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Curvature in the balance: the Weyl functional and scalar curvature of 4-manifolds

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Abstract: The infimum of the Weyl functional is shown to be surprisingly small on many compact 4-manifolds that admit positive-scalar-curvature metrics. Results are also proved that systematically compare the scalar and self-dual Weyl curvatures of certain almost-Kähler 4-manifolds.

Keywords: Four-manifold, Weyl functional, scalar curvature.

1. Introduction

The curvature tensor of an oriented Riemannian 4-manifold (M^4, g) may be invariantly decomposed into exactly four independent pieces

$$\mathcal{R} = s \oplus \mathring{r} \oplus W_+ \oplus W_-,$$

where s is the scalar curvature, where $\mathring{r}$ is the trace-free Ricci curvature, and where $W_\pm$ are respectively the self-dual and anti-self-dual Weyl curvatures. This happens because the curvature tensor $\mathcal{R}$ at a point may naturally be thought of as an element of $\odot^2\Lambda^2 \cap (\Lambda^4)^\perp$, and decomposing this vector space into irreducible $\mathbf{SO}(n)$-modules splits it into exactly four factors when $n = 4$. Four dimensions is completely anomalous in this respect; by contrast, the curvature consists of just three invariant pieces when $n > 4$, of just two pieces when $n = 3$, and of only a single piece when $n = 2$.

Assuming henceforth that M^4 is compact and without boundary, we now obtain four basic quadratic curvature functionals on the space of Riemannian metrics on M by taking the L^2-norm-squared of each of our curvature pieces

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$$\begin{aligned} g &\longmapsto \int_M s^2 d\mu_g \\ g &\longmapsto \int_M |\mathring{r}|^2 \, d\mu_g \\ g &\longmapsto \int_M |W_+|^2 d\mu_g \\ g &\longmapsto \int_M |W_-|^2 d\mu_g \end{aligned}$$

and any other quadratic curvature functional is then a linear combination of these four. Each of these functionals is invariant under constant rescalings $g \rightsquigarrow cg$, for $c \in \mathbb{R}^+$, and the last two functionals are actually *conformally invariant*, in the sense that they are unaltered by arbitrary conformal rescalings $g \rightsquigarrow ug$, where $u : M \to \mathbb{R}^+$ is a smooth positive function.

On the other hand, these four functionals are not genuinely independent, because the 4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g \tag{1}$$

and Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu_g \tag{2}$$

express two important homotopy invariants of the compact oriented 4-manifold M as linear combinations of these basic curvature functionals. For metrics on a fixed oriented 4-manifold M, the two functionals

$$g \longmapsto \int_M \frac{s^2}{24} d\mu_g, \qquad g \longmapsto \int_M |W_+|^2 d\mu_g, \tag{3}$$

therefore completely determine every other quadratic curvature functional.

The main theme of this article concerns a question of balance: how do the two functionals (3) compare in size, for specific types of metrics on interesting classes 4-manifolds?

One source of motivation for this question stems from the Kähler case. Suppose that g is a Kähler metric on (M, J), and that M is given the orientation determined by the complex-structure tensor J. We then have the point-wise identity

$$|W_+|^2 = \frac{s^2}{24},$$

and our two basic functionals (3) therefore coincide on Kähler metrics.

Einstein metrics provide a particularly compelling context for this issue. Recall [3] that a Riemannian metric g is said to be Einstein if its Ricci tensor satisfies $r = \lambda g$ for some constant λ. In any dimension $n > 2$, this is equivalent to requiring the trace-free Ricci tensor $\mathring{r} = r - \frac{s}{n}g$ to vanish, and the scalar curvature s of such a metric then coincides with n times the Einstein constant λ. When $n = 4$, our balance question turns out to be highly relevant to the study of Einstein metrics, but the direction in which the balance tips critically depends on the sign of the Einstein constant. For example, when the scalar curvature is positive, the self-dual Weyl curvature almost always outweighs the scalar curvature [13, 14]:

Theorem (Gursky)**.** *Let (M^4, g) be a compact oriented Einstein 4-manifold with $s > 0$ that is not an irreducible symmetric space. Then*

$$\int_M |W_+|^2 d\mu_g \geqslant \int_M \frac{s^2}{24} d\mu_g,$$

with equality iff g is locally Kähler-Einstein.

By contrast, in the negative-scalar-curvature setting, there are large classes of 4-manifolds where the balance tips in the opposite direction [20, 21]:

Theorem (L)**.** *Let M be a smooth compact 4-manifold that admits a symplectic form, but does not admit an Einstein metric with $s > 0$. Then, with respect to the symplectic orientation, any Einstein metric g on M satisfies*

$$\int_M \frac{s^2}{24} d\mu_g \geqslant \int_M |W_+|^2 d\mu_g,$$

with equality iff g is a Kähler-Einstein metric.

(Here the assumption that the symplectic manifold M admits an Einstein metric, but does not admit an Einstein metric of positive scalar curvature, guarantees that the symplectic form ω satisfies $c_1 \cdot [\omega] \leqslant 0$ and $c_1^2 \geqslant 0$. A result of Taubes [35] thus implies that, for the spin^c structure determined by the symplectic form, the unperturbed Seiberg-Witten equations admit a solution for every metric, and the desired inequality then follows from the Weitzenböck formula for the Seiberg-Witten equations. By contrast, the assumption in Gursky's theorem that (M, g) is not an irreducible symmetric space implies, by a result of Hitchin [3, Theorem 13.30], that $W_+ \not\equiv 0$; using the fact that the second Bianchi identity implies that $\delta W_+ = 0$ on any Einstein 4-manifold, the theorem is then deduced from the corresponding Weitzenböck formula for W_+ via a clever conformal-rescaling argument.)

Given these results about the Einstein case, it might now seem tempting to inquire about the balance between our two basic functionals (3) for general Riemannian metrics on a smooth compact oriented 4-manifold. However, this naïve form of the question is simply silly, because $\int |W_+|^2 d\mu$ is conformally invariant, while $\int s^2 d\mu$ varies wildly in any conformal class!

Example. Let (M, g) be a compact oriented Riemannian 4-manifold, and consider arbitrary conformal rescalings $\hat{g} = u^2 g$, where $u : M \to \mathbb{R}^+$ is a smooth positive function. The Yamabe functional of such a conformally rescaled metric is then given by

$$\mathscr{E}(\hat{g}) := \frac{\int_M s_{\hat{g}} d\mu_{\hat{g}}}{\sqrt{\int_M d\mu_{\hat{g}}}} = \frac{\int_M \left[6|\nabla u|^2 + su^2\right] d\mu_g}{\sqrt{\int_M u^4 d\mu_g}}$$

and, since there are functions u that are C^0 close to 1, but which are wildly oscillatory on a microscopic scale, one immediately sees that there are sequences of metrics $\hat{g}_j$ in the given conformal class $[g] := \{u^2 g\}$ that remain C^0 close to g, but have $\mathscr{E}(\hat{g}_j) \to +\infty$. But since $\int s^2 d\mu_{\hat{g}} \geqslant [\mathscr{E}(\hat{g})]^2$ by the Cauchy-Schwarz inequality, this means that $\int s^2 d\mu \to +\infty$ among metrics in our arbitrary conformal class $[g]$. In particular, this shows that there exist metrics on any 4-manifold M for which

$$\int \frac{s^2}{24} d\mu > \int |W_+|^2 d\mu$$

and, indeed, that there exist such metrics in any conformal class. ♢

Example. Let g be a J-compatible Kähler metric of non-constant scalar curvature on a compact complex surface (M^4, J); a generic Kähler metric in any Kähler class will have this property [6]. By the solution of the Yamabe problem [24, 31], there exists a constant-scalar-curvature metric $\breve{g} = u^2 g$ conformal to g that minimizes the Yamabe functional $\mathscr{E}$ in the conformal class $[g]$; and, because M has real dimension 4, such a Yamabe metric $\breve{g}$ also minimizes $\int s^2 d\mu$ in its conformal class [4, Proposition 2.1]. Thus $\breve{g}$ must satisfy

$$\int \frac{s^2}{24} d\mu < \int |W_+|^2 d\mu$$

because equality is already achieved by the higher-energy metric g. Thus, any compact 4-manifold that admits a complex structure of Kähler type will admit metrics for which self-dual Weyl outweighs the scalar curvature. ♢

Thus, our question of balance only becomes reasonable if we somehow turn it into a conformally invariant question, or else narrow the scope of the question in a way that effectively precludes conformal rescaling. One particularly nice such modification, which coincides with the original question in the Einstein case, is to ask whether

$$\int_M |W_+|^2 d\mu \stackrel{?}{\geqslant} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu, \tag{4}$$

since the Gauss-Bonnet formula (1) implies that the right-hand side is also conformally invariant. By combining Gauss-Bonnet with the Thom-Hirzebruch signature formula (2), it is now easy to see that this modified question is exactly equivalent to asking when

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu \stackrel{?}{\geqslant} \frac{1}{3}(2\chi + 3\tau)(M). \tag{5}$$

Asking whether such an inequality holds for *all* metrics on a given M is then a question about the infimum of the *Weyl functional*

$$\mathscr{W}([g]) := \int_M \left(|W_+|^2 + |W_-|^2 \right) d\mu_g,$$

which measures the deviation of a conformal class $[g]$ from local conformal flatness. Since equation (2) implies that

$$\mathscr{W}([g]) = -12\pi^2 \tau(M) + 2 \int_M |W_+|^2 d\mu_g,$$

knowing the infimum of $\mathscr{W}$ is equivalent to understanding the differential-topological invariant $\inf_g \int_M |W_+|^2 d\mu_g$, and for our purposes this will be the more convenient formulation of the problem.

The infimum of the Weyl functional seems to have been first discussed by Atiyah, Hitchin, and Singer [1], who discovered that the infimum is achieved on $\mathbb{CP}_2$ by the Fubini-Study metric; indeed, they more generally observed that (2) implies that any metric on a compact oriented 4-manifold M satisfies

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu_g \geqslant 3\tau(M), \tag{6}$$

with equality iff $W_- \equiv 0$. This seems to have then inspired Osamu Kobayashi [16] to examine the key example of $M = S^2 \times S^2$, where inequality (6) just

becomes the trivial statement that $\int_M |W_+|^2 d\mu \geqslant 0$, but where this lower bound is impossible to achieve, since a theorem of Kuiper [18] guarantees that S^4 is the only simply-connected 4-manifold that admits a metric with $W_+ = W_- \equiv 0$. Kobayashi conjectured that the infimum on $S^2 \times S^2$ is achieved by the Kähler-Einstein metric arising as the Riemannian product of two round 2-spheres of the same radius. Kobayashi's evidence for this conjecture was modest, but interesting; by calculating the second variation of $\mathscr{W}$, he proved that this standard Einstein metric is a local minimum of the Weyl functional, and he also checked that it is the unique global minimizer of the restriction of $\mathscr{W}$ to the Kähler metrics on $\mathbb{CP}_1 \times \mathbb{CP}_1$.

While Kobayashi's evidence was admittedly fragmentary, Matthew Gursky later discovered a beautiful general result [12] that puts the question on an entirely different footing:

Theorem (Gursky)**.** *Let M be a compact oriented 4-manifold such that $b_+(M) \neq 0$, and let $[g]$ be any conformal class with $Y_{[g]} > 0$. Then*

$$\frac{1}{4\pi^2}\int_M |W_+|^2 d\mu_g \geqslant \frac{1}{3}(2\chi + 3\tau)(M), \tag{7}$$

with equality iff $[g]$ is the conformal class of a Kähler-Einstein metric.

Here we recall that the *Yamabe constant* $Y_{[g]}$ of a conformal class $[g]$ is by definition the infimum of the Yamabe functional $\mathscr{E}(\hat{g})$ over $\hat{g} \in [g]$, and that $Y_{[g]}$ is positive iff $[g]$ contains a metric of positive scalar curvature. If M is any compact oriented 4-manifold, also recall that $b_+(M)$ is defined to be the dimension of any maximal subspace of $H^2(M, \mathbb{R})$ on which the intersection pairing $H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R}$ is positive definite; and since, for any Riemannian metric g on M, the self-dual/anti-self-dual decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

of the bundle of 2-forms induces an intersection-form-adapted decomposition

$$\mathcal{H}_g = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$$

of the harmonic 2-forms $\mathcal{H}_g \cong H^2(M, \mathbb{R})$ into eigenspaces of the Hodge star operator $\star : \mathcal{H}_g \to \mathcal{H}_g$, it easily follows that $b_+(M)$ is exactly the dimension of the space $\mathcal{H}_g^+$ of harmonic self-dual 2-forms on (M, g). Gursky's argument uses the Weitzenböck formula for self-dual harmonic 2-forms, together with a conformal rescaling argument, to show that every conformal class on any M^4

with $b_+ \neq 0$ contains a metric with $2\sqrt{6}|W_+| \geqslant s$ everywhere. Integrating and applying Cauchy-Schwarz, one then concludes that

$$\int_M |W_+|^2 d\mu \geqslant \frac{1}{24} \left(Y_{[g]}\right)^2 \tag{8}$$

whenever $Y_{[g]} \geqslant 0$. The result then follows, because the right-hand side of (4) at a Yamabe metric is certainly less than (or equal to) the right-hand side of (8) for any conformal class.

While Gursky's theorem certainly seems like a huge step in the direction of answering Kobayashi's question, Gursky's method unfortunately cannot provide any information at all about conformal classes with $Y_{[g]} < 0$; and, for better or worse, "most" conformal classes on any 4-manifold inevitably have negative Yamabe constant. Fortunately, an entirely different method does allow us to plunge into this Yamabe-negative realm; but this method only works on those rare 4-manifolds with $2\chi + 3\tau > 0$ that admit both symplectic structures and Riemannian metrics with $s > 0$. These are exactly [7, 21] the previously-mentioned manifolds that carry both a symplectic structure and a $\lambda > 0$ Einstein metric. Equivalently, they are the smooth 4-manifolds underlying the *del Pezzo surfaces*, meaning the compact complex surfaces that have ample anti-canonical line bundle K^{-1}. Up to oriented diffeomorphism, there are exactly ten of these manifolds, namely $S^2 \times S^2$ and the connected sums $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k = 0, 1, \ldots, 8$. Most of these actually carry Kähler-Einstein metrics, and one could hope, in the spirit of Kobayashi's conjecture, that their conformal classes might exactly minimize the Weyl functional.

Because each del Pezzo 4-manifold M has $b_+(M) = 1$, there is, up to a multiplicative constant, a unique self-dual harmonic 2-form ω on M for each Riemannian metric g, and this ω moreover only depends on the conformal class $[g]$ of the given metric. When this ω is everywhere non-zero, it is automatically a symplectic form, and one therefore says that $[g]$ is a conformal class of *symplectic type*. Like Gursky's condition $Y_{[g]} > 0$, this new condition is open in the C^2 topology; however, it is also genuinely different, because one can construct sequences of conformal classes $[g_j]$ of symplectic type with $Y_{[g_j]} \to -\infty$. Nonetheless, inequality (7) can still be shown [23] to hold for such conformal classes, even in these Yamabe-negative depths:

Theorem (L)**.** *Let M^4 be the underlying smooth compact oriented manifold of a del Pezzo surface. Then any conformal class $[g]$ of* symplectic type *on M satisfies*

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu_g \geqslant \frac{1}{3}(2\chi + 3\tau)(M),$$

with equality iff $[g]$ contains a Kähler-Einstein metric g with $s > 0$.

Here the method of proof focuses on choosing a representative g for the conformal class $[g]$ for which the harmonic 2-form ω has pointwise norm $|\omega| \equiv \sqrt{2}$; this makes (M, g, ω) into an *almost-Kähler* manifold with $c_1 \cdot [\omega] > 0$, and this then gives rise to sharp lower bounds for the Weyl functional. It is also worth mentioning that there are two del Pezzo manifolds, $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$, on which the associated Einstein metrics are only *conformally* Kähler, but nevertheless still appear to minimize the Weyl functional [23].

The upshot is that Kobayashi's conjecture seems increasingly plausible for $S^2 \times S^2$ and its del Pezzo cousins. But could the lower bound (7) also hold in the Yamabe-negative realm on many other manifolds? One might first worry about Taubes' theorem [34], asserting that, for any compact oriented 4-manifold M, the connected sum $M\#k\overline{\mathbb{CP}}_2$ will admit metrics with $W_+ \equiv 0$ for astronomically large $k \gg 0$; however, $2\chi+3\tau \ll 0$ for these examples, so (7) becomes a tautology in this context. More pertinently, compact hyperbolic 4-manifolds have $W_+ \equiv 0$ and $2\chi+3\tau > 0$, and so certainly violate (7); but these examples never admit positive-scalar-curvature metrics [11], and thus do not carry any metrics to which Gursky's result applies. It is thus more relevant to ask whether there are 4-manifolds that *do* carry positive-scalar-curvature metrics, but which also carry some Yamabe-negative conformal classes for which (7) fails. In the simply connected case, our first main result shows that this always eventually happens after one first "stabilizes" the manifold by taking its connected sum with sufficiently many copies of $S^2 \times S^2$:

Theorem A. *For any sufficiently large integer m, the smooth compact simply-connected zero-signature spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2)\# \cdots \#(S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[h]$ that satisfy

$$\frac{1}{4\pi^2}\int_M |W_+|^2 d\mu_h < \frac{1}{3}(2\chi + 3\tau)(M). \tag{9}$$

Similarly, for any sufficiently large integer m, together with any second integer n such that $\frac{n}{m}$ is sufficiently close to 1, the smooth compact simply-connected non-spin manifold

$$M = m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2 := \underbrace{\mathbb{CP}_2\# \cdots \#\mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2\# \cdots \#\overline{\mathbb{CP}}_2}_{n}$$

admits conformal classes $[h]$ that satisfy inequality (9).

The proof of this result can be found in §2 below.

On the other hand, given the role of almost-Kähler geometry in the above discussion, it also seems natural to explore our question of balance in the almost-Kähler context. Here, the scales can tip either way. Indeed, if we choose to impose one of the two key conditions that played a role in our previous discussion, systematic but opposing patterns emerge:

Theorem B. *If (M, g, ω) is a compact almost-Kähler 4-manifold for which $\delta W_+ = 0$, where δ denotes the divergence operator, then*

$$\int_M \frac{s^2}{24} d\mu_g \geqslant \int_M |W_+|^2 d\mu_g,$$

with equality iff (M, g, ω) is Kähler. By contrast, if (M, g, ω) is instead a compact almost-Kähler 4-manifold with scalar curvature $s \geqslant 0$, then

$$\int_M |W_+|^2 d\mu_g \geqslant \int_M \frac{s^2}{24} d\mu_g,$$

again with equality iff (M, g, ω) is Kähler. In particular, any compact almost-Kähler 4-manifold (M, g, ω) with $\delta W_+ = 0$ and $s \geqslant 0$ is necessarily Kähler.

For the proof, see §3 below.

2. Curvature and connected sums

The constructions in this section will depend on the existence of simply-connected minimal complex surfaces (X^4, J) of general type with $\tau(X) > 0$, where the signature $\tau(X) = b_+(X) - b_-(X)$ is understood to be computed with respect to the complex orientation of X^4. Recall [2] that a complex surface (X, J) is said to be minimal if it contains no holomorphically embedded $\mathbb{CP}_1$ of homological self-intersection -1, and that (X^4, J) is said to be of general type if $h^0(X, \mathcal{O}(K^{\otimes j}))$ grows quadratically in j for $j \gg 0$, where $K = \Lambda^{2,0}_X$ denotes the canonical line bundle of X. Minimal, simply-connected complex surfaces of general type exist in abundance; indeed, every smooth complete-intersection surface in $\mathbb{CP}_n$ of degree $\geqslant 9$ is an example. However, constructing such complex surfaces with $\tau(X) > 0$ is surprisingly difficult and subtle, and no examples were known prior to the trailblazing work of Miyaoka [26] and Moishezon-Teicher [27]. A plethora of non-spin examples were then constructed by Chen [8], after which Persson, Peters, and Xiao [28] proceeded to show that spin examples exist in similar profusion. Much more recently,

Roulleau and Urzúa [30] settled a celebrated problem in complex-surface geography by showing that there exist sequences of such X with $c_1^2(X)/c_2(X) \to 3$; moreover, one can either do this while insisting that these 4-manifolds X be spin, or while instead insisting that that they be non-spin. In terms of the signature $\tau = (c_1^2 - 2c_2)/3$ and topological Euler characteristic $\chi = c_2$, Roulleau and Urzúa's construction yields sequences of simply-connected complex surfaces with $\tau(X)/\chi(X) \to 1/3$.

These complex surfaces X will eventually become essential building blocks in our construction. To make good use of them, however, we will first need to introduce some basic differential-geometric tricks.

Lemma 2.1. *Let $\epsilon > 0$ be given. Then, for any smooth compact oriented Riemannian 4-manifold (Y, g_0), there is a Riemannian metric g_ϵ on Y which is flat on some tiny ball, but which also satisfies*

$$\frac{1}{4\pi^2}\int_Y |W_+|^2 d\mu_{g_\epsilon} < \frac{1}{4\pi^2}\int_Y |W_+|^2 d\mu_{g_0} + \epsilon. \tag{10}$$

Similarly, if (Y^4, g_0) is a compact Riemannian orbifold with only isolated singularities, there exists an orbifold metric g_ϵ on Y which is flat in a small neighborhood of each orbifold singularity, but also satisfies (10).

Proof. In geodesic normal coordinates about some point $p \in Y$,

$$g_0 = \boldsymbol{\delta} + O(\varrho^2),$$

where $\boldsymbol{\delta}$ is the flat Euclidean metric associated with the coordinate system, and ϱ is the Euclidean radius. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing smooth function which is identically 0 on $(-\infty, \frac{1}{2})$ and identically 1 on $(1, \infty)$, and, for each sufficiently small $t > 0$, set

$$h_t = \boldsymbol{\delta} + \phi(\frac{\varrho}{t})[g_0 - \boldsymbol{\delta}],$$

so that h_t coincides with g_0 for $\varrho > t$, but is flat for $\varrho < t/2$. We now extend this to a metric on all of Y by setting $h_t = g_0$ outside our coordinate chart. In the transition region $\varrho \in (t/2, t)$, for any small t, one then has

$$\|h_t - \boldsymbol{\delta}\| \leqslant Ct^2, \qquad \|\mathbb{D}h_t\| \leqslant Ct, \qquad \|\mathbb{D}^2 h_t\| \leqslant C,$$

where $\mathbb{D}$ is the Euclidean derivative operator associated with the given coordinate system, and where the constant C is independent of t. In this transition

annulus, the norm-square $|\mathcal{R}|^2$ of the curvature tensor of h_t is therefore everywhere less than a constant C' independent of t, and the same is therefore true of $|W_+|^2 \leqslant |\mathcal{R}|^2$. Since the volume of this transition annulus is less than a constant times t^4, the effect of this modification on $\|W_+\|^2_{L^2}$ is less than a constant times t^4. We can therefore achieve our goal by setting $g_\epsilon = h_t$ for some sufficiently small t.

In the orbifold case, the proof is the same, except that, instead of altering the given metric on a single ball, we instead change it in the above manner on a finite number of neighborhoods modeled on B^4/Γ_j for appropriate finite subgroups $\Gamma_j \subset \mathbf{SO}(4)$. □

Lemma 2.2. *Let (Y_1, g_1) and (Y_2, g_2) be two compact oriented 4-manifolds, where each one is conformally flat on some small open set. Then the connected sum $Y_1 \# Y_2$ admits a conformal class of metrics $[g]$ such that*

$$\int_{Y_1\#Y_2} |W_+|^2 d\mu_g = \int_{Y_1} |W_+|^2 d\mu_{g_1} + \int_{Y_2} |W_+|^2 d\mu_{g_2}. \tag{11}$$

Similarly, if (Y_1, g_1) and (Y_2, g_2) are oriented orbifolds which contain flat neighborhoods modeled on B^4/Γ and $\overline{B^4/\Gamma}$, respectively, where the bar is used to indicate reversal of orientation, then the generalized connected sum $Y_1 \#_\Gamma Y_2$ admits a conformal class of orbifold metrics $[g]$ for which $\|W_+\|^2_{L^2}$ is exactly additive, in the sense that a perfect analog of (11) holds.

Proof. We may delete a tiny round ball from the conformally flat region of each manifold, and then form the connected sum by identifying conformally-flat annular regions near the boundary spheres by an orientation-preserving conformal inversion. Since these gluing maps have been chosen to preserve the conformal structure, there is then a well-defined conformal structure induced on the connected sum. Since $\|W_+\|^2_{L^2}$ is conformally invariant, the conformally flat balls we have deleted have no impact whatsoever on the curvature integral, and integrals therefore coincide with the sum of the integrals in the ball-complements. This moreover works equally well in the orbifold case, with only cosmetic changes. □

With these lemmata in hand, we can now prove the following:

Proposition 2.3. *Let (X, J) be a minimal complex surface of general type, and let $\epsilon > 0$ be given. Then the smooth compact oriented 4-manifold X admits a conformal class $[g_\epsilon]$ of Riemannian metrics such that*

$$\frac{1}{4\pi^2} \int_Y |W_+|^2 d\mu_{g_\epsilon} < \frac{1}{3} c_1^2(X) + \epsilon. \tag{12}$$

Proof. Let $\mathscr{X}$ be the pluricanonical model of (X, J), which is obtained by collapsing all (-2)-curves in X. Since $\mathscr{X}$ has only rational double-point singularities, we may choose to view it as a complex orbifold that has only A-D-E singularities. This orbifold then has $c_1 < 0$, and the usual Aubin-Yau proof [38] therefore implies [36] that it carries an orbifold Kähler-Einstein metric $\check{g}$. This Kähler-Einstein metric then satisfies

$$\frac{1}{4\pi^2}\int_{\mathscr{X}} |W_+|^2 d\mu_{\check{g}} = \frac{1}{4\pi^2}\int_{\mathscr{X}} \frac{s^2}{24} d\mu_{\check{g}} = \frac{1}{3}c_1^2(\mathscr{X}) = \frac{1}{3}c_1^2(X),$$

where the last equality reflects the fact that $X \to \mathscr{X}$ is a crepant resolution. If $\mathscr{X} = X$, we are already done. Otherwise, for each singular point p_j with orbifold group $\Gamma_j \subset \mathbf{SU}(2)$, let $(Y_j, [g_j])$ be the one-point conformal compactification of one of Kronheimer's gravitational instanton metrics [17] on the minimal resolution of $\mathbb{C}^2/\Gamma_j$; this is a smooth compact orbifold with $W_+ \equiv 0$ that has a single, isolated singularity modeled on $\overline{B^4/\Gamma_j}$. Flattening these orbifolds slightly at their singular points, in accordance with Lemma 2.1, and then performing generalized connected sums, in accordance with Lemma 2.2, we then obtain a conformal metric on $X = \mathscr{X}\#_{\Gamma_1}Y_1\#_{\Gamma_2}\cdots\#_{\Gamma_k}Y_k$ with $\|W_+\|^2_{L^2}$ as close as we like to its value for the orbifold $(\mathscr{X}, [\check{g}])$. □

This now puts us in a position to prove one of our key results:

Theorem 2.4. *For any sufficiently large integer m, the smooth compact simply-connected spin 4-manifold $M = m(S^2 \times S^2)$ admits conformal classes $[h]$ that satisfy inequality (9).*

Proof. There are [28, 30] infinitely many simply-connected compact complex surfaces X of general type with signature $\tau(X) > 0$ that are spin (and so, in particular, minimal). Choose such a complex surface X, and, for a small ϵ we will specify later, equip X, per Lemma 2.1 and Proposition 2.3, with a metric g_ϵ that satisfies (12) and is flat on four tiny balls. We also equip the orientation-reversed version $\overline{X}$ of X with mirror-image versions of this g_ϵ, and finally equip $S^2 \times S^2$ with a modification $\mathbf{g}_\epsilon$ of its standard product Kähler-Einstein metric that is flat on a tiny ball and has

$$\frac{1}{4\pi^2}\int_{S^2\times S^2} |W_+|^2 d\mu_{\mathbf{g}_\epsilon} < \frac{c_1^2}{3}(\mathbb{CP}_1 \times \mathbb{CP}_1) + \epsilon = \frac{8}{3} + \epsilon.$$

By removing round, conformally flat balls and gluing as in Lemma 2.2, we thus obtain a conformal class $[h_\epsilon]$ on

$$M_{k,\ell} := (k+\ell)[X\#\overline{X}]\#(2k+\ell)[S^2 \times S^2]$$

that satisfies

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{M_{k,\ell}} |W_+|^2 d\mu_{h_\epsilon} \\
&\quad = \frac{k+\ell}{4\pi^2}\left[\int_X |W_+|^2 d\mu_{g_\epsilon} + \int_{\overline{X}} |W_+|^2 d\mu_{g_\epsilon}\right] + \frac{2k+\ell}{4\pi^2}\int_{S^2\times S^2} |W_+|^2 d\mu_{\mathbf{g}_\epsilon} \\
&\quad = \frac{k+\ell}{4\pi^2}\left[2\int_X |W_+|^2 d\mu_{g_\epsilon} - 12\pi^2\tau(X)\right] + \frac{2k+\ell}{4\pi^2}\int_{S^2\times S^2} |W_+|^2 d\mu_{\mathbf{g}_\epsilon} \\
&\quad < (k+\ell)\left[\frac{2}{3}(2\chi+3\tau)(X) - 3\tau(X) + 2\epsilon\right] + \frac{8}{3}(2k+\ell) + (2k+\ell)\epsilon \\
&\quad \leqslant (k+\ell)\left[\frac{4}{3}\chi(X) - \tau(X) + \frac{16}{3} + 4\epsilon\right].
\end{aligned}$$

By contrast,

$$\begin{aligned}
\frac{1}{3}(2\chi+3\tau)(M_{k,\ell}) = \frac{2}{3}\chi(M_{k,\ell}) &= \frac{2}{3}\left[2 + b_2(M_{k,\ell})\right] \\
&= \frac{4}{3} + \frac{2}{3}(k+\ell)[b_2(X) + b_2(\overline{X})] + \frac{4}{3}(2k+\ell) \\
&= \frac{4}{3} + \frac{4}{3}(k+\ell)b_2(X) + \frac{4}{3}(2k+\ell) \\
&> \frac{4}{3}(k+\ell)[b_2(X)+1] \\
&= (k+\ell)\left[\frac{4}{3}\chi(X) - \frac{4}{3}\right].
\end{aligned}$$

Taking $\epsilon \in (0, \frac{1}{12})$, we thus deduce that

$$\frac{1}{3}(2\chi+3\tau)(M_{k,\ell}) - \frac{1}{4\pi^2}\int_{M_{k,\ell}} |W_+|^2 d\mu_{h_\epsilon} > (k+\ell)\left[\tau(X) - 7\right]. \tag{13}$$

But since X is a spin manifold, Rokhlin's Theorem [19, 29] tells us that $16|\tau(X)$, so our $\tau(X) > 0$ hypothesis therefore implies that $\tau(X) \geqslant 16$. Thus, the right-hand-side of (13) is automatically positive, and we have therefore produced conformal classes $[h]$ on $M_{k,\ell}$ that satisfy (9).

Now notice that, since $X\#\overline{X}$ is a simply connected spin manifold of signature zero, Wall's stable classification [37] via h-cobordism implies that there exists some large integer $\mathfrak{p}$ such that $X\#\overline{X}\#(k+\ell)(S^2\times S^2)$ is diffeomorphic to a connected sum $\mathfrak{q}(S^2\times S^2)$ for any $(k+\ell) \geqslant \mathfrak{p}$. By induction on the number of $X\#\overline{X}$ summands, and then adding k additional $(S^2\times S^2)$

summands, we thus see that $M_{k,\ell} = (k+\ell)(X\#\overline{X})\#(2k+\ell)(S^2\times S^2)$ is diffeomorphic to $m(S^2\times S^2)$ whenever $(k+\ell)\geqslant \mathfrak{p}$, where we have set $m := k[b_2(X)+2]+\ell[b_2(X)+1]$.

On the other hand, any integer $m \geqslant b_2(X)[b_2(X)+1]$ can be expressed as $k[b_2(X)+2]+\ell[b_2(X)+1]$ for some integers $k,\ell\geqslant 0$; this elementary fact is actually a special case of Sylvester's solution [33] of the Frobenius two-coin problem. But since this expression for m implies that $(k+\ell)[b_2(X)+2]\geqslant m$, we also have $(k+\ell)\geqslant\mathfrak{p}$ whenever $m\geqslant\chi(X)\mathfrak{p}$. Thus, whenever m exceeds $\chi(X)\max(\mathfrak{p}(X),\chi(X))$, the connected sum $m(S^2\times S^2)$ can also be expressed as $M_{k,\ell}$ for some k and ℓ, and consequently admits a conformal class $[h]$ that satisfies (9). □

Of course, by the Gromov-Lawson surgery theorem [10, Theorem A], the smooth 4-manifolds $m(S^2\times S^2)$ all admit metrics of positive scalar curvature. and Gursky's inequality (7) then gives an interesting lower bound on the Weyl functional on these Yamabe-positive conformal classes. The point of Theorem 2.4, however, is that the infimum of the Weyl functional is usually considerably lower than one might guess without taking a plunge into the Yamabe-negative depths.

Essentially the same phenomenon also occurs on non-spin 4-manifolds:

Theorem 2.5. *Choose any $\varepsilon\in(0,\frac{1}{5})$. Then, for every sufficiently large integer m, and for any integer n satisfying $(\frac{4}{5}+\varepsilon)m < n < (2-\varepsilon)m$, the smooth compact simply-connected non-spin 4-manifold $M = m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ admits conformal classes $[h]$ that satisfy inequality (9).*

Proof. By imitating the proof of Theorem 2.4, we first handle the case where $n=m$. Thus, we begin by considering

$$M_{k,\ell} = (k+\ell)[X\#\overline{X}]\#(2k+\ell)[S^2\times S^2],$$

but now take X to be a simply connected non-spin minimal complex surface of general type with signature $\tau(X)\geqslant 8$. In this setting, Wall's stable classification [37] implies that there is some $\mathfrak{p}$ such that the simply-connected zero-signature non-spin 4-manifold $M_{k,\ell}$ is diffeomorphic to $m\mathbb{CP}_2\#m\overline{\mathbb{CP}}_2$ whenever $(k+\ell)\geqslant\mathfrak{p}$, where $m := k\chi(X)+\ell[\chi(X)-1]$. Since any integer $m\geqslant[\chi(X)]^2$ can be expressed as $k\chi(X)+\ell[\chi(X)-1]$ for integers $k,\ell\geqslant 0$, and since these integers will then satisfy $(k+\ell)\geqslant m/\chi(X)$, it in particular follows that $(k+\ell)\geqslant\mathfrak{p}$ whenever $m\geqslant\chi(X)\mathfrak{p}$. Hence $m\mathbb{CP}_2\#m\overline{\mathbb{CP}}_2$ is diffeomorphic to some $M_{k,\ell}$ whenever $m\geqslant\chi(X)\max(\mathfrak{p},\chi(X))$. On the other

hand, our previous gluing construction now yields conformal metrics $[h]$ on $M_{k,\ell}$ which satisfy

$$\frac{1}{3}(2\chi+3\tau)(M_{k,\ell})-\frac{1}{4\pi^2}\int_{M_{k,\ell}}|W_+|^2d\mu_h>(k+\ell)\left[\tau(X)-7\right]\geqslant m\frac{\tau(X)-7}{\chi(X)}.$$

Thus, whenever m is large, $m\mathbb{CP}_2\#m\overline{\mathbb{CP}}_2\approx M_{k,\ell}$ admits a conformal class that not only satisfies (9), but for which we actually have a lower bound for the gap in terms of m and the homeotype of our chosen building-block X.

We next consider the manifolds

$$\widehat{M}_{j,k,\ell}=M_{k,\ell}\#j\overline{\mathbb{CP}}_2,$$

and notice that $m\mathbb{CP}_2\#(m+j)\overline{\mathbb{CP}}_2$ is then diffeomorphic to some such $\widehat{M}_{j,k,\ell}$ whenever m is sufficiently large. But since the mirror-image Fubini-Study metric on $\overline{\mathbb{CP}}_2$ has $W_+\equiv 0$, Lemma 2.1 guarantees that this reverse-oriented version of $\mathbb{CP}_2$ carries conformal classes with $\frac{1}{4\pi^2}\int|W_+|^2d\mu<\epsilon$ that are conformally flat on some tiny ball. Since the constructed conformal classes $[h]$ on $M_{k,\ell}$ also contain tiny conformally-flat regions, gluing per Lemma 2.2 thus produces conformal classes on $\widehat{M}_{j,k,\ell}$ with

$$\frac{1}{4\pi^2}\int_{\widehat{M}_{j,k,\ell}}|W_+|^2d\mu<\frac{1}{4\pi^2}\int_{M_{k,\ell}}|W_+|^2d\mu+j\epsilon$$

for ϵ as small as we like. On the other hand,

$$\frac{1}{3}(2\chi+3\tau)(\widehat{M}_{j,k,\ell})=\frac{1}{3}(2\chi+3\tau)(M_{k,\ell})-\frac{j}{3}$$

so that

$$\frac{1}{3}(2\chi+3\tau)(\widehat{M}_{j,k,\ell})-\frac{1}{4\pi^2}\int_{\widehat{M}_{j,k,\ell}}|W_+|^2d\mu>m\frac{\tau(X)-7}{\chi(X)}-\frac{j}{3}-j\epsilon$$

and, by taking ϵ sufficiently small, our construction therefore produces conformal classes $[h]$ on $\widehat{M}_{j,k,\ell}$ satisfying (9) whenever

$$0\leqslant j<3m\frac{\tau(X)-7}{\chi(X)}.$$

Setting $n=m+j$, our construction therefore yields conformal classes on

$m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ that satisfy (9), provided that $m \geqslant \chi(X)\max(\mathfrak{p}, \chi(X))$ and

$$m \leqslant n < \left(1 + 3\frac{\tau(X) - 7}{\chi(X)}\right) m. \tag{14}$$

We can similarly construct controlled conformal classes on

$$\widetilde{M}_{j,k,\ell} = j\mathbb{CP}_2\#M_{k,\ell}$$

by instead conformally gluing in our mild modifications of the Fubini-Study metric in a way that is compatible with the standard orientation of $\mathbb{CP}_2$. While each copy of $\mathbb{CP}_2$ now contributes a substantial additional amount additional self-dual Weyl curvature, we still have

$$\frac{1}{4\pi^2}\int_{\widetilde{M}_{j,k,\ell}} |W_+|^2 d\mu < \frac{1}{4\pi^2}\int_{M_{k,\ell}} |W_+|^2 d\mu + 3j + j\epsilon.$$

This is mitigated by the fact that each added $\mathbb{CP}_2$ also increases $2\chi + 3\tau$:

$$\frac{1}{3}(2\chi + 3\tau)(\widetilde{M}_{j,k,\ell}) = \frac{1}{3}(2\chi + 3\tau)(M_{k,\ell}) + \frac{5}{3}j.$$

After the slight change of notation of now setting $n = k\chi(X) + \ell[\chi(X) - 1]$, we thus have

$$\frac{1}{3}(2\chi + 3\tau)(\widetilde{M}_{j,k,\ell}) - \frac{1}{4\pi^2}\int_{\widetilde{M}_{j,k,\ell}} |W_+|^2 d\mu > n\frac{\tau(X) - 7}{\chi(X)} - \frac{4}{3}j - j\epsilon$$

so that, for sufficiently small ϵ, our constructed conformal classes satisfy (9) whenever

$$0 \leqslant j < n\frac{3}{4}\frac{\tau(X) - 7}{\chi(X)}.$$

Setting $m = n + j$, we have thus constructed conformal classes on $\widetilde{M}_{j,k,\ell}$ that satisfy (9) whenever

$$\frac{m}{1 + \frac{3}{4}\frac{\tau(X)-7}{\chi(X)}} < n \leqslant m. \tag{15}$$

However, since X is of general type, it satisfies [2, 25, 38] the Miyaoka-Yau inequality $\chi(X) \geqslant 3\tau$, and (15) therefore implies that $n > \frac{4}{5}m$. Whenever inequality (15) holds and $m \geqslant \frac{5}{4}\chi(X)\max(\mathfrak{p}, \chi(X))$, our previous arguments

therefore imply that $m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ is diffeomorphic to some $\widetilde{M}_{j,k,\ell}$ on which our construction yields conformal metrics that satisfy (9).

Up until this point, our discussion has in principle worked for essentially any non-spin simply connected minimal complex surface X of positive signature. To optimize our conclusion, however, we now invoke the beautiful and surprising theorem of Roulleau and Urzúa [30], which asserts that there exist sequences of such X such that $c_1^2(X)/c_2(X) \to 3$. Now notice that the Euler characteristic $\chi(X) = c_2(X)$ must tend to infinity for any such sequence, since the Miyaoka-Yau inequality $c_1^2 \leqslant 3c_2$ is only saturated [38] by ball quotients, which are of course never simply-connected. Consequently,

$$\frac{\tau(X)-7}{\chi(X)} = \frac{1}{3}\left[\frac{c_1^2(X)}{c_2(X)} - 2 - \frac{21}{c_2(X)}\right] \to \frac{1}{3}$$

for any such sequence. Given any $\varepsilon \in (0, \frac{1}{5})$, we can therefore choose such an X such that

$$\frac{\tau(X)-7}{\chi(X)} > \frac{1-\varepsilon}{3}$$

and this choice will then satisfy both

$$1 + 3\frac{\tau(X)-7}{\chi(X)} > 2-\varepsilon \qquad \text{and} \qquad \frac{1}{1+\frac{3}{4}\frac{\tau(X)-7}{\chi(X)}} < \frac{4}{5} + \varepsilon.$$

Thus, if $m \geqslant \frac{5}{4}\chi(X)\max(\mathfrak{p}(X), \chi(X))$ and $(\frac{4}{5}+\varepsilon)m < n < (2-\varepsilon)m$, either (14) or (15) must hold for m and n with this choice of X, and $m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ is consequently diffeomorphic to one of the manifolds $\check{X}_{j,k,\ell}$ or $\hat{X}_{j,k,\ell}$ on which have constructed a conformal class satisfying inequality (9). □

Together, Theorems 2.4 and 2.5 now imply Theorem A.

Of course, the Gromov-Lawson surgery theorem [10, Theorem A] again implies that the connected sums $m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2$ all admit metrics of positive scalar curvature, and Gursky's inequality (7) then gives us an interesting lower bound for the Weyl functional of all such metrics. On the other hand, Theorem 2.5 produces metrics which violate this inequality, thus showing that the infimum of the Weyl functional is actually unexpectedly small. This is only possible because the constructed conformal classes $[h]$ all have very negative Yamabe constants $Y_{[h]}$; specifically, these conformal classes $[h]$ must necessarily satisfy

$$Y_{[h]} < -2\sqrt{6}\,\|W_+\|_{L^2,h}. \tag{16}$$

Indeed, let h be a Yamabe metric in $[h]$, and notice that (9) can be re-expressed as

$$\frac{3}{4\pi^2}\int_M |W_+|^2 d\mu_h < (2\chi+3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_h.$$

It therefore follows that

$$\int_M \frac{s^2}{24} d\mu_h > \int_M |W_+|^2 d\mu_h,$$

and since h has been chosen to be a Yamabe metric, this means that

$$[Y_{[h]}]^2 = \left(s_h \operatorname{Vol}_h^{1/2}\right)^2 > 24\, \|W_+\|^2_{L^2,h}. \tag{17}$$

But since the constructed conformal classes live on 4-manifolds with $b_+ \neq 0$, Gursky's theorem [12, Theorem 1] assures us that (9) can only happen for a conformal class $[h]$ with $Y_{[h]} < 0$. Thus, in the present context, (17) automatically forces inequality (16) to hold.

3. Goldberg variations

We now wrap up our exploration of the balance between scalar curvature and self-dual Weyl curvature by examining those almost-Kähler 4-manifolds that have harmonic self-dual Weyl curvature, in the sense that

$$\delta W_+ := -\nabla \cdot W_+ = 0. \tag{18}$$

Since any 4-dimensional Einstein manifold satisfies (18) by the second Bianchi identity, this question offers a possible source of insight into the so-called *Goldberg conjecture* [9], which claims that any compact almost-Kähler Einstein manifold should be Kähler-Einstein.

We begin by observing that any 4-dimensional almost-Kähler manifold (M, g, ω) satisfies

$$\Lambda^+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus K \oplus \overline{K},$$

where $K = \Lambda^{2,0}_J$ is the canonical line bundle of the almost-complex structure $J_a{}^b = \omega_{ac} g^{bc}$ on M. Locally choosing a unit section φ of K, we thus have

$$\nabla\omega = \theta \otimes \varphi + \bar{\theta} \otimes \bar{\varphi}$$

for a unique 1-form $\theta \in \Lambda^{1,0}_J$, since $\omega \perp \nabla\omega$, and $\nabla \wedge \omega = 0$. If

$$\circledast : \Lambda^+ \times \Lambda^+ \to \odot^2_0 \Lambda^+$$

denotes the symmetric trace-free product, we therefore have

$$(\nabla_e \omega) \circledast (\nabla^e \omega) = 2|\theta|^2 \varphi \circledast \bar{\varphi} = -\frac{1}{4}|\nabla\omega|^2 \omega \circledast \omega$$

and we thus deduce that

$$\begin{aligned}
\langle W_+, \nabla^*\nabla(\omega \otimes \omega)\rangle &= 2W_+(\omega, \nabla^*\nabla\omega) - 2W_+(\nabla_e\omega, \nabla^e\omega) \\
&= 2W_+(\omega, \nabla^*\nabla\omega) + \frac{1}{2}|\nabla\omega|^2 W_+(\omega,\omega) \\
&= 2W_+(\omega, 2W_+(\omega) - \frac{s}{3}\omega) + \Big[W_+(\omega,\omega) - \frac{s}{3}\Big]W_+(\omega,\omega) \\
&= -\frac{2}{3}sW_+(\omega,\omega) + 4|W_+(\omega)|^2 + \Big[W_+(\omega,\omega) - \frac{s}{3}\Big]W_+(\omega,\omega) \\
&= [W_+(\omega,\omega)]^2 + 4|W_+(\omega)|^2 - sW_+(\omega,\omega),
\end{aligned}$$

where we have used the Weitzenböck formula

$$0 = \nabla^*\nabla\omega - 2W_+(\omega) + \frac{s}{3}\omega \tag{19}$$

for the harmonic self-dual 2-form ω, as well as its consequence

$$\frac{1}{2}|\nabla\omega|^2 = W_+(\omega,\omega) - \frac{s}{3}, \tag{20}$$

arising from the fact that $|\omega|^2 \equiv 2$. But if $\delta W_+ = 0$, we also have the Weitzenböck formula

$$0 = \nabla^*\nabla W_+ + \frac{s}{2}W_+ - 6W_+ \circ W_+ + 2|W_+|^2 I,$$

and for M compact this therefore implies that

$$\begin{aligned}
0 &= \int_M \Big\langle \Big(\nabla^*\nabla W_+ + \frac{s}{2}W_+ - 6W_+ \circ W_+ + 2|W_+|^2 I\Big), \omega \otimes \omega \Big\rangle d\mu \\
&= \int_M \Big[\langle W_+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W_+(\omega,\omega) - 6|W_+(\omega)|^2 + 2|W_+|^2|\omega|^2\Big] d\mu \\
&= \int_M \Big[[W_+(\omega,\omega)]^2 - \frac{s}{2}W_+(\omega,\omega) - 2|W_+(\omega)|^2 + 4|W_+|^2\Big] d\mu.
\end{aligned}$$

Hence any compact almost-Kähler (M^4, g, ω) with $\delta W_+ = 0$ satisfies

$$\int s W_+(\omega,\omega) d\mu = \int \Big[8|W_+|^2 - 4|W_+(\omega)|^2 + 2[W_+(\omega,\omega)]^2 \Big] d\mu. \tag{21}$$

This has an amusing application to our question of balance:

Theorem 3.1. *If a compact almost-Kähler 4-manifold (M, g, ω) satisfies $\delta W_+ = 0$, then*

$$\int_M \frac{s^2}{24} d\mu_g \geqslant \int_M |W_+|^2 d\mu_g, \tag{22}$$

with equality iff (M, g, ω) is a constant-scalar-curvature Kähler manifold.

Proof. To better understand the meaning of (21), let us express W_+ at an arbitrary point in an orthonormal basis $\{\mathsf{e}_j\}_{j=1}^3$ for Λ^+ in which $\omega = \sqrt{2}\mathsf{e}_1$, and in which $W_+(\omega)$ is orthogonal to e_3. Then, in this basis,

$$W_+ = \begin{bmatrix} \alpha & \gamma & \\ \gamma & \beta & \\ & & -(\alpha+\beta) \end{bmatrix}$$

for suitable real numbers α, β, γ. In terms of these components,

$$|W_+|^2 = 2\alpha^2 + 2\beta^2 + 2\alpha\beta + 2\gamma^2,$$
$$[W_+(\omega,\omega)]^2 = 4\alpha^2,$$

and

$$|W_+(\omega)|^2 = 2\alpha^2 + 2\gamma^2.$$

We therefore have

$$\begin{aligned} 4|W_+|^2 - 4|W_+(\omega)|^2 + 2[W_+(\omega,\omega)]^2 &= 8\alpha^2 + 8\beta^2 + 8\alpha\beta \\ &= 6\alpha^2 + 8\left(\frac{\alpha}{2} + \beta\right)^2 \\ &\geqslant 6\alpha^2 = \frac{3}{2}[W_+(\omega,\omega)]^2. \end{aligned}$$

Thus (21) implies that

$$\int s W_+(\omega,\omega) d\mu \geqslant \int \Big[4|W_+|^2 + \frac{3}{2}[W_+(\omega,\omega)]^2 \Big] d\mu,$$

or in other words that

$$\frac{3}{8}\int\left[\frac{2s}{3}-W_+(\omega,\omega)\right]W_+(\omega,\omega)\;d\mu\geqslant\int|W_+|^2d\mu.$$

Substituting $W_+(\omega,\omega)=\frac{1}{2}|\nabla\omega|^2+\frac{s}{3}$ from (20), we thus have

$$\frac{3}{8}\int\left[\frac{s}{3}-\frac{1}{2}|\nabla\omega|^2\right]\left[\frac{s}{3}+\frac{1}{2}|\nabla\omega|^2\right]\;d\mu\geqslant\int|W_+|^2d\mu$$

and algebraic simplification therefore yields

$$\int_M\frac{s^2}{24}d\mu-\frac{3}{32}\int_M|\nabla\omega|^4d\mu\geqslant\int_M|W_+|^2d\mu. \tag{23}$$

Hence any compact almost-Kähler manifold (M^4,g,ω) with $\delta W_+=0$ must satisfy (22), with equality iff (M,g,ω) is Kähler. The claim therefore follows, because a Kähler surface (M^4,g,J) satisfies $\delta W_+=0$ if and only if its scalar curvature s is constant. □

On the other hand, the balance tips in the opposite direction for any almost-Kähler manifold of non-negative scalar curvature:

Proposition 3.2. *If (M,g,ω) is a compact almost-Kähler 4-manifold with scalar curvature $s\geqslant 0$, then*

$$\int_M|W_+|^2d\mu_g\geqslant\int_M\frac{s^2}{24}d\mu_g,$$

with equality iff (M,g,ω) is a Kähler manifold.

Proof. By the Weitzenböck formula (19), we have

$$2\sqrt{\frac{2}{3}}|W_+|\geqslant W_+(\omega,\omega)=\frac{s}{3}+\frac{1}{2}|\nabla\omega|^2$$

where the inequality results from the fact that W_+ is trace-free and $|\omega|^2\equiv 2$. Consequently, any almost-Kähler (M^4,g,ω) satisfies

$$|W_+|\geqslant\frac{s}{2\sqrt{6}},$$

with equality only at points where $\nabla\omega=0$. When $s\geqslant 0$, squaring both sides and integrating thus yields the desired result. □

Theorem 3.1 and Proposition 3.2 now imply Theorem B, along with:

Corollary 3.3. *Any compact almost-Kähler manifold (M^4, g, ω) with $s \geqslant 0$ and $\delta W_+ = 0$ is actually a constant-scalar-curvature Kähler manifold.*

In the special case where g is an Einstein metric, this gives a different proof of Sekigawa's partial solution [32] of the 4-dimensional Goldberg conjecture. For related results, see [22].

By contrast, however, if we drop the assumption that $s \geqslant 0$, there are explicit examples of compact almost-Kähler 4-manifolds with $\delta W_+ = 0$ that are manifestly non-Kähler. In particular, one can construct [5, 15] explicit compact, strictly almost-Kähler manifolds that are anti-self-dual; and since these have $W_+ \equiv 0$, they obviously satisfy $\delta W_+ = 0$, too. Because these anti-self-dual examples have scalar curvature $s = -\frac{3}{2}|\nabla\omega|^2 \leqslant 0$, with strict inequality at most points, they inhabit outlands that lie well beyond the reach of Corollary 3.3. In particular, these examples show that one naïve generalization of the Goldberg conjecture is certainly false.

Finally, we recall that, in the Kähler case, the first Chern class is represented by $\frac{1}{2\pi}\rho$, where $\rho = r(J\cdot, \cdot)$ is the Ricci form. As a consequence,

$$c_1^2(M) = \int_M \frac{\rho}{2\pi} \wedge \frac{\rho}{2\pi} = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{4} - |\mathring{r}|^2 \right) d\mu \tag{24}$$

for any Kähler manifold of real dimension 4. However, since

$$c_1^2(M) = (2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g,$$

for any Riemannian metric, equation (24) can instead be explained by the fact that $|W_+|^2 = \frac{s^2}{24}$ in the Kähler case. This latter way of understanding (24) has the advantage of making it clear that generalizations of this formula to other contexts must hinge on our familiar question of balance. For example, in the almost-Kähler context, Proposition 3.2 and Theorem 3.1 immediately imply the following result:

Corollary 3.4. *Let (M, g, ω) be a compact almost-Kähler 4-manifold.*

(i) *If g has scalar curvature $s \geqslant 0$, then*

$$\frac{1}{8\pi^2} \int_M \left(\frac{s^2}{4} - |\mathring{r}|^2 \right) d\mu \leqslant c_1^2(M),$$

with equality iff (M, g, ω) is Kähler.

(ii) *If, instead, g satisfies $\delta W_+ = 0$, then*

$$\frac{1}{8\pi^2}\int_M \left(\frac{s^2}{4} - |\mathring{r}|^2\right) d\mu \geqslant c_1^2(M),$$

again with equality iff (M, g, ω) is Kähler.

This once again illustrates the degree to which the question of balance consistently plays a natural role in understanding the relationship between curvature and the topology of smooth compact Riemannian 4-manifolds.

4. Unanswered questions

On simply connected compact 4-manifolds that carry metrics of positive scalar curvature, we have just seen that the infimum of the Weyl functional is often substantially smaller than one might have guessed on the basis of Gursky's inequality (7). In particular, Theorem 2.4 asserts that there exists an integer m_0 such that $m(S^2 \times S^2)$ carries a conformal class satisfying (9) whenever $m \geqslant m_0$. But the method of proof used here does not actually display a concrete m_0 with this property; nor does it even hint at what might happen when m is reasonably small. Thus, while one might hope that this phenomenon already occurs when, say, $m = 2$, proving or disproving such a statement would require an entirely different set of ideas. Moreover, the present method only gives us a crude upper bound for the infimum of the Weyl functional, and does not begin to hint at its actual value. For example, while Kuiper's theorem [18] implies that $m(S^2 \times S^2)$ cannot admit a metric with $W_+ \equiv 0$, it doesn't guarantee that $\inf \int |W_+|^2 d\mu$ of such a manifold could never equal zero. Proving that this infimum is actually positive would be an interesting accomplishment in itself!

Our lack of effective estimates for m_0 becomes even more severe in the non-spin setting of Theorem 2.5. Given a closed interval $I \subset (\frac{4}{5}, 2)$, we have seen that there is an integer m_0 so that a metric satisfying (9) can be found on $m\mathbb{CP}_2 \# n\overline{\mathbb{CP}}_2$ whenever $m \geqslant m_0$ and $\frac{n}{m} \in I$. However, the value of m_0 produced by the proof is astronomical in practice, and in any case tends to infinity when, for example, the lower endpoint of I approaches $\frac{4}{5}$. Another possible objection is that inequality (9) depends on a choice of orientation. This, however, is not really a serious issue, because the construction produces metrics that satisfy (9) for both orientations if $I \subset (\frac{4}{5}, \frac{5}{4})$.

Finally, the almost-Kähler version of our question of balance has only been touched on here in a very preliminary way, and there could be many interesting things that remain to be discovered in this setting. For example,

while we have seen that the direction in which the balance tips is different for two interesting classes of almost-Kähler manifolds, it is possible that the patterns we have noticed might hold for larger phyla of almost-Kähler metrics. For example, if an almost-Kähler metric has positive scalar curvature, it then follows that $c_1 \cdot [\omega] > 0$; and, conversely, this last condition certainly implies that W_+ has relatively large L^2-norm. Is there a version of the almost-Kähler balance story that only depends on the sign of $c_1 \cdot [\omega]$? A clean result along these lines would certainly shed interesting new light on the subject.

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References

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, **362** (1978), pp. 425–461. MR0506229

[2] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, vol. 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1984. MR0749574

[3] A. L. Besse, *Einstein Manifolds, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Springer-Verlag, Berlin, 1987. MR0867684

[4] G. Besson, G. Courtois, and S. Gallot, *Volume et entropie minimale des espaces localement symétriques*, Inv. Math., **103** (1991), pp. 417–445. MR1085114

[5] C. J. Bishop and C. LeBrun, *Anti-self-dual 4-manifolds, quasi-Fuchsian groups, and almost-Kähler geometry*, Comm. Anal. Geom., **28** (2020), pp. 745–780. MR4165309

[6] E. Calabi, *Extremal Kähler metrics*, in: Seminar on Differential Geometry, vol. 102 of Ann. Math. Studies, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR0645743

[7] X. X. Chen, C. LeBrun, and B. Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc., **21** (2008), pp. 1137–1168. MR2425183

[8] Z. J. Chen, *On the geography of surfaces. Simply connected minimal surfaces with positive index*, Math. Ann., **277** (1987), pp. 141–164. MR0884652

[9] S. I. Goldberg, *Integrability of almost Kaehler manifolds*, Proc. Amer. Math. Soc., **21** (1969), pp. 96–100. MR0238238

[10] M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. Math., **111** (1980), pp. 423–434. MR0577131

[11] M. Gromov and H. B. Lawson, *Spin and scalar curvature in the presence of the fundamental group*, Ann. Math., **111** (1980), pp. 209–230. MR0569070

[12] M. J. Gursky, *The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics*, Ann. of Math. (2), **148** (1998), pp. 315–337. MR1652920

[13] M. J. Gursky, *Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere*, Math. Ann., **318** (2000), pp. 417–431. MR1800764

[14] M. J. Gursky and C. LeBrun, *On Einstein manifolds of positive sectional curvature*, Ann. Global Anal. Geom., **17** (1999), pp. 315–328. MR1705915

[15] I. Kim, *Almost-Kähler anti-self-dual metrics*, Ann. Global Anal. Geom., **49** (2016), pp. 369–391. MR3510523

[16] O. Kobayashi, *On a conformally invariant functional of the space of Riemannian metrics*, J. Math. Soc. Japan, **37** (1985), pp. 373–389. MR0792982

[17] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom., **29** (1989), pp. 665–683. MR0992334

[18] N. H. Kuiper, *On conformally-flat spaces in the large*, Ann. of Math. (2), **50** (1949), pp. 916–924. MR0031310

[19] H. B. Lawson and M. Michelsohn, *Spin Geometry*, Princeton University Press, 1989. MR1031992

[20] C. LeBrun, *Einstein metrics and Mostow rigidity*, Math. Res. Lett., **2** (1995), pp. 1–8. MR1312972

[21] C. LeBrun, *Einstein metrics, complex surfaces, and symplectic 4-manifolds*, Math. Proc. Cambridge Philos. Soc., **147** (2009), pp. 1–8. MR2507306

[22] C. LeBrun, *Einstein metrics, harmonic forms, and symplectic four-manifolds*, Ann. Global Anal. Geom., **48** (2015), pp. 75–85. MR3351078

[23] C. LeBrun, *Weyl curvature, Del Pezzo surfaces, and almost-Kähler geometry*, J. Geom. Anal., **25** (2015), pp. 1744–1772. MR3358072

[24] J. Lee and T. Parker, *The Yamabe problem*, Bull. Am. Math. Soc., **17** (1987), pp. 37–91. MR0888880

[25] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math., **42** (1977), pp. 225–237. MR0460343

[26] Y. Miyaoka, *Algebraic surfaces with positive indices*, in: Classification of Algebraic and Analytic Manifolds (Katata, 1982), vol. 39 of Progr. Math., Birkhäuser Boston, Boston, MA, 1983, pp. 281–301. MR0728611

[27] B. Moishezon and M. Teicher, *Simply-connected algebraic surfaces of positive index*, Invent. Math., **89** (1987), pp. 601–643. MR0903386

[28] U. Persson, C. Peters, and G. Xiao, *Geography of spin surfaces*, Topology, **35** (1996), pp. 845–862. MR1404912

[29] V. A. Rohlin, *New results in the theory of four-dimensional manifolds*, Doklady Akad. Nauk SSSR (N.S.), **84** (1952), pp. 221–224. MR0052101

[30] X. Roulleau and G. Urzúa, *Chern slopes of simply connected complex surfaces of general type are dense in* $[2, 3]$, Ann. of Math. (2), **182** (2015), pp. 287–306. MR3374961

[31] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., **20** (1984), pp. 478–495. MR0788292

[32] K. Sekigawa, *On some* 4*-dimensional compact Einstein almost Kähler manifolds*, Math. Ann., **271** (1985), pp. 333–337. MR0787184

[33] J. J. Sylvester, *On subvariants, i.e. semi-invariants to binary quantics of an unlimited order*, Amer. J. Math., **5** (1882), pp. 79–136. MR1505319

[34] C. H. Taubes, *The existence of anti-self-dual conformal structures*, J. Differential Geom., **36** (1992), pp. 163–253. MR1168984

[35] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett., **1** (1994), pp. 809–822. MR1306023

[36] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann., **281** (1988), pp. 123–133. MR0944606

[37] C. T. C. Wall, *On simply connected 4-manifolds*, J. Lond. Math. Soc., **39** (1964), pp. 141–149. MR0163324

[38] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A., **74** (1977), pp. 1798–1799. MR0451180

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