A note on pseudorandom Ramsey graphs

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Abstract

For fixed $s \geq 3$, we prove that if optimal K_s-free pseudorandom graphs exist, then the Ramsey number $r(s,t) = t^{s-1+o(1)}$ as $t \to \infty$. Our method also improves the best lower bounds for $r(C_\ell, t)$ obtained by Bohman and Keevash from the random C_ℓ-free process by polylogarithmic factors for all odd $\ell \geq 5$ and $\ell \in \{6, 10\}$. For $\ell = 4$ it matches their lower bound from the C_4-free process.

We also prove, via a different approach, that $r(C_5, t) > (1 + o(1))t^{11/8}$ and $r(C_7, t) > (1 + o(1))t^{11/9}$. These improve the exponent of t in the previous best results and appear to be the first examples of graphs F with cycles for which such an improvement of the exponent for $r(F, t)$ is shown over the bounds given by the random F-free process and random graphs.

1 Introduction

The *Ramsey number* $r(F, t)$ is the minimum N such that every F-free graph on N vertices has an independent set of size t. When $F = K_s$ we simply write $r(s, t)$ instead of $r(F, t)$. Improving on earlier results of Spencer [33] and the classical Erdős-Szekeres [16] theorem on Ramsey numbers, Ajtai, Komlós and Szemerédi [1] proved the following upper bound on $r(s, t)$, and Bohman and Keevash [9] proved the lower bound by considering the *random K_s-free process*: consequently for $s \geq 3$, there exist constants $c_1(s), c_2(s) > 0$ such that

$$c_1(s) \frac{t^{\frac{s+1}{2}}}{(\log t)^{\frac{s+1}{2} - \frac{1}{s-2}}} \leq r(s,t) \leq c_2(s) \frac{t^{s-1}}{(\log t)^{s-2}}. \tag{1}$$

For $s = 3$, the lower bound was proved in a celebrated paper of Kim [23] and the upper bound was proved by Shearer [32] with $c_2(3) = 1 + o(1)$. In particular, recent results of Bohman and Keevash [8] and Fiz Pontiveros, Griffiths and Morris [17] together with the bound of Shearer show

$$(\tfrac{1}{4} - o(1)) \cdot \frac{t^2}{\log t} \leq r(3,t) \leq (1 + o(1)) \cdot \frac{t^2}{\log t} \tag{2}$$

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as $t \to \infty$. There have been no improvements in the exponents in (1) for any $s \geq 4$ for many decades. In this note, we show that if certain density-optimal K_s-free pseudorandom graphs exist, then $r(s,t) = t^{s-1+o(1)}$. This approach suggests that pseudorandom graphs may be the central tool required to determine classical graph Ramsey numbers.

An (n,d,λ) *graph* is an n-vertex d-regular graph such that the absolute value of every eigenvalue of its adjacency matrix, besides the largest one, is at most λ. Constructions of (n,d,λ)-graphs arise from a number of sources, including Cayley graphs, projective geometry and strongly regular graphs – we refer the reader to Krivelevich and Sudakov [26] for a survey of (n,d,λ)-graphs. Sudakov, Szabo and Vu [35] show that a K_s-free (n,d,λ)-graph satisfies

$$\lambda = \Omega(d^{s-1}/n^{s-2}) \tag{3}$$

as $n \to \infty$. For $s = 3$, if G is any triangle-free (n,d,λ)-graph with adjacency matrix A, then

$$0 = \mathrm{tr}(A^3) \geq d^3 - \lambda^3(n-1). \tag{4}$$

If $\lambda = O(\sqrt{d})$, then this gives $d = O(n^{2/3})$ matching (3). Alon [2] constructed a triangle-free pseudorandom graph attaining this bound, and Conlon [13] more recently analyzed a randomized construction with the same average degree. A similar argument to (4) shows that a K_s-free (n,d,λ)-graph with $\lambda = O(\sqrt{d})$ has $d = O(n^{1-\frac{1}{2s-3}})$. The Alon-Boppana Bound [30, 31] shows that $\lambda = \Omega(\sqrt{d})$ for every (n,d,λ)-graph provided d/n is bounded away from 1. Sudakov, Szabo and Vu [35] raised the question of the existence of optimal pseudorandom K_s-free graphs for $s \geq 4$, namely (n,d,λ)-graphs achieving the bound in (3) with $\lambda = O(\sqrt{d})$ and $d = \Omega(n^{1-\frac{1}{2s-3}})$. We show that a positive answer to this question gives the exponent of the Ramsey numbers $r(s,t)$ via a short proof of the following general theorem, based on ideas of Alon and Rödl [5]:

Theorem 1. *Let F be a graph, n, d, λ be positive integers with $d \geq 1$ and $\lambda > 1/2$ and let $t = \lceil 2n \log^2 n/d \rceil$. If there exists an F-free (n,d,λ)-graph, then*

$$r(F,t) > \frac{n}{20\lambda} \log^2 n. \tag{5}$$

Theorem 1 provides good bounds whenever we have an F-free (n,d,λ)-graph with many edges and good pseudorandom properties (meaning that d is large and λ is small). For example, we immediately obtain the following consequence.

Corollary 2. *If K_s-free (n,d,λ)-graphs exist with $d = \Omega(n^{1-\frac{1}{2s-3}})$ and $\lambda = O(\sqrt{d})$, then as $t \to \infty$,*

$$r(s,t) = \Omega\Big(\frac{t^{s-1}}{\log^{2s-4} t}\Big). \tag{6}$$

Corollary 2 follows from (5) using $F = K_s$. To see this, from $t = \lceil 2n \log^2 n/d \rceil$ we obtain $d =$

$\Theta(t^{2s-4}/(\log t)^{2(2s-4)})$. Inserting this in (5) with $\lambda = O(\sqrt{d})$ gives (6).

Alon and Krivelevich [4] gave a construction of K_s-free (n,d,λ)-graphs with $d = \Omega(n^{1-1/(s-2)})$ and $\lambda = O(\sqrt{d})$ for all $s \geq 3$, and this was slightly improved by Bishnoi, Ihringer and Pepe [7] to obtain $d = \Omega(n^{1-1/(s-1)})$. This is the current record for the degree of a K_s-free (n,d,λ)-graph with $\lambda = O(\sqrt{d})$. The problem of obtaining optimal K_s-free pseudorandom constructions in the sense (3) with $\lambda = O(\sqrt{d})$ for $s \geq 4$ seems difficult and is considered to be a central open problem in pseudorandom graph theory. The problem of determining the growth rate of $r(s,t)$ is classical and much older, and it wasn't completely clear whether the upper or lower bound in (1) was closer to the truth. Based on Theorem 1, it seems reasonable to conjecture that if $s \geq 4$ is fixed, then $r(s,t) = t^{s-1+o(1)}$ as $t \to \infty$.

We next consider cycle-complete Ramsey numbers. The cycle complete Ramsey numbers $r(C_\ell, t)$ appear to be very difficult to determine – the best upper bounds are provided by Sudakov [34] for odd cycles and Caro, Li, Rousseau and Zhang [11] for even cycles. The best lower bound for fixed $\ell \geq 4$ is

$$r(C_\ell, t) = \Omega\left(\frac{t^{(\ell-1)/(\ell-2)}}{\log t}\right) \tag{7}$$

due to Bohman and Keevash [9] by analyzing the C_ℓ-free process. A generalization of the optimal triangle-free (n,d,λ)-graphs constructed by Alon [2] to optimal pseudorandom C_ℓ-free graphs for odd $\ell \geq 5$ was given by Alon and Kahale [3], and gives an (n,d,λ)-graph with $d = \Theta(n^{2/\ell})$ and $\lambda = O(\sqrt{d})$. Using this construction, Theorem 1 gives the following on odd-cycle complete Ramsey numbers, which gives a polylogarithmic improvement over (7):

Corollary 3. *Let $\ell \geq 3$ be an odd integer. Then as $t \to \infty$,*

$$r(C_\ell, t) = \Omega\Big(\frac{t^{(\ell-1)/(\ell-2)}}{\log^{2/(\ell-2)} t}\Big). \tag{8}$$

Note when $\ell = 3$, this matches the lower bound of Spencer [33] from the local lemma. Applying Theorem 1 when F is bipartite can give lower bounds on $r(F,t)$ that are better than those obtained from the F-free process. We can see this when $F = C_\ell$ and $\ell \in \{6, 10\}$. To apply Theorem 1 when $F = C_4$, we may consider polarity graphs of projective planes to be (n,d,λ)-graphs with $n = q^2+q+1$, $d = q+1$ and $\lambda = \sqrt{q}$ (see [29] for a detailed study of independent sets in such graphs). Theorem 1 then gives $r(C_4,t) = \Omega(t^{3/2}/\log t)$ which matches (7). It is a wide open conjecture of Erdős that $r(C_4,t) \leq t^{2-\epsilon}$ for some $\epsilon > 0$.

For $\ell \in \{6, 10\}$ Theorem 1 provides results that exceed the previous best known bounds of (7) from the random C_ℓ-free process.

Corollary 4. *As* $t \to \infty$*,*

$$r(C_6, t) = \Omega\left(\frac{t^{5/4}}{\log^{1/2} t}\right) \qquad \text{and} \qquad r(C_{10}, t) = \Omega\left(\frac{t^{9/8}}{\log^{1/4} t}\right).$$

These results are obtained by considering polarity graphs of *generalized quadrangles* and *generalized hexagons*. For certain prime powers q, generalized quadrangles are (n, d, λ)-graphs with $n = q^3 + q^2 + q + 1$, $d = q + 1$ and $\lambda = \sqrt{2q}$, and generalized hexagons are (n, d, λ)-graphs with $n = q^5 + q^4 + q^3 + q^2 + q + 1$, $d = q + 1$ and $\lambda = \sqrt{3q}$. For the existence of such graphs, we refer the reader to Brouwer, Cohen and Neumaier [10] and Lazebnik, Ustimenko and Woldar [27]. We can then apply Theorem 1 and obtain the desired result by (5).

Our next result uses a completely different construction than that in Theorem 1 for $F = C_5$ and $F = C_7$. For these two cases, we are able to improve the exponents in the lower bounds given by Corollary 3. Our approach here is to use a random block construction. This idea was used in [15] and [24] and also recently in [13] to construct triangle-free pseudorandom graphs.

Theorem 5. *As* $t \to \infty$*,*

$$\begin{aligned} r(C_5, t) &\geq (1 + o(1))t^{11/8} \\ r(C_7, t) &\geq (1 + o(1))t^{11/9}. \end{aligned}$$

This appears to be the first instance of a graph F containing cycles for which random graphs do not supply the right exponent for $r(F, t)$.

2 Proof of Theorem 1

The proof of Theorem 1 uses the following property of independent sets in (n, d, λ)-graphs, due to Alon and Rödl [5] (we give a slightly stronger statement below):

Theorem 6. (Alon-Rödl [5]) *Let G be an (n, d, λ)-graph with $d \geq 1$, $\lambda > 1/2$, and let $t \geq 2n \log^2 n/d$ be an integer. Then the number of independent sets of size t in G is at most $(\frac{2e^2\lambda}{\log^2 n})^t$.*

Proof. Alon and Rödl (Theorem 2.1 in [5]) proved that the number Z of independent sets of size t in G is at most

$$\frac{1}{t!}\binom{t}{\ell} n^{\ell} \left(\frac{2\lambda n}{d}\right)^{t-\ell}$$

where $\ell = t/\log n$. Using $\binom{t}{\ell} \leq 2^t$ and $t! \geq (t/e)^t$,

$$Z \leq \left(\frac{2e}{t}\right)^t \cdot n^{\ell} \cdot \left(\frac{2\lambda n}{d}\right)^t \cdot \left(\frac{d}{2\lambda n}\right)^{\ell} = \left(\frac{4e\lambda n}{dt}\right)^t \cdot \left(\frac{d}{2\lambda}\right)^{t/\log n}.$$

Since $\lambda > 1/2$, we obtain $d/2\lambda \le d \le n$ and therefore $(d/2\lambda)^{t/\log n} \le n^{t/\log n} \le e^t$. Using $t \ge 2n(\log^2 n)/d$,

$$Z \le \Big(\frac{4e^2\lambda n}{dt}\Big)^t \le \Big(\frac{2e^2\lambda}{\log^2 n}\Big)^t$$

and the proof is complete. □

Proof of Theorem 1. Let G be an F-free (n,d,λ)-graph and let U be a random set of vertices of G where each vertex is chosen independently with probability $p = \log^2 n/2e^2\lambda$. Let Z be the number of independent sets of size $t = \lceil 2n\log^2 n/d\rceil$ in the induced subgraph $G[U]$. Then by Theorem 6 and the choice of p,

$$E(|U| - |Z|) \ge pn - p^t\Big(\frac{2e^2\lambda}{\log^2 n}\Big)^t = pn - 1.$$

Therefore there is a set $U \subset V(G)$ such that if we remove one vertex from every independent set in U, the remaining set T has $|T| \ge pn - 1$ and $G[T]$ has no independent set of size t. It follows that

$$r(F,t) \ge pn > \frac{n}{20\lambda}\log^2 n.$$

This completes the proof. □

3 Proof of Theorem 5

A key ingredient in the proof of Theorem 5 is the existence of dense bipartite graphs G of high girth. For the first statement in the theorem, we let G be a bipartite graph of girth at least twelve with parts U and V of sizes $m = (q+1)(q^8+q^4+1)$ and $n = (q^3+1)(q^8+q^4+1)$ such that every vertex of V has degree $q+1$ and every vertex of U has degree q^3+1 – these are the incidence graphs of generalized hexagons of order (q,q^3) (see [18, 36] or [6] page 115 Corollary 5.38 for details about these constructions).

For each $u \in U$, let (A_u, B_u) be a random partition of $N_G(u)$, independently for $u \in U$. Let H be the random graph with $V(H) = V$ obtained by placing a complete bipartite graph with parts A_u and B_u inside $N_G(u)$ for each $u \in U$. It is evident that H is C_5-free since G has girth twelve.

Now we show every independent set in H has size at most $(1+o(1))q^8$. This is sufficient to show $r(C_5,t) > n = (1+o(1))t^{11/8}$. Let I be a set of t vertices in H. If $|I \cap N_G(u)| = t_u$ for $u \in U$, then

$$P(e(I \cap N_G(u)) = 0) = 2^{1-t_u}. \tag{9}$$

Since the partitions (A_u, B_u) are independent over different $u \in U$ and the sum of t_u is $(q+1)t$,

$$P(e(I) = 0) = \prod_{u\in U} P(e(I \cap N_G(u)) = 0) = \prod_{u\in U} 2^{1-t_u} = 2^{m-(q+1)t}. \tag{10}$$

There are $\binom{n}{t}$ choices of I, so the expected number of independent sets of size t in H is

$$\binom{n}{t}2^{m-(q+1)t} \leq 2^{t\log_2 n+m-(q+1)t} = 2^{m-(q+o(q))t}. \tag{11}$$

Since $m = (1+o(1))q^9$, we make take $t = (1+o(1))q^8$ so that the above expression decays to zero. Consequently, with high probability, every independent set in H has less than t.

For the second statement of Theorem 5, the Ree-Tits octagons [18, 36] supply requisite bipartite graphs of girth at least sixteen – these graphs have parts of sizes $m = (q+1)(q^9+q^6+q^3+1)$ and $n = (q^2+1)(q^9+q^6+q^3+1)$ with all vertices in the larger part of degree $q+1$. We omit the details for this case, which are almost identical to the above. □

4 Random block constructions

Theorem 5 may be generalized as follows. Let F be a graph and let $\mathcal{P} = (P_1, P_2, \ldots, P_k)$ be a partition of $E(F)$ into bipartite graphs with at least one edge each. Let $X = \{x_1, x_2, \ldots, x_k\}$ be new vertices, and let $F_{\mathcal{P}}$ be the graph with $V(F_{\mathcal{P}}) = V(F) \cup X$ and edge set

$$E(F_{\mathcal{P}}) = \bigcup_{i=1}^{k} \{\{x_i, y\} : y \in V(P_i)\}. \tag{12}$$

Let $L(F)$ be the family of all graphs $F_{\mathcal{P}}$ taken over partitions $\mathcal{P}$ of $E(F)$ into paths with at least one edge each. For instance, when F is a triangle, then $L(F)$ consists of C_4 plus a pendant edge and C_6. If F is a pentagon then every member of $L(F)$ is a cycle of length at most ten plus a set of pendant edges. Let G be a bipartite graph with parts U and V containing no member of $L(F)$ and such that every vertex of V has degree d. We form a new graph H with $V(H) = V$ by taking for each $u \in U$ independently a random partition (A_u, B_u) of $N_G(u)$ and then adding a complete bipartite graph with parts A_u and B_u. By definition, H does not contain F. Then the proof of the following is the same as the proof of Theorem 5:

Theorem 7. *Let F be a graph and let G be an $L(F)$-free bipartite graph with parts U and V such that $|U| = m$ and $|V| = n$ and every vertex of V has degree at least d. If $dt > m + t\log n$, then*

$$r(F, t) > n. \tag{13}$$

If $F = K_4$, then a C_4-free graph containing no 1-subdivision of K_4 is $L(F)$-free. It is possible to show that any graph not containing a 1-subdivision of K_4 has $O(n^{7/5})$ edges (see Conlon and Lee [14], and Janzer [22]). If there is a d-regular graph containing no 1-subdivision of K_4 with n vertices and with $d = \Omega(n^{2/5})$ even, then one can produce a random graph H as above that is d^2-regular, and has a chance to be an (n, d, λ)-graph with $\lambda = d^{1/2+o(1)}$ as in the work of Conlon [13]. Via Theorem 1, this would then show $r(4, t) = t^{3-o(1)}$. However, the best construction of an n-vertex graph with no subdivision of K_4 has only $O(n^{4/3})$ edges.

5 Concluding remarks

• Although the construction in Theorem 1 starting with Alon's [2] pseudorandom triangle-free graph provides slightly worse bounds than the known random constructions for $r(3, t)$, the number of random bits used is less than the known random constructions [8, 33], which use roughly $t^{4+o(1)}$ bits. The same observation applies to the case $r(C_\ell, t)$ for $\ell \in \{4, 6, 10\}$ where we match or exceed the best known construction obtained via the C_ℓ-free process using fewer random bits.

• It would be interesting to see if the choice of the random subset U in the proof of Theorem 1 can be made explicit; for instance, the best explicit construction [25] of a K_4-free graph without independent sets of size t only gives $r(4, t) = \Omega(t^{8/5})$, as compared to random graphs which give $r(4, t) = \Omega^*(t^{5/2})$.

• If we apply the proof of Theorem 1 to Paley graphs of order q, which are (n, d, λ)-graphs with $d = (q - 1)/2$ and $\lambda = \frac{1}{2}(\sqrt{q} \pm 1)$ where q is a prime power congruent to 1 mod 4, we find almost all subsets of $\Omega(\sqrt{q} \log^2 q)$ vertices have no independent set or clique of size more than $2(\log q)^2$. In fact, Noga Alon (personal communication) had already observed a stronger statement in 1991, that one can randomly take q^α vertices for suitable α and the resulting induced subgraph has clique and independence number $O(\log q)$. It would be interesting to know if this can be done without randomness. It is a major open question (see Croot and Lev [12]) to determine, when q is prime, the maximum size of independent sets and cliques in the Paley graph. These were shown to be at least $\Omega(\log q \log \log q)$ by Montgomery [28] under GRH and at least $\Omega(\log q \log \log \log q)$ unconditionally by Graham and Ringrose [19]. The current best upper bound is $\sqrt{q/2} + 1$ by Hanson and Petridis [20].

• In order to improve the exponent in the lower bound (1) using Theorem 1, one could try to find a K_s-free (n, d, λ)-graph with $n/\lambda \geq (n/d)^{(q+1)/2}$ for some $q > s$, so as to obtain $r(s, t) = \Omega(t^{(q+1)/2})$. In the case $\lambda = O(\sqrt{d})$, it is sufficient that $d = \Omega(n^{1-1/q})$.

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