Bounded Output Feedback Control of Planar Systems With Unknown Nonlinear Structures and Application to Output Consensus

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***Abstract*—This letter addresses the problem of output feedback stabilization for a class of uncertain planar systems with unknown nonlinear structures. The underlying philosophy behind the proposed approach is primarily to revamp/advance the classical lead compensator with an arctangent function-based mechanism that assures bounded control magnitudes, leading to a new design methodology not only conquering the obstruction in constructing state observers but also directing the design of a bounded output feedback stabilizer. Inheriting and leveraging the stability-increasing capability offered by lead compensators, the resultant controller is capable of dealing with systems even suffering from unknown nonlinear structures and measurements concurrently. The strategy presented is further expanded to formulate a bounded output feedback output consensus protocol for uncertain planar two-agent systems with unknown nonlinear heterogeneous dynamics.**

***Index Terms*—Planar systems, unknown nonlinear structure, output feedback, bounded control.**

I. Introduction

In this letter, we primarily focus on the issue of attaining global stabilization via bounded output feedback for a class of uncertain planar systems with unknown nonlinear structure described by

$$\dot{x}_1 = g(x_2), \quad \dot{x}_2 = \theta u, \quad y = x_1 \tag{1}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$, $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are respectively the system state, output, and control input. The control coefficient $\theta \in \mathbb{R}_{>0}$ is an *unknown* constant, and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is an *unknown* continuous function. In the field of nonlinear control, planar nonlinear systems play a crucial role as fundamental for comprehending and describing the dynamic behaviors of diverse practical real-world systems, such as pendulums, robotic manipulators, and chemical reactions [1]. Attaining stabilization of planar nonlinear systems is of paramount significance as an initial step, serving as a fundamental precursor for subsequent control objectives including regulation and/or trajectory tracking [2], [3].

In the literature, most achievements in state feedback stabilization for planar or higher-order nonlinear systems often rely on the assumption of securing prior knowledge of system structures. (see, e.g., [4], [5], [6], [7], [8], [9], [10]); however, this assumption is challenged by practical systems, such as the boiler turbine unit in [11] whose simplified dynamic model takes the form of system (1) with $g(x_2) = sign(x_2)|x_2|^p$ and $\theta = 1$ where p is an unknown power parameter fluctuating under various operational conditions, thus posing difficulties in identifying and acquiring the system structure necessary for designing an effective state feedback controller. Recently, the work [12] presented an interval homogeneity-based control method mainly to address the global state feedback stabilization for the systems in the same form of the boiler turbine unit (i.e., system (1) with $g(x_2) = sign(x_2)|x_2|^p$ and $\theta = 1$), assuming that the power p is an unknown fixed constant lying within a known interval. Building upon the idea of [12], the study [13] delves deeper into the state feedback stabilization issue for the same type of systems under a more general circumstance that the power p is a time-varying scalar with known bounds. Later, several advances dedicated to state feedback stabilization have been subsequently achieved for this kind of systems with multiple unknown powers p's based on similar assumptions [14], [15], [16].

Without a doubt, the challenge of the output feedback stabilization problem for planar or general nonlinear systems far surpasses the one of state feedback stabilization due to the inapplicability of the separation principle for nonlinear systems [17], [18]. Over the past two decades, numerous works leveraging high gain feedback design have emerged in the literature (see, for instance, [17], [18], [19], [20], [21], [22], [23], [24]), demonstrating success in achieving output feedback stabilization of systems with clearly known structures subject to diverse structural requirements, including linear [19], [20], homogeneous [21], [22], and

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polynomial [23], [24] growth conditions. In addition, the utilization of high gain feedback has also led to solutions [25], [26] for output feedback stabilization of systems having known structures but suffering from an uncertain measurement $y = h(x_1)$ where $h(\cdot)$ is often assumed to a differentiable function with bounded derivatives, representing the relation between the output y and the state x_1. Notably, in scenarios when system structures are unknown, the output feedback stabilization of nonlinear systems is significantly formidable owing to the intractable quandary/hurdle stemming from handling and coping with the unknown structures in constructing a suitable state observer capable of acquiring unmeasurable states for feedback purposes, and thus preventing the applications of the existing results [17], [18], [19], [20], [21], [22], [23], [24]. Furthermore, when uncertain/unknown measurements are also involved, the task of stabilizing nonlinear systems with unknown structures using output feedback becomes immensely insurmountable and inevitably inhibits the feasibility of the strategies in [25], [26], mainly due to the absence of applicable state observers/estimators, as well as the unavailability of precise state information required for estimating unmeasurable states.

In this letter, we concentrate on designing an output feedback controller to globally asymptotically stabilize the planar system (1) with unknown nonlinear structures (i.e., $g(\cdot)$ and θ are unknown). The underlying philosophy behind our approach is chiefly to revamp the classical lead compensator by incorporating an arctangent function-based mechanism that guarantees bounded control magnitudes, thereby offering a novel design methodology not only triumphing over the obstruction associated with the construction of state observers but also providing an explicit design of a bounded output feedback stabilizer, as described in Theorem 1, that contains a tunable/assignable at will control gain so that magnitude constraints/limitations imposed on system (1) can be efficiently yet straightforwardly satisfied (see Remark 2). Also, technically inheriting and leveraging the stability-increasing capability offered by lead compensators, the resultant output feedback stabilizer, as substantiated in Theorem 2, is capable of dealing with system (1) even in the presence of unknown nonlinear structures (namely, $g(\cdot)$ and θ_i) and measurement $y = h(x_1)$ concurrently, without requiring the differentiability of $h(\cdot)$, and thus enjoying wider applicability as delineated in [25], [26]. With the capability to handle unknown nonlinear structures, the presented methodology (design framework) is further expanded, as depicted in Theorem 3, to formulate a bounded output feedback output consensus protocol for uncertain planar two-agent systems with unknown nonlinear heterogeneous dynamics, each of which possesses a distinct dynamic model and behavior.

II. Main Results

Owing to the continuity exhibited by the unknown function $g(\cdot)$, the closed-loop system (1) is only ensured to maintain continuity while also harboring the possibility of non-unique solutions, even when a smooth feedback controller $u(\cdot)$ is applied. To lay a theoretical foundation for our analysis and investigation, we hereby recall an essential lemma that elaborates on an extension of LaSalle's invariance principle, thoughtfully tailored to accommodate continuous systems, and skillfully sidesteps the prerequisite of the uniqueness of solutions/trajectories starting from arbitrary initial states.

Lemma 1 ([27], [28]): Consider a nonlinear system

$$\dot{\eta} = \phi(\eta), \ \ \eta(0) \in \mathbb{R}^n \tag{2}$$

with $\phi : \mathbb{R}^n \to \mathbb{R}^n$ being a continuous function. Suppose that $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a positive definite, continuously differentiable and proper function such that $\dot{V}(\eta) := (\partial V(\eta)/\partial\eta)\phi(\eta) \leq 0$ for all $\eta \in \mathbb{R}^n$. Let $\mathbb{H} = \{\eta \in \mathbb{R}^n | \dot{V}(\eta) = 0\} \subseteq \mathbb{R}^n$ and $\mathbb{M} \subseteq \mathbb{H}$ be the union of all forward complete solutions $\eta(t)$'s of system (2) that remain in[1] $\mathbb{H}$. Then, for any initial state $\eta(0) \in \mathbb{R}^n$, every solution $\eta(t)$ of system (2) with respect to $\eta(0)$ is defined on $[0, \infty)$ and fulfilling $\eta(t) \to \mathbb{M}$ as $t \to \infty$.

Remark 1: Note that, the proof of Lemma 1 has been solidly substantiated in [27, Th. 2, p. 62] or [28, Th. 3.2, Corollary 3.3, p. 243]. Due to the continuity of $\phi(\cdot)$, system (2) in Lemma 1 essentially exhibits the non-uniqueness in the solutions emanating from a given initial state; therefore, Lemma 1 serves as a crucial tool that aids in systematically assessing the convergence behavior of the multiple solutions $\eta(t)$'s to the set $\mathbb{M}$, starting at any initial state $\eta(0) \in \mathbb{R}^n$.

A. Bounded Output Feedback Controller for System (1)

We now impose an assumption on the unknown nonlinear structure $g(\cdot)$ of system (1) and present our first result.

Assumption 1: The unknown continuous function $g(\cdot)$ is strictly increasing and satisfies $g(0) = 0$.

Theorem 1: Under Assumption 1, the following dynamic bounded output feedback controller

$$u = -L\sigma(y) - L\sigma(z), \quad \dot{z} = u - M\sigma(z) \tag{3}$$

with any constants $L, M \in \mathbb{R}_{>0}$ globally asymptotically stabilizes system (1), where $\sigma : \mathbb{R} \to \mathbb{R}$ is defined as $\sigma(s) = \tan^{-1}(s)/\pi$ for all $s \in \mathbb{R}$.

Proof: By letting $e = x_2 - \theta z$, it can be deduced that

$$\dot{e} = \theta u - \theta(u - M\sigma(z)) = \theta M\sigma\left(\frac{x_2 - e}{\theta}\right) \tag{4}$$

and the output feedback controller (3) becomes

$$u = -L\sigma(x_1) - L\sigma\left(\frac{x_2 - e}{\theta}\right). \tag{5}$$

Substituting (5) into system (1) and involving (4) yields

$$\begin{aligned}\dot{x}_1 &= g(x_2)\\ \dot{x}_2 &= -\theta L\sigma(x_1) - \theta L\sigma\left(\frac{x_2 - e}{\theta}\right)\\ \dot{e} &= \theta M\sigma\left(\frac{x_2 - e}{\theta}\right)\end{aligned} \tag{6}$$

whose right-hand side is only continuous for all $(x, e) \in \mathbb{R}^2 \times \mathbb{R}$; thus, system (6) might have non-unique (multiple) solutions with respect to any initial state $(x(0), e(0)) \in \mathbb{R}^2 \times \mathbb{R}$.

[1] That is, the set $\mathbb{M}$ is the union of all solutions $\eta(t)$'s of system (2) initiating at $\eta(0) \in \mathbb{H}$ and satisfying the properties that $\eta(t)$ are defined on $[0, \infty)$ and $\eta(t) \in \mathbb{H}$ for all $t \in [0, \infty)$.

Now, we denote $\eta = (x, e) \in \mathbb{R}^2 \times \mathbb{R} =: \mathbb{R}^3$ and consider a function $V : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ of the form

$$V(\eta) = \theta L \int_0^{x_1} \sigma(s)ds + \int_0^{x_2} g(s)ds + \frac{L}{M}\int_0^{e} g(s)ds$$

which is positive definite as both $\sigma(\cdot)$ and $g(\cdot)$ are functions in the first and third quadrants. Furthermore, by the monotone properties of $g(\cdot)$ and $\sigma(\cdot)$, it is also clear that $V(\eta) \to \infty$ as $\|\eta\| \to \infty$ which reveals that $V(\eta)$ is proper. Because $V(\eta)$ is continuously differentiable, the time derivative of $V(\cdot)$ along system (6) is

$$\begin{aligned}\dot{V}(\eta) &:= \frac{\partial V(\cdot)}{\partial x_1}\dot{x}_1 + \frac{\partial V(\cdot)}{\partial x_2}\dot{x}_2 + \frac{\partial V(\cdot)}{\partial e}\dot{e} \\ &= \theta L\sigma(x_1)g(x_2) - \theta L\sigma(x_1)g(x_2) \\ &\quad - \theta L\sigma\left(\frac{x_2 - e}{\theta}\right)g(x_2) + \frac{L}{M}g(e)\theta M\sigma\left(\frac{x_2 - e}{\theta}\right) \\ &= -\theta L(g(x_2) - g(e))\sigma\left(\frac{x_2 - e}{\theta}\right) \leq 0 \end{aligned} \tag{7}$$

for all $\eta \in \mathbb{R}^3$. Observing the inequality (7) and noting Assumption 1 (i.e., the strict monotonicity of $g(\cdot)$), one can derive $\mathbb{H} := \{\eta \in \mathbb{R}^3 | \dot{V}(\eta) = 0\} \subseteq \mathbb{R}^3$ as $\mathbb{H} = \{(x, e) \in \mathbb{R}^3 | x_2 - e = 0\}$.

On the other hand, considering the continuity of system (6) and Peano's theorem [29], we let $\eta : [0, t_s) \to \mathbb{R}^3$ having the form $\eta(t) = (x(t), e(t))$ with $t_s \in \mathbb{R}_{>0}$ be any saturation (uncontinuable) solution of system (6) with respect to any initial state $\eta(0) = (x(0), e(0)) \in \mathbb{R}^3$. It follows from (7) that $dV(\eta(t))/dt \leq 0$ for all $t \in [0, t_s)$; this implies that $V(\eta(t)) \leq V(\eta(0)) < \infty$ for all $t \in [0, t_s)$ which along with Assumption 1 ensures that $\eta(t) = (x(t), e(t))$ is bounded uniformly in $t \in [0, t_s)$. Hence, in accordance with the continuation theorem [29], each solution $\eta(t) = (x(t), e(t))$ of system (6) in regard to any initial state $\eta(0) = (x(0), e(0)) \in \mathbb{R}^3$ is bounded uniformly and defined on $[0, \infty)$. With this in mind, it is evident that for all $t \in [0, \infty)$ and for every solution $\eta(t) = (x(t), e(t))$ of system (6) originating from $\eta(0) = (x(0), e(0)) \in \mathbb{R}^3$, there hold $\dot{x}_1(t) = g(x_2(t))$ and

$$\begin{aligned}\dot{x}_2(t) - \dot{e}(t) = &-\theta L\sigma(x_1(t)) - \theta L\sigma\left(\frac{x_2(t) - e(t)}{\theta}\right) \\ &- \theta M\sigma\left(\frac{x_2(t) - e(t)}{\theta}\right)\end{aligned}$$

for all $t \in [0, \infty)$, which together with Assumption 1 directly show that the union of all forward complete solutions $\eta(t)$'s of system (6) that remain in $\mathbb{H}$ is $\mathbb{M} := \{\eta(t) \in \mathbb{R}^3 | \dot{x}_2(t) - \dot{e}(t) = 0 \text{ for all } t \in [0, \infty)\} \subseteq \mathbb{H}$ which takes the form

$$\mathbb{M} = \left\{(x(t), e(t)) \in \mathbb{R}^3 \middle| x_1(t) = x_2(t) = e(t) = 0 \text{ for all } t \in [0, \infty)\right\}.$$

As a results, it can be concluded from Lemma 1 that each solution $\eta(t) = (x(t), e(t))$ of system (6) starting at any initial state $\eta(0) = (x(0), e(0)) \in \mathbb{R}^3$ fulfills $\eta(t) = (x(t), e(t)) \to \mathbb{M}$ as $t \to \infty$; that is, $\eta(t) = (x(t), e(t)) \to 0$ as $t \to \infty$. This completes the proof. ■

Remark 2: It is interesting to mention that, due to the distinctive but simple structure, the controller $u(\cdot)$ given by (3), to a certain extent, can be thought of as a similar form of second-order sliding mode controllers (e.g., the twisting algorithm [30]) with a (saturation) arctangent function $\sigma(\cdot)$ acting as a smoother to effectively increase the smoothness of the control signal, at the expense of compromising the robustness to persistent (or fast oscillations) bounded external disturbances. In addition, owing to the use of the (saturation) arctangent function $\sigma(\cdot)$ in the construction, the magnitude of $u(\cdot)$ is uniformly bounded by the limit of $L \in \mathbb{R}_{>0}$, i.e., $|u(\cdot)| \leq L$ holds throughout control operations. Because $L \in \mathbb{R}_{>0}$ is a freely tunable/assignable constant, it technically enables and facilitates the efficient yet straightforward satisfaction of control magnitude constraints/limitations imposed on system (1) when utilizing the proposed controller (3). Such an idea of employing saturation (bounded) control was also presented in the recent work [31] for addressing feedforward systems with nonlinear parametrization and delays.

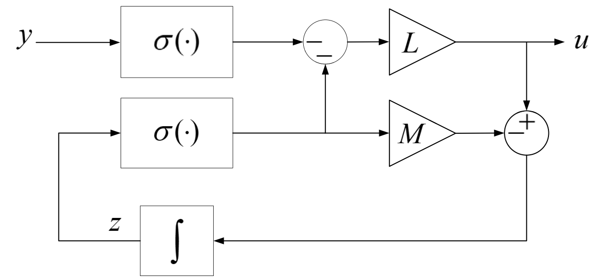

Fig. 1. Block diagram of the proposed controller (3).

Remark 3: It is noteworthy to highlight that the underlying philosophy behind the bounded output feedback controller (3), having the architecture depicted in Fig. 1, is the construction of a revamped lead compensator adeptly equipped with a mechanism securing bounded control magnitudes. To be specific, upon removing the function $\sigma(\cdot) = \tan^{-1}(\cdot)/\pi$ from the controller (3), which corresponds to the elimination of the mechanism responsible for confining control magnitudes, it can be noticed that the output feedback controller (3) undergoes a direct degeneration into

$$u = -Ly - Lz, \quad \dot{z} = u - Mz \tag{8}$$

whose Laplace transforms are

$$U(s) = -LY(s) - LZ(s), \quad sZ(s) = U(s) - MZ(s).$$

A simple calculation shows that the transfer function of the output feedback controller (8) is

$$U(s) = L\left(\frac{s + M}{s + L + M}\right)(-Y(s))$$

with $L, M \in \mathbb{R}_{>0}$, which in turn exposes that the degenerated controller (8) functions primarily as a lead compensator. Building upon the fact that the lead compensator is renowned for its remarkable ability to increase/enhance the stability of linear systems [32], [33], as extensively recognized in classic control theory, Theorem 1 sheds further light on the revelation that the proposed bounded output feedback controller (3), which intrinsically inherits the superior nature of increasing stability from the lead compensator, is capable of stabilizing the uncertain planar nonlinear system (1).

Remark 4: It is worth noting that Assumption 1 is somewhat necessary, as demonstrated by the system $\dot{x}_1 = \sin(x_2)$, $\dot{x}_2 = u$, $y = x_1$ which is in the same structure as system (1) with $g(x_2) = \sin(x_2)$ failing to satisfy Assumption 1. We assert that this system cannot be stabilized by any bounded continuous output feedback controller $u(y, z)$ with $\dot{z} = \psi(y, z, u)$. In fact, when $x_1(0) = 0$, $x_2(0) = \pi$ and $z(0) = 0$, it follows that $u(t) = 0$ and thus $x_1(t) = 0$ and $x_2(t) = \pi$ for all $t \in [0, \infty)$; i.e., the stabilization task is not achievable for this system.

B. Extension to System (1) With Unknown Measurement

Based on the analysis and proof of Theorem 1, we next show that the presented bounded output feedback controller (3), enjoying the stability-increasing capability intrinsically, can be also utilized to handle system (1) with an *unknown* measurement depicted by

$$\dot{x}_1 = g(x_2), \quad \dot{x}_2 = \theta u, \quad y = h(x_1) \tag{9}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$, $y \in \mathbb{R}$ and $u \in \mathbb{R}$ represent, respectively, the system state, output, and control input. As described in system (1), the control coefficient $\theta \in \mathbb{R}_{>0}$ and the continuous nonlinearity $g : \mathbb{R} \to \mathbb{R}$ are *unknown*; in addition, the measurement function $h : \mathbb{R} \to \mathbb{R}$ is an *unknown* continuous function complying with the assumption below.

Assumption 2: The unknown continuous function $h(\cdot)$ satisfies the following

1) $h(0) = 0$ and $h(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ with $\xi \neq 0$
2) $\int_0^{\xi} \sigma(h(\alpha))d\alpha > 0$ for all $\xi \in \mathbb{R}$ with $\xi \neq 0$
3) $\lim_{|\xi|\to+\infty} \int_0^{\xi} \sigma(h(\alpha))d\alpha = +\infty$.

Remark 5: It is apparent that Assumption 2 encompasses a broader class of functions, such as $h(x_1) = sign(x_1)\min\{|x_1|, \beta\}$ and $h(x_1) = \beta x_1/(1 + x_1^2)$ for some unknown constant $\beta \in \mathbb{R}_{>0}$, both of which are not strictly increasing and even exhibit saturation. Undoubtedly, the saturation phenomenon of the measurement output y substantially challenges the feasibility and validity of output feedback design using conventional observer-based methods, due to the inherent difficulty in attaining and/or extracting a usable state value from the measured reading. Such an obstacle is in fact a particular case of systems suffering from unknown measurement $y = h(x_1)$, which can be effectively surmounted through the presented controller (3), as shown in the theorem below.

Theorem 2: Under Assumptions 1 and 2, the dynamic bounded output feedback controller (3) with any constants $L, M \in \mathbb{R}_{>0}$ globally asymptotically stabilizes system (9).

Proof: Following the argument performed in the proof of Theorem 1 and using $y = h(x_1)$, one has

$$\begin{aligned} \dot{x}_1 &= g(x_2) \\ \dot{x}_2 &= -\theta L\sigma(h(x_1)) - \theta L\sigma\left(\frac{x_2 - e}{\theta}\right) \\ \dot{e} &= \theta M\sigma\left(\frac{x_2 - e}{\theta}\right). \end{aligned} \tag{10}$$

By considering the function $V : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ as below

$$V(\eta) = \theta L \int_0^{x_1} \sigma(h(s))ds + \int_0^{x_2} g(s)ds + \frac{L}{M}\int_0^{e} g(s)ds$$

with $\eta = (x, e) \in \mathbb{R}^3$ and $e = x_2 - \theta z$, which, in view of Assumptions 1 and 2 together with the strict increasingness of $\sigma(\cdot)$, is positive definite, continuously differentiable and proper, it follows readily from (3) and (10) that

$$\dot{V}(\eta) = -\theta L(g(x_2) - g(e))\sigma\left(\frac{x_2 - e}{\theta}\right) \leq 0$$

for all $\eta \in \mathbb{R}^3$, which is completely identical to (7). The remaining part of the proof can be straightforwardly derived in a manner consistent with that of Theorem 1, thereby proving that (3) is a global asymptotic stabilizer for system (9). ■

We now proceed by providing an illustrative example to demonstrate that the bounded output feedback controller (3) is capable of attaining global asymptotic stabilization for both systems (1) and (9).

Example 1: Consider the following system

$$\dot{x}_1 = 3x_2^{\frac{5}{3}}, \quad \dot{x}_2 = \theta u, \quad y = sign(x_1)\min(|x_1|, \beta) \tag{11}$$

where $\theta = 2$ and $\beta = 1$. Clearly, system (11) has the same form as system (9) with $g(x_2) = 3x_2^{5/3}$ and $h(x_1) = sign(x_1)\min(|x_1|, \beta)$; hence, Assumptions 1 and 2 are evidently fulfilled. It is worth noting that the measurement y of system (11) becomes saturated as $|x_1|$ of system (11) approaches one; if the measurement y of system (11) is replaced by $y = x_1$, which corresponds to the scenario with $\beta \to \infty$ (i.e., no saturation in the measurement), system (11) is exactly of the same form as system (1). By choosing the gains $L = 8$ and $M = 1.3$ for the output feedback controller (3), along with the initial state $(x_1(0), x_2(0), z(0)) = (7.5, 4, 2)$, the simulation results for system (11) with two different measurements $y = h(x_1) = sign(x_1)\min(|x_1|, \beta)$ (red line) and $y = x_1$ (blue line) are depicted in Fig. 2. These results disclose the effectiveness of the output feedback controller (3) as well as its boundedness (i.e., $|u(\cdot)| \leq 8$).

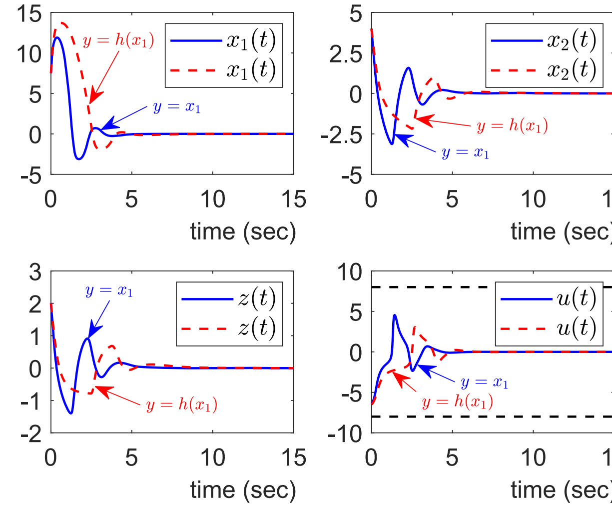

Fig. 2. Timing responses of $x_1(t)$, $x_2(t)$, $z(t)$ and $u(t)$ of systems (11).

C. Bounded Output Feedback Output Consensus Protocol

As an immediate consequence of Theorem 1, the methodology and framework for devising the dynamic bounded output feedback (3) can be readily extended to construct a bounded output feedback consensus protocol for uncertain planar heterogeneous two-agent systems, each of which owns the same structure as system (1). To demonstrate this aspect, we consider a two-agent system described by

$$\dot{w}_i = g_i(v_i), \quad \dot{v}_i = \theta_i u_i, \quad y_i = w_i \tag{12}$$

with $i = 1, 2$, where $(w_i, v_i) \in \mathbb{R}^2$, $y_i \in \mathbb{R}$, and u_i denote the system state, output, and control input, respectively. The control coefficient $\theta_i \in \mathbb{R}_{>0}$ and the continuous nonlinearity $g_i : \mathbb{R} \to \mathbb{R}$ are *unknown*. We also assume that the adjacency elements between the two agents are $a_{12} = a_{21} = 1$. In what follows, the leaderless output consensus problem of system (12) is said to be achieved by a protocol u_i with $i = 1, 2$ if for all $i = 1, 2$ and for any initial states $(w_i(0), v_i(0))$, every solution $(w_i(t), v_i(t))$ of system (12) commencing at $(w_i(0), v_i(0))$ is defined on $[0, \infty)$ and realizes $\lim_{t\to\infty} w_i(t) - w_j(t) = 0$ for all $i, j = 1, 2$ and $i \neq j$.

Theorem 3: If the unknown continuous function $g_i(\cdot)$ of system (12) satisfies Assumption 1 for all $i = 1, 2$, then the following dynamic bounded output feedback protocol

$$\begin{aligned} u_i &= -L\sigma\big(y_i - y_j\big) - L\sigma(z_i) \\ \dot{z}_i &= u_i - M\sigma(z_i), \quad i, j = 1, 2 \text{ and } i \neq j \end{aligned} \tag{13}$$

with any constants $L, M \in \mathbb{R}_{>0}$ achieves the output consensus of system (12).

Proof: The idea introduced for proving Theorem 1 can be mostly carried over into this proof.

Letting $e_i = v_i - \theta_i z_i$ and $\varepsilon_i = w_i - w_j$ for all $i, j = 1, 2$ and $i \neq j$, one instantly obtains

$$\begin{aligned} \dot{\varepsilon}_i &= g_i(v_i) - g_j\big(v_j\big) \\ \dot{v}_i &= -\theta_i L\sigma(\varepsilon_i) - \theta_i L\sigma\left(\frac{v_i - e_i}{\theta_i}\right) \\ \dot{e}_i &= \theta_i M\sigma\left(\frac{v_i - e_i}{\theta_i}\right) \end{aligned} \tag{14}$$

with $i, j = 1, 2$ and $i \neq j$. We denote $\eta = (\varepsilon_1, \varepsilon_2, v_1, v_2, e_1, e_2) \in \mathbb{R}^6$ and select a function $V : \mathbb{R}^6 \to \mathbb{R}_{\geq 0}$ as

$$\begin{aligned} V(\eta) = L&\left(\sum_{k=1}^{2}\int_0^{\varepsilon_k}\theta_k\sigma(s)ds\right) \\ &+\left(\sum_{k=1}^{2}\int_0^{v_k}\left(\frac{\theta_1+\theta_2}{\theta_k}\right)g_k(s)ds\right) \\ &+\frac{L}{M}\left(\sum_{k=1}^{2}\int_0^{e_k}\left(\frac{\theta_1+\theta_2}{\theta_k}\right)g_k(s)ds\right). \end{aligned}$$

Because of the strict increasingness of $\sigma(\cdot)$ and Assumption 1, it is obvious that $V(\eta)$ is positive definite, continuously differentiable and proper. From (14), we have

$$\begin{aligned} \dot{V}(\eta) &= \theta_1 L\sigma(\varepsilon_1)(g_1(v_1) - g_2(v_2)) \\ &\quad + \theta_2 L\sigma(\varepsilon_2)(g_2(v_2) - g_1(v_1)) \\ &\quad - (\theta_1+\theta_2)L\sum_{k=1}^{2} g_k(v_k)\left(\sigma(\varepsilon_k) + \sigma\left(\frac{v_k - e_k}{\theta_k}\right)\right) \\ &\quad + (\theta_1+\theta_2)L\sum_{k=1}^{2} g_k(e_k)\sigma\left(\frac{v_k - e_k}{\theta_k}\right) \end{aligned}$$

for all $\eta \in \mathbb{R}^6$. Because $\sigma(\cdot)$ is an odd function, leading to $\sigma(\varepsilon_1) = -\sigma(\varepsilon_2)$ for all $\eta \in \mathbb{R}^6$, it follows that

$$\begin{aligned} \dot{V}(\eta) = &-(\theta_1+\theta_2)L(g_1(v_1) - g_1(e_1))\sigma\left(\frac{v_1 - e_1}{\theta_1}\right) \\ &-(\theta_1+\theta_2)L(g_2(v_2) - g_2(e_2))\sigma\left(\frac{v_2 - e_2}{\theta_2}\right) \leq 0 \end{aligned}$$

for all $\eta \in \mathbb{R}^6$. Similarly, in view of Assumption 1 and the definition of $\sigma(\cdot)$, the set $\mathbb{H} = \{\eta \in \mathbb{R}^6 | \dot{V}(\eta) = 0\}$ is of the form $\mathbb{H} = \{(\varepsilon_1, \varepsilon_2, v_1, v_2, e_1, e_2) \in \mathbb{R}^6 | v_i - e_i = 0 \text{ for all } i = 1, 2\}$.

In line with the derivations performed in the proof of Theorem 1, we immediately know that each solution $\eta(t) = (\varepsilon_1(t), \varepsilon_2(t), v_1(t), v_2(t), e_1(t), e_2(t))$ of system (14) in respect of any initial state $\eta(t) = (\varepsilon_1(0), \varepsilon_2(0), v_1(0), v_2(0), e_1(0), e_2(0)) \in \mathbb{R}^6$ is indeed bounded uniformly and defined on $[0, \infty)$. This in conjunction with the definition of $\mathbb{H}$ indicates that the union of all forward complete solutions $\eta(t)$'s of system (14) always staying in $\mathbb{H}$ is $\mathbb{M} = \{\eta(t) \in \mathbb{R}^6 | v_i(t) - e_i(t) = 0 \text{ for all } i = 1, 2 \text{ and } t \in [0, \infty)\} \subseteq \mathbb{H}$, which on the basis of the relations below

$$\begin{aligned} \dot{v}_i(t) - \dot{e}_i(t) = &-\theta_i L\sigma(\varepsilon_i(t)) - \theta_i L\sigma\left(\frac{v_i(t) - e_i(t)}{\theta_i}\right) \\ &-\theta_i M\sigma\left(\frac{v_i(t) - e_i(t)}{\theta_i}\right) \\ \dot{\varepsilon}_i(t) = &\; g_i(v_i) - g_j\big(v_j(t)\big) \end{aligned}$$

for all $i, j = 1, 2$ with $i \neq j$ and $t \in [0, \infty)$, becomes

$$\begin{aligned} \mathbb{M} = \Big\{&(\varepsilon_1(t), \varepsilon_2(t), v_1(t), v_2(t), e_1(t), e_2(t)) \in \mathbb{R}^6 \,\Big| \\ &\varepsilon_i(t) = 0, v_i(t) - e_i(t) = 0 \text{ and } g_i(v_i(t)) = g_j\big(v_j(t)\big) \\ &\text{for all } i, j = 1, 2 \text{ with } i \neq j \text{ and } t \in [0, \infty)\Big\}. \end{aligned}$$

Therefore, from Lemma 1 we can conclude that each solution $\eta(t) = (\varepsilon_1(t), \varepsilon_2(t), v_1(t), v_2(t), e_1(t), e_2(t))$ of system (14) starting at any initial state $\eta(0) = (\varepsilon_1(0), \varepsilon_2(0), v_1(0), v_2(0), e_1(0), e_2(0)) \in \mathbb{R}^6$ satisfies $\eta(t) \to \mathbb{M}$ as $t \to \infty$, which implies that the output consensus of system (12) can be successfully accomplished. The proof is completed. ■

We present an example below to showcase the effectiveness of the bounded output feedback protocol (13) in achieving the output consensus task for system (12).

Example 2: Consider two systems as below

$$Sys_1 : \begin{cases} \dot{w}_1 = 2v_1 \\ \dot{v}_1 = \theta_1 u_1 \\ y_1 = w_1 \end{cases} \quad Sys_2 : \begin{cases} \dot{w}_2 = v_2^3 \\ \dot{v}_2 = \theta_2 u_2 \\ y_2 = w_2 \end{cases} \tag{15}$$

where $\theta_1 = 2.3$ and $\theta_2 = 1.5$. Both of them constitute system (12) with $g_1(v_1) = 2v_1$ and $g_2(v_2) = v_2^3$, satisfying 1. The simulation result shown in Fig. 3 is conducted using the gains $L = 2.5$ and $M = 1$ for the output feedback protocol (13), together with the initial state $(w_1(0), w_2(0), v_1(0), v_2(0), z_1(0), z_2(0)) = (4, 3, 1, -8, -2, -3)$. It has been demonstrated that the output consensus problem of system (12) can be reliably carried out by the output feedback protocol (13) with the bounded magnitudes $|u_i(\cdot)| \leq 2.5$ for all $i = 1, 2$.

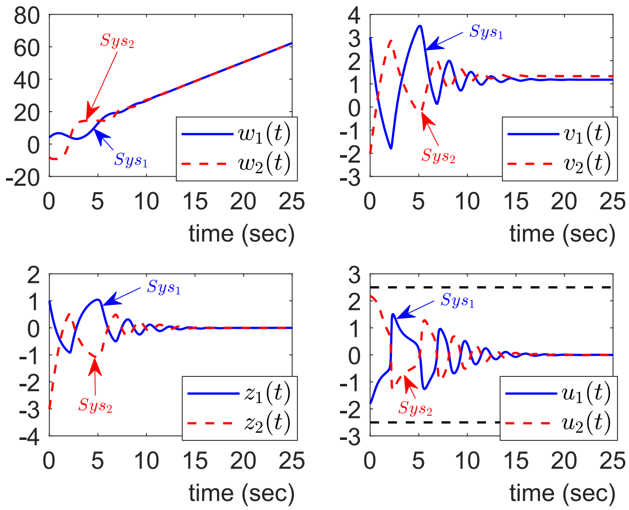

Fig. 3. Timing responses of $w_i(t)$, $v_i(t)$, $z_i(t)$ and $u_i(t)$ of system (15).

III. Conclusion

This letter has introduced a novel approach to address the issue of output feedback stabilization for a class of uncertain planar systems with unknown nonlinear structures. The proposed method involves an innovative revamp of the classical lead compensator, incorporating an arctangent function-based mechanism that guarantees bounded control magnitudes, thus not only surmounting the obstacle in constructing state observers but also clarifying the design of a bounded output feedback stabilizer. Based on the stability-increasing nature inherited from lead compensators, the resultant controller was also proved to be effective in handling systems afflicted by both unknown nonlinear structures and measurements simultaneously. Finally, the applicability of the proposed strategy was extended to the development of a bounded output feedback output consensus protocol for uncertain planar two-agent systems with unknown nonlinear heterogeneous/distinct dynamics. An intriguing issue for future research lies in the extension of the proposed approach to higher dimensional cases by integrating the nested saturation scheme [31].

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