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The Mabuchi geometry of low energy classes

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Abstract
Let (X, ω) be a Kähler manifold and $\psi : \mathbb{R} \to \mathbb{R}_+$ be a concave weight. We show that $\mathcal{H}_\omega$ admits a natural metric d_ψ whose completion is the low energy space $\mathcal{E}_\psi$, introduced by Guedj–Zeriahi. As d_ψ is not induced by a Finsler metric, the main difficulty is to show that the triangle inequality holds. We study properties of the resulting complete metric space $(\mathcal{E}_\psi, d_\psi)$.

1 Introduction

Let (X, ω) be a compact connected Kähler manifold. A basic problem in Kähler geometry is to find various canonical metrics among the Kähler metrics ω' that are in the same de Rham cohomology class as ω [12, 44]. Due to Hodge theory, such ω' can be written as $\omega' = \omega + i\partial\bar\partial u$, where u is a smooth function from the space of Kähler potentials:

$$\mathcal{H}_\omega := \{v \in C^\infty(X) \;\; \omega_v := \omega + i\partial\bar\partial v > 0\}.$$

When studying weak notions of Kähler metrics, or degenerations of smooth ones, a natural space to consider is the space of ω-plurisubharmonic (ω-psh) functions $\mathrm{PSH}(X, \omega)$. With slight abuse of precision, we say that $v : X \to [-\infty, \infty)$ is ω-psh if it is usc, integrable and $\omega_v := \omega + i\partial\bar\partial v \geq 0$ in the sense of currents.

As pointed out in a series of works by Guedj–Zeriahi, and their collaborators [2, 11, 31, 34] the space of full mass potentials $\mathcal{E} = \{v \in \mathrm{PSH}(X, \omega) \;\; \int_X \omega_v^n = \int_X \omega^n\}$ has a prominent role in the study of weak solutions to complex Monge–Ampère equations with measure theoretic right hand side. Here ω_v^n is the non-pluripolar complex Monge–Ampere measure of v, extending the interpetation of Bedford–Taylor and Cegrell from

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the local case [1, 13]. To study $\mathcal{E}$, it is feasible to consider various weights $\phi : \mathbb{R} \to \mathbb{R}^+$ and consider the subspaces with finite ϕ-energy:

$$\mathcal{E}_\phi := \left\{ v \in \mathcal{E} \mid E_\phi(v) := \int_X \phi(v)\omega_v^n < \infty \right\}.$$

One important point is that $\mathcal{E}$ can be exhausted by a special class of finite energy subspaces [34, Proposition 2.2]:

$$\mathcal{E} = \bigcup_{\psi \in \mathcal{W}^-} \mathcal{E}_\psi. \tag{1}$$

Here $\mathcal{W}^-$ is the space of weights $\phi : [-\infty, \infty] \to [0, \infty]$ that are even and continuous on $\mathbb{R}$, in addition to being smooth, concave, and strictly increasing on $(0, \infty)$, normalized by $\psi(0) = 0$ and $\psi(\pm\infty) = \infty$. Following terminology of [34], we will call elements of $\mathcal{W}^-$ *concave weights*, and the spaces $\mathcal{E}_\psi$ *low energy classes* (see Remark 2.1 for superficial differences in our discussion compared to [34]).

As noticed in [4, Section 2], the subspace $\mathcal{E}_1 = \{v \in \mathcal{E} \ \int_X |v|\omega_v^n < \infty\}$ has a complete metric topology. This was refined further in [20], where it was noticed that the more general *high energy classes* $\mathcal{E}_\chi$ are the metric completions of an appropriately defined Orlicz–Finsler metric structure on the smooth space $\mathcal{H}_\omega$. Recall that high energy classes $\mathcal{E}_\chi$ are given by weights $\chi : \mathbb{R} \to \mathbb{R}^+$, that are even convex functions satisfying $\chi(0) = 0$, $\chi'(1) = 1$ and a growth estimate $t\chi'(t) \leq p\chi(t)$ (notation: $\chi \in \mathcal{W}_p^+$).

Moreover, in [20] it was also pointed out the the resulting metric spaces $(\mathcal{E}_\chi, d_\chi)$ admit geodesic segments connecting arbitrary points. This latter fact had a wide range of applications, including energy properness [5, 14, 15, 26], K-stability [6], convergence and existence the weak Calabi flow [3, 41], etc.

Unfortunately, for all $\chi \in \mathcal{W}_p^+$ we have the inclusions $\mathcal{E}_\chi \subseteq \mathcal{E}_1 \subsetneq \mathcal{E}$. As a result, it is natural to ask if subspaces of $\mathcal{E}$ not included in $\mathcal{E}_1$ can be naturally topologized/geometrized as well. One may even ask: does $\mathcal{E}$ admit a natural topology/geometry? Revisiting (1), one is tempted to first find a natural metric topology on the low energy spaces $\mathcal{E}_\psi$, as these exhaust $\mathcal{E}$. This is what we accomplish in this paper.

Not much is known about the metric geometry of low energy spaces, despite their wast array of applications to weak solutions of complex Monge–Ampère equations [11, 34]. The only related result seems to be [36, Theorem 1.6], implying existence of a metrizable uniform space topology on $\mathcal{E}_p := \{u \in \mathcal{E}, \int_X |u|^p\omega_u^n < \infty\}, \ p \in (0, 1)$.

To start, let $\mathcal{H}_\omega^\Delta := \mathrm{PSH}(X, \omega) \cap C^{1,\bar{1}}$, where by $C^{1,\bar{1}}$ we denote functions on X with bounded mixed second partial derivatives. Equivalently, $\mathcal{H}_\omega^\Delta$ is the space of ω-psh potentials with bounded Laplacian. Given $u_0, u_1 \in \mathcal{H}_\omega$, let $[0, 1] \ni t \to u_t \in \mathcal{H}_\omega^\Delta$ be Chen's weak geodesic joining u_0, u_1 [17, 18]. We introduce the following candidate metric on $\mathcal{H}_\omega$:

$$d_\psi(u_0, u_1) := \int_X \psi(\dot{u}_0)\omega_{u_0}^n. \tag{2}$$

The above definition of d_ψ bears superficial similarities with the one in [20, (2)], dealing with the case of high energy classes. However it is not difficult to see that d_ψ is not induced by the length metric of a Finsler structure, contrasting with [20]. Thus one has to work hard to prove the triangle inequality, this being the first main result of this paper:

Theorem 1.1 *$(\mathcal{H}_\omega, d_\psi)$ is a metric space.*

Hoping for further analogies with the case of high energy classes [20], one might mistakenly expect that $(\mathcal{H}_\omega, d_\psi)$ is at least a length space, and the weak geodesic $t \to u_t$ appearing in (2) is a metric d_ψ-geodesic connecting u_0, u_1. This is unfortunately not the case either. In fact, when $\psi(t) = |t|^\alpha$, $\alpha \in (0, 1)$, one can easily verify that the d_ψ-length of smooth curves inside $\mathcal{H}_\omega$ is always zero. In addition, this also confirms that d_ψ can not be induced by a Finsler metric on $\mathcal{H}_\omega$.

In fact, the right analogy to follow here is the one coming from the case of toric Kähler manifolds (X_T, ω_T), and restricting d_ψ to the torus invariant potentials $\mathcal{H}^T_\omega$. As is well known, the Legendre transform $\mathcal{L}$ transforms $\mathcal{H}^T_\omega$ bijectively into the space $Conv_\omega(P)$ of smooth convex functions on a Delzant polytope $P \subset \mathbb{R}^n$ with specific asymptotics near ∂P (for details see [32, Section 4] or [19]). By the same calculations as in [32, Proposition 4.3], we obtain that for $u_0, u_1 \in \mathcal{H}^T_\omega$ we have

$$d_\psi(u_0, u_1) = \int_P \psi(\mathcal{L}(u_0) - \mathcal{L}(u_1)) d\mu,$$

with μ being the Lebesque measure on P. By Lemma 2.6 below we immediately see that in this case d_ψ satisfies the triangle inequality trivially.

In addition, the d_ψ-completion of $\mathcal{H}^T_\omega$ will be $L^\psi(P) \cap Conv(P)$, the space of convex functions on P that have finite ψ-integral. This space is exactly the Legendre dual of $\mathcal{E}^T_\psi$, the set of torus invariant potentials in $\mathcal{E}_\psi$ [32, Proposition 4.5].

With the toric analogies in mind, our reader is perhaps less surprised by the statement of Theorem 1.1 above, and might also expect that the metric completion of $(\mathcal{H}_\omega, d_\psi)$ equals $\mathcal{E}_\psi$, even in the absence of toric symmetries. This is confirmed in our next main result.

Theorem 1.2 *The metric d_ψ extends to $\mathcal{E}_\psi$, making $(\mathcal{E}_\psi, d_\psi)$ a complete metric space, that is the metric completion of $(\mathcal{H}_\omega, d_\psi)$.*

This result is analogous to [20, Theorem 2] that deals with the case of high energy classes. The similarities don't stop here. Paralleling [20, Theorem 3], the d_ψ metric is comparable to a concrete analytic expression:

Theorem 1.3 *For any $u_0, u_1 \in \mathcal{E}_\psi$ we have*

$$d_\psi(u_0, u_1) \le \int_X \psi(u_0 - u_1)\omega^n_{u_0} + \int_X \psi(u_0 - u_1)\omega^n_{u_1} \le 2^{2n+5} d_\psi(u_0, u_1). \quad (3)$$

This result implies that the expression $I_\psi(u_0, u) := \int_X \psi(u_0 - u_1)\omega^n_{u_0} + \int_X \psi(u_0 - u_1)\omega^n_{u_1}$ satisfies a quasi-triangle inequality, a result of independent interest. Previously this was obtained using analytic methods for the weights $\psi(t) = |t|^p,\ p \in (0,1)$ in [36, Theorem 1.6].

As pointed about above, the absence of a background Finsler structure requires a new approach to the proof of Theorem 1.1. However once the triangle inequality is obtained, many pluripotential theoretic arguments can be used from [20], and this will be apparent in the proofs of Theorems 1.2 and 1.3.

Contrasting with the case of high energy classes explored in [20], our methods suggest that the metric d_ψ is somehow positively curved (see Proposition 4.4 and Corollary 4.3). However it remains to be seen if such a notion can be defined for non-geodesic metric spaces, as it is the case here. Since weak geodesic segments are used to define d_ψ in (2), it could be beneficial to understand what role these curves play from a metric/geometric point of view. In a different direction, it would be interesting to extend our results to more singular spaces. There has been a flurry of activity in this latter area recently, focusing on the high energy case [29, 42, 43]. Lastly, we are curious if the quantization scheme of the high energy spaces from [27] has an analogue in our low energy context. We hope to investigate these questions, as well as applications in future works.

Organization. In Sect. 2 we recall known results about finite energy classes, and obtain the second order variation of low energy weak quasi-norms. After some preliminary results on our candidate metric d_ψ in Sect. 3, we prove the triangle inequality (and Theorem 1.1) in Sect. 4. Theorem 1.2 is proved in Sect. 5. Finally, Theorem 1.3 is proved in Sect. 6.

2 Preliminaries

Most of our notation and terminology builds on that of [20, 34] and the survey [23]. We refer our reader to these works for a detailed background. Below we only recall the basics, adapted to our specific context.

2.1 Finite energy classes

We recall here some basic facts about the class $\mathcal{E} \subset \text{PSH}(X,\omega)$ and its subspaces. We refer to the original papers [11, 34] and the recent book [35] for a complete picture. For $v \in \text{PSH}(X,\omega)$, the canonical cutoffs $v_h,\ h \in \mathbb{R}$ are given by the formula $v_h := \max(-h, v)$. By an application of the comparison principle of Bedford–Taylor theory, it follows that the Borel measures $\mathbb{1}_{\{v>-h\}}(\omega + i\partial\bar\partial v_h)^n$ are increasing in h. As a result, one can make make sense of $(\omega + i\partial\bar\partial v)^n$ as the limit of these increasing measures, even if v is unbounded:

$$\omega_v^n := (\omega + i\partial\bar\partial v)^n = \lim_{h\to\infty} \mathbb{1}_{\{v>-h\}}(\omega + i\partial\bar\partial v_h)^n. \tag{4}$$

With this definition, ω_v^n is called the non-pluripolar Monge–Ampère measure of v. It follows from (4) that

$$\int_X \omega_v^n \leq \int_X \omega^n =: V,$$

bringing us to the class of full mass potentials $\mathcal{E}$. By definition, $v \in \mathcal{E}$ if

$$\int_X \omega_v^n = \lim_{h\to\infty} \int_X \mathbb{1}_{\{v>-h\}}(\omega + i\partial\bar{\partial}v_h)^n = V. \tag{5}$$

Suppose $\phi : [-\infty, \infty] \to [0, \infty]$ is a continuous even function, with $\phi(0) = 0$ and $\phi(\pm\infty) = \infty$. Such ϕ is referred to as a *weight*. The set of all weights is denoted by $\mathcal{W}$. By definition, for $v \in \mathcal{E}$ we have $v \in \mathcal{E}_\phi$ if

$$E_\phi(v) := \int_X \phi(v)\omega_v^n < \infty.$$

The two special classes of weights that are most interesting in the theory are:

$$\mathcal{W}^- = \{\psi \in \mathcal{W} \mid \psi \text{ is concave, strictly increasing, and smooth on } (0,\infty)\},$$
$$\mathcal{W}_p^+ = \{\chi \in \mathcal{W} \mid \chi \text{ is convex and } t\chi'(t) \leq p\chi(t),\ t \in \mathbb{R}\},$$

where $p \geq 1$. We note the sign difference between our convention for $\mathcal{W}^-$, $\mathcal{W}_p^+$, and the ones in [34] and [22, Section 2.3].

Of particular importance are the weights $\chi_p(t) = |t|^p$, $p > 0$, and the associated classes $\mathcal{E}_p := \mathcal{E}_{\chi_p}$. Note that $\chi_p \in \mathcal{W}_p^+$ for $p \geq 1$ and $\chi_p \in \mathcal{W}^-$ for $0 < p \leq 1$. The case $p = 1$ interpolates between convex and concave energy classes since

$$\mathcal{E}_\chi \subset \mathcal{E}_1 \subset \mathcal{E}_\psi,$$

for any $\chi \in \mathcal{W}_p^+$ and $\psi \in \mathcal{W}^-$.

In this work we will be focusing on the concave weights $\mathcal{W}^-$. As mentioned in the introduction, the interest in them comes from the following fact [34, Proposition 2.2]:

$$\mathcal{E} = \{v \in \mathcal{E}_\psi \mid \psi \in \mathcal{W}^-\}. \tag{6}$$

Remark 2.1 To be precise, in [34] the authors proved (6) for concave weights ψ that are not necessarily smooth on $(0, \infty)$. However it is elementary to see that for a non-smooth concave weight ψ we can find another concave weight $\tilde{\psi}$, smooth on $(0, \infty)$, such that $\mathcal{E}_\psi = \mathcal{E}_{\tilde{\psi}}$. Indeed, one can even make sure that $\psi - \tilde{\psi}$ is bounded. Because of this, very little is gained from working with more general concave weights. For sake of brevity we leave it to the interested reader to work out the details of our results in the case when the weights $\psi \in \mathcal{W}^-$ are not assumed to be smooth on $(0, \infty)$. This can be carried out using approximation, much in the same way as it is done in [20].

The following result is sometimes called the “fundamental estimate”:

Proposition 2.2 [34, Lemma 2.3, Lemma 3.5] *Let* $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+,\ p \geq 1$. *If* $u, v \in \mathcal{E}_\phi$ *with* $u \leq v \leq 0$ *then*

$$E_\phi(v) \leq C E_\phi(u).$$

Here $C > 0$ *depends only on* p.

If $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+,\ p \geq 1$ then the ϕ-energy has a very useful continuity property:

Proposition 2.3 [34, Proposition 5.6] *Let* $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+$ *and* $\{u_j\}_{j\in\mathbb{N}} \subset \mathrm{PSH}(X,\omega) \cap L^\infty$ *is a sequence decreasing to* $u \in \mathrm{PSH}(X,\omega)$. *If* $\sup_j E_\phi(u_j) < \infty$ *then* $u \in \mathcal{E}_\phi$. *Moreover we have*

$$E_\phi(u) = \lim_{j\to\infty} E_\phi(u_j).$$

Using the canonical cutoffs, the last two results imply the very important “monotonicity property”:

Corollary 2.4 *Let* $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+,\ p \geq 1$. *If* $u \leq v$ *and* $u \in \mathcal{E}_\phi$ *then* $v \in \mathcal{E}_\phi$.

We note that the continuity property of the Monge–Ampère operator from Bedford–Taylor theory [1] is also preserved in this more general setting:

Proposition 2.5 [11, Theorem 2.17] *Suppose* $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{E}(X,\omega)$ *decreases (increases a.e.) to* $v \in \mathcal{E}(X,\omega)$. *Then* $\omega_{v_k}^n \to \omega_v^n$ *weakly.*

A more general weak convergence result is proved in [23, Proposition 2.20], and the remark following it.

2.2 The L^2 metric and weak geodesics

As introduced by Mabuchi, and independently by Semmes and Donaldson, $\mathcal{H}_\omega$ can be endowed with a natural infinite dimensional L^2-type Riemannian metric:

$$\langle \alpha, \beta \rangle_u = \frac{1}{\int_X \omega^n} \int_X \alpha\beta\omega_u^n, \quad \alpha, \beta \in T_u\mathcal{H}_\omega = C^\infty(X). \tag{7}$$

One can compute the Levi–Civita connection $\nabla_{(\cdot)}(\cdot)$ of this inner-product and the associated geodesic equation. For a thorough discussion of the L^2 Mabuchi–Semmes–Donaldson geometry, as well as its Levi-Civita connection, we refer to the surveys [8, Section 4], [23, Section 3.1], as well as the original papers [17, 30, 39, 40].

Unfortunately smooth geodesics connecting arbitrary $u_0, u_1 \in \mathcal{H}_\omega$ don’t exist, but a weak notion of geodesic was studied by Chen [17]. His construction can be generalized to construct weak geodesic segments connecting points of $\mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Following Berndtssson, we recall how this argument works.

As before, let $S \subset \mathbb{C}$ be the strip $\{0 < \text{Re } s < 1\}$ and $\tilde{\omega}$ be the pullback of ω to the product $S \times X$. As argued in [7, Section 2.1], for $u_0, u_1 \in \text{PSH}(X, \omega) \cap L^\infty(X)$ the following Dirichlet problem has a unique solution:

$$\begin{aligned} & u \in \text{PSH}(S \times X, \tilde{\omega}) \cap L^\infty(S \times X) \\ & (\tilde{\omega} + i\partial\bar{\partial}u)^{n+1} = 0 \\ & u(t + ir, x) = u(t, x)\ \forall x \in X, t \in (0, 1), r \in \mathbb{R} \\ & \lim_{t \to 0,1} u(t, x) = u_{0,1}(x), \forall x \in X. \end{aligned} \tag{8}$$

Since the solution u invariant in the imaginary direction, we denote it by $[0, 1] \ni t \to u_t \in \text{PSH}(X, \omega) \cap L^\infty$ and call it the weak geodesic joining u_0 and u_1.

In case $u_0, u_1 \in \mathcal{H}_\omega$ in [17] it was proved that $\Delta u \in L^\infty(S \times X)$. Such a curve $[0, 1] \ni t \to u_t \in \text{PSH}(X, \omega) \cap C^{1,\bar{1}} =: \mathcal{H}_\omega^\Delta$ is called a $C^{1,\bar{1}}$-geodesic.

A curve $[0, 1] \ni t \to v_t \in \text{PSH}(X, \omega)$ is called a subgeodesic if $v(s, x) := v_{\text{Re}s}(x) \in \text{PSH}(S \times X, \tilde{\omega})$. We recall that the solution u of (8) is constructed as the upper envelope

$$u = \sup_{v \in \mathcal{S}} v, \tag{9}$$

where $\mathcal{S}$ is the following set of weak subgeodesics:

$$\mathcal{S} = \left\{ (0, 1) \ni t \to v_t \in \text{PSH}(X, \omega) \text{ is a subgeodesic with } \lim_{t \to 0,1} v_t \le u_{0,1} \right\}.$$

For a thorough discussion of weak geodesics we refer to Sect. 3 in the survey [23].

2.3 First and second order variation of weak quasi-norms

To start, we observe that concave weights are subadditive:

Lemma 2.6 *Let $\psi \in \mathcal{W}^-$. Then ψ is subadditive, i.e., $\psi(a+b) \le \psi(a)+\psi(b)$, $a, b \in \mathbb{R}$.*

Proof Since ψ is increasing on $[0, \infty)$ we have that $\psi(a + b) = \psi(|a + b|) \le \psi(|a| + |b|)$. On the other hand, since $\partial_+\psi$ is decreasing on $(0, \infty)$, we can finish the proof in the following way:

$$\begin{aligned} \psi(|a|) = \psi(|a|) - \psi(0) &= \int_0^{|a|} \partial_+\psi(t)dt \\ &\ge \int_{|b|}^{|a|+|b|} \partial_+\psi(t)dt = \psi(|a| + |b|) - \psi(|b|). \end{aligned}$$

□

To any concave weight $\psi \in \mathcal{W}^-$, and a finite measure space (Y, μ), one can associate the space L^ψ_μ. These will be μ-measurable functions f, such that

$\int_Y \psi(f)d\mu < \infty$. For such functions f, we can associate the a *weak quasi-norm* that is only homogeneous, and typically does not satisfy (even weaker versions of) the triangle inequality:

$$\|f\|_{\psi,\mu} := \inf\{N > 0 \text{ s.t. } \int_Y \psi(f/N)d\mu \leq 1\}. \tag{10}$$

When $\psi \in \mathcal{W}_p^+$, the above quantity does define a bona-fide norm, and these are used in the Kähler geometry literature for approximation of L^p Finsler metrics [20, 24]. Despite the fact that in our case the triangle inequality fails, these weak quasi-norms will still be important in our discussion. To note, compared to [20, (13)], our definition in (10) is slightly different. There, to obtain the Hölder inequality [20, (14)], we needed a version of (10) that is invariant with respect to taking scalar multiples of ψ. Our definition here is intentionally not scale invariant, since we need exactly this property in the last step of the proof of Proposition 4.4 below.

Given $u \in \mathcal{H}_\omega$ and $f \in L^{\psi}_{\omega_u^n}$, we will denote $\|f\|_{\psi,\omega_u^n}$ simply as $\|f\|_{\psi,u}$. To start, we note the following elementary convergence result.

Lemma 2.7 *Let μ and μ_k be finite Borel measures on Y. Let $\psi \in \mathcal{W}^-$, and f_k, f be bounded functions that are μ_k-measurable and μ-measurable respectively. If $\int_Y \psi(cf_k)d\mu_k \to \int_Y \psi(cf)d\mu$ for all $c \in [0,\infty)$, then $\|f_k\|_{\psi,\mu_k} \to \|f\|_{\psi,\mu}$.*

Proof We can assume that $f \not\equiv 0$ (a.e. with respect to μ). In this case $[0,\infty) \ni c \to \int_Y \psi(cf)d\mu \in \mathbb{R}$ is strictly increasing and continuous (the latter by the dominated convergence theorem). As a result, for any $\varepsilon > 0$ there exists $\delta^1_\varepsilon, \delta^2_\varepsilon > 0$ such that:

$$\begin{aligned} 1 + \frac{\varepsilon}{2} \leq \int_Y \psi\Big(\frac{f}{\|f\|_{\psi,\mu} - \delta^1}\Big)d\mu \\ \leq 1 + \varepsilon \text{ and } 1 - \varepsilon \leq \int_Y \psi\Big(\frac{f}{\|f\|_{\psi,\mu} + \delta^2}\Big)d\mu \leq 1 - \frac{\varepsilon}{2}. \end{aligned}$$

In addition, $\delta^1_\varepsilon, \delta^2_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$.

By our assumption we have that $\int_Y \psi\big(f_k/(\|f\|_{\psi,\mu} - \delta^1)\big)d\mu_k \to \int_Y \psi(f/(\|f\|_{\psi,\mu} - \delta^1))d\mu$ and $\int_Y \psi(f_k/(\|f\|_{\psi,\mu} + \delta^2))d\mu_k \to \int_Y \psi(f/(\|f\|_{\psi,\mu} + \delta^2))d\mu$. By definition of our weak quasi-norm we conclude that $\|f\|_{\psi,\mu} - \delta^1_\varepsilon \leq \liminf_k \|f_k\|_{\psi,\mu_k} \leq \limsup_k \|f_k\|_{\psi,\mu_k} \leq \|f\|_{\psi,\mu} + \delta^2_\varepsilon$, finishing the proof. Letting $\varepsilon \searrow 0$, the result follows. □

Lemma 2.8 *Let f_k, f be continuous functions on a compact topological space Y and $f_k \to f$ uniformly. Let μ and μ_k be Borel measures on Y with finite mass such that $\mu_k \to \mu$ weakly. Then $\int_Y \psi(cf_k)d\mu_k \to \int_Y \psi(cf)d\mu$ for any $c \in [0,\infty)$ and $\|f_k\|_{\psi,\mu_k} \to \|f\|_{\psi,\mu}$.*

Proof For any $c > 0$ we have that $\psi(cf_k) \to \psi(cf)$ uniformly. Since we are dealing with finite measure spaces and Y is compact, it follows that $\int_Y \psi(cf_k)d\mu_k \to \int_Y \psi(cf)d\mu$. The last claim follows from Lemma 2.7. □

In case we have smooth maps $[0,1] \ni t \to v_t \in \mathcal{H}_\omega$, $[0,1] \ni t \to f_t \in C^\infty$ with $f_t > 0$, it is easy to see that $t \to \|f_t\|_{\psi,v_t}$ is smooth. Indeed, since $\psi|_{(0,\infty)}$ is smooth, the arguments of [20, Proposition 3.1] carry over without change, and we have the following precise formula for the first derivative:

Proposition 2.9 *Suppose $\psi \in \mathcal{W}^-$. Given a smooth curve $(0,1) \ni t \to u_t \in \mathcal{H}$, i.e. $u(t,x) := u_t(x) \in C^\infty((0,1)\times X)$, and a positive smooth vector field $(0,1) \ni t \to f_t \in C^\infty(X)$ along this curve, the following formula holds:*

$$\partial_t \|f_t\|_{\psi,u_t} = \frac{\int_X \psi'\Big(\frac{f_t}{\|f_t\|_{\psi,u_t}}\Big)\nabla_{\dot{u}_t} f_t \omega^n_{u_t}}{\int_X \psi'\Big(\frac{f_t}{\|f_t\|_{\psi,u_t}}\Big)\frac{f_t}{\|f_t\|_{\psi,u_t}}\omega^n_{u_t}}, \tag{11}$$

where ∇ is the covariant derivative of the L^2 Mabuchi–Semmes–Donaldson metric (7).

Recall that Chen's ε-geodesics are smooth curves $t \to u_t$ that solve the following elliptic equation [17]:

$$\nabla_{\partial_t u}\partial_t u \omega_u^n = \varepsilon\omega^n. \tag{12}$$

As pointed out in [17], the advantage of ε-geodesics is that they are smooth, and approximate uniformly the weak $C^{1\bar{1}}$-geodesic connecting $u_0, u_1 \in \mathcal{H}_\omega$ that solves (8).

For this paper we need to compute the second order variation of the length of very special vector fields across ε-geodesics (c.f. [38, Section 4] and [16, Section 5]):

Proposition 2.10 *Suppose $\psi \in \mathcal{W}^-$. Let $[0,1]^2 \ni (s,t) \to u(s,t) \in \mathcal{H}_\omega$ be smooth and an ε-geodesic in each t-direction, such that $\partial_s u > 0$. The following formula holds:*

$$\partial_t^2 \|\partial_s u\|_{\psi,u} = \frac{\int_X \psi''(\eta)\Big(\|\partial_s u\|_{\psi,u}\big(\nabla_{\partial_t u}\eta\big)^2 + \frac{1}{\|\partial_s u\|_{\psi,u}}\{\partial_s u, \partial_t u\}^2_{\omega_u} + \frac{\varepsilon}{\|\partial_s u\|_{\psi,u}}\langle \nabla_{\omega_u}\partial_s u, \nabla_{\omega_u}\partial_s u\rangle \frac{\omega^n}{\omega_u^n}\Big)\omega_u^n}{\int_X \psi'(\eta)\eta\omega_u^n}, \tag{13}$$

where $\{\cdot,\cdot\}^2_{\omega_u}$ is the Poisson bracket of ω_u, and we introduced $\eta := \frac{\partial_s u}{\|\partial_s u\|_{\psi,u}}$, for simplicity. In particular, for fixed s, the map $t \to \|\partial_s u\|_{\psi,u}$ is concave.

Proof The proof is a careful calculation of the derivative of the right hand side of (11) in case $s \in [0,1]$ is fixed and $f_t := \partial_s u(s,t)$, $u_t := u(s,t)$.

We start with some side calculations, and put things together in the end. Since $\int_X \psi(\eta)\omega_u^n = 1$, the product rule for the Levi–Civita connection gives:

$$\int_X \psi'(\eta)\nabla_{\partial_t u}\eta\,\omega_u^n = \partial_t \int_X \psi(\eta)\omega_u^n = 0. \tag{14}$$

Using this identity and the product rule of the Levi–Civita connection again, we can differentiate the denominator of the right hand side of (11) and obtain:

$$\partial_t \int_X \psi'(\eta)\eta\, \omega_u^n = \int_X \psi''(\eta)\, \eta\, \nabla_{\partial_t u}(\eta)\omega_u^n. \tag{15}$$

Next we turn to the numerator of the right hand side of (11). The product rule again gives:

$$\begin{aligned}\partial_t \int_X \psi'(\eta)\nabla_{\partial_t u}\partial_s u\, \omega_u^n &= \int_X \psi''(\eta)\nabla_{\partial_t u}(\eta)\nabla_{\partial_t u}\partial_s u\, \omega_u^n + \int_X \psi'(\eta)\nabla_{\partial_t u}\nabla_{\partial_t u}\partial_s u\\ &= \int_X \psi''(\eta)\nabla_{\partial_t u}(\eta)\nabla_{\partial_t u}\partial_s u\, \omega_u^n + \int_X \psi'(\eta)\nabla_{\partial_t u}\nabla_{\partial_s u}\partial_t u\, \omega_u^n.\end{aligned} \tag{16}$$

For the last term on the right of (16) we make the following side computation:

$$\begin{aligned}&\int_X \psi'(\eta)\nabla_{\partial_t u}\nabla_{\partial_t u}\partial_s u\omega_u^n = \int_X \psi'(\eta)R(\partial_t u, \partial_s u)\partial_t u\, \omega_u^n + \int_X \psi'(\eta)\nabla_{\partial_s u}\nabla_{\partial_t u}\partial_t u\, \omega_u^n\\ &= \frac{1}{\|\partial_s u\|_{\psi,u}} \int_X \psi''(\eta)\{\partial_s u, \partial_t u\}^2\, \omega_u^n + \varepsilon \int_X \psi'(\eta)\nabla_{\partial_s}\left(\frac{\omega^n}{\omega_u^n}\right)\omega_u^n\\ &= \frac{1}{\|\partial_s u\|_{\psi,u}} \int_X \psi''(\eta)\{\partial_s u, \partial_t u\}^2\, \omega_u^n\\ &\quad - \varepsilon \int_X \psi'(\eta)\left(\Delta_{\omega_u}\partial_s u \cdot \left(\frac{\omega^n}{\omega_u^n}\right) + \langle \nabla_{\omega_u}\left(\frac{\omega^n}{\omega_u^n}\right), \nabla_{\omega_u}\partial_s u\rangle_{\omega_u}\right)\omega_u^n\\ &= \frac{1}{\|\partial_s u\|_{\psi,u}} \int_X \psi''(\eta)\{\partial_s u, \partial_t u\}^2\, \omega_u^n + \frac{\varepsilon}{\|\partial_s u\|_{\psi,u}} \int_X \psi''(\eta)\langle \nabla_{\omega_u}\partial_s u, \nabla_{\omega_u}\partial_s u\rangle_{\omega_u}\omega^n,\end{aligned} \tag{17}$$

where in the second we used the precise formula curvature $R(\cdot, \cdot)(\cdot)$ (computed in [16, (5.13)] or [8, Theorem 5]) and (12), in the third line we used the formula for the Levi–Civita connection, and in the last line we used integration by parts.

For the first term of (16) we use that $\partial_s u = \|\partial_s u\|_{\psi,u}\eta$, and the product rule for the Levi–Civita connection:

$$\begin{aligned}&\int_X \psi''(\eta)\nabla_{\partial_t u}(\eta)\nabla_{\partial_t u}\partial_s u\, \omega_u^n\\ &\quad = \|\partial_s u\|_{\psi,u} \int_X \psi''(\eta)\big(\nabla_{\partial_t u}(\eta)\big)^2\, \omega_u^n + \partial_t \|\partial_s u\|_{\psi,u} \int_X \psi''(\eta)\eta\nabla_{\partial_t u}(\eta)\, \omega_u^n\end{aligned} \tag{18}$$

Substituting (17) and (18) into (16) we arrive at

$$\partial_t \int_X \psi'(\eta)\nabla_{\partial_t u}\partial_s u\, \omega_u^n = \int_X \psi''(\eta)\Big(\partial_t \|\partial_s u\|_{\psi,u}\eta\nabla_{\partial_t u}(\eta) + \|\partial_s u\|_{\psi,u}\big(\nabla_{\partial_t u}\eta\big)^2$$

$$+\frac{1}{\|\partial_s u\|_{\psi,u}}\{\partial_s u,\partial_t u\}^2+\frac{\varepsilon}{\|\partial_s u\|_{\psi,u}}\langle\nabla_{\omega_u}\partial_s u,\nabla_{\omega_u}\partial_s u\rangle\frac{\omega^n}{\omega_u^n}\Big)\omega_u^n \tag{19}$$

Differentiating (11), we bring the above calculations together:

$$\begin{aligned}\partial_t^2\|\partial_s u\|_{\psi,u}&=\frac{\partial_t\int_X\psi'(\eta)\nabla_{\partial_t u}\partial_s u\omega_u^n}{\int_X\psi'(\eta)\eta\omega_u^n}-\partial_t\|\partial_s u\|_{\psi,u}\frac{\partial_t\int_X\psi'(\eta)\eta\omega_u^n}{\int_X\psi'(\eta)\eta\omega_u^n}\\&=\frac{\partial_t\int_X\psi'(\eta)\nabla_{\partial_t u}\partial_s u\omega_u^n}{\int_X\psi'(\eta)\eta\omega_u^n}-\partial_t\|\partial_s u\|_{\psi,u}\frac{\int_X\psi''(\eta)\eta\nabla_{\partial_t u}(\eta)\omega_u^n}{\int_X\psi'(\eta)\eta\omega_u^n},\end{aligned} \tag{20}$$

where in the last line we used (15). Next, in the numerator of the first fraction we now substitute (19) and notice that the last term on the right hand side of (20) will cancel with the first term on the right hand side of (19), ultimately yielding (13). □

3 The candidate metric d_ψ

We start with a preliminary discussion of our candidate metric d_ψ, defined in (2). By He's theorem [37, Theorem 1.1], we know that for $u_0,u_1\in\mathcal{H}_\omega^\Delta:=\mathrm{PSH}(X,\omega)\cap\{\Delta_\omega v\in L^\infty\}=\mathrm{PSH}(X,\omega)\cap C^{1\bar{1}}$, we also have $u_t\in\mathcal{H}_\omega^\Delta,\ t\in[0,1]$, where $t\to u_t$ is the weak geodesic connecting u_0,u_1.

It is not yet known if $t\to u_t$ is C^1 in the t-direction when $u_0,u_1\in\mathcal{H}_\omega^{1,\bar{1}}$. However, since $t\to u_t$ is t-convex, it makes sense to define $\dot{u}_0$ as the right derivative at $t=0$ and $\dot{u}_1$ as the left derivative at $t=1$. As a result, it is possible to extend the definition (2) to potentials with bounded Laplacian:

$$d_\psi(u_0,u_1):=\int_X\psi(\dot{u}_0)\omega_{u_0}^n. \tag{21}$$

This will be helpful since many operations on potentials are stable in the class $\mathcal{H}_\omega^\Delta$, and are not stable in $\mathcal{H}_\omega$. For example, by [25, Theorem 2.5], we know that $u,v\in\mathcal{H}_\omega^\Delta$ implies $P(u,v)\in\mathcal{H}_\omega^\Delta$. The same property is not true for potentials of $\mathcal{H}_\omega$.

In addition, we also introduce

$$\hat{d}_\psi(u_0,u_1)=\|\dot{u}_0\|_{\psi,u_0}, \tag{22}$$

where the term on the right hand side is the weak quasi-norm of $\dot{u}_0$ with respect to the weight ψ and the measure $\omega_{u_0}^n$ (10).

In case of $u_0,u_1\in\mathcal{H}_\omega$, by [20, Lemma 4.10] (slightly extending [7, Proposition 2.2]) we obtain that $t\to\int_X\psi(c\dot{u}_t)\omega_{u_t}^n$ is constant for any $c\in\mathbb{R}_+$. By definition of the weak quasi-norms, this immediately gives that $t\to\|\dot{u}_t\|_{\psi,u_t}$ is constant as well, hence in this case:

$$\hat{d}_\psi(u_0,u_1)=\|\dot{u}_t\|_{\psi,u_t},\quad\text{for any } t\in[0,1]. \tag{23}$$

$$d_\psi(u_0,u_1)=\int_X \psi(\dot u_t)\omega^n_{u_t}, \quad \text{for any } t\in[0,1]. \tag{24}$$

Though we will not need it, by [20, Lemma 4.10] the same formulas hold in case $u_0,u_1\in\mathcal H^\Delta_\omega$ as well. However in this case one has to clarify what $\dot u_t$ means for $t\in(0,1)$. As follows from [20, Lemma 4.10] (and its proof), $\partial_t^+ u_t=\partial_t^- u_t$ a.e with respect to $\omega^n_{u_t}$. As a result, $\dot u_t$ is a.e. well defined with respect to $\omega^n_{u_t}$, allowing to make sense of the right hand side of (23) and (24) in this more general situation as well.

First we prove an approximation result for the above introduced notions:

Proposition 3.1 *Let $u_0,u_1\in\mathcal H^\Delta_\omega$ and $u_0^k,u_1^k\in\mathcal H^\Delta_\omega$ such that $u_0^k\to u_0$ and $u_1^k\to u_1$ uniformly. Then we have that $d_\psi(u_0^k,u_1^k)\to d_\psi(u_0,u_1)$ and $\hat d_\psi(u_0^k,u_1^k)\to \hat d_\psi(u_0,u_1)$.*

Proof First we show that $d_\psi(u_0^k,u_1^k)\to d_\psi(u_0,u_1)$. Let $[0,1]\ni t\to u_t,u_t^k\in\mathcal H^\Delta_\omega$ be the weak geodesic joining u_0,u_1 and u_0^k,u_1^k respectively. We first claim that the push-forward measures $|\dot u_0^k|_*\omega^n_{u_0^k}$ weakly converge to $|\dot u_0|_*\omega^n_{u_0}$.

Assuming the claim, since $\dot u_0^k,\dot u_0$ are uniformly bounded [21, Theorem 1], we can apply this to ψ to arrive at the conclusion:

$$d_\psi(u_0^k,u_1^k)=\int_X\psi(\dot u_0^k)\omega^n_{u_0^k}=\int_X\psi(|\dot u_0^k|)\omega^n_{u_0^k}\to\int_X\psi(|\dot u_0|)\omega^n_{u_0}=d_\psi(u_0,u_1).$$

Now we prove the claim. From [20, Theorem 3] and the triangle ineqality for d_p we know that $d_p(u_0^k,u_1^k)\to d_p(u_0,u_1)$ for all $p\geq 1$. By [20, Lemma 4.11] this is equivalent with $\int_X|\dot u_0^k|^p\omega^n_{u_0^k}\to\int_X|\dot u_0|^p\omega^n_{u_0}$.

Since the global masses of the pushforward measures $|\dot u_0^k|_*\omega^n_{u_0^k}$, $|\dot u_0|_*\omega^n_{u_0^k}$ are finite, and $\dot u_0^k,\dot u_0$ are uniformly bounded, the Stone–Weierstrass theorem implies that $\int_X\alpha(|\dot u_0^k|)\omega^n_{u_0^k}\to\int_X\alpha(|\dot u_0|)\omega^n_{u_0}$ for any $\alpha\in C(\mathbb R)$. This is equivalent with $|\dot u_0^k|_*\omega^n_{u_0^k}\to|\dot u_0|_*\omega^n_{u_0}$, as desired.

We can repeat the above for $\psi(ct)$ instead of $\psi(t)$ for any $c\in[0,\infty)$, and conclude that $\hat d_\psi(u_0^k,u_1^k)\to\hat d_\psi(u_0,u_1)$ via Lemma 2.7. □

Next, we point out that an analogue of the Pythagorean identity holds for d_ψ:

Lemma 3.2 *Let $\psi\in\mathcal W^-$ and $u,v\in\mathcal H^\Delta_\omega$. Then for $P(u,v)\in\mathcal H^\Delta_\omega$ we have that*

$$d_\psi(u,v)=d_\psi(u,P(u,v))+d_\psi(v,P(u,v)).$$

Proof This is a consequence of [20, Proposition 4.13] for $f:=\psi$. □

Next we point out that the operator $u\to P(u,w)$ is d_ψ-shrinking:

Proposition 3.3 *Let $\psi\in\mathcal W_-$ and $u,v,w\in\mathcal H^\Delta_\omega$. Then we have*

$$d_\psi(P(u,w),P(v,w))\leq d_\psi(u,v).$$

Proof The proof is exactly the same as that of [22, Proposition 8.2], where one replaces the convex weight $|t|^2$ with our weight $\psi \in \mathcal{W}^-$. □

Using the argument of [20, Lemma 4.2] we note the following lemma:

Lemma 3.4 *Let $\alpha, \beta, \gamma \in \mathcal{H}_\omega^\Delta$ such that $\alpha \geq \beta \geq \gamma$. Then $d_\psi(\alpha, \beta) \leq d_\psi(\alpha, \gamma)$ and $\hat{d}_\psi(\alpha, \beta) \leq \hat{d}_\psi(\alpha, \gamma)$. Analogously, $d_\psi(\gamma, \beta) \leq d_\psi(\gamma, \alpha)$ and $\hat{d}_\psi(\gamma, \beta) \leq \hat{d}_\psi(\gamma, \alpha)$.*

Proof Let $[0, 1] \ni t \to u_t, v_t \in \mathcal{H}_\omega^\Delta$ be the weak geodesics connecting α, β and α, γ respectively. We notice that they are both decreasing, satisfy $u_t \geq v_t$ by the comparison principle, and $u_0 = v_0 = \alpha$. From this it follows that $0 \geq \dot{u}_0 \geq \dot{v}_0$. Using this, (21) and (22) yield that $d_\psi(\alpha, \beta) \leq d_\psi(\alpha, \gamma)$ and $\hat{d}_\psi(\alpha, \beta) \leq \hat{d}_\psi(\alpha, \gamma)$. The last sentence is proved analogously, using two weak geodesics meeting at γ. □

4 The triangle inequality

First we obtain the triangle inequality for d_ψ in a special case (Proposition 4.4), and then derive the general version from this using the Pythagorean identity for d_ψ.

We start with the analogue of [16, Lemma 5.2] in our setting, that will only hold in the particular case of increasing smooth curves (c.f. [38, Theorem 1.2]).

Proposition 4.1 *Let $\varepsilon > 0$ and $[0, 1] \ni s \to u_{0,s}, u_{1,s} \in \mathcal{H}_\omega$ be smooth curves satisfying $\partial_s u_{0,s} > 0, \partial_s u_{1,s} > 0$. For fixed s, let $[0, 1] \ni t \to u^\varepsilon_{t,s} \in \mathcal{H}_\omega$ be the ε-geodesic connecting $u_{0,s}, u_{1,s}$. Then $t \to \int_0^1 \|\partial_s u^\varepsilon_{t,s}\|_{\psi, u^\varepsilon_{t,s}} ds$ is concave.*

Proof By assumption, $\partial_s u_{0,s}, \partial_s u_{1,s} \geq \delta > 0$ for some constant δ. By the proof of [23, Corollary 3.4] we get that $\partial_s u^\varepsilon_{t,s} \geq \delta > 0$ for any $t, s \in [0, 1]$ and $\varepsilon > 0$. In particular, $s \to \|\partial_s u\|_{\psi,u}$ is smooth and the results of Sect. 2.3 are applicable. In particular, since $\psi|_{[0,\infty)}$ is concave, Proposition 2.10 gives:

$$\frac{d^2}{dt^2} \int_0^1 \|\partial_s u^\varepsilon_{t,s}\|_{\psi, u_{t,s}} ds = \int_0^1 \frac{d^2}{dt^2} \|\partial_s u^\varepsilon_{t,s}\|_{\psi, u_{t,s}} ds \leq 0.$$

This is equivalent to concavity of $t \to \int_0^1 \|\partial_s u^\varepsilon_{t,s}\|_{\psi, u^\varepsilon_{t,s}} ds$. □

Proposition 4.2 *Let $u_0, u_1 \in \mathcal{H}_\omega$ with $u_0 < u_1$. Let $[0, 1] \ni s \to u_s \in \mathcal{H}_\omega^\Delta$ be the (increasing) weak geodesic joining u_0, u_1. Then $\hat{d}_\psi(u_0, u_1) = \int_0^1 \|\dot{u}_s\|_{\psi, u_s} ds \geq \int_0^1 \|\dot{\zeta}_s\|_{\psi, \zeta_s} ds$, where $t \to \zeta_t$ is any smooth increasing curve ($\dot{\zeta}_s > 0$) joining $\zeta_0 := u_0$ and $\zeta_1 := u_1$.*

The following argument is due to Lempert.

Proof Let $\delta > 0$ such that $u_1 - u_0 > \delta$ and $\dot{\zeta}_s > \delta$ for all $s \in [0, 1]$. From (9) we obtain that $u_t \geq u_0 + \delta t$. Since $t \to u_t$ is t-convex, we obtain that $\dot{u}_t \geq \dot{u}_0 \geq \delta$.

By (23) we know that $s \to \|\dot{u}_s\|_{\psi, u_s}$ is constant equal to $c > 0$. Since ψ is concave and smooth on $(\delta/2c, \infty)$, it admits a concave extension $\tilde{\psi}$ to $(-\infty, \infty)$ such that $\tilde{\psi}|_{(\delta/2c,\infty)} = \psi|_{(\delta/2c,\infty)}$. Such extension of course is non-unique.

As $\dot{\zeta}_s > 0$, Proposition 2.9 implies that $s \to \|\dot{\zeta}_s\|_{\psi,\zeta_s}$ is smooth. Since weak quasi-norms are homogeneous, it is possible to reparametrize $[0,1] \ni s \to \zeta_s \in \mathcal{H}_\omega$ to a smooth curve $[0,1] \ni s \to \tilde{\zeta}_s \in \mathcal{H}_\omega$ such that $s \to \|\dot{\tilde{\zeta}}_s\|_{\psi,\zeta_s}$ is constant and the ψ-arclength does not change:

$$\int_0^1 \|\dot{\zeta}_s\|_{\psi,\zeta_s} ds = \int_0^1 \|\dot{\tilde{\zeta}}_s\|_{\psi,\tilde{\zeta}_s} ds = \|\dot{\tilde{\zeta}}_l\|_{\psi,\tilde{\zeta}_l}, \quad l \in [0,1].$$

Since $t \to -\tilde{\psi}(t/c)$ is convex, [38, Theorem 1.1] implies that

$$\begin{aligned} 1 &= \int_0^1 \int_X \psi\Big(\frac{\dot{u}_s}{c}\Big)\omega^n_{u_s} ds = \int_0^1 \int_X \tilde{\psi}\Big(\frac{\dot{u}_s}{c}\Big)\omega^n_{u_s} ds \\ &\geq \int_0^1 \int_X \tilde{\psi}\Big(\frac{\dot{\tilde{\zeta}}_s}{c}\Big)\omega^n_{\tilde{\zeta}_s} ds = \int_0^1 \int_X \psi\Big(\frac{\dot{\tilde{\zeta}}_s}{c}\Big)\omega^n_{\tilde{\zeta}_s} ds. \end{aligned}$$

In particular, by the mean value theorem, we obtain that $\int_X \psi(\frac{\dot{\tilde{\zeta}}_t}{c})\omega^n_{\tilde{\zeta}_t} \leq 1$ for some $t \in [0,1]$. By the definition of the weak quasi-norm, we get that $\|\dot{\tilde{\zeta}}_t\|_{\psi,\tilde{\zeta}_t} \leq c$. But since $s \to \|\dot{\tilde{\zeta}}_s\|_{\psi,\tilde{\zeta}_s}$ is constant, we actually get that $\|\dot{\tilde{\zeta}}_s\|_{\psi,\tilde{\zeta}_s} \leq c = \|\dot{u}_s\|_{\psi,u_s}$ for all $s \in [0,1]$. Integrating this inequality on $[0,1]$ yields the desired estimate. □

As a corollary of the above two results, we obtain the following:

Corollary 4.3 *Suppose we are given $\alpha, \beta, \gamma \in \mathcal{H}^\Delta_\omega$ such that $\alpha \geq \beta \geq \gamma$. Let $[0,1] \ni t \to \alpha_t, \gamma_t \in \mathcal{H}^\Delta_\omega$ be the weak geodesic joining $\alpha_0 := \beta, \alpha_1 := \alpha$ and $\gamma_0 := \beta, \gamma_1 := \gamma$ respectively. Then the function $t \to \hat{d}_\psi(\alpha_t, \gamma_t)$ is concave.*

Proof Using Proposition 3.1 and [28], we can assume that $\alpha, \beta, \gamma \in \mathcal{H}_\omega$ and $\alpha > \beta > \gamma$.

As in the first step of the proof of Proposition 4.2, there exists $\delta > 0$ such that $\dot{\alpha}_t > \delta$ and $\dot{\gamma}_t < -\delta$ for all $t \in [0,1]$. Let $[0,1] \ni t \to \alpha^\varepsilon_t, \gamma^\varepsilon_t \in \mathcal{H}_\omega$ be the ε-geodesic joining $\alpha^\varepsilon_0 := \beta, \alpha^\varepsilon_1 := \alpha$ and $\gamma^\varepsilon_0 := \beta, \gamma^\varepsilon_1 := \gamma$ respectively. As $\varepsilon-$geodesics converge to weak geodesics in the $C^{1,\alpha}$-topology, for small enough ε, we also have $\dot{\alpha}^\varepsilon_t > \delta$ and $\dot{\gamma}^\varepsilon_t < -\delta$. In particular, $t \to \alpha^\varepsilon_t$ is strictly increasing and $t \to \gamma^\varepsilon_t$ is strictly decreasing

Let $t, t' \in (0,1]$. Now let $\varepsilon' > 0$ be small enough so that both ε'-geodesics $[0,1] \ni s \to v^{\varepsilon,\varepsilon',t}_s, v^{\varepsilon,\varepsilon',t'}_s \in \mathcal{H}_\omega$ joining $v^{\varepsilon,\varepsilon',t}_0 := \gamma^\varepsilon_t$ and $v^{\varepsilon,\varepsilon',t}_1 := \alpha^\varepsilon_t$, respectively $v^{\varepsilon,\varepsilon',t'}_0 := \gamma^\varepsilon_{t'}$ and $v^{\varepsilon,\varepsilon',t'}_1 := \alpha^\varepsilon_{t'}$ are strictly increasing (i.e. $\partial_s v^{\varepsilon,\varepsilon',t}_s, \partial_s v^{\varepsilon,\varepsilon',t'}_s > 0$).

For $s \in [0,1]$ fixed, let $[0,1] \ni \lambda \to \eta^{\varepsilon',\varepsilon}(\lambda, s)$ be the ε-geodesic joining $\eta^{\varepsilon',\varepsilon}(0,s) := v^{\varepsilon,\varepsilon',t}_s$ and $\eta^{\varepsilon,\varepsilon',\varepsilon}(1,s) := v^{\varepsilon,\varepsilon',t'}_s$. Notice that $\eta^{\varepsilon',\varepsilon}(\lambda,1) = \alpha^\varepsilon_{(1-\lambda)t+\lambda t'}$ and $\eta^{\varepsilon',\varepsilon}(\lambda,0) = \gamma^\varepsilon_{(1-\lambda)t+\lambda t'}$.

We fix $\lambda \in [0,1]$. Combining previous results we can finish the proof:

$$(1-\lambda)\hat{d}_\psi(\alpha_t,\gamma_t) + \lambda\hat{d}_\psi(\alpha_{t'},\gamma_{t'}) = \lim_{\varepsilon\to 0}\Big((1-\lambda)\hat{d}_\psi(\alpha^\varepsilon_t,\gamma^\varepsilon_t) + \lambda\hat{d}_\psi(\alpha^\varepsilon_{t'},\gamma^\varepsilon_{t'})\Big)$$

$$= \lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} \left(\int_0^1 \left((1-\lambda) \| \partial_s v^{\varepsilon,\varepsilon',t} \|_{\psi, v^{\varepsilon,\varepsilon'}} \omega^n_{v^{\varepsilon,\varepsilon',t}} + \lambda \| \partial_s v^{\varepsilon,\varepsilon',t'} \|_{\psi, v^{\varepsilon,\varepsilon'}} \right) ds \right)$$
$$\leq \lim_{\varepsilon \to 0} \int_0^1 \| \partial_s \eta^{\varepsilon',\varepsilon}((1-\lambda)t + \lambda t', s) \|_{\psi, \eta^{\varepsilon',\varepsilon}((1-\lambda)t+\lambda t', s)} ds$$
$$\leq \lim_{\varepsilon \to 0} \hat{d}_\psi (\alpha^\varepsilon_{(1-\lambda)t+\lambda t'}, \gamma^\varepsilon_{(1-\lambda)t+\lambda t'})$$
$$= \hat{d}_\psi (\alpha_{(1-\lambda)t+\lambda t'}, \gamma_{(1-\lambda)t+\lambda t'}),$$

where in the first line we have used Proposition 3.1, in the second line we have used (23) and Lemma 2.8, in the third line we have used Proposition 4.1, in the fourth line we have used Proposition 4.2, and in the last line we have used Proposition 3.1 again. □

Proposition 4.4 *Given $\alpha, \beta, \gamma \in \mathcal{H}^\Delta_\omega$ such that $\alpha \geq \beta \geq \gamma$, we have that*

$$d_\psi(\alpha, \gamma) \leq d_\psi(\alpha, \beta) + d_\psi(\beta, \gamma). \tag{25}$$

Proof Using Proposition 3.1, we can assume that α, β, γ are smooth Kähler potentials, moreover $\alpha > \beta > \gamma$.

Let $[0, 1] \ni t \to u_t, v_t \in \mathcal{H}^\Delta_\omega$ be the weak geodesics connecting $u_0 := \beta$ and $u_1 := \alpha$, respectively $v_0 := \beta$ and $v_1 := \gamma$. By Corollary 4.3 we get that $t \to \hat{d}_\psi(u_t, v_t)$ is concave. Hence, since $\hat{d}_\psi(u_0, v_0) = 0$, $t \to \hat{d}_\psi(u_t, v_t)/t$ is decreasing.

Let $[0, 1] \ni t \to \eta_t$ be the weak geodesic connecting $\eta_0 = \alpha$ and $\eta_1 = \gamma$. We can use the ψ-version of [20, Lemma 4.1] (whose proof is identical) to write:

$$\|\dot{\eta}_0\|_{\psi,\alpha} = \hat{d}_\psi(\alpha, \gamma) = \hat{d}_\psi(u_1, v_1) \leq \lim_{t \to 0} \frac{\hat{d}_\psi(u_t, v_t)}{t} \leq \limsup_{t \to 0} \frac{\|u_t - v_t\|_{\psi, v_t}}{t}$$
$$= \limsup_{t \to 0} \left\| \frac{u_t - v_t}{t} \right\|_{\psi, v_t} = \|\dot{u}_0 - \dot{v}_0\|_{\psi,\beta},$$

where in the last step we have used that $\omega^n_{v_t} \to \omega^n_\beta$ weakly, moreover $(u_t - v_t)/t \to \dot{u}_0 - \dot{v}_0$ uniformly, as $t \to 0$. Indeed, this allows an application of Lemma 2.8 to conclude.

Finally, if we replace $\psi(t)$ with $\tilde{\psi}(t) := \psi(t)/\int_X \psi(\dot{\eta}_0)\omega^n_\alpha \in \mathcal{W}^-$, the same inequality as above implies that, $1 = \|\dot{\eta}_0\|_{\tilde{\psi},\alpha} \leq \|\dot{u}_0 - \dot{v}_0\|_{\tilde{\psi},\beta}$, i.e., $\int_X \tilde{\psi}(\dot{u}_0 - \dot{v}_0)\omega^n_\beta \geq 1$, i.e., $\int_X \psi(\dot{u}_0 - \dot{v}_0)\omega^n_\beta \geq \int_X \psi(\dot{\eta}_0)\omega^n_\alpha$. Using Lemma 2.6 we now conclude that

$$d_\psi(\alpha, \beta) + d_\psi(\beta, \gamma) = \int_X \psi(\dot{u}_0)\omega^n_\beta + \int_X \psi(\dot{v}_0)\omega^n_\beta$$
$$\geq \int_X \psi(\dot{u}_0 - \dot{v}_0)\omega^n_\beta \geq \int_X \psi(\dot{\eta}_0)\omega^n_\alpha = d_\psi(\alpha, \gamma).$$

□

We are ready to prove the general case of the triangle inequality.

Theorem 4.5 *Let $u, v, w \in \mathcal{H}_\omega^\Delta$. Then we have*

$$d_\psi(u, w) \leq d_\psi(u, v) + d_\psi(v, w). \tag{26}$$

Proof The triangle inequality follows from the following sequence of inequalities:

$$\begin{aligned}
d_\psi(u, v) + d_\psi(v, w) &= d_\psi(u, P(u, v)) + d_\psi(P(u, v), v) \\
&\quad + d_\psi(v, P(v, w)) + d_\psi(P(v, w), w) \\
&\geq d_\psi(u, P(u, v)) + d_\psi(P(v, w), P(u, v, w)) \\
&\quad + d_\psi(P(u, v), P(u, v, w)) + d_\psi(P(v, w), w) \\
&= d_\psi(u, P(u, v)) + d_\psi(P(u, v), P(u, v, w)) \\
&\quad + d_\psi(w, P(v, w)) + d_\psi(P(v, w), P(u, v, w)) \\
&\geq d_\psi(u, P(u, v, w)) + d_\psi(w, P(u, v, w)) \\
&\geq d_\psi(u, P(u, w)) + d_\psi(w, P(u, w)) \\
&= d_\psi(u, w),
\end{aligned}$$

where in the first and last line we have used the Pythagorean identity for d_ψ (Lemma 3.2), in the second line we have used Proposition 3.3 for the second and third terms, in the fourth line we have used twice the particular case of the triangle inequality obtained in Proposition 4.4, and in the fifth line we used Lemma 3.4. □

Corollary 4.6 *$(\mathcal{H}_\omega^\Delta, d_\psi)$ is a metric space.*

Proof By the previous result, we only need to argue that $d_\psi(u_0, u_1) = 0$ implies $u_0 = u_1$.

If $d_\psi(u_0, u_1) = 0$, by Lemma 3.2 we have $d_\psi(u_0, P(u_0, u_1)) = 0$ and also $d_\psi(u_1, P(u_0, u_1)) = 0$. By the first estimate of Proposition 4.7 below, it follows that $u_0 = P(u_0, u_1)$ a.e. with respect to $\omega^n_{P(u_0,u_1)}$, and similarly, $u_1 = P(u_0, u_1)$ a.e. with respect to $\omega^n_{P(u_0,u_1)}$. We can now use the domination principle of full mass potentials due to Dinew [10, Proposition 5.9] to obtain that $u_0 \leq P(u_0, u_1)$ and $u_1 \leq P(u_0, u_1)$. As the reverse inequalities are trivial, we get that $u_0 = P(u_0, u_1) = u_1$. □

Proposition 4.7 *Suppose $u, v \in \mathcal{H}_\omega^\Delta$ with $u \leq v$. Then we have:*

$$\max\left(\frac{1}{2^{n+1}} \int_X \psi(v - u)\omega_u^n, \int_X \psi(v - u)\omega_v^n\right) \leq d_\psi(u, v) \leq \int_X \psi(v - u)\omega_u^n. \tag{27}$$

Proof Using Proposition 3.1 we can assume that u and v are smooth. Suppose $[0, 1] \ni t \to w_t \in \mathcal{H}_\omega^\Delta$ is the weak geodesic segment joining $w_0 = u$ and $w_1 = v$. By (24) we have

$$d_\psi(u, v) = \int_X \psi(\dot{w}_0)\omega_u^n = \int_X \psi(\dot{w}_1)\omega_v^n.$$

Since $u \leq v$, we have that $u \leq w_t$, as follows from the comparison principle. Since $(t, x) \to w_t(x)$ is convex in the t variable, we get $0 \leq \dot{w}_0 \leq v - u \leq \dot{w}_1$, and together with the above identity we obtain part of (27):

$$\int_X \psi(v-u)\omega_v^n \leq d_\psi(u, v) \leq \int_X \psi(v-u)\omega_u^n. \tag{28}$$

Now we prove the rest of (27). Using $\omega_u^n \leq 2^n \omega_{(u+v)/2}^n$ and concavity of ψ on $[0, \infty)$, we obtain that

$$\frac{1}{2^{n+1}} \int_X \psi(v-u)\omega_u^n \leq \int_X \psi\Big(\frac{u+v}{2} - u\Big)\omega_{(u+v)/2}^n.$$

Since $u \leq (u+v)/2$, the first estimate of (28) allows to write:

$$\frac{1}{2^{n+1}} \int_X \psi(v-u)\omega_u^n \leq d_\psi\Big(\frac{u+v}{2}, u\Big).$$

Finally, Lemma 3.4 implies that $d_\psi((u+v)/2, u) \leq d_\psi(v, u)$, giving the remaining estimate in (27). □

5 Extending d_ψ to $\mathcal{E}_\psi$ and completeness

Given $u_0, u_1 \in \mathcal{E}_\psi(X, \omega)$, by a classical result of Demailly [28] (see [9] for a short argument) there exists decreasing sequences $u_0^k, u_1^k \in \mathcal{H}_\omega$ such that $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$. We propose to extend d_ψ to $\mathcal{E}_\psi$ in the following way:

$$d_\psi(u_0, u_1) = \lim_{k\to\infty} d_\psi(u_0^k, u_1^k). \tag{29}$$

Very similar to the high energy case [20], we will show that the limit on the right hand side exists and is independent of the approximating sequences. For this, we first prove the next lemma:

Lemma 5.1 *Suppose $u \in \mathcal{E}_\psi$ and $\{u_k\}_k \subset \mathcal{H}_\omega^\Delta$ is a sequence decreasing to u. Then $d_\psi(u_l, u_k) \to 0$ as $l, k \to \infty$.*

Proof We can suppose that $l \leq k$. Then $u_k \leq u_l$, hence by Proposition 4.7 we have:

$$d_\psi(u_l, u_k) \leq \int_X \psi(u_k - u_l)\omega_{u_k}^n.$$

Let us fix l momentarily, and let $\{v_j\}_j \in \mathcal{H}_\omega$ be such that $v_j \searrow u_l$. Then $u - v_j, u_k - v_j \in \mathcal{E}_\psi(X, \omega_{v_j})$ and $u - v_j \leq u_k - v_j \leq 0$. Hence, applying Proposition 2.2 for the class $\mathcal{E}_\psi(X, \omega_{u_l})$ we obtain

$$d_\psi(u_l, u_k) \leq \int_X \psi(u_k - u_l)\omega_{u_k}^n \leq \lim_j \int_X \psi(u_k - v_j)\omega_{u_k}^n$$

$$\leq C \lim_j \int_X \psi(u - v_j)\omega_u^n = C \int_X \psi(u - u_l)\omega_u^n. \tag{30}$$

As u_l decreases to $u \in \mathcal{E}_\psi$, by the dominated convergence theorem we have $d_\psi(u_l, u_k) \to 0$ as $l, k \to \infty$. □

Our next lemma confirms that the way we proposed to extend the d_ψ metric to $\mathcal{E}_\psi$ in (29) is consistent.

Lemma 5.2 *Given $u_0, u_1 \in \mathcal{E}_\psi$, the limit in (29) is finite and independent of the approximating sequences $u_0^k, u_1^k \in \mathcal{H}_\omega^\Delta$.*

Proof By Proposition 3.1 we can assume that the approximating sequences are smooth. By the triangle inequality and Lemma 5.1 we can write:

$$|d_\psi(u_0^l, u_1^l) - d_\psi(u_0^k, u_1^k)| \leq d_\psi(u_0^l, u_0^k) + d_\psi(u_1^l, u_1^k) \to 0, \ l, k \to \infty,$$

proving that $d_\psi(u_0^k, u_1^k)$ is indeed convergent.

Now we prove that the limit in (29) is independent of the choice of approximating sequences. Let $v_0^l, v_1^l \in \mathcal{H}_\omega$ be different approximating sequences. By adding small constants we arrange that the sequences u_0^l, u_1^l, respectively v_0^l, v_1^l, are strictly decreasing to u_0, u_1.

Fixing k for the moment, the sequence $\{\max\{u_0^{k+1}, v_0^j\}\}_{j\in\mathbb{N}}$ decreases pointwise to u_0^{k+1}. By Dini's lemma there exists $j_k \in \mathbb{N}$ such that for any $j \geq j_k$ we have $v_0^j < u_0^k$. By repeating the same argument we can also assume that $v_1^j < u_1^k$ for any $j \geq j_k$. By the triangle inequality again

$$|d_\psi\left(u_0^k, u_1^k\right) - d_\psi\left(v_0^j, v_1^j\right)| \leq d_\psi\left(u_0^k, v_0^j\right) + d_\psi\left(u_1^k, v_1^j\right), \ j \geq j_k.$$

From (30) it follows that for k big enough $d_\psi(u_0^j, v_0^k), d_\psi(u_1^j, v_1^k), \ j \geq j_k$ are arbitrarily small. As a result, $d_\psi(u_0, u_1)$ is independent of the choice of approximating sequences.

When $u_0, u_1 \in \mathcal{H}_\omega^\Delta$, one can approximate with the constant sequence, hence the restriction to $\mathcal{H}_\omega$ of the extended d_ψ from (29) coincides with the original definition (21). □

By the above result, [23, Proposition 2.20], and the remark following it, many properties of d_ψ extend to $\mathcal{E}_\psi$, in particular the triangle inequality, the Pythagorean formula, etc. We list these in the proposition below and leave the standard proofs to the interested reader.

Proposition 5.3 *Let $\psi \in \mathcal{W}^-$. Then the following hold:*

(i) *$d_\psi : \mathcal{E}_\psi \times \mathcal{E}_\psi \to \mathbb{R}$ satisfies the triangle inequality.*
(ii) *If $u, v \in \mathcal{E}_\psi$ then $P(u, v) \in \mathcal{E}_\psi$ and $d_\psi(u, v) = d_\psi(u, P(u, v)) + d_\psi(v, P(u, v))$.*

(iii) *Suppose $u, v \in \mathcal{E}_\psi$ with $u \le v$. Then we have:*

$$\max\left(\frac{1}{2^{n+1}}\int_X \psi(v-u)\omega_u^n, \int_X \psi(v-u)\omega_v^n\right) \le d_\psi(u,v) \le \int_X \psi(v-u)\omega_u^n.$$

(iv) *For $u, v, w \in \mathcal{E}_\psi$ we have $d_\psi(P(u,w), P(v,w)) \le d_\psi(u,v)$.*

We now argue that non-degeneracy of d_ψ on $\mathcal{E}_\psi$ holds as well:

Proposition 5.4 *Given $u_0, u_1 \in \mathcal{E}_\psi$ if $d_\psi(u_0,u_1) = 0$ then $u_0 = u_1$. In particular, $(\mathcal{E}_\psi, d_\psi)$ is a metric space.*

Proof We can repeat the argument of Corollary 4.6. By Proposition 5.3(ii) it follows that $d_\psi(u_0, P(u_0,u_1)) = 0$ and also $d_\psi(u_1, P(u_0,u_1)) = 0$. By Proposition 5.3(iii), it follows that $u_0 = P(u_0,u_1)$ a.e. with respect to $\omega^n_{P(u_0,u_1)}$, and similarly, $u_1 = P(u_0,u_1)$ a.e. with respect to $\omega^n_{P(u_0,u_1)}$. We can now use the domination principle of full mass potentials due to Dinew [10, Proposition 5.9] to obtain that $u_0 \le P(u_0,u_1)$ and $u_1 \le P(u_0,u_1)$. As the reverse inequalities are trivial, we get that $u_0 = P(u_0,u_1) = u_1$. □

Corollary 5.5 *If $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\psi$ decreases or increases a.e. to $w \in \mathcal{E}_\psi$ then $d_\psi(w_k, w) \to 0$.*

Proof By Proposition 5.3(iii), we have $d_\psi(w, w_k) \le \int_X \psi(w - w_k)(\omega^n_{w_k} + \omega^n_w)$. We can use [23, Proposition 2.20] (and the remark following it) to conclude that $d_\psi(w, w_k) \to 0$. □

Lemma 5.6 *Suppose $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\psi$ be an increasing d_ψ–bounded sequence. Then $\sup_X u_k$ is a bounded sequence.*

Proof Using Theorem 6.1 from below we have that $\int_X \psi(\max(u_k,0))\omega^n \le \int_X \psi(u_k)\omega^n \le 2^{2n+5} d_\psi(u_k, 0) \le C$, for some $C > 0$. Let $v := \lim_k \max(0,u_k))$, a measurable function on X. By the monotone convergence theorem we obtain that $\int_X \psi(v)\omega^n \le C$. This implies that for some $d > 0$ the set $K := \{v \le d\}$ has non-zero Lebesgue measure, hence K is also non-pluripolar.

On K we have that $u_k \le d$. As a result, due to [33, Corollary 4.3] we obtain that $\{u_k\}_k \subset \mathrm{PSH}(X,\omega)$ is relatively L^1-compact, hence $\sup_X u_k$ can not converge to ∞, finishing the proof. □

Next we argue that bounded monotone sequences in $\mathcal{E}_\psi$ have limits inside $\mathcal{E}_\psi$. Using the previous lemma, the proof of this result is very similar to [23, Lemma 3.34]:

Lemma 5.7 *Suppose $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\psi$ is a decreasing/increasing d_ψ-bounded sequence. Then $u = \lim_{k\to\infty} u_k \in \mathcal{E}_\psi$ and additionally $d_\psi(u, u_k) \to 0$.*

Proof Due to the previous lemma, after subtracting a constant, we can assume without loss of generality that $u_k \le 0$.

Let us assume that $\{u_k\}_k$ is decreasing. From Proposition 5.3(iii) we have that $\int_X \psi(u_k)\omega^n_{u_k}$ is uniformly bounded. Due to [34, Proposition 5.6] we get that $u = \lim_k u_k \in \mathcal{E}_\psi(X,\omega)$. Corollary 5.5 implies that $d_\psi(u_k, u) \to 0$.

Now us assume that $\{u_k\}_k$ is increasing. Due to the previous lemma, there exists $u \in \mathcal{E}_\psi(X, \omega)$ such that $u_k \nearrow u$. By Corollary 5.5 again, $d_\psi(u_k, u) \to 0$. □

Finally, we argue completeness of $(\mathcal{E}_\psi, d_\psi)$:

Theorem 5.8 *$(\mathcal{E}_\psi, d_\psi)$ is a complete metric space, that is the metric completion of $(\mathcal{H}_\omega, d_\psi)$.*

Proof By Corollary 5.5 and [28] $\mathcal{H}_\omega$ is a d_ψ-dense subset of $\mathcal{E}_\psi$. We need to argue completeness, which can be done identically as in [20], due to Proposition 3.3.

Indeed, suppose $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{E}_\psi$ is a d_ψ-Cauchy sequence. We will prove that there exists $v \in \mathcal{E}_\psi$ such that $d_\psi(u_k, v) \to 0$. After passing to a subsequence we can assume that

$$d_\psi(u_l, u_{l+1}) \leq 1/2^l, \ l \in \mathbb{N}.$$

By [22, Theorem 3.6] we can introduce $v^k_l = P(u_k, u_{k+1}, \ldots, u_{k+l}) \in \mathcal{E}_\psi, \ l, k \in \mathbb{N}$. We argue first that each decreasing sequence $\{v^k_l\}_{l\in\mathbb{N}}$ is d_ψ-Cauchy. We observe that $v^k_{l+1} = P(v^k_l, u_{k+l+1})$ and $v^k_l = P(v^k_l, u_{k+l})$. Using this and Proposition 5.3(iv) we can write:

$$d_\psi(v^k_{l+1}, v^k_l) = d_\psi(P(v^k_l, u_{k+l+1}), P(v^k_l, u_{k+l})) \leq d_\psi(u_{k+l+1}, u_{k+l}) \leq \frac{1}{2^{k+l}}.$$

From Lemma 5.7 it follows now that each sequence $\{v^k_l\}_{l\in\mathbb{N}}$ is d_ψ-convergening to some $v^k \in \mathcal{E}_\psi$. By the same trick as above, we can write:

$$\begin{aligned}
d_\psi(v^k, v^{k+1}) &= \lim_{l\to\infty} d_\psi\left(v^k_{l+1}, v^{k+1}_l\right) = \lim_{l\to\infty} d_\psi\left(P(u_k, v^{k+1}_l), P(u_{k+1}, v^{k+1}_l)\right)\\
&\leq d_\psi(u_k, u_{k+1}) \leq \frac{1}{2^k},\\
d_\psi(v^k, u_k) &= \lim_{l\to\infty} d_\psi(v^k_l, u_k) = \lim_{l\to\infty} d_\psi\left(P(u_k, v^{k+1}_{l-1}), P(u_k, u_k)\right)\\
&\leq \lim_{l\to\infty} d_\psi(v^{k+1}_{l-1}, u_k)\\
&= \lim_{l\to\infty} d_\psi\left(P(u_{k+1}, v^{k+2}_{l-2}), u_k\right)\\
&\leq \lim_{l\to\infty} d_\psi\left(P(u_{k+1}, v^{k+2}_{l-2}), u_{k+1}\right) + d_\psi(u_{k+1}, u_k)\\
&\leq \lim_{l\to\infty} \sum_{j=k}^{l+k} d_\psi(u_j, u_{j+1}) \leq \frac{1}{2^{k-1}}.
\end{aligned}$$

Consequently, $\{v^k\}_{k\in\mathbb{N}}$ is an increasing d_ψ-bounded d_ψ-Cauchy sequence that is equivalent to $\{u_k\}_{k\in\mathbb{N}}$. By Lemma 5.7 there exists $v \in \mathcal{E}_\psi$ such that $d_\psi(v_k, v) \to 0$, which in turn implies that $d_\psi(u_k, v) \to 0$, finishing the proof. □

6 An analytic expression governing the d_ψ metric

As another application of the Pythagorean formula we will show that the d_ψ metric is comparable to a concrete analytic expression, reminiscent of the analogous result for high energy classes [20, Theorem 3]:

Theorem 6.1 *For any $u_0, u_1 \in \mathcal{E}_\psi$ we have*

$$d_\psi(u_0, u_1) \le \int_X \psi(u_0 - u_1)\omega_{u_0}^n + \int_X \psi(u_0 - u_1)\omega_{u_1}^n \le 2^{2n+5} d_\psi(u_0, u_1). \quad (31)$$

Proof To obtain the first estimate we use the triangle inequality and Proposition 5.3(iii):

$$\begin{aligned} d_\psi(u_0, u_1) &\le d_\psi(u_0, \max(u_0, u_1)) + d_\psi(\max(u_0, u_1), u_1) \\ &\le \int_X \psi(u_0 - \max(u_0, u_1))\omega_{u_0}^n + \int_X \psi(\max(u_0, u_1) - u_1)\omega_{u_1}^n \\ &= \int_{\{u_1 > u_0\}} \psi(u_0 - u_1)\omega_{u_0}^n + \int_{\{u_0 > u_1\}} \psi(u_0 - u_1)\omega_{u_1}^n \\ &\le \int_X \psi(u_0 - u_1)\omega_{u_0}^n + \int_X \psi(u_0 - u_1)\omega_{u_1}^n. \end{aligned}$$

Now we deal with the second estimate in (31). By the next result, Proposition 5.3(ii) and Proposition 5.3(iii) we can write

$$\begin{aligned} 2^{n+2} d_\psi(u_0, u_1) &\ge d_\psi\Big(u_0, \frac{u_0 + u_1}{2}\Big) \ge d_\psi\Big(u_0, P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big) \\ &\ge \int_X \psi\Big(u_0 - P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big)\omega_{u_0}^n. \end{aligned}$$

By a similar reasoning as above, and the fact that $2^n \omega_{(u_0+u_1)/2}^n \ge \omega_{u_0}^n$ we can write:

$$\begin{aligned} 2^{n+2} d_\psi(u_0, u_1) &\ge d_\psi\Big(u_0, \frac{u_0 + u_1}{2}\Big) \ge d_\psi\Big(\frac{u_0 + u_1}{2}, P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big) \\ &\ge \int_X \psi\Big(\frac{u_0 + u_1}{2} - P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big)\omega_{(u_0+u_1)/2}^n \\ &\ge \frac{1}{2^n} \int_X \psi\Big(\frac{u_0 + u_1}{2} - P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big)\omega_{u_0}^n. \end{aligned}$$

Adding the last two estimates, and using sublinearity and concavity of ψ (Lemma 2.6) we obtain:

$$\begin{aligned} 2^{2n+3} d_\psi(u_0, u_1) \ge \int_X &\psi\Big(u_0 - P\Big(u_0, \frac{u_0 + u_1}{2}\Big)\Big) \\ &+ \psi\Big(P\Big(u_0, \frac{u_0 + u_1}{2}\Big) - \frac{u_0 + u_1}{2}\Big)\Big)\omega_{u_0}^n \end{aligned}$$

$$\geq \int_X \psi\Big(\frac{u_0-u_1}{2}\Big)\omega_{u_0}^n \geq \frac{1}{2}\int_X \psi(u_0-u_1)\omega_{u_0}^n.$$

By symmetry we also have $2^{2n+4}d_\psi(u_0,u_1) \geq \int_X \psi(u_0-u_1)\omega_{u_1}^n$, and adding these last two estimates together the second inequality in (31) follows. □

Lemma 6.2 *Suppose $u_0, u_1 \in \mathcal{E}_\psi$. Then we have*

$$d_\psi\left(u_0, \frac{u_0+u_1}{2}\right) \leq 2^{n+2} d_\psi(u_0,u_1).$$

Proof Using Proposition 5.3(ii) and (iii) we can start writing:

$$\begin{aligned}
&d_\psi\Big(u_0, \frac{u_0+u_1}{2}\Big) = d_\psi\Big(u_0, P\Big(u_0, \frac{u_0+u_1}{2}\Big)\Big) + d_\psi\Big(\frac{u_0+u_1}{2}, P\Big(u_0, \frac{u_0+u_1}{2}\Big)\Big)\\
&\leq d_\psi(u_0, P(u_0,u_1)) + d_\psi\Big(\frac{u_0+u_1}{2}, P(u_0,u_1)\Big)\\
&\leq \int_X \psi(u_0 - P(u_0,u_1))\omega_{P(u_0,u_1)}^n + \int_X \psi\Big(\frac{u_0+u_1}{2} - P(u_0,u_1)\Big)\omega_{P(u_0,u_1)}^n\\
&\leq \int_X \psi(u_0 - P(u_0,u_1))\omega_{P(u_0,u_1)}^n + \int_X \psi(\max(u_1,u_0) - P(u_0,u_1))\omega_{P(u_0,u_1)}^n\\
&\leq \int_X \Big((1+\mathbb{1}_{\{u_0\geq u_1\}})\psi(u_0 - P(u_0,u_1)) + \mathbb{1}_{\{u_1\geq u_0\}}\psi(u_1 - P(u_0,u_1))\Big)\omega_{P(u_0,u_1)}^n\\
&\leq 2^{n+2}\big(d_\psi(u_0, P(u_0,u_1)) + d_\psi(u_1, P(u_0,u_1))\big) = 2^{n+2} d_\psi(u_0,u_1),
\end{aligned}$$

where in the second line we have used the first claim of Lemma 3.4 and the fact that $P(u_0,u_1) \leq P(u_0,(u_0+u_1)/2)$, in the third and sixth line Proposition 5.3(iii), and in the last equality we have used Proposition 5.3(ii). □

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