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Optimal asymptotic of the *J* functional with respect to the d_1 metric

Tamás Darvas[1] · Erin George[2] · Kevin Smith[3]

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Abstract
We obtain sharp inequalities between the large scale asymptotic of the J functional with respect to the d_1 metric on the space of Kähler metrics. Applications regarding the initial value problem for geodesic rays are presented.

Keywords Kahler metrics · J functional · Optimal inequality · Convex functions

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1 Introduction and main results

Let (X, ω) be a Kähler manifold of dimension n. We consider the space of Kähler metrics $\tilde{\omega}$ that are cohomologous to ω:

$$\mathcal{H} := \{\tilde{\omega} \text{ Kähler on } X \text{ and } [\tilde{\omega}]_{dR} = [\omega]_{dR}\}$$

By the $\partial\bar{\partial}$-lemma of Hodge theory, for all $\tilde{\omega} \in \mathcal{H}$ there exists $u \in C^\infty(X)$, unique up to a constant, such that $\tilde{\omega} = \omega_u := \omega + i\partial\bar{\partial}u$. Consequently, instead of looking at $\mathcal{H}$ directly, it makes sense to work with the space of *Kähler potentials* instead:

$$\mathcal{H}_\omega := \{u \in C^\infty(X) \text{ s.t. } \omega + i\partial\bar{\partial}u > 0\}.$$

✉ Tamás Darvas
tdarvas@umd.edu

Erin George
egeo@math.ucla.edu

Kevin Smith
kjs@math.columbia.edu

1 University of Maryland, College Park, USA

2 University of California, Los Angeles, USA

3 Columbia University, New York, USA

Clearly $\mathcal{H}_\omega \subset \mathrm{PSH}(X,\omega)$, hence all Kähler potentials are ω-plurisubharmonic (ω-psh). For a comprehensive treatment of ω-psh functions we refer to the recent book [25].

For a quick introduction to the basics of pluripotential theory in the context of Kähler geometry we refer to [13, Appendix A.1], [6, Section 2] or [25, Chapter 8].

A main direction of research is to find Kähler structures such that $\mathcal{H}_\omega$ admits constant scalar curvature Kähler (csck) metrics. Such metrics are minimizers of Mabuchi's K-energy functional $\mathcal{K} : \mathcal{H}_\omega \to \mathbb{R}$ [27], that can be defined by the Chen–Tian formula [8,34]:

$$\mathcal{K}(u) := \frac{1}{V}\int_X \left[\log\left(\frac{\omega_u^n}{\omega^n}\right)\omega_u^n - u\sum_{j=0}^{n-1}\mathrm{Ric}\,\omega\wedge\omega_u^j\wedge\omega^{n-j-1}\right] + \bar{S}I(u),$$

where $V = \int_X \omega^n$ is the total volume, $\mathrm{Ric}\,\omega$ is the Ricci curvature of the background metric ω, and $\bar{S} = \frac{1}{V}\int_X S_\omega\omega^n$ is the average scalar curvature of ω, which is also an invariant of the class $\mathcal{H}_\omega$ (see [13, eq. (4.40)]). Lastly, $I : \mathcal{H}_\omega \to \mathbb{R}$ is the *Monge–Ampère energy* (also called Aubin–Mabuchi or Aubin–Yau energy), one of the most basic functionals of Kähler geometry:

$$I(u) := \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X u\omega^j\wedge\omega_u^{n-j}.$$

For a more detailed analysis of I we refer to [6, page 111] and [13, Section 3.7]. Closely related to I, the J functional $J : \mathcal{H}_\omega \to \mathbb{R}$ is defined as follows:

$$J(u) = \frac{1}{V}\int_X u\omega^n - I(u).$$

Using Stokes' theorem, it can be showed that $J(u) \geq 0$, and in many ways J acts as a norm-like expression on $\mathcal{H}_\omega$. This aspect will be featured prominently in this work.

By definition, the space of Kähler potentials $\mathcal{H}_\omega$ is a convex open subset of $C^\infty(X)$, hence one can think of it as a trivial "Fréchet manifold". As such, one can introduce on $\mathcal{H}_\omega$ an L^1 type Finsler metric with relevant underlying geometry [11]. If $u \in \mathcal{H}_\omega$ and $\xi \in T_u\mathcal{H}_\omega \simeq C^\infty(X)$, then the L^1-length of ξ is given by the following expression:

$$\|\xi\|_u = \frac{1}{V}\int_X |\xi|\omega_u^n. \tag{1}$$

The corresponding L^2 type metric recovers the Riemannian geometry of Mabuchi [27] (independently discovered by Semmes [32] and Donaldson [22], studied later by Chen [7]). For more details we refer to [13, Chapter 3].

To the Finsler metric in (1) one associates a path length pseudo-metric $d_1(\cdot,\cdot)$. As proved in [11], d_1 is actually a metric and $(\mathcal{H}_\omega, d_1)$ is a geodesic metric space, whose abstract completion can be identified with $(\mathcal{E}^1, d_1)$, where $\mathcal{E}^1 \subset \mathrm{PSH}(X,\omega)$ is a space

of potentials introduced by Guedj–Zeriahi [24], with connections to earlier work of Cegrell.

Let us assume that momentarily that (X, J) does not admit global holomorphic vectorfieds. Tian conjectured that existence of csck metrics in $\mathcal{H}_\omega$ is equivalent to J-properness of the K-energy functional [35,36]:

$$\mathcal{K}(u) \geq C_J J(u) + D_J, \quad u \in \mathcal{H}_\omega \cap I^{-1}(0), \tag{2}$$

where C_J, D_J are some positive contants dependent only on (X, ω). Necessity of (2) was pointed out in [5], building on techniques of [18]. Chen and Cheng proved that existence of csck metrics in $\mathcal{H}_\omega$ is equivalent to

$$\mathcal{K}(u) \geq C_d d_1(0, u) + D_d, \quad u \in \mathcal{H}_\omega \cap I^{-1}(0), \tag{3}$$

where C_d, D_d are some positive contants dependent on (X, ω) [9]. As pointed out in [11,18], this last estimate is equivalent to (2), since there exist constants $m, M, D > 0$ such that

$$mJ(u) - D \leq d_1(0, u) \leq MJ(u) + D, u \in \mathcal{H}_\omega \cap I^{-1}(0). \tag{4}$$

By the analysis of [11] one can choose $M = 2$ and $m = 2^{-2n-6}$ in the above inequality (see Proposition 2.2 below). Since the C_J and C_d are linked to the uniform version of K-stability [3], it interesting to know what the optimal conversion rate is between these two constants. For this the optimal values of m and M need to be found in (4), and this is what we investigate in this paper, first on toric Kähler manifolds (X, ω):

Theorem 1.1 *Let (X, ω) be a toric Kähler manifold. Then there exists a constant $D > 0$ such that*

$$\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n J(u) - D \leq d_1(0, u) \leq 2J(u) + D, \quad u \in \mathcal{H}_\omega^T \cap I^{-1}(0), \tag{5}$$

and the constants multiplying $J(u)$ are sharp.

By the above result, at least in the toric case, the optimal constant m in (4) has linear decay with respect to dim X, and not exponential, as previously thought. Surprisingly, neither of the optimal constants depend on the choice of Kähler metric ω.

Similar flavour results were recently obtained by Sjöström Dyrefelt, who proved optimal inequalities between the asymptotics of other "J-type" functionals, when the so called J-equation has no solution [33, Theorem 1.2]. As pointed out in this latter work, inequalities like the ones in (5) often give new existence criteria for canonical metrics, allowing to sharpen the results of [9], something we hope to investigate in the future.

Using Legendre transforms the inequality in the above result can be transformed into a pair of sharp integral inequalities involving convex functions defined on a convex domain, that we now present:

Theorem 1.2 *Let $P \subset \mathbb{R}^n$ be a bounded open convex set. Then for any convex $\phi \in L^1(P)$, satisfying $\int_P \phi = 0$, the following sharp inequality holds:*

$$-\frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n \inf_P \phi \leq \frac{1}{\mu(P)}\int_P |\phi| d\mu \leq -2\inf_P \phi, \tag{6}$$

where all integrations are in terms of the Lebesque measure.

We prove sharpness of (6) by providing concrete extremizing potentials, and it would be interesting to characterize all such extremizers. Another interesting question would be to find the optimal constants in (6) for any fixed P. As these questions have potential applications in Kähler geometry, we hope to return to them in the future.

Despite its basic nature, we could not find the above inequality in the written convex geometry literature, even after consulting with experts [28]. To be clear, the emphasis here is not on novelty, but rather on the significance of (6) in the context of Kähler geometry.

We conjecture that Theorem 1.1 holds in case of general Kähler manifolds as well. To provide strong evidence for this, we prove the radial version of this expected result. For this we recall some terminology from [17] first.

Let $\{u_t\} \in \mathcal{R}^1$ be a geodesic ray $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ of $(\mathcal{E}^1, d_1)$, emanating from $u_0 = 0 \in \mathcal{H}_\omega$, and normalized by $I(u_t) = 0$, $t \geq 0$. Since the J functional is convex along geodesics, one can define its slope along geodesic rays (informally called the *radial J functional*):

$$J\{u_t\} := \lim_{t\to\infty} \frac{J(u_t)}{t}.$$

The L^1 speed of a ray $\{u_t\}_t$ is simply the quantity $d_1(0,u_1)$. We prove the following sharp inequality between the L^1 speed and the radial J functional:

Theorem 1.3 *Suppose that (X,ω) is a compact Kähler manifold and $\{u_t\}_t \in \mathcal{R}^1$. Then the following sharp inequality holds:*

$$\frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n J\{u_t\} \leq d_1(0,u_1) \leq 2J\{u_t\}. \tag{7}$$

Since rays are constant speed, notice that the middle term in the above inequality could have been replaced by the limit $\lim_{t\to\infty}\frac{d_1(0,u_t)}{t}$. In particular, (7) would instantly follow from the conjectured inequality (5) for general Kähler manifolds.

Remarkably, the proof of (7) rests on the ideas yielding (6), despite the fact that it works for general (non-toric) Kähler manifolds. We refer to Sect. 5 for more details.

Finally, we give an application for Theorem 1.3 regarding the initial value problem for geodesic rays in $\mathcal{H}_\omega$. In case ω is real analytic, and one is given a real analytic function $v : X \to \mathbb{R}$, by an application of the Cauchy-Kovalevskaya theorem, there exists a smooth geodesic $[0,\varepsilon_v) \ni t \to u_t \in \mathcal{H}_\omega$ such that $u_0 = 0$ and $\dot{u}_0 = v$ [30,31]. Of course, $t \to u_t$ is real analytic too, and it is still not known if such $t \to u_t$ can be extended to a geodesic ray (in the metric sense). In some instances, this can be

done, as pointed out in [1]. However, as we confirm below, there is plenty of analytic initial data, for which this fails to happen. What is more, we give a general condition that initial data of L^∞ rays $\{u_t\}_t \in \mathcal{R}^\infty$ (rays from $\mathcal{R}^1$ with bounded potenials) need to satisfy:

Corollary 1.4 *Suppose that $\{u_t\}_t \in \mathcal{R}^\infty$ is a geodesic ray, with bounded potentials. Then the initial tangent vector $v := \lim_{t\to 0} \frac{u_t}{t} \in L^\infty(X)$ satisfies the following sharp inequalities:*

$$\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \sup_X v \leq \int_X |v|\omega^n \leq 2 \sup_X v. \tag{8}$$

Since real analytic functions are dense among smooth ones [26, Proposition 2.1], one can find plenty of real analytic v for which either inequality in (8) fails, in particular such v can not be the initial tangent vector for a geodesic ray.

Interestingly, in the above result we can not pinpoint the time where the geodesic segment with (real analytic) initial tangent v can not be continued anymore. Related to this, in [15] the author has provided non-smooth geodesic segments that can not be continued at a specific point, however the information there could not be linked to the initial tangent.

2 The L^1 geometry of the space of Kähler potentials

In this short section we recall basic facts about the path length metric d_1 associated to the Finsler metric (1), with focus on the relationship with the J functional. For a survey on this topic, we refer to [13, Section 3.7]

Let us recall the following comparison theorem for the d_1 metric [13, Theorem 3.32]:

Theorem 2.1 *For any $u_0, u_1 \in \mathcal{H}_\omega$ we have*

$$d_1(u_0, u_1) \leq \frac{1}{V}\int_X |u_0 - u_1|\omega_{u_0}^n + \frac{1}{V}\int_X |u_0 - u_1|\omega_{u_1}^n \leq 2^{2n+6} d_1(u_0, u_1). \tag{9}$$

We also recall the following concrete formula for the path length metric d_1 ([13, Proposition 3.43]):

$$d_1(u, v) = I(u) + I(v) - 2I(P(u, v)), \quad u, v \in \mathcal{H}_\omega, \tag{10}$$

where $P(u, v)$ is the following "rooftop" envelope:

$$P(u, v) = \sup\{w \in \mathrm{PSH}(X, \omega) \text{ s.t. } w \leq u \text{ and } w \leq v\}.$$

More concretely, $P(u, v)$ is the greatest ω-psh function that lies below u and v. For properties of $P(u, v)$, we refer to [13, Section 2.4].

As already suggested by (10), there is an intimate relationship between the metric d_1 and the I functional. By inspection, $I(u+c) = I(u) + c$ for any $u \in \mathcal{H}_\omega$ and $c \in \mathbb{R}$. This allows for the following bijection between metrics and potentials:

$$\mathcal{H}_\omega \cap I^{-1}(0) \simeq \mathcal{H}.$$

What is more, the hypersurface $\mathcal{H}_\omega \cap I^{-1}(0)$ is totally geodesic within $\mathcal{H}_\omega$ (see the discussion near [13, (3.67)]). Finally, let us recall [13, Proposition 3.44], giving the best available asymptotic comparison between the J functional and the d_1 metric from the literature:

Proposition 2.2 *There exists $C = C(X,\omega) > 1$ such that*

$$2^{-2n-6} J(u) - C \leq d_1(0,u) \leq 2J(u) + C, \quad u \in \mathcal{H}_\omega \cap I^{-1}(0). \tag{11}$$

Proof Let $u \in \mathcal{H}_\omega \cap I^{-1}(0)$. By Theorem 2.1 we have

$$J(u) = \frac{1}{V}\int_X u\omega^n \leq \frac{1}{V}\int_X |u|\omega^n \leq 2^{2n+6} d_1(0,u),$$

implying the first estimate in (11). For the second estimate, since $I(u) = 0$, we have that $\sup_X u \geq 0$ and

$$d_1(0,u) = -2I(P(0,u)). \tag{12}$$

Clearly, $u - \sup_X u \leq \min(0,u)$, so $u - \sup_X u \leq P(0,u)$. Thus, $-\sup_X u = I(u - \sup_X u) \leq I(P(0,u))$. Combined with (12), we obtain that $d_1(0,u) = -2I(P(0,u)) \leq 2\sup_X u$. Finally, it is well known that $\sup_X u \leq \frac{1}{V}\int_X u\omega^n + C'$ for some $C'(X,\omega) > 1$ [13, Lemma 3.45], finishing the proof. □

3 Analysis on toric Kähler manifolds

In this short section we point out how the inequalities (5) and (6) are related. Much of the material here is based on [20, Section 6] and [38], and we invite the reader to consult these works for a more thorough treatment.

We say that (X,ω) is a toric Kähler manifold of complex dimension n if one can embed $(\mathbb{C}^*)^n$ into X such that the complement of $(\mathbb{C}^*)^n$ inside X is Zariski closed. Additionally, we ask that the trivial action of $\mathbb{T}^n := (S^1)^n$ on $(\mathbb{C}^*)^n$ extends to X, and the Kähler form ω is $\mathbb{T}^n$-invariant.

Using the fact that ω is $\mathbb{T}^n$-invariant we get that

$$\omega = i\partial\bar{\partial}(\psi_0 \circ L) \text{ on } (\mathbb{C}^*)^n, \tag{13}$$

where $\psi_0 \in C^\infty(\mathbb{R}^n)$ and $L(z_1, z_2, \ldots, z_n) = (\log|z_1|, \log|z_2|, \ldots, \log|z_n|) \in \mathbb{R}^n$. Since $\psi_0 \circ L$ is psh on $(\mathbb{C}^*)^n$, it follows that ψ_0 has to be strictly convex on $\mathbb{R}^n$ (see (16) below), and we may choose $\psi_0(0) = 0$.

By $\mathcal{H}^T$ we will denote the metrics $\omega' \in \mathcal{H}$ that are torus invariant, i.e., $(S^1)^n$ acts by isometries on ω'. The corresponding space of toric potentials will be denoted by $\mathcal{H}^T_\omega$.

Given $u \in \mathcal{H}^T_\omega$, comparing with (13), we can introduce the following potential

$$\psi_u := \psi_0 + u \circ E,$$

where $E(x) = E(x_1, x_2, \ldots, x_n) := (e^{x_1}, e^{x_2}, \ldots, e^{x_n})$, $x \in \mathbb{R}^n$. The point here is that $\omega_u = i\partial\bar{\partial}\psi_u \circ L$ on $(\mathbb{C}^*)^n$.

The Legendre transform Given $\omega_u \in \mathcal{H}^T_\omega$, it follows from a result of Atiyah–Guillemin—(see [10, Chapter 27]) that the "moment map" $\nabla\psi_u : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and sends $\mathbb{R}^n$ to $P := \text{Im}\,\nabla\psi_u$, which is a convex bounded polytope, independent of u, that can be described in the following manner:

$$P := \{l_j(s) \geq 0,\ 1 \leq j \leq d\} \subset \mathbb{R}^n,$$

where $l_j(s) = \langle s, v_j\rangle - \lambda_j$ are affine functions that determine the sides of P.

Though we will not use it, by a theorem of Delzant, P satisfies a number of properties (it is simple, i.e., there are n edges meeting at each vertex; it is rational, i.e., the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $0 \leq t < \infty$, where $u_i \in \mathbb{Z}^n$; it is smooth, i.e., these $u_1, \ldots, u_n$ can be chosen to be a basis of $\mathbb{Z}^n$). In fact, such Delzant polytopes P determine toric Kähler structures (X, ω) uniquely (see [2] and [10, Chapter 28]).

Since $\psi_u : \mathbb{R}^n \to \mathbb{R}$ is convex, we can take the Legendre transform of ψ_u and obtain another convex function ϕ_u, with possible values equal to $+\infty$:

$$\phi_u(s) = \psi_u^*(s) := \sup_{x\in\mathbb{R}^n} \big(\langle s, x\rangle - \psi_u(x)\big),\quad s \in \mathbb{R}^n.$$

Since $\text{Im}\,\nabla\psi_u = P$, it follows that $\phi_u(s)$ is finite if and only if $s \in P$. Also, by the involutive property of Legendre transforms, for all $x \in \mathbb{R}^n$ we will have

$$\phi_u^* = \psi_u^{**} = \psi_u,$$
$$\phi_u(\nabla\psi_u(x)) = \langle x, \nabla\psi_u(x)\rangle - \psi_u(x) \quad \text{and} \quad \nabla\psi_u(x) = s \Leftrightarrow \nabla\phi_u(s) = x.$$

Summarizing, the Legendre transform $u \to \psi_u^* = \phi_u$ gives a one-to-one correspondence between elements of $\mathcal{H}^T_\omega$ and the class $\mathcal{C}(P)$:

$$\mathcal{C}(P) := \{f : P \to \mathbb{R} \text{ is convex and } f - \phi_0 \in C^\infty(\overline{P})\},$$

where $\phi_0 = \psi_0^*$. In addition, as pointed out in [23, Proposition 4.5], there is a one-to-one correspondence between $(S^1)^n$-invariant elements of $\text{PSH}(X, \omega) \cap L^\infty$ and convex functions $f : P \to \mathbb{R}$ for which $f - \phi_0$ is only bounded on P. In particular, $\phi_0 \in L^\infty(P)$.

The L^1 Finsler geometry of toric metrics We now describe the L^1 geometry of $\mathcal{H}_\omega$ restricted to $\mathcal{H}_u^T$, the space of potentials for $\mathbb{T}^n$-invariant Kähler metrics.

Let $[0,1] \ni t \to u_t \in \mathcal{H}_\omega^T$ be a smooth curve connecting $u_0, u_1 \in \mathcal{H}_\omega^T$. Taking the Legendre transform of the potentials ψ_{u_t} we arrive at

$$\phi_{u_t}(s) := \sup_{x\in\mathbb{R}^n} \left\{\langle x, s\rangle - \psi_{u_t}(x)\right\} = \langle x_t, s\rangle - \psi_{u_t}(x_t),\ s \in \mathbb{R}^n, \tag{14}$$

where $x_t = x_t(s)$ is such that $\nabla\psi_{u_t}(x_t) = s$. Taking derivatives of this identity with respect to t yields

$$\nabla^2\psi_{u_t}\cdot\dot{x}_t = -\nabla\dot{\psi}_{u_t}.$$

Taking t-derivative of (14) and using this formula we arrive at

$$\dot{\phi}_{u_t}(s) = -\dot{\psi}_{u_t}(x_t). \tag{15}$$

To continue, we observe that

$$\omega_{u_t} = \frac{\partial^2(\psi_{u_t}\circ L)}{\partial z_i \partial\overline{z_j}} = \frac{1}{4}\frac{1}{z_i\overline{z_j}}\cdot\frac{\partial^2\psi_{u_t}}{\partial x_i\partial x_j}\circ L \ \text{ on } \ (\mathbb{C}^*)^n. \tag{16}$$

Thus on $(\mathbb{C}^*)^n$ we have

$$\omega_{u_t}^n = (i\partial\bar\partial\psi_{u_t}\circ L)^n = n!\det\left(\frac{\partial^2(\psi_{u_t}\circ L)}{\partial z_i\partial\overline{z_j}}\right) i^n(dz_1\wedge d\bar z_1)\wedge\cdots\wedge(dz_n\wedge d\bar z_n) =$$
$$= \frac{n!}{2^n}\frac{1}{\Pi_j|z_j|^2}\cdot\big(MA_{\mathbb{R}}(\psi_{u_t})\circ L\big)(dx_1\wedge dy_1)\wedge\cdots\wedge(dx_n\wedge dy_n), \tag{17}$$

where $MA_{\mathbb{R}}(\psi_{u_t}) = \det\left(\frac{\partial^2\psi_{u_t}}{\partial x_i\partial x_j}\right)$ denotes the real Monge-Ampère measure of the (smooth) convex function ψ_{u_t}. As a result, after using polar coordinates in each $\mathbb{C}^*$ component ($dx\wedge dy = rdr\wedge d\theta$), we conclude that

$$\int_X|\dot u_t|\omega_{u_t}^n = \int_{(\mathbb{C}^*)^n}|\dot\psi_{u_t}\circ L|(i\partial\bar\partial\psi_{u_t}\circ L)^n = \pi^n n!\int_{\mathbb{R}^n}|\dot\psi_{u_t}|^p MA_{\mathbb{R}}(\psi_{u_t}).$$

By (15), $\dot\psi_{u_t}(x) = -\dot\phi_{u_t}(\nabla\psi_{u_t}(x))$. Moreover , $MA_{\mathbb{R}}(\psi_{u_t}) = (\nabla\psi_{u_t})^* d\mu(s)$, where μ is the Lebesgue measure. Therefore, a change of variables $s = \nabla\psi_{u_t}(x)$ yields

$$\int_X|\dot u_t|\omega_{u_t}^n = \pi^n n!\int_{\mathbb{R}^n}|\dot\psi_{u_t}|MA_{\mathbb{R}}(\psi_{u_t}) = \pi^n n!\int_P|\dot\phi_{u_t}(s)|d\mu(s), \tag{18}$$

Hence the Legendre transform sends the L^1 geometry of $\mathcal{H}_\omega^T$ to the flat L^1 geometry of convex functions on P. In the particular case when $u_t := t$, we obtain the following

useful formula about volumes:

$$\int_X \omega^n = V = \pi^n n! \int_P d\mu. \tag{19}$$

Regarding the underlying path length metrics, (18) has the following important consequence:

Theorem 3.1 *Suppose* $u_0, u_1 \in \mathcal{H}^T_\omega$. *Then*

$$d_1(u_0, u_1) = \frac{1}{\mu(P)} \int_P |\phi_{u_0}(s) - \phi_{u_1}(s)| d\mu(s), \tag{20}$$

where $\mu(P)$ *is the Lebesgue measure of* P.

Proof By definition $d_1(u_0, u_1)$ is the infimum of the L^1 Mabuchi length of smooth curves $t \to u_t$ connecting u_0 and u_1:

$$d_1(u_0, u_1) = \inf_{t \to u_t} \frac{1}{V} \int_0^1 \int_X |\dot{u}_t| \omega^n_{u_t}.$$

Similarly, the integral $\int_P |\phi_{u_0} - \phi_{u_1}| d\mu$ is equal to the infimum of flat L^1 length of smooth curves $t \to \phi_t$ of convex functions on P, connecting ϕ_{u_0} and ϕ_{u_1}:

$$\int_P |\phi_{u_0} - \phi_{u_1}| d\mu = \inf_{t \to \phi_t} \int_0^1 \int_P |\dot{\phi}_t| d\mu.$$

Comparing with (18) and (19), the identity (20) follows. □

The I and J functionals of toric metrics In this paragraph we analyze the I and the J functionals in terms of the Legendre transform.

As it turns out, the Monge–Ampère energy is essentially the Lebesgue integral, after applying the Legendre transform. Indeed, let $[0, 1] \ni t \to v_t \in \mathcal{H}^T_\omega$ be any smooth curve connecting $v_0 = 0$ and $v_1 = u$. We then obtain the following formula:

$$\begin{aligned} I(u) = I(u) - I(0) &= \int_0^1 \frac{d}{dt} I(v_t) dt = \int_0^1 \frac{1}{V} \int_X \dot{v}_t \omega^n_{v_t} dt = \int_0^1 \frac{1}{V} \int_{(\mathbb{C}^*)^n} \dot{v}_t \omega^n_{v_t} dt. \\ &= \int_0^1 \int_{(\mathbb{C}^*)^n} \frac{1}{V} \dot{\psi}_{v_t} \circ L (i\partial\bar{\partial}\psi_{v_t} \circ L)^n dt \\ &= \frac{1}{\mu(P)} \int_0^1 \int_{\mathbb{R}^n} \dot{\psi}_{v_t} MA_{\mathbb{R}}(\psi_{v_t}) dt \\ &= \frac{-1}{\mu(P)} \int_0^1 \int_P \dot{\phi}_{v_t}(s) d\mu(s) dt = \frac{-1}{\mu(P)} \int_P (\phi_u(s) - \phi_0(s)) d\mu(s), \end{aligned} \tag{21}$$

where in the second line we used (17), in the third line we used again the change of variables $s = \nabla\psi_{v_t}$, similar to (18), and in the last line we used (15). Since

$\phi_0 \in L^\infty(P)$ ([23, Proposition 4.5]), we conclude that there exists $C = C(X, \omega) > 0$ such that

$$\frac{-1}{\mu(P)} \int_P \phi_u d\mu - C \leq I(u) \leq \frac{-1}{\mu(P)} \int_P \phi_u d\mu + C, \quad u \in \mathcal{H}^T_\omega. \tag{22}$$

A closed formula for the J energy is likely not available in terms of the Legendre transform, but we can express its magnitude in relatively simple terms, which is sufficient for our later analysis:

Proposition 3.2 *There exists $C := C(X, \omega) > 0$ such that for all $u \in \mathcal{H}^T_\omega$ with $I(u) = 0$ we have*

$$-\inf_P \phi_u - C \leq J(u) = \frac{1}{V} \int_X u\omega^n \leq -\inf_P \phi_u + C.$$

The argument is adapted from [38, Lemma 2.2].

Proof Notice that $-\inf_P \phi_u = \psi_u(0)$, hence it is enough to prove existence of $C = C(X, \omega) > 0$ such that:

$$\psi_u(0) - C \leq \frac{1}{V} \int_{(\mathbb{C}^*)^n} u\omega^n \leq \psi_u(0) + C.$$

Since $\psi_u(0) = \psi_0(0) + u(E(0)) = u(E(0)) \leq \sup_X u$, by [13, Lemma 3.45], the first estimate is trivial.

Next we show that there exists $C := C(X, \omega) > 0$ such that for any $v \in \mathrm{PSH}(X, \omega)$ and constant $h > 0$ we have

$$\int_{\{v < \sup_X v - h\}} \omega^n \leq \frac{C}{h}. \tag{23}$$

Indeed, from [13, Lemma 3.45] we have that

$$\begin{aligned} V(\sup_X v - C(X, \omega)) &\leq \int_X v\omega^n = \int_{\{v < \sup_X v - h\}} v\omega^n + \int_{\{v \geq \sup_X v - h\}} v\omega^n \\ &\leq (\sup_X v - h) \int_{\{v < \sup_X v - h\}} \omega^n + \sup_X v \int_{\{v \geq \sup_X v - h\}} \omega^n. \end{aligned}$$

This implies (23). Next, since $\mathrm{Im}\, \nabla\psi_u = \mathrm{Im}\, \nabla\psi_0 = P$ and $\nabla u \circ E = \nabla\psi_u - \nabla\psi_0$, it follows that $\nabla u \circ E$ is uniformly bounded on the unit ball $B(0, 1) \subset \mathbb{R}^n$. This implies that

$$|u(E(x)) - u(E(0))| \leq C(P), \quad x \in B(0, 1). \tag{24}$$

By (23), there exists a constant $h := h(X, \omega)$ such that $\{u \circ E \geq \sup_X u - h\}$ intersects $B(0, 1)$. This together with (24), gives the desired inequality: $\psi_u(0) = u(E(0)) \geq \sup_X u - C \geq \frac{1}{V} \int_X u\omega^n - C$. □

4 A sharp inequality for convex functions

In this section we prove the following sharp double inequality about convex functions, stated in Theorem 1.2:

Theorem 4.1 *Let $P \subset \mathbb{R}^n$ be a bounded open convex set. Then for $f \in L^1(P)$ convex and satisfying $\int_P \phi d\mu = 0$, the following sharp inequalities hold:*

$$-\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \inf_P \phi \leq \frac{1}{\mu(P)} \int_P |\phi| d\mu \leq -2 \inf_P \phi, \tag{25}$$

where the integration is in terms of the Lebesgue measure.

Proof First we argue the second inequality. Let

$$P_- := \{x \in P : \phi \leq 0\} \quad \text{and} \quad P_+ := \{x \in P : \phi > 0\} \tag{26}$$

Since $\int_P \phi = \int_{P_-} \phi + \int_{P_+} \phi = 0$, we have

$$\int_P |\phi| = -\int_{P_-} \phi + \int_{P_+} \phi = -2 \int_{P_-} \phi.$$

Furthermore, $\int_{P_-} \phi \geq \mu(P_-) \inf_{P_-} \phi \geq \mu(P) \inf_P \phi$, and the second estimate follows.

We move on to the first estimate. For all $a \in \mathbb{R}$, let

$$P_a := \{x \in P : \phi(x) \leq a\}.$$

Through scaling of ϕ, we can assume without loss of generality that $\inf_P \phi = -1$. We first claim that

$$\int_P |\phi| \geq -\frac{2}{n+1} \mu(P_-) \inf_P \phi. \tag{27}$$

Through translation of ϕ and P, we can momentarily assume that $\phi(0) = -1 + \varepsilon$ for an arbitrarily small $\varepsilon > 0$. By convexity of ϕ, if $-1 + \varepsilon < a < b$, we have

$$P_b \subset \frac{b+1-\varepsilon}{a+1-\varepsilon} P_a \text{ and } \mu(P_b) \leq \left(\frac{b+1-\varepsilon}{a+1-\varepsilon}\right)^n \mu(P_a).$$

But translating ϕ and P does not change the measure of any of the sets involved, so the above relation holds for all $\varepsilon > 0$ regardless of the actual value of $\phi(0)$. Thus,

$$\mu(P_b) \leq \left(\frac{1+b}{1+a}\right)^n \mu(P_a) \text{ for any } -1 < a < b. \tag{28}$$

Since $\int_P \phi = \int_{P_+} \phi + \int_{P_-} \phi = 0$, we have $\int_P |\phi| = 2\int_{P_-} |\phi|$, allowing us to bound $\int_P |\phi|$ in the following manner, proving the claim:

$$\int_P |\phi| = 2\int_{P_-} |\phi| = 2\int_{-1}^{0} \mu\left(P_x\right) dx \geq 2\int_{-1}^{0} (1+x)^n \mu\left(P_-\right) dx = \frac{2}{n+1}\mu(P_-),$$

where we used that $\int f d\mu = \int_0^{+\infty} \mu\{f \geq t\}dt$ for any non-negative μ-measurable f, estimate (28) for $-1 < x < 0$, and the fact that $P_0 = P_-$.

Let $b > 0$. Next we estimate $\int_{P_+} \phi$ similarly, applying (28) for $-1 < 0 < x$:

$$\begin{aligned}\int_{P_+} \phi \geq \int_0^b \mu\left(P \setminus P_x\right) dx &= \int_0^b \mu\left(P\right) - \mu\left(P_x\right) dx \\ &\geq \int_0^b \mu\left(P\right) - (1+x)^n \mu\left(P_-\right) dx \\ &= b\mu(P) - \frac{(b+1)^{n+1}}{n+1}\mu(P_-) + \frac{1}{n+1}\mu(P_-)\end{aligned}$$

We then let $b = \frac{1}{n}$ and $\frac{1}{2}\int_P |\phi| = \int_{P_+} |\phi| = \int_{P_-} |\phi| = A\mu(P_-)$ for some $A > 0$. This gives

$$A\mu(P_-) = \int_{P_+} \phi \geq \frac{1}{n}\mu(P) - \frac{1}{n} \cdot \left(\frac{n+1}{n}\right)^n \mu(P_-) + \frac{1}{n+1}\mu(P_-),$$

implying

$$\int_P |\phi| = 2A\mu(P_-) \geq \frac{2A}{nA + \left(\frac{n+1}{n}\right)^n - \frac{n}{n+1}}\mu(P).$$

The right-hand side is an increasing function of A and by (27) we know $A \geq \frac{1}{n+1}$. This means the right hand side is minimized at this value, so

$$\int_P |\phi| \geq \frac{\frac{2}{n+1}}{\left(\frac{n+1}{n}\right)^n}\mu(P) = \frac{2}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^n \mu(P).$$

□

Finally, we address the tightness of the bounds in the previous theorem.

Proposition 4.2 *For all $n > 0$ there exists an open bounded convex subset $P \subset \mathbb{R}^n$ and convex function $\phi : P \to \mathbb{R}$ integrating to zero such that*

$$\frac{1}{\mu(P)}\int_P |\phi| d\mu = -\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \inf_P \phi.$$

Proof Fix n. Take $P \subset \mathbb{R}^n$ to be the interior of the simplex with vertex set

$$\left\{(0,\ldots,0),\left(\frac{n+1}{n},0,\ldots,0\right),\ldots,\left(0,\ldots,0,\frac{n+1}{n}\right)\right\}$$

and $\phi : P \to \mathbb{R}$ to be the function defined by

$$\phi(x_1,\ldots,x_n) = -1 + x_1 + \cdots + x_n.$$

It is clear that P and ϕ are convex and that $\inf_P \phi = -1$. To evaluate the integrals of ϕ and $|\phi|$, first let S_x to be the simplex with vertices

$$\{(0,\ldots,0),(x,0,\ldots,0),\ldots,(0,\ldots,0,x)\}.$$

The volume of S_x is just $\frac{x^n}{n!}$. This is the region where ϕ takes values less than or equal to $x-1$. We can use the same technique from the previous theorem to show

$$\begin{aligned}\int_{P_+}\phi &= \int_1^{\frac{n+1}{n}}\mu(P\setminus S_x)dx = \int_1^{\frac{n+1}{n}}\frac{1}{n!}\left(\left(\frac{n+1}{n}\right)^n - x^n\right)dx\\ &= \frac{1}{n}\cdot\frac{1}{n!}\left(\frac{n+1}{n}\right)^n - \frac{1}{(n+1)!}\left(\frac{n+1}{n}\right)^{n+1} + \frac{1}{(n+1)!}\\ &= \frac{1}{(n+1)!}\end{aligned}$$

and

$$\int_{P_-} -\phi = \int_0^1 \mu(S_{(1-x)})dx = \int_0^1 \frac{(1-x)^n}{n!}dx = \frac{1}{(n+1)!},$$

where P_+ and P_- are again the portions of the domain where ϕ is positive and negative, respectively. Combined these yield $\int_P \phi = 0$ and $\int_P |\phi| = \frac{2}{(n+1)!}$.

As $\mu(P)$ is just the volume of $S_{\frac{n+1}{n}}$, it follows that

$$\frac{1}{\mu(P)}\int_P |\phi| = \frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n.$$

□

Proposition 4.3 *For all $n > 0$ there exists an open bounded convex set $P \subset \mathbb{R}^n$ such that*

$$\sup_\phi \frac{\int_P |\phi|}{-\mu(P)\inf_P \phi} = 2.$$

Proof First, suppose $n = 1$ and let $P = (0, 1)$. Now, for all integers m, let

$$\phi_m(x) = \begin{cases} 2m - 1 - 2m^2 x & \text{if } x < \frac{1}{m} \\ -1 & \text{if } x \geq \frac{1}{m}. \end{cases}$$

ϕ_m is convex, and it can be seen that $\int_P \phi_m = 0$. It is also the case that

$$-\frac{\int_P |\phi_m|}{\mu(P) \inf_P \phi_m} = -2 \int_{P_-} \phi_m = 2 - \frac{2}{m}.$$

Letting $m \to \infty$ shows that the supremum for $n = 1$ is indeed 2.

For $n > 1$, let $P_n = (0, 1)^n$ and $\phi_{n,m}(x_1, \ldots, x_n) = \phi_m(x_1)$. $\phi_{n,m}$ is still convex and we still have that $\int_{P_n} \phi_{n,m} = 0$ and $\int_{P_n} |\phi_{n,m}| = 2 - \frac{2}{m}$. Again, we can let $m \to \infty$ to show that the supremum is 2 for all n. □

5 Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1 Using Theorem 3.1 and Proposition 3.2 the inequalities of (5) are equivalent to the inequalities of (6). Theorem 1.1 is now a consequence of Theorem 4.1 □

In proving Theorem 1.3 we heavily rely on the formalism developed in [17] regarding the metric space of geodesic rays.

By $\mathcal{E}^1 \subset \mathrm{PSH}(X, \theta)$ we denote the space of finite energy pontentials: $u \in \mathcal{E}^1$ if $\int_X \theta_u^n = \int_X \omega^n$ (where θ_u^n is the non-pluripolar complex Monge–Ampère measure, defined in [24, Section 1]), moreover $\int_X |u| \theta_u^n < \infty$.

By $\mathcal{R}^1$ we denote the space of L^1 Mabuchi geodesic rays $[0, \infty) \ni t \to u_t \in \mathcal{E}^1$ that are normalized by $u_0 = 0$ and $I(u_t) = 0$, $t \geq 0$.

It was shown in [4, Proposition 5.1] that $t \to d_1(u_t, v_t)$ is convex for any $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}^1$. In [17] this was used to definte the following metric:

$$d_1^c(\{u_t\}_t, \{v_t\}_t) = \lim_{t \to \infty} \frac{d_1(u_t, v_t)}{t}.$$

We know that $(\mathcal{R}^1, d_1^c)$ is complete [17, Theorem 1.3 and 1.4], moreover the space of normalized bounded geodesic rays $\mathcal{R}^\infty$ is dense in $\mathcal{R}^1$ [17, Theorem 1.5].

Due to this and the next result it is enough to prove Theorem 1.3 for bounded geodesic rays:

Lemma 5.1 *Suppose that $\{u_t^j\}_t, \{u_t\}_t \in \mathcal{R}^1$ such that $d_1^c(\{u_t^j\}, \{u_t\}_t) \to 0$. Then $J\{u_t^j\} \to J\{u_t\}$.*

Proof By the lemma below we have that $J\{u_t\} = \sup_X u_1$ and $J\{u_t^j\} = \sup_X u_1^j$. Since $t \to d_1(u_t^j, u_t)$ is convex, we have that $d_1(u_1^j, u_1) \leq d_1^c(\{u_t^j\}_t, \{u_t^j\}_t) \to 0$.

[11, Theorem 5] gives that $\|u_1^j - u_1\|_{L^1} \to 0$. Hartogs' lemma [25, Proposition 8.4] now implies that $J\{u_t^j\} = \sup_X u_1^j \to \sup_X u_1 = J\{u_t\}$. □

The radial J functional can be expressed in very simple terms:

Lemma 5.2 *For $\{u_t\}_t \in \mathcal{R}^1$ we have that $J\{u_t\} = \frac{\sup_X u_l}{l}$ for any $l > 0$. Moreover, in case $\{u_t\}_t \in \mathcal{R}^\infty$, we also have that $J\{u_t\} = \sup_X \dot{u}_0$, where $\dot{u}_0 := \lim_{t\to 0} \frac{u_t}{t}$.*

Proof We have that $J(u_t) = \frac{1}{V}\int_X u_t\omega^n - I(u_t) = \frac{1}{V}\int_X u_t\omega^n$. By [17, Lemma 2.2] we obtain that

$$J\{u_t\} = \lim_{t\to\infty} \frac{J(u_t)}{t} = \lim_{t\to\infty} \frac{\sup_X u_t}{t}.$$

Since $t \to \sup_X u_t$ is well known to be linear [3], [19, Lemma 3.2], the first statement follows.

In case $\{u_t\}_t \in \mathcal{R}^\infty$, by [12, Theorem 1, Lemma 3.1] know that $\sup_X \dot{u}_t^- = \frac{\sup_X u_t}{t} = \sup_X u_1,\ t > 0$, proving the second statement.

By t-convexity we have $\sup_X \dot{u}_0 \le \sup_X \dot{u}_t^- = \sup_X u_1,\ t > 0$. Let $x \in X$ such that $u_1(x) = \sup_X u_1$. By t-convexity we have that

$$\sup_X u_1 = u_1(x) - u_0(x) = \int_0^1 \dot{u}_t^-(x)dt.$$

Since $t \to \dot{u}_t^-(x)$ is increasing, and $\dot{u}_t^- \le \sup_X u_1$, we obtain that $u_t(x) = t\sup_X u_1,\ t \in [0, 1]$, giving $\sup_X \dot{u}_0 = \sup_X u_1$. □

Lemma 5.3 *For $\{u_t\}_t \in \mathcal{R}^1$ we have that $d_1(0, u_1) = \int_X |\dot{u}_0|\omega^n$ and $0 = I(u_1) = \int_X \dot{u}_0\omega^n$.*

Proof That $d_1(0, u_1) = \int_X |\dot{u}_0|\omega^n$ follows from [5, Lemma 3.4]. The argument of [5, Lemma 3.4] is seen to imply $I(u_1) = \int_X \dot{u}_0\omega^n$. □

Given $v \in \mathrm{PSH}(X, \theta)$, we say that v is a *model potential* if $v = P[v]$, where

$$P[v] := \mathrm{usc}(\sup\{h \in \mathrm{PSH}(X, \omega),\ h \le 0,\ h \le v + C,\ C \in \mathbb{R}\}).$$

For more on model potentials we refer to [14], as well as [19, Theorem 3.7], where a connection with finite energy rays is made, used in the argument of the next lemma.

Lemma 5.4 *For any $\{u_t\}_t \in \mathcal{R}^1$ and $b \le a \le \sup_X \dot{u}_0$ we have that*

$$\int_{\{\dot{u}_0 \ge a\}} \omega^n \le \int_{\{\dot{u}_0 \ge b\}} \omega^n \le \frac{(\sup_X \dot{u}_0 - b)^n}{(\sup_X \dot{u}_0 - a)^n} \int_{\{\dot{u}_0 \ge a\}} \omega^n.$$

Proof The argument uses the formalism of Legendre transforms for geodesic rays going back to [29], further developed in [19].

The first estimate is trivial. For the second estimate we consider the Legendre transform of $\{u_t\}_t \in \mathcal{R}^1$:

$$\hat{u}_\tau := \inf_{t\geq 0}(u_t - t\tau), \quad \tau \in \mathbb{R}.$$

It is shown in [19, Theorem 3.7] that $\hat{u}_\tau \in \mathrm{PSH}(X,\omega)$ is a model potential. As a result, by [21, Theorem 1], [14, Theorem 3.8] we have that $\int_X \omega^n_{\hat{u}_\tau} = \int_{\{\hat{u}_\tau=0\}} \omega^n$. Moreover, due to basic properties of Legendre transforms $\{\dot{u}_0 \geq \tau\} = \{\hat{u}_\tau = 0\}$, in particular,

$$\int_X \omega^n_{\hat{u}_\tau} = \int_{\{\dot{u}_0\geq\tau\}} \omega^n. \tag{29}$$

Let now $b \leq a \leq \sup_X \dot{u}_0$. Since $\tau \to \hat{u}_\tau$ is concave, we know that $\tau \to \left(\int_X \omega^n_{\hat{u}_\tau}\right)^{\frac{1}{n}}$ is concave as well ([16, Theorem B] and [37, Theorem 1.2]). As a result, we can write that

$$\frac{(\sup_X \dot{u}_0 - a)}{(\sup_X \dot{u}_0 - b)}\left(\int_X \omega^n_{\hat{u}_b}\right)^{\frac{1}{n}} + \frac{(a-b)}{(\sup_X \dot{u}_0 - b)}\left(\int_X \omega^n_{\hat{u}_{\sup_X \dot{u}_0}}\right)^{\frac{1}{n}} \leq \left(\int_X \omega^n_{\hat{u}_a}\right)^{\frac{1}{n}}.$$

Comparing with (29), the result follows. □

Finally we arrive at the main result of this section, whose proof will be reminiscent to that of Theorem 1.2:

Theorem 5.5 *Suppose that (X,ω) is a compact Kähler manifold and $\{u_t\}_t \in \mathcal{R}^1$. Then the following sharp inequality holds:*

$$\frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n J\{u_t\} \leq d_1(0,u_1) \leq 2J\{u_t\}. \tag{30}$$

Proof Due to Lemma 5.1 and [17, Theorem 1.5], it is enough to prove the inequalities for $\{u_t\}_t \in \mathcal{R}^\infty$ satisfying $I\{u_t\} = 0$. Due to Lemma 5.2, for such rays we have to argue the following estimates:

$$\frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n \sup_X \dot{u}_0 \leq \frac{1}{V}\int_X |\dot{u}_0|\omega^n \leq 2\sup_X \dot{u}_0.$$

Using re-scaling in time, we can further assume that $\sup_X \dot{u}_0 = 1$, hence it is enough to argue that

$$\frac{2}{n+1}\cdot\left(\frac{n}{n+1}\right)^n \leq \frac{1}{V}\int_X |\dot{u}_0|\omega^n \leq 2. \tag{31}$$

We first argue the second estimate which is much more simple. Let $X_- := \{\dot{u}_0 < 0\}$ and $X_+ := \{\dot{u}_0 \geq 0\}$. Since $I(u_1) = \int_X \dot{u}_0 \omega^n$ we have that

$$\frac{1}{V}\int_X |\dot{u}_0|\omega^n = \frac{2}{V}\int_{X_+} \dot{u}_0\omega^n \leq 2\sup_X \dot{u}_0.$$

To address the first estimate, we make the following preliminary calculation:

$$\begin{aligned}\int_X |\dot{u}_0|\omega^n &= 2\int_{X^+} \dot{u}_0\omega^n = 2\int_0^1 \int_{\{\dot{u}_0\geq x\}} \omega^n dx \\ &\geq 2\int_0^1 (1-x)^n \int_{\{\dot{u}_0\geq 0\}} \omega^n dx \\ &= \frac{2}{n+1}\int_{\{\dot{u}_0\geq 0\}} \omega^n, \end{aligned} \tag{32}$$

where we used that $\int f d\mu = \int_0^{+\infty} \mu\{f \geq t\}dt$ for any non-negative μ-measurable f and Lemma 5.4 for the parameters $0 \leq x \leq \sup_X \dot{u}_0 = 1$.

Let $b > 0$. To estimate $\int_{X_-} |\dot{u}_0|\omega^n$, we can use a similar technique to the above:

$$\begin{aligned}\int_{X_-} |\dot{u}_0|\omega^n &\geq \int_0^b \int_{X\setminus\{\dot{u}_0\geq x\}} \omega^n dx = \int_0^b \left(V - \int_{\{\dot{u}_0\geq x\}} \omega^n\right)dx \\ &\geq \int_0^b \left(V - (1+x)^n \int_{\{\dot{u}_0\geq 0\}} \omega^n\right)dx \\ &= bV - \frac{(b+1)^{n+1}}{n+1}\int_{\{\dot{u}_0\geq 0\}} \omega^n + \frac{1}{n+1}\int_{\{\dot{u}_0\geq 0\}} \omega^n,\end{aligned}$$

where in the second line we used Lemma 5.4 again, for the parameters $-x \leq 0 \leq \sup_X \dot{u}_0 = 1$.

We now let $b = \frac{1}{n}$ and $\frac{1}{2}\int_X |\dot{u}_0|\omega^n = \int_{X_+} |\dot{u}_0|\omega^n = -\int_{X_-} \dot{u}_0\omega^n = A\int_{\{\dot{u}_0\geq 0\}} \omega^n$ for some $A > 0$. This gives

$$A\int_{\{\dot{u}_0\geq 0\}} \omega^n \geq \frac{1}{n}V - \frac{1}{n}\cdot\left(\frac{n+1}{n}\right)^n \int_{\{\dot{u}_0\geq 0\}} \omega^n + \frac{1}{n+1}\int_{\{\dot{u}_0\geq 0\}} \omega^n,$$

implying

$$\int_X |\dot{u}_0|\omega^n = 2A\int_{\{\dot{u}_0\geq 0\}} \omega^n \geq \frac{2A}{nA + \left(\frac{n+1}{n}\right)^n - \frac{n}{n+1}}V.$$

The right-hand side is an increasing function of A and by (32) we know $A \geq \frac{1}{n+1}$. This means the right hand side is minimized at this value, so

$$\int_X |\dot{u}_0|\omega^n \geq \frac{\frac{2}{n+1}}{\left(\frac{n+1}{n}\right)^n} V = \frac{2}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^n V.$$

This finishes the proof of (31). □

Remark 5.6 The inequalities of the above theorem are sharp due to the examples produced in the toric case. Indeed, if we take $\phi(x_1, \ldots, x_n) = -1 + x_1 + \cdots + x_n$, as considered in Proposition 4.2, then $t \to t\phi$ will give a toric ray confirming the optimality of the constant $\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$ in (30).

Similarly, taking $\phi_{n,m}$ as in the proof of Proposition 4.3, the correspondence $t \to t\phi_{n,m}$ gives a toric ray that confirms optimality of the constant 2 in the second ineqality of (30).

Finally we give an application regarding the initial value problem for geodesic rays, proving Corollary 1.4 in the process.

As is well known, the initial value problem for smooth geodesics in the space of Kähler metrics is not well posed [31]. More specifically, in [31, Theorem 1.1] the authors characterize C^3 functions v that can arise as the initial tangent vectors of geodesics existing for time at least $T > 0$, in terms of analytic extension of the Hamiltonian flow of v. For the the initial value problem of weak geodesic rays we have below a necessary integral condition instead. To what extent our integral condition below is sufficient remains to be seen. We hope to return to this problem in the future.

Corollary 5.7 *Let $\{u_t\}_t \in \mathcal{R}^\infty$. Then the initial tangent vector $v := \dot{u}_0 \in L^\infty(X)$ satisfies the following sharp estimates:*

$$\frac{2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \sup_X v \leq \int_X |v|\omega^n \leq 2 \sup_X v. \tag{33}$$

Proof By Lemmas 5.2 and 5.3 we know that $d_1(0, u_1) = \int_X |\dot{u}_0|\omega^n$ and $J\{u_t\} = \sup_X \dot{u}_0$. (33) now follows directly from (30). □

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