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THE TROPICAL CRITICAL POINTS OF AN AFFINE MATROID*

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Abstract. We prove that the number of tropical critical points of an affine matroid (M,e) is equal to the beta invariant of M. Motivated by the computation of maximum likelihood degrees, this number is defined to be the degree of the intersection of the Bergman fan of (M,e) and the inverted Bergman fan of $N=(M/e)^{\perp}$, where e is an element of M that is neither a loop nor a coloop. Equivalently, for a generic weight vector w on $E-e$, this is the number of ways to find weights $(0,x)$ on M and y on N with $x+y=w$ such that, on each circuit of M (resp., N), the minimum x-weight (resp., y-weight) occurs at least twice. This answers a question of Sturmfels.

Key words. Bergman fan, matroids, beta invariant, tautological classes

MSC codes. 05B35, 51M20, 14C15

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1. Introduction.

During the Workshop on Nonlinear Algebra and Combinatorics from Physics at the Center for the Mathematical Sciences and Applications at Harvard University in April 2022, Bernd Sturmfels [Stu22] posed one of those combinatorial problems that is deceivingly simple to state, but whose answer requires a deeper understanding of the objects at hand.

CONJECTURE 1.1 ([Stu22]). *Let M be a matroid on E, and let $e\in E$ be an element that is neither a loop nor a coloop. Let M/e be the contraction of M by e, and let $N=(M/e)^{\perp}$ be its dual matroid.*

1. (Combinatorial version) *Given a vector $\mathsf{w}\in\mathbb{R}^{E-e}$, we wish to find weight vectors $(0,\mathsf{x})\in\mathbb{R}^{E}$ on M (where e has weight 0) and $\mathsf{y}\in\mathbb{R}^{E-e}$ on N such that,*
   - *on each circuit of M, the minimum x-weight occurs at least twice;*
   - *on each circuit of $N=(M/e)^{\perp}$, the minimum y-weight occurs at least twice; and*
   - $\mathsf{w}=\mathsf{x}+\mathsf{y}$.

   For generic w, the number of solutions is the beta invariant $\beta(M)$.
2. (Geometric version) *The degree of the stable intersection of the Bergman fan $\Sigma_{(M,e)}$ and the inverted Bergman fan $-\Sigma_N=-\Sigma_{(M/e)^{\perp}}$ is*

$$\deg\left(\Sigma_{(M,e)}\cdot -\Sigma_{(M/e)^{\perp}}\right)=\beta(M)\,.$$

The goal of this paper is to prove this conjecture.

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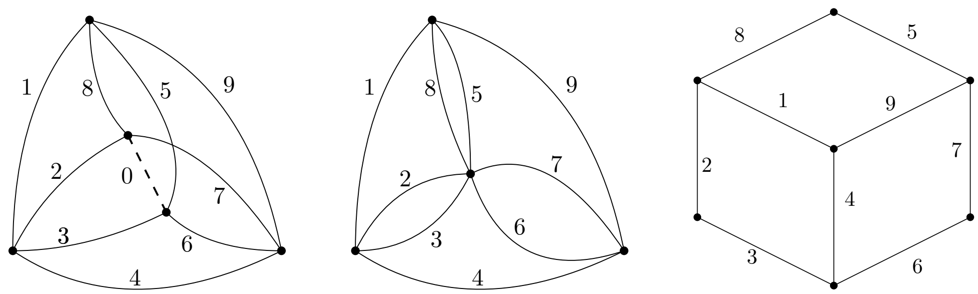

FIG. 1. *A graph G, its contraction $G/0$, and its dual $H = (G/0)^{\perp}$.*

THEOREM 1.2. *Versions* 1. *and* 2. *of Conjecture* 1.1 *are true.*

The affine matroid (M, e) is the matroid M with a special chosen element e. The Bergman fan of (M, e) is the Bergman fan of M intersected with the hyperplane $x_e = 0$. The other relevant definitions are given in section 2.3. The combinatorial and geometric formulations of Conjecture 1.1 are equivalent because, in the stable intersection above, all intersection points have multiplicity 1 [ABF$^+$23, Lemma 7.4]. In the proof of Theorem 1.2 we will follow closely the matroids in Figure 1 for sake of example.

The results of this paper were motivated by the problem of computing maximum likelihood degrees in algebraic statistics, pioneered by Catanese, Khetan, Hoşten, and Sturmfels [CHKS06]. For linear models, Varchenko showed that the maximum likelihood degree equals the beta invariant of the corresponding matroid; see [Var95, Zas75] and [CHKS06, Theorem 13].

Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, and Telen first encountered a special case of Conjecture 1.1 in [ABF$^+$23]. Using algebro-geometric results of Huh and Sturmfels [HS14], which built on earlier work of Varchenko [Var95], they proved Theorem 1.2 for matroids realizable over the real numbers [ABF$^+$23, Theorem 7.1]. In a related setting of linear Gaussian models, the maximum likelihood degrees were shown to be matroid invariants of the linear subspace [SU10, EFSS21].

We prove Theorem 1.2 for all matroids. Following the original motivation, we call the solutions to Conjecture 1.1.1 the *tropical critical points* of the affine matroid; our main result is that they are counted by Crapo's beta invariant $\beta(M)$. We do something stronger. Agostini et al. write,

> "we would like to describe the multivalued map that takes any tropical data vector w to the set of its critical points" [ABF$^+$23, section 7].

We give an explicit formula for this map for all w that are rapidly increasing under any order $<$ on the ground set E.

In section 3, we prove Theorem 1.2.1 combinatorially, relying on the tropical geometric fact that the number of solutions is the same for all generic[1] w. We show that, when the entries of w are rapidly increasing with respect to some order $<$ on E, the solutions to Conjecture 1.1.1 are naturally in bijection with the β-nbc-bases of the matroid with respect to $<$. It is known that the number of such bases is the beta invariant of the matroid, regardless of the order $<$.

In section 4, we sketch a proof of Theorem 1.2.2 that relies on the theory of *tautological classes of matroids* of Berget, Eur, Spink, and Tseng [BEST23]. This

[1] We will say that a property holds for generic $\mathsf{w} \in \mathbb{R}^n$ if it holds for all w outside of a polyhedral complex of dimension smaller than n.

proof is not combinatorial; it relies on computations in the equivariant Chow ring of the permutahedral variety initiated in [BEST23] and extended here. We do not know of a direct relationship between our two proofs. For a survey of the relationships and differences between these proof techniques in a similar setting, see [AM24].

2. Notation and preliminaries.

2.1. The lattice of set partitions. A set partition $\boldsymbol{\lambda}$ of a set E is a collection of subsets, called blocks, of E, say, $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_\ell\}$, whose union is E and whose pairwise intersections are empty. We write $\boldsymbol{\lambda} \models E$. We let $|\boldsymbol{\lambda}| = \ell$ be the number of blocks of $\boldsymbol{\lambda}$. If $e \in E$ and $\boldsymbol{\lambda} \models E$, we write $\boldsymbol{\lambda}(e)$ for the block of $\boldsymbol{\lambda}$ that contains e.

We define the *linear space of a set partition* $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_\ell\} \models E$ to be

$$\begin{aligned} \mathsf{L}(\boldsymbol{\lambda}) &:= \operatorname{span}\{\mathsf{e}_{\lambda_1}, \dots, \mathsf{e}_{\lambda_\ell}\} \subseteq \mathbb{R}^E \\ &= \{\mathsf{x} \in \mathbb{R}^E \mid x_i = x_j \text{ whenever } i, j \text{ are in the same block of } \boldsymbol{\lambda}\}, \end{aligned}$$

where $\{\mathsf{e}_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$ and $\mathsf{e}_S = \sum_{s \in S} e_s$ for $S \subseteq E$. Notice that $\dim \mathsf{L}(\boldsymbol{\lambda}) = |\boldsymbol{\lambda}|$. The map $\boldsymbol{\lambda} \mapsto \mathsf{L}(\boldsymbol{\lambda})$ is a bijection between the set partitions of E and the flats of the *braid arrangement*, which is the hyperplane arrangement in $\mathbb{R}^E$ given by the hyperplanes $x_i = x_j$ for $i \neq j$ in E.

If $e \in E$, then we write $\mathsf{L}(\boldsymbol{\lambda})|_{x_e=0} = \{\mathsf{x} \in \mathbb{R}^{E-e} : (0, \mathsf{x}) \in \mathsf{L}(\boldsymbol{\lambda}) \subseteq \mathbb{R}^E\}$.

2.2. The intersection graph of two set partitions. We denote $[a,b] := \{a, a+1, \dots, b-1, b\}$ and $[n] := [1,n]$. The following construction from [AE21] will play an important role.

DEFINITION 2.1. *Let $\boldsymbol{\lambda} \models [0,n]$ and $\boldsymbol{\mu} \models [n]$ be set partitions. The* intersection graph $\Gamma = \Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *is the bipartite graph with vertex set $\boldsymbol{\lambda} \sqcup \boldsymbol{\mu}$ and edge set $[n]$, where the edge labeled e connects the parts $\lambda(e)$ of $\boldsymbol{\lambda}$ and $\mu(e)$ of $\boldsymbol{\mu}$ containing e. The vertex $\lambda(0)$ is marked with a hollow point.*

The intersection graph may have several parallel edges connecting the same pair of vertices. Notice that the label of a vertex in Γ is just the set of labels of the edges incident to it. Therefore, we can remove the vertex labels and simply think of Γ as a bipartite multigraph on edge set $[n]$. This is illustrated in Figure 2.

LEMMA 2.2. *Let $\boldsymbol{\lambda} \models [0,n]$ and $\boldsymbol{\mu} \models [n]$ be set partitions and $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ be their intersection graph.*

1. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ has a cycle, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$ for generic $\mathsf{w} \in \mathbb{R}^n$.*
2. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point for any $\mathsf{w} \in \mathbb{R}^n$.*
3. *If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is a point for any $\mathsf{w} \in \mathbb{R}^n$.*

Proof. Let $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$. Write $x_{\lambda(i)} := x_i$ and $y_{\mu(i)} := y_i$ for $i \in [n]$. The subspace $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ can be naturally regarded as living in $\mathbb{R}^{\boldsymbol{\lambda} \sqcup \boldsymbol{\mu}}$, where it is cut out by the equalities

$$\begin{aligned} x_{\lambda(i)} + y_{\mu(i)} &= w_i \qquad \text{for } i \in [n], \\ x_{\lambda(0)} &= 0. \end{aligned}$$

This system has $n+1$ equations and $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ independent unknowns. The linear dependences among these equations are controlled by the cycles of the graph $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. More precisely, the first n linear functionals $\{x_{\lambda(i)} + y_{\mu(i)} : i \in [n]\}$ on $\mathbb{R}^{\boldsymbol{\lambda} \sqcup \boldsymbol{\mu}}$ give a realization of the graphical matroid of $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. The last equation is clearly linearly independent from the others.

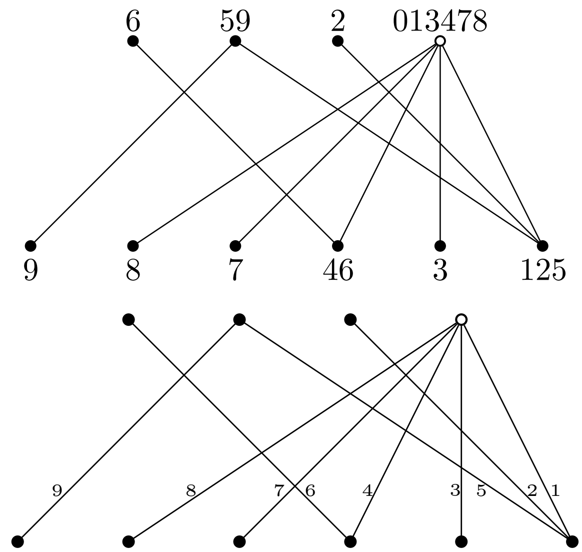

FIG. 2. *The intersection graph of* $\boldsymbol{\lambda} = \{6, 59, 2, 013478\} \models [0,9]$ *and* $\boldsymbol{\mu} = \{9, 8, 7, 46, 3, 125\} \models [9]$. *We omit brackets for legibility. Left: The vertices are labeled by the blocks of the set partitions. Right: The edges are labeled by the elements of* $[9]$.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ has a cycle with edges $i_1, i_2, \ldots, i_{2k}$ in that order, then the above equalities imply that $w_{i_1} - w_{i_2} + w_{i_3} - \cdots - w_{i_{2k}} = 0$. For generic w, this equation does not hold, so we have $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu})) = \emptyset$.

If $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is disconnected, let A be the set of edges in a connected component not containing the vertex $\lambda(0)$. If $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ satisfy $\mathsf{x} + \mathsf{y} = \mathsf{w}$ and $x_0 = 0$, then $\mathsf{x} + r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\lambda})$ and $\mathsf{y} - r\mathsf{e}_A \in \mathsf{L}(\boldsymbol{\mu})$ also satisfy those equations for any real number r. Therefore, $\mathsf{L}(\boldsymbol{\lambda})|_{x_0=0} \cap (\mathsf{w} - \mathsf{L}(\boldsymbol{\mu}))$ is not a point.

Finally, if $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, then its number of vertices is one more than the number of edges (that is, $n + 1 = |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$), so the system of equations has equally many equations and unknowns. Also, these equations are linearly independent since $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree. It follows that the system has a unique solution. ∎

When $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is a tree, we call $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ an *arboreal pair.*

LEMMA 2.3. *Let* $\boldsymbol{\lambda} \models [0, n]$ *and* $\boldsymbol{\mu} \models [n]$ *be an arboreal pair of set partitions, and let* $\Gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *be their intersection tree. Let* $\mathsf{w} \in \mathbb{R}^n$. *The unique vectors* $\mathsf{x} \in \mathsf{L}(\boldsymbol{\lambda})$ *and* $\mathsf{y} \in \mathsf{L}(\boldsymbol{\mu})$ *such that* $\mathsf{x} + \mathsf{y} = \mathsf{w}$ *and* $x_0 = 0$ *are given by*

$$x_{\lambda_i} = w_{e_1} - w_{e_2} + \cdots \pm w_{e_k}, \qquad \text{where } e_1 e_2 \ldots e_k \text{ is the unique path from } \lambda_i \text{ to } \lambda(0),$$
$$y_{\mu_j} = w_{f_1} - w_{f_2} + \cdots \pm w_{f_l}, \qquad \text{where } f_1 f_2 \ldots f_l \text{ is the unique path from } \mu_j \text{ to } \lambda(0)$$

for any i *and* j.

Proof. This follows readily from the fact that, for each $1 \leq i \leq k$, the values of $x_{\lambda(e_i)}$ and $y_{\mu(e_i)}$ on the vertices incident to edge i have to add up to w_{e_i}. ∎

Example 2.4. Let $\boldsymbol{\lambda} = \{6, 59, 2, 013478\} \models [0,9]$ and $\boldsymbol{\mu} = \{9, 8, 7, 46, 3, 125\} \models [9]$. These set partitions form an arboreal pair, as evidenced by their intersection tree, shown in Figure 2. We have, for example, $y_9 = w_9 - w_5 + w_1$ because the path from $\mu(9) = \{9\}$ to $\lambda(0) = \{013478\}$ uses edges $9, 5, 1$ in that order. The remaining values are

$$x_6 = w_6 - w_4, \quad x_{59} = w_5 - w_1, \quad x_2 = w_2 - w_1, \quad x_{013478} = 0,$$
$$y_9 = w_9 - w_5 + w_1, \quad y_8 = w_8, \quad y_7 = w_7, \quad y_{46} = w_4, \quad y_3 = w_3, \quad y_{125} = w_1.$$

The tropical critical points of a matroid are better behaved for the following family of vectors.

DEFINITION 2.5. *A vector* $\mathsf{w} \in \mathbb{R}^n$ *is* rapidly increasing *if* $w_{i+1} > 3w_i > 0$ *for* $1 \le i \le n-1$.

The next lemma is readily verified.

LEMMA 2.6. *Let* w *be rapidly increasing. For any* $1 \le a < b \le n$ *and any choice of* ϵ_i*s and* δ_i*s in* $\{-1,0,1\}$*, we have* $w_a + \sum_{i=1}^{a-1} \epsilon_i w_i < w_b + \sum_{j=1}^{b-1} \delta_j w_j$.

DEFINITION 2.7. *Given a rapidly increasing vector* $\mathsf{w} \in \mathbb{R}^n$ *and a real number* x*, we will say* x is near w_i *and write* $x \approx w_i$ *if* $w_i - (w_1 + \cdots + w_{i-1}) \le x \le w_i + (w_1 + \cdots + w_{i-1})$ *for* $i = 1, \ldots, n$*. By Lemma* 2.6*, if* $x \approx w_i$ *and* $y \approx w_j$ *for* $i < j$*, then* $x < y$.

2.3. Matroids, Bergman fans, and tropical geometry. In what follows, we will assume familiarity with basic notions in matroid theory; for definitions and proofs, see [Oxl06, Wel76]. We also state here some facts from tropical geometry that we will need; see [MS15, MR10] for a thorough introduction.

Let M be a matroid on E of rank $r+1$. The *dual matroid* $M^\perp$ is the matroid on E whose set of bases is $\{B^\perp \mid B \text{ is a basis of } M\}$, where $B^\perp := E - B$. The following lemma is useful to see how M and $M^\perp$ interact; see [ADH22, Lemma 3.14] and [Oxl06, Proposition 2.1.11] for proofs.

LEMMA 2.8. *If* F *is a flat of* M *and* G *is a flat of* $M^\perp$*, then* $|F \cup G| \neq |E| - 1$.

DEFINITION 2.9 ([Cra67]). *The* beta invariant *of* M *is defined to be* $\beta(M) := (-1)^r \frac{d\chi_M(t)}{dt}\Big|_{t=1}$*, where* $\chi_M(t)$ *is the* characteristic polynomial *of* M*:*

$$\chi_M(t) := \sum_{X \subseteq E} (-1)^{|X|} t^{r(M)-r(X)}.$$

DEFINITION 2.10. *Fix a linear order* $<$ *on* M*. A* broken circuit *is a set of the form* $C - \min_< C$ *where* C *is a circuit of* M*. An* nbc-basis *of* M *is a basis of* M *that contains no broken circuits. A* β-nbc-basis *of* M *is an nbc-basis* B *such that* $B^\perp \cup 0 \setminus 1$ *is an nbc-basis of* $M^\perp$.

THEOREM 2.11 ([Zie92]). *For any linear order* $<$ *on* E*, the number of* β*-nbc-bases of* M *is equal to the beta invariant* $\beta(M)$.

The *closure* of a set $A \subseteq E$, denoted by $\mathrm{cl}_M(A)$, is the smallest flat F containing A. For each basis $B = \{b_1 > \ldots > b_r > b_{r+1}\}$ of the matroid M, we define the complete flag of flats

$$\mathcal{F}_M(B) := \{\emptyset \subsetneq \mathrm{cl}_M\{b_1\} \subsetneq \mathrm{cl}_M\{b_1, b_2\} \subsetneq \cdots \subsetneq \mathrm{cl}_M\{b_1, \ldots, b_r\} \subsetneq E\}.$$

The following characterization of nbc-bases will be useful.

LEMMA 2.12 ([Bjö92, (7.30), (7.31)]). *Let* M *be a matroid of size* $n+1$ *and rank* $r+1$*, and let* B *a basis of* M*. Then,* B *is an nbc-basis of* M *if and only if* $b_i = \min F_i$ *for* $i = 1, \ldots, r+1$.

An *affine matroid* (M, e) on E is a matroid M on E with a chosen element $e \in E$ [Zie92]. The set E is also called the ground set of (M, e).

DEFINITION 2.13 ([Stu02]). *The* Bergman fan *of a matroid M on E is*

$$\Sigma_M = \{\mathsf{x} \in \mathbb{R}^E \mid \min_{c \in C} x_c \textit{ is attained at least twice for any circuit } C \textit{ of } M\}.$$

The Bergman fan of an affine matroid (M,e) *on E is*

$$\Sigma_{(M,e)} = \{\mathsf{x} \in \mathbb{R}^{E-e} \mid (0,\mathsf{x}) \in \Sigma_M\} = \Sigma_M|_{x_e=0}.$$

Remark 2.14. The Bergman fan contains the lineality space $\mathbb{1}\mathbb{R}$. Taking the quotient by this space or intersecting with a coordinate linear hyperplane will give the same result, and typically, the (projective) Bergman fan is defined in the quotient vector space $\mathbb{R}^E/\mathbb{1}\mathbb{R}$ in the literature.

The motivation for this definition comes from tropical geometry. A subspace $V \subset \mathbb{R}^E$ determines a matroid M_V on E, and the tropicalization of V is precisely the Bergman fan of M_V. Similarly, an affine subspace $W \subset \mathbb{R}^{E-e}$ determines an affine matroid (M_W, e) on E, where e represents the hyperplane at infinity. The tropicalization of W is the Bergman fan $\Sigma_{(M_W,e)}$.

THEOREM 2.15 ([AK06]). *The Bergman fan of a matroid M is equal to the union of the cones*

$$\begin{aligned}\sigma_{\mathcal{F}} &= \textit{cone}(\mathsf{e}_{F_1}, \dots, \mathsf{e}_{F_{r+1}}) + \mathbb{R}\mathbb{1} \\ &= \{\mathsf{x} \in \mathbb{R}^E \mid x_a \geq x_b \textit{ whenever } a \in F_i \textit{ and } b \in F_j \textit{ for some } 1 \leq i \leq j \leq r+1\}\end{aligned}$$

for the complete flags $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\}$ of flats of M. It is a tropical fan with weights $w(\mathcal{F}) = 1$ for all $\mathcal{F}$.

If Σ_1 and Σ_2 are tropical fans of complementary dimensions, then Σ_1 and $v + \Sigma_2$ intersect transversally at a finite set of points for any sufficiently generic vector $v \in \mathbb{R}^n$. Furthermore, each intersection point p is equipped with a multiplicity $w(p)$ that depends on the respective intersecting cones in such a way that the quantity

$$\deg(\Sigma_1 \cdot \Sigma_2) := \sum_{p \in \Sigma_1 \cap (v + \Sigma_2)} w(p)$$

is constant for generic v [MR10, Proposition 4.3.3, 4.3.6]; this is called the *degree* of the intersection.

In all the tropical intersections that arise in this paper, it was verified in [ABF$^+$23, Lemma 7.4] that the multiplicity index $w(p)$ is 1. This also follows readily from the fact that every such intersection comes from an arboreal pair $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ by Lemma 2.2, as explained in the next section. Therefore, the degree of the intersection will be simply the number of intersection points

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_{(M/e)^\perp}) = |\Sigma_{(M,e)} \cap (v - \Sigma_{(M/e)^\perp})|$$

for generic $v \in \mathbb{R}^{E-e}$. This explains the equivalence of the two versions of Conjecture 1.1 and Theorem 1.2.

3. Proof of the main theorem via basis activities.

Let M be a matroid on $[0,n]$ of rank $r+1$ such that 0 is not a loop nor a coloop. Then, $M/0$ has rank r, and $N = (M/0)^\perp$ has rank $n-r$. For any basis B of M containing 0, $B^\perp = [0,n] - B$ is a basis of $N = (M/0)^\perp$. Conversely, every basis of N equals $B^\perp$ for a basis B of M containing 0.

Let us construct an intersection point in $\Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$ for each β-nbc-basis of M.

LEMMA 3.1. *Let M be a matroid on $E=[0,n]$ of rank $r+1$ such that 0 is not a coloop, and let $N=(M/0)^\perp$. Let $\mathsf{w}\in\mathbb{R}^n$ be rapidly increasing. For any β-nbc-basis B of M, there exist unique vectors $(0,\mathsf{x})\in\sigma_{\mathcal{F}_M(B)}$ and $\mathsf{y}\in\sigma_{\mathcal{F}_N(B^\perp)}$ such that $\mathsf{x}+\mathsf{y}=\mathsf{w}$.*

Proof. A flag $\{\emptyset\subsetneq F_1\subsetneq\cdots\subsetneq F_k\subsetneq E\}$ of subsets of E gives rise to a set partition $\{F_1,F_2-F_1,\dots,E-F_k\}$ of E. First, we show that the set partitions $\boldsymbol{\pi}$ and $\boldsymbol{\pi}^\perp$ corresponding to the flags $\mathcal{F}=\mathcal{F}_M(B)$ and $\mathcal{F}^\perp=\mathcal{F}_N(B^\perp)$ form an arboreal pair. Since they have sizes $|B|=r+1$ and $|B^\perp|=n-r$, respectively, their intersection graph has $n+1$ vertices and n edges. Therefore, it is sufficient to prove that the intersection graph $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$ is connected; this implies that it is a tree.

Assume contrariwise, and let A be a connected component not containing the edge 1. Let $a>1$ be the smallest edge in A. Then, a is the smallest element of its part $\boldsymbol{\pi}(a)$ in $\boldsymbol{\pi}$, and, since B is nbc-basis in M, this implies $a\in B$. Similarly, since $B^\perp$ is nbc-basis in N, this also implies $a\in B^\perp$. This contradicts Lemma 2.12.

It follows from Lemma 2.2 that there exist unique $(0,\mathsf{x})\in\mathsf{L}(\boldsymbol{\pi})$ and $\mathsf{y}\in\mathsf{L}(\boldsymbol{\pi}^\perp)$ such that $\mathsf{x}+\mathsf{y}=\mathsf{w}$. It remains to show that $(0,\mathsf{x})\in\sigma_{\mathcal{F}}$ and $\mathsf{y}\in\sigma_{\mathcal{F}^\perp}$.

Lemma 2.3 provides formulas for x and y in terms of the paths from the various vertices of the tree of $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$ to $\pi(0)$. To understand those paths, let us give each edge e an orientation as follows:

$$\pi(e)\longrightarrow\pi^\perp(e) \text{ if } \min\pi(e)>\min\pi^\perp(e),$$
$$\pi(e)\longleftarrow\pi^\perp(e) \text{ if } \min\pi(e)<\min\pi^\perp(e).$$

We never have $\min\pi(e)=\min\pi^\perp(e)$ because, as above, that would imply that $e\in B\cap B^\perp$.

We claim that every vertex other than $\pi(0)$ has an outgoing edge under this orientation. Consider a part $\pi_i\neq\pi(0)$ of $\boldsymbol{\pi}$; let $\min\pi_i=b$. Edge b connects $\pi_i=\pi(b)$ to $\pi^\perp(b)\ni b$, and we cannot have $\min\pi^\perp(b)>b=\min\pi(b)$, so we must have $\pi_i\to\pi^\perp(b)$. The same argument works for any part $\pi_j^\perp$ of $\boldsymbol{\pi}^\perp$.

Now, since B is an nbc-basis of M, every element $b\in B$ is minimum in $\pi(b)$, so there is a directed path that starts at $\pi(b)$ and can only end at $\pi(0)$, and its first edge is b. Furthermore, by the definition of the orientation, the labels of the edges decrease along this path. Thus, in the alternating sum $x_b=w_b\pm\cdots$ given by Lemma 2.3, the first term dominates, and $x_b\approx w_b$ for $b\in B\setminus 0$, whereas $x_0=0$. Similarly, since $B^\perp$ is an nbc-basis of N, $y_c\approx w_c$ for all $c\in B^\perp$.

Therefore, if we write $B=\{b_1>\dots>b_r>b_{r+1}=0\}$, since w is rapidly increasing, it follows that $x_{b_1}>x_{b_2}>\cdots>x_{b_r}>x_{b_{r+1}}=0$; so, from Theorem 2.15, we have $(0,\mathsf{x})\in\sigma_{\mathcal{F}}$. Similarly, if we write $B^\perp=E-B=\{c_1>\dots>c_{n-r}=1\}$, then $y_{c_1}>y_{c_2}>\cdots>y_{c_{n-r}}$, so $\mathsf{y}\in\sigma_{\mathcal{F}^\perp}$. The desired result follows. ∎

Example 3.2. The graphical matroid M of the graph G in Figure 1 has six β-nbc-bases: 0256, 0257, 0259, 0368, 0378, 0379. Let us compute the intersection point in $\Sigma_{(M,0)}\cap(\mathsf{w}-\Sigma_N)$ associated to 0257 for the rapidly increasing vector $\mathsf{w}=(10^0,10^1,\dots,10^8)\in\mathbb{R}^9$.

For $B=0257$, we have $B^\perp=134689$. The flags they generate in M and N are

$$\mathcal{F}_M(B)=\{\emptyset\subsetneq 7\subsetneq 57\subsetneq 2457\subsetneq 0123456789\},$$
$$\mathcal{F}_N(B^\perp)=\{\emptyset\subsetneq 9\subsetneq 89\subsetneq 689\subsetneq 46789\subsetneq 346789\subsetneq 123456789\},$$

which give rise to the corresponding set partitions

$$\boldsymbol{\pi}=7|5|24|013689, \qquad \boldsymbol{\pi}^\perp=9|8|6|47|3|125.$$

This is indeed an arboreal pair, as evidenced by their intersection graph in Figure 3.

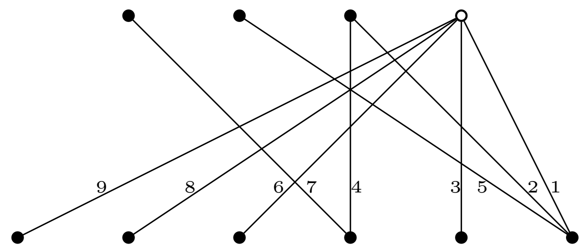

FIG. 3. *The intersection graph of* $\boldsymbol{\pi} = 7|5|24|13689$ *and* $\boldsymbol{\pi}^\perp = 9|8|6|47|3|125$.

Lemma 3.1 gives us the unique points $(0, \mathsf{x}) \in \mathcal{F}_{\boldsymbol{\pi}}$ and $\mathsf{y} \in \mathcal{F}_{\boldsymbol{\tau}}$ such that $\mathsf{x} + \mathsf{y} = \mathsf{w}$; they are given by the paths to the special vertex $\pi(0)$ in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\pi}^\perp}$. For example $x_7 = 10^6 - 10^3 + 10^1 - 10^0 = 999009$ and $y_4 = 10^3 - 10^1 + 10^0 = 991$ are given by the paths 7421 and 421 from $\pi(7) = \pi_1$ and $\pi^\perp(7) = \pi_4^\perp$ to $\pi(0)$, respectively. In this way, we obtain

$$\begin{array}{rrrrrrrrrr} \mathsf{x} = & 0 & 9 & 0 & 9 & 9999 & 0 & 999009 & 0 & 0, \\ \mathsf{y} = & 1 & 1 & 100 & 991 & 1 & 100000 & 991 & 10000000 & 100000000, \\ \mathsf{w} = & 1 & 10 & 100 & 1000 & 10000 & 100000 & 1000000 & 10000000 & 100000000, \end{array}$$

and $\mathsf{x} \in \Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$. We invite the reader to record the weights $(0, \mathsf{x})$ and y in the graphs G and H of Figure 1 and verify that, in each cycle, the minimum weight appears at least twice.

Conversely, the following lemma shows that any intersection point between $\Sigma_{(E,e)}$ and $v - \Sigma_N$ is of the form constructed in Lemma 3.1; that is, it comes from a β-nbc-basis.

LEMMA 3.3. *Let M be a matroid on $E = [0, n]$ of rank $r + 1$ such that 0 is not a loop or a coloop and $N = (M/0)^\perp$. Let $\mathsf{w} \in \mathbb{R}^n$ be generic and rapidly increasing. Let*

$$\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\},$$
$$\mathcal{G} = \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{n-r-1} \subsetneq G_{n-r} = E - 0\}$$

be complete flags of the matroids M and N, respectively, such that $\Sigma_{(M,0)}$ and $\mathsf{w} - \Sigma_N$ intersect at $\sigma_{\mathcal{F}}$ and $\mathsf{w} - \sigma_{\mathcal{G}}$. Then, there exists a β-nbc-basis B of M such that $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(B^\perp)$.

Proof. If $\sigma_{\mathcal{F}}$ and $\mathsf{w} - \sigma_{\mathcal{G}}$ intersected at more than one point, their intersection would contain a line segment, so $\Sigma_{(M,0)} \cap (\mathsf{w} - \Sigma_N)$ would be infinite. Since $\Sigma_{(M,0)}$ and $-\Sigma_N$ have complementary dimensions, this would contradict the genericity of w.

Therefore, $\sigma_{\mathcal{F}} \cap (\mathsf{w} - \sigma_{\mathcal{G}})$ is a point, and Lemma 2.2 implies that the set partitions $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ of $\mathcal{F}$ and $\mathcal{G}$ form an arboreal pair; that is, $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$ is a tree. In particular, $\pi_a \cap \tau_b = (F_a - F_{a-1}) \cap (G_b - G_{b-1})$ cannot have more than one element for any a and b. We proceed in several steps.

1. Our first step will be to show that, in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$, the top right vertex π_{r+1} contains 0 and 1, the bottom right vertex τ_{n-r} contains 1, and thus, the edge 1 connects these two rightmost vertices.

The matroid $N = (M/0)^\perp = M^\perp - 0$ can be obtained by deleting the element 0 from the matroid $M^\perp$. Each G_i is a flat of N, so $G_i^\bullet := \mathrm{cl}_{M^\perp}(G_i) \in \{G_i, G_i \cup 0\}$ is a flat of $M^\perp$. Consider the flag of flats of $M^\perp$

$$\mathcal{G}^\bullet := \{\emptyset = G_0^\bullet \subsetneq G_1^\bullet \subsetneq \cdots \subsetneq G_{n-r-1}^\bullet \subsetneq G_{n-r}^\bullet = E\},$$

where $G^\bullet_{n-r} = E$ because 0 is not a coloop of $M^\perp$ and $G^\bullet_0 = \emptyset$ because 0 is not a loop of $M^\perp$. Let m be the minimal index such that $0 \in G^\bullet_m$, so

$$\mathcal{G}^\bullet := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{m-1} \subsetneq G_m \cup 0 \subsetneq \cdots \subsetneq G_{n-r-1} \cup 0 \subsetneq G_{n-r} \cup 0 = E\}.$$

Consider the unions of the flat F_r with the coflats in $\mathcal{G}^\bullet$; let j be the index such that

$$F_r \cup G^\bullet_{j-1} \neq E, \qquad F_r \cup G^\bullet_j = E.$$

The former cannot have size $|E| - 1$ because it is the union of a flat and a coflat. Therefore,

$$(1) \qquad (F_r \cup G^\bullet_j) - (F_r \cup G^\bullet_{j-1}) = (E - F_r) \cap (G^\bullet_j - G^\bullet_{j-1}) \text{ has size at least 2.}$$

But $\mathcal{F}$ and $\mathcal{G}$ are arboreal, so

$$(2) \qquad \pi_{r+1} \cap \tau_j = (E - F_r) \cap (G_j - G_{j-1}) \text{ has size at most 1.}$$

Now, observe that $G^\bullet_j - G^\bullet_{j-1}$ and $G_j - G_{j-1}$ can only differ by $\{0\}$, so (1) and (2) imply that they must differ by $\{0\}$; furthermore, the differing element 0 must be in $E - F_r$. We conclude the following:

a) $G^\bullet_j = G_j \cup 0$ and $G^\bullet_{j-1} = G_{j-1}$; that is, $j = m$.

b) $0 \in E - F_r = \pi_{r+1}$.

Similarly, consider the union of the coflat $G^\bullet_{n-r-1}$ with the flats in $\mathcal{F}$; let i be the unique index such that

$$F_{i-1} \cup G^\bullet_{n-r-1} \neq E, \qquad F_i \cup G^\bullet_{n-r-1} = E.$$

An analogous argument shows that $(F_i - F_{i-1}) \cap (E - G^\bullet_{n-r-1})$ has size at least 2, whereas $\pi_i \cap \tau_{n-r} = (F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1})$ has size at most 1. This has three consequences:

c) $G^\bullet_{n-r-1} = G_{n-r-1}$; that is, $m = n - r$.

d) $0 \in F_i - F_{i-1}$, which, in light of b), implies that $i = r + 1$.

e) $(F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1}) = \pi_{r+1} \cap \tau_{n-r} = \{e\}$ for some element $e \in E - 0$.

But $e \in \pi_{r+1}$ means that $x_e = 0$ is minimum among all x_is for any $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$, and $e \in \tau_{n-r}$ means that y_e is minimum among all y_is for any $\mathsf{y} \in \sigma_{\mathcal{G}}$ by Theorem 2.15. Since $\mathsf{w} = \mathsf{x} + \mathsf{y}$ for some such x and y, $w_e = x_e + y_e$ is minimum among all w_is, and since w is rapidly increasing, $e = 1$.

It follows that, in the intersection tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$, the top right vertex π_{r+1} contains 0 and 1 by d) and e), the bottom right vertex τ_{n-r} contains 1 by e), and thus, 1 connects them.

2. Next, we claim that, for any path in the tree $\Gamma_{\boldsymbol{\pi},\boldsymbol{\tau}}$ directed towards and ending at edge 1, the first edge has the largest label.[2] Assume contrariwise, and consider a containment-minimal path P that does not satisfy this property; its edges must have labels satisfying $e < f > f_2 > \cdots > f_k$ sequentially. If edge e goes from $\pi(e)$ to $\tau(e)$, Lemma 2.3 gives $x_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < 0 = x_1$, contradicting that $(0, \mathsf{x}) \in \sigma_{\mathcal{F}}$. If e goes from $\tau(e)$ to $\pi(e)$, we get $y_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < w_1 = y_1$, contradicting that $\mathsf{y} \in \sigma_{\mathcal{G}}$.

[2]This implies that the edge labels decrease along any such path, but we will not use this in the proof.

3. Now, define

$$b_i := \min(F_i - F_{i-1}) \text{ for } i = 1, \dots, r+1,$$
$$c_j := \min(G_j - G_{j-1}) \text{ for } j = 1, \dots, n-r.$$

Then, $B := \{b_1, \dots, b_{r+1}\}$ and $C := \{c_1, \dots, c_{n-r}\}$ are bases of M and N, and $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(C)$. We will show that B is an β-nbc-basis and $C = B^\perp$.

To do so, we first notice that the path from vertex $\pi_i = F_i - F_{i-1}$ (resp., $\tau_j = G_j - G_{j-1}$) to edge 1 must start with edge b_i (resp., c_j). Indeed, if it started with some other (necessarily larger) edge $b' \in F_i - F_{i-1}$, then the path from edge b_i to edge 1 would include edge b' and hence would not start with the largest edge, contradicting 2. This has two consequences.

f) The sets B and C are disjoint. If we had $b_i = c_j = e$, then edge e, which connects vertices $\pi_i = F_i - F_{i-1}$ and $\tau_j = G_j - G_{j-1}$, would have to be the first edge in the paths from both of these vertices to edge 1; this is impossible in a tree. We conclude that B and C are disjoint. Since $|B| = r+1$ and $|C| = n-r$, we have $C = B^\perp$.

g) For each i, we have $x_{b_i} \approx w_{b_i}$ because the path from τ_i to vertex 0—which is the path from τ_i to edge 1, with edge 1 possibly removed—starts with the largest edge b_i, so Lemma 2.3 gives $x_{b_i} = w_{b_i} \pm$ (smaller terms) $\approx w_{b_i}$. Similarly, $y_{c_i} \approx w_{c_i}$. Now, $(0, \mathrm{x}) \in \sigma_{\mathcal{F}}$ gives $x_{b_1} > \cdots > x_{b_{r+1}}$, which implies $w_{b_1} > \cdots > w_{b_{r+1}}$, which, in turn, gives

$$b_1 > \cdots > b_r > b_{r+1} \qquad \text{and, analogously,} \qquad c_1 > \cdots > c_{n-r-1} > c_{n-r} = 1.$$

The former implies that B is nbc-basis in M by Lemma 2.12. The latter, combined with c), implies that $c_1 > \cdots > c_{n-r-1} > 0$, respectively, are the minimum elements of $G_1^\bullet, \dots, G_{n-r-1}^\bullet, G_{n-r}^\bullet = E$ that they sequentially generate, so $C \cup 0 \setminus 1 = B^\perp \cup 0 \setminus 1$ is nbc-basis in $M^\perp$. It follows that B is β-nbc-basis in M.

We conclude that B is β-nbc-basis in M, $\mathcal{F} = \mathcal{F}_M(B)$, and $\mathcal{G} = \mathcal{F}_N(B^\perp)$, as desired. ∎

Proof of Theorem 1.2.1. This follows by combining the previous two lemmas. ∎

4. Proof of the main theorem via torus-equivariant geometry. In this section, we give a proof of Theorem 1.2.2 using the framework of tautological classes of matroids of Berget, Eur, Spink, and Tseng. See [BEST23] for details on what follows. Recall that M is a matroid on E of rank $r+1$.

In this framework, one works with the Chow ring of the permutohedral fan Σ_E, which is the Bergman fan of the Boolean matroid on E whose only basis is E. Its lattice of flats is the poset of subsets of E, and its set of maximal cones is in bijection with the set $\mathfrak{S}_E$ of permutations of E. Let $S = \mathbb{Z}[t_i : i \in E]$; we can think of it as the ring of polynomials on $\mathbb{R}^E$ with integer coefficients. Then, $S^{\mathfrak{S}_E}$ is the ring of $|E|!$-tuples of polynomials in S, one polynomial f_σ for each permutation σ of E, or equivalently, one polynomial f_σ for each chamber σ of Σ_E.[3] We are interested in the $|E|!$-tuples for which the function $f : \mathbb{R}^E \to \mathbb{R}$ given by $f(x) = f_\sigma(x)$ for $x \in \sigma$ is well-defined.

The *Chow ring* $A^\bullet(\Sigma_E)$ of Σ_E has the following description.

[3]We caution that $S^{\mathfrak{S}_E}$ does *not* denote the ring of $\mathfrak{S}_E$-invariants of S, despite notational similarity.

DEFINITION 4.1. *Let $A_T^\bullet(\Sigma_E)$ be the subring of $S^{\mathfrak{S}_E}$ defined by*

$$A_T^\bullet(\Sigma_E) = \{\text{continuous piecewise polynomials with integer coefficients supported on } \Sigma_E\}$$
$$= \left\{ (f_\sigma)_{\sigma\in\mathfrak{S}_E} \in S^{\mathfrak{S}_E} \;\middle|\; \begin{array}{c}\text{for any } \sigma,\sigma'\in\mathfrak{S}_E, \text{the polynomials } f_\sigma \text{ and } f_{\sigma'}\\ \text{agree as functions on } \sigma\cap\sigma'\subseteq\mathbb{R}^E\end{array}\right\}.$$

Let I be the ideal of $A_T^\bullet(\Sigma_E)$ generated by the global linear functions. Then,

$$A^\bullet(\Sigma_E) = A_T^\bullet(\Sigma_E)/I.$$

One can associate to the fans $\Sigma_{(M,e)}$ and $-\Sigma_{(M/e)^\perp}$ certain elements $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$ of $A^\bullet(\Sigma_E)$ as follows. First, per Remark 2.14, the fan Σ_E in $\mathbb{R}^E$ contains the linear space $\mathbb{1}\mathbb{R}$, and the quotient fan $\Sigma_E/\mathbb{1}\mathbb{R}$ has a natural unimodular isomorphirm to the *affine braid fan* $\Sigma_{E,e} = \Sigma_E|_{x_e=0}$ in $\mathbb{R}^{E-e}$, whose $|E|!$ chambers correspond to the possible orders of $\{x_f : f\in E-e\}\cup\{0\}$. This is the affine Bergman fan of the Boolean matroid with special element e.

Then, the fans $\Sigma_{(M,e)}$ and $-\Sigma_{(M/e)^\perp}$ are subfans of $\Sigma_{E,e}$, and they are tropical fans in the sense that they satisfy the balancing condition (see, for instance, [AHK18, Definition 5.1]). Via the theory of Minkowski weights [FS97], they consequently define elements $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$ of the Chow ring $A^\bullet(\Sigma_{E,e}) \cong A^\bullet(\Sigma_E)$. Moreover, the ring $A^\bullet(\Sigma_E)$ is equipped with a degree map $\deg : A^\bullet(\Sigma_E)\to\mathbb{Z}$, which agrees with the map deg in Theorem 1.2 in the sense that

$$\deg(\Sigma_{(M,e)}\cap -\Sigma_{(M/e)^\perp}) = \deg_{\Sigma_E}([\Sigma_{(M,e)}]\cdot[-\Sigma_{(M/e)^\perp}]). \tag{3}$$

For a survey of these facts, see [Huh18, section 4], [AHK18, section 5], or [BEST23, section 7.1].

We now describe how [BEST23] provided a distinguished representative in $A_T^\bullet(\Sigma_E)$ of the class $[\Sigma_{(M,e)}]\in A^\bullet(\Sigma_E) = A_T^\bullet(\Sigma_E)/I$, and similarly for the class $[-\Sigma_{(M/e)^\perp}]$. For a matroid M on E, consider the following elements of the rings $A_T^\bullet(\Sigma_E)$ and $A^\bullet(\Sigma_E)$, modeled after the geometry of torus-equivariant vector bundles from realizable matroids. For each permutation $\sigma\in\mathfrak{S}_E$, let $B_\sigma(M)$ be the lexicographically first basis of M with respect to the ordering $\sigma(1)<\cdots<\sigma(n)$ of the ground set.

DEFINITION 4.2 ([BEST23, Definition 3.9]). *Let M be a matroid of rank $r+1$ on a ground set E of size $n+1$. Its* torus-equivariant tautological Chern classes *are the elements $\{c_i^T(\mathcal{S}_M^\vee)\}_{i=0,\dots,r+1}$ and $\{c_j^T(\mathcal{Q}_M)\}_{j=0,\dots,n-r}$ in $A_T^\bullet(\Sigma_E)$ defined by*

$$c_i^T(\mathcal{S}_M^\vee)_\sigma = \text{the } i\text{th elementary symmetric polynomial in } \{t_k : k\in B_\sigma(M)\} \text{ and}$$
$$c_j^T(\mathcal{Q}_M)_\sigma = \text{the } j\text{th elementary symmetric polynomial in } \{-t_\ell : \ell\in E\setminus B_\sigma(M)\}$$

for any permutation $\sigma\in\mathfrak{S}_E$. Their images in the quotient $A^\bullet(\Sigma_E)$, denoted $c_i(\mathcal{S}_M^\vee)$ and $c_j(\mathcal{Q}_M)$, are called the tautological Chern classes *of M.*

[BEST23, Proposition 3.8] shows that these elements are well-defined. The results of [BEST23] yield the following representatives in $A_T^\bullet(\Sigma_E)$ of the elements $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]\in A^\bullet(\Sigma_E)$. Let $M/e\oplus U_{0,e}$ be the matroid on E obtained from M/e by adding back the element e as a loop. This matroid has rank r.

LEMMA 4.3. *Let M be a matroid of rank $r+1$ on a ground set E of size $n+1$. Define elements $[\Sigma_{(M,e)}]^T$ and $[-\Sigma_{(M/e)^\perp}]^T$ in $A_T^\bullet(\Sigma_E)$ by $[\Sigma_{(M,e)}]^T = c_{n-r}^T(\mathcal{Q}_M)$ and $[-\Sigma_{(M/e)^\perp}]^T = c_r^T(\mathcal{S}^\vee_{M/e\oplus U_{0,e}})$, or explicitly,*

$$[\Sigma_{(M,e)}]^T_\sigma = \prod_{i\in E\setminus B_\sigma(M)}(-t_i) \quad \text{and} \quad [-\Sigma_{(M/e)^\perp}]^T_\sigma = \prod_{i\in B_\sigma(M/e\oplus U_{0,e})} t_i \quad \text{for all } \sigma\in\mathfrak{S}_E.$$

Then, their images in the quotient $A^\bullet(\Sigma_E)$ *are exactly* $[\Sigma_{(M,e)}]$ *and* $[-\Sigma_{(M/e)^\perp}]$*, respectively.*

Proof. The first equality is a restatement of [BEST23, Theorem 7.6] when one notes that the choice of $e \in E$ induces an isomorphism $\mathbb{R}^E/\mathbb{R}(1,\dots,1) \simeq \mathbb{R}^{E-e}$. The second statement also follows from that theorem when one combines it with [BEST23, Propositions 5.11, 5.13], which describe how tautological Chern classes behave with respect to matroid duality and direct sums, respectively. ∎

Proof of Theorem 1.2.2. We begin with [BEST23, Theorem 6.2], which states that

$$\deg_{\Sigma_E}\left([\Sigma_{(M,e)}] \cdot c_r(\mathcal{S}_M^\vee)\right) = \beta(M).$$

In light of (3), the desired statement $\deg(\Sigma_{(M,e)} \cap -\Sigma_{(M/e)^\perp}) = \beta(M)$ will follow once we show that $[\Sigma_{(M,e)}] \cdot \left(c_r(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]\right) = 0$ in $A^\bullet(\Sigma_E)$.

Towards this end, we look at the distinguished representative of this product in $A_T^\bullet(\Sigma_E)$ and show that the variable t_e divides $[\Sigma_{(M,e)}]_\sigma^T \cdot \left(c_r^T(\mathcal{S}_M^\vee)_\sigma - [-\Sigma_{(M/e)^\perp}]_\sigma^T\right)$ for any $\sigma \in \mathfrak{S}_E$, as follows.

- If $e \notin B_\sigma(M)$, then $[\Sigma_{(M,e)}]_\sigma^T = \prod_{i \in E \setminus B_\sigma(M)} (-t_i)$ is divisible by t_e.
- If $e \in B_\sigma(M)$, then $B_\sigma(M/e \oplus U_{0,e}) = B_\sigma(M) \setminus e$, and hence,

$$\begin{aligned} c_r^T(\mathcal{S}_M^\vee)_\sigma - [-\Sigma_{(M/e)^\perp}]_\sigma^T &= \mathrm{Elem}_r(\{t_k : k \in B_\sigma(M)) - \prod_{j \in B_\sigma(M)\setminus e} t_j \\ &= \sum_{i \in B_\sigma(M)} \left(\prod_{j \in B_\sigma(M)\setminus i} t_j\right) - \prod_{j \in B_\sigma(M)\setminus e} t_j \\ &= \sum_{i \in B_\sigma(M)\setminus e} \left(\prod_{j \in B_\sigma(M)\setminus i} t_j\right) \end{aligned}$$

  is divisible by t_e.

This means that $[\Sigma_{(M,e)}]^T \cdot \left(c_r^T(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]^T\right)$ is a multiple of the global polynomial t_e and hence is in the ideal I of Definition 4.1. Therefore, $[\Sigma_{(M,e)}] \cdot \left(c_r(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]\right) = 0$ in the quotient $A^\bullet(\Sigma_E)$, as desired. ∎

Remark 4.4. Since Theorem 1.2.2 was established for matroids realizable over $\mathbb{R}$ in [ABF+23], one may attempt to give yet another proof of Theorem 1.2.2 via the following property of matroid valuations [DF10]: If two functions $f(M)$ and $g(M)$ coincide for all matroids M that are realizable over $\mathbb{R}$, and if the functions f and g are *valuative* under matroid subdivisions [AFR10, Definition 3.10], then $f(M)$ and $g(M)$ coincide for general, not necessarily realizable, matroids M. The right-hand side of Theorem 1.2.2, the beta invariant $\beta(M)$, is valuative [AFR10]. For the left-hand side, however, while the maps $M \mapsto \Sigma_{(M,e)}$ and $M \mapsto -\Sigma_{(M/e)^\perp}$ are each valuative, products of valuative functions are in general not valuative. Thus, it is a priori unclear why the map $f : M \mapsto \deg(\Sigma_{(M,e)} \cdot -\Sigma_{(M/e)^\perp})$ is valuative. We do not know any argument that establishes the valuativity of the left-hand side of Theorem 1.2.2 independently of the theorem.

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