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A complete metric topology on relative low energy spaces

Prakhar Gupta[1]

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Abstract
In this paper, we show that the low energy spaces in the prescribed singularity case $\mathcal{E}_\psi(X, \theta, \phi)$ have a natural topology which is completely metrizable. This topology is stronger than convergence in capacity.

Keywords Kähler manifolds · Pluripotential theory · Monge–Ampère measures · Finite energy classes · Kähler–Ricci flow

Mathematics Subject Classification Primary 32U05; Secondary 32Q15 · 53E30 · 54E35

Contents

1 Introduction

Let (X, ω) be a compact Kähler manifold. By the dd^c-lemma, any Kähler metric cohomologous to ω is of the form $\omega_u := \omega + dd^c u$. This led to studying the space

$$\mathcal{H} = \{u \in C^\infty(X) : \omega_u := \omega + dd^c u > 0\}$$

✉ Prakhar Gupta
pgupta8@umd.edu

1 Department of Mathematics, University of Maryland, College Park, MD, USA

of smooth functions on X to find a canonical metric in the same cohomology class as ω. Mabuchi [26], Semmes [28], and Donaldson [17] independently found a Riemannian structure on $\mathcal{H}$ given by

$$\langle \phi, \psi \rangle_u := \frac{1}{\mathrm{Vol(X)}} \int_X \phi \psi \omega_u^n$$

for $u \in \mathcal{H}$ and $\phi, \psi \in T_u\mathcal{H} = C^\infty(X)$. Later, Chen [7] showed that this Riemannian structure makes $\mathcal{H}$ a metric space $(\mathcal{H}, d_2)$.

Darvas [9] showed that the completion $\overline{(\mathcal{H}, d_2)}$ can be identified with $(\mathcal{E}^2(X, \omega), d_2)$, where $\mathcal{E}^2(X, \omega)$ is the space of finite energy introduced by Guedj–Zeriahi [21]. Upon introducing the Finsler type structure on $\mathcal{H}$, Darvas [8] introduced metrics d_p on $\mathcal{H}$ for $p \geq 1$ and showed that their completions are $(\mathcal{E}^p(X, \omega), d_p)$. The metric $(\mathcal{H}, d_1)$ and its completion $(\mathcal{E}^1(X, \omega), d_1)$ were useful in studying special metrics on Kähler manifolds [3, 5, 6, 18].

This led to further attempts to find natural complete metrics on the various subspaces of $\mathrm{PSH}(X, \theta)$ for varying θ. Recall that, for a smooth closed $(1, 1)$-form θ we say that an integrable function $u : X \to \mathbb{R} \cup \{-\infty\}$ is θ-psh if locally u can be written as a sum of a smooth function and a plurisubharmonic function and $\theta + dd^c u \geq 0$ in the sense of currents. The following list of works illustrates the interest in finding natural metrics on these spaces.

(1) Darvas–Di Nezza–Lu [16] found that the space $\mathcal{E}^1(X, \theta)$ has a complete metric for θ merely big.
(2) Using approximation by Kähler classes Di Nezza–Lu found that the spaces $\mathcal{E}^p(X, \beta)$ for $p \geq 1$ have a complete metric for $\{\beta\}$ a nef and big class in $H^{1,1}(X, \mathbb{R})$.
(3) In [30], Trusiani showed that the space $\mathcal{E}^1(X, \omega, \phi)$ has a complete metric topology where ω is a Kähler form and $\phi \in \mathrm{PSH}(X, \omega)$ is a model potential. The space $\mathcal{E}^1(X, \omega, \phi)$ consists of potentials more singular than ϕ and having finite energy relative to ϕ. See Sect. 2 for relevant definitions and results about model potentials and spaces with prescribed singularities.
(4) Xia [32] extended this result to show that the spaces $\mathcal{E}^p(X, \theta, \phi)$ have complete metric space structure for θ having big cohomology class and $\phi \in \mathrm{PSH}(X, \theta)$ a model potential.
(5) Most recently, Darvas [10] showed that the space $\mathcal{E}_\psi(X, \omega)$ has a natural complete metric, when ω is Kähler and ψ is a low energy weight as introduced by Guedj–Zeriahi [21]. In the process, Darvas used the geodesics on $\mathcal{H}$ introduced by [17, 26, 28].

Note that only [10] deals with the low energy weights. Working with the low energy space is desirable because

$$\mathcal{E}(X, \omega) = \bigcup_\psi \mathcal{E}_\psi(X, \omega)$$

where the union is over all low energy weights (see [21, Proposition 2.2]). However, all the finite energy spaces $\mathcal{E}^p(X, \omega)$ are contained in $\mathcal{E}^1(X, \omega) \subsetneq \mathcal{E}(X, \omega)$. Another method of measuring distance between potentials $u, v \in \mathcal{E}(X, \omega)$ is proposed by Lempert [24] where he measures the distance $\rho(u, v)$ by a function $\rho(u, v) : (0, V) \to \mathbb{R}$ where $V = \int_X \omega^n$ is the volume of (X, ω).

In [10], the author noted that the metric d_ψ on the space $\mathcal{E}_\psi(X, \omega)$ satisfies

$$d_\psi(u, v) \leq \int_X \psi(u - v)(\omega_u^n + \omega_v^n) \leq 2^{2n+5} d_\psi(u, v)$$

for any $u, v \in \mathcal{E}_\psi(X, \omega)$. In [10], the author asked if the central expression in the above equation can be shown to satisfy a quasi-triangle inequality, without constructing the metric

d_ψ using geodesics in $\mathcal{H}$, and thus show that the spaces $\mathcal{E}_\psi(X,\theta)$ have a quasi-metric space structure. This is the question we answer in this paper, in a more general context of low energy spaces in the prescribed singularity setting. Note that in the big case and in the prescribed singularity case, there are no $C^{1,1}$ geodesics, and hence the methods of [10] do not work.

Theorem 1.1 *Let θ be a closed smooth $(1,1)$-form whose cohomology class is big. Let $\phi \in PSH(X,\theta)$ be a model potential such that $\int_X \theta_\phi^n > 0$. Then for any $u, v \in \mathcal{E}_\psi(X,\theta,\phi)$,*

$$I_\psi(u,v) := \int_X \psi(u-v)(\theta_u^n + \theta_v^n) \tag{1}$$

is a quasi-metric. Moreover, the topology induced on $\mathcal{E}_\psi(X,\theta,\phi)$ by I_ψ is completely metrizable.

Here we say a few words about the prescribed singularity setting. Let $\phi \in \text{PSH}(X,\theta)$ with $\int_X \theta_\phi^n > 0$. We denote by $\text{PSH}(X,\theta,\phi)$ the set of θ-psh functions u that are more singular than ϕ, meaning $u \le \phi + C$ for some constant C. In particular, $\text{PSH}(X,\theta) = \text{PSH}(X,\theta,V_\theta)$. The set of relatively full mass potentials is given by

$$\mathcal{E}(X,\theta,\phi) = \left\{ u \in \text{PSH}(X,\theta,\phi) : \int_X \theta_u^n = \int_X \theta_\phi^n \right\}$$

and the set of relatively finite energy is given by

$$\mathcal{E}_\psi(X,\theta,\phi) = \left\{ u \in \mathcal{E}(X,\theta,\phi) : \int_X \psi(u-\phi)\theta_u^n < \infty \right\}.$$

See Sect. 2.1 to learn more about potentials in prescribed singularity setting.

After this we study some properties of the new topology on $\mathcal{E}_\psi(X,\theta,\phi)$. The usual topology on $\mathcal{E}_\psi(X,\theta,\phi)$ given by $L^1(\omega^n)$ is not satisfactory for the purposes of studying Monge–Ampère equation primarily because for $u_k, u \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $u_k \to u$ in $L^1(\omega^n)$ does not imply that the non-pluripolar Monge–Ampère measures satisfy $\theta_{u_k}^n \to \theta_u^n$. Hence the following result shows that the new topology is stronger and more natural.

Theorem 1.2 *If $u_k, u \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $I_\psi(u_k,u) \to 0$ as $k \to \infty$ then $u_k \to u$ in capacity and hence $\theta_{u_k}^n \to \theta_u^n$ weakly as measures. In particular, $\int_X |u_k - u|\omega^n \to 0$ as $k \to \infty$ as well.*

This result is new in the Kähler case when $\psi(t)$ grows slower than $|t|$. When $\psi(t) = |t|$ and $\phi = V_\theta$ the model potential with minimal singularity, in [1] the authors show that a closely related functional,

$$\tilde{I}(u,v) = \int_X (u-v)(\theta_u^n - \theta_V^n)$$

satisfy the same properties as in Theorem 1.2. Moreover, in [29] the author shows the same results using $\tilde{I}$ for $\psi(t) = |t|$, θ a Kähler class and ϕ any model prescribed singularity.

At the end we give an application to the Kähler–Ricci flow. Guedj–Zeriahi [15, 22] showed that given a potential $u \in \text{PSH}(X,\omega)$ such that u has zero Lelong numbers we have smooth function u_t for $t > 0$ such that

$$\frac{\partial u_t}{\partial t} = \log\left[\frac{(\omega + dd^c u_t)^n}{\omega^n}\right], \quad u_t \to u \text{ in } L^1(\omega^n) \text{ as } t \to 0. \tag{2}$$

If $\omega_t := \omega + dd^c u_t$, then the above equation is equivalent to

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \text{Ric}(\omega).$$

Moreover, these functions satisfy $u_t \to u$ in capacity as $t \to 0$. In case $u \in \mathcal{E}_\psi(X, \omega)$ we show a stronger convergence $u_t \to u$ as $t \to 0$ in the following theorem.

Theorem 1.3 *If $u \in \mathcal{E}_\psi(X, \omega)$ and u_t satisfy (2) then $I_\psi(u_t, u) \to 0$ and thus by Theorem 1.2, we recover that $u_t \to u$ in capacity and $\omega_{u_t}^n \to \omega_u^n$ weakly.*

Here we point out that Theorem 1.3 shows that any non-pluripolar measure can be approximated by measures with smooth density using Kähler–Ricci flow.

Organization

In Sect. 2, we setup the notation and mention all the relevant results required for the theorems we prove. In Sect. 3, we show that I_ψ is a quasi-metric on $\mathcal{E}_\psi(X, \theta, \phi)$. In Sect. 4, we show that the induced topology on $\mathcal{E}_\psi(X, \theta, \phi)$ is completely metrizable thereby completing the proof of Theorem 1.1. In Sect. 5, we discuss some relevant properties of the new topology and prove Theorem 1.2. In Sect. 6, we discuss an application to Kähler Ricci flow and prove Theorem 1.3.

2 Preliminaries

In this section, we fix the notations and give relevant definitions and results.

We work on a fixed Kähler manifold (X, ω). Let θ be a closed smooth $(1, 1)$-form on X. An integrable function $u : X \to \mathbb{R} \cup \{-\infty\}$ is a θ-psh if locally u can be written as a sum of a smooth function and a plurisubharmonic function and $\theta + dd^c u \geq 0$ in the sense of currents. We denote by $\text{PSH}(X, \theta)$ the set of all θ-psh functions. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be the cohomology class represented by θ. We say α is big if there exists $\varepsilon > 0$ and $u \in \text{PSH}(X, \theta - \varepsilon\omega)$. See [4] to learn more about pluripotential theory on compact Kähler manifolds in big cohomology classes. In particular, op. cit. describes how to define the non-pluripolar Monge–Ampère measure θ_u^n for any $u \in \text{PSH}(X, \theta)$.

2.1 Prescribed singularity setting

In [11, 13] the authors developed the theory of pluripotential theory in prescribed singularity setting. For two potentials $u, v \in \text{PSH}(X, \theta)$, we say that u is *more singular than* v if $u \leq v + C$ for some constant C. We denote by $[u] = [v]$ the fact that u and v have the same singularity type. Given a potential $\phi \in \text{PSH}(X, \theta)$, we denote by $\text{PSH}(X, \theta, \phi)$ the set of all potentials $v \in \text{PSH}(X, \theta)$ such that v is more singular than ϕ. To solve the Monge–Ampère equation with prescribed singularity, Darvas–Di Nezza–Lu [11] defined the space of relatively full mass potentials as

$$\mathcal{E}(X, \theta, \phi) := \left\{ u \in \text{PSH}(X, \theta, \phi) : \int_X \theta_u^n = \int_X \theta_\phi^n] \right\}.$$

Next, we want to define some subspaces of $\mathcal{E}(X, \theta, \phi)$ which consists of potentials having relatively finite 'ψ-energy' for some weight ψ. By a weight ψ, we mean a function $\psi : \mathbb{R} \to$

$\mathbb{R}$ such that ψ is even, continuous, $\psi(0) = 0$, $\psi(\pm\infty) = \infty$, and on $(0, \infty)$ ψ is smooth and increasing. We say ψ is *low energy* if ψ is concave on $(0, \infty)$ and ψ is *high energy* if ψ is convex on $(0, \infty)$. We denote by $\mathcal{W}^-$ the set of all low energy weights. For example, the weight $\psi(t) = |t|^p$ is high energy for $p \geq 1$ and low energy for $0 < p \leq 1$. For each weight $\psi \in \mathcal{W}^-$, we define the space of potentials with finite ψ-energy relative to ϕ as

$$\mathcal{E}_\psi(X, \theta, \phi) = \left\{u \in \mathcal{E}(X, \theta, \phi) : \int_X \psi(u - \phi)\theta_u^n < \infty\right\}.$$

As in the Kähler case, in the prescribed singularity case we also have

$$\mathcal{E}(X, \theta, \phi) = \bigcup_{\psi \in \mathcal{W}^-} \mathcal{E}_\psi(X, \theta, \phi).$$

We list a few results about the spaces $\mathcal{E}_\psi(X, \theta, \phi)$ that we use frequently in the rest of the paper.

Lemma 2.1 (Comparison principle) [11, Corollary 3.6] *Let $\phi \in PSH(X, \theta)$ and $u, v \in \mathcal{E}(X, \theta, \phi)$. Then*

$$\int_{\{u<v\}} \theta_v^n \leq \int_{\{u<v\}} \theta_u^n.$$

Lemma 2.2 (The Fundamental inequality) [31, Proposition 3.3] *If $u, v \in PSH(X, \theta, \phi)$ are such that $u \leq v \leq \phi$ then*

$$\int_X \psi(v - \phi)\theta_v^n \leq 2^{n+1} \int_X \psi(u - \phi)\theta_u^n.$$

Thus for any $u \in \mathcal{E}_\psi(X, \theta, \phi)$ and $v \in PSH(X, \theta, \phi)$ such that $u \leq v$ we have $v \in \mathcal{E}_\psi(X, \theta, \phi)$.

Proof We give a simplified proof based on the argument of [13, Lemma 2.4].

$$\begin{aligned}\int_X \psi(v - \phi)\theta_v^n &= \int_0^\infty \psi'(t)\theta_v^n(v < \phi - t)dt \\ &= 2\int_0^\infty \psi'(2t)\theta_v^n(v < \phi - 2t)dt.\end{aligned}$$

Now observe that $\{v < \phi - 2t\} \subset \{u < \frac{v+\phi}{2} - t\} \subset \{u < \phi - t\}$, thus using Lemma 2.1 and the fact that $\theta_v^n \leq 2^n\theta_{(v+\phi)/2}^n$

$$\begin{aligned}&\leq 2\int_0^\infty \psi'(2t)\theta_v^n\left(u < \frac{v+\phi}{2} - t\right)dt \\ &\leq 2^{n+1}\int_0^\infty \psi'(2t)\theta_{(v+\phi)/2}^n\left(u < \frac{v+\phi}{2} - t\right)dt \\ &\leq 2^{n+1}\int_0^\infty \psi'(2t)\theta_u^n(u < \phi - t)dt.\end{aligned}$$

Since ψ is concave, we get $\psi'(2t) \leq \psi'(t)$.

$$\begin{aligned}&\leq 2^{n+1}\int_0^\infty \psi'(t)\theta_u^n(u < \phi - t)dt \\ &= 2^{n+1}\int_X \psi(u - \phi)\theta_u^n.\end{aligned}$$

□

Lemma 2.3 (Integrability) *Given $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$ we have*

$$\int_X \psi(u-\phi)\theta_v^n < +\infty.$$

In particular, if $u, v \le \phi$, then

$$\int_X \psi(u-\phi)\theta_v^n \le 2\int_X \psi(u-\phi)\theta_u^n + 2\int_X \psi(v-\phi)\theta_v^n.$$

Proof The proof again generalizes the proof of [13, Lemma 2.5] and [21, Proposition 2.5].

$$\begin{aligned}\int_X \psi(u-\phi)\theta_v^n &= \int_0^\infty \psi'(t)\theta_v^n\{t < \phi - u\}dt\\ &= 2\int_X \psi'(2t)\theta_v^n\{2t < \phi - u\}dt\\ &\le 2\int_X \psi'(t)\theta_v^n\{u < \phi - 2t\}dt.\end{aligned}$$

Notice that $\{u < \phi - 2t\} \subset \{u < -t + v\} \cup \{v < \phi - t\}$. This gives

$$\theta_v^n\{u < \phi - 2t\} \le \theta_v^n\{u < -t + v\} + \theta_v^n\{v < \phi - t\}$$

and Lemma 2.1 gives

$$\le \theta_u^n\{u < -t + v\} + \theta_v^n\{v < \phi - t\}.$$

Notice that $\{u < -t+v\} = \{u-\phi < -t+v-\phi\}$. Since $v - \phi \le 0$, we get $\{u < -t+v\} \subset \{u - \phi < -t\}$. This gives us

$$\le \theta_u^n\{u - \phi < -t\} + \theta_v^n\{v - \phi < -t\}.$$

Using this we get

$$\begin{aligned}&\le 2\int_X \psi'(t)(\theta_u^n\{u-\phi < -t\} + \theta_v^n\{v - \phi < -t\})\\ &= 2\int_X \psi(u-\phi)\theta_u^n + 2\int_X \psi(v-\phi)\theta_v^n.\end{aligned}$$

Now assume $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$ then for some C, we have $u, v \le \phi + C$. Let $\tilde{u} = u - C$ and $\tilde{v} = v - C$, so $\tilde{u}, \tilde{v} \le \phi$. Then,

$$\begin{aligned}\int_X \psi(u-\phi)\theta_v^n &= \int_X \psi(\tilde{u} - \phi + C)\theta_{\tilde{v}}^n\\ &\le \int_X \psi(\tilde{u}-\phi)\theta_{\tilde{v}}^n + \int_X \psi(C)\theta_{\tilde{v}}^n < \infty.\end{aligned}$$

□

Corollary 2.4 *The proposed quasi-metric I_ψ (see Eq. (1)) on $\mathcal{E}_\psi(X, \theta, \phi)$ is always finite.*

Proof Let $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$. Then

$$\begin{aligned}I_\psi(u, v) &= \int_X \psi(u-v)(\theta_u^n + \theta_v^n)\\ &\le \int_X (\psi(u-\phi) + \psi(v-\phi))(\theta_u^n + \theta_v^n).\end{aligned}$$

Here we used [10, Lemma 2.6] which states that for any $a, b \in \mathbb{R}$, and any $\psi \in \mathcal{W}^-$, we have $\psi(a+b) \leq \psi(a) + \psi(b)$. By Lemma 2.3 all the terms in the expression above are finite. □

Lemma 2.5 (Domination principle) [11, Proposition 3.11]. *Let $\phi \in PSH(X, \theta)$ such that $\int_X \theta_\phi^n > 0$. Then for $u, v \in \mathcal{E}(X, \theta, \phi)$, if $\theta_u^n(\{u < v\}) = 0$, then $u \geq v$.*

We also need the following slight generalization of [14, Theorem 2.2], removing the assumption of uniform boundedness of χ_k.

Lemma 2.6 *Let θ^j, $j \in \{1, \ldots, n\}$ be smooth closed $(1,1)$-forms on X whose cohomology classes are big. Let $u_j, u_j^k \in PSH(X, \theta^j)$ are such that $u_j^k \to u_j$ in capacity as $k \to \infty$. Let $\chi_k, \chi \geq 0$ be quasi-continuous functions such that $\chi_k \to \chi$ in capacity as $k \to \infty$. Then*

$$\int_X \chi \theta_{u_1}^1 \wedge \theta_{u_2}^2 \wedge \cdots \wedge \theta_{u_n}^n \leq \liminf_{k \to \infty} \int_X \chi_k \theta_{u_1^k}^1 \wedge \theta_{u_2^k}^2 \wedge \cdots \wedge \theta_{u_n^k}^n.$$

Proof Consider $\chi_{k,C} = \min(\chi_k, C)$. Then $\chi_{k,C}$ are uniformly bounded and quasi-continuous and $\chi_{k,C} \to \chi_C$ in capacity. Therefore, using [14, Theorem 2.2]

$$\int_X \chi_C \theta_{u_1}^1 \wedge \cdots \wedge \theta_{u_n}^n \leq \liminf_{k \to \infty} \int_X \chi_{k,C} \theta_{u_1^k}^1 \wedge \cdots \wedge \theta_{u_n^j}^n.$$

Since $\chi_{k,C} \leq \chi_k$, we have

$$\int_X \chi_C \theta_{u_1}^1 \wedge \cdots \wedge \theta_{u_n}^n \leq \liminf_{k \to \infty} \int_X \chi_{k,C} \theta_{u_1^k}^1 \wedge \cdots \wedge \theta_{u_n^k}^n \leq \liminf_{k \to \infty} \int_X \chi_k \theta_{u_1^k}^1 \wedge \cdots \wedge \theta_{u_n^k}^n.$$

As $\chi_C \nearrow \chi$ as $C \to \infty$ and applying monotone convergence theorem, we get

$$\int_X \chi \theta_{u_1}^1 \wedge \cdots \wedge \theta_{u_n}^n \leq \liminf_{k \to \infty} \int_X \chi_k \theta_{u_1^k}^1 \wedge \cdots \wedge \theta_{u_n^k}^n.$$

□

The following result is mentioned in [31, Corollary 3.4]. We present a proof here for completeness.

Lemma 2.7 *Let $\phi \in PSH(X, \theta)$ such that $\int_X \theta_\phi^n > 0$. Let $u_j \in \mathcal{E}_\psi(X, \theta, \phi)$ such that*

$$\sup_j \int_X \psi(u_j - \phi) \theta_{u_j}^n < \infty.$$

If $u_j \to u$ in $L^1(\omega^n)$ for some $u \in PSH(X, \theta)$, then $u \in \mathcal{E}_\psi(X, \theta, \phi)$.

Proof Since there exists A such that $\sup_X u_j \leq A$ for all j (see [23, Proposition 8.4]), and the fact that $\psi(a+b) \leq \psi(a) + \psi(b)$ for any $a, b \in \mathbb{R}$ we can assume without loss of generality that $u_j, u \leq \phi$ for all j.

First, we assume that $u_j \searrow u$. This implies that $u_j \to u$ in capacity. Also, $\psi(u_k - \phi) \geq 0$ are quasicontinuous functions which converge in capacity to $\psi(u - \phi)$. Thus by Theorem 2.6, we get that

$$\int_X \psi(u - \phi) \theta_u^n \leq \liminf_{k \to \infty} \int_X \psi(u_k - \phi) \theta_{u_k}^n < \infty.$$

In general, if $u_j \to u$ in $L^1(\omega^n)$ and if $v_j := (\sup_{k\geq j} u_k)^*$, then $u_j \leq v_j \leq \phi$. Thus, by Lemma 2.2 we get

$$\sup_j \int_X \psi(v_j - \phi)\theta^n_{v_j} \leq 2^{n+1} \sup_j \int_X \psi(u_j - \phi)\theta^n_{u_j} < \infty.$$

Since $v_j \searrow u$, by the argument above, we get that

$$\int_X \psi(u-\phi)\theta^n_u < \liminf_{j\to\infty} \int_X \psi(v_j - \phi)\theta^n_{v_j} < \infty.$$

Although this shows that u has finite energy, we are yet to show that u has full mass. By [11, Lemma 3.4] we just need to show that $\int_{\{u\leq\phi-C\}} \theta^n_{u^C} \to 0$ as $C \to \infty$. Here $u^C = \max(u, \phi - C)$. First, notice that $v^C_j \searrow u^C$ in $j \to \infty$.

$$\int_X \psi(u^C - \phi)\theta^n_{u^C} \leq \liminf_j \int_X \psi(v^C_j - \phi)\theta^n_{v^C_j} \leq 2^{n+1} \liminf_j \int_X \psi(u_j - \phi)\theta^n_{u_j}.$$

Thus

$$\int_{\{u\leq\phi-C\}} \theta^n_{u^C} \leq \frac{1}{\psi(C)} \int_X \psi(u^C - \phi)\theta^n_{u^C} \leq \frac{2^{n+1}}{\psi(C)} \sup_j \int_X \psi(u_j - \phi)\theta^n_{u_j} \to 0$$

as $C \to \infty$. This finishes the proof that $u \in \mathcal{E}_\psi(X, \theta, \phi)$. □

2.2 The operator $P_\theta(u, v)$

The notion of model potential and model type singularity was introduced by Darvas–Di Nezza–Lu [11] to solve the Monge–Ampère equation in the prescribed singularity setting. In particular, in [13], the authors showed that if ϕ is a model potential and if μ is a non-pluripolar positive measure such that $\mu(X) = \int_X \theta^n_\phi > 0$, then there unique (up to a constant) $u \in \mathcal{E}(X, \theta, \phi)$ such that $\theta^n_u = \mu$.

Recall that for an upper semicontinuous function f, we define $P_\theta(f)$ to be the largest θ-psh function lying below f. In particular,

$$P_\theta(f) := \sup\{v \in \mathrm{PSH}(X, \theta) : v \leq f\}.$$

When $u, v \in \mathrm{PSH}(X, \theta)$, we say $P_\theta(u, v) := P_\theta(\min(u, v))$. For $\phi \in \mathrm{PSH}(X, \theta)$, the envelope of its singularity type is defined as

$$P_\theta[\phi] := \sup\{v \in \mathrm{PSH}(X, \theta, \phi) : v \leq 0\}.$$

A potential $\phi \in \mathrm{PSH}(X, \theta)$ is called a *model potential* if $\phi = P_\theta[\phi]$. Their importance in understanding the space $\mathcal{E}(X, \theta, \phi)$ and $\mathrm{PSH}(X, \theta)$ and solving the Monge–Ampère equations is described in [11, 13]. In the rest of the section ϕ is a model potential.

The following part is adapted from [12]. We will use the following lemma repeatedly in the arguments ahead.

Lemma 2.8 *Let μ be a non-pluripolar measure such that $0 < \mu(X) < \infty$ and $\lambda > 0$. Let $u, v \in \mathcal{E}(X, \theta, \phi)$ satisfy*

$$\theta^n_u \geq e^{\lambda u - w}\mu, \theta^n_v \leq e^{\lambda v - w}\mu$$

for some Borel measurable function $w : X \to \mathbb{R} \cup \{-\infty\}$ that is bounded from above. Then $u \leq v$ on X.

Proof The proof is same as in [12, Lemma 2.5], adapted to the prescribed singularity case. Using the comparison principle, we have

$$\int_{\{v<u\}} e^{\lambda u - w}\mu \le \int_{\{v<u\}} \theta_u^n \le \int_{\{v<u\}} \theta_v^n \le \int_{\{v<u\}} e^{\lambda v - w}\mu \le \int_{\{v<u\}} e^{\lambda u - w} d\mu.$$

Thus all the expressions are equal and we get

$$\int_{\{v<u\}} \left(e^{\lambda u} - e^{\lambda v}\right) e^{-w}\mu = 0.$$

Since w is bounded from above, e^{-w} is never 0. Thus $\mu\{v < u\} = 0$ and hence $\theta_v^n(\{v < u\}) = 0$. Now the domination principle (see Lemma 2.5) implies that $u \le v$. □

Theorem 2.9 *Let $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$. Then*

$$P_\theta(u, v) := \sup\{w \in PSH(X, \theta) : w \le \min(u, v)\} \in \mathcal{E}_\psi(X, \theta, \phi).$$

In particular, if $u, v \in \mathcal{E}(X, \theta, \phi)$, then $P_\theta(u, v) \in \mathcal{E}(X, \theta, \phi)$.

Proof Let $u_j = \max(u, \phi - j)$ and $v_j = \max(v, \phi - j)$. Then u_j and v_j have the same singularity type as ϕ. Thus by Lemma 2.10 below, there is a unique function $\varphi_j \in \mathcal{E}(X, \theta, \phi)$ with the same singularity type as ϕ such that

$$\theta_{\varphi_j}^n = e^{\varphi_j - u_j}\theta_{u_j}^n + e^{\varphi_j - v_j}\theta_{v_j}^n. \tag{3}$$

Notice that $\theta_{\varphi_j}^n \ge e^{\varphi_j - u_j}\theta_{u_j}^n$. Defining $\mu = e^{-u_j}\theta_{u_j}^n$, we see that Lemma 2.8 implies that $\varphi_j \le u_j$, and similarly, $\varphi_j \le v_j$. Therefore, $\varphi_j \le \min(u_j, v_j)$. Consequently, $\varphi_j \le P_\theta(u_j, v_j)$. Now we claim that

$$\sup_j \int_X \psi(\varphi_j - \phi)\theta_{\varphi_j}^n < \infty. \tag{4}$$

By Eq. (3), it is enough to show that

$$\sup_j \int_X \psi(\varphi_j - \phi)e^{\varphi_j - u_j}\theta_{u_j}^n < \infty. \tag{5}$$

Again, using the fact that $\psi(a + b) \le \psi(a) + \psi(b)$, we get

$$\int_X \psi(\varphi_j - \phi)e^{\varphi_j - u_j}\theta_{u_j}^n \le \int_X \psi(\varphi_j - u_j)e^{\varphi_j - u_j}\theta_{u_j}^n + \int_X \psi(u_j - \phi)e^{\varphi_j - u_j}\theta_{u_j}^n.$$

Since $\psi(t)e^t \le C$ for some fixed C and for all $t \le 0$, observing $\varphi_j \le u_j$ we get that

$$\le C\int_X \theta_\phi^n + \int_X \psi(u_j - \phi)\theta_{u_j}^n.$$

As $u \in \mathcal{E}_\psi(X, \theta, \phi)$, we get Eq. (5) and consequently Eq. (4).

Since $\varphi_j \le u_j \le u_1$, we get that $\sup_X \varphi_j$ is uniformly bounded. By the proof of Lemma 5.1, we also get that $\varphi_j \not\to -\infty$ uniformly. Thus up to choosing a subsequence, we get that there exists $\varphi \in \mathrm{PSH}(X, \theta)$ such that $\varphi_j \to \varphi$ in $L^1(\omega^n)$. By Lemma 2.7 we get that $\varphi \in \mathcal{E}_\psi(X, \theta, \phi)$. Since $\varphi_j \le P_\theta(u_j, v_j)$, we get that $\varphi \le P_\theta(u, v)$. Thus Lemma 2.2 tells us that $P_\theta(u, v) \in \mathcal{E}_\psi(X, \theta, \phi)$. □

Lemma 2.10 *Let $u, v \in PSH(X,\theta,\phi)$ such that u, v have the same singularity type as ϕ. Then there exists a unique $\varphi \in PSH(X,\theta,\phi)$ with the same singularity type as ϕ such that*

$$\theta_\varphi^n = e^{\varphi-u}\theta_u^n + e^{\varphi-v}\theta_v^n.$$

Proof First we show uniqueness. Let $\tilde{\varphi}$ be another solution. Let $\mu = e^u\theta_v^n + e^v\theta_u^n$ and $w = u+v$. Then $\mu(X) < \infty$ and w is bounded from above. Notice that $\theta_\varphi^n = e^{\varphi-w}\mu$ and $\theta_{\tilde{\varphi}}^n = e^{\tilde{\varphi}-w}\mu$. By Lemma 2.8 we get that $\varphi = \tilde{\varphi}$.

Let $u_j = \max(u, -j)$. Note that u_j is no longer a θ-psh function, but now it is a bounded function. Consider the measure

$$\mu_j = e^{-u_j}\theta_u^n + e^{-v_j}\theta_v^n.$$

Then μ_j is a non-pluripolar positive measure. By [11, Theorem 4.23], we get that there exists a unique $\varphi_j \in \mathcal{E}^1(X,\theta,\phi)$ such that

$$\theta_{\varphi_j}^n = e^{\varphi_j}\mu_j.$$

Since u and v have the same singularity type, there exists C such that $\sup_X |u-v| \le 2C$. Consider the function

$$w = \frac{u+v}{2} - C - (n+1)\ln 2.$$

Observing $\theta_u^n, \theta_v^n \le 2^n\theta_w^n$, we get

$$\theta_w^n \ge e^w\mu_j.$$

Now Lemma 2.8 implies that $w \le \varphi_j$. As w has relative minimal singularity type we obtain φ_j has relative minimal singularity type as well. Notice that for $j \ge k$, we have $\mu_j \ge \mu_k$. Thus,

$$\theta_{\varphi_j}^n \ge e^{\varphi_j}\mu_k \quad \text{and} \quad \theta_{\varphi_k}^n = e^{\varphi_k}\mu_k.$$

Lemma 2.8 again shows that $\varphi_k \ge \varphi_j$. Thus φ_j is decreases as $j \to \infty$. Let $\varphi = \lim_j \varphi_j$. Then $w \le \varphi$ thus φ has the relative minimal singularity type and by continuity of non-pluripolar Monge–Ampère operator under decreasing sequences, we get that

$$\theta_\varphi^n = e^{\varphi-u}\theta_u^n + e^{\varphi-v}\theta_v^n.$$

□

2.3 Metrics from a quasi-metric

If we relax the condition of the triangle inequality from the definition of a metric space, we obtain what we call a quasi-metric space.

Definition 2.11 (*Quasi-metric space*) Given a set X a function $\rho : X \times X \to [0,\infty)$ is a quasi-metric if it satisfies

(1) (Non-degeneracy) For any $x, y \in X$, $\rho(x,y) = 0$ iff $x = y$.
(2) (Symmetry) For all $x, y \in X$, $\rho(x,y) = \rho(y,x)$.
(3) (Quasi-triangle inequality)There exists $C \ge 1$ such that $\rho(x,y) \le C(\rho(x,z) + \rho(y,z))$ for all $x, y, z \in X$.

In [27] the authors show that given any quasi-metric, we can construct a metric comparable to the quasi-metric using a p-chain approach. In particular, if we consider $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = \inf \left\{ \sum_{i=0} \rho(x_n, x_{n+1})^p : x_0, \ldots, x_{n+1} \in X \text{ such that } x = x_0,\, y = x_{n+1} \right\}$$

then d_p is symmetric, satisfies triangle inequality, but in general $d_p(x, y) = 0$ even if $x \neq y$. But if $0 < p \leq 1$ is chosen such that $(2C)^p = 2$, then d_p is non-degenerate and moreover,

$$d_p(x, y) \leq \rho(x, y)^p \leq 4d_p(x, y).$$

This shows that if for $x_n, x \in X$ we have $\rho(x_n, x) \to 0$ iff $d_p(x_k, x) \to 0$. Thus ρ and d_p induce the same topology.

3 Quasi-metric on $\mathcal{E}_\psi(X, \theta, \phi)$

In this section we prove that the expression

$$I_\psi(u, v) = \int_X \psi(u - v)(\theta_u^n + \theta_v^n)$$

is a quasi-metric on $\mathcal{E}_\psi(X, \theta, \phi)$ where $\phi \in \mathrm{PSH}(X, \theta)$ is such that $\int_X \theta_\phi^n > 0$.

Theorem 3.1 (Quasi-triangle inequality) *Let $\phi \in PSH(X, \theta)$ be such that $\int_X \theta_\phi^n > 0$, then for any $u, v, w \in \mathcal{E}_\psi(X, \theta, \phi)$, the functional*

$$I_\psi(u, v) = \int_X \psi(u - v)(\theta_u^n + \theta_v^n)$$

satisfies

$$I_\psi(u, v) \leq C(I_\psi(u, w) + I_\psi(v, w))$$

for $C = 8 \cdot 3^{n+1}$.

Proof The proof is inspired from the proof in [19, Theorem 1.6]. We start with

$$\int_X \psi(u - v)\theta_u^n = \int_0^\infty \theta_u^n(\{\psi(u - v) > t\})dt. \tag{6}$$

Changing the variable $t = \psi(2s)$, we get $dt = 2\psi'(2s)ds$. Thus

$$= 2\int_0^\infty \psi'(2s)\theta_u^n(\{\psi(u - v) > \psi(2s)\})ds \tag{7}$$

$$= 2\int_0^\infty \psi'(2s)\theta_u^n(\{|u - v| > 2s\})ds. \tag{8}$$

Observe that $\{w - s \leq u < v - 2s\} \subset \{w < \frac{u+2v}{3} - \frac{s}{3}\}$ and $\{u < v - 2s\} \subset \{u < w - s\} \cup \{w < \frac{u+2v}{3} - \frac{s}{3}\}$ to obtain

$$\theta_u^n(\{u < v - 2s\}) \leq \theta_u^n(\{u < w - s\}) + \theta_u^n\left(\left\{w < \frac{u + 2v}{3} - \frac{s}{3}\right\}\right). \tag{9}$$

Since $\theta_u^n \leq 3^n \theta^n_{\frac{u+2v}{3}}$ and $\mathcal{E}(X, \theta, \phi)$ is convex [11, Corollary 3.15], we get

$$\leq \theta_u^n(\{u < w - s\}) + 3^n \theta^n_{\frac{u+2v}{3}}\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right). \tag{10}$$

Now Lemma 2.1 gives

$$\leq \theta_u^n(\{u < w - s\}) + 3^n \theta^n_{w}\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right). \tag{11}$$

Again by the comparison principle we have

$$\theta_u^n(\{v < u - 2s\}) \leq \theta_v^n(\{v < u - 2s\}).$$

By a similar computation as in Eq. (11) we get

$$\leq \theta_v^n(\{v < w - s\}) + 3^n \theta^n_{w}\left(\left\{w < \frac{v+2u}{3} - \frac{s}{3}\right\}\right). \tag{12}$$

Combining Eqs. (11) and (12) we get

$$\begin{aligned}
\theta_u^n(\{|u - v| > 2s\}) &\leq \theta_u^n(\{u < v - 2s\}) + \theta_u^n(\{v < u - 2s\}) \\
&\leq \theta_u^n(\{u < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right) \\
&\quad + \theta_v^n(\{v < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{v+2u}{3} - \frac{s}{3}\right\}\right) \\
&\leq \theta_u^n(\{|u - w| > s\}) + 3^n \theta_w^n\left(\left\{\left|w - \frac{u+2v}{3}\right| > \frac{s}{3}\right\}\right) \\
&\quad + \theta_v^n(\{|v - w| > s\}) + 3^n \theta_w^n\left(\left\{\left|w - \frac{v+2u}{3}\right| > \frac{s}{3}\right\}\right).
\end{aligned} \tag{13}$$

Combining Eqs. (8) and (13), we get

$$\begin{aligned}
\int_X \psi(u - v)\theta_u^n &\leq 2\int_0^\infty \psi'(2s)\theta_u^n(\{|u - w| > s\})ds \\
&\quad + 2\int_0^\infty \psi'(2s)\theta_v^n(\{|v - w| > s\})ds \\
&\quad + 2 \cdot 3^n \int_0^\infty \psi'(2s)\theta_w^n\left(\left\{\left|w - \frac{u+2v}{3}\right| > \frac{s}{3}\right\}\right)ds \\
&\quad + 2 \cdot 3^n \int_0^\infty \psi'(2s)\theta_w^n\left(\left\{\left|w - \frac{v+2u}{3}\right| > \frac{s}{3}\right\}\right)ds.
\end{aligned} \tag{14}$$

Since ψ is concave, therefore the slope is decreasing. Hence $\psi'(2s) \le \psi'(s)$ and $\psi'(2s) < \psi'(s/3)$. Using this in Eq. (14) we get

$$\begin{aligned}
&\le 2\int_0^\infty \psi'(s)\theta_u^n(\{|u-w|>s\})ds \\
&\quad + 2\int_0^\infty \psi'(s)\theta_v^n(\{|v-w|>s\})ds \\
&\quad + 2\cdot 3^n \int_0^\infty \psi'(s/3)\theta_w^n\left(\left\{\left|w-\frac{u+2v}{3}\right|>\frac{s}{3}\right\}\right)ds \\
&\quad + 2\cdot 3^n \int_0^\infty \psi'(s/3)\theta_w^n\left(\left\{\left|w-\frac{v+2u}{3}\right|>\frac{s}{3}\right\}\right)ds.
\end{aligned} \tag{15}$$

Changing the variable again $t=\psi(s)$ in the first two terms and $t=\psi(s/3)$ in the last two terms, we get

$$\begin{aligned}
&= 2\int_0^\infty \theta_u^n(\{\psi(u-w)>t\})dt \\
&\quad + 2\int_0^\infty \theta_v^n(\{\psi(v-w)>t\})dt \\
&\quad + 2\cdot 3^{n+1} \int_0^\infty \theta_w^n\left(\left\{\psi\left(w-\frac{u+2v}{3}\right)>t\right\}\right)dt \\
&\quad + 2\cdot 3^{n+1} \int_0^\infty \theta_w^n\left(\left\{\psi\left(w-\frac{v+2u}{3}\right)>t\right\}\right)dt.
\end{aligned} \tag{16}$$

$$\begin{aligned}
\int_X \psi(u-v)\theta_u^n &\le 2\int_X \psi(u-w)\theta_u^n + 2\int_X \psi(v-w)\theta_v^n \\
&\quad + 2\cdot 3^{n+1}\int_X \psi\left(w-\frac{u+2v}{3}\right)\theta_w^n \\
&\quad + 2\cdot 3^{n+1}\int_X \psi\left(w-\frac{v+2u}{3}\right)\theta_w^n.
\end{aligned} \tag{17}$$

Using the fact that $\psi(a+b) \le \psi(a)+\psi(b)$ for any $a, b \in \mathbb{R}$, (see [10, Lemma 2.6]) we get

$$\begin{aligned}
&\le 2\int_X \psi(u-w)\theta_u^n + 2\int_X \psi(v-w)\theta_v^n \\
&\quad + 2\cdot 3^{n+1}\int_X \left(\psi\left(\frac{w-u}{3}\right)+\psi\left(\frac{2(w-v)}{3}\right)\right)\theta_w^n \\
&\quad + 2\cdot 3^{n+1}\int_X \left(\psi\left(\frac{w-v}{3}\right)+\psi\left(\frac{2(w-u)}{3}\right)\right)\theta_w^n.
\end{aligned} \tag{18}$$

Since ψ is increasing in $(0,\infty)$ and symmetric, we get that $\psi(x/3) \le \psi(x)$ and $\psi(2x/3) \le \psi(x)$ for any $x \in \mathbb{R}$, thus

$$\begin{aligned}
&\le 2\int_X \psi(u-w)\theta_u^n + 2\int_X \psi(v-w)\theta_v^n \\
&\quad + 4\cdot 3^{n+1}\int_X \psi(w-u)\theta_w^n + 4\cdot 3^{n+1}\int_X \psi(w-v)\theta_w^n
\end{aligned} \tag{19}$$

$$\le 4\cdot 3^{n+1}(I(u,w)+I(v,w)). \tag{20}$$

A similar computation for $\int_X \psi(u-v)\theta_v^n$ gives

$$I(u, v) \leq 8 \cdot 3^{n+1}(I(u, w) + I(v, w)). \tag{21}$$

This finishes the proof of quasi-triangle inequality. □

Theorem 3.2 (Non-degeneracy) *If* $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$, *such that* $I_\psi(u, v) = 0$, *then* $u = v$.

Proof Let $I_\psi(u, v) = 0$. Then

$$\int_X \psi(u-v)\theta_u^n + \int_X \psi(u-v)\theta_v^n = 0.$$

Thus we have $u = v$ almost everywhere with respect to θ_u^n and θ_v^n. Using Lemma 2.5, we get that $u = v$. □

The symmetry of $I_\psi(u, v)$ follows from the fact that the weight function is an even function. This finishes the proof of the fact that I_ψ is a quasi-metric on $\mathcal{E}_\psi(X, \theta, \phi)$.

4 Completeness

In this section we finish the proof of Theorem 1.1 by showing that the quasi-metric is complete.

We need the following lemma to construct a monotone sequence and show that it converges.

Theorem 4.1 (Pythagorean identity) *Let* $\phi \in PSH(X, \theta)$ *and let* $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$. *We know that* $\max(u, v) \in \mathcal{E}_\psi(X, \theta, \phi)$. *Moreover, they satisfy*

$$I_\psi(u, v) = I_\psi(\max(u, v), u) + I_\psi(\max(u, v), v).$$

Proof

$$\begin{aligned} I_\psi(\max(u, v), u) &= \int_X \psi(\max(u, v) - u)(\theta^n_{\max(u,v)} + \theta_u^n) \\ &= \int_{\{v>u\}} \psi(v-u)(\theta^n_{\max(u,v)} + \theta_u^n). \end{aligned}$$

Since $u \mapsto \theta_u^n$ is plurifine local, we have $\mathbb{1}_{\{v>u\}}\theta^n_{\max(u,v)} = \mathbb{1}_{\{v>u\}}\theta_v^n$

$$= \int_{\{v>u\}} \psi(v-u)(\theta_v^n + \theta_u^n).$$

Similarly,

$$I_\psi(\max(u, v), v) = \int_X \psi(\max(u, v) - v)(\theta^n_{\max(u,v)} + \theta_v^n)$$

and the same computation as before gives

$$= \int_{\{u>v\}} \psi(u-v)(\theta_u^n + \theta_v^n).$$

Adding the two gives

$$\begin{aligned} I_\psi(\max(u, v), u) + I_\psi(\max(u, v), v) &= \int_X \psi(u-v)(\theta_u^n + \theta_v^n) \\ &= I_\psi(u, v). \end{aligned}$$

□

Proposition 4.2 *If $u_j, u \in \mathcal{E}_\psi(X, \theta, \phi)$ such that $u_j \searrow u$ or $u_j \nearrow u$, then*

$$\int_X \psi(u_j - u)\theta_{u_j}^n \to 0.$$

Consequently, by the monotone convergence theorem $I_\psi(u_j, u) \to 0$.

Proof Without loss of generality, we can assume that $u_j, u \leq \phi \leq 0$. First, we assume that $\phi - L \leq u_j, u \leq \phi$. We also assume that $u_j \searrow u$ and the proof for $u_j \nearrow u$ is similar. Thus $0 \leq u_j - u \leq L$, which implies $0 \leq \psi(u_j - u) \leq \psi(L)$. For some large enough A, $\theta \leq A\omega$. Thus $\mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$, and we get that $u_j, u \in \mathrm{PSH}(X, A\omega)$. Hence they are quasi-continuous (at least with respect to Cap_ω). So we can find an open set O such that u_j, u are continuous on $X \setminus O$ and $\mathrm{Cap}_\omega(O) < \varepsilon$. The next part follows the proof in [12, Corollary 2.12] adapted to our case. We have

$$\int_X \psi(u_j - u)\theta_{u_j}^n = \int_{X\setminus O} \psi(u_j - u)\theta_{u_j}^n + \int_O \psi(u_j - u)\theta_{u_j}^n.$$

Since $X \setminus O$ is compact and $\psi(u_j - u)$ are continuous functions decreasing to 0, by Dini's theorem, we get that the convergence is uniform. So for large enough j, $\psi(u_j - u) \leq \varepsilon$ on $X \setminus O$.

Moreover,

$$\begin{aligned}\int_O \psi(u_j - u)\theta_{u_j}^n &\leq \psi(L) \int_O \theta_{u_j}^n \\ &\leq \psi(L)L^n \mathrm{Cap}_{\theta,\phi}(O) \\ &\leq \psi(L)L^n f(\mathrm{Cap}_\omega(O)) \\ &\leq \psi(L)L^n f(\varepsilon).\end{aligned}$$

Here we used [25, Theorem 1.1] with $(\theta_1, \psi_1) = (\theta, \phi)$ and $(\theta_2, \psi_2) = (\omega, 0)$ and thus $\mathrm{Cap}_{\theta,\phi} \leq f(\mathrm{Cap}_\omega)$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(0) = 0$. Combining these two results we get that for large enough j,

$$\int_X \psi(u_j - u)\theta_{u_j}^n \leq \varepsilon \mathrm{vol}(\theta^n) + \tilde{C} f(\varepsilon).$$

Therefore, $\lim_{j\to\infty} \int_X \psi(u_j - u)\theta_{u_j}^n \to 0$.

Now we remove the assumption that $\phi - L \leq u_j \leq \phi$. Let $u_j^L = \max\{u_j, \phi - L\}$ and $u^L = \max\{u, \phi - L\}$. We want to show that

$$\left| \int_X \psi(u_j - u)\theta_{u_j}^n - \int_X \psi(u_j^L - u^L)\theta_{u_j^L}^n \right| \to 0 \quad \text{as } L \to \infty$$

uniformly with respect to j. Since $u_j \searrow u$, we have that $u_j \geq u$. On the set $\{u > \phi - L\}$, we have $u^L = u$ and $u_j^L = u_j$. Using the fact that Monge–Ampère measures are plurifine, we get that

$$\int_{\{u > \phi - L\}} \psi(u_j - u)\theta_{u_j}^n = \int_{\{u > \phi - L\}} \psi(u_j^L - u^L)\theta_{u_j^L}^n.$$

Thus we need to show that

$$\left| \int_{\{u \leq \phi - L\}} \psi(u_j - u)\theta_{u_j}^n - \int_{\{u \leq \phi - L\}} \psi(u_j^L - u^L)\theta_{u_j^L}^n \right| \to 0.$$

We will show that both the terms go to 0 as $L \to \infty$ independent of j.

Notice that since $u_j \geq u$ we have $u_j - u = (\phi - u) - (\phi - u_j) \leq \phi - u$. Since ψ is increasing in $(0, \infty)$ we have $\psi(u_j - u) \leq \psi(\phi - u) = \psi(u - \phi)$. Now we can construct a weight $\tilde{\psi}$ such that $\tilde{\psi}$ is still even, $\tilde{\psi}(0) = 0$ and $\tilde{\psi}_{|(0,\infty)}$ is smooth increasing and concave such that $\psi(t) \leq \tilde{\psi}(t)$ and $\psi(t)/\tilde{\psi}(t) \searrow 0$ as $t \to \infty$. Moreover, we require that $u \in \mathcal{E}_{\tilde{\psi}}(X, \theta, \phi)$. Since $u_j \geq u$, we have $u_j \in \mathcal{E}_{\tilde{\psi}}(X, \theta, \phi)$ as well. Now

$$\begin{aligned}\int_{\{u \leq \phi - L\}} \psi(u_j - u)\theta_{u_j}^n &\leq \int_{\{u \leq \phi - L\}} \psi(u - \phi)\theta_{u_j}^n \\ &\leq \frac{\psi(L)}{\tilde{\psi}(L)} \int_{\{u \leq \phi - L\}} \tilde{\psi}(u - \phi)\theta_{u_j}^n.\end{aligned}$$

Since $u, u_j \in \mathcal{E}_{\tilde{\psi}}(X, \theta, \phi)$, Lemma 2.3 tells us

$$\leq 2\frac{\psi(L)}{\tilde{\psi}(L)} \left(\int_X \tilde{\psi}(u - \phi)\theta_u^n + \int_X \tilde{\psi}(u_j - \phi)\theta_{u_j}^n \right).$$

Using Lemma 2.2 and noticing that $u \leq u_j \leq \phi$ we get that

$$\leq 2(2^{n+1} + 1)\frac{\psi(L)}{\tilde{\psi}(L)} \int_X \tilde{\psi}(u - \phi)\theta_u^n.$$

Since u has finite $\tilde{\psi}$ energy, $\int_X \tilde{\psi}(u - \phi)\theta_u^n$ is finite and $\psi(L)/\tilde{\psi}(L) \to 0$ as $L \to \infty$ independent of j. The same proof shows that $\int_{\{u \leq \phi - L\}} \psi(u_j^L - u^L)\theta_{u_j^L}^n \to 0$ independent of j.

We can also modify the proof to show that it works when $u_j \nearrow u$. □

Lemma 4.3 *Let $u_k \in \mathcal{E}_\psi(X, \theta, \phi)$ be a decreasing that is I_ψ-Cauchy: for any $\varepsilon > 0$, there exists N such that $I_\psi(u_j, u_k) \leq \varepsilon$ for $j, k \geq N$. Then $\lim_{k \to \infty} u_k =: u \in \mathcal{E}_\psi(X, \theta, \phi)$ and $I_\psi(u_k, u) \to 0$.*

Proof Find a subsequence of $\{u_k\}$ and still denote it by $\{u_k\}$ such that

$$I_\psi(u_k, u_{k+1}) \leq C^{-2k}$$

where C is the same constant as in Theorem 3.1. Let $u = \lim_k u_k$. We have to show $u_k \not\equiv -\infty$. We have $I_\psi(u_k, \phi)$ is bounded as $k \to \infty$. To see this, repeated application of Theorem 3.1 gives

$$\begin{aligned}I_\psi(\phi, u_k) &\leq C(I_\psi(\phi, u_1) + I_\psi(u_1, u_k)) \\ &\leq C I_\psi(\phi, u_1) + C^2 I_\psi(u_1, u_2) + C^2 I_\psi(u_2, u_k).\end{aligned}$$

Doing this k times we get

$$\begin{aligned}
&\leq C I_\psi(\phi, u_1) + \sum_{j=2}^{k} C^j I_\psi(u_{j-1}, u_j)\\
&\leq C I_\psi(\phi, u_1) + \sum_{j=2}^{k} C^j C^{-2j+2}\\
&\leq C I_\psi(\phi, u_1) + C^2 \sum_{j=2}^{\infty} C^{-j}\\
&= C I_\psi(\phi, u_1) + \frac{C}{C-1} = \tilde{C}.
\end{aligned}$$

This gives us

$$\int_X \psi(u_k - \phi)\theta_\phi^n \leq I(\phi, u_k) \leq \tilde{C}.$$

Without loss of generality we can assume that $u_1 \leq \phi$ so $u_k \leq u_1 \leq \phi$. Since u_k is decreasing, we have $u_k - \phi$ is decreasing, thus $\psi(u_k - \phi)$ is increasing. Applying monotone convergence theorem we get that

$$\int_X \psi(u - \phi)\theta_\phi^n \leq \tilde{C}.$$

Thus $u \not\equiv -\infty$. Hence $u \in \mathrm{PSH}(X, \theta)$. Since $u_j \searrow u$ and

$$\int_X \psi(u_k - \phi)\theta_{u_k}^n \leq I_\psi(u_k, \phi) \leq \tilde{C}$$

using Lemma 2.7 we get that $u \in \mathcal{E}_\psi(X, \theta)$. Now we need to show that $I_\psi(u_k, u) \to 0$. This means

$$\int_X \psi(u_k - u)\theta_u^n + \int_X \psi(u_k - u)\theta_{u_k}^n \to 0.$$

For the first integral, we notice that $u_k \geq u$, thus $u_k - u \geq 0$ and $u_k - u \searrow 0$ and thus $\psi(u_k - u) \searrow 0$. Thus we can apply the monotone convergence theorem to get

$$\int_X \psi(u_k - u)\theta_u^n \to 0.$$

To show that

$$\int_X \psi(u_k - u)\theta_{u_k}^n \to 0$$

we use Proposition 4.2. Thus $I_\psi(u_k, u) \to 0$. □

Lemma 4.4 *Let $\{u_k\} \in \mathcal{E}_\psi(X, \theta, \phi)$ be such that*

$$I_\psi(u_k, u_{k+1}) \leq C^{-2k}.$$

Then u_k are uniformly bounded from above.

Proof Let

$$v_k = \max\{u_1, \dots, u_k\}.$$

Then repeated application of Theorems 3.1 and 4.1 gives

$$\begin{aligned} I_\psi(\phi, v_k) &\le C I_\psi(\phi, u_1) + C I_\psi(u_1, \max\{u_1, \dots, u_k\}) \\ &\le C I_\psi(\phi, u_1) + C I_\psi(u_1, \max\{u_2, \dots, u_k\}) \\ &\le C I_\psi(\phi, u_1) + C^2 I_\psi(u_1, u_2) + C^2 I_\psi(u_2, \max\{u_2, \dots, u_k\}) \\ &\le C I_\psi(\phi, u_1) + C^2 I_\psi(u_1, u_2) + C^2 I_\psi(u_2, \max\{u_3, \dots, u_k\}). \end{aligned}$$

Doing this k times we get

$$\begin{aligned} &\le C I_\psi(\phi, u_1) + C^2 I_\psi(u_1, u_2) + \dots + C^k I_\psi(u_{k-1}, u_k) \\ &\le \tilde{C}. \end{aligned}$$

Therefore,

$$\int_X \psi(v_k - \phi)\theta_\phi^n \le \tilde{C}.$$

Let $w_k = \max\{\phi, v_k\} \in \mathcal{E}_\psi(X, \theta)$. Then $w_k - \phi \ge 0$ and as v_k is pointwise increasing, we get that $w_k - \phi$ is pointwise increasing. As ψ is increasing on $[0, \infty)$, we get that $\psi(w_k - \phi)$ is pointwise increasing. Let $w := \lim_{k\to\infty} w_k$. Then $\psi(w - \phi) = \lim_{k\to\infty} \nearrow \psi(w_k - \phi)$. The monotone convergence theorem implies

$$\int_X \psi(w - \phi)\theta_\phi^n = \lim_{k\to\infty} \int_X \psi(w_k - \phi)\theta_\phi^n \le \int_X \psi(v_k - \phi)\theta_\phi^n \le \tilde{C}.$$

Thus the set $K = \{\psi(w - \phi) \le (\tilde{C} + 1)/\int_X \theta_\phi^n\}$ has positive θ_ϕ^n-measure. Thus K is not $\mathrm{PSH}(X, \omega)$-polar. We can find a constant A such that $\theta \le A\omega$. Then $\mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$.

Consider the set

$$\mathcal{F}_K = \left\{\varphi \in \mathrm{PSH}(X, A\omega) : \sup_K \varphi = 0\right\}.$$

Since K is not $\mathrm{PSH}(X, A\omega)$-polar, we know that $\mathcal{F}_K$ is compact in $L^1(\omega^n)$-topology. This follows from [20, Theorem 4.7]. Let $\alpha_k = \sup_K v_k$. Let e be such that $\psi(e) = (\tilde{C}+1)/\int_X \theta_\phi^n$. Then $K = \{w - \phi \le e\}$. Notice that $\phi \le 0$ gives

$$\alpha_k = \sup_K v_k \le \sup_K (v_k - \phi) \le \sup_K (w - \phi) \le e.$$

Define $\tilde{v}_k = v_k - \alpha_k$. Then $\tilde{v}_k \in \mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$. Since $\sup_K \tilde{v}_k = 0$, we have $\tilde{v}_k \in \mathcal{F}_K$. Since $\mathcal{F}_K$ is relatively compact, we have $\{\tilde{v}_k\}$ is relatively compact in $L^1(X)$ topology and hence uniformly bounded from above. Therefore $v_k = \tilde{v}_k + \alpha_k$ are uniformly bounded from above and hence $u_k \le v_k$ are uniformly bounded from above. □

The following theorem is the last step in proving completeness of the quasi-metric.

Theorem 4.5 *Let $\{u_j\} \in \mathcal{E}_\psi(X, \theta, \phi)$ be an I_ψ-Cauchy sequence. Then there exists $u \in \mathcal{E}_\psi(X, \theta, \phi)$ such that $I_\psi(u_j, u) \to 0$.*

Proof Extract a subsequence and still denote it by $\{u_j\}$ such that

$$I_\psi(u_j, u_{j+1}) \leq C^{-2j}$$

where C is the same constant as in Theorem 3.1. By Lemma 4.4 $\{u_j\}$ are uniformly bounded from above. Thus

$$v_j := (\sup_{k\geq j} u_k)^* \in \mathrm{PSH}(X, \theta).$$

Since $u_j \leq v_j$ we know using Lemma 2.2 that $v_j \in \mathcal{E}_\psi(X, \theta, \phi)$. Define

$$v_j^l := \max\{u_j, u_{j+1}, \ldots, u_{j+l}\}.$$

Then $v_j^l \in \mathcal{E}_\psi(X, \theta, \phi)$ and $v_j^l \nearrow v_j$ almost everywhere. Lemma 2.7 implies $I_\psi(v_j^l, v_j) \to 0$ as $l \to \infty$. Using quasi-triangle inequality twice we get that $I_\psi(v_j, v_{j+1}) \leq C^2 \lim_{l\to\infty} I_\psi(v_j^{l+1}, v_{j+1}^l)$. Now,

$$I_\psi(v_j^{l+1}, v_{j+1}^l) = I_\psi(\max\{u_j, v_{j+1}^l\}, v_{j+1}^l).$$

Using Theorem 4.1, we get

$$\leq I_\psi(u_j, v_{j+1}^l).$$

Using Theorem 3.1, we get

$$\leq C I_\psi(u_j, u_{j+1}) + C I_\psi(u_{j+1}, \max\{u_{j+1}, v_{j+2}^{l-1}\}).$$

This again by Theorem 4.1 gives us

$$\leq C I_\psi(u_j, u_{j+1}) + C I_\psi(u_{j+1}, v_{j+2}^{l-1}).$$

Applying this l times we get

$$\begin{aligned} &\leq \sum_{k=1}^{l} C^k I_\psi(u_{j+k-1}, u_{j+k}) \\ &\leq \sum_{k=1}^{l} C^k C^{-2j-2k+2} \\ &\leq \sum_{k=1}^{\infty} C^{-2j-k+2} \\ &\leq C^{-2j+2} \frac{1}{C-1} = C^{-2j} \frac{C^2}{C-1}. \end{aligned}$$

Thus we obtain that

$$I_\psi(v_j, v_{j+1}) \leq C^{-2j} \frac{C^4}{C-1}.$$

This shows that $\{v_j\}$ is a decreasing I_ψ-Cauchy sequence. Thus by Lemma 4.3, we get that there exists $v \in \mathcal{E}_\psi(X, \theta, \phi)$ such that $I_\psi(v_j, v) \to 0$.

Now we want to show that $\{u_j\}$ and $\{v_j\}$ are equivalent sequences, i.e., $I_\psi(u_j, v_j) \to 0$. Then $I_\psi(u_j, v) \leq C I_\psi(u_j, v_j) + C I_\psi(v_j, v)$. As both the terms go to 0, we get that $I_\psi(u_j, v) \to 0$.

Now,

$$\begin{aligned}I_\psi(u_j, v_j^l) &= I_\psi(u_j, \max\{u_j, v_{j+1}^{l-1}\})\\ &\le I_\psi(u_j, v_{j+1}^{l-1})\\ &\le CI_\psi(u_j, u_{j+1}) + CI_\psi(u_{j+1}, v_{j+1}^{l-1})\\ &\le CI_\psi(u_j, u_{j+1}) + CI_\psi(u_{j+1}, v_{j+2}^{l-2})\\ &\le CI_\psi(u_j, u_{j+1}) + C^2 I_\psi(u_{j+1}, u_{j+2}) + C^2 I_\psi(u_{j+2}, v_{j+2}^{l-2}).\end{aligned}$$

Doing this l times we get

$$\begin{aligned}&\le CI_\psi(u_j, u_{j+1}) + C^2 I_\psi(u_{j+1}, u_{j+2}) + \dots C^l I_\psi(u_{j+l-1}, u_{j+l})\\ &\le \sum_{k=1}^{l} C^k C^{-2k-2j+2}\\ &\le C^{-2j}\frac{C^2}{C-1}.\end{aligned}$$

Thus

$$I_\psi(u_j, v_j) \le CI_\psi(u_j, v_j^l) + CI_\psi(v_j^l, v_j) \le C^{-2j}\frac{C^3}{C-1}$$

by taking limit $l \to \infty$. Thus $I_\psi(u_j, v_j) \to 0$. Hence we have found $v \in \mathcal{E}_\psi(X, \theta, \phi)$ such that $I_\psi(u_j, v) \to 0$. Thus $(\mathcal{E}_\psi(X, \theta, \phi), I_\psi)$ is a completely metrizable topological space when topologized with the quasi-metric I_ψ. □

This finishes the proof for completeness for of the quasi-metric I_ψ on the space $\mathcal{E}_\psi(X, \theta, \phi)$.

5 Properties of the new topology

The following lemma generalizes [19, Lemma 1.5] and Lemma 2.7.

Lemma 5.1 *Let $u_j \in \mathcal{E}_\psi(X, \theta, \phi)$ be a decreasing sequence and $\varphi \in \mathcal{E}_\psi(X, \theta, \phi)$ be such that*

$$\sup_{j\in\mathbb{N}} \int_X \psi(u_j - \varphi)\theta_{u_j}^n < \infty.$$

Then $u = \lim_j u_j \in \mathcal{E}_\psi(X, \theta, \phi)$.

Proof First we show that $u_j \not\to -\infty$ uniformly. If, on the contrary, $u \to \infty$ uniformly, up to choosing a subsequence and relabeling, we can assume that $u_j < -j$. Choose an A such that $\theta_\varphi^n\{\varphi > -A\} > 0$. Now

$$\begin{aligned}\int_X \psi(u_j-\varphi)\theta^n_{u_j} &= \int_0^\infty \theta^n_{u_j}\{\psi(u_j-\varphi)>t\}dt\\ &= \int_0^\infty \psi'(s)\theta^n_{u_j}\{|u_j-\varphi|>s\}ds\\ &= \int_0^\infty \psi'(s)\theta^n_{u_j}\{\varphi<u_j-s\}ds\\ &\quad+\int_0^\infty \psi'(s)\theta^n_{u_j}\{u_j<\varphi-s\}ds.\end{aligned}$$

Using comparison principle and the fact that $\psi'(s)\geq 0$, we get

$$\geq \int_0^\infty \psi'(s)\theta_\varphi\{u_j<\varphi-s\}ds.$$

Since $u_j<-j$, we get that $\{-j<\varphi-s\}\subset\{u_j<\varphi-s\}$. Thus,

$$\geq \int_0^\infty \psi'(s)\theta^n_\varphi\{-j<\varphi-s\}ds.$$

Choosing $j>A$, we can write

$$\geq \int_0^{j-A} \psi'(s)\theta^n_\varphi\{\varphi>s-j\}ds.$$

Again, we notice that $s<j-A$ implies that $\{\varphi>-A\}\subset\{\varphi>s-j\}$, which gives

$$\begin{aligned}&\geq \int_0^{j-A}\psi'(s)\theta^n_\varphi\{\varphi>-A\}ds\\ &=\theta^n_\varphi\{\varphi>-A\}\psi(j-A)\to\infty\end{aligned}$$

as $j\to\infty$. This is a contradiction to the fact that $\int_X\psi(u_j-\varphi)\theta^n_{u_j}$ is bounded in j.

Moreover we observe that $I_\psi(u_j,\varphi)$ is bounded. Notice that, like in the previous argument, we have

$$\int_X\psi(u_j-\varphi)\theta^n_\varphi=\int_0^\infty\psi'(s)\theta^n_\varphi\{u_j<\varphi-s\}ds+\int_0^\infty\psi'(s)\theta^n_\varphi\{\varphi<u_j-s\}.$$

Using comparison principle for the first expression and the fact that $u_j\leq u_1$ in the second expression, we get that

$$\begin{aligned}&\leq\int_0^\infty\psi'(s)\theta^n_{u_j}\{u_j<\varphi-s\}ds+\int_0^\infty\psi'(s)\theta^n_\varphi\{\varphi<u_1-s\}ds\\ &\leq\int_X\psi(u_j-\varphi)\theta^n_{u_j}+\int_X\psi(u_1-\varphi)\theta^n_\varphi.\end{aligned}$$

Thus $\int_X\psi(u_j-\varphi)\theta^n_\varphi$ is bounded from above independent of j. Now notice that

$$\int_X\psi(u_j-\phi)\theta^n_{u_j}\leq I_\psi(\phi,u_j).$$

By Quasi-triangle inequality for I_ψ, we get

$$\begin{aligned}&\leq C(I_\psi(\phi,\varphi)+I_\psi(u_j,\varphi))\\ &\leq C\left(I_\psi(\phi,\varphi)+\int_X\psi(u_j-\varphi)\theta^n_{u_j}+\int_X\psi(u_j-\varphi)\theta^n_\varphi\right).\end{aligned}$$

Thus $\int_X \psi(u_j - \phi)\theta^n_{u_j}$ is bounded from above independent of j and $u = \lim_j u_j \in \mathrm{PSH}(X,\theta)$, thus by Lemma 2.7 we get that $u \in \mathcal{E}_\psi(X,\theta,\phi)$. □

The following theorem is the generalization of [2, Proposition 2.6]. Also when $\psi(t) = |t|$, the following theorem appears in [29, Proposition 5.7].

Theorem 5.2 *Let $u_j, u \in \mathcal{E}_\psi(X,\theta,\phi)$ be such that $I_\psi(u_j, u) \to 0$. Then there exits a subsequence still denoted by u_j and $v_j, w_j \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $v_j \le u_j \le w_j$ and v_j increase to u a.e. and w_j decrease to u. Thus, by Proposition 4.2 and monotone convergence theorem, $I_\psi(v_j, u) \to 0$ and $I_\psi(w_j, u) \to 0$.*

Proof We can pass to a subsequence of (u_j) such that $I_\psi(u_j, u) \le C^{-2j}$ where C is the same constant as in Theorem 3.1. By quasi-triangle inequality, $I_\psi(u_j, u_{j+1}) \le C^{-2j+2}$. By Lemma 4.4, u_j are uniformly bounded from above. Thus

$$w_j := (\sup_{k\ge j} u_k)^* \in \mathcal{E}_\psi(X,\theta,\phi).$$

Moreover, by the proof of Theorem 4.5 we get that w_j is a decreasing sequence in $\mathcal{E}_\psi(X,\theta)$ and is equivalent to u_j. Hence $w_j \ge u_j$ and $w_j \searrow u$ and thus $I_\psi(w_j, u) \to 0$.

For $j < k$, define $v^k_j = P_\theta(\min(u_j, \dots, u_k))$. By Theorem 2.9, $v^k_j \in \mathcal{E}_\psi(X,\theta,\phi)$. Moreover, by [11, Lemma 3.7],

$$\theta^n_{v^k_j} \le \sum_{l=j}^{k} \mathbb{1}_{\{v^k_j = u_l\}} \theta^n_{u_l}.$$

Therefore

$$\int_X \psi(u - v^k_j)\theta^n_{v^k_j} \le \sum_{l=j}^{k} \int_X \psi(u - u_l)\theta^n_{u_l} \le \sum_{l=j}^{k} I_\psi(u, u_l) \le C^{-2j+2}.$$

Also v^k_j is decreasing as $k \to \infty$, thus $v_j := \lim_k v^k_j \in \mathcal{E}_\psi(X,\theta,\phi)$ by Lemma 5.1 $v_j \in \mathcal{E}_\psi(X,\theta,\phi)$.

Since v^k_j decreases to v_j, we get that $v^k_j \to v_j$ in capacity. Moreover, the functions $\psi(u - v^k_j) \to \psi(u - v_j)$ in capacity. Using Lemma 2.6, we get that

$$\int_X \psi(u - v_j)\theta^n_{v_j} \le \liminf_{k\to\infty} \int_X \psi(u - v^k_j)\theta^n_{v^k_j} \le C^{-2j+2}.$$

Since $(\sup_{k\ge j} u_k)^* = w_j$, therefore, $\sup_{k\ge j} u_k = w_j$ a.e. As $w_j \searrow u$, we get that $\limsup_k u_k = u$ a.e. For any v_j, we have $v_j \le u_k$ for $k \ge j$. Taking limsup we get $v_j \le \limsup_{k\ge j} u_k = u$. Therefore, $v = (\lim_j v_j)^* \le u$.

By the same argument as before, we have,

$$\int_X \psi(u - v)\theta^n_v \le \liminf_{j\to\infty} \int_X \psi(u - v_j)\theta^n_{v_j} \le \liminf_{j\to\infty} C^{-2j+2} = 0.$$

Thus $\theta^n_v(\{u \ne v\} = 0$. Again using the domination principle (see Lemma 2.5), we get that $u \le v$ everywhere. This shows that $u = v$.

Thus we found an increasing sequence $v_j \le u_j$ increasing to u. □

Corollary 5.3 *The I_ψ-topology on $\mathcal{E}_\psi(X,\theta,\phi)$ is stronger than the usual $L^1(\omega^n)$ topology on $\mathcal{E}_\psi(X,\theta,\phi)$. More precisely, if $u_k, u \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $I_\psi(u_k,u) \to 0$, then*

$$\int_X |u_k - u|\omega^n \to 0$$

as $k \to \infty$.

Proof It is enough to show $L^1(\omega^n)$ convergence for a subsequence. Let u_{j_k} be the subsequence provided by Theorem 5.2 and v_{j_k} and w_{j_k} be corresponding monotone sequences. Then $v_{j_k} \le u_{j_k} \le w_{j_k}$ and $v_{j_k} \le u \le w_{j_k}$. Then

$$\begin{aligned}\int_X |u_{j_k} - u|\omega^n &\le \int_X (w_{j_k} - v_{j_k})\omega^n \\ &\le \int_X (w_{j_k} - u)\omega^n + \int_X (u - v_{j_k})\omega^n \\ &\to 0\end{aligned}$$

by the monotone convergence theorem. Thus the new topology is stronger and has more open, thus closed sets. □

Theorem 5.4 *If $u_j, u \in \mathcal{E}_\psi(X,\theta,\phi)$, such that $I_\psi(u_j,u) \to 0$ as $j \to \infty$, then $u_j \to u$ in capacity.*

Proof It is enough to show the convergence in capacity for a subsequence. Let u_{j_k} be a subsequence as provided by Theorem 5.2 and v_{j_k} and w_{j_k} are corresponding monotone sequences converging to u. We get that $v_{j_k} \to u$ and $w_{j_k} \to u$ in capacity. We want to claim that $u_{j_k} \to u$ in capacity as well. Fix $\varepsilon > 0$.

$$\{|u_{j_k} - u| > \varepsilon\} \subset \{w_{j_k} - v_{j_k} > \varepsilon\} \subset \{w_{j_k} - u > \varepsilon/2\} \cup \{u - v_{j_k} > \varepsilon/2\}.$$

Taking capacity of above sets we get

$$\mathrm{Cap}_\omega\{|u_{j_k} - u| > \varepsilon\} \le \mathrm{Cap}_\omega\{w_{j_k} - u > \varepsilon/2\} + \mathrm{Cap}_\omega\{u - v_{j_k} > \varepsilon/2\}$$

and taking limit $k \to \infty$ we get $\lim_{k\to\infty} \mathrm{Cap}_\omega\{|u_{j_k} - u| > \varepsilon\} = 0$.

Since for any subsequence u_{j_k} of u_j we can find another subsequence $u_{j_{k_l}}$ for which $\lim_{l\to\infty} \mathrm{Cap}_\omega\{|u_{j_{k_l}} - u| > \varepsilon\} = 0$, we get that $\lim_{j\to\infty} \mathrm{Cap}_\omega\{|u_j - u| > \varepsilon\} = 0$. Thus I_ψ convergence implies convergence in capacity. □

Corollary 5.5 (Weak convergence of measures) *If $u_k, u \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $I_\psi(u_k,u) \to 0$, then $\theta^n_{u_k} \to \theta^n_u$ weakly as measures.*

Proof Since $u_k \to u$ in I_ψ, Theorem 5.4 shows that $u_k \to u$ in capacity. Since u_k, u have full mass, [11, Theorem 2.3] (see also [34, Theorem 1] and [33, Theorem 1]) implies that $\theta^n_{u_k} \to \theta^n_u$ weakly. □

6 Kähler Ricci flow

In [22], authors showed that we can start Kähler Ricci flow from any potential φ with zero Lelong numbers. They showed that if $\varphi \in \mathrm{PSH}(X,\omega)$ has zero Lelong numbers then there exist smooth potentials φ_t for small time such that

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right], \quad \varphi_t \to \varphi \text{ in } L^1(\omega^n) \text{ as } t \to 0. \tag{22}$$

Di Nezza–Lu [15, Corollary 5.2] further showed that $\varphi_j \to \varphi$ in capacity.

Without loss of generality, we can assume that $\varphi \leq -1$. In [22, Lemma 2.9], the authors showed that (for $\beta = 1$ and $\alpha = 1$) for a continuous ω-psh function u, satisfying

$$(\omega + dd^c u)^n = e^{u-2\varphi}\omega^n$$

we have

$$(1-2t)\varphi(x) + tu(x) + n(t\ln t - t) \leq \varphi_t.$$

Since $\varphi \leq -1$, we have $\varphi \leq (1-2t)\varphi$ for small t and since u is continuous there exists some constant C such that $C \leq u$. Combining these, we get

$$\varphi(x) + tC + n(t\ln t - t) \leq \varphi_t$$

or

$$\varphi(x) \leq \varphi_t - tC - n(t\ln t - t).$$

Notice that $-tC - n(t\ln t - t) = f(t)$ satisfies $f(t) \to 0$ as $t \to 0$.

Since $\varphi_t \to \varphi$ in $L^1(\omega^n)$ as $t \to 0$, we also obtain that $\varphi_t + f(t)$ converge to φ in $L^1(\omega^n)$ as $t \to 0$.

Lemma 6.1 *Given $\varphi \in \mathcal{E}_\psi(X,\omega)$ and a sequence $\varphi_j \in \mathcal{E}_\psi(X,\omega)$ such that*

$$\varphi \leq \varphi_j$$

and $\varphi_j \to \varphi$ in $L^1(\omega^n)$ as $j \to \infty$, then $I_\psi(\varphi_j,\varphi) \to 0$ as $j \to \infty$.

Proof Since $\varphi_j \to \varphi$ in $L^1(\omega^n)$, we get a subsequence φ_{j_k} such that $\varphi_{j_k} \to \varphi$, a.e. Consider the functions

$$v_j = \sup_{k\geq j} \varphi_{j_k}.$$

Then $v_j^* \in \mathrm{PSH}(X,\omega)$ and $v_j^* = v_j$ except on a pluripolar set. Moreover, since $v_j^* \searrow \varphi$ ω^n almost everywhere, we get that $v_j^* \searrow \varphi$ everywhere.

This shows that $v_j \searrow \varphi$ except on a pluripolar set. This means $\lim_{j\to\infty}\sup_{k\geq j}\varphi_{k_j} = \varphi$ except on a pluripolar set. Therefore

$$\limsup_{k\to\infty} \varphi_{j_k} = \varphi$$

except on a pluripolar set. Since $\varphi_{j_k} \geq \varphi$, we automatically have

$$\liminf_{k\to\infty} \varphi_{j_k} \geq \varphi$$

everywhere.

Therefore, $\varphi_{j_k} \to \varphi$ except on a pluripolar set. Using $\varphi \leq \varphi_{j_k} \leq v_k$ and that $v_k \searrow \varphi$ as $k \to \infty$, we get

$$\lim_{k\to\infty}\int_X \psi(\varphi_{j_k} - \varphi)\omega_\varphi^n \leq \lim_{k\to\infty}\int_X \psi(v_k - \varphi)\omega_\varphi^n = 0. \tag{23}$$

Moreover, since $\varphi_{j_k} \geq \varphi$, we have

$$\int_X \psi(\varphi_{j_k} - \varphi)\omega_{\varphi_{j_k}}^n = \int_0^\infty \psi'(s)\omega_{\varphi_{j_k}}^n(\varphi_{j_k} > \varphi + s)ds.$$

Using comparison theorem, we get

$$\leq \int_0^\infty \psi'(s)\omega_\varphi^n(\varphi_{j_k} > \varphi + s)ds$$
$$= \int_X \psi(\varphi_{j_k} - \varphi)\omega_\varphi^n.$$

Taking limit we get

$$\lim_{k\to\infty} \int_X \psi(\varphi_{j_k} - \varphi)\omega_{\varphi_{j_k}}^n \leq \lim_{k\to\infty} \int_X \psi(\varphi_{j_k} - \varphi)\omega_\varphi^n = 0.$$

Combining these two we get

$$\lim_{k\to\infty} I_\psi(\varphi_{j_k}, \varphi) = \lim_{k\to\infty} \int_X \psi(\varphi_{j_k} - \varphi)(\omega_{\varphi_{j_k}}^n + \omega_\varphi^n) = 0.$$

Since this holds for any subsequence of φ_j, we get that

$$I_\psi(\varphi_j, \varphi) \to 0.$$

□

The following corollary generalizes the [22, Proposition 5.2] where they show that $\varphi_t \to \varphi$ in energy if $\varphi \in \mathcal{E}^1(X, \omega)$.

Corollary 6.2 *Let $\varphi \in \mathcal{E}_\psi(X, \omega)$. Let φ_t be the solution to Eq. (22). Then $I_\psi(\varphi_t, \varphi) \to 0$ as $t \to 0$.*

Proof We saw earlier that the assumptions imply that

$$\varphi \leq \varphi_t + f(t)$$

where $f(t) \to 0$ as $t \to 0$. Since $\varphi_t + f(t)$ are also θ-psh functions which converge in $L^1(\omega^n)$ to φ as $t \to 0$, using Lemma 6.1, we get that $I_\psi(\varphi_t, \varphi) \to 0$ as $t \to 0$. □

Note that using Theorem 5.4 and Corollary 5.5 we obtain that $\varphi_t \to \varphi$ in capacity and $\omega_{\varphi_t}^n \to \omega_\varphi^n$ as $t \to 0$, thus finishing the proof of Theorem 1.3.

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