ON THE GENERALIZED HAMMING WEIGHTS OF HYPERBOLIC CODES

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*Dedicated to Joachim Rosenthal on the occasion of his sixtieth birthday.
We thank Prof. Rosenthal for his selfless, continuous, and endless
support to shape the coding theory and cryptography community.*

Abstract. A hyperbolic code is an evaluation code that improves a Reed-Muller code because the dimension increases while the minimum distance is not penalized. We give necessary and sufficient conditions, based on the basic parameters of the Reed-Muller code, to determine whether a Reed-Muller code coincides with a hyperbolic code. Given a hyperbolic code $\mathcal{C}$, we find the largest Reed-Muller code contained in $\mathcal{C}$ and the smallest Reed-Muller code containing $\mathcal{C}$. We then prove that similar to Reed-Muller and affine Cartesian codes, the r-th generalized Hamming weight and the r-th footprint of the hyperbolic code coincide. Unlike for Reed-Muller and affine Cartesian codes, determining the r-th footprint of a hyperbolic code is still an open problem. We give upper and lower bounds for the r-th footprint of a hyperbolic code that, sometimes, are sharp.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime. An $[n, k, \delta]$ *linear code* $\mathcal{C}$ over $\mathbb{F}_q$ is a subspace $\mathcal{C} \subseteq \mathbb{F}_q^n$ with $\mathbb{F}_q$-dimension k and minimum distance $\delta := \min\{d_H(\mathbf{c}, \mathbf{c}') : \mathbf{c}, \mathbf{c}' \in \mathcal{C}, \mathbf{c} \neq \mathbf{c}'\}$, where $d_H(\cdot, \cdot)$ denotes the *Hamming distance*.

The Generalized Hamming weights (GHWs) for linear codes, a natural generalization of the minimum distance, were introduced by Wei in 1992 [18]. Wei showed in the same work [18] that the GHWs completely characterize the performance of a linear code when used on the wire-tap channel of type II. The GHWs are also related to resilient functions and trellis, or branch complexity of linear codes [17]. The precise definition is the following. For a nonnegative integer s, we set $[s] := \{1, 2, ..., s\}$. The support of a subspace $\mathcal{D} \subseteq \mathbb{F}_q^n$ is defined by $\chi(\mathcal{D}) := \{i \in [n] \ : \ \text{there is } (x_1, \ldots, x_n) \text{ in } \mathcal{D} \text{ with } x_i \neq 0\}$. For an integer $1 \leq r \leq k$, the r-th *generalized Hamming weight* of $\mathcal{C}$ is given by

$$\delta_r(\mathcal{C}) := \min\{ \ |\chi(\mathcal{D})| \ : \ \mathcal{D} \subseteq \mathcal{C}, \dim(\mathcal{D}) = r\}.$$

Note that $\delta_1(\mathcal{C})$ is the minimum distance of $\mathcal{C}$.

This work will focus on evaluation codes whose evaluation points are the points in $\mathcal{P} := \mathbb{F}_q^m$. Throughout this paper, $\mathbb{N}$ will represent the set of non-negative integers. For $A \subseteq \mathbb{N}^m$, let $\mathbb{F}_q[A]$

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be the subspace of polynomials in $\mathbb{F}_q[\mathbf{X}] := \mathbb{F}_q[X_1, \ldots, X_m]$ with $\mathbb{F}_q$-basis $\left\{ \mathbf{X}^{\mathbf{i}} := X_1^{i_1} \cdots X_m^{i_m} \ : \ \mathbf{i} = (i_1, \ldots, i_m) \in A \right\}$. Write $\mathcal{P} = \{P_1, \ldots, P_n\}$, where $n := |\mathcal{P}| = q^m$. Define the following evaluation map

$$\begin{array}{rccc} \mathrm{ev}_{\mathcal{P}}: & \mathbb{F}_q[X_1, \ldots, X_m] & \longrightarrow & \mathbb{F}_q^n \\ & f & \longmapsto & (f(P_1), \ldots, f(P_n)). \end{array}$$

The *evaluation or monomial code* associated with A is denoted and defined by

$$\mathcal{C}_A := \mathrm{ev}_{\mathcal{P}}(\mathbb{F}_q[A]) = \{\mathrm{ev}_{\mathcal{P}}(f) \ : \ f \in \mathbb{F}_q[A]\}.$$

For $a, b \in \mathbb{R}$ and $a \leq b$, we denote by $[\![a, b]\!]$ the integer interval $[a, b] \cap \mathbb{Z}$. Recall $A \subseteq \mathbb{N}^m$. As $\alpha^q = \alpha$ for every $\alpha \in \mathbb{F}_q$, one can find a unique set $B \subseteq [\![0, q-1]\!]^m$ such that $\mathcal{C}_A = \mathcal{C}_B$. In what follows, if a set $A \subseteq \mathbb{N}^m$ defines the code $\mathcal{C}_A$, we are assuming that $A \subseteq [\![0, q-1]\!]^m$.

Observe that the length and dimension of the evaluation code $\mathcal{C}_A$ are q^m and $|A|$, respectively. The minimum distance of $\mathcal{C}_A$ has been studied in terms of the footprint that we now define. The *footprint* of the evaluation code $\mathcal{C}_A$ is the integer

$$\mathrm{FB}(\mathcal{C}_A) := \min \left\{ \prod_{j=1}^{m} (q - i_j) \ : \ (i_1, \ldots, i_m) \in A \right\}.$$

The footprint matters because the minimum distance $\delta_1(\mathcal{C}_A)$ of $\mathcal{C}_A$ is lower bounded by the footprint bound [10]: $\mathrm{FB}(\mathcal{C}_A) \leq \delta_1(\mathcal{C}_A)$. The footprint bound has been extensively studied in the literature. See, for example, [1, 4, 8, 15] and the references therein.

The families of Reed-Muller and hyperbolic codes that we describe below are particular cases of evaluation codes. Let $s \geq 0, m \geq 1$ be integers and take

$$R := \left\{ \mathbf{i} = (i_1, \ldots, i_m) \in [\![0, q-1]\!]^m \ : \ \sum_{j=1}^{m} i_j \leq s \right\}.$$

The evaluation code $\mathcal{C}_R$, denoted by $\mathrm{RM}_q(s, m)$, is called *Reed-Muller code* over $\mathbb{F}_q$ of order s with m variables.

The hyperbolic code is defined as follows. Let $d, m \geq 1$ be integers and take

$$H := \left\{ \mathbf{i} = (i_1, \ldots, i_m) \in [\![0, q-1]\!]^m \ : \ \prod_{j=1}^{m} (q - i_j) \geq d \right\}.$$

The evaluation code $\mathcal{C}_H$, denoted by $\mathrm{Hyp}_q(d, m)$, is called the *hyperbolic code* over $\mathbb{F}_q$ of order d with m variables.

A hyperbolic code is an evaluation code designed to improve the dimension of a Reed-Muller code while the minimum distance is not penalized. The hyperbolic codes were first introduced in [16] as hyperbolic cascade Reed-Solomon codes, and they were later generalized by Feng and Rao as improved Goppa codes in [6]. Høholdt and Pellikaan in [14] mentioned another generalization from order functions and estimated the minimum distance using the order bound (also known as the Feng-Rao bound). In [13], Geil and Høholdt used footprints to estimate the minimum distance of several evaluation codes, and later in [11], the same authors used the results of footprints to prove that the designed distance of hyperbolic codes (coming from the order bound) it is, in fact, the actual minimum distance. The strategies in this paper are similar to those from [11], and so it is our primary reference.

Remark 1.1. We denote the hyperbolic code over $\mathbb{F}_q$ of order d with m variables by $\mathrm{Hyp}_q(d,m)$. In [11], the authors use the notation $\mathrm{Hyp}_q(s,m)$, where $s = q^m - d$. Since we are interested in the minimum distance and the generalized Hamming weights, we prefer to emphasize the dependence with respect to the minimum distance rather than the cardinality of the corresponding footprint $s = q^m - d$. In [11], the authors also defined $\mathrm{Hyp}_q(d,m)$ as the dual of the evaluation code coming from $H^\perp = \{\mathbf{i} = (i_1,\dots,i_m) \in [\![0,q-1]\!]^m \,:\, \prod_{j=1}^m (i_j+1) < d\}$. From [11], we have that both definitions coincide.

Note that the hyperbolic code $\mathrm{Hyp}_q(d,m)$ has been designed to be the code with the largest possible dimension among those monomial codes $\mathcal{C}_A$ such that $\mathrm{FB}(\mathcal{C}_A) \geq d$ by [11]. There are instances where the hyperbolic codes improve the Reed-Muller codes, meaning that the dimension has increased [14]. But sometimes, the hyperbolic and Reed-Muller codes coincide. In this paper, we give necessary and sufficient conditions to determine whether a Reed-Muller code is hyperbolic; those conditions are provided in terms of the basic parameters of the Reed-Muller code. Given a hyperbolic code, we find the largest (respectively smallest) Reed-Muller code contained in (respectively that contains) the hyperbolic.

The GHWs have been studied for many well-known families of codes. Heijnen and Pellikaan introduced in [12], in a general setting, the order bound on GHWs of codes on varieties to compute the GHWs of Reed-Muller codes. Beelen and Datta used a similar approach of the order bound in [2] to calculate the GHWs of affine Cartesian codes. Jaramillo et al. introduced in [15] the r-th footprint to bound the GHWs for any evaluation code. This paper proves that the r-th generalized Hamming weight and the r-th footprint of a hyperbolic code coincide.

The outline of this paper is as follows. In Section 2, we determine when a Reed-Muller code is hyperbolic. Thus, we indicate when the hyperbolic code of order d has a greater dimension concerning a Reed-Muller code with the same minimum distance. Given a hyperbolic code $\mathrm{Hyp}_q(d,m)$, we find in Section 3 the smallest Reed-Muller code $\mathrm{RM}_q(s',m)$ that contains $\mathrm{Hyp}_q(d,m)$. In Section 4, we find the largest Reed-Muller code $\mathrm{RM}_q(s,m)$ contained in $\mathrm{Hyp}_q(d,m)$. In other words, in Section 3 and 4 we find the largest s and the smallest s' such that

$$\mathrm{RM}_q(s,m) \subseteq \mathrm{Hyp}_q(d,m) \subseteq \mathrm{RM}_q(s',m).$$

In Section 5, we prove that similar to Reed-Muller and affine Cartesian codes, the r-th generalized Hamming weight and the r-th footprint of the hyperbolic code coincide. Unlike for Reed-Muller and affine Cartesian codes, determining the r-th footprint of a hyperbolic code is still an open problem. We use the results from Sections 2, 3, and 4 to provide upper and lower bounds for the r-th footprint of a hyperbolic code that, sometimes, are sharp.

2. When hyperbolic and Reed-Muller codes coincide

We determine in this section when a Reed-Muller code is a hyperbolic code. In other words, for a Reed-Muller code $\mathrm{RM}_q(s,m)$, we give necessary and sufficient conditions over q, s, m and its minimum distance to determine if $\mathrm{RM}_q(s,m)$ is a hyperbolic code.

Remark 2.1. From now on, we assume that $s < m(q-1)$. Note that when $s = m(q-1)$, the corresponding hyperbolic code is a Reed-Muller code because

$$\mathrm{Hyp}_q(1,m) = \mathbb{F}_q^n = \mathrm{RM}_q(m(q-1),m).$$

Proposition 2.2. *Assume $s = mt + r$, where $t, r \in \mathbb{N}$ and $0 \le r \le m-1$. Then*

$$\max\left\{ \prod_{j=1}^{m} (q - i_j) \;:\; \sum_{j=1}^{m} i_j = s,\; 0 \le i_j \le q-1 \right\} = (q-t-1)^r (q-t)^{m-r}.$$

Proof. Consider $\mathbf{i} = (i_1, \ldots, i_m)$ such that $\prod_{j=1}^{m}(q - i_j)$ reaches the maximum value. If all the i_j's are equal, then $i_j = \frac{s}{m}$ and we have the result ($r = 0$). If they are not equal, we can assume by symmetry that $i_1 > i_2$, and then we would have that

$$(q - i_1 + 1)(q - i_2 - 1) \prod_{j=3}^{m} (q - i_j) - \prod_{j=1}^{m} (q - i_j) > 0$$

if and only if $i_1 - i_2 - 1 > 0$. Since we have chosen $\mathbf{i}$ to be maximum, then $i_1 - i_2 - 1 = 0$ and therefore $i_1 = i_2 + 1$. This means that $i_1 = \ldots = i_r = t + 1$ and $i_{r+1} = \ldots = i_m = t$ for some r and t (and then $s = mt + r$) and thus the conclusion follows. □

We come to one of the main results of this section.

Theorem 2.3. *Let $m \ge 1$ and $0 \le s < m(q-1)$. The Reed-Muller code $\mathrm{RM}_q(s,m)$ with minimum distance δ is a hyperbolic code if and only if*

$$(q-t-1)^r (q-t)^{m-r} < \delta,$$

where $s + 1 = mt + r$ and $0 \le r < m$. Even more, in this case we have $\mathrm{RM}_q(s,m) = \mathrm{Hyp}_q(\delta, m)$.

Proof. Define the sets

$$R = \left\{ \mathbf{i} \in [\![0, q-1]\!]^m \;:\; \sum_{j=1}^{m} i_j \le s \right\} \text{ and } H = \left\{ \mathbf{i} \in [\![0, q-1]\!]^m \;:\; \prod_{j=1}^{m} (q - i_j) \ge \delta \right\}.$$

By [3, Theorem 3.9 (iii)], we know that the minimum distance of the Reed-Muller code $\mathrm{RM}q(s,m)$ is δ. Therefore, for every vector $\mathbf{i} = (i_1, \ldots, i_m)$ such that $\sum_{j=1}^{m} i_j \le s$, we have that $\prod_{j=1}^{m}(q - i_j) \ge \delta$. This implies that $R \subseteq H$. Thus, by definition of hyperbolic code, the Reed-Muller code $\mathrm{RM}_q(s,m)$ is a hyperbolic code if and only $H \subseteq R$. Define $\mathbf{i} = (i_1, \ldots, i_m)$ such that $\sum_{j=1}^{m} i_j \ge s + 1 = mt + r$. By Proposition 2.2, $\prod_{j=1}^{m}(q - i_j) \le (q-t-1)^r (q-t)^{m-r}$. We conclude $R = H$ if and only if $(q-t-1)^r (q-t)^{m-r} < \delta$. In this case, we see that $\mathcal{C}_H$ is the hyperbolic code $\mathrm{Hyp}_q(\delta, m)$. □

Theorem 2.3 was previously proved in [7, Proposition 2] for the case when $m = 2$. Even when $m = 2$, we can observe that, in most nontrivial cases, the hyperbolic code outperforms the corresponding Reed-Muller code with the same minimum distance.

Example 2.4. Take $q = 9$. By Theorem 2.3, we have the following inequality for the dimensions of the Reed-Muller and the hyperbolic codes

$$\dim(\mathrm{RM}_9(s,2)) < \dim(\mathrm{Hyp}_9(\delta, 2))$$

for all $s \in [\![5, 13]\!]$. The dimensions are equal for $s \in [\![0, 4]\!] \cup [\![14, 16]\!]$.

Corollary 2.5. *In the binary case, the Reed-Muller code $\mathrm{RM}_2(s,m)$ of order $s < m$ coincides with the hyperbolic code $\mathrm{Hyp}_2(2^{m-s}, m)$ of order equal to the minimum distance of $\mathrm{RM}_2(s,m)$.*

Proof. If $s < m-1$, take $r = s+1$. By Theorem 2.3, since

$$2^{m-r} < 2^{m-s} = \delta(\mathrm{RM}_2(s,m)),$$

we have that $\mathrm{RM}_2(s,m) = \mathrm{Hyp}_2(2^{m-s}, m)$.

If $s = m-1$, then $\delta(\mathrm{RM}_2(s,m)) = 2$. Note that the element $(1,\dots,1)$ is not in the set $H = \left\{(i_1,\dots,i_m) \in [\![0, 2-1]\!]^m \ : \ \prod_{j=1}^m (2-i_j) \geq 2\right\}$. because $\prod_{j=1}^m (2-1) = 1$. So, the evaluation of the monomial $X_1 \cdots X_m$ is not in the hyperbolic code $\mathrm{Hyp}_2(2^{m-s}, m)$, meaning $\dim \mathrm{Hyp}_2(2^{m-s}, m) \leq 2^m - 1 = \dim \mathrm{RM}_2(s,m)$. By the fact that any Reed-Muller code of minimum distance d is contained in the Hyperbolic code of designed distance d, the result follows. □

3. The smallest Reed-Muller code

Given the hyperbolic code of order d with m variables $\mathrm{Hyp}_q(d,m)$, we will now find the smallest degree s such that $\mathrm{Hyp}_q(d,m) \subseteq \mathrm{RM}_q(s,m)$. We will use the following notation. The symbol $\lfloor a \rfloor$ denotes the integer part of the real number a, which is the nearest and smaller integer of a, and $\{a\}$ is the fractional part of a, defined by the formula $\{a\} = a - \lfloor a \rfloor$.

Remark 3.1. By Remark 2.1, in the following results we just consider $d > 1$.

We start with $m = 2$, the case of two variables.

Proposition 3.2. *Given $d \in \mathbb{N}$, with $d > 1$, define $a := q - \sqrt{d}$ and $s := \lfloor 2a \rfloor$. Then $\mathrm{Hyp}_q(d,2) \subseteq \mathrm{RM}_q(s,2)$. Moreover, s is the smallest integer with this property, that is*

$$\mathrm{Hyp}_q(d,2) \not\subseteq \mathrm{RM}_q(s-1,2).$$

Proof. Let $H, R_1, R_2 \subseteq [\![0, q-1]\!]^m$ be the sets defining the codes $\mathrm{Hyp}_q(d,2)$, $\mathrm{RM}_q(s,2)$ and $\mathrm{RM}_q(s-1,2)$, respectively. We show first that $\mathrm{Hyp}_q(d,2) \subseteq \mathrm{RM}_q(s,2)$. We will use the following fact:

$$\min\{a_1 + a_2 \ : \ a_1, a_2 \in \mathbb{R}_{\geq 0}, a_1 a_2 = d\} = 2\sqrt{d}. \tag{3.1}$$

For every $\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2$ such that $\mathbf{i} \in H$, we have that $(q-i_1)(q-i_2) \geq d$. By Eq. (3.1), $(q-i_1) + (q-i_2) \geq 2\sqrt{d}$, i.e. $i_1 + i_2 \leq 2a$. Moreover, since $i_1, i_2 \in \mathbb{N}$, then $i_1 + i_2 \leq \lfloor 2a \rfloor = s$. Thus $\mathbf{i} \in R_1$, which proves the first statement.

We show now that $\mathrm{Hyp}_q(d,2) \not\subseteq \mathrm{RM}(s-1,2)$. We separate it into two cases.

- Case 1. $0 \leq \{a\} < \frac{1}{2}$. Take $\mathbf{a} = (\lfloor a \rfloor, \lfloor a \rfloor) \in \mathbb{N}^2$. As $(q - \lfloor a \rfloor)^2 \geq (q-a)^2 = d$, $\mathbf{a}$ belongs to H. Observe that $\lfloor a \rfloor + \lfloor a \rfloor = 2\lfloor a \rfloor = \lfloor 2a \rfloor = s > s-1$, thus $\mathbf{a} \notin R_2$.
- Case 2. $\frac{1}{2} \leq \{a\} < 1$. Take $\mathbf{a} = (\lfloor a \rfloor, \lfloor a \rfloor + 1) \in \mathbb{N}^2$. Since $\{a\} \geq \frac{1}{2}$, then $q - a = q - \lfloor a \rfloor - \{a\} \leq q - \lfloor a \rfloor - \frac{1}{2}$. Thus, the equation $\left(q - \lfloor a \rfloor - \frac{1}{2}\right)^2 = (q - \lfloor a \rfloor)(q - \lfloor a \rfloor - 1) + \frac{1}{4}$, implies that $(q - \lfloor a \rfloor)(q - \lfloor a \rfloor - 1) = \left\lfloor \left(q - \lfloor a \rfloor - \frac{1}{2}\right)^2 \right\rfloor \geq \lfloor (q-a)^2 \rfloor = \lfloor d \rfloor = d$. This means that $\mathbf{a}$ belongs to H. As $\lfloor a \rfloor + \lfloor a \rfloor + 1 = 2\lfloor a \rfloor + 1 = \lfloor 2a \rfloor = s > s-1$, we have that $\mathbf{a} \notin R_2$.

Hence, the proof is complete. □

The trivial generalization to m variables of Proposition 3.2 is not valid. As the following example shows, it is not true that in general the code $\mathrm{RM}_q(s,m)$ is the smallest Reed-Muller code that contains the hyperbolic code $\mathrm{Hyp}_q(d,m)$, where $a = q - \sqrt[m]{d}$ and $s = \lfloor ma \rfloor$.

Example 3.3. Take $q = 27$, $m = 3$ and $d = 37$. Then $a = q - \sqrt[3]{d} = 27 - \sqrt[3]{37} \approx 23.667$, and $s = \lfloor 3a \rfloor = 71$. It is computationally easy to check that if i_1, i_2 and i_3 are integers such that $(q - i_1)(q - i_2)(q - i_3) \geq 37$, then $i_1 + i_2 + i_3 \leq 70$. Thus $\mathrm{Hyp}_q(d, m) \subseteq \mathrm{RM}_q(s - 1, m)$.

We have the following result as the first generalization of Proposition 3.2.

Proposition 3.4. *Given $d \in \mathbb{N}$, with $d > 1$, define $a := q - \sqrt[m]{d}$ and $s := \lfloor ma \rfloor$. Then $\mathrm{Hyp}_q(d, m) \subseteq \mathrm{RM}_q(s, m)$. Moreover, s is the smallest integer with this property if $\{a\} < \frac{1}{m}$.*

Proof. Let $H \subseteq [\![0, q-1]\!]^m$ be the set defining the hyperbolic code $\mathrm{Hyp}_q(d, m)$. We will use the following fact:

$$\min \left\{ \sum_{j=1}^{m} a_j \ : \ a_j \in \mathbb{R}_{\geq 0} \text{ and } \prod_{j=1}^{m} a_j = d \right\} = m \sqrt[m]{d}. \tag{3.2}$$

For every $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m$ such that $\mathbf{i} \in H$, we have that $\prod_{j=1}^{m} (q - i_j) \geq d$. By Equation (3.2), we obtain $\sum_{j=1}^{m} (q - i_j) \geq m \sqrt[m]{d}$, i.e. $\sum_{j=1}^{m} i_j \leq ma$. Since $i_j \in \mathbb{N}$ for $j \in \{1, \ldots, m\}$, $\sum_{j=1}^{m} i_j \leq \lfloor ma \rfloor = s$, which proves that $\mathrm{Hyp}_q(d, m) \subseteq \mathrm{RM}_q(s, m)$.

Let $R \subseteq [\![0, q-1]\!]^m$ be the set defining the code $\mathrm{RM}_q(s-1, m)$. If $\{a\} < \frac{1}{m}$, then $\lfloor ma \rfloor = m \lfloor a \rfloor$. Consider $\mathbf{a} = (\lfloor a \rfloor, \ldots, \lfloor a \rfloor) \in \mathbb{N}^m$. It is easy to check that $\mathbf{a} \in H$, but $\mathbf{a} \notin R$. □

Remark 3.5. Given $d \in \mathbb{N}$, with $d > 1$, take $a := q - \sqrt[m]{d}$. Observe that $\mathrm{Hyp}_q(d, m) \not\subseteq \mathrm{RM}_q(m\lfloor a \rfloor - 1, m)$. Indeed, let $H, R \subseteq [\![0, q-1]\!]^m$ be the sets defining the codes $\mathrm{Hyp}_q(d, m)$ and $\mathrm{RM}_q(m\lfloor a \rfloor - 1, m)$, respectively. Consider $\mathbf{a} := (\lfloor a \rfloor, \ldots, \lfloor a \rfloor) \in \mathbb{N}^m$. It is easy to check that $\mathbf{a} \in H$, but $\mathbf{a} \notin R$.

We now come to one of the main results of this section.

Theorem 3.6. *Given $d \in \mathbb{N}$ with $d > 1$, define $a := q - \sqrt[m]{d}$. Then $\mathrm{Hyp}_q(d, m) \subseteq \mathrm{RM}_q(s, m)$, where*

$$s = m \lfloor a \rfloor + r \quad \text{and} \quad r = \left\lfloor \frac{m \log\left(\frac{q-a}{q-\lfloor a \rfloor}\right)}{\log\left(\frac{q-\lfloor a \rfloor - 1}{q - \lfloor a \rfloor}\right)} \right\rfloor.$$

Even more, s is the smallest integer with this property, that is

$$\mathrm{Hyp}_q(d, m) \not\subseteq \mathrm{RM}_q(s - 1, m).$$

Proof. Let $H, R_1, R_2 \subset [\![0, q-1]\!]^m$ be the sets defining the codes $\mathrm{Hyp}_q(d, m)$, $\mathrm{RM}_q(s, m)$, and $\mathrm{RM}_q(s-1, m)$, respectively. First note that, by the definition of r, we know that $r \in \{0, \ldots, m-1\}$ is the largest integer such that

$$(q - \lfloor a \rfloor)^{m-r} (q - \lfloor a \rfloor - 1)^r \geq (q - a)^m = d.$$

Indeed, the last inequality is satisfied if and only if $\left(\frac{q-\lfloor a \rfloor - 1}{q - \lfloor a \rfloor}\right)^r \geq \left(\frac{q-a}{q-\lfloor a \rfloor}\right)^m$, or equivalently, if $r \leq m \frac{\log\left(\frac{q-a}{q-\lfloor a \rfloor}\right)}{\log\left(\frac{q-\lfloor a \rfloor-1}{q-\lfloor a \rfloor}\right)}$ as $\log\left(\frac{q-\lfloor a \rfloor - 1}{q-\lfloor a \rfloor}\right) < 0$. Observe that $r = 0$ if a is an integer.

Thus, if we consider

$$\mathbf{a} = (\underbrace{\lfloor a \rfloor + 1, \ldots, \lfloor a \rfloor + 1}_{r}, \underbrace{\lfloor a \rfloor, \ldots, \lfloor a \rfloor}_{m-r}) \in \mathbb{N}^m,$$

then it is easy to check that $\mathbf{a} \in H$ but $\mathbf{a} \notin R_2$. Thus, $\mathrm{Hyp}_q(d, m) \not\subseteq \mathrm{RM}_q(s - 1, m)$.

Let $\overline{R_1}$ be the complement of R_1 in $[\![0, q-1]\!]^m$, *i.e.*

$$\overline{R_1} = \left\{ \mathbf{i} \in [\![0, q-1]\!]^m \ : \ \sum_{j=1}^{m} i_j \geq s+1 \right\}.$$

We will show that $H \cap \overline{R_1} = \emptyset$. First note that the point

$$\mathbf{b} = (\underbrace{\lfloor a \rfloor + 1, \ldots, \lfloor a \rfloor + 1}_{r+1}, \underbrace{\lfloor a \rfloor, \ldots, \lfloor a \rfloor}_{m-r-1}) \in \mathbb{N}^m,$$

satisfies that $\mathbf{b} \in \overline{R_1}$ but $\mathbf{b} \notin H$. A similar situation happens with any point obtained by a permutation of the entries of $\mathbf{b}$. Now let $\mathbf{i} = (i_1, \ldots, i_m) \in \overline{R_1}$ such that $\sum_{j=1}^{m} i_j = s+1$. If there exists an index l such that $i_l > \lfloor a \rfloor + 1$, there must be an index ℓ such that $i_\ell \leq \lfloor a \rfloor$ (by Dirichlet's principle). Take $\mathbf{a}_1 = \mathbf{i} - \mathbf{e}_l + \mathbf{e}_\ell$, where $\mathbf{e}_i$ denotes the i-th standard vector in $\mathbb{N}^m$. Define the function

$$f(X_1, \ldots, X_m) = \prod_{j=1}^{m} (q - X_j).$$

It is easy to check that $f(\mathbf{i}) < f(\mathbf{a}_1)$. Indeed,

$$\begin{aligned} f(\mathbf{i}) < f(\mathbf{a}_1) &\iff (q - i_l)(q - i_\ell) < (q - a_{1,l})(q - a_{1,\ell}) \\ &\iff i_\ell < i_l - 1. \end{aligned}$$

Now, if there exists again an index l_2 such that $a_{1,l_2} > \lfloor a \rfloor + 1$, then there must exists an index ℓ_2 such that $a_{1,\ell_2} \leq \lfloor a \rfloor$. Then we can repeat the process until we reach a permutation of the entries of the point $\mathbf{b}$. Thus, we get a set of points $\{\mathbf{a}_i\}_{i=1,\ldots,t}$ such that $f(\mathbf{i}) < f(\mathbf{a}_1) < \ldots < f(\mathbf{a}_t) < f(\mathbf{b}) < d$. That is $\mathbf{i} \in \overline{R_1}$, but $\mathbf{i} \notin H$. □

Example 3.7. The Reed-Muller code $\mathrm{RM}_9(s, 2)$ is a hyperbolic code for $s \leq 4$ and $s \geq 14$ by Theorem 2.3.

We close this section with an example that shows the smallest Reed-Muller code that contains a hyperbolic code.

Example 3.8. The lattice points under the red curve of Figure 1(A) define the hyperbolic code $\mathrm{Hyp}_9(27, 2)$. By Theorem 3.6, we have that $\mathrm{Hyp}_9(27, 2) \subseteq \mathrm{RM}_9(s, 2)$ when $s \geq 7$. The lattice points under the blue curve of Figure 1(A) define the Reed-Muller hyperbolic code $\mathrm{RM}_9(7, 2)$, which is the smallest Reed-Muller code that contains $\mathrm{Hyp}_9(27, 2)$.

Example 3.9. The lattice points under the red curve of Figure 1(B) define the hyperbolic code $\mathrm{Hyp}_9(9, 2)$. By Theorem 3.6, we have that $\mathrm{Hyp}_9(9, 2) \subseteq \mathrm{RM}_9(s, 2)$ when $s \geq 12$. The lattice points under the blue curve of Figure 1(B) define the Reed-Muller hyperbolic code $\mathrm{RM}_9(12, 2)$, which is the smallest Reed-Muller code that contains $\mathrm{Hyp}_9(9, 2)$.

4. The largest Reed-Muller code

Given the hyperbolic code over $\mathbb{F}_q$ of order d with m variables $\mathrm{Hyp}_q(d, m)$, we now find the largest degree s such that $\mathrm{RM}_q(s, m) \subseteq \mathrm{Hyp}_q(d, m)$. We first recall the minimum distance of a Reed-Muller code.

Proposition 4.1. *([9]) Take $s \leq (q-1)m$. Write $s = t(q-1) + r$, where $t, r \in \mathbb{N}$ and $0 \leq r < q-1$. The minimum distance of the Reed-Muller code $\mathrm{RM}_q(s, m)$ is $\delta = (q-r)q^{m-1-t}$.*

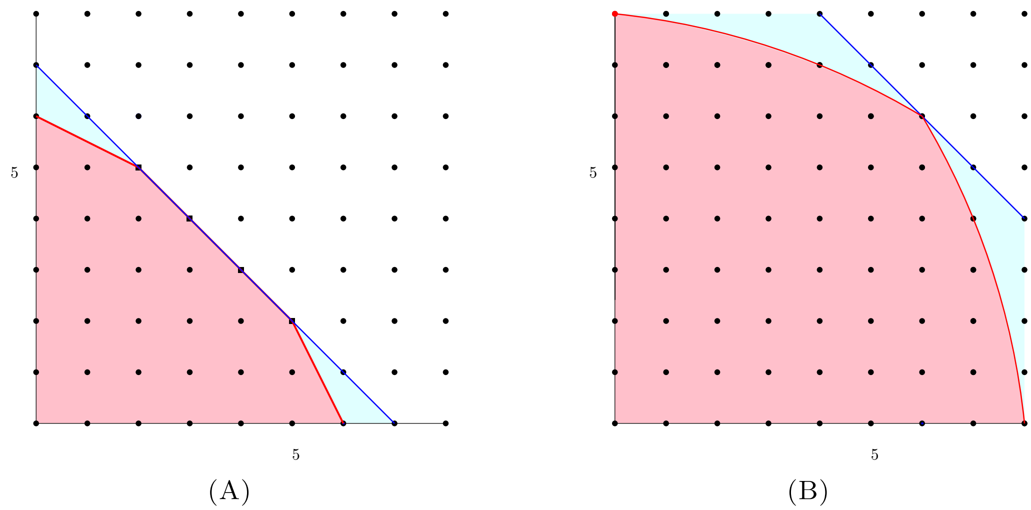

FIGURE 1. (A) The lattice points under the red curve define $\mathrm{Hyp}_9(27, 2)$. The lattice points under the blue curve define $\mathrm{RM}_9(7, 2)$, the smallest Reed-Muller code that contains $\mathrm{Hyp}_9(27, 2)$. (B) The lattice points under the red curve define $\mathrm{Hyp}_9(9, 2)$. The lattice points under the blue curve define $\mathrm{RM}_9(12, 2)$, the smallest Reed-Muller code that contains $\mathrm{Hyp}_9(9, 2)$.

Proposition 4.2. *Let $d \in \mathbb{N}$, with $d > 1$. The minimum distance $\delta(\mathrm{RM}_q(s, m)) \geq d$ if and only if*

$$s \leq (m-c)(q-1) + q - \left\lceil \frac{d}{q^{c-1}} \right\rceil, \text{ where } c := \lceil \log_q(d) \rceil .$$

Proof. Write $s = t(q-1) + r$, where $t, r \in \mathbb{N}$ and $0 \leq r < q-1$. By Proposition 4.1,

$$\delta(\mathrm{RM}_q(s, m)) = (q-r)q^{m-1-t}.$$

($\Rightarrow$) Assume that $\delta(\mathrm{RM}_q(s, m)) \geq d$. Then we have that $q^{m-t} \geq (q-r)q^{m-1-t} \geq d$. Because of the properties of the logarithm, $m - t \geq \lceil \log_q(d) \rceil$ and $t \leq m - \lceil \log_q(d) \rceil = m - c$. If $t < m - c$ we have that $s < (t+1)(q-1) \leq (m-c)(q-1)$ and the result follows. Moreover, for $t = m - c$, then $q - r \geq \lceil d/q^{m-1-t} \rceil = \lceil d/q^{c-1} \rceil$, and hence, $r \leq q - \left\lceil \frac{d}{q^{c-1}} \right\rceil$. Putting all together, we have that $s \leq (m-c)(q-1) + q - \left\lceil \frac{d}{q^{c-1}} \right\rceil$.

($\Leftarrow$) Conversely let $u = (m-c)(q-1) + q - \left\lceil \frac{d}{q^{c-1}} \right\rceil$ and let $s \leq u$. Then,

$$\mathrm{RM}_q(s, m) \subseteq \mathrm{RM}_q(u, m).$$

As $c = \lceil \log_q(d) \rceil$, observe that $q - \left\lceil \frac{d}{q^{c-1}} \right\rceil < q - 1$, hence,

$$\delta(\mathrm{RM}_q(s, m)) \geq \delta(\mathrm{RM}_q(u, m)) = \left(q - \left(q - \left\lceil \frac{d}{q^{c-1}} \right\rceil\right)\right) q^{m-1-(m-c)} = \left\lceil \frac{d}{q^{c-1}} \right\rceil q^{c-1} \geq d.$$

This completes the proof. $\square$

We come to one of the main results of this section, which helps to find the largest Reed-Muller code inside of a hyperbolic code.

Theorem 4.3. *Let* $d \in \mathbb{N}$*, with* $d > 1$*. Then* $\mathrm{RM}_q(s, m) \subseteq \mathrm{Hyp}_q(d, m)$ *if and only if*

$$s \leq (m - c)(q - 1) + q - \left\lceil \frac{d}{q^{c-1}} \right\rceil, \text{ where } c := \left\lceil \log_q(d) \right\rceil .$$

Proof. This is a direct consequence of Proposition 4.2. □

Example 4.4. The lattice points under the red curve of Figure 2(A) define the hyperbolic code $\mathrm{Hyp}_9(27, 2)$. By Theorem 4.3, we have that $\mathrm{RM}_9(s, 2) \subseteq \mathrm{Hyp}_9(27, 2)$ when $s \leq 6$. The lattice points under the black curve of Figure 2(A) define the Reed-Muller hyperbolic code $\mathrm{RM}_9(6, 2)$, which is the largest Reed-Muller code in $\mathrm{Hyp}_9(27, 2)$.

Example 4.5. The lattice points under the red curve of Figure 2(B) define the hyperbolic code $\mathrm{Hyp}_9(9, 2)$. By Theorem 4.3, we have that $\mathrm{RM}_9(s, 2) \subseteq \mathrm{Hyp}_9(9, 2)$ when $s \leq 8$. The lattice points under the black curve of Figure 2(B) define the Reed-Muller hyperbolic code $\mathrm{RM}_9(8, 2)$, which is the largest Reed-Muller code in $\mathrm{Hyp}_9(9, 2)$.

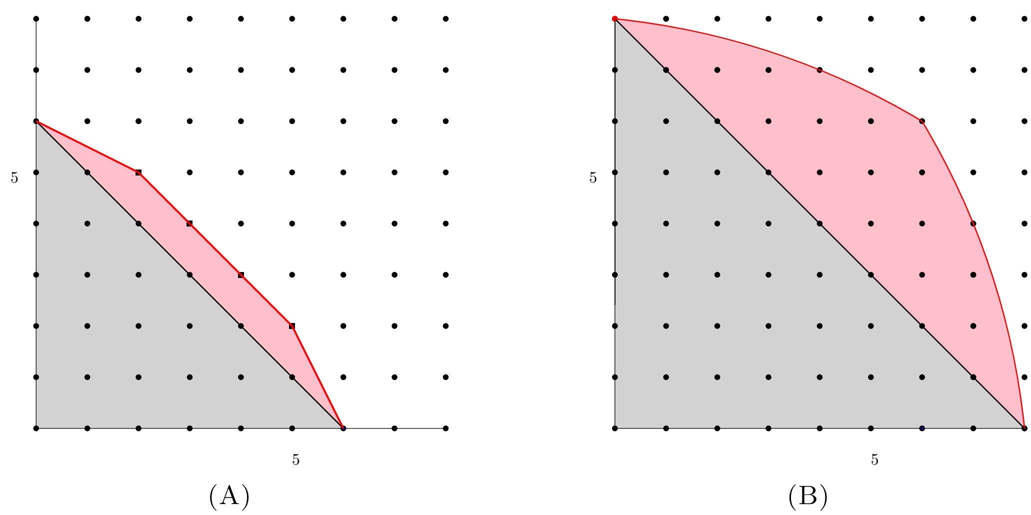

FIGURE 2. (A) The lattice points under the red curve define $\mathrm{Hyp}_9(27, 2)$. The lattice points under the black curve define $\mathrm{RM}_9(6, 2)$, the largest Reed-Muller code in $\mathrm{Hyp}_9(27, 2)$. (B) The lattice points under the red curve define $\mathrm{Hyp}_9(9, 2)$. The lattice points under the black curve define $\mathrm{RM}_9(8, 2)$, the largest Reed-Muller code in $\mathrm{Hyp}_9(9, 2)$.

5. Generalized Hamming weights

This section proves that, similar to Reed-Muller and affine Cartesian codes, the r-th generalized Hamming weight and the r-th footprint of the hyperbolic code coincide. Unlike for Reed-Muller [12] and affine Cartesian [2] codes, determining the r-th footprint of a hyperbolic code is still an open problem. We give upper and lower bounds for the r-th footprint of a hyperbolic code $\mathrm{Hyp}_q(d, m)$ in terms of the largest Reed-Muller code $\mathrm{RM}_q(s, m)$ contained in $\mathrm{Hyp}_q(d, m)$ and the smallest Reed-Muller code $\mathrm{RM}_q(s', m)$ that contains $\mathrm{Hyp}_q(d, m)$. These bounds sometimes are sharp.

Recall that the monomial code associated with $A \subseteq [\![0, q-1]\!]^m$ is given by

$$\mathcal{C}_A = \mathrm{ev}_{\mathcal{P}}(\mathbb{F}_q[A]) = \{\mathrm{ev}_{\mathcal{P}}(f) \ : \ f \in \mathbb{F}_q[A]\} .$$

For an integer $1 \leq r \leq |A|$, the r-th generalized Hamming weight of $\mathcal{C}_A$ is given by

$$\delta_r(\mathcal{C}_A) = \min\{ \ |\chi(\mathcal{D})| \ : \ \mathcal{D} \subseteq \mathcal{C}_A, \ \dim(\mathcal{D}) = r\},$$

where $\chi(\mathcal{D}) := \{i \in [n] \ : \text{there is } \mathbf{x} \in \mathcal{D} \text{ with } x_i \neq 0\}$. We now explain how to bound the r-th generalized Hamming weight in terms of the footprint. For $\mathbf{i} = (i_1, \ldots, i_m) \in [\![0, q-1]\!]^m$, we define the set

$$\nabla(\mathbf{i}) := [\![i_1, q-1]\!] \times \cdots \times [\![i_m, q-1]\!].$$

Note that $|\nabla(\mathbf{i})| = \prod_{j=1}^m (q - i_j)$. We can rewrite the footprint bound of the code $\mathcal{C}_A$ as

$$\mathrm{FB}(\mathcal{C}_A) = \min\{|\nabla(\mathbf{i})| \ : \ \mathbf{i} \in A\}.$$

The minimum distance $\delta(\mathcal{C}_A)$ of $\mathcal{C}_A$ is lower bounded by the footprint bound [10]: $\mathrm{FB}(\mathcal{C}_A) \leq \delta(\mathcal{C}_A)$. Jaramillo et al. generalized in [15] the footprint bound to the *r-th footprint*:

$$\mathrm{FB}_r(\mathcal{C}_A) := \min\left\{ \left| \bigcup_{j=1}^{r} \nabla(\mathbf{i}_j) \right| \ : \ \mathbf{i}_j \in A, \ \mathbf{i}_\ell \neq \mathbf{i}_j \text{ for } \ell, j \in [\![1, r]\!] \right\}.$$

Similar to the minimum distance, the r-th generalized Hamming weight is lower bounded by the r-th footprint [15, Theorem 3.9]:

$$\mathrm{FB}_r(\mathcal{C}_A) \leq \delta_r(\mathcal{C}_A). \tag{5.1}$$

The r-th footprint is sharp for Reed-Muller and affine Cartesian codes by [12] and [2], respectively. We now extend the result by proving that the r-th footprint is sharp for hyperbolic codes. Recall that the hyperbolic code $\mathrm{Hyp}_q(d, m)$ depends on the set

$$H = \{\mathbf{i} \in [\![0, q-1]\!]^m \ : \ |\nabla(\mathbf{i})| \geq d\}.$$

We come to one of the main results of this section.

Theorem 5.1. *Let $1 \leq r \leq \dim(\mathrm{Hyp}_q(d, m))$. Then, the r-th generalized Hamming weight of a hyperbolic code $\mathrm{Hyp}_q(d, m)$ is given by the r-th footprint:*

$$\delta_r(\mathrm{Hyp}_q(d, m)) = \mathrm{FB}_r(\mathrm{Hyp}_q(d, m)) := \min\left\{ \left| \bigcup_{j=1}^{r} \nabla(\mathbf{i}_j) \right| \ : \ \mathbf{i}_j \in H, \ \mathbf{i}_\ell \neq \mathbf{i}_j \textit{ for } \ell, j \in [\![1, r]\!] \right\}.$$

Proof. By Equation (5.1), $\mathrm{FB}_r(\mathrm{Hyp}_q(d, m)) \leq \delta_r(\mathrm{Hyp}_q(d, m))$.

To prove the inequality $\delta_r(\mathrm{Hyp}_q(d, m)) \leq \mathrm{FB}_r(\mathrm{Hyp}_q(d, m))$, we construct r elements in $\mathrm{Hyp}_q(d, m)$ that generate a subspace in $\mathrm{Hyp}_q(d, m)$ of dimension r and support length precisely $\mathrm{FB}_r(\mathrm{Hyp}_q(d, m))$. Let γ be a primitive element of $\mathbb{F}_q$. For a nonnegative integer ℓ, we define the polynomial $f(\ell, x)$ in $\mathbb{F}_q[x]$ of degree ℓ as

$$f(\ell, x) := \begin{cases} 1 & \text{if } \ell = 0 \\ x & \text{if } \ell = 1 \\ (x)(x-\gamma)\cdots(x-\gamma^{\ell-1}) & \text{if } \ell > 1. \end{cases}$$

Let $\mathbf{i}_1, \ldots, \mathbf{i}_r$ be elements in H such that $\mathrm{FB}_r(\mathrm{Hyp}_q(d, m)) = \left| \bigcup_{j=1}^r \nabla(\mathbf{i}_j) \right|$. For every $1 \leq j \leq r$, assume $\mathbf{i}_j = (i_{j1}, \ldots, i_{jm})$, and define the polynomial

$$f_j := f(i_{j1}, x_1) \cdots f(i_{jm}, x_m).$$

Denote by $Z(f_j)$ the set of zeros of f_j in $\mathbb{F}_q^m$. Note that

$$\mathbb{F}_q^m \setminus Z(f_j) = \left\{ (\gamma^{a_1}, \ldots, \gamma^{a_m}) \in \mathbb{F}_q^m \ : \ (a_1, \ldots, a_m) \in \nabla(\mathbf{i}_j) \right\},$$

which implies that $\mathrm{ev}_{\mathbb{F}_q^m}(f_j) \in \mathrm{Hyp}_q(d,m)$, since $\mathbf{i}_j \in H$. Let $Z(f_1,\ldots,f_r)$ be the set of common zeros of $f_1,\ldots,f_r$ in $\mathbb{F}_q^m$. As $Z(f_1,\ldots,f_r) = \bigcap_{j=1}^r Z(f_j)$, then $\mathbb{F}_q^m \setminus Z(f_1,\ldots,f_r) = \bigcup_{j=1}^r \left(\mathbb{F}_q^m \setminus Z(f_j)\right)$. Thus, if $\mathcal{D}_r := \mathrm{Span}_{\mathbb{F}_q}\left\{\mathrm{ev}_{\mathbb{F}_q^m}(f_1),\ldots,\mathrm{ev}_{\mathbb{F}_q^m}(f_r)\right\} \subseteq \mathrm{Hyp}_q(d,m)$, then

$$|\chi(\mathcal{D}_r)| = |\mathbb{F}_q^m \setminus Z(f_1,\ldots,f_r)| = \left|\bigcup_{j=1}^r \nabla(\mathbf{i}_j)\right| = \mathrm{FB}_r(\mathrm{Hyp}_q(d,m)).$$

We conclude that $\delta_r(\mathrm{Hyp}_q(d,m)) \leq \mathrm{FB}_r(\mathrm{Hyp}_q(d,m))$. □

We now use Theorem 5.1 to bound the GHWs of hyperbolic codes in terms of the r-th footprint.

Corollary 5.2. *Let $\mathbf{i}_1,\ldots,\mathbf{i}_r \in H$ be the first r elements of H in descending lexicographical order. Then*

$$\delta_r(\mathrm{Hyp}_q(d,m)) \leq \left|\bigcup_{j=1}^r \nabla(\mathbf{i}_j)\right|.$$

Proof. This is a direct consequence of Theorem 5.1. □

Heijnen and Pellikaan proved in [12, Theorem 5.10] that the bound of Corollary 5.2 is sharp for a Reed-Muller code. Even more, Heijnen and Pellikaan explicitly described the r-th generalized Hamming weight in terms of the r-th element in $[\![0,q-1]\!]^m$ in the lexicographic order. Note that Theorem 5.1 gives an expression to compute the GHWs of a hyperbolic code in terms of finding the minimum on a set. Naturally, when the hyperbolic code coincides with a Reed-Muller code, we obtain a closed formula for the GHWs of some hyperbolic codes.

Theorem 5.3. *Take $m \geq 1$. Let d be such that $(q-t-1)^r(q-t)^{m-r} < d$, where $s+1 = mt+r$, $0 \leq s < m(q-1)$, and $0 \leq r < m$. The r-th generalized Hamming weight of the hyperbolic code $\mathrm{Hyp}_q(d,m)$ is given by:*

$$\delta_r(\mathrm{Hyp}_q(d,m)) = \sum_{i=1}^m a_{m-i+1}q^{i-1} + 1,$$

where $\mathbf{a} = (a_1,\ldots,a_m)$ is the r-th element in $[\![0,q-1]\!]^m$ in the lexicographic order with the property that $\deg(\mathbf{a}) > (q-1)m - s - 1$.

Proof. By Theorem 2.3, the hyperbolic code $\mathrm{Hyp}_q(d,m)$ coincides with the Reed-Muller code $\mathrm{RM}_q(s,m)$. By [12, Theorem 5.10], the r-th generalized Hamming weight is given by $\delta_r(\mathrm{RM}_q(s,m)) = \sum_{i=1}^m a_{m-i+1}q^{i-1} + 1$. □

We also have the following bounds for the GHWs of an arbitrary hyperbolic code $\mathrm{Hyp}_q(d,m)$ in terms of the GHWs of some Reed-Muller codes.

Corollary 5.4. *Let $\mathrm{Hyp}_q(d,m)$ be a hyperbolic code. Define $s' := m\lfloor a\rfloor + r$, where $a = q - \sqrt[m]{d}$ and $r = \left\lfloor \frac{m\log\left(\frac{q-a}{q-\lfloor a\rfloor}\right)}{\log\left(\frac{q-\lfloor a\rfloor-1}{q-\lfloor a\rfloor}\right)} \right\rfloor$. Let s be the maximum integer such that $s \leq (m-c)(q-1)+q-\left\lceil \frac{d}{q^{c-1}}\right\rceil$, where $c := \left\lceil \log_q(d)\right\rceil$. Then*

$$\delta_r(\mathrm{RM}_q(s',m)) \leq \delta_r(\mathrm{Hyp}_q(d,m)) \leq \delta_r(\mathrm{RM}_q(s,m)),$$

where the first inequality is valid for any $1 \leq r \leq \dim(\mathrm{RM}_q(s,m))$ and the second inequality is true for any $1 \leq r \leq \dim(\mathrm{Hyp}_q(d,m))$.

Proof. The result follows from Theorems 3.6 and 4.3, where we prove $\mathrm{RM}_q(s,m) \subseteq \mathrm{Hyp}_q(d,m) \subseteq \mathrm{RM}_q(s',m)$. □

Example 5.5. From Examples 3.8 and 4.4, we have that $\mathrm{RM}_9(6,2) \subseteq \mathrm{Hyp}_9(27,2) \subseteq \mathrm{RM}_9(7,2)$. Thus,

$$26 = \delta_2(\mathrm{RM}_9(7,2)) \leq \delta_2(\mathrm{Hyp}_9(27,2)) \leq \delta_2(\mathrm{RM}_9(6,2)) = 35.$$

Using computational software and Theorem 5.1, we can see that the actual value is $\delta_2(\mathrm{Hyp}_9(27,2)) = \mathrm{FB}_2(\mathrm{Hyp}_9(27,2)) = 32$ (see Figure 3).

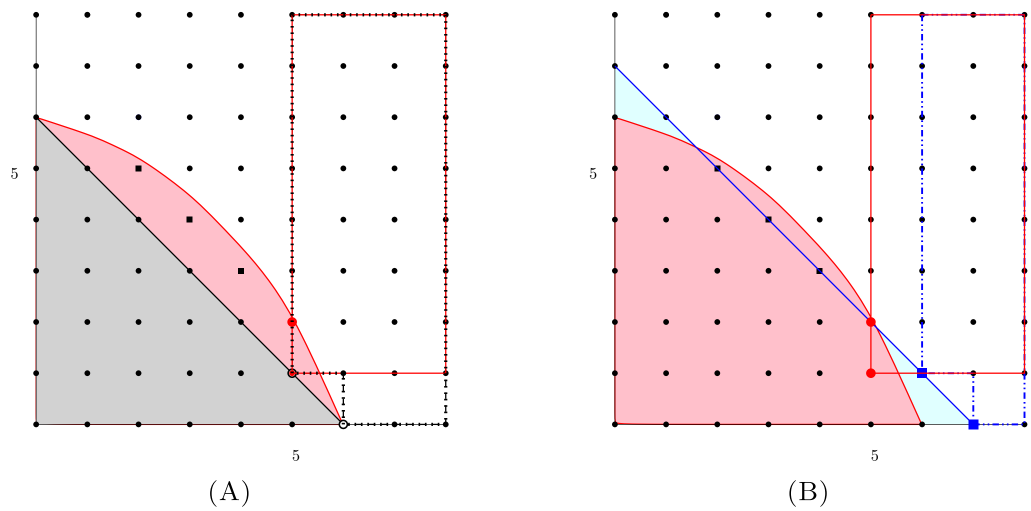

FIGURE 3. We observe that $\mathrm{RM}_9(6,2) \subseteq \mathrm{Hyp}_9(27,2) \subseteq \mathrm{RM}_9(7,2)$. The boxes represent the lattice points that help to compute the second GHWs. The number of lattice points inside: the red box is equal to $\delta_2(\mathrm{Hyp}_9(27,2))$, the black box is equal to $\delta_2(\mathrm{RM}_9(6,2))$, and the blue box is equal to $\delta_2(\mathrm{RM}_9(7,2))$.

Example 5.6. From Examples 3.9 and 4.5, we have that $\mathrm{RM}_9(8,2) \subseteq \mathrm{Hyp}_9(9,2) \subseteq \mathrm{RM}_9(12,2)$. Thus,

$$9 = \delta_2(\mathrm{RM}_9(12,2)) \leq \delta_2(\mathrm{Hyp}_9(9,2)) \leq \delta_2(\mathrm{RM}_9(8,2)) = 17.$$

Using computational software and Theorem 5.1, we can see that the actual value is $\delta_2(\mathrm{Hyp}_9(9,2)) = \mathrm{FB}_2(\mathrm{Hyp}_9(9,2)) = 12$ (see Figure 4).

The following example shows that the bounds of Corollary 5.2 may be sharp for some of the GHWs of a hyperbolic code.

Example 5.7. Let $q = 9$ and $H = \{(i_1,i_2) \in [\![0,8]\!] \mid (9-i_1)(9-i_2) \geq 27\}$. We found with the computational software Octave [5] that the element in H that minimizes the set $\{|\nabla(\mathbf{i})| : \mathbf{i} \in H\}$ coincides with the first element of H in descending lexicographical order. This first element is $(6,0)$; see Figure 5. As $|\nabla(6,0)| = 27$, we obtain that the first generalized Hamming weight, which is the minimum distance, is given by $\delta_1(\mathrm{Hyp}_9(27,2)) = 27$.

The first two elements in descending lexicographical order in H are $(6,0)$ and $(5,2)$. See Figure 6. We obtain $|\nabla(6,0) \cup \nabla(5,2)| = 34$. However, $\delta_2(\mathrm{Hyp}_9(27,2)) = 32$ by Example 5.5, which means that the first two elements do not give the second weight in descending lexicographical order.

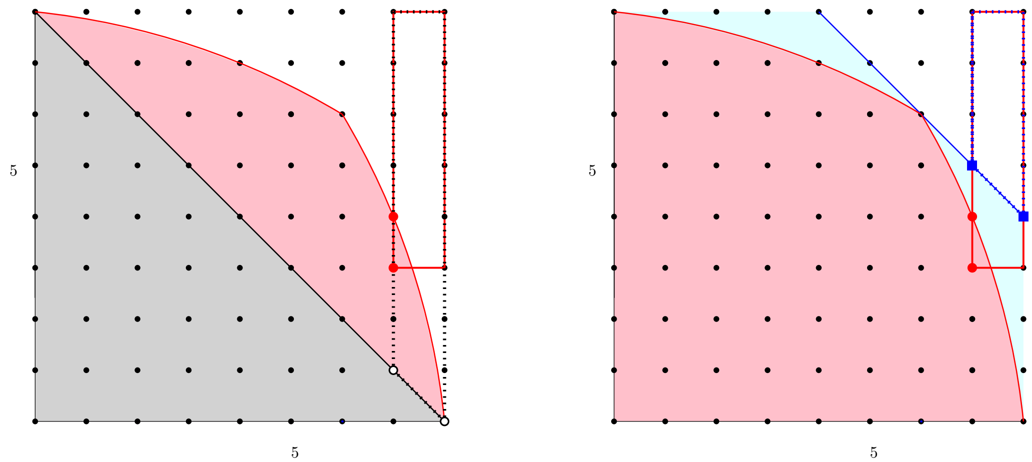

FIGURE 4. We observe that $\mathrm{RM}_9(8,2) \subseteq \mathrm{Hyp}_9(9,2) \subseteq \mathrm{RM}_9(12,2)$. The boxes represent the lattice points that help to compute the second GHWs. The number of lattice points inside: the red box is equal to $\delta_2(\mathrm{Hyp}_9(9,2))$, the black box is equal to $\delta_2(\mathrm{RM}_9(8,2))$, and the blue box is equal to $\delta_2(\mathrm{RM}_9(12,2))$.

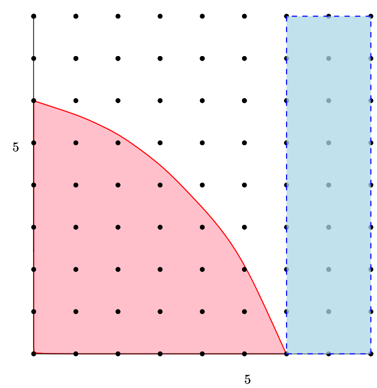

FIGURE 5. The number of lattice points inside of the blue box equals $\delta_1(\mathrm{Hyp}_9(27,2))$.

The first four elements in descending lexicographical order in H are $(6,0),(5,2),(5,1)$, and $(5,0)$. See Figure 7. The first three and the first four elements, respectively, give the third and fourth GHWs:

$$\delta_3(\mathrm{Hyp}_9(27,2)) = |\nabla(6,0) \cup \nabla(5,2) \cup \nabla(5,1)| = 35$$

and,

$$\delta_4(\mathrm{Hyp}_9(27,2)) = |\nabla(6,0) \cup \nabla(5,2) \cup \nabla(5,1) \cup \nabla(5,0)| = 36.$$

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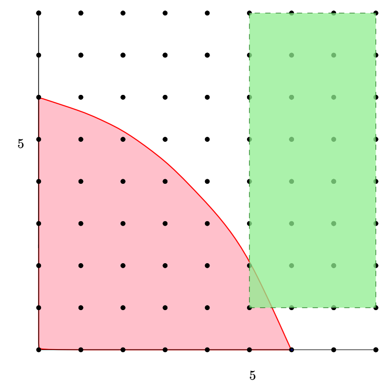

FIGURE 6. The number of lattice points inside of the green box equals $\delta_2(\mathrm{Hyp}_9(27,2))$.

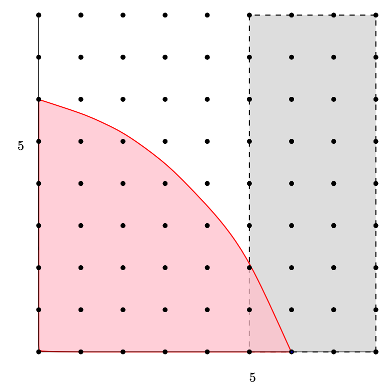

FIGURE 7. The number of lattice points inside of the grey box equals $\delta_4(\mathrm{Hyp}_9(27,2)) = |\nabla(6,0) \cup \nabla(5,2) \cup \nabla(5,1) \cup \nabla(5,0)| = 36$.

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