson bounds imply the existence of a decay function G_{α} so that for all $X, Y \in \mathcal{P}_0(\Gamma)$,

$$(3.27) \quad \| [\alpha_s^{\Lambda}(A), B] \| \leq s \|A\| \|B\| |X| G_{\alpha}(d(X, Y))$$

for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $s \in \mathbb{R}$. G_{α} is independent of $\Lambda \in \mathcal{P}_0(\Gamma)$ and has finite moments of all orders.

3.2. In the GNS space. The spectral perturbation arguments are carried out in the GNS representation of the reference state ω_0 . Recall that $(\mathcal{H}, \pi_0, \Omega)$ is our notation for the corresponding GNS triple. In this subsection, the quasi-local maps discussed previously are lifted to the GNS space. Here, and in what follows, we will use the notation $\tilde{A} = \pi_0(A) \in \mathcal{B}(\mathcal{H})$ to describe the GNS representative of an observable $A \in \mathcal{A}$. We now present the necessary properties we will need in this setting.

3.2.1. *Dynamics.* As discussed in Section 2.1, the unperturbed dynamics $\tau_t^{(0)}$ is implemented in the GNS representation of ω_0 by the GNS Hamiltonian H_0 , as in (2.20). We further show that the perturbed dynamics $\tau_t^{(\Lambda,s)}$ is implemented in the GNS representation of ω_0 by the Hamiltonian

$$(3.28) \quad H(\Lambda, s) = H_0 + \pi_0(sV_\Lambda), s \in \mathbb{R},$$

from (2.31). Specifically,

$$(3.29) \quad \pi_0(\tau_t^{(\Lambda,s)}(A)) = e^{itH(\Lambda,s)}\pi_0(A)e^{-itH(\Lambda,s)} \quad \text{for all } A \in \mathcal{A} \text{ and } t \in \mathbb{R}.$$

Applying the GNS representation to the interaction picture representation (3.10) gives

$$(3.30) \quad \pi_0(\tau_t^{(\Lambda,s)}(A)) = (\pi_0(K_t^{(\Lambda,s)}))^* e^{itH_0} \pi_0(A) e^{-itH_0} \pi_0(K_t^{(\Lambda,s)}).$$

Then, (3.29) follows by observing that

$$(3.31) \quad e^{-itH(\Lambda,s)} = e^{-itH_0} \pi_0(K_t^{(\Lambda,s)}),$$

as by (3.11) $\tilde{K}_t^{(\Lambda,s)} := \pi_0(K_t^{(\Lambda,s)})$ is the unique, unitary solution of

$$(3.32) \quad \frac{d}{dt} \tilde{K}_t^{(\Lambda,s)} = -ie^{itH_0} \pi_0(sV_\Lambda) e^{-itH_0} \tilde{K}_t^{(\Lambda,s)} \quad \text{with} \quad \tilde{K}_0^{(\Lambda,s)} = \mathbb{1}.$$

3.2.2. *Weighted Integral Operators.* For any $\gamma > 0$, $\Lambda \in \mathcal{P}_0(\Gamma)$, and $s \in \mathbb{R}$ we map the weighted integral operators of (3.14) to the GNS space by defining $\tilde{\mathcal{F}}_s^\Lambda$ and $\tilde{\mathcal{G}}_s^\Lambda$ by

$$(3.33) \quad \tilde{\mathcal{F}}_s^\Lambda(A) = \int_{-\infty}^{\infty} e^{itH(\Lambda,s)} A e^{-itH(\Lambda,s)} w_\gamma(t) dt \quad \text{and} \quad \tilde{\mathcal{G}}_s^\Lambda(A) = \int_{-\infty}^{\infty} e^{itH(\Lambda,s)} A e^{-itH(\Lambda,s)} W_\gamma(t) dt$$

for all $A \in \mathcal{B}(\mathcal{H})$. Using (3.29), it is clear that

$$(3.34) \quad \pi_0(\mathcal{F}_s^\Lambda(A)) = \tilde{\mathcal{F}}_s^\Lambda(\pi_0(A)) \quad \text{and} \quad \pi_0(\mathcal{G}_s^\Lambda(A)) = \tilde{\mathcal{G}}_s^\Lambda(\pi_0(A)) \quad \text{for all } A \in \mathcal{A}.$$

3.2.3. *The Spectral Flow.* For fixed $\Lambda \in \mathcal{P}_0(\Gamma)$, following [47, Section VI.A] we define a norm-continuous family of unitaries $\tilde{U}(\Lambda, s) \in \mathcal{B}(\mathcal{H})$ as the unique solution of

$$(3.35) \quad \frac{d}{ds} \tilde{U}(\Lambda, s) = -i\tilde{D}(\Lambda, s) \tilde{U}(\Lambda, s) \quad \text{with} \quad \tilde{U}(\Lambda, 0) = \mathbb{1},$$

where

$$(3.36) \quad \tilde{D}(\Lambda, s) = \int_{-\infty}^{\infty} e^{itH(\Lambda,s)} \pi_0(V_\Lambda) e^{-itH(\Lambda,s)} W_\gamma(t) dt.$$

The *spectral flow* associated with $H(\Lambda, s)$ is the family of automorphisms of $\mathcal{B}(\mathcal{H})$ defined by

$$(3.37) \quad \tilde{\alpha}_s^\Lambda(A) = \tilde{U}(\Lambda, s)^* A \tilde{U}(\Lambda, s).$$

By (3.34) it is clear that $\tilde{D}(\Lambda, s) = \pi_0(D(\Lambda, s))$ with $D(\Lambda, s)$ as in (3.18) and, hence, by the uniqueness of the unitary solution of (3.35), $\tilde{U}(\Lambda, s) = \pi_0(U(\Lambda, s))$, where $U(\Lambda, s)$ is as in (3.21). Therefore, π_0 lifts the spectral flow in \mathcal{A} to the GNS space:

$$(3.38) \quad \pi_0(\alpha_s^\Lambda(A)) = \tilde{\alpha}_s^\Lambda(\pi_0(A)) \quad \text{for all } A \in \mathcal{A}.$$

Recall that $E(\Lambda, s)$ denotes the ground state energy of $H(\Lambda, s)$. Under our assumptions the ground state space of $H(\Lambda, 0) = H_0$ is one-dimensional by Proposition A.1, and $\gamma_0 := \text{gap}(H_0)$ is strictly positive. By standard results, see e.g [33], for $|s|$ sufficiently small the kernel of $H(\Lambda, s) - E(\Lambda, s)\mathbb{1}$ is one-dimensional and the ground state gap does not immediately close. More precisely, for any $\gamma \in (0, \gamma_0)$, there is $s_0^\Lambda(\gamma) > 0$ so that

$$(3.39) \quad \text{gap}(H(\Lambda, s)) := \sup\{\delta > 0 : (E(\Lambda, s), E(\Lambda, s) + \delta) \cap \text{spec}(H(\Lambda, s)) = \emptyset\} \geq \gamma,$$

for all $|s| \leq s_0^\Lambda(\gamma)$. Although the existence of $s_0^\Lambda(\gamma) > 0$ is trivial, the main objective for proving stability is to establish the existence of a Λ -independent $s_0(\gamma)$, e.g. as in the statement of Theorem 2.7. Given this, an application of [47, Theorem 6.3] shows that

$$(3.40) \quad \tilde{\alpha}_s^\Lambda(P^\Lambda(s)) = P^\Lambda(0) \quad \text{whenever } |s| \leq s_0^\Lambda(\gamma),$$

where by $P^\Lambda(s)$ we denote the orthogonal projection onto the ground state space of $H(\Lambda, s)$.

For any $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in \mathbb{R}$, the state ω_s^Λ given by

$$(3.41) \quad \omega_s^\Lambda(A) = \omega_0(\alpha_s^\Lambda(A)) \quad \text{for all } A \in \mathcal{A}$$

is a vector state in the GNS space:

$$(3.42) \quad \omega_s^\Lambda(A) = \langle \Omega(\Lambda, s), \pi_0(A) \Omega(\Lambda, s) \rangle \quad \text{for all } A \in \mathcal{A},$$

where $\Omega(\Lambda, s) = \tilde{U}(\Lambda, s)\Omega \in \mathcal{H}$. By our assumptions, $P^\Lambda(0) = |\Omega\rangle\langle\Omega|$. An application of (3.40) then shows that

$$(3.43) \quad P^\Lambda(s) = \tilde{U}(\Lambda, s)|\Omega\rangle\langle\Omega|\tilde{U}(\Lambda, s)^* = |\Omega(\Lambda, s)\rangle\langle\Omega(\Lambda, s)| \quad \text{for all } |s| \leq s_0^\Lambda(\gamma),$$

and thus $\Omega(\Lambda, s)$ is the ground state of $H(\Lambda, s)$.

Finally, we recall that with the parameters γ and s as above that the weighted integral operator $\tilde{\mathcal{F}}_s^\Lambda$ from (3.33) satisfies the relation

$$(3.44) \quad \left[\tilde{\mathcal{F}}_s^\Lambda(A), |\Omega(\Lambda, s)\rangle\langle\Omega(\Lambda, s)| \right] = 0 \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

See, e.g. [47, Lemma 6.8], for a proof of this property.

3.3. On Domains. Recall that (Γ, d) is a ν -regular metric space. Let F be an F -function on (Γ, d) , and Φ an interaction with $\|\Phi\|_F < \infty$. As in Section 2, let δ^Φ be the closed derivation with dense domain $\text{dom}(\delta^\Phi) \subset \mathcal{A}$, and which satisfies

$$(3.45) \quad \delta^\Phi(A) = \sum_{\substack{Y \in \mathcal{P}_0(\Gamma): \\ Y \cap X \neq \emptyset}} [\Phi(Y), A] \quad \text{for any } A \in \mathcal{A}_X.$$

Although the sum on the right-hand-side above may be infinite, it is absolutely convergent when Φ has a finite F -norm. In fact, δ^Φ is locally bounded:

$$(3.46) \quad \|\delta^\Phi(A)\| \leq 2\|F\|\|\Phi\|_F\|X\|\|A\| \quad \text{for all } X \in \mathcal{P}_0(\Gamma), A \in \mathcal{A}_X,$$

see Example 4.7 of [47, Section IV.B.1]. We have the following lemma.

Lemma 3.1. *Let (Γ, d) be ν -regular, F an F -function on (Γ, d) , and g a decay function with a finite ν -moment, i.e.,*

$$(3.47) \quad \sum_{n=1}^{\infty} (n+1)^\nu g(n) < \infty.$$

For any interaction Φ on Γ with $\|\Phi\|_F < \infty$, we have that $\mathcal{A}^g \subset \text{dom}(\delta^\Phi)$.

Proof. For $n \geq 1$, and $A \in \mathcal{A}^g$, for some $x \in \Gamma$, and observables $A_n \in \mathcal{A}_{b_x(n)}$ satisfying $\|A - A_n\| \leq \|A\|_{x,g}g(n)$. In this case, the bound $\|A_{n+1} - A_n\| \leq 2\|A\|_{x,g}g(n)$ is clear. Using (3.46) and ν -regularity of Γ , we conclude

$$(3.48) \quad \|\delta^\Phi(A_{n+1}) - \delta^\Phi(A_n)\| \leq 4\|F\|\|\Phi\|_F\kappa\|A\|_{x,g}(n+1)^\nu g(n).$$

Thus, for all $m < n$,

$$(3.49) \quad \|\delta^\Phi(A_n) - \delta^\Phi(A_m)\| \leq 4\kappa\|A\|_{x,g}\|F\|\|\Phi\|_F \sum_{k=m}^{n-1} (k+1)^\nu g(k).$$

Since we assumed that g has a finite ν -moment, this implies that $\delta^\Phi(A_n)$ is a Cauchy sequence. Since $A_n \rightarrow A$ and \mathcal{A}^{loc} is a core for δ^Φ , it follows that $A \in \text{dom}(\delta^\Phi)$. \square

Given the assumptions of Theorem 2.7, Lemma 3.1 clearly applies to the derivation δ_0 . Using that $H_0\Omega = 0$, one readily checks the relation

$$(3.50) \quad e^{itH_0}\pi_0(A)\Omega = \pi_0(\tau_t^{(0)}(A))\Omega,$$

from which the inclusion $\pi_0(\text{dom}(\delta_0))\Omega \subset \text{dom}(H_0)$ is clear. As a result, if g is a decay function with a finite ν -moment, then $\pi_0(\mathcal{A}^g)\Omega \subset \text{dom}(H_0) = \text{dom}(H(\Lambda, s))$ for any $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in \mathbb{R}$. Since $U(\Lambda, s) \in \mathcal{A}^g$ for some g with finite moments of all of orders by (3.22), it follows that $\pi_0(AU(\Lambda, s))\Omega \in \text{dom}(H(\Lambda, s))$, for any $A \in \mathcal{A}^{\text{loc}}$, $s \in \mathbb{R}$.

A consequence of this is a gap inequality for the perturbed ground state ω_s^Λ from (3.41). Namely, we show that for $\gamma \in (0, \gamma_0)$, $\Lambda \in \mathcal{P}_0(\Gamma)$, and $|s| \leq s_0^\Lambda(\gamma)$:

$$(3.51) \quad \omega_s^\Lambda(A^* \delta_s^\Lambda(A)) \geq \gamma \omega_s^\Lambda(A^* A), \text{ for all } A \in \mathcal{A}^{\text{loc}} \text{ with } \omega_s^\Lambda(A) = 0.$$

To see this, fix $|s| \leq s_0^\Lambda(\gamma)$. Since $\Omega(\Lambda, s)$ is the unique ground state of $H(\Lambda, s)$,

$$(3.52) \quad \langle \psi, (H(\Lambda, s) - E(\Lambda, s)\mathbb{1})\psi \rangle \geq \gamma \langle \psi, \psi \rangle$$

for all $\psi \in \text{dom}(H_0)$ with $\langle \Omega(\Lambda, s), \psi \rangle = 0$. In particular, if $\omega_s^\Lambda(A) = 0$ for some $A \in \mathcal{A}^{\text{loc}}$, then (3.52) holds for $\psi = \pi_0(A)\Omega(\Lambda, s) = \pi_0(AU(\Lambda, s))\Omega$ since

$$\langle \Omega(\Lambda, s), \psi \rangle = \langle \Omega(\Lambda, s), \pi_0(A)\Omega(\Lambda, s) \rangle = \omega_s^\Lambda(A).$$

Then (3.51) follows from rewriting (3.52).

It will be important that on an appropriate dense domain, the action of the unbounded Hamiltonians can be expressed as a limit of finite-volume quantities. This is the content of the next lemma.

Lemma 3.2. *Let $(\mathcal{H}, \pi_0, \Omega)$ be the GNS representation of ω_0 , an infinite-volume, zero energy, ground state of a frustration free model as in Assumption 2.1. For any decay function g with a finite ν -moment and any IAS (Λ_n) ,*

$$(3.53) \quad \lim_{n \rightarrow \infty} \pi_0(H_{\Lambda_n})\psi = H_0\psi \quad \text{for all } \psi \in \pi_0(\mathcal{A}^g)\Omega,$$

where $H_{\Lambda_n} \in \mathcal{A}_{\Lambda_n}$ is as in (2.13) and H_0 is the GNS Hamiltonian.

Proof. Note that (3.53) is trivially satisfied for $\psi = \pi_0(A)\Omega$, for $A \in \mathcal{A}^{\text{loc}}$ since

$$(3.54) \quad \lim_{n \rightarrow \infty} \pi_0(H_{\Lambda_n})\pi_0(A)\Omega = \lim_{n \rightarrow \infty} \pi_0([H_{\Lambda_n}, A])\Omega = \pi_0(\delta_0(A))\Omega = H_0\pi_0(A)\Omega.$$

For the first equality we used $\pi_0(H_{\Lambda_n})\Omega = 0$, which is a consequence of the frustration-free property. Then, by the finite-range condition on the unperturbed model, $[H_{\Lambda_n}, A]$ becomes constant for n sufficiently large.

Take $\psi = \pi_0(A)\Omega$ for any $A \in \mathcal{A}^g$. By the definition of \mathcal{A}^g , there exists $x \in \Gamma$ and observables $A_m \in \mathcal{A}_{b_x(m)}$ so that $\|A - A_m\| \leq \|A\|_{x,g}g(m)$ for all $m \geq 1$, and so the vectors $\psi_m := \pi_0(A_m)\Omega$ satisfy

$$(3.55) \quad \|\psi - \psi_m\| \leq \|A\|_{x,g}g(m).$$

Moreover, since the interaction h is uniformly bounded with range R , it follows from (3.45) and ν -regularity that for any $k \geq 1$,

$$\|\delta_0(A_{\ell+1} - A_\ell)\| \leq \sum_{x \in b_x(\ell+1+R)} \| [h_x, A_{\ell+1} - A_\ell] \| \leq 2\kappa^2 R^\nu (\ell+1)^\nu \|h\|_\infty \|A_{\ell+1} - A_\ell\|$$

where we use the bound $|b_x(n+m)| \leq |b_x(n)||b_x(m)|$. Then, by the last equality of (3.54), one finds that $(H_0\psi_m)_{m \in \mathbb{N}}$ is Cauchy as

$$(3.56) \quad \|H_0\psi_m - H_0\psi_k\| \leq \|\delta_0(A_m) - \delta_0(A_k)\| \leq D \sum_{\ell=k}^{m-1} (\ell+1)^\nu g(\ell).$$

where we set $D = 4\kappa^2 R^\nu \|h\|_\infty \|A\|_{x,g}$. Since H_0 is closed, and $\psi \in \text{dom}(H_0)$ by Lemma 3.1 and the subsequent discussion, the bound

$$(3.57) \quad \|H_0\psi - H_0\psi_k\| \leq D \sum_{\ell=k}^{\infty} (\ell+1)^\nu g(\ell)$$

follows immediately from (3.55)-(3.56).

In the case of a local Hamiltonian, using again the first equality in (3.54), a similar argument shows that for all $n \geq 1$,

$$(3.58) \quad \begin{aligned} \|\pi_0(H_{\Lambda_n})\psi - \pi_0(H_{\Lambda_n})\psi_k\| &= \lim_{m \rightarrow \infty} \|\pi_0(H_{\Lambda_n})\psi_m - \pi_0(H_{\Lambda_n})\psi_k\| \\ &\leq \lim_{m \rightarrow \infty} \| [H_{\Lambda_n}, A_m - A_k] \| \leq D \sum_{\ell=k}^{\infty} (\ell+1)^\nu g(\ell). \end{aligned}$$

Putting all of this together, one finds that for any $n \geq 1$ and each $k \geq 1$,

$$(3.59) \quad \begin{aligned} \|\pi_0(H_{\Lambda_n})\psi - H_0\psi\| &\leq \|\pi_0(H_{\Lambda_n})\psi - \pi_0(H_{\Lambda_n})\psi_k\| + \|\pi_0(H_{\Lambda_n})\psi_k - H_0\psi_k\| + \|H_0\psi_k - H_0\psi\| \\ &\leq \|\pi_0(H_{\Lambda_n})\psi - \pi_0(H_{\Lambda_n})\psi_k\| + \|H_0\psi_k - H_0\psi\| \end{aligned}$$

For $k \geq 1$ sufficiently large, (3.57) and (3.58) guarantee that the first and last term above can be made arbitrarily small. Given such a k , the middle term vanishes for n sufficiently large, see the comment following (3.54). This completes the proof. \square

Lemma 3.2 also trivially applies to the perturbed system in the GNS space. In fact, for $\Lambda \in \mathcal{P}_0(\Gamma)$ and $s \in \mathbb{R}$, under assumptions as above, a direct application of Lemma 3.2 shows that we also have

$$(3.60) \quad \lim_n \pi_0(H_{\Lambda_n} + sV_\Lambda)\psi = H_0\psi + s\pi_0(V_\Lambda)\psi = H(\Lambda, s)\psi \quad \text{for all } \psi \in \pi_0(\mathcal{A}^g)\Omega.$$

Remark 3.3. An analogue of Lemma 3.2 holds more generally. In fact, if F is an F -function with a finite ν -moment, then for any frustration free interaction Φ with $\|\Phi\|_F < \infty$, the GNS Hamiltonian again satisfies (3.53). The argument is identical to the above except that one uses the more general estimate in Lemma 3.1 and bounds the middle term in (3.59) by

$$(3.61) \quad \sum_{\substack{X \in \mathcal{P}_0(\Gamma): \\ X \cap b_x(k) \neq \emptyset, X \cap \Lambda_n^c \neq \emptyset}} \|\pi_0(\Phi(X))\psi_k\| \leq \|\Phi\|_F (1 + g(0)) \|A\|_{x,g} \sum_{y \in b_x(k), z \in \Lambda_n^c} F(d(y, z))$$

For fixed k , the above is the sum of finitely many ‘tails’ of the uniformly summable function F .

We now investigate how the weighted integral operator $\tilde{\mathcal{F}}_s^\Lambda$ from (3.33) can be applied to the unbounded Hamiltonian $H(\Lambda, s)$. To begin, we prove an analogue of the desired statement for the unperturbed dynamics; this is Lemma 3.4 below. To this end, assume $w \in L^1(\mathbb{R})$ satisfies

$$(3.62) \quad \int_{\mathbb{R}} w(u) du = 1 \quad \text{and} \quad \int_{\mathbb{R}} |u|^\nu |w(u)| du < \infty,$$

and define a weighted integral operator $\tilde{\mathcal{F}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by setting

$$(3.63) \quad \tilde{\mathcal{F}}(A) = \int_{-\infty}^{\infty} e^{iuH_0} A e^{-iuH_0} w(u) du \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

To simplify notation, let us also write

$$(3.64) \quad \tilde{\tau}_u^{(0)}(A) = e^{iuH_0} A e^{-iuH_0}.$$

Our first result is as follows.

Lemma 3.4. *Let (Γ, d) be ν -regular, g be a decay function with a finite 2ν -moment, and $w \in L^1(\mathbb{R})$ satisfies (3.62). For each choice of IAS (Λ_n) , the weighted integral operator $\tilde{\mathcal{F}}$ from (3.63) satisfies*

$$(3.65) \quad \lim_{n \rightarrow \infty} \tilde{\mathcal{F}}(\pi_0(H_{\Lambda_n}))\psi = H_0\psi \quad \text{for all } \psi \in \pi_0(\mathcal{A}^g)\Omega,$$

where $H_{\Lambda_n} \in \mathcal{A}_{\Lambda_n}$ is as in (2.13).

Proof. Fix an IAS (Λ_n) and take $\psi = \pi_0(A)\Omega$ for some $A \in \mathcal{A}^g$. We can rewrite the convergence claimed in (3.65) as the convergence of integrals of a sequence of functions $f_n : \mathbb{R} \rightarrow \mathcal{H}$ given by

$$(3.66) \quad f_n(u) = w(u)\tilde{\tau}_u^{(0)}(\pi_0(H_{\Lambda_n}))\psi \quad \text{for all } u \in \mathbb{R}.$$

Since $H_0\Omega = 0$, the above can be re-written as

$$(3.67) \quad f_n(u) = w(u)e^{iuH_0}\pi_0(H_{\Lambda_n})\pi_0(\tau_{-u}^{(0)}(A))\Omega$$

using (2.20). We claim that there is a decay function g_τ with a finite ν -moment such that $\tau_{-u}^{(0)}(A) \in \mathcal{A}^{g_\tau}$ for all $u \in \mathbb{R}$. Given this, Lemma 3.2 applies and we find that

$$(3.68) \quad \lim_{n \rightarrow \infty} f_n(u) = w(u)e^{iuH_0}H_0\pi_0(\tau_{-u}^{(0)}(A))\Omega = w(u)H_0\psi \quad \text{for all } u \in \mathbb{R}.$$

By (3.62), the integral of this limit coincides with the right-hand-side of (3.65). Therefore, to complete the proof we only need to justify an application of dominated convergence.

Let us first prove the existence of a decay function g_τ as claimed. Fix $A \in \mathcal{A}^g$. In this case, there is $x \in \Gamma$, $C \geq 0$, and observables $A_m \in \mathcal{A}_{b_x(m)}$ for which $\|A - A_m\| \leq Cg(m)$ for all $m \in \mathbb{N}$. Let $u \in \mathbb{R}$ and for any $n \in \mathbb{N}$, set

$$(3.69) \quad A_n(u) = \Pi_n(\tau_u^{(0)}(A_{\lceil n/2 \rceil})) \in \mathcal{A}_{b_x(n)}$$

where, to ease notation, we have written $\Pi_n = \Pi_{b_x(n)}$, for the conditional expectation from Section 3.1.1. A straightforward estimate shows that for any $\mu > 0$,

$$(3.70) \quad \begin{aligned} \|\tau_u^{(0)}(A) - A_n(u)\| &\leq \|\tau_u^{(0)}(A) - \tau_u^{(0)}(A_{\lceil n/2 \rceil})\| + \|\tau_u^{(0)}(A_{\lceil n/2 \rceil}) - A_n(u)\| \\ &\leq Cg(n/2) + \kappa C_\mu e^{\mu(v_\mu|u|+1)}(1+g(0))\|A\|_{x,g}(n/2+1)^\nu e^{-\mu n/2} \end{aligned}$$

where we used (3.8) and (3.4) for the final bound. The existence of the decay function g_τ follows from the moment condition on g and the decay of the exponential term..

We now turn to finding a dominating function for f_n . Recall that for any $m_0 \in \mathbb{N}$, A can be written as an absolutely convergent, telescopic sum:

$$(3.71) \quad A = A_{m_0} + \sum_{k=m_0+1}^{\infty} B_k \quad \text{where} \quad B_k = A_k - A_{k-1} \quad \text{and} \quad \|B_k\| \leq 2Cg(k-1).$$

Inserting this decomposition of A into (3.67), we find that for any $n \in \mathbb{N}$ and each $u \in \mathbb{R}$:

$$(3.72) \quad \|f_n(u)\| \leq |w(u)| \left(\|\pi_0(H_{\Lambda_n})\pi_0(\tau_{-u}^{(0)}(A_{m_0}))\Omega\| + \sum_{k=m_0+1}^{\infty} \|\pi_0(H_{\Lambda_n})\pi_0(\tau_{-u}^{(0)}(B_k))\Omega\| \right).$$

Now, by the zero-energy property of the ground state we find the bound

$$(3.73) \quad \|\pi_0(H_{\Lambda_n})\pi_0(A)\Omega\| \leq \kappa(k+R)^\nu \|h\|_\infty \|A\| \quad \text{for all } A \in \mathcal{A}_{b_x(k)},$$

which we stress is uniform in n . This suggests a mechanism for bounding the first term in (3.72). Let $\ell_0 \geq m_0$ and write

$$(3.74) \quad \tau_{-u}^{(0)}(A_{m_0}) = \sum_{\ell \geq \ell_0} \Delta_{\ell_0}^\ell(\tau_{-u}^{(0)}(A_{m_0})),$$

where we have used the short-hand Δ_m^ℓ for $\Delta_{b_x(m)}^\ell$ as in (3.7). For $\ell = \ell_0$, the bound

$$\begin{aligned} \|\pi_0(H_{\Lambda_n})\pi_0(\Delta_{\ell_0}^{\ell_0}(\tau_{-u}^{(0)}(A_{m_0})))\Omega\| &= \|\pi_0(H_{\Lambda_n})\pi_0(\Pi_{\ell_0}(\tau_{-u}^{(0)}(A_{m_0})))\Omega\| \\ (3.75) \qquad \qquad \qquad &\leq \kappa\|h\|_\infty(\ell_0 + R)^\nu\|A_{m_0}\| \end{aligned}$$

follows from (3.73). For $\ell \geq \ell_0$, the estimate

$$\begin{aligned} \|\pi_0(H_{\Lambda_n})\pi_0(\Delta_{\ell_0}^\ell(\tau_{-u}^{(0)}(A_{m_0})))\Omega\| &\leq \kappa\|h\|_\infty(\ell + R)^\nu\|\Delta_{\ell_0}^\ell(\tau_{-u}^{(0)}(A_{m_0}))\| \\ (3.76) \qquad \qquad \qquad &\leq 2\kappa^2\|h\|_\infty C_\mu m_0^\nu(\ell + R)^\nu\|A_{m_0}\|e^{-\mu(\ell-m_0-1-v_\mu|u|)} \end{aligned}$$

follows from another application of (3.73) and the quasi-locality estimate for the unperturbed dynamics in combination with (3.6). We conclude that

$$\begin{aligned} \|\pi_0(H_{\Lambda_n})\pi_0(\tau_{-u}^{(0)}(A_{m_0}))\Omega\| &\leq \sum_{\ell \geq \ell_0} \|\pi_0(H_{\Lambda_n})\pi_0(\Delta_{\ell_0}^\ell(\tau_{-u}^{(0)}(A_{m_0})))\Omega\| \\ (3.77) \qquad \qquad \qquad &\leq \kappa\|h\|_\infty\|A_{m_0}\| \left((\ell_0 + R)^\nu + 2\kappa C_\mu m_0^\nu \sum_{\ell \geq \ell_0+1} (\ell + R)^\nu e^{-\mu(\ell-m_0-1-v_\mu|u|)} \right). \end{aligned}$$

If we now take $\ell_0 = \lceil v_\mu|u| + m_0 \rceil$, then we have found that there is $K \geq 0$ for which

$$(3.78) \qquad \|\pi_0(H_{\Lambda_n})\pi_0(\tau_{-u}^{(0)}(A_{m_0}))\Omega\| \leq K\|h\|_\infty\|A_{m_0}\| (m_0^{2\nu} + |u|^\nu + (m_0|u|)^\nu + 1)$$

and here $K = K(\kappa, \mu, \nu, R)$.

The terms B_k in (3.72) can be estimated similarly. Regarding k as m_0 and arguing as in (3.74) - (3.77) with some $\ell_0 \geq k$, a bound analogous to (3.78) can be found. Of course, here one replaces $\|A_{m_0}\|$ with $\|B_k\|$. Since $\|B_k\| \leq 2Cg(k-1)$ and g has a finite 2ν -moment, we have obtained a bound on the right-hand-side of (3.72) of the form:

$$(3.79) \qquad \|f_n(u)\| \leq \tilde{K}\|h\|_\infty(1 + |u|^\nu)|w(u)| \quad \text{for all } u \in \mathbb{R}.$$

By the assumption on w , i.e. (3.62), the above is a dominating function for the sequence f_n . This justifies dominated convergence and completes the proof. \square

We will also need a version of Lemma 3.4 for the perturbed system. Recall that for any $\gamma > 0$, $s \in \mathbb{R}$, and $\Lambda \in \mathcal{P}_0(\Gamma)$, the weighted integral operator $\tilde{\mathcal{F}}_s^\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ are defined by

$$(3.80) \qquad \tilde{\mathcal{F}}_s^\Lambda(A) = \int_{-\infty}^{\infty} e^{itH(\Lambda, s)} A e^{-itH(\Lambda, s)} w_\gamma(t) dt \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

We note that w_γ from [47, Section VI.B] satisfies (3.62). It is clear that

$$(3.81) \qquad \tilde{\mathcal{F}}_s^\Lambda(e^{iuH(\Lambda, s)}) = e^{iuH(\Lambda, s)} \quad \text{for all } u, s \in \mathbb{R},$$

since the dynamics leaves this bounded operator invariant and w_γ integrates to 1. Lemma 3.5 provides a differential version of this fact.

Lemma 3.5. *Let (Γ, d) be ν -regular, g be a decay function with a finite 2ν -moment. Let $\Lambda \in \mathcal{P}_0(\Gamma)$ and take $s \in \mathbb{R}$. For each choice of IAS (Λ_n) , consider the weighted integral operator $\tilde{\mathcal{F}}_s^\Lambda$, as in (3.80), with arbitrary $w \in L^1(\mathbb{R})$ satisfying (3.62). Then*

$$(3.82) \qquad \lim_{n \rightarrow \infty} \tilde{\mathcal{F}}_s^\Lambda(\pi_0(H_{\Lambda_n} + sV_\Lambda))\psi = H(\Lambda, s)\psi \quad \text{for all } \psi \in \pi_0(\mathcal{A}^g)\Omega,$$

with $H_{\Lambda_n} \in \mathcal{A}_{\Lambda_n}$ as in (2.13) and V_Λ as in (2.32).

Proof. Fix an IAS (Λ_n) where we assume for convenience that $\Lambda \subset \Lambda_1$. As in the proof of Lemma 3.4, take $\psi = \pi_0(A)\Omega$ with $A \in \mathcal{A}^g$, and for each $n \in \mathbb{N}$, consider $f_n : \mathbb{R} \rightarrow \mathcal{H}$ given by

$$(3.83) \quad f_n(u) = w(u)\tilde{\tau}_u^{(\Lambda,s)}(\pi_0(H_{\Lambda_n} + sV_{\Lambda}))\pi_0(A)\Omega \quad \text{for all } u \in \mathbb{R},$$

where, in analogy to (3.64), we have set

$$(3.84) \quad \tilde{\tau}_t^{(\Lambda,s)}(A) = e^{itH(\Lambda,s)}Ae^{-itH(\Lambda,s)} \quad \text{for all } A \in \mathcal{B}(\mathcal{H}) \text{ and } t \in \mathbb{R}.$$

Using (3.29), (3.10), and (2.20), we may write

$$(3.85) \quad \tilde{\tau}_u^{(\Lambda,s)}(\pi_0(H_{\Lambda_n} + sV_{\Lambda})) = \pi_0(K_u^{(\Lambda,s)})^*e^{iuH_0}\pi_0(H_{\Lambda_n} + sV_{\Lambda})e^{-iuH_0}\pi_0(K_u^{(\Lambda,s)})$$

for all $u \in \mathbb{R}$. In this case, we find that

$$(3.86) \quad f_n(u) = w(u)\pi_0(K_u^{(\Lambda,s)})^*e^{iuH_0}\pi_0(H_{\Lambda_n} + sV_{\Lambda})\pi_0(\tau_{-u}^{(0)}(K_u^{(\Lambda,s)}A))\Omega.$$

Following a similar argument and using (3.13), one shows that there is a decay function g' with a finite ν -moment such that $\tau_{-u}^{(0)}(K_u^{(\Lambda,s)}A) \in \mathcal{A}^{g'}$. As a result, the point-wise limit

$$(3.87) \quad \lim_n f_n(u) = w(u)\pi_0(K_u^{(\Lambda,s)})^*e^{iuH_0}(H_0 + s\pi_0(V_{\Lambda}))\pi_0(\tau_{-u}^{(0)}(K_u^{(\Lambda,s)}A))\Omega = w(u)H(\Lambda, s)\psi$$

is clear from properties of the interaction picture dynamics, see the discussion following (3.10).

The argument demonstrating that we can apply the dominated convergence theorem also proceeds as in the proof of Lemma 3.4. Since the differences stemming from the presence of the u -dependence in the operators A_{m_0} and B_k are minor, we leave the details to the reader. \square

4. CONSTRUCTION OF A UNITARILY EQUIVALENT PERTURBED SYSTEM

The crux of the stability strategy, as introduced in [12], is to use the spectral flow (aka quasi-adiabatic evolution) to construct a unitarily equivalent perturbed system for which one can prove a relative form bound using quasi-locality estimates and LTQO. In the infinite-system setting, this begins by justifying that the unbounded Hamiltonian $H(\Lambda, s)$ from (2.31) can be transformed by the spectral flow defined in the GNS space, see (3.37). To this end, note that in Section 3.1.4 we proved that $U(\Lambda, s) \in \mathcal{A}^g$ for some g with finite moments of all orders, and thus, an application of Lemma 3.2 shows that $\tilde{U}(\Lambda, s)\pi_0(A)\Omega \in \text{dom } H(\Lambda, s)$ for $A \in \mathcal{A}^{\text{loc}}$. As a consequence, one may write

$$(4.1) \quad \tilde{U}(\Lambda, s)^*H(\Lambda, s)\tilde{U}(\Lambda, s)\psi = H_0\psi + W(\Lambda, s)\psi + E(\Lambda, s)\psi \quad \text{for all } \psi \in \pi_0(\mathcal{A}^{\text{loc}})\Omega,$$

where $E(\Lambda, s)$ the ground state energy of $H(\Lambda, s)$ from (2.34), and $W(\Lambda, s)$ is well-defined since all other quantities in (4.1) are well-defined. Our goal now is to show that this defines $W(\Lambda, s)$ as a bounded operator with an explicit, Λ -independent form-bound with respect to H_0 .

In fact, the proof of Theorem 2.7 follows as a consequence of two results. The first, Theorem 4.1, establishes that $W(\Lambda, s)$ is indeed bounded and can be decomposed in a way that is suitable for deriving a relative form bound. The second, Theorem 5.1 in Section 5, is the relative form bound itself.

Theorem 4.1. *Suppose Assumptions 2.1–2.2 and 2.4–2.5 hold, and fix $\Lambda \in \mathcal{P}_0(\Gamma)$. Then, for any $\gamma \in (0, \gamma_0)$ and $|s| \leq s_0^{\Lambda}(\gamma)$, there is a family of self-adjoint observables $\Phi^{(2)}(x, m, s) \in \mathcal{A}_{b_x(m)}$, for each $x \in \Gamma$ and $m \geq R$, with the following properties:*

- (i) $\Phi^{(2)}(x, m, s)P_{b_x(m)} = P_{b_x(m)}\Phi^{(2)}(x, m, s) = 0$;
- (ii) $\|\Phi^{(2)}(x, m, s)\| \leq 2sG_{\Lambda}^{(2)}(x, m)$ with

$$(4.2) \quad G_{\Lambda}^{(2)}(x, m) = G_{\Lambda}(x, m/2) + 2G_{\Lambda}^{(1)}(x, \lceil m/2 \rceil) + 2G_{\Lambda}^{(1)}(x, R)\sqrt{\lceil m/2 \rceil^{\nu}G_0(m/2)}$$

where $P_{b_x(m)}$ is the ground state projection associated to $H_{b_x(m)}$, $G_{\Lambda}(x, m)$ is as in Theorem 4.2, $G_{\Lambda}^{(1)}(x, m) = \sum_{n \geq m} G_{\Lambda}(x, n)$, and G_0 is from Assumption 2.4. Furthermore, $W(\Lambda, s)$ is given by the absolutely convergent sum

$$(4.3) \quad W(\Lambda, s) = \sum_{x \in \Gamma} \sum_{m \geq R} \pi_0(\Phi^{(2)}(x, m, s)).$$

Note that the operator $W(\Lambda, s)$ is *a priori* defined in the GNS representation. *A posteriori*, however, (4.3) implies that $W(\Lambda, s)$ is the image of a quasi-local observable in \mathcal{A} .

The decomposition from Theorem 4.1 is proved in two steps. The first uses quasi-locality and conditional expectations to prove that for all $|s| \leq s_0^{\Lambda}(\gamma)$, the action of the spectral flow on the GNS Hamiltonian $H(\Lambda, s)$ can be again realized as a perturbation of H_0 . Namely, we show that for all $\psi \in \pi_0(\mathcal{A}_{\text{loc}})\Omega$

$$(4.4) \quad \tilde{U}(\Lambda, s)^* H(\Lambda, s) \tilde{U}(\Lambda, s) \psi = H_0 \psi + \sum_{x \in \Gamma} \sum_{m \geq R} \tilde{\Phi}^{(1)}(x, m, s) \psi$$

where the perturbation terms $\tilde{\Phi}^{(1)}(x, m, s) \in \pi_0(\mathcal{A}_{b_x(m)})$ are self-adjoint, satisfy a norm bound that is linear in s , and are absolutely summable over $x \in \Gamma$ and $m \geq R$. This is accomplished in Theorem 4.2 of Section 4.1 below.

In the second step, carried out in Section 4.2, the final form of (4.3) from Theorem 4.1 is proved using the frustration-free and LTQO ground state properties to produce a refined decomposition of the perturbation terms from (4.4).

4.1. Quasilocal decomposition of the transformed perturbation. We now turn to establishing the first decomposition (4.4), which is the content of the following theorem.

Theorem 4.2. *Under the conditions of Theorem 4.1, there exists a function $G_{\Lambda} : \Gamma \times [0, \infty) \rightarrow [0, \infty)$ for which*

$$(4.5) \quad \sum_{x \in \Gamma} \sum_{m \geq R} G_{\Lambda}(x, m) < \infty$$

and a self-adjoint operator $\tilde{\Phi}^{(1)}(x, m, s)^* = \tilde{\Phi}^{(1)}(x, m, s) \in \pi_0(\mathcal{A}_{b_x(m)})$ for each $x \in \Gamma$ and $m \geq R$, such that $\|\tilde{\Phi}^{(1)}(x, m, s)\| \leq sG_{\Lambda}(x, m)$ and

$$(4.6) \quad W(\Lambda, s) + E(\Lambda, s)\mathbb{1} = \sum_{x \in \Gamma} \sum_{m \geq R} \tilde{\Phi}^{(1)}(x, m, s).$$

Moreover, for each $x \in \Gamma$, the operator $\tilde{\Phi}^{(1)}(x, s) := \sum_{m \geq R} \tilde{\Phi}^{(1)}(x, m, s)$ belongs to $\pi_0(\mathcal{A})$ and commutes with the ground state projection $|\Omega\rangle\langle\Omega|$.

The global term $\tilde{\Phi}^{(1)}(x, s)$ above will result from applying quasi-local maps $\mathcal{K}_s^{i, \Lambda}$, $i = 1, 2$, to the interaction and perturbation terms associated to the site x . These maps are defined in terms of the examples introduced in Section 3.1, and emerge from fixing any IAS (Λ_n) and then applying Lemmas 3.4-3.5 to rewrite

$$(4.7) \quad (W(\Lambda, s) + E(\Lambda, s)\mathbb{1})\psi = \lim_{n \rightarrow \infty} \left(\tilde{U}(\Lambda, s)^* \tilde{\mathcal{F}}_s^{\Lambda}(\pi_0(H_{\Lambda_n} + sV_{\Lambda})) \tilde{U}(\Lambda, s)\psi - \tilde{\mathcal{F}}(\pi_0(H_{\Lambda_n}))\psi \right)$$

where we choose $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_0^{\Lambda}$. As the argument in the above limit is a finite sum of bounded operators, the various relationships (3.34)-(3.38) between the quasi-local maps in the GNS representation to

those on the C^* -algebra implies that for each n :

$$(4.8) \quad \begin{aligned} \tilde{\alpha}_s^\Lambda \circ \tilde{\mathcal{F}}_s^\Lambda(\tilde{H}_{\Lambda_n} + s\tilde{V}_\Lambda) - \tilde{\mathcal{F}}(\tilde{H}_{\Lambda_n}) &= \sum_{x \in \Lambda_n} \pi_0(\alpha_s^\Lambda \circ \mathcal{F}_s^\Lambda(h_x) - \mathcal{F}(h_x)) \\ &+ \sum_{x \in \Lambda} \sum_{\substack{k \geq R: \\ b_x(k) \subseteq \Lambda}} \pi_0(s\alpha_s^\Lambda \circ \mathcal{F}_s^\Lambda(\Phi(x, k))). \end{aligned}$$

Given this, for $i = 1, 2$ the map $\mathcal{K}_s^{i, \Lambda} : \mathcal{A} \rightarrow \mathcal{A}$ are defined by

$$(4.9) \quad \mathcal{K}_s^{1, \Lambda}(A) = \alpha_s^\Lambda(\mathcal{F}_s^\Lambda(A)) - \mathcal{F}(A) \quad \text{and} \quad \mathcal{K}_s^{2, \Lambda}(A) = s\alpha_s^\Lambda(\mathcal{F}_s^\Lambda(A)).$$

It was proved, e.g. in [48, Lemma 4.4], that both of these maps satisfy a local bound and quasi-local estimate that is independent of the finite volume Λ . Specifically, for each $i = 1, 2$ there are non-negative numbers p_i, q_i and C_i , and a decay function $G_{\mathcal{K}^i}$ (all independent of Λ) such that

$$(4.10) \quad \|\mathcal{K}_s^{i, \Lambda}(A)\| \leq sC_i|X|^{p_i}\|A\|$$

$$(4.11) \quad \|[\mathcal{K}_s^{i, \Lambda}(A), B]\| \leq s|X|^{q_i}\|A\|\|B\|G_{\mathcal{K}^i}(d(X, Y))$$

hold for any $X, Y \in \mathcal{P}_0(\Gamma)$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $s \in \mathbb{R}$. In fact, one can take $p_1 = q_1 = 2$, $p_2 = 0$ and $q_2 = 1$ and make explicit estimates on the decay function, see e.g. [48, Remark 4.7]. However, it suffices to note that each $G_{\mathcal{K}^i}$ have finite moments of all orders in the sense of (3.17).

However, as $\Lambda_n \uparrow \Gamma$ when $n \rightarrow \infty$, to prove that the decomposition in (4.6) is absolutely summable, we will need refinements of (4.10)-(4.11) for $\mathcal{K}_s^{1, \Lambda}$ that also decay in the distance $d(X, \Lambda)$. Both of these bounds will be a consequence of the perturbation V_Λ being locally supported, which implies that the spectral flow α_s^Λ is approximately the identity far from Λ . The necessary bounds are the content of Lemmas 4.3 and 4.4 below.

Lemma 4.3 (Distance Locality Bound for \mathcal{K}_s^1). *There exists a decay function $F_{\mathcal{K}^1}$, with finite moments of all orders for which, given any $X, \Lambda \in \mathcal{P}_0(\Gamma)$ with $d(X, \Lambda) > 0$, $A \in \mathcal{A}_X$, and any $s \in \mathbb{R}$, the following local bound holds:*

$$(4.12) \quad \|\mathcal{K}_s^{1, \Lambda}(A)\| \leq s|X|^2\|A\|F_{\mathcal{K}^1}(d(X, \Lambda)).$$

It is easy to check that for fixed $\epsilon \in (0, 1)$ and any decay function F with finite ν -moment, the function $M_F^\epsilon : [0, \infty) \rightarrow [0, \infty)$ defined by

$$(4.13) \quad M_F^{(\epsilon)}(r) = \sum_{n \geq r} (n+1)^\nu F(\epsilon n)$$

is also a decay function. The proof of Lemma 4.3 shows that one may take

$$(4.14) \quad F_{\mathcal{K}^1}(r) = 2\kappa\|\Psi\|_{G_\Psi} \left(M_{G_\Psi}^{(1-\epsilon)}(r) + 2G_\Psi(R)M_{G_F}^{(\epsilon)}(r) \right)$$

where G_F and G_Ψ are the decay functions previously discussed in (3.16) and (3.26). Since G_F and G_Ψ both have finite moments of all orders, the same is true for $F_{\mathcal{K}^1}$.

The proof of Lemma 4.3 will also make use of the following bound, which holds for any F and ϵ as in (4.13), and $\Lambda, X \in \mathcal{P}_0(\Gamma)$ such that $d(X, \Lambda) > 0$:

$$(4.15) \quad \sum_{z \in \Lambda} F(\epsilon d(z, X)) \leq \kappa|X|M_F^{(\epsilon)}(d(X, \Lambda)).$$

This follows from the following simple calculation

$$(4.16) \quad \begin{aligned} \sum_{z \in \Lambda} F(\epsilon d(z, X)) &\leq \sum_{\substack{z \in \Gamma: \\ d(z, X) \geq d(X, \Lambda)}} F(\epsilon d(z, X)) \leq \sum_{n \geq d(X, \Lambda)} \sum_{\substack{z \in \Gamma: \\ n \leq d(z, X) < n+1}} F(\epsilon d(z, X)) \\ &\leq \kappa|X| \sum_{n \geq d(X, \Lambda)} (n+1)^\nu F(\epsilon n) \end{aligned}$$

where the last inequality uses that $|X(n)| \leq \kappa n^\nu |X|$ for any $n \geq 1$ by ν -regularity, see (2.1).

Proof of Lemma 4.3: Fix $X, \Lambda \in \mathcal{P}_0(\Gamma)$ such that $X \cap \Lambda = \emptyset$, and let $A \in \mathcal{A}_X$ be arbitrary. Recall that $\mathcal{K}_s^{1,\Lambda}$ is as defined in (4.9), and that $D(\Lambda, s)$ from (3.18) is the generator of the spectral flow. Then, since $\alpha_0 = \text{id}$ and $\mathcal{F} = \mathcal{F}_0^\Lambda$, it follows that

$$(4.17) \quad \mathcal{K}_s^{1,\Lambda}(A) = \int_0^s \frac{d}{dr} \alpha_r^\Lambda(\mathcal{F}_r^\Lambda(A)) dr = i \int_0^s \alpha_r^\Lambda([D(\Lambda, r), \mathcal{F}_r^\Lambda(A)]) dr$$

where one uses (3.19) and [47, Equation (6.37)] to obtain

$$\begin{aligned} \frac{d}{ds} \mathcal{F}_s^\Lambda(A) &= i \int_{-\infty}^\infty \int_0^t \tau_r^{(\Lambda,s)}([V_\Lambda, \tau_{t-r}^{(\Lambda,s)}(A)]) dr w_\gamma(t) dt \\ &= i \int_{-\infty}^\infty \int_0^t \tau_r^{(\Lambda,s)}([V_\Lambda, \tau_{t-r}^{(\Lambda,s)}(A)]) dr \left(-\frac{d}{dt} W_\gamma(t) + \delta_0(t) \right) dt \\ (4.18) \quad &= i \int_{-\infty}^\infty \tau_t^{(\Lambda,s)}([V_\Lambda, A]) W_\gamma(t) dt = 0. \end{aligned}$$

Here, the final two equalities follow from integration by parts, and the fact that the supports of V_Λ and A are disjoint.

Returning to (4.17), we expand the generator as in (3.25) to write

$$(4.19) \quad [D(\Lambda, r), \mathcal{F}_r^\Lambda(A)] = \sum_{z \in \Lambda} \sum_{n \geq R} [\Psi_\Lambda(z, n, r), \mathcal{F}_r^\Lambda(A)]$$

Fix $\epsilon \in (0, 1)$, and for each $z \in \Lambda$, set $k_z(\epsilon) = \epsilon d(z, X)$. For each term in (4.19), we approximate $\mathcal{F}_r^\Lambda(A)$ with a strictly local approximation:

$$(4.20) \quad [\Psi_\Lambda(z, n, r), \mathcal{F}_r^\Lambda(A)] = [\Psi_\Lambda(z, n, r), \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A))] + (\mathcal{F}_r^\Lambda(A) - \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A)))$$

where one uses conditional expectation associated with the inflated set $X(k_z(\epsilon))$, see (2.1)-(2.2). For the second term, one can apply the quasi-local bound for \mathcal{F}_r^Λ from (3.16) coupled with (3.4) to produce

$$(4.21) \quad \|[\Psi_\Lambda(z, n, r), \mathcal{F}_r^\Lambda(A) - \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A))]\| \leq 4\|A\| |X| \|\Psi_\Lambda(z, n, r)\| G_{\mathcal{F}}(k_z(\epsilon)).$$

Then, summing over $z \in \Lambda$ and $n \geq R$, and applying (3.26) and (4.15) gives the final estimate

$$(4.22) \quad \sum_{z \in \Lambda} \sum_{n \geq R} \|[\Psi_\Lambda(z, n, r), \mathcal{F}_r^\Lambda(A) - \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A))]\| \leq 4\kappa \|A\| |X|^2 \|\Psi\|_{G_\Psi} G_\Psi(R) M_{G_{\mathcal{F}}}^{(\epsilon)}(d(X, \Lambda)).$$

To estimate the remaining terms in (4.20), note that for each $z \in \Lambda$, $b_z(n) \cap X(k_z(\epsilon)) \neq \emptyset$ only when $n \geq k_z(1 - \epsilon)$. As a result, arguments similar to the prior estimate produce the bound

$$\begin{aligned} \sum_{z \in \Lambda} \sum_{n \geq R} \|[\Psi_\Lambda(z, n, r), \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A))]\| &= \sum_{z \in \Lambda} \sum_{n \geq k_z(1 - \epsilon)} \|[\Psi_\Lambda(z, n, r), \Pi_{X(k_z(\epsilon))}(\mathcal{F}_r^\Lambda(A))]\| \\ &\leq 2\|A\| \|\Psi\|_{G_\Psi} \sum_{z \in \Lambda} G_\Psi(k_z(1 - \epsilon)) \\ (4.23) \quad &\leq 2\kappa \|A\| |X| \|\Psi\|_{G_\Psi} M_{G_\Psi}^{(1-\epsilon)}(d(X, \Lambda)). \end{aligned}$$

Recalling the specific decay function from (4.14), the bound claimed in (4.12) now follows by inserting (4.19) into (4.17) and using the estimates found in (4.22) and (4.23) above. \square

By combining the estimate in Lemma 4.3 and the original quasi-locality bound from (4.11), one arrives that the following quasi-locality bound for $\|[\mathcal{K}_s^{1,\Lambda}(A), B]\|$, which decays in both the distance between $X = \text{supp}(A)$ and $Y = \text{supp}(B)$ as well as the distance between Λ and X . This is the content of the next lemma.

Lemma 4.4 (Distance Quasi-Locality for \mathcal{K}_1). *There exists a function $G : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, non-increasing in both variables, such that given any $\Lambda, X, Y \in \mathcal{P}_0(\Gamma)$ with $d(X, \Lambda) > 0$, the bound*

$$(4.24) \quad \|[\mathcal{K}_s^{1,\Lambda}(A), B]\| \leq s\|A\|\|B\||X|^2G(d(X, \Lambda), d(X, Y))$$

holds for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $s \in \mathbb{R}$. More precisely, for any $\delta \in (0, 1)$, one may choose

$$(4.25) \quad G(m, n) = \max \left\{ 2F_{\mathcal{K}^1}^\delta(m)F_{\mathcal{K}^1}^{1-\delta}(n), G_{\mathcal{K}^1}^\delta(m)G_{\mathcal{K}^1}^{1-\delta}(n) \right\}$$

where $F_{\mathcal{K}^1}$ and $G_{\mathcal{K}^1}$ are the decay functions from Lemma 4.3 and (4.11), and $G^\delta(m) := (G(m))^\delta$.

In applications, it can be convenient to bound $G(n, m)$ by a function that separates over the two arguments. In this case, taking δ as in (4.25),

$$(4.26) \quad G(m, n) \leq 2F_\delta(m) \cdot F_{1-\delta}(n) \quad \text{with} \quad F_\delta(m) = \max\{F_{\mathcal{K}^1}^\delta(m), G_{\mathcal{K}^1}^\delta(m)\}.$$

Proof. Fix $0 < \delta < 1$. In the case that $d(X, \Lambda) \leq d(X, Y)$, the quasi-locality estimate (4.11) shows that

$$(4.27) \quad \|[\mathcal{K}_s^{1,\Lambda}(A), B]\| \leq s|X|^2\|A\|\|B\|G_{\mathcal{K}^1}^\delta(d(X, \Lambda))G_{\mathcal{K}^1}^{1-\delta}(d(X, Y))$$

where we have used that $G_{\mathcal{K}^1}$ is non-increasing.

Alternatively, if $d(X, \Lambda) > d(X, Y)$, the local bound from Lemma 4.3 implies

$$(4.28) \quad \|[\mathcal{K}_s^{1,\Lambda}(A), B]\| \leq 2\|\mathcal{K}_s^{1,\Lambda}(A)\|\|B\| \leq 2s|X|^2\|A\|\|B\|F_{\mathcal{K}^1}(d(X, \Lambda)).$$

Since $F_{\mathcal{K}^1}$ is also non-increasing, the bound $F_{\mathcal{K}^1}(d(X, \Lambda)) \leq F_{\mathcal{K}^1}^\delta(d(X, \Lambda))F_{\mathcal{K}^1}^{1-\delta}(d(X, Y))$ follows. The bound (4.24) is then a consequence of (4.27) and (4.28). \square

We conclude this section by proving Theorem 4.2, which will proceed as follows. We first define the global terms $\tilde{\Phi}^{(1)}(x, s) \in \pi_0(\mathcal{A})$ and show that they commute with the ground state projection $|\Omega\rangle\langle\Omega|$. Then, we use the localizing operators from (3.7) to define the local terms $\Phi^{(1)}(x, m, s)$ for $m \geq R$ and show that they formally satisfy (4.6). The third and final step of the proof uses Lemmas 4.3-4.4 to show that these new local interaction terms satisfy the desired norm bound for a function $G_\Lambda(x, m)$ satisfying (4.5). This justifies the above-mentioned formal equality, and will be a consequence of considering the cases $x \in \Lambda(R)$ and $x \in \Gamma \setminus \Lambda(R)$ separately.

Proof of Theorem 4.2. Fix $\gamma \in (0, \gamma_0)$, $\Lambda \in \mathcal{P}_0(\Gamma)$, and take any IAS (Λ_n) such that $\Lambda \subseteq \Lambda_n$ for all n . Define the spectral flow α_s^Λ and the weighted integral operators \mathcal{F}_s^Λ , $\mathcal{F} = \mathcal{F}_0^\Lambda$ with respect to the choices of γ and Λ as in (3.23) and (3.14), and then take $\mathcal{K}_s^{i,\Lambda}$, $i = 1, 2$, as defined (4.9).

For the first step, let χ_Λ be the characteristic function of $\Lambda \subset \Gamma$. Then, for each $x \in \Gamma$ and $s \in \mathbb{R}$ such that $|s| \leq s_0^\Lambda(\gamma)$, the self-adjoint operator $\tilde{\Phi}^{(1)}(x, s) = \pi_0(\Phi^{(1)}(x, s)) \in \mathcal{B}(\mathcal{H})$ is defined by

$$(4.29) \quad \Phi^{(1)}(x, s) = \mathcal{K}_s^{1,\Lambda}(h_x) + \chi_\Lambda(x) \sum_{\substack{k \geq R: \\ b_x(k) \subset \Lambda}} \mathcal{K}_s^{2,\Lambda}(\Phi(x, k)) \in \mathcal{A}.$$

To show that each $\tilde{\Phi}^{(1)}(x, s)$ commutes with the ground state projection $|\Omega\rangle\langle\Omega|$, recall that the ground state of the perturbed system is $\Omega(\Lambda, s) = \tilde{U}(\Lambda, s)\Omega$ if $|s| \leq s_\gamma^\Lambda$. Then, recalling the relations (3.34)-(3.38), a simple calculation shows that for all $A \in \mathcal{A}$

$$(4.30) \quad \begin{aligned} [\pi_0(\alpha_s^\Lambda(\mathcal{F}_s^\Lambda(A))), |\Omega\rangle\langle\Omega|] &= \left[\tilde{U}(\Lambda, s)^* \tilde{\mathcal{F}}_s^\Lambda(\pi_0(A)) \tilde{U}(\Lambda, s), |\Omega\rangle\langle\Omega| \right] \\ &= \tilde{U}(\Lambda, s)^* \left[\tilde{\mathcal{F}}_s^\Lambda(\pi_0(A)), |\Omega(\Lambda, s)\rangle\langle\Omega(\Lambda, s)| \right] \tilde{U}(\Lambda, s) = 0, \end{aligned}$$

where the final equality uses that (3.44) holds since $|s| \leq s_0^\Lambda(\gamma)$. Since (4.30) trivially holds for $s = 0$, considering (4.9), the above implies that

$$(4.31) \quad [\pi_0(\mathcal{K}_s^{1,\Lambda}(A)), |\Omega\rangle\langle\Omega|] = [\pi_0(\mathcal{K}_s^{2,\Lambda}(A)), |\Omega\rangle\langle\Omega|] = 0.$$

Hence, $[\tilde{\Phi}^{(1)}(x, s), |\Omega\rangle\langle\Omega|] = 0$ for all $x \in \Gamma$ and $|s| \leq s_0^\Lambda(\gamma)$ as claimed.

We now turn to the second step of the proof. To establish (4.6), use the conditional expectations from (3.7) to decompose each $\tilde{\Phi}^{(1)}(x, s)$ as

$$(4.32) \quad \tilde{\Phi}^{(1)}(x, s) = \sum_{m \geq R} \tilde{\Phi}^{(1)}(x, m, s)$$

where $\tilde{\Phi}^{(1)}(x, m, s) = \pi_0(\Phi^{(1)}(x, m, s)) \in \pi_0(\mathcal{A}_{b_x(m)})$ is defined for each $m \geq R$ by

$$(4.33) \quad \Phi^{(1)}(x, m, s) := \Delta_{b_x(R)}^m(\mathcal{K}_s^{1,\Lambda}(h_x)) + \chi_\Lambda(x) \sum_{\substack{R \leq k \leq m: \\ b_x(k) \subseteq \Lambda}} \Delta_{b_x(k)}^m(\mathcal{K}_s^{2,\Lambda}(\Phi(x, k))).$$

With respect to this notation, (4.7), (4.8) and (4.29) show that for all $\psi \in \pi_0(\mathcal{A}^{\text{loc}})\Omega$,

$$(4.34) \quad \begin{aligned} (W(\Lambda, s) + E(\Lambda, s)\mathbb{1})\psi &= \lim_{n \rightarrow \infty} \sum_{x \in \Lambda_n} \tilde{\Phi}^{(1)}(x, s)\psi \\ &= \lim_{n \rightarrow \infty} \sum_{x \in \Lambda_n} \sum_{m \geq R} \tilde{\Phi}^{(1)}(x, m, s)\psi. \end{aligned}$$

Since this $\pi_0(\mathcal{A}^{\text{loc}})\Omega \subseteq \mathcal{H}$ is dense, the equality in (4.6) follows from establishing absolute summability of the terms $\tilde{\Phi}^{(1)}(x, m, s)$. This is achieved by defining a function

$$(4.35) \quad G_\Lambda(x, m) = \chi_{\Lambda(R)}(x)G_1(m) + \chi_{\Gamma \setminus \Lambda(R)}(x)G_2(d(b_x(R), \Lambda), m)$$

which bounds the norms of these terms and satisfies (4.5). Here, we note that $\Lambda(R)$ is as in (2.1), and the functions $G_1 : [0, \infty) \rightarrow [0, \infty)$ and $G_2 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ will be independent of Λ .

We now proceed to the final step of the proof. First, suppose $x \in \Gamma \setminus \Lambda(R)$. As $R \geq 0$ is the finite range of the unperturbed interaction, (4.33) simplifies to

$$(4.36) \quad \Phi^{(1)}(x, m, s) = \Delta_{b_x(R)}^m(\mathcal{K}_s^{1,\Lambda}(h_x)).$$

Then, applying Lemmas 4.3 and 4.4 with the local approximation bound (3.6) one finds

$$(4.37) \quad \|\tilde{\Phi}^{(1)}(x, m, s)\| = \|\Phi^{(1)}(x, m, s)\| \leq sG_2(d(b_x(R), \Lambda), m)$$

where for any fixed $\delta \in (0, 1)$, the function G_2 can be taken to be

$$(4.38) \quad G_2(l, m) = C \cdot \begin{cases} F_{\mathcal{K}^1}(l), & \text{if } m = R, \\ 4F_\delta(l) \cdot F_{1-\delta}(m - R - 1), & \text{if } m \geq R + 1. \end{cases}$$

Here, $C = \kappa^2 R^{2\nu} \|h\|_\infty$, $F_{\mathcal{K}^1}$ is the function from Lemma 4.3, and

$$(4.39) \quad F_\delta(l) = \max \left\{ (F_{\mathcal{K}^1}(l))^\delta, (G_{\mathcal{K}^1}(l))^\delta \right\}.$$

More specifically, the bound in (4.37) for $m = R$ is a direct application of Lemma 4.3 while the bound for $m \geq R + 1$ follows from the quasi-local estimate in Lemma 4.4 and the subsequent bound (4.26) coupled with (3.3)-(3.6).

Given (4.35), the summability of G_Λ over the sites $x \in \Gamma \setminus \Lambda(R)$ follows from observing that

$$(4.40) \quad \begin{aligned} \sum_{x \in \Lambda(R)^c} \sum_{m \geq R} G_2(d(b_x(R), \Lambda), m) &= C \sum_{x \in \Lambda(R)^c} F_{\mathcal{K}^1}(d(b_x(R), \Lambda)) \\ &\quad + 4C \sum_{x \in \Lambda(R)^c} F_\delta(d(b_x(R), \Lambda)) \sum_{m \geq R+1} F_{1-\delta}(m - R - 1) < \infty \end{aligned}$$

as both $F_{\mathcal{K}^1}$ and $G_{\mathcal{K}^1}$ (and, thus, F_δ) have finite moments of all orders. In particular, for any decay function $F : [0, \infty) \rightarrow [0, \infty)$ with a finite ν -moment,

$$\sum_{x \in \Lambda(R)^c} F(d(b_x(R), \Lambda)) \leq \sum_{n \geq R} \sum_{x \in \Lambda(n+1) \setminus \Lambda(n)} F(n-R) \leq \kappa |\Lambda| \sum_{n \geq R} (n+1)^\nu F(n-R) < \infty.$$

We now turn to the sites $x \in \Lambda(R)$, for which we demonstrate that

$$(4.41) \quad \|\Phi^{(1)}(x, m, s)\| \leq sG_1(m)$$

where G_1 is a summable function. First consider (4.33) when $m = R$. Combining the local bounds (4.10), the uniform bound (2.12), and the interaction bound in Assumption 2.5, one produces the x -independent bound

$$(4.42) \quad \begin{aligned} \|\Phi^{(1)}(x, R, s)\| &\leq \|\mathcal{K}^{1,\Lambda}(h_x)\| + \|\mathcal{K}^{2,\Lambda}(\Phi(x, R))\| \\ &\leq sC_1 \kappa^2 R^{2\nu} \|h\|_\infty + sC_2 \|\Phi\| e^{-aR^\theta}. \end{aligned}$$

Alternatively, for $m \geq R+1$, (4.33) can be estimated as

$$(4.43) \quad \begin{aligned} \|\Phi^{(1)}(x, m, s)\| &\leq \|\Delta_{b_x(R)}^m(\mathcal{K}_s^{1,\Lambda}(h_x))\| + \chi_\Lambda(x) \sum_{\substack{R \leq k \leq m: \\ b_x(k) \subseteq \Lambda}} \|\Delta_{b_x(k)}^m(\mathcal{K}_s^{2,\Lambda}(\Phi(x, k)))\| \\ &\leq 2s\kappa^2 R^{2\nu} \|h\|_\infty G_{\mathcal{K}^1}(m-R-1) + 2s\kappa \sum_{\substack{R \leq k \leq m: \\ b_x(k) \subseteq \Lambda}} k^\nu \|\Phi(x, k)\| G_{\mathcal{K}^2}(m-k-1) \end{aligned}$$

where one uses the quasi-local estimates from (4.11) and the local approximation bound in (3.6). Given Assumption 2.5, the final sum above can be further estimated as

$$(4.44) \quad \begin{aligned} \sum_{\substack{R \leq k \leq m: \\ b_x(k) \subseteq \Lambda}} k^\nu \|\Phi(x, k)\| G_{\mathcal{K}^2}(m-k-1) &\leq \|\Phi\| \sum_{k=R}^m k^\nu e^{-ak^\theta} G_{\mathcal{K}^2}(m-k-1) \\ &\leq \|\Phi\| \left(G_{\mathcal{K}^2}(m/2) \sum_{k=R}^{m/2-1} k^\nu e^{-ak^\theta} + G_{\mathcal{K}^2}(0) \sum_{k \geq m/2} k^\nu e^{-ak^\theta} \right). \end{aligned}$$

To simplify notation, let $M_\Phi(r) := \sum_{k \geq r} k^\nu e^{-ak^\theta}$ denote the ν -th moment of the decay function associated with the perturbation Φ from Assumption 2.5. Then, in summary, one has that for $x \in \Lambda(R)$, (4.41) holds for the decay function G_1 defined by

$$(4.45) \quad G_1(R) = C_1 \kappa^2 R^{2\nu} \|h\|_\infty + C_2 \|\Phi\| e^{-aR^\theta}$$

and for $m \geq R+1$,

$$(4.46) \quad G_1(m) = 2CG_{\mathcal{K}^1}(m-R-1) + 2\kappa \|\Phi\| (M_\Phi(R) G_{\mathcal{K}^2}(m/2) + G_{\mathcal{K}^2}(0) M_\Phi(m/2)).$$

Since each of the decay functions in (4.46) has finite moments of all orders, it is clear that $\sum_{m \geq R} G_1(m) < \infty$. As a consequence, G_Λ as in (4.35) satisfies

$$\sum_{x \in \Lambda(R)} \sum_{m \geq R} G_\Lambda(x, m) \leq \kappa R^\nu |\Lambda| \sum_{m \geq R} G_1(m) < \infty.$$

This demonstrates absolute summability of the terms in (4.6), and hence, completes the proof of Theorem 4.2. \square

4.2. The final decomposition of the transformed perturbation via LTQO. We now turn our attention to proving Theorem 4.1, which is a consequence of one last decomposition of the transformed perturbation from Theorem 4.2, i.e.

$$\left\{ \tilde{\Phi}^{(1)}(x, m, s) \in \pi_0(\mathcal{A}_{b_x(m)}) : x \in \Gamma, m \geq R \right\}.$$

The key component for proving the desired norm bounds for this final decomposition is Lemma 4.5 below, and it is in the proof of this result where one needs the LTQO property from Assumption 2.4. To this end, we first shift the transformed perturbation terms by their expectation in the ground state Ω , as this will put us in the appropriate setting to apply LTQO.

Throughout this section, we assume $\gamma \in (0, \gamma_0)$ is fixed and that $s \in \mathbb{R}$ is such that $|s| \leq s_0^\Lambda(\gamma)$. As such, $\Omega(\Lambda, s) = \tilde{U}(\Lambda, s)\Omega$ is the ground state of $H(\Lambda, s)$, and one finds that $W(\Lambda, s)\Omega = 0$ from considering (4.1) in the case $\psi = \Omega$. Thus, Theorem 4.2 implies that for any $\psi \in \pi_0(\mathcal{A}^{\text{loc}})\Omega$

$$(4.47) \quad W(\Lambda, s)\psi = \tilde{U}(\Lambda, s)^*H(\Lambda, s)\tilde{U}(\Lambda, s)\psi - H_0\psi - E(\Lambda, s)\psi = \sum_{x \in \Gamma} \sum_{m \geq R} \tilde{\Phi}_\omega^{(1)}(x, m, s)\psi,$$

where the (self-adjoint) observables $\tilde{\Phi}_\omega^{(1)}(x, m, s) \in \pi_0(\mathcal{A}_{b_x(m)})$ are defined by

$$(4.48) \quad \tilde{\Phi}_\omega^{(1)}(x, m, s) = \tilde{\Phi}^{(1)}(x, m, s) - \langle \Omega, \tilde{\Phi}^{(1)}(x, m, s)\Omega \rangle \mathbb{1}$$

and normalized to have zero ground state expectation: $\langle \Omega, \tilde{\Phi}_\omega^{(1)}(x, m, s)\Omega \rangle = 0$. For the proofs of Lemma 4.5 and Theorem 4.1, it is also convenient to set

$$(4.49) \quad \tilde{\Phi}_\omega^{(1)}(x, s) := \sum_{m \geq R} \tilde{\Phi}_\omega^{(1)}(x, m, s), \quad x \in \Gamma$$

which belongs to $\pi_0(\mathcal{A})$ by Theorem 4.2.

Lemma 4.5. *Let $\tilde{P}_{b_x(n)} = \pi_0(P_{b_x(n)}) \in \mathcal{B}(\mathcal{H})$ denote the representation of the ground state projection $P_{b_x(n)}$ in the GNS space. Then, under the assumptions of Theorem 4.1, the bound*

$$(4.50) \quad \left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{P}_{b_x(n)} \right\| \leq 2s \left(G_\Lambda^{(1)}(x, m+1) + G_\Lambda^{(1)}(x, R) \sqrt{(1+m)^\nu G_0(n-m)} \right)$$

holds where G_0 is the decay function from Assumption 2.4, and $G_\Lambda^{(1)}(x, m) = \sum_{k \geq m} G_\Lambda(x, k)$ with G_Λ as in Theorem 4.2.

Proof. To begin, one uses the LTQO property (2.27) to show that

$$(4.51) \quad \left| \|\tilde{A}\tilde{P}_{b_x(n)}\| - \|\tilde{A}P_\Omega\| \right| \leq \|A\| \sqrt{(1+m)^\nu G_0(n-m)} \quad \text{for all } A \in \mathcal{A}_{b_x(m)}$$

where $P_\Omega = |\Omega\rangle\langle\Omega|$ and $\tilde{A} = \pi_0(A)$. To see this, first note that the inequality $|a - b|^2 \leq |a^2 - b^2|$ for any $a, b \geq 0$, implies that

$$\left| \|\tilde{A}\tilde{P}_{b_x(n)}\| - \|\tilde{A}P_\Omega\| \right|^2 \leq \left| \|\tilde{A}\tilde{P}_{b_x(n)}\|^2 - \|\tilde{A}P_\Omega\|^2 \right|.$$

Recalling that $(\mathcal{H}, \pi_0, \Omega)$ is the GNS representation of the unperturbed ground state ω_0 , the second term on the right-hand-side above is simply

$$\|\tilde{A}P_\Omega\|^2 = \langle \Omega, \pi_0(A^*A)\Omega \rangle \|P_\Omega\| = \omega_0(A^*A) \|P_{b_x(n)}\|.$$

Here we find it convenient to use that π_0 is norm-preserving. From this, it follows that

$$\|\tilde{A}\tilde{P}_{b_x(n)}\|^2 = \|\pi_0(AP_{b_x(n)})\|^2 = \|P_{b_x(n)}A^*AP_{b_x(n)}\|.$$

Given these observations, Assumption 2.4 then implies that

$$\begin{aligned}
\left| \|\tilde{A}\tilde{P}_{b_x(n)}\| - \|\tilde{A}P_\Omega\| \right|^2 &\leq \left| \|\tilde{P}_{b_x(n)}A^*AP_{b_x(n)}\| - \omega_0(A^*A)\|\tilde{P}_{b_x(n)}\| \right| \\
&\leq \|\tilde{P}_{b_x(n)}A^*AP_{b_x(n)} - \omega_0(A^*A)\tilde{P}_{b_x(n)}\| \\
(4.52) \quad &\leq \|A\|^2(1+m)^\nu G_0(n-m),
\end{aligned}$$

which establishes (4.51).

Now using (4.51) with $\tilde{A} = \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) \in \pi_0(\mathcal{A}_{b_x(m)})$, one finds that for any $n \geq m$,

$$\begin{aligned}
\left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{P}_{b_x(n)} \right\| &\leq \left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) P_\Omega \right\| + \left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) \right\| \sqrt{(1+m)^\nu G_0(n-m)} \\
(4.53) \quad &\leq \left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) P_\Omega \right\| + 2sG_\Lambda^{(1)}(x, R) \sqrt{(1+m)^\nu G_0(n-m)}
\end{aligned}$$

where the last inequality follows from Theorem 4.2 as $\|\tilde{\Phi}_\omega^{(1)}(x, k, s)\| \leq 2\|\tilde{\Phi}^{(1)}(x, k, s)\| \leq 2sG_\Lambda(x, k)$.

The remaining operator norm from (4.53) can then be trivially bounded in terms of $\tilde{\Phi}_\omega^{(1)}(x, s)$ from (4.49) as follows:

$$(4.54) \quad \left\| \sum_{k=R}^m \tilde{\Phi}_\omega^{(1)}(x, k, s) P_\Omega \right\| \leq \left\| \tilde{\Phi}_\omega^{(1)}(x, s) P_\Omega \right\| + \sum_{k=m+1}^{\infty} \left\| \tilde{\Phi}_\omega^{(1)}(x, k, s) P_\Omega \right\|.$$

Once again applying Theorem 4.2 then shows that

$$(4.55) \quad \sum_{k=m+1}^{\infty} \left\| \tilde{\Phi}_\omega^{(1)}(x, k, s) P_\Omega \right\| \leq 2 \sum_{k=m+1}^{\infty} \left\| \tilde{\Phi}^{(1)}(x, k, s) \right\| \leq 2sG_\Lambda^{(1)}(x, m+1),$$

and, moreover, $[\tilde{\Phi}_\omega^{(1)}(x, s), P_\Omega] = [\tilde{\Phi}^{(1)}(x, s), P_\Omega] = 0$. As a result, $\left\| \tilde{\Phi}_\omega^{(1)}(x, s) P_\Omega \right\| = 0$ since

$$(4.56) \quad \tilde{\Phi}_\omega^{(1)}(x, s) P_\Omega = P_\Omega \tilde{\Phi}_\omega^{(1)}(x, s) P_\Omega = \langle \Omega, \tilde{\Phi}_\omega^{(1)}(x, s) \Omega \rangle P_\Omega = 0,$$

where the last equality follows from (4.48)-(4.49). Thus, inserting (4.55) into (4.53) proves (4.50). \square

We now prove Theorem 4.1, which uses both Lemma 4.5 and the frustration-free property.

Proof of Theorem 4.1. Fix $x \in \Gamma$ and recall that $P_\Omega = |\Omega\rangle\langle\Omega|$. Since $[\tilde{\Phi}_\omega^{(1)}(x, s), P_\Omega] = 0$, one can write

$$\tilde{\Phi}_\omega^{(1)}(x, s) = P_\Omega \tilde{\Phi}_\omega^{(1)}(x, s) P_\Omega + (\mathbb{1} - P_\Omega) \tilde{\Phi}_\omega^{(1)}(x, s) (\mathbb{1} - P_\Omega) = (\mathbb{1} - P_\Omega) \tilde{\Phi}_\omega^{(1)}(x, s) (\mathbb{1} - P_\Omega),$$

where the last equality uses (4.56). The terms $\tilde{\Phi}^{(2)}(x, m, s)$ are defined by decomposing $\mathbb{1} - P_\Omega$ in terms of the finite volume ground state projections $\tilde{P}_n := \tilde{P}_{b_x(n)} \in \pi_0(\mathcal{A}_{b_x(n)})$.

First, note that \tilde{P}_n converges strongly to P_Ω for all $\psi \in \mathcal{H}$ by the frustration-free and LTQO properties. As a consequence, the collection of operators

$$(4.57) \quad \tilde{E}_n = \begin{cases} \mathbb{1} - \tilde{P}_R, & n = R \\ \tilde{P}_{n-1} - \tilde{P}_n, & n \geq R+1 \end{cases}$$

forms a family of orthogonal projections that are mutually orthogonal and sum to $\mathbb{1} - P_\Omega$. That is,

$$(4.58) \quad \tilde{E}_n^* = \tilde{E}_n, \quad \tilde{E}_n \tilde{E}_m = \delta_{m,n} \tilde{E}_n, \quad \text{and} \quad (\mathbb{1} - P_\Omega)\psi = \sum_{n \geq R} \tilde{E}_n \psi \quad \forall \psi \in \mathcal{H},$$

where the second equality holds since the frustration-free property implies $\tilde{P}_n \tilde{P}_m = \tilde{P}_m$ for $m \geq n$. Moreover, it trivially holds that

$$(4.59) \quad \mathbb{1} - \tilde{P}_m = \sum_{n=R}^m \tilde{E}_n.$$

Using (4.49), the above properties imply that for all $\psi, \phi \in \mathcal{H}$,

$$(4.60) \quad \langle \phi, (\mathbb{1} - P_\Omega) \tilde{\Phi}_\omega^{(1)}(x, s) (\mathbb{1} - P_\Omega) \psi \rangle = \sum_{k, m, n \geq R} \langle \phi, \tilde{E}_n \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{E}_m \psi \rangle$$

We note that the triple sum of operators actually converges absolutely in norm, and so the operator equality holds in the norm sense.

Each term $\Phi^{(2)}(x, m, s) \in \mathcal{A}_{b_x(m)}$ will be defined as a sum of two self-adjoint terms

$$\pi_0(\Phi^{(2)}(x, m, s)) = \Theta_1(x, m, s) + \Theta_2(x, m, s)$$

each of which is annihilated by the ground state projection \tilde{P}_m . Fix $k \geq R$, and use the properties in (4.58)-(4.59) to write

$$(4.61) \quad \sum_{n, m \geq R} \tilde{E}_n \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{E}_m = (\mathbb{1} - \tilde{P}_{2k}) \tilde{\Phi}_\omega^{(1)}(x, k, s) (\mathbb{1} - \tilde{P}_{2k}) + \sum_{m > 2k} \tilde{\Phi}_{k, m},$$

where $\tilde{\Phi}_{k, m} = \tilde{\Phi}_{k, m}^*$ is defined by

$$(4.62) \quad \tilde{\Phi}_{k, m} = \tilde{E}_m \tilde{\Phi}_\omega^{(1)}(x, k, s) (\mathbb{1} - \tilde{P}_{m-1}) + (\mathbb{1} - \tilde{P}_m) \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{E}_m \in \pi_0(\mathcal{A}_{b_x(m)}).$$

Self-adjointness follows from noting that $\mathbb{1} - \tilde{P}_m = \mathbb{1} - \tilde{P}_{m-1} + \tilde{E}_m$.

For each $m \geq R$, define $\Theta_1(x, m, s) \in \pi_0(\mathcal{A}_{b_x(m)})$, by

$$(4.63) \quad \Theta_1(x, m, s) = \begin{cases} (\mathbb{1} - \tilde{P}_m) \tilde{\Phi}_\omega^{(1)}(x, m/2, s) (\mathbb{1} - \tilde{P}_m), & m \text{ even} \\ 0, & m \text{ odd} \end{cases}$$

These operators are self-adjoint, satisfy $\Theta_1(x, m, s) \tilde{P}_m = 0$, and Theorem 4.2 implies that their norm is bounded from above by $2sG_\Lambda(x, m/2)$ as for m even:

$$(4.64) \quad \|\Theta_1(x, m, s)\| \leq \|\Phi_\omega^{(1)}(x, m/2, s)\| \leq 2sG_\Lambda(x, m/2).$$

For the Θ_2 terms, one sums the remaining terms $\sum_{m > 2k} \tilde{\Phi}_{k, m}$ over k , and then uses the indicator function $\chi_{m > 2k}$ to exchange the summations as follows:

$$\begin{aligned} \sum_{k \geq R} \sum_{m > 2k} \tilde{\Phi}_{k, m} &= \sum_{k \geq R} \sum_{m > 2R} \tilde{\Phi}_{k, m} \chi_{\{m > 2k\}} \\ &= \sum_{m > 2R} \sum_{R \leq k < m/2} \tilde{\Phi}_{k, m} \\ &= \sum_{m > 2R} \Theta_2(x, m, s) \end{aligned}$$

where, for $m > 2R$ one recalls (4.62) and defines

$$(4.65) \quad \Theta_2(x, m, s) = \sum_{k=R}^{\lceil m/2 \rceil - 1} \tilde{E}_m \tilde{\Phi}_\omega^{(1)}(x, k, s) (\mathbb{1} - \tilde{P}_{m-1}) + (\mathbb{1} - \tilde{P}_m) \tilde{\Phi}_\omega^{(1)}(x, k, s) \tilde{E}_m.$$

This definition is extended by setting $\Theta_2(x, m, s) = 0$ for $R \leq m \leq 2R$. One sees that $\Theta_2(x, m, s) \tilde{P}_m = 0$ by applying $\tilde{P}_m \tilde{P}_{m-1} = \tilde{P}_m$ to verify

$$\tilde{E}_m \tilde{P}_m = (\tilde{P}_m - \tilde{P}_{m-1}) \tilde{P}_m = 0.$$

Finally, applying Lemma 4.5,

$$\begin{aligned}
\|\Theta_2(x, m, s)\| &\leq 2 \left\| \sum_{k=R}^{\lceil m/2 \rceil - 1} \tilde{\Phi}_\omega^{(1)}(x, k, s) (\mathbb{1} - \tilde{P}_{m-1}) \right\| \\
(4.66) \quad &\leq 4s \left(G_\Lambda^{(1)}(x, \lceil m/2 \rceil) + G_\Lambda^{(1)}(x, R) \sqrt{\lceil m/2 \rceil^\nu G_0(m/2)} \right)
\end{aligned}$$

where $G_\Lambda^{(1)}(x, m) = \sum_{n \geq m} G_\Lambda(x, n)$.

Therefore, setting $\tilde{\Phi}^{(2)}(x, m, s) = \Theta_1(x, m, s) + \Theta_2(x, m, s)$ for all m , we have constructed self-adjoint operators such that

$$(4.67) \quad W(\Lambda, s) = \sum_{x \in \Gamma} \sum_{m \geq R} \tilde{\Phi}^{(2)}(x, m, s).$$

Moreover, these operators satisfy that for all $m \geq R$ and $x \in \Gamma$,

$$\tilde{\Phi}^{(2)}(x, m, s) \tilde{P}_m = \tilde{P}_m \tilde{\Phi}^{(2)}(x, m, s) = 0$$

and, combining (4.64) and (4.66), their norms can be bounded above by

$$(4.68) \quad \|\tilde{\Phi}^{(2)}(x, m, s)\| \leq 2sG_\Lambda^{(2)}(x, m)$$

where

$$(4.69) \quad G_\Lambda^{(2)}(x, m) = G_\Lambda(x, m/2) + 2G_\Lambda^{(1)}(x, \lceil m/2 \rceil) + 2G_\Lambda^{(1)}(x, R) \sqrt{\lceil m/2 \rceil^\nu G_0(m/2)}.$$

The absolute summability of the series in (4.67) is a direct consequence of G_0 being summable as well as that both G_Λ and $G_\Lambda^{(1)}$ satisfy (4.5). For $G_\Lambda^{(1)}$ this can easily be seen from the fact that G_Λ is a combination of functions with finite moments of all orders, see specifically (4.35), (4.38) and (4.46). \square

5. PROVING THEOREMS 2.7 AND 2.8 VIA A FORM BOUND FOR THE GNS HAMILTONIAN

In this section, we generalize [48, Theorem 3.8], which was itself based off [41, Proposition 2], so that it is applicable to the setting of infinite systems in their GNS representation. Afterwards, we apply the form bound in conjunction with Theorem 4.1 to prove Theorem 2.7. We then conclude with the proof of Theorem 2.8, which follows as a consequence of Theorem 2.7.

5.1. The Michalakis-Zwolak relative form bound. The form bound result is stated in the setting described in Section 2 and, in particular, under Assumption 2.3. Thus, there is a family of sets

$$\mathcal{S} = \{\Lambda(x, n) \mid x \in \Gamma, n \geq 0 \text{ s.t. } b_x(n) \subseteq \Lambda(x, n)\},$$

accompanied with a family $\mathcal{T} = \{\mathcal{T}_n : n \geq 0\}$ of separating partitions of (c, ζ) -polynomial growth, for which the associated finite volume Hamiltonians satisfy

$$(5.1) \quad H_{\Lambda(x, n)} \geq \gamma(n) P_{\Lambda(x, n)}, \text{ for all } n \geq R.$$

Moreover, the local gaps are further assumed to satisfy $\gamma(n) \geq \gamma_1 n^\alpha$ for some $\gamma_1 > 0$ and $\alpha \geq 0$.

Theorem 5.1 (Michalakis-Zwolak [41]). *Let H_0 be the GNS Hamiltonian associated with a zero-energy ground state of an initial system satisfying Assumptions 2.1 and 2.3, and let $V \in \mathcal{A}$ be a perturbation associated with an absolutely-summable, anchored interaction on (Γ, d) . That is, there exist $\Phi(x, n)^* = \Phi(x, n) \in \mathcal{A}_{b_x(n)}$ for all $x \in \Gamma$ and $n \geq R$ such that*

$$(5.2) \quad V = \sum_{x \in \Gamma} \sum_{n \geq R} \Phi(x, n), \quad \sum_{x \in \Gamma} \sum_{n \geq R} \|\Phi(x, n)\| < \infty.$$

In addition, assume that terms of V annihilate the finite-volume ground state projections of the initial system, i.e.

$$(5.3) \quad \Phi(x, n)P_{b_x(n)} = P_{b_x(n)}\Phi(x, n) = 0 \quad \text{for all } x \in \Gamma \text{ and } n \geq R,$$

and that there is a decay function G with finite $(\zeta + \alpha)$ -moment such that $\sup_{x \in \Gamma} \|\Phi(x, n)\| \leq G(n)$. Then, for all $\psi \in \text{dom } H_0$,

$$(5.4) \quad |\langle \psi, \pi_\omega(V) \psi \rangle| \leq \beta \langle \psi, H_0 \psi \rangle \quad \text{where } \beta = c \sum_{n \geq R} \frac{n^\zeta G(n)}{\gamma(n)} \leq \frac{c}{\gamma_1} \sum_{n \geq R} n^{\zeta + \alpha} G(n).$$

In keeping with the notation from the previous sections, denote by $\tilde{A} = \pi_0(A)$ the image of any observable $A \in \mathcal{A}$ under the GNS representation $(\mathcal{H}, \pi_0, \Omega)$ of the zero-energy ground state ω_0 . The proof of Theorem 5.1 follows closely the argument proving [48, Theorem 3.8], with the proviso that one must check that the infinite operator sums replacing the finite operator sums from [48] are well-defined.

To this end, let $n \geq R$, and consider the n -th separating partition $\mathcal{T}_n = \{\mathcal{T}_n^i : i \in \mathcal{I}_n\}$. For each $i \in \mathcal{I}_n$ and any choice of $x, y \in \mathcal{T}_n^i$, one has that

$$(5.5) \quad [\tilde{H}_{\Lambda(x, n)}, \tilde{H}_{\Lambda(y, n)}] = 0 \quad \text{and} \quad [\tilde{P}_{\Lambda(x, n)}, \tilde{P}_{\Lambda(y, n)}] = 0.$$

This follows since the corresponding algebra elements $H_{\Lambda(x, n)}, H_{\Lambda(y, n)} \in \mathcal{A}$ are supported on disjoint sets (and similarly for the ground state projections), which carries over to the GNS space by the homomorphism property.

With $n \geq R$ and $i \in \mathcal{I}_n$ fixed, denote by C_n^i the collection of all configurations associated to \mathcal{T}_n^i . More precisely,

$$(5.6) \quad C_n^i = \{0, 1\}^{\mathcal{T}_n^i} = \{\sigma : \sigma = \{\sigma_x\} \text{ where } \sigma_x \in \{0, 1\} \text{ for all } x \in \mathcal{T}_n^i\}.$$

For each $\sigma \in C_n^i$, we define $|\sigma|$ by

$$(5.7) \quad |\sigma| = \sum_{x \in \mathcal{T}_n^i} \sigma_x.$$

Recall that $\pi_0(\mathcal{A}^{\text{loc}})\Omega$ is a dense subspace of \mathcal{H} . Let $\psi = \tilde{A}\Omega$ for some $A \in \mathcal{A}_X$, and set $\tilde{Q}_{\Lambda(x, n)} = \mathbb{1} - \tilde{P}_{\Lambda(x, n)}$. In this case, one has that

$$(5.8) \quad \tilde{P}_{\Lambda(x, n)}\psi = \psi \quad \text{and} \quad \tilde{Q}_{\Lambda(x, n)}\psi = 0$$

whenever $\Lambda(x, n)$ satisfies $\Lambda(x, n) \cap X = \emptyset$. It is also clear that for any $X \in \mathcal{P}_0(\Gamma)$, the set of sites $\{x \in \mathcal{T}_n^i : \Lambda(x, n) \cap X \neq \emptyset\}$ is finite. Thus, for any $\psi \in \pi_0(\mathcal{A}^{\text{loc}})\Omega$, define the operator $S(\sigma)$ by

$$(5.9) \quad S(\sigma)\psi = \begin{cases} \prod_{x \in \mathcal{T}_n^i} [\sigma_x \tilde{Q}_{\Lambda(x, n)} + (1 - \sigma_x) \tilde{P}_{\Lambda(x, n)}] \psi & \text{if } |\sigma| < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if $|\sigma| < \infty$, then at most finitely many of these factors act non-trivially, and moreover, by (5.5), all factors above commute. Since $\pi_0(\mathcal{A}^{\text{loc}})\Omega$ is dense, there is a unique extension of $S(\sigma)$ to an element of $\mathcal{B}(\mathcal{H}_\omega)$ for each $\sigma \in C_n^i$. One checks that these operators satisfy:

$$(5.10) \quad S(\sigma)^* = S(\sigma), \quad S(\sigma)S(\sigma') = \delta_{\sigma, \sigma'}S(\sigma), \quad \text{and} \quad \sum_{\sigma \in C_n^i} S(\sigma) = \mathbb{1}.$$

Since $\tilde{Q}_{\Lambda(x, n)} = \mathbb{1} - \tilde{P}_{\Lambda(x, n)}$, by (5.5) one clearly has that

$$(5.11) \quad [\tilde{Q}_{\Lambda(x, n)}, \tilde{P}_{\Lambda(y, n)}] = 0 \quad \text{for all } x, y \in \mathcal{T}_n^i$$

and as a result, also

$$(5.12) \quad [\tilde{Q}_{\Lambda(x, n)}, S(\sigma)] = 0 \quad \text{for each } x \in \mathcal{T}_n^i \text{ and all } \sigma \in C_n^i.$$

For $\sigma \in C_n^i$ with $|\sigma| < \infty$, (5.11) implies $0 = [\tilde{Q}_{\Lambda(x,n)}, S(\sigma)]\psi$ for all $\psi \in \pi_0(\mathcal{A}^{\text{loc}})\Omega$ and, hence, for all $\psi \in \mathcal{H}$. If $|\sigma| = \infty$, then the commutator is zero by definition. Combining (5.10) and (5.12), we find that

$$(5.13) \quad S(\sigma)\tilde{Q}_{\Lambda(x,n)}S(\sigma') = \delta_{\sigma,\sigma'}\tilde{Q}_{\Lambda(x,n)}S(\sigma) = \delta_{\sigma,\sigma'}(\mathbb{1} - \tilde{P}_{\Lambda(x,n)})S(\sigma) = \delta_{\sigma,\sigma'}\sigma_x S(\sigma)$$

for all $\sigma, \sigma' \in C_n^i$ and each $x \in \mathcal{T}_n^i$.

We use these families of orthogonal projections in the following proof.

Proof of Theorem 5.1. In the GNS representation, the terms of \tilde{V} can be rearranged using the family of separating partitions from Assumption 2.3 as

$$(5.14) \quad \tilde{V} = \sum_{x \in \Gamma} \sum_{n \geq R} \tilde{\Phi}(x, n) = \sum_{n \geq R} \sum_{i \in \mathcal{I}_n} \tilde{V}_n^i, \quad \tilde{V}_n^i := \sum_{x \in \mathcal{T}_n^i} \tilde{\Phi}(x, n)$$

since V is absolutely summable. As a result, for any $\psi \in \mathcal{H}$, one has the bound

$$(5.15) \quad |\langle \psi, \tilde{V}\psi \rangle| \leq \sum_{n \geq R} \sum_{i \in \mathcal{I}_n} |\langle \psi, \tilde{V}_n^i \psi \rangle| \quad \text{with} \quad |\langle \psi, \tilde{V}_n^i \psi \rangle| \leq \sum_{x \in \mathcal{T}_n^i} |\langle \psi, \tilde{\Phi}(x, n) \psi \rangle|.$$

Now, since $b_x(n) \subset \Lambda(x, n)$ for all $\Lambda(x, n) \in \mathcal{S}$, the frustration free property implies

$$(5.16) \quad P_{\Lambda(x,n)} = P_{b_x(n)}P_{\Lambda(x,n)} = P_{\Lambda(x,n)}P_{b_x(n)},$$

and so by (5.3) it follows that for each $x \in \mathcal{T}_n^i$, the term $\tilde{\Phi}(x, n)$ satisfies

$$(5.17) \quad [\tilde{\Phi}(x, n), \tilde{P}_{\Lambda(y,n)}] = 0 \quad \text{for all } y \in \mathcal{T}_n^i.$$

Arguing as in (5.12)-(5.13) above, one then finds that for all $x \in \mathcal{T}_n^i$ and $\sigma, \sigma' \in C_n^i$

$$(5.18) \quad [\tilde{\Phi}(x, n), S(\sigma)] = 0$$

and, moreover,

$$(5.19) \quad \begin{aligned} S(\sigma)\tilde{\Phi}(x, n)S(\sigma') &= \delta_{\sigma,\sigma'}S(\sigma)\tilde{\Phi}(x, n) \\ &= \delta_{\sigma,\sigma'}\sigma_x S(\sigma)\tilde{\Phi}(x, n) \\ &= \delta_{\sigma,\sigma'}S(\sigma)\tilde{Q}_{\Lambda(x,n)}\tilde{\Phi}(x, n)\tilde{Q}_{\Lambda(x,n)}S(\sigma). \end{aligned}$$

As a consequence, one can use (5.10) and (5.19) to bound

$$(5.20) \quad \begin{aligned} |\langle \psi, \tilde{\Phi}(x, n) \psi \rangle| &\leq \sum_{\sigma, \sigma'} |\langle \psi, S(\sigma)\tilde{\Phi}(x, n)S(\sigma') \psi \rangle| = \sum_{\sigma} |\langle \psi, S(\sigma)\tilde{Q}_{\Lambda(x,n)}\tilde{\Phi}(x, n)\tilde{Q}_{\Lambda(x,n)}S(\sigma) \psi \rangle| \\ &\leq G(n) \sum_{\sigma} \|\tilde{Q}_{\Lambda(x,n)}S(\sigma)\psi\|^2. \end{aligned}$$

Summing these orthogonal projections again, one obtains

$$(5.21) \quad \sum_{\sigma} \|\tilde{Q}_{\Lambda(x,n)}S(\sigma)\psi\|^2 = \sum_{\sigma} \langle \psi, \tilde{Q}_{\Lambda(x,n)}S(\sigma)\psi \rangle = \langle \psi, \tilde{Q}_{\Lambda(x,n)}\psi \rangle \leq \frac{1}{\gamma(n)} \langle \psi, \tilde{H}_{\Lambda(x,n)}\psi \rangle$$

where, the final bound follows since the representation preserves positivity and the operator inequality $\gamma(n)Q_{\Lambda(x,n)} \leq H_{\Lambda(x,n)}$ holds in the algebra.

Given the above, one concludes that for any $\psi \in \text{dom}(H_0)$,

$$(5.22) \quad |\langle \psi, \tilde{V}_n^i \psi \rangle| \leq \sum_{x \in \mathcal{T}_n^i} |\langle \psi, \tilde{\Phi}(x, n) \psi \rangle| \leq \frac{G(n)}{\gamma(n)} \sum_{x \in \mathcal{T}_n^i} \langle \psi, \tilde{H}_{\Lambda(x,n)} \psi \rangle \leq \frac{G(n)}{\gamma(n)} \langle \psi, H_0 \psi \rangle$$

where the last inequality uses that $\Lambda(x, n) \cap \Lambda(y, n) = \emptyset$ for distinct $x, y \in \mathcal{T}_n^i$. Returning to (5.15), since these partitions satisfy the (c, ζ) -polynomial growth bound it is clear that

$$(5.23) \quad |\langle \psi, \tilde{V}\psi \rangle| \leq \sum_{n \geq R} \sum_{i \in \mathcal{I}_n} |\langle \psi, \tilde{V}_n^i \psi \rangle| \leq \beta \langle \psi, H_0 \psi \rangle,$$

with β as in (5.4). This completes the proof. \square

5.2. Proofs of Theorems 2.7 and 2.8. Theorem 4.1 showed that for every finite $\Lambda \subset \Gamma$, the Hamiltonian

$$H(\Lambda, s) = H_0 + s\pi_0(V_\Lambda) = H_0 + s \sum_{b_x(n) \subset \Lambda} \pi_0(\Phi(x, n))$$

transforms under the spectral flow unitary as follows

$$\tilde{U}(\Lambda, s)^* H(\Lambda, s) \tilde{U}(\Lambda, s) \psi - E(\Lambda, s) \psi = H_0 \psi + \sum_{x \in \Gamma} \sum_{m \geq R} \pi_0(\Phi_\Lambda^{(2)}(x, m, s)) \psi,$$

where $E(\Lambda, s)$ is the ground state energy of $H(\Lambda, s)$, and $\Phi_\Lambda^{(2)}(x, m, s)$ is a balled interaction satisfying the conditions of Theorem 5.1 with norm bounds that are linear in $|s|$ and given in terms of a Λ -dependent decay function. However, we will show below that the constant β from Theorem 5.1 can be taken independent of Λ . Theorem 2.7 will then follow from applying [48, Corollary 3.3]. In our context, the latter result states the following. Suppose that H_0 is a self-adjoint, positive operator on a Hilbert space \mathcal{H} with $\min \text{spec} H_0 = 0$ and $(0, \gamma_0) \cap \text{spec}(H_0) = \emptyset$. Then, for any $V = V^* \in \mathcal{B}(\mathcal{H})$ such that there exists $0 \leq \beta < 1$ for which

$$|\langle \psi, V\psi \rangle| \leq \beta \langle \psi, H_0 \psi \rangle \quad \forall \psi \in \text{dom}(H_0),$$

one has that

$$(5.24) \quad \text{spec}(H_0 + V) \cap (0, (1 - \beta)\gamma_0) = \emptyset.$$

Proof of Theorem 2.7. Fix $\gamma \in (0, \gamma_0)$ where γ_0 is as in Assumption 2.2, and let $\Lambda \in \mathcal{P}_0$ be arbitrary. Given Theorem 4.1, it is clear that Theorem 5.1 applies and produces a non-trivial form bound (see (5.4)) so long as

$$G_\Lambda^{(2)}(m) := \sup_{x \in \Gamma} G_\Lambda^{(2)}(x, m)$$

has a finite $(\zeta + \alpha)$ -moment where, as stated in Assumption 2.3, ζ is the polynomial growth of the separating partitions and $\gamma(n) \geq \gamma_1 n^{-\alpha}$ is the lower bound on the local gaps. Moreover, to show that

$$(5.25) \quad s_0(\gamma) := \inf_{\Lambda \in \mathcal{P}_0} s_0^\Lambda(\gamma) > 0,$$

see (3.39), we wish to show this form bound is uniform in Λ .

To this end, recall that G_Λ from (4.35) is defined in terms of two functions G_1, G_2 that are independent of Λ and decay faster than any polynomial. It is then clear that for all $\Lambda \in \mathcal{P}_0, x \in \Gamma$ and $m \geq R$,

$$G_\Lambda(x, m) \leq G_1(m) + G_2(0, m) =: G(m),$$

and similarly, $G_\Lambda^{(1)}(x, m) \leq G^{(1)}(m)$ where $G^{(1)}(m) = \sum_{m \geq R} G(m)$. It follows immediately that

$$\sup_{\Lambda \in \mathcal{P}_0} G_\Lambda^{(2)}(m) \leq G^{(2)}(m) := G(m/2) + 2G^{(1)}(m/2) + 2G^{(1)}(R)\sqrt{(1 + m)^\nu G_0(m/2)}$$

and that $G^{(2)}$ has a finite $(\zeta + \alpha)$ -moment as long as G_0 satisfies (2.26).

Given the norm bound from Theorem 4.1, it follows from Theorem 5.1 that for all $\psi \in \text{dom}(H_0)$

$$\langle \psi, W(\Lambda, s)\psi \rangle \leq s\beta \langle \psi, H_0 \psi \rangle \quad \text{where} \quad \beta = 2c \sum_{m \geq R} \frac{n^\zeta G^{(2)}(m)}{\gamma(m)}.$$

Thus, $\text{gap}(H(\Lambda, s)) \geq \gamma_0 - s\beta\gamma_0$ by (5.24) and, in particular, (5.25) holds since this implies

$$s_0^\Lambda(\gamma) \geq \frac{\gamma_0 - \gamma}{\beta\gamma_0}.$$

□

We conclude with using the uniform estimate from Theorem 2.7 to establish the claimed lower bound estimate on the gap of the extensively perturbed system from Theorem 2.8.

Proof of Theorem 2.8. Let $0 < \gamma < \gamma_0$ and consider $|s| \leq s_0(\gamma)$. Recall that for any IAS (Λ_n) , the following limits hold in \mathcal{A} as $n \rightarrow \infty$:

$$(5.26) \quad \alpha_s^{\Lambda_n}(A) \rightarrow \alpha_s(A), \quad \text{for all } A \in \mathcal{A}$$

$$(5.27) \quad \delta_s^{\Lambda_n}(A) \rightarrow \delta_s(A), \quad \text{for all } A \in \mathcal{A}_{\text{loc}},$$

see (2.37) and (2.38). As a consequence, $\omega_s^{\Lambda_n}(A) = \omega_0 \circ \alpha_s^{\Lambda_n}(A) \rightarrow \omega_s(A)$ for all $A \in \mathcal{A}$ and, moreover, ω_s is a ground state of δ_s .

Now consider $A \in \mathcal{A}^{\text{loc}}$ for which $\omega_s(A) = 0$. Given Theorem 2.7, the GNS Hamiltonian $H(\Lambda_n, s)$ along any IAS (Λ_n) has a gap above its unique ground state lower bounded by γ for all $|s| \leq s_0(\gamma)$. Therefore, $\omega_s^{\Lambda_n}$ satisfies (2.22) and, in particular, the inequality

$$(5.28) \quad \omega_s^{\Lambda_n}(B_n^* \delta_s^{\Lambda_n}(B_n)) \geq \gamma \omega_s^{\Lambda_n}(B_n^* B_n)$$

holds for the observable $B_n = A - \omega_s^{\Lambda_n}(A)\mathbb{1} \in \mathcal{A}^{\text{loc}}$. Combining (5.27) and the local uniform convergence of $\alpha_s^{\Lambda_n}$, the limit $n \rightarrow \infty$ can be taken on both sides of (5.28) to obtain (2.39). The remaining claims follow as in (2.22). □

APPENDIX A. LTQO IMPLIES UNIQUENESS OF THE GROUND STATE

In this section, we prove a simple, yet general, uniqueness result within the framework of Section 2. To this end, let \mathcal{A} be a C^* -algebra of quasi-local observables associated to a quantum spin system defined over a ν -regular, discrete metric space (Γ, d) . Let Φ be a non-negative, frustration-free interaction on (Γ, d) for which the corresponding derivation δ , defined initially on the local observable algebra as in (2.18), is a generator of a strongly continuous dynamics, denoted by $\{\tau_t \mid t \in \mathbb{R}\}$. Let ω be an infinite-volume, zero-energy ground state associated to δ , i.e. ω satisfies (2.19) and $\omega(\Phi(X)) = 0$ for all finite $X \subset \Gamma$. Denote by $(\mathcal{H}, \pi, \Omega)$ the GNS triple associate to ω . To simplify notation, let us set $\tilde{A} := \pi(A)$ the representative of $A \in \mathcal{A}$ in the GNS space. We say that ω satisfies LTQO if there is a function $G : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{m \rightarrow \infty} G(m) = 0$ for which: given any choices $m \geq k \geq 0$, $x \in \Gamma$, and $A \in \mathcal{A}_{b_x(k)}$,

$$(A.1) \quad \|\tilde{P}_{b_x(m)} \tilde{A} \tilde{P}_{b_x(m)} - \langle \Omega, \tilde{A} \Omega \rangle \tilde{P}_{b_x(m)}\| \leq \|A\| (1+k)^\nu G(m-k).$$

Note that the above is simply the reformulation of (2.27) in $\mathcal{B}(\mathcal{H})$. The following result holds.

Proposition A.1. *Let ω be an infinite-volume, zero-energy ground state of a frustration-free model. If ω satisfies LTQO, then the kernel of the GNS Hamiltonian H is one-dimensional.*

Proof. By way of contradiction, assume there is a unit vector $\psi \in \ker(H)$ with $\langle \psi, \Omega \rangle = 0$. First, since $\pi(\mathcal{A}^{\text{loc}})\Omega$ is dense in \mathcal{H} , there is $x \in \Gamma$ and operators $A_n \in \mathcal{A}_{b_x(n)}$ for which $\psi_n = \tilde{A}_n \Omega \rightarrow \psi$. In this case,

$$(A.2) \quad \lim_{n \rightarrow \infty} \langle \Omega, \tilde{A}_n \Omega \rangle = \lim_{n \rightarrow \infty} \langle \Omega, \psi_n \rangle = \langle \Omega, \psi \rangle = 0.$$

Thus, there is $n_1 \in \mathbb{N}$ so that $|\langle \Omega, \tilde{A}_n \Omega \rangle| \leq 1/3$ for all $n \geq n_1$.

Next, since $\ker(H) \subset \ker(\tilde{H}_{b_x(m)})$ for all $m \geq 0$ and $x \in \Gamma$, we have

$$(A.3) \quad \langle \psi, \tilde{P}_{b_x(m)} \tilde{A}_n \tilde{P}_{b_x(m)} \Omega \rangle = \langle \psi, \tilde{A}_n \Omega \rangle = \langle \psi, \psi_n \rangle$$

for all $m \geq 0$. Since $1 = \lim_{n \rightarrow \infty} \langle \psi, \psi_n \rangle$, there is some $n_2 \in \mathbb{N}$ for which $|\langle \psi, \psi_n \rangle| \geq 2/3$ for all $n \geq n_2$. Combining this with (A.3), we find that for any $m \geq n \geq n_2$,

$$(A.4) \quad 2/3 \leq \|\tilde{P}_{b_x(m)} \tilde{A}_n \tilde{P}_{b_x(m)}\|.$$

It follows that for $n \geq \max\{n_1, n_2\}$,

$$(A.5) \quad 1/3 \leq \|\tilde{P}_{b_x(m)} \tilde{A}_n \tilde{P}_{b_x(m)} - \langle \Omega, \tilde{A}_n \Omega \rangle \tilde{P}_{b_x(m)}\| \leq \|A_n\| (1+n)^\nu G(m-n),$$

where we have applied (A.1). This is a contradiction as the decay function G satisfies $G(m-n) \rightarrow 0$ as $m \rightarrow \infty$. \square

APPENDIX B. PROOF OF BOUNDED DIFFERENTIABILITY OF THE SPECTRAL FLOW

In this section, we demonstrate that the derivative of the spectral flow is a well-defined bounded operator when restricted to appropriate collections of quantifiably local observables $\mathcal{A}^{(\Lambda_n),g}$. A related result also appears in [42, Appendix].

To state this result, we consider models of quantum spin systems defined over a ν -regular, discrete metric space (Γ, d) . More specifically, we investigate differentiable curves of interactions $[0, 1] \ni s \mapsto \Phi(s)$, i.e. for each finite $X \subset \Gamma$ and $0 \leq s \leq 1$, $\Phi(X, s)^* = \Phi(X, s) \in \mathcal{A}_X$ and $\Phi(X, \cdot) : [0, 1] \rightarrow \mathcal{A}_X$ is differentiable. We assume that these curves have a finite norm of the following form:

$$(B.1) \quad \|\Phi\|_F^1 = \sup_{s \in [0, 1]} \sup_{x, y \in \Gamma} \frac{1}{F(d(x, y))} \sum_{\substack{X \in \mathcal{P}_0(\Gamma): \\ x, y \in X}} (\|\Phi(X, s)\| + |X| \|\Phi'(X, s)\|)$$

where F is an F -function on (Γ, d) of stretched exponential decay, i.e. as in (2.11). In this case, it was shown in [47, Section VI.E.2] that for any differentiable curves of interactions Φ with finite norm $\|\Phi\|_F^1$, there is an F -function \tilde{F} and an s -dependent interaction $\Psi(s)$ with

$$\|\Psi\|_{\tilde{F}} := \sup_{s \in [0, 1]} \|\Psi(s)\|_{\tilde{F}} < \infty,$$

for which $\Psi(s)$ is a generator of the infinite volume spectral flow automorphisms α_s . We further know by [47, Theorem 3.9], that this infinite volume spectral flow is differentiable on $\mathcal{A}_\Gamma^{\text{loc}}$ with

$$(B.2) \quad \frac{d}{ds} \alpha_s(A) = i \sum_{X \in \mathcal{P}_0(\Gamma)} \alpha_s([\Psi(X, s), A]) \quad \text{for all } A \in \mathcal{A}_\Gamma^{\text{loc}}.$$

Our aim is to show that

$$\alpha'_s := \frac{d}{ds} \alpha_s$$

is a bounded map when defined on a suitable Banach algebra of g -local observables, in the sense discussed in Section 2.1.

Theorem B.1 (Bounded Differentiability). *Let $g : [0, \infty) \rightarrow [0, \infty)$ be non-increasing and satisfy $\lim_{n \rightarrow \infty} g(n) = 0$. Let (Λ_n) be any increasing and absorbing sequence of finite subsets of Γ satisfying*

$$(B.3) \quad \sum_{n \geq 1} |\Lambda_{n+1}| g(n) < \infty.$$

Then, the derivative α'_s of the spectral flow α_s is a well-defined bounded linear map $\mathcal{A}^{(\Lambda_n),g} \rightarrow \mathcal{A}$, satisfying

$$(B.4) \quad \|\alpha'_s(A)\| \leq 2\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 \left(|\Lambda_1| + 2 \sum_{n=1}^{\infty} |\Lambda_{n+1}| g(n) \right) \|A\|_{(\Lambda_n),g},$$

with $\|\tilde{F}\|_1$ as defined in (2.9).

Proof. We start by noting that for any local observable $A \in \mathcal{A}_Y$ with $Y \subset \Gamma$ finite, we have the following estimate:

$$\begin{aligned}
 \left\| \frac{d}{ds} \alpha_s(A) \right\| &\leq \sum_{\substack{X: \\ X \cap Y \neq \emptyset}} \|[\Psi(X, s), \alpha_s(A)]\| \leq 2\|A\| \sum_{y \in Y} \sum_{z \in \Gamma} \sum_{\substack{X: \\ y, z \in X}} \|\Psi(X, s)\| \\
 &\leq 2\|\Psi\|_{\tilde{F}} \|A\| \sum_{y \in Y} \sum_{z \in \Gamma} \tilde{F}(d(y, z)) \\
 (B.5) \quad &\leq 2\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 |Y| \|A\|
 \end{aligned}$$

Let $A \in \mathcal{A}^{(\Lambda_n), g}$. By (2.6),

$$(B.6) \quad \|A - \Pi_n(A)\| \leq \|A\|_{(\Lambda_n), g} g(n)$$

which implies that the sequence $\{\Pi_n(A)\}_{n=1}^\infty$ converges to A (in norm) at a rate governed by g . Define a map $\alpha'_s : \mathcal{A}^{(\Lambda_n), g} \rightarrow \mathcal{A}$ by setting

$$(B.7) \quad \alpha'_s(A) = \lim_{n \rightarrow \infty} \frac{d}{ds} \alpha_s(\Pi_n(A)) \quad \text{for all } A \in \mathcal{A}^{(\Lambda_n), g}.$$

Note that for any strictly local observable A , $\Pi_n(A) = A$ if n is sufficiently large, and thus this definition agrees with the standard definition of the derivative of $\alpha_s(A)$ for $A \in \mathcal{A}^{\text{loc}}$.

Now, consider integers $1 \leq M < N < \infty$. For any observable A ,

$$(B.8) \quad \Pi_N(A) - \Pi_M(A) = \sum_{n=M}^{N-1} (\Pi_{n+1}(A) - \Pi_n(A))$$

and for $A \in \mathcal{A}^{(\Lambda_n), g}$, we also have that for any $n \geq 1$,

$$\begin{aligned}
 \|\Pi_{n+1}(A) - \Pi_n(A)\| &\leq \|\Pi_{n+1}(A) - A\| + \|A - \Pi_n(A)\| \\
 (B.9) \quad &\leq 2\|A\|_{(\Lambda_n), g} g(n)
 \end{aligned}$$

where we have used (B.6) and the fact that g is non-increasing. We conclude that

$$\begin{aligned}
 \|\alpha'_s(\Pi_N(A)) - \alpha'_s(\Pi_M(A))\| &\leq \sum_{n=M}^{N-1} \|\alpha'_s(\Pi_{n+1}(A) - \Pi_n(A))\| \\
 &\leq 2\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 \sum_{n=M}^{N-1} |\Lambda_{n+1}| \|\Pi_{n+1}(A) - \Pi_n(A)\| \\
 (B.10) \quad &\leq 4\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 \|A\|_{(\Lambda_n), g} \sum_{n=M}^{N-1} |\Lambda_{n+1}| g(n).
 \end{aligned}$$

Here, for the second inequality above we used (B.5), and the final bound comes from (B.9). We conclude that whenever g satisfies (B.3), the sequence $\{\alpha'_s(\Pi_n(A))\}_{n=1}^\infty$ is norm Cauchy; hence, norm convergent. This shows that α'_s as given in (B.7) is well-defined and equals the derivative of $\alpha_s(A)$, for $A \in \mathcal{A}^{(\Lambda_n), g}$. Moreover, for any $N \geq 1$,

$$\begin{aligned}
 \|\alpha'_s(\Pi_N(A))\| &\leq \|\alpha'_s(\Pi_1(A))\| + \|\alpha'_s(\Pi_N(A)) - \alpha'_s(\Pi_1(A))\| \\
 (B.11) \quad &\leq 2\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 |\Lambda_1| \|\Pi_1(A)\| + 4\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 \|A\|_{(\Lambda_n), g} \sum_{n=1}^{N-1} |\Lambda_{n+1}| g(n)
 \end{aligned}$$

and therefore,

$$(B.12) \quad \|\alpha'_s(A)\| = \lim_{N \rightarrow \infty} \|\alpha'_s(\Pi_N(A))\| \leq 2\|\Psi\|_{\tilde{F}} \|\tilde{F}\|_1 \left(|\Lambda_1| + 2 \sum_{n=1}^{\infty} |\Lambda_{n+1}| g(n) \right) \|A\|_{(\Lambda_n), g}$$

as claimed. \square

As an application of this result we can establish a relation between spectral gap stability, automorphic equivalence, and differentiability of the ground state. In particular, we find that Assumption 1.2 (vii) in [43], i.e. uniform differentiability of the ground state, follows from automorphic equivalence. Therefore, for perturbation that leave the gap stable, the differentiability assumption is always satisfied under the other items in [43, Assumption 1.2] (see also [9, 51]). Given [43, Theorem 1.3], one can also regard Assumption 1.2 (vii) as *equivalent* to automorphic equivalence. To clarify the relation between differentiability of the ground state and automorphic equivalence, we first define a notion of stable perturbations.

As before in this appendix, consider differentiable curves of interactions Φ . Let ω_0 be a ground state of the derivation defined by $\Phi(\cdot, 0)$ as in (2.18)-(2.19), and let H_0 denote the GNS Hamiltonian of ω_0 . In this discussion we assume that all interactions have stretched exponential decay.

Definition B.2. We say that Φ is stable with respect to ω_0 if there exists an IAS (Λ_n) and a sequence of differentiable interactions Φ_n converging locally in F -norm to Φ (see [47, Definition 3.7]), such that sequence of perturbed Hamiltonians of the form

$$(B.13) \quad H(\Lambda_n, s) = H_0 + \pi_0(V_{\Lambda_n}(s)), \quad \text{with } V_{\Lambda_n}(s) = \sum_{X \subset \Lambda_n} \Phi_n(X, s)$$

has a positive spectral gap above the ground state uniformly in n and $s \in [0, s_0]$, for some $s_0 > 0$. In this situation we refer to $[0, s_0]$ as the stability range.

The main result of this work establishes conditions under which an interaction $\Phi(\cdot, s)$ is stable for sufficiently small values of s , meaning there exists $s_0 > 0$ such that $\Phi(\cdot, s)$ is stable with respect to ω_0 with stability range $[0, s_0]$.

To make the connection with the results in [43], assume that $\Phi(\cdot, s)$ has a unique ground state ω_s for $s \in [0, s_0]$ and that Φ is stable with respect to ω_0 with stability range $[0, s_0]$. By automorphic equivalence ([7, Theorem 5.5], applied to the GNS representation), the spectral flows $\alpha_s^{(\Lambda_n)}$ associated with the perturbations $V_{\Lambda_n}(s)$ satisfy the following properties:

(i) they converge strongly to a co-cycle of automorphisms in the thermodynamic limit:

$$\alpha_s = \lim_{n \rightarrow \infty} \alpha_s^{(\Lambda_n)}, s \in [0, s_0];$$

(ii) $\omega_s = \lim_{n \rightarrow \infty} \omega_0 \circ \alpha_s^{(\Lambda_n)} = \omega_s \circ \alpha_s, s \in [0, s_0]$;

(iii) Theorem B.1 applies to $\alpha_s, s \in [0, s_0]$.

Therefore, differentiability of the ground state follows from the differentiability of α_s provided by Theorem B.1. As a result, [43, Assumption 1.2 (vii)] holds in the stability range.

We note that in the same way one obtains the extension of the invariants constructed by Ogata in [49, 51, 52] from models with finite-range interactions to the setting with interactions of stretched exponential decay.

Using methods as in [47], a similar result is seen to hold for more general quantum lattice models. In fact, equipped with these methods, see specifically [47, Section VI], minor modifications of the above proof demonstrate boundedness of the derivative of the spectral flow for models with unbounded on-sites perturbed by smooth bounded interactions.

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On behalf of all authors, the corresponding author states that there is no conflict of interest.

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