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ABSTRACT

This work advances the understanding of oscillator Ising machines (OIMs) as a nonlinear dynamic system for solving computationally hard problems. Specifically, we classify the infinite number of all possible equilibrium points of an OIM, including non-$0/\pi$ ones, into three types based on their structural stability properties. We then employ the stability analysis techniques from control theory to analyze the stability property of all possible equilibrium points and obtain the necessary and sufficient condition for their stability. As a result of these analytical results, we establish, for the first time, the threshold of the binarization in terms of the coupling strength and strength of the second harmonic signal. Furthermore, we provide an estimate of the domain of attraction of each asymptotically stable equilibrium point by employing the Lyapunov stability theory. Finally, we illustrate our theoretical conclusions by numerical simulation.

Ising machines are actively being investigated as alternate compute engines to efficiently solve large-scale combinatorial optimization problems, many of which are NP-hard problems. The oscillator Ising machine (OIM), as an analog circuit design solution, is one such promising candidate. Many experimental OIM prototypes show a common phenomenon wherein phases of coupled oscillators bifurcate and converge to either 0 or π if the effect of sub-harmonic injection locking is sufficiently strong. These phase clusters then represent the corresponding spin assignments. As a dynamic system, this phenomenon, roughly speaking, is equivalent to the asymptotic stability property of an equilibrium point of the spin-based dynamics of the Ising machine. However, much of the existing work does not explain how an oscillator Ising machine works from a nonlinear control theoretic perspective, and the analysis is mainly based on experimental observations, which pertain to specific Ising machines. Consequently, there is a lack of a more comprehensive insight into the operation of OIMs. For example, the equilibrium points related to $0/\pi$ have been the focus of most of the existing work, and little attention has been paid to equilibrium points that consist of non-$0/\pi$ phases in the dynamics of the OIM, which also affects its computation results. In this paper, we carry out a comprehensive analysis of the equilibrium points and their structural and stability properties and quantify the influence of the second harmonic signal on the stability of phase dynamics of an OIM.

I. INTRODUCTION

The von Neumann architecture-based digital computers have become the workhorse of modern information processing. Despite their immense success, there exist problems in computing that digital computers and algorithms still struggle to compute efficiently. A case in point, and the beneficiary of the current work, is the category of NP (non-deterministic polynomial time)-hard combinatorial optimization problems (COPs). Computing their solutions using digital computers requires exponentially increasing computing resources with the increase in problem size, making even moderate problems difficult to solve effectively. Moreover, such problems are not just theoretical constructs but find immense practical applications across a wide spectrum of areas ranging from communication, interpretable machine learning, software, and data analytics to maintaining supply chain in the manufacturing industry. For example, decoding noisy multi-user MIMO signals in modern communication is an application that can be directly mapped to a modified version of the computationally intractable MaxCut problem. Consequently, there has been increasing interest in exploring alternate computational paradigms to solve such problems.

Ising machines provide a promising physics-inspired computing paradigm to solve such challenging COPs.[1,2] The concept of the Ising machine is based on the archetypal Ising model used to investigate the properties of spin glasses. In the Ising model, each spin can take two states (± 1) and interact with the neighboring spins. Subsequently, the system evolves to a configuration that minimizes the system energy, specified by the Ising Hamiltonian $H = -\sum_{i,j} J_{ij} s_i s_j$, where s_i and s_j are the states of the spins i and j, respectively, and J_{ij} represents the connection weights between them. From a computational standpoint, the relevance of the energy-minimization property is that many challenging COPs (e.g., MaxCut and Satisfiability[3–6]) can be mapped to such an H, and the solution to these problems can be expressed in terms of the minimization of H.[7] Therefore, there has been active research in dynamic system implementations, aka Ising machines, whose dynamics can naturally evolve to minimize H. Several hardware implementations spanning from degenerate optical parametric oscillator (DOPO)-based coherent Ising machines,[8–17] mechanical resonator-based Ising machines,[18–21] to electronic oscillator-based Ising machines[22–25] have been proposed, each with their benefits and shortcomings. Here, we focus specifically on the electronic oscillator Ising machine (OIM) owing to its scalability as well as compatibility with the CMOS process technology.

The basic architecture of the OIM consists of a coupled network of oscillators under second harmonic injection locking. The network has the same topological structure as the input graph with each oscillator representing a node of the input graph (spin) and the coupling elements representing the edges (interactions among the spins). It can be shown that the minima of the cost function of the oscillator system can be equivalent to the minimum of the Hamiltonian. While much work has been done on the OIM, prior work[26–31] primarily focuses on the computational properties or the hardware implementation. There has been less emphasis on understanding the dynamic behavior of OIMs, which can also be very consequential to their computational properties.

The original work of Wang *et al.*[26,27] showed that the derivative of the cost function along the trajectories of the OIM dynamics is non-positive, which reveals the evolution trend of the cost function but does not indicate the stability properties of the equilibrium points of the OIM dynamics that correspond to the minimum points of the cost function. Reference 32 indicated that a sufficiently strong coupling with the second harmonic signal, referred to as SYNC, is needed for a binarized state of oscillators to emerge, even though the threshold for such coupling strength seems to be difficult to determine. Recently, Bashar *et al.*[33] carried out a stability analysis of the equilibrium points of an OIM. By numerically calculating the eigenvalues of the Jacobian matrix at each equilibrium point, they determined local asymptotic stability of the equilibrium points.

This paper builds on the work in Ref. 33 and significantly advances the understanding of OIM from a control theoretic perspective. Unlike many existing works that are based on experimental observations and numerical simulation pertaining to specific Ising OIMs, this paper carries out stability analysis for general OIMs from a control theoretic approach.

The main contributions of this paper are listed as follows.

- *The classification of equilibrium points and stability analysis.* We identify all possible equilibrium points and classify them into three types. We further identify Type I equilibrium points that are always unstable and establish conditions in terms of the coupling strength and strength of SYNC under which the rest of the Type I equilibrium points are asymptotically stable or unstable. We prove that all Type II equilibrium points are unstable. We establish conditions in terms of the coupling strength and strength of SYNC under which a Type III equilibrium point is asymptotically stable or unstable.
- *The threshold for binarization.* As a result of our stability analysis, we show that as long as the strength of SYNC ensures that at least one Type I equilibrium point is asymptotically stable, a stable binarized state of oscillators will emerge. We establish, for the first time, the threshold of binarization in terms of the coupling strength and strength of SYNC. In addition, we examine the influence of noise in the strength of SYNC on binarization.
- *The estimate of the domain of attraction.* For each asymptotically stable equilibrium point, an estimate of its domain of attraction is provided.

We note that, as in Ref. 33, we will restrict our consideration to the case of unweighted graphs where the coupling weights among the spins are either -1 or 0.

Notation: Throughout the paper, we will use standard notation. We use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ to denote the sets of all real numbers, complex numbers, and integers, respectively. Given vectors $x_1, x_2, \ldots, x_N$, $\text{col}\{x_1, x_2, \ldots, x_N\} = \left[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Let $\mathbf{1}_N$ denote $\text{col}\{1, 1, \ldots, 1\}$. Let $\Re(\lambda)$ denote the real part of $\lambda \in \mathbb{C}$. For a matrix M, $[M]_{ij}$ is its (i, j) entry and $\lambda(M)$ denotes the set of its eigenvalues.

The remainder of the paper is organized as follows. Section II introduces the problem we are to study in this paper. Our main results are presented in Sec. III. We will start some technical preparation in Sec. III A, proceed to stability analysis of equilibrium points in Sec. III B, and complete the section with the estimation of the domains of attraction of asymptotically stable equilibrium points in Sec. III C. Section IV presents numerical examples to illustrate the theoretical results of the paper. Section V concludes the paper.

II. MOTIVATION AND PROBLEM FORMULATION

A combinatorial optimization problem (COP) is usually formulated as the problem of finding an optimal combination of integers that minimizes the energy of coupled spins s_i, $i = 1, 2, \ldots, N$. The energy is equivalent to the Hamiltonian function given as follows:

$$H = -\sum_{i,j=1,i<j}^{N} J_{ij} s_i s_j,$$

where spins $s_i, s_j \in \{-1, 1\}$ represent the states of oscillators i and j, corresponding to phases of 0 and π, respectively, and J_{ij} denotes the connection weight between oscillators i and j with $J_{ij} = J_{ji}$. In particular, $J_{ij} = -1$ if oscillators i and j are connected, and $J_{ij} = 0$ otherwise.

There exists an interesting relationship, namely, $s_i s_j = \cos(\phi_i - \phi_j)$, where ϕ_i and ϕ_j take the value of 0 or π. Note that ϕ_i and ϕ_j settle to 0 or π, corresponding to s_i, s_j equaling to 1 or -1. Taking advantage of this relationship, we can map a bounded cost function into the Hamiltonian function with a constant offset as follows:[26]

$$E = -K \sum_{i,j=1,i\neq j}^{N} J_{ij} \cos(\phi_i - \phi_j) - K_s \sum_{i=1}^{N} \cos(2\phi_i),$$

where K and K_s are two positive constants that we will later further discuss. Therefore, E takes the minimum value, at a point $(\phi_1, \phi_2, \ldots, \phi_N)$, $\phi_i \in \{0, \pi\}$, which is equal to $2K$ times the minimum value of H with a constant offset NK_s. Consequently, a typical objective is to design the phase dynamics for the oscillators such that the cost function E decreases along the trajectories of the dynamics, namely,

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq 0. \tag{1}$$

Note that $\frac{\mathrm{d}E}{\mathrm{d}t}$ can be written as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\partial E}{\partial \phi_i} \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\partial E}{\partial \phi_i} \dot{\phi}_i.$$

Thus, the dynamics of ϕ_i given as $\dot{\phi}_i = -\frac{1}{2}\frac{\partial E}{\partial \phi_i}$ would satisfy (1). By calculating $\frac{\partial E}{\partial \phi_i}$, we yield

$$\dot{\phi}_i = -K \sum_{j=1, j\neq i}^{N} J_{ij} \sin(\phi_i - \phi_j) - K_s \sin(2\phi_i), \tag{2}$$

where $K > 0$ represents the coupling strength and $K_s > 0$ represents the strength of the coupling from the SYNC.

However, decreasing of the cost function along a trajectory the Ising machine approaches an equilibrium point does not imply the stability of that equilibrium point as the cost function is not positive definite with respect to its value at that equilibrium, which is required by the Lyapunov stability theory. In the absence of this positive definiteness property of the cost function E, the trajectory could pass through the equilibrium point as E continues to decrease. As a result, an additional understanding of the stability of all equilibrium points or solutions is needed. In our recent work,[33] we resorted to numerical calculation of the eigenvalues of the Jacobian matrix at each equilibrium point to determine the local asymptotic stability of the equilibrium points for various values of the coupling strengths K and K_s. Strong effects of the coupling strength with the SYNC on the stability of the equilibrium points were also observed.

In this paper, our objective is to carry out a comprehensive analysis of the equilibrium points and their structural and stability properties. In particular, we first classify all equilibrium points into three types based on their structural stability. Then we carry out stability analysis for all equilibrium points by Lyapunov stability theory. We next establish the threshold of binarization in terms of the ratio of K_s and K. Meanwhile, we examine the influence of noise of in the value of K_s on binarization. In addition, we provide the estimates of the domains of attraction for all asymptotically stable equilibrium points. Finally, numerical simulation illustrates our analytical results.

III. MAIN RESULTS

A. Technical lemmas

To develop our main results, we need some preliminary technical lemmas.

Lemma 1 (Ref. 34): *Consider a nonlinear system*

$$\dot{x} = f(x), \ x \in \mathbb{R}^N,$$

where $f: \mathscr{D} \to \mathbb{R}^N$ is continuously differentiable and $\mathscr{D}$ is a neighborhood of its equilibrium point $x = x^\star$. Let

$$A = \left.\frac{\partial f}{\partial x}(x)\right|_{x=x^\star}.$$

Then,

1. *The equilibrium point $x = x^\star$ is asymptotically stable if $\Re(\lambda_i) < 0$ for all eigenvalues λ_i of A.*
2. *The equilibrium point $x = x^\star$ is unstable if $\Re(\lambda_i) > 0$ for one or more of the eigenvalues λ_i of A.*

Lemma 2: *For any positive number $m \in (0, \pi)$, the inequality $x\sin(x) \geq \frac{\sin(m)}{m}x^2$ holds for $x \in [-m, m]$.*

Proof: *As shown in Fig. 1, $\sin x \geq \frac{\sin(m)}{m}x$ for $x \in [0, m]$, and the equality holds for $x = 0$ and $x = m$. Thus, we have $x\sin(x) \geq \frac{\sin(m)}{m}x^2$ for $x \in [0, m]$. Noting that $x\sin(x)$ and x^2 are both even functions, we see that $x\sin(x) \geq \frac{\sin(m)}{m}x^2$ also holds for $x \in [-m, 0]$.*

Lemma 3: *For any $x, \delta \in \mathbb{R}$, $|\cos(x+\delta) - \cos(x)| \leq |\delta|$.*

Proof: *The largest and smallest slopes of $\cos(x)$ equal 1 and -1, respectively. By the mean value theorem, for any $\delta \neq 0$, we have*

$$\left|\frac{\cos(x+\delta) - \cos(x)}{\delta}\right| < 1,$$

and, hence, $|\cos(x+\delta) - \cos(x)| \leq |\delta|$ for any $x, \delta \in \mathbb{R}$. This completes the proof.

B. Stability analysis

The number of equilibrium points of system (2), some of which are composed of phases other than 0 and π, is infinite.

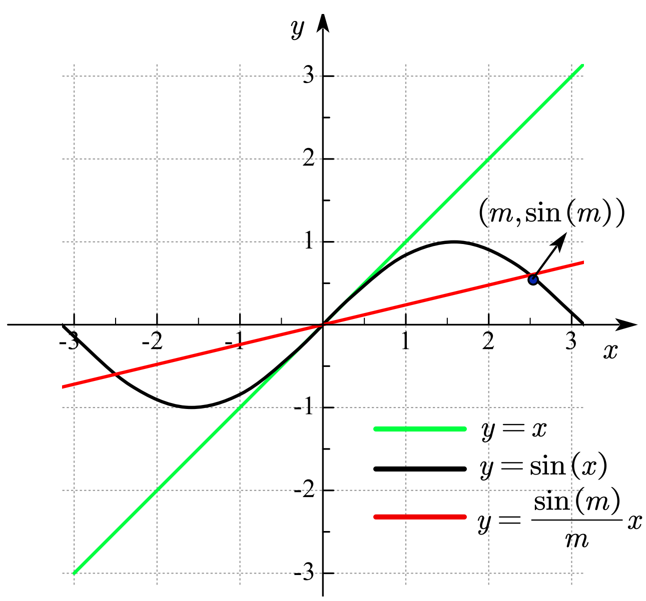

FIG. 1. x, $\sin(x)$, and $\frac{\sin(m)}{m}x$.

These equilibrium points can be classified into three types. The first type of equilibrium point always exists no matter how the system parameters or topology vary. Such equilibrium points are called Type I (structurally stable) equilibrium points. The other two types of equilibrium points change once the system parameters or topology change and are referred to as Type II and III (structurally unstable) equilibrium points. We study the stability of these three types of equilibrium points separately. First, we identify the Type I equilibrium points in Theorem 1 and analyze their stability in Theorem 2. We will give the stability analysis for Type II and III equilibrium points in Theorems 3 and 4, respectively.

Definition 1: *A Type I equilibrium point $\phi^\star = \mathrm{col}\{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\}$ satisfies $\phi_i^\star \in \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ for all i, and it remains unchanged no matter how the system parameters and topology vary.*

Theorem 1: *Let $\phi^\star = \mathrm{col}\{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\} \in \mathbb{R}^N$ be a Type I equilibrium point of (2). Then, either $\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k}{2} \in \mathbb{Z}\right\}$ for all i or $\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k+1}{2} \in \mathbb{Z}\right\}$ for all i.*

Proof: *The equilibrium point $\phi^\star$ satisfies the phase dynamics (2), that is,*

$$K_s \sin(2\phi_i^\star) = -K \sum_{j=1, j\neq i}^{N} J_{ij} \sin\left(\phi_i^\star - \phi_j^\star\right). \tag{3}$$

Since $\phi^\star$ remains unchanged as the system parameters vary, Eq. (3) holds for all values of K and K_s, which implies that

$$\begin{cases} \sin(2\phi_i^\star) = 0, \ i = 1, 2, \ldots, N, \\ \sum_{j=1, j\neq i}^{N} J_{ij} \sin(\phi_i^\star - \phi_j^\star) = 0, \ i = 1, 2, \ldots, N. \end{cases} \tag{4}$$

In addition, since (4) holds for all possible values of J_{ij}, we have

$$\begin{cases} 2\phi_i^\star = k\pi, \ i = 1, 2, \ldots, N, \\ \phi_i^\star - \phi_j^\star = k\pi, \ i, j = 1, 2, \ldots, N, \end{cases}$$

where $k \in \mathbb{Z}$. It then follows that either

$$\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k}{2} \in \mathbb{Z}\right\}$$

for all i or

$$\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k+1}{2} \in \mathbb{Z}\right\}$$

for all i. This completes the proof.

Theorem 2: *Consider a Type I equilibrium point $\phi^\star = \mathrm{col}\{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\} \in \mathbb{R}^N$. If $\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k+1}{2} \in \mathbb{Z}\right\}$ for all i, then, $\phi^\star$ is unstable for any values of the parameters K and K_s. If $\phi_i^\star \in \left\{\frac{k\pi}{2} : \frac{k}{2} \in \mathbb{Z}\right\}$ for all i, then, $\phi^\star$ is asymptotically stable for $K_s > \frac{K\lambda_N(D(\phi^\star))}{2}$ or $K_s/K > \frac{\lambda_N(D(\phi^\star))}{2}$ and unstable for $K_s < \frac{K\lambda_N(D(\phi^\star))}{2}$ or $K_s/K < \frac{\lambda_N(D(\phi^\star))}{2}$, where $\lambda_N(D(\phi^\star))$ denotes the maximum eigenvalue of matrix $D(\phi^\star)$, shown in (5).*

$$D(\phi^\star) = \begin{bmatrix} -\sum_{j=1, j\neq 1}^{N} J_{1j} \cos(\phi_1^\star - \phi_j^\star) & J_{12} \cos(\phi_1^\star - \phi_2^\star) & \cdots & J_{1N} \cos(\phi_1^\star - \phi_N^\star) \\ J_{21} \cos(\phi_2^\star - \phi_1^\star) & -\sum_{j=1, j\neq 2}^{N} J_{2j} \cos(\phi_2^\star - \phi_j^\star) & \cdots & J_{2N} \cos(\phi_2^\star - \phi_N^\star) \\ \vdots & \vdots & \ddots & \vdots \\ J_{N1} \cos(\phi_N^\star - \phi_1^\star) & J_{N2} \cos(\phi_N^\star - \phi_2^\star) & \cdots & -\sum_{j=1, j\neq N}^{N} J_{Nj} \cos(\phi_N^\star - \phi_j^\star) \end{bmatrix}. \tag{5}$$

Proof: *Let $f(\phi) = \mathrm{col}\{f_1(\phi), f_2(\phi), \ldots, f_N(\phi)\}$, where*

$$f_i(\phi) = -K \sum_{j=1, j\neq i}^{N} J_{ij} \sin(\phi_i - \phi_j) - K_s \sin(2\phi_i).$$

The Jacobian matrix of $f(\phi)$ at the equilibrium point $\phi = \phi^\star$ is evaluated as

$$A(\phi^\star) = \left.\frac{\partial f}{\partial \phi}\right|_{\phi = \phi^\star} = KD(\phi^\star) - 2K_s \begin{bmatrix} \cos(2\phi_1^\star) & 0 & \cdots & 0 \\ 0 & \cos(2\phi_2^\star) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cos(2\phi_N^\star) \end{bmatrix},$$

where $D(\phi^\star)$ is as shown in (5).

Let

$$M_1 = \left\{\phi = \mathrm{col}\{\phi_1, \phi_2, \ldots, \phi_N\} : \phi_i \in \left\{\frac{k\pi}{2} : \frac{k+1}{2} \in \mathbb{Z}\right\}\right\}$$

and

$$M_2 = \left\{\phi = \mathrm{col}\{\phi_1, \phi_2, \ldots, \phi_N\} : \phi_i \in \left\{\frac{k\pi}{2} : \frac{k}{2} \in \mathbb{Z}\right\}\right\}.$$

If $\phi^\star \in M_1$, then the Jacobian matrix at $\phi = \phi^\star$ is given as

$$A(\phi^\star) = KD(\phi^\star) + 2K_s I_N.$$

It can be verified that $D(\phi^\star)$ has the following property:

$$D(\phi^\star)\mathbf{1}_N = 0,$$

which means that $D(\phi^\star)$ has a 0 eigenvalue with an associated eigenvector $\mathbf{1}_N$. Thus, we can conclude that $A(\phi^\star)$ has at least one eigenvalue that is equal to $2K_s > 0$. By using Lemma 1, $\phi = \phi^\star$ is unstable.

If $\phi^\star \in M_2$, then the Jacobian matrix at $\phi^\star$ is given as

$$A(\phi^\star) = KD(\phi^\star) - 2K_s I_N.$$

Let $\lambda_N(D(\phi^\star))$ be the maximum eigenvalue of matrix $D(\phi^\star)$. Then, all eigenvalues of $A(\phi^\star)$ are negative if

$$K_s > \frac{K\lambda_N(D(\phi^\star))}{2}.$$

In this case, by Lemma 1, $\phi = \phi^\star$ is asymptotically stable. On the other hand, at least one eigenvalue of $A(\phi^\star)$ is positive if

$$K_s < \frac{K\lambda_N(D(\phi^\star))}{2}.$$

In this case, again by Lemma 1, $\phi = \phi^\star$ is unstable. This completes the proof.

Remark 1: *It is noted that $D(\phi^\star)$ is a symmetric matrix with off-diagonal elements belonging to $\{-1, 0, 1\}$. In some literature,[35,36] $D(\phi^\star)$ is dubbed as "Net Laplacian," "Repelling Laplacian," or "Signed Laplacian." Determining the eigenvalues of $D(\phi^\star)$ will incur a large computational cost as the number of oscillators N increases. Although an upper bound of its maximum eigenvalue can be estimated in terms of its matrix norm $\|D(\phi^\star)\|_p$, such an estimate could be very conservative. Consequently, a selection of the parameters K and K_s based on this estimate could be conservative. Recently, a relatively less conservative upper bound was given in Ref. 37 as*

$$\lambda_N\big(D(\phi^\star)\big) \le \max_{1\leqslant i\leqslant N}\left\{\frac{1}{2}\left(d_i^\pm + \sqrt{{d_i^\pm}^2 + 8d_i m_i}\right)\right\},$$

where $d_i^\pm = d_i^+ - d_i^-$, $m_i = \frac{\sum_{j\in\mathcal{N}_i} d_j^\star}{d_i}$, and $d_j^\star = \max\{d_j^+, d_j^-\}$. Here, d_i^+ (respectively, d_i^-) represents the number of vertices adjacent to i by a positive (respectively, negative) edge, and $\mathcal{N}_i$ represents the index set of vertex i's neighbors.

Remark 2: *An alternative way to determine a precise range of K_s/K within which $A(\phi^\star)$ has all negative eigenvalues is through verification of a linear matrix inequality (LMI).[38] The symmetric matrix $A(\phi^\star)$ will have all negative eigenvalues if and only if there is a positive definite matrix $P \in \mathbb{R}^{N\times N}$ such that the following Lyapunov matrix inequality holds:*

$$A^{\mathrm{T}}(\phi^\star)P + PA(\phi^\star) < 0$$

or

$$\big(D(\phi^\star) - 2\alpha I_N\big)^{\mathrm{T}}P + P\big(D(\phi^\star) - 2\alpha I_N\big) < 0, \tag{6}$$

where $\alpha = K_s/K > 0$. Thus, the range of α satisfying (6) can be obtained by solving the following LMI optimization problem:

$$\begin{gathered}\min_{P>0,\alpha>0} \alpha,\\ \text{s.t. Inequality (6).}\end{gathered} \tag{7}$$

Since Inequality (6) is an LMI for each fixed value of α, the optimization problem (7) can be solved by a line search over $\alpha \in (0, \infty)$.

Remark 3: *Theorem 2 establishes the condition for stability of the equilibrium points that belong to set M_2. This condition is necessary and sufficient except for the critical case of $K_s = \frac{K\lambda_N(D(\phi^\star))}{2}$. In this case, the eigenvalues of the Jacobian matrix are all non-positive but at least one of them is zero. As a result, the linearization fails to determine the stability of the equilibrium point in question and further analysis is entailed. We give two examples to show that the equilibrium point can be either stable or unstable in this critical case.*

Example 1 (Stable equilibrium): *Consider two coupled oscillators. The dynamics $\dot\phi = f(\phi)$ are given as*

$$\begin{cases}\dot\phi_1 = K\sin(\phi_1 - \phi_2) - K_s\sin(2\phi_1),\\ \dot\phi_2 = K\sin(\phi_2 - \phi_1) - K_s\sin(2\phi_2).\end{cases} \tag{8}$$

Consider the equilibrium point $\phi^\star = \mathrm{col}(0, 0)$, for which

$$D(\phi^\star) = \begin{bmatrix} 1 & -1\\ -1 & 1\end{bmatrix}$$

and $\lambda(D(\phi^\star)) = \{0, 2\}$. Thus, the critical case is $K_s = K$. In this critical case, the Jacobian matrix at equilibrium point $\phi^\star = \mathrm{col}(0, 0)$ is

$$A = \begin{bmatrix} -K & -K\\ -K & -K\end{bmatrix},$$

with $\lambda(A) = \{-2K, 0\}$. Hence, no conclusion can be drawn on the stability of the equilibrium point. We will resort to the center manifold theory[34] to analyze the stability. To do so, we first write (8) as

$$\dot\phi = A\phi + \tilde f(\phi),$$

where $\tilde f(\phi) = f(\phi) - A\phi$. For brevity and without loss of generality, we let $K = 1$. We next carry out a state transformation $\theta = \mathrm{col}(\theta_1, \theta_2) = T\phi$, with

$$T = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}.$$

It is easy to verify that

$$\begin{cases}\dot\theta_1 = g_1(\theta_1, \theta_2),\\ \dot\theta_2 = -2\theta_2 + g_2(\theta_1, \theta_2),\end{cases} \tag{9}$$

where

$$g_1(\theta_1, \theta_2) = \sin(2\theta_1) - \frac{1}{2}\sin(2\theta_1 - 2\theta_2) - \frac{1}{2}\sin(2\theta_1 + 2\theta_2),$$

$$g_2(\theta_1, \theta_2) = \frac{1}{2}\sin(2\theta_1 - 2\theta_2) - \frac{1}{2}\sin(2\theta_1 + 2\theta_2) + 2\theta_2.$$

Let $\theta_2 = h(\theta_1)$ be a center manifold. Then, the function $h(\theta_1)$ satisfies the partial differential equation

$$\begin{aligned}\mathcal{M}(h(\theta_1)) = \frac{\partial h}{\partial\theta_1}(\theta_1)\big(A_1\theta_1 + g_1(\theta_1, h(\theta_1))\big) - A_2 h(\theta_1)\\ - g_2(\theta_1, h(\theta_1)) = 0,\end{aligned} \tag{10}$$

with the boundary conditions

$$h(0) = 0, \ \frac{\partial h}{\partial \theta_1}(0) = 0.$$

It is noted that $h(\theta_1) = 0$ *satisfies (10). Hence, the reduced system can be obtained from (9) as*

$$\dot{\theta}_1 = 0.$$

Consider $V(\theta_1) = \theta_1^2$. *The derivative of* V *along the trajectory of the reduced system is given by*

$$\dot{V}(\theta_1) = 2\theta_1\dot{\theta}_1 = 0.$$

Thus, by Corollary 8.1 of Ref. 34, system (8) is stable at the equilibrium point $\phi^\star = \mathrm{col}(0,0)$.

In fact, we also note that system (8) has a continuum of equilibrium points on $\phi_1 + \phi_2 = 0$ *when* $K_s = K$. *Since* $\theta_2 = \frac{1}{2}(\phi_1 + \phi_2)$, *every point on the center manifold* $\theta_2 = 0$ *is an equilibrium point. We will show that each of these equilibrium points is stable. To see this, we consider* $V(\theta_2) = \theta_2^2$. *The derivative of* $V(\theta_2)$ *along the trajectory of (9) can be evaluated as*

$$\begin{aligned}\dot{V}(\theta_2) &= \frac{\partial V}{\partial \theta_2}\dot{\theta}_2 = -K\theta_2\big(\sin(2\phi_1) + \sin(2\phi_2)\big)\\ &= -K(\phi_1 + \phi_2)\big(\sin(2\phi_1) + \sin(2\phi_2)\big).\end{aligned}$$

We consider the region of $|\phi_1| < \frac{\pi}{4}, |\phi_2| < \frac{\pi}{4}$ *in the following two cases.*

Case 1: $\phi_1 + \phi_2 \geq 0$. *In this case,* $\phi_1 \geq -\phi_2$ *and* $2\phi_1, 2\phi_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then,* $\sin(2\phi_1) \geq \sin(-2\phi_1)$ *or* $\sin(2\phi_1) + \sin(2\phi_2) \geq 0$. *Consequently,* $\dot{V}(\theta_2) \leq 0$.

Case 2: $\phi_1 + \phi_2 < 0$. *In this case,* $\phi_1 < -\phi_2$ *and* $2\phi_1, 2\phi_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then,* $\sin(2\phi_1) < \sin(-2\phi_2)$ *or* $\sin(2\phi_1) + \sin(2\phi_2) < 0$. *Consequently,* $\dot{V}(\theta_2) < 0$.

Based on the above analysis, we conclude that $\dot{V}(\theta_2) < 0$ *in the region of* $|\phi_1| < \frac{\pi}{4}, |\phi_2| < \frac{\pi}{4}$, $\phi_1 + \phi_2 \neq 0$. *Thus, all trajectories converge to the manifold* $\theta_2 = 0$ *asymptotically, and on the manifold,* $\dot{\theta}_1 = 0$. *This shows that every point on the manifold is stable, but not asymptotically stable. A phase portrait is shown in Fig. 2 to illustrate our analytical conclusion.*

Example 2 (Unstable equilibrium): *Consider three coupled oscillators, whose connectivity topology is shown in Fig. 3. The dynamics* $\dot{\phi} = f(\phi)$ *are given as*

$$\begin{cases}\dot{\phi}_1 = K\sin(\phi_1 - \phi_2) - K_s\sin(2\phi_1),\\ \dot{\phi}_2 = K\sin(\phi_2 - \phi_1) + K\sin(\phi_2 - \phi_3) - K_s\sin(2\phi_2),\\ \dot{\phi}_3 = K\sin(\phi_3 - \phi_2) - K_s\sin(2\phi_3).\end{cases} \tag{11}$$

Consider the equilibrium point $\phi^\star = \mathrm{col}\{0,0,0\}$, *for which*

$$D(\phi^\star) = \begin{bmatrix} 1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

and $\lambda(D(\phi^\star)) = \{0,1,3\}$. *Thus, the critical case is* $K_s = 1.5\,\mathrm{K}$. *In this critical case, the Jacobian matrix at equilibrium point* $\phi^\star = \mathrm{col}\{0,0,0\}$ *is*

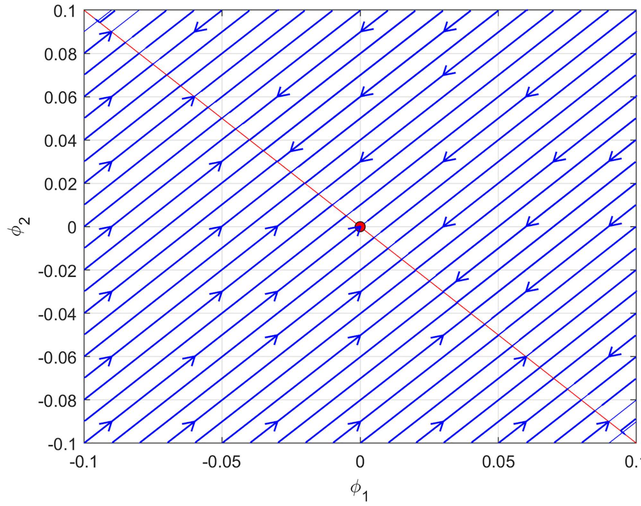

FIG. 2. A phase portrait showing that all points on the manifold $\phi_1 + \phi_2 = 0$ is a stable, but not asymptotically stable, equilibrium point when $K = K_s = 1$.

$$A = \begin{bmatrix} -2K & -K & 0\\ -K & -K & -K\\ 0 & -K & -2K\end{bmatrix},$$

with $\lambda(A) = \{-3K, -2K, 0\}$. *Hence, no conclusion can be drawn on the stability of the equilibrium point. We will again resort to the center manifold theory to analyze the stability. First, we write (11) as*

$$\dot{\phi} = A\phi + \tilde{f}(\phi),$$

where $\tilde{f}(\phi) = f(\phi) - A\phi$. *For brevity and without loss of generality, we let* $K = 1$. *We next carry out a state transformation* $\theta = \mathrm{col}\{\theta_1, \theta_2\} = T\phi$, $\theta_2 = \mathrm{col}\{\theta_{2,1}, \theta_{2,2}\}$, *with*

$$T = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}.$$

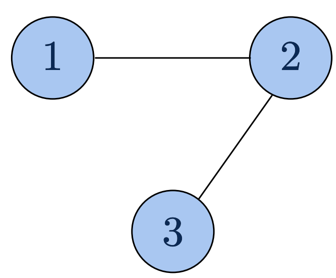

FIG. 3. A topology of three coupled oscillators.

It can be verified that

$$\dot{\theta}_1 = A_1\theta_1 + g_1(\theta_1, \theta_2),$$
$$\dot{\theta}_2 = A_2\theta_2 + g_2(\theta_1, \theta_2), \tag{12}$$

where $g_2 = \mathrm{col}\{g_{2,1}, g_{2,2}\}$,

$$A_1 = 0, A_2 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix},$$

and

$$\begin{aligned} g_1(\theta_1,\theta_2) &= \frac{1}{2}\big(\sin(3\theta_1 - \theta_{2,2}) + \sin(3\theta_1 + \theta_{2,2}) \\ &\quad + \sin(-4\theta_1 + 2\theta_{2,1})\big) - \frac{1}{4}\big(\sin(2\theta_1 + 2\theta_{2,1} - 2\theta_{2,2}) \\ &\quad + \sin(2\theta_1 + 2\theta_{2,1} + 2\theta_{2,2})\big), \\ g_{2,1}(\theta_1,\theta_2) &= 3\theta_{2,1} - \frac{1}{2}\big(\sin(2\theta_1 + 2\theta_{2,1} - 2\theta_{2,2}) \\ &\quad + \sin(2\theta_1 + 2\theta_{2,1} + 2\theta_{2,2}) + \sin(-4\theta_1 + 2\theta_{2,1})\big), \\ g_{2,2}(\theta_1,\theta_2) &= 2\theta_{2,2} - \frac{1}{2}\big(\sin(3\theta_1 - \theta_{2,2}) - \sin(3\theta_1 + \theta_{2,2})\big) \\ &\quad + \frac{3}{4}\big(\sin(2\theta_1 + 2\theta_{2,1} - 2\theta_{2,2}) \\ &\quad - \sin(2\theta_1 + 2\theta_{2,1} + 2\theta_{2,2})\big). \end{aligned}$$

Let $\theta_2 = h(\theta_1) : \mathbb{R} \to \mathbb{R}^2$ *be a center manifold. Then, the function* $h(\theta_1)$ *satisfies the partial differential equation*

$$\begin{aligned}\mathcal{M}(h(\theta_1)) &= \frac{\partial h}{\partial \theta_1}(\theta_1)\big(A_1\theta_1 + g_1(\theta_1, h(\theta_1))\big) - A_2 h(\theta_1) \\ &\quad - g_2(\theta_1, h(\theta_1)) = 0,\end{aligned}$$

with the boundary conditions

$$h(0) = 0, \ \frac{\partial h}{\partial \theta_1}(0) = 0.$$

In this case, the solution $h(\theta_1)$ *to the partial differential equation is hard to obtain. We will approximate it with its Taylor expansion series* $\psi(\theta_1)$. *We start with* $\psi(\theta_1) = 0$ *and verify that* $\mathcal{M}(\psi(\theta_1)) = o(\theta_1^3)$ *in the neighborhood of* $\theta_1 = 0$. *Thus, by Ref. 34 [Theorem 8.3],* $h(\theta_1) = \psi(\theta_1) + o(\theta_1^3)$ *and the reduced system can be obtained from (12) as*

$$\dot{\theta}_1 = \sin(3\theta_1) - \frac{1}{2}\sin\left(4\theta_1 - 2o(\theta_1^3)\right) - \frac{1}{2}\sin\left(2\theta_1 + 2o(\theta_1^3)\right). \tag{13}$$

Recalling the Taylor expansion $\sin x$ *at* $x = 0$,

$$\sin x = x - \frac{1}{6}x^3 + o(x^5),$$

we can rewrite the reduced system (13) as

$$\dot{\theta}_1 = \frac{3}{2}\theta_1^3 + o(\theta_1^5).$$

which is unstable at $\theta_1 = 0$. *By the Center Manifold Theorem, the equilibrium point* $\phi^\star = \mathrm{col}\{0, 0, 0\}$ *of the original system (11) is unstable. The evolution of* ϕ_1, ϕ_2, *and* ϕ_3 *from a set of small initial values is shown in Fig. 4 to illustrate our analytical conclusion.*

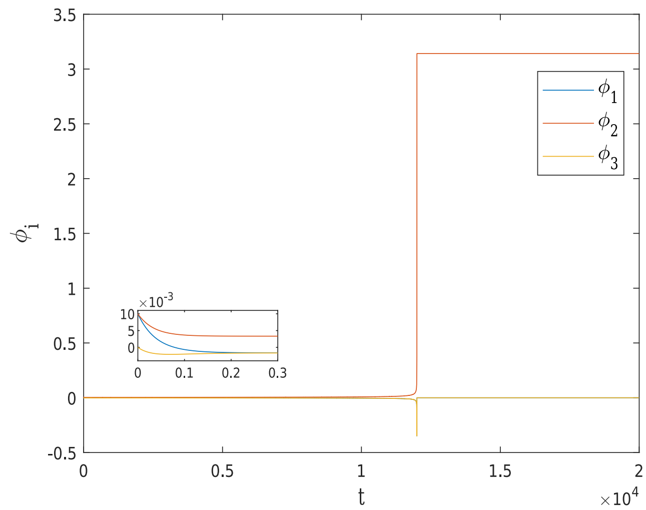

FIG. 4. The evolution of the system trajectory from $\phi_1(0) = 0.01, \phi_2(0) = 0.01, \phi_3(0) = 0$, showing that the equilibrium point $\phi^\star = \mathrm{col}\{0, 0, 0\}$ is unstable when $K = 10$ and $K_s = 15$.

We will next consider other types of equilibrium points, Type II equilibrium points, and Type III equilibrium points.

Definition 2: *A Type II equilibrium point* $\phi^\star = \mathrm{col}\{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\}$ *satisfies* $\phi_i^\star \in \{\frac{k\pi}{2}, k \in \mathbb{Z}\}$ *for all i, and there exists a set of values* J_{ij}'*s such that (4) does not hold.*

Definition 3: *A Type III equilibrium point* $\phi^\star = \mathrm{col}\{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\}$ *is one for which* $\phi_i^\star \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ *for at least one* $i, i = 1, 2 \cdots, N$.

Remark 4: *It is noted that the union of Type I and II equilibrium points is* $\{\phi = \{\phi_1^\star, \phi_2^\star, \ldots, \phi_N^\star\} : \phi_i^\star \in \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ *for all i}. As a result, the union of Type I, II, and III equilibrium points contains all possible equilibrium points.*

Remark 5: *A Type II equilibrium point may fail to satisfy (4) as the topology varies. A Type III equilibrium point satisfies (3) for specific values of the system parameters K and* K_s *but does not satisfy (4). Type II and Type III equilibrium points are structurally unstable. On the contrary, Type I equilibrium points are structurally stable.*

We will first examine the stability property of Type II equilibrium points.

Theorem 3: *Any Type II equilibrium point* $\phi = \phi^\star$ *is unstable.*

Proof: *For a given Type II equilibrium point, it satisfies (4) with some given values of* J_{ij}'*s, which means that each spin's phase has to belong to* $\{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ *since*

$$\sin(2\phi_i) = 0, \ i = 1, 2, \ldots, N. \tag{14}$$

However, it is noted that (4) does not hold for all values of J_{ij}'*s. Hence,*

$$\sum_{j=1, j\neq i}^{N} J_{ij}\sin(\phi_i^\star - \phi_j^\star) = 0, \ i = 1, 2, \ldots, N \tag{15}$$

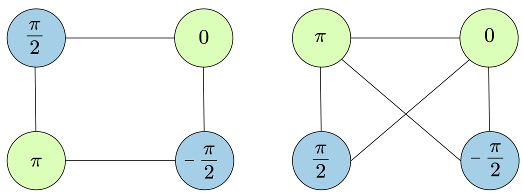

FIG. 5. Type II equilibrium points.

implies that there must exist at least two phase differences of coupled spins that do not equal $k\pi, k \in \mathbb{Z}$, and these phase differences can result in the corresponding terms in (15) canceling each other. Thus, the only one possible that a Type II equilibrium point satisfies (14) and (15) is that $0 + 2k\pi, \pi + 2k\pi, \frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$ appears simultaneously in the configuration. Additionally, any spin in phases $0 + 2k\pi$ or $\pi + 2k\pi$ must be connected to at least two points in phases $\frac{\pi}{2} + 2k\pi$ and $-\frac{\pi}{2} + 2k\pi$, vice versa since such configurations will result in the emergence of terms $\sin(\frac{\pi}{2} + 2k\pi), \sin\left(-\frac{\pi}{2} + 2k\pi\right)$ in (15). Figure 5 gives two examples of possible configurations of Type II equilibrium points.

The Jacobian matrix at $\phi = \phi^\star$ is given by

$$
\begin{aligned}
A(\phi^\star) &= \frac{\partial f}{\partial \phi}\bigg|_{\phi=\phi^\star} = KD(\phi^\star) \\
&\quad - 2K_s \begin{bmatrix} \cos(2\phi_1^\star) & 0 & \cdots & 0 \\ 0 & \cos(2\phi_2^\star) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cos(2\phi_N^\star) \end{bmatrix} \\
&= KD(\phi^\star) - 2K_s \Delta(\phi^\star),
\end{aligned}
$$

where $\cos(2\phi_i^\star) \in \{+1, -1\}, i = 1, 2, \ldots, N$. We will show that $A(\phi^\star)$ has at least one positive eigenvalue. To do so, we only need to find a vector $x = \mathrm{col}\{x_1, x_2, \ldots, x_N\} \neq 0$ such that $x^{\mathrm{T}} A(\phi^\star) x > 0$. Define four sets

$$
\begin{aligned}
\mathscr{Q} &= \left\{ q : \phi_q^\star = -\frac{\pi}{2} + 2k\pi \right\}, \\
\mathscr{P} &= \left\{ p : \phi_p^\star = \frac{\pi}{2} + 2k\pi \right\}, \\
\mathscr{S} &= \left\{ s : \phi_s^\star = 0 + 2k\pi \right\}, \\
\mathscr{U} &= \left\{ u : \phi_u^\star = \pi + 2k\pi \right\}.
\end{aligned}
$$

Note that these sets are all non-empty and form a partition of the set $\{1, 2, \ldots, N\}$. Based on these sets, we can then evaluate $x^{\mathrm{T}} A(\phi^\star) x$ as follows:

$$
\begin{aligned}
x^{\mathrm{T}} A(\phi^\star) x &= K x^{\mathrm{T}} D(\phi^\star) x - 2K_s x^{\mathrm{T}} \Delta(\phi^\star) x \\
&= K \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} J_{ij}(x_i - x_j)^2 + K \sum_{i \in \mathscr{S}, j \in \mathscr{U}} J_{ij}(x_i - x_j)^2 \\
&\quad - K \sum_{i,j \in \mathscr{Q}, i \neq j} J_{ij}(x_i - x_j)^2 - K \sum_{i,j \in \mathscr{P}, i \neq j} J_{ij}(x_i - x_j)^2 \\
&\quad - K \sum_{i,j \in \mathscr{S}, i \neq j} J_{ij}(x_i - x_j)^2 - K \sum_{i,j \in \mathscr{U}, i \neq j} J_{ij}(x_i - x_j)^2 \\
&\quad + 2K_s \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} (x_i^2 + x_j^2) - 2K_s \sum_{i \in \mathscr{S}, j \in \mathscr{U}} (x_i^2 + x_j^2),
\end{aligned}
$$

where we have used the fact that the (i, j) entry of $D(\phi^\star)$ equals to 0 if $i \in \mathscr{Q} \bigcup \mathscr{P}$ and $j \in \mathscr{S} \bigcup \mathscr{U}$ since $\cos(\frac{\pi}{2} + k\pi) = 0, k \in \mathbb{Z}$. Noticing some of the terms in $x^{\mathrm{T}} A(\phi^\star) x$ are non-negative, we have

$$
\begin{aligned}
x^{\mathrm{T}} A(\phi^\star) x \geq K &\sum_{i \in \mathscr{Q}, j \in \mathscr{P}} J_{ij}(x_i - x_j)^2 + K \sum_{i \in \mathscr{S}, j \in \mathscr{U}} J_{ij}(x_i - x_j)^2 \\
&+ 2K_s \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} (x_i^2 + x_j^2) - 2K_s \sum_{i \in \mathscr{S}, j \in \mathscr{U}} (x_i^2 + x_j^2).
\end{aligned}
$$

Let $x_i = x_j = 0$ for all $i \in \mathscr{S}$ and all $j \in \mathscr{U}$. We have

$$
\begin{aligned}
x^{\mathrm{T}} A(\phi^\star) x &\geq K \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} J_{ij}(x_i - x_j)^2 + 2K_s \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} (x_i^2 + x_j^2) \\
&\geq -K \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} (x_i - x_j)^2 + 2K_s \sum_{i \in \mathscr{Q}, j \in \mathscr{P}} (x_i^2 + x_j^2),
\end{aligned}
$$

which is positive if we choose $x_i = x_j \neq 0$ for all $i \in \mathscr{Q}$ and all $j \in \mathscr{P}$. Thus, there is a vector $x \neq 0$ such that $x^{\mathrm{T}} A(\phi^\star) x > 0$, which implies that $A(\phi^\star)$ is non-negative definite. Therefore, $A(\phi^\star)$ has at least one positive eigenvalue, and, hence, the Type II equilibrium point $\phi = \phi^\star$ is unstable. This completes the proof.

We next examine some key properties of Type III equilibrium points. We will first characterize their stability property with respect to the values of the system parameters K and K_s.

Theorem 4: *Consider a Type III equilibrium point $\phi = \phi^\star$ corresponding to specific values of the parameters $K = \bar{K}$ and $K_s = \bar{K}_s$. It is asymptotically stable if $\bar{K}_s > \frac{\bar{K}\lambda_N(\bar{D}(\phi^\star))}{2}$ and unstable if $\bar{K}_s < \frac{\bar{K}\lambda_N(\bar{D}(\phi^\star))}{2}$, where*

$$
\bar{D}(\phi^\star) = D(\phi^\star) + \delta \begin{bmatrix} \varphi_1 & 0 & \cdots & 0 \\ 0 & \varphi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_N \end{bmatrix},
$$

with

$$
\begin{aligned}
\varphi_i &= -\sin^2(\phi_i^\star), \ i = 1, 2, \ldots, N, \\
\delta &= \frac{4 \sum_{j=1, j \neq \kappa}^{N} J_{\kappa j} \sin(\phi_\kappa^\star - \phi_j^\star)}{\sin(2\phi_\kappa^\star)} = \frac{-4\bar{K}_s}{\bar{K}},
\end{aligned}
$$

where $\phi_\kappa^\star \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$.

Remark 6: *A Type III equilibrium point satisfies (3) with given values of the parameters $\bar{K}$ and $\bar{K}_s$, which implies that*

$$
\bar{K}_s \sin(2\phi_\kappa^\star) = -\bar{K} \sum_{j=1, j \neq \kappa}^{N} J_{\kappa j} \sin(\phi_\kappa^\star - \phi_j^\star)
$$

or

$$
\frac{\sum_{j=1, j \neq \kappa}^{N} J_{\kappa j} \sin(\phi_\kappa^\star - \phi_j^\star)}{\sin(2\phi_\kappa^\star)} = \frac{-\bar{K}_s}{\bar{K}}
$$

for all κ such that $\phi_\kappa \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. Thus, the value of δ remains unchanged for any κ such that $\phi_\kappa \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$.

Proof: *Consider the Jacobian matrix at $\phi = \phi^\star$,*

$$\begin{aligned}
A(\phi^\star) &= \frac{\partial f}{\partial \phi}\bigg|_{\phi=\phi^\star} \\
&= \bar{K}D(\phi^\star) \\
&\quad - 2\bar{K}_s \begin{bmatrix} \cos(2\phi_1^\star) & 0 & \cdots & 0 \\ 0 & \cos(2\phi_2^\star) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cos(2\phi_N^\star) \end{bmatrix} \\
&= \bar{K}D(\phi^\star) - 2\bar{K}_s \begin{bmatrix} \kappa_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_N \end{bmatrix} \\
&= \bar{K}\bar{D}(\phi^\star) - 2\bar{K}_s I_N,
\end{aligned}$$

where $\cos(2\phi_i^\star) = 1 - 2\sin^2(\phi_i^\star), i = 1, 2, \ldots, N$. The remaining proof is the same as in the proof of Theorem 2.

We finally examine the phases of the spins for Type III equilibrium points.

Theorem 5: *Suppose that there exists a Type III equilibrium $\phi = \phi^\star$. Let the set of all phases of all Type III equilibrium points be $\Theta = \{\theta_1 + 2k_1\pi, \theta_2 + 2k_2\pi, \ldots, \theta_m + 2k_m\pi : k_1, k_2, \ldots, k_m \in \mathbb{Z}\}$, where θ_i and θ_j are non-equivalent of each other for all $i \neq j$, that is, $\theta_i - \theta_j \neq 2k\pi$ for any $k \in \mathbb{Z}$. Then, $m \geq 3$.*

Proof: *We first observe that $m \neq 1$. Otherwise, $\Theta = \{\theta_1 + 2k_1\pi : k_1 \in \mathbb{Z}\}$. By the definition of the Type III equilibrium point, there exists at least one $\phi_i^\star \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. Therefore, we have $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. In this case, however, the right-hand side of (3) equals zero but its left-hand side is not. This contradicts the fact that $\phi = \phi^\star$ is an equilibrium point.*

We next show that $m \neq 2$. Otherwise, $\Theta = \{\theta_1 + 2k_1\pi, \theta_2 + 2k_2\pi : k_1, k_2 \in \mathbb{Z}\}$. Let us first consider the case that all $\phi_i^\star$ are equivalent. In this case $\phi_i^\star \in \{\theta_1 + 2k_1\pi : k_1 \in \mathbb{Z}\}$ for all i or $\phi_i^\star \in \{\theta_2 + 2k_2\pi : k_2 \in \mathbb{Z}\}$ for all i, which, in view of (3), implies that $\phi_i^\star \in \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ for all i. This contradicts the fact that $\phi^\star$ is a Type III equilibrium point.

We now consider the case that not all $\phi_i^\star$ are equivalent, that is, there exist $\phi_{i_1}^\star \in \{\theta_1 + 2k\pi : k \in \mathbb{Z}\}$ and $\phi_{i_2}^\star \in \{\theta_2 + 2k\pi : k \in \mathbb{Z}\}$. Since $\phi = \phi^\star$ is a Type III equilibrium point, $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ or/and $\theta_2 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. In what follows, we assume that $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. We will analyze this case in the following two separate scenarios.

The first scenario is that $\theta_1 + \theta_2 \neq 2k\pi, k \in \mathbb{Z}$. Note that the equilibrium point $\phi = \phi^\star$, with $\phi_i^\star \in \Theta$, satisfies (3). Since both sides of (3) are odd functions, there is another Type III equilibrium point $\phi^\dagger = -\phi^\star$, with $\phi_i^\dagger = -\phi_i^\star \in \{-\theta_1 + 2k_1\pi, -\theta_2 + 2k_2\pi : k_1, k_2 \in \mathbb{Z}\}$, that satisfies (3). The fact that $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$, along with the fact that $\theta_1 + \theta_2 \neq 2k\pi$, imply that there are at least three non-equivalent phases $\{\theta_1 + 2k_1\pi, \theta_2 + 2k_2\pi, -\theta_1 + 2k_1\pi\}$, which contradicts the assumption that $m = 2$.

The second scenario is that $\theta_1 + \theta_2 = 2k\pi, k \in \mathbb{Z}$. In this scenario, $\phi_i^\star \in \{\theta_1 + 2k_1\pi, -\theta_1 + 2k_2\pi : k_1, k_2 \in \mathbb{Z}\}$ for all i and Eq. (3) simplifies to

$$\bar{K}_s \sin(2\theta_1) = \bar{K} l_i \sin(2\theta_1), \; i = 1, 2, \ldots, N, \tag{16}$$

where l_i denotes the number of spins in the opposite phase of spin i that are connected to spin i. Equation (16) indicates that $l_1 = l_2 = \cdots = l_N$. As a result, (16) still holds if θ_1 is alternated by any $\tilde{\theta}_1 \in \mathbb{R}$. This implies that there are infinity many Type III equilibrium points of the form $\phi_i^\dagger = \phi_i^\star + \mathrm{sign}(\phi_i^\star)\delta, \delta \neq 2k\pi$, $i = 1, 2, \ldots, N$ and, consequently, there are infinitely many non-equivalent phases among Type III equilibrium points, contradicting the assumption that $m = 2$. This completes the proof.

Theorem 6: *Consider a Type III equilibrium point $\phi = \phi^\star$. If $\phi = \phi^\star$ is asymptotically stable, there exists another asymptotically stable Type III equilibrium point $\phi = \phi^\dagger$ such that there are at least three non-equivalent phases in $\phi^\star$ and $\phi^\dagger$.*

Proof: *For a Type III equilibrium point $\phi = \phi^\star$, there are at least two non-equivalent phases in $\phi^\star$. Otherwise, all $\phi_i^\star$ are equivalent, implying, in view of (3), that $\phi_i^\star \in \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ for all i. This contradicts the fact that $\phi = \phi^\star$ is a Type III equilibrium point.*

First, our objective is to find such a $\phi^\dagger$ when there are exactly two non-equivalent phases in $\phi^\star$, that is, $\phi_i^\star \in \{\theta_1 + 2k_1\pi, \theta_2 + 2k_2\pi : k_1, k_2 \in \mathbb{Z}\}$ for all i, and there exist $\phi_{i_1}^\star \in \{\theta_1 + 2k\pi : k \in \mathbb{Z}\}$ and $\phi_{i_2}^\star \in \{\theta_2 + 2k\pi : k \in \mathbb{Z}\}$, $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ or/and $\theta_2 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. Without loss of generality, we assume that $\theta_1 \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$. We will analyze this case in the following two separate scenarios.

The first scenario is that $\theta_1 + \theta_2 \neq 2k\pi, k \in \mathbb{Z}$. We can find $\phi^\dagger = -\phi^\star$ that is asymptotically stable since the Jacobian matrix at $\phi = \phi^\star$ and $\phi = -\phi^\star$ are the same and all eigenvalues of the Jacobian matrix at $\phi = \phi^\star$ have a negative real part. In addition, we see that there are at least three non-equivalent phases $\{\theta_1 + 2k_1\pi, \theta_2 + 2k_2\pi, -\theta_1 + 2k\pi : k_1, k_2 \in \mathbb{Z}\}$ in $\phi^\star$ and $\phi^\dagger$.

The second scenario is that $\theta_1 + \theta_2 = 2k\pi, k \in \mathbb{Z}$. Following an argument similar to one in the proof of Theorem 5, we can find $\phi^\dagger$ with $\phi_i^\dagger = \phi_i^\star + \mathrm{sign}(\phi_i^\star)\delta, \delta = \pi$. We will show that $\phi = \phi^\dagger$ is asymptotically stable. Note that the Jacobian matrix at $\phi = \phi^\star$ or $\phi = \phi^\dagger$ takes the following form:

$$\begin{aligned}
A(\phi) &= \bar{K}D - 2\bar{K}_s\cos(2\theta)I_N \\
&= \bar{K}\cos(2\theta)\mathscr{L}_1 + \bar{K}\mathscr{L}_2 - 2\bar{K}_s\cos(2\theta)I_N,
\end{aligned}$$

where $\theta \in \{\theta_1, \theta_1 + \pi\}$ and $\mathscr{L}_1$ and $\mathscr{L}_2$ are two constant matrices. It is noted that $A(\phi^\star) = A(\phi^\dagger)$. The fact that all eigenvalues of $A(\phi^\star)$ have a negative real part then implies that $\phi = \phi^\dagger$ is asymptotically stable. In addition, there are at least three non-equivalent phases $\{\theta_1 + 2k_1\pi, -\theta_1 + 2k_2\pi, \theta_1 + \pi + 2k_3\pi : k_1, k_2, k_3 \in \mathbb{Z}\}$ in $\phi^\star$ and $\phi^\dagger$.

Finally, when there are at least three non-equivalent phases in $\phi^\star$. Recall that $\phi = \phi^\dagger = -\phi^\star$ is also asymptotically stable. Clearly, there are at least three non-equivalent phases in $\phi^\star$ and $\phi^\dagger$. This completes the proof.

Remark 7: *In many previous works (see, for example, Refs. 26, 27, 32, 33, and 33), the threshold of SYNC that binarizes the spin* *phases is difficult to determine. Our Theorem 2 provides such a threshold. In particular, by Theorem 2, for given set of J_{ij}'s and a fixed value of K, at least one Type I equilibrium point is asymptotically stable if $K_s > \min_{\phi^\star}\left\{\frac{K\lambda_N(D(\phi^\star))}{2}\right\}$, where $\phi = \phi^\star$ is any Type I equilibrium point, and the phases of all spins in asymptotically stable Type I equilibrium points will be $0 + 2k\pi$ or $\pi + 2k\pi$. Theorem 2 also indicates that no Type I equilibrium points are asymptotically stable if $K_s < \min_{\phi^\star}\left\{\frac{K\lambda_N(D(\phi^\star))}{2}\right\}$. On the other hand, Theorem 6 indicates that Type III equilibrium points do not binarize even if there are asymptotically stable Type III equilibrium points. We would like to point out that there is a situation where asymptotically stable Type I and Type III equilibrium points coexist with fixed $\bar{K}$ and K_s that satisfy $K_s > \min_{\phi^\star}\left\{\frac{\bar{K}\lambda_N(D(\phi^\star))}{2}\right\}$. In this situation, the spin phases are still not binarized. However, we can perturb K_s with a positive random noise $\xi(t)$. With this perturbation to K_s, Type III equilibrium points will disappear as they are structurally unstable in terms of both J_{ij}'s and the parameter K_s, as discussed in Remark 5, and, hence, facilitate the spin phases binarized. In summary, the threshold for binarization is given by*

$$K_s = \min_{\phi^\star}\left\{\frac{K\lambda_N(D(\phi^\star))}{2}\right\},$$

where $\phi = \phi^\star$ is any Type I equilibrium point.

C. Domains of attractions

In this subsection, we will provide an estimate of the domains of attraction for asymptotically stable equilibrium points. We first recall the definition of the domain of attraction of an equilibrium point of a nonlinear dynamic system.

Definition 4 (Ref. 34): *Let $x = x^\star$ be an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = f(x),\ x \in \mathscr{D}, \tag{17}$$

where $f : \mathscr{D} \to \mathbb{R}^N$ is locally Lipschitz and $\mathscr{D}$ is a domain where the system operates. Let $\phi(t; x)$ be the solution of (17) that starts at initial state x at time $t = 0$. The domain of attraction of $x = x^\star$, denoted by $\mathscr{D}_A$, is defined by

$$\mathscr{D}_A = \left\{x \in \mathscr{D} : \lim_{t\to\infty} \phi(t; x) = x^\star\right\}.$$

Theorem 7: *Consider an asymptotically stable Type I equilibrium point $\phi = \phi^\star$, where $\phi_i^\star \in \{\frac{k\pi}{2} : \frac{k}{2} \in \mathbb{Z}\}$ for all $i = 1, 2, \ldots, N$. Let*

$$\mathscr{M} = \Big\{\phi : J_{ij}(\phi_i - \phi_j) = \pi + 2k\pi, k \in \mathbb{Z},\ \text{for all } i \text{ and } j \text{ such that } J_{ij} \neq 0\Big\},$$

and

$$\overline{\mathscr{M}} = \Big\{\phi : J_{ij}(\phi_i - \phi_j) = 2k\pi, k \in \mathbb{Z},\ \text{for some } i \text{ and } j \text{ such that } J_{ij} \neq 0\Big\}.$$

Also, let

$$\mathscr{I}_1 = \left\{(i,j) : J_{ij}(\phi_i^\star - \phi_j^\star) = \pi + 2k\pi, i < j, J_{ij} \neq 0\right\},$$
$$\mathscr{K}_1 = \left\{i, j : (i,j) \in \mathscr{I}_1\right\}$$

and

$$\mathscr{I}_2 = \left\{(i,j) : J_{ij}(\phi_i^\star - \phi_j^\star) = 2k\pi, i < j, J_{ij} \neq 0\right\},$$
$$\mathscr{K}_2 = \left\{i, j : (i,j) \in \mathscr{I}_2\right\} = \mathscr{K}_2^0 \cup \mathscr{K}_2^\pi,$$
$$\mathscr{K}_2^0 = \left\{i, j : (i,j) \in \mathscr{I}_2, \phi_i, \phi_j \in \{2k\pi\}\right\} = \{l_1^0, l_2^0, \ldots, l_p^0\},$$
$$\mathscr{K}_2^\pi = \left\{i, j : (i,j) \in \mathscr{I}_2, \phi_i, \phi_j \in \{\pi + 2k\pi\}\right\} = \{l_1^\pi, l_2^\pi, \ldots, l_q^\pi\}.$$

If $\phi^\star \in \mathscr{M}$, then an estimate of its domain of attraction is given by

$$\widehat{\mathscr{D}}_A = \left\{\phi : \|\phi - \phi^\star\|_2 < \frac{\pi}{2}\right\}.$$

If $\phi^\star \in \overline{\mathscr{M}}$, then an estimate of its domain of attraction is given by

$$\widehat{\mathscr{D}}_A = \left\{\phi : \|\phi - \phi^\star\|_2 < \frac{\beta\pi}{2}\right\},$$

where $0 < \beta < 1$ is such that matrix $Q^* \in \mathbb{R}^{n\times n}$ for $* = 0, n = p$ and $* = \pi, n = q$, shown in (18) below, is negative definite.

$$Q^* = \begin{bmatrix} -4\frac{\sin(\beta\pi)}{\beta\pi}K_s - 2K\sum\limits_{j\in\mathscr{K}_2^*, j\neq l_1^*} J_{l_1^* j} & 2KJ_{l_1^* l_2^*} & \cdots & 2KJ_{l_1^* l_n^*} \\ 2KJ_{l_2^* l_1^*} & -4\frac{\sin(\beta\pi)}{\beta\pi}K_s - 2K\sum\limits_{j\in\mathscr{K}_2^*, j\neq l_2^*} J_{l_2^* j} & \cdots & 2KJ_{l_2^* l_n^*} \\ \vdots & \vdots & \ddots & \vdots \\ 2KJ_{l_n^* l_1^*} & 2KJ_{l_n^* l_2^*} & \cdots & -4\frac{\sin(\beta\pi)}{\beta\pi}K_s - 2K\sum\limits_{j\in\mathscr{K}_2^*, j\neq l_n^*} J_{l_n^* j} \end{bmatrix}. \tag{18}$$

Proof: *Let $z = \mathrm{col}\{z_1, z_2, \ldots, z_N\} = \phi - \phi^\star$. Consider the Lyapunov function candidate $V(z) = z^{\mathrm{T}} z$. The derivative of V along the trajectory of (2) can be evaluated as*

$$
\begin{aligned}
\dot{V} &= 2\dot{z}^{\mathrm{T}} z \\
&= 2\begin{bmatrix} -K\sum_{j=1,j\neq 1}^{N} J_{1j}\sin(\phi_1-\phi_j) - K_{\mathrm{s}}\sin(2\phi_1) \\ -K\sum_{j=1,j\neq 2}^{N} J_{2j}\sin(\phi_2-\phi_j) - K_{\mathrm{s}}\sin(2\phi_2) \\ \vdots \\ -K\sum_{j=1,j\neq N}^{N} J_{Nj}\sin(\phi_N-\phi_j) - K_{\mathrm{s}}\sin(2\phi_N) \end{bmatrix}^{\mathrm{T}} z \\
&= -2K\sum_{i=1}^{N} z_i \sum_{j=1,j\neq i}^{N} J_{ij}\sin\left(z_i - z_j + \phi_i^\star - \phi_j^\star\right) \\
&\quad - 2K_{\mathrm{s}}\sum_{i=1}^{N} z_i \sin(2z_i),
\end{aligned}
$$

where we used the fact that $2\phi_i^\star = 2k\pi, k \in \mathbb{Z}$, for all i.

Noting that $J_{ij} = J_{ji}$, we have

$$
J_{ij}\sin(z_i - z_j + \phi_i^\star - \phi_j^\star) = -J_{ji}\sin(z_j - z_i + \phi_j^\star - \phi_i^\star),
$$

and, hence,

$$
\begin{aligned}
\dot{V} &= -2K\sum_{i,j=1,i<j}^{N} J_{ij}(z_i - z_j)\sin(z_i - z_j + \phi_i^\star - \phi_j^\star) \\
&\quad - 2K_{\mathrm{s}}\sum_{i=1}^{N} z_i\sin(2z_i) \\
&= W_1 + W_2,
\end{aligned}
$$

where, in view of the four index sets $\mathscr{I}_1, \mathscr{K}_1, \mathscr{I}_2$, and $\mathscr{K}_2$,

$$
\begin{aligned}
W_1 &= -2K\sum_{(i,j)\in\mathscr{I}_1} J_{ij}(z_i - z_j)\sin(z_i - z_j + \phi_i^\star - \phi_j^\star) \\
&\quad - 2K_{\mathrm{s}}\sum_{i\in\mathscr{K}_1, i\notin\mathscr{K}_2} z_i\sin(2z_i), \\
W_2 &= -2K\sum_{(i,j)\in\mathscr{I}_2} J_{ij}(z_i - z_j)\sin(z_i - z_j + \phi_i^\star - \phi_j^\star) \\
&\quad - 2K_{\mathrm{s}}\sum_{i\in\mathscr{K}_2} z_i\sin(2z_i).
\end{aligned}
$$

In view of the definition of $\mathscr{I}_1$, W_1 can be simplified as

$$
\begin{aligned}
W_1 &= 2K\sum_{(i,j)\in\mathscr{I}_1}^{N} J_{ij}(z_i - z_j)\sin(z_i - z_j) \\
&\quad - 2K_{\mathrm{s}}\sum_{i\in\mathscr{K}_1, i\notin\mathscr{K}_2}^{N} z_i\sin(2z_i).
\end{aligned}
$$

It is clear that $W_1 < 0$ if $|2z_i| < \pi$ and $z_i \neq 0$ for all $i \in \mathscr{K}_1, i \notin \mathscr{K}_2$ and $|z_i - z_j| < \pi$ for all $(i,j) \in \mathscr{I}_1$.

Similarly, in view of the definition of $\mathscr{I}_2$ and for $|z_i - z_j| < \pi$ for $(i,j) \in \mathscr{I}_2$, W_2 can be simplified as

$$
\begin{aligned}
W_2 &= -2K\sum_{(i,j)\in\mathscr{I}_2}^{N} J_{ij}(z_i - z_j)\sin(z_i - z_j) \\
&\quad - 2K_{\mathrm{s}}\sum_{i\in\mathscr{K}_2}^{N} z_i\sin(2z_i) \\
&\leq -2K\sum_{(i,j)\in\mathscr{I}_2}^{N} J_{ij}(z_i - z_j)^2 - 2K_{\mathrm{s}}\sum_{i\in\mathscr{K}_2}^{N} z_i\sin(2z_i) \\
&\leq -2K\sum_{(i,j)\in\mathscr{I}_2}^{N} J_{ij}(z_i - z_j)^2 - 4\frac{\sin(\beta\pi)}{\beta\pi}K_{\mathrm{s}}\sum_{i\in\mathscr{K}_2}^{N} z_i^2 \\
&= z_{\mathscr{K}_2^0}^{\mathrm{T}} Q^0 z_{\mathscr{K}_2^0} + z_{\mathscr{K}_2^\pi}^{\mathrm{T}} Q^\pi z_{\mathscr{K}_2^\pi},
\end{aligned}
$$

where $z_{\mathscr{K}_2^0} = \mathrm{col}\{z_{l_1^0}, z_{l_2^0}, \ldots, z_{l_p^0}\}$, $z_{\mathscr{K}_2^\pi} = \mathrm{col}\{z_{l_1^\pi}, z_{l_2^\pi}, \ldots, z_{l_q^\pi}\}$, $Q^* \in \mathbb{R}^{n\times n}$ with $* = 0, n = p$ or $* = \pi, n = q$ is given in (18), and we have used Lemma 2 with $x = 2z_i$ and $m = \beta\pi, \beta \in (0,1)$.

Clearly, $W_2 < 0$ if $|2z_i| < \beta\pi$ for all $i \in \mathscr{K}_2$, $|z_i - z_j| < \pi$ for all $(i,j) \in \mathscr{I}_2$, and $\beta \in (0,1)$ is such that $Q^*(Q^0, Q^\pi)$ is negative definite.

Now, if $\phi^\star \in \mathscr{M}$, then $\mathscr{I}_2 = \emptyset$, $\mathscr{K}_2 = \emptyset$, and, hence, $\dot{V} = W_1$. By the analysis above, $\dot{V} < 0$ if $0 < |2z_i| < \pi$ and $|z_i - z_j| < \pi$ for all $i \in \mathscr{K}_1$ and all $(i,j) \in \mathscr{I}_1$, which is implied by $\phi \in \mathscr{D}_{\mathrm{A}} = \left\{\phi : \|\phi - \phi^\star\|_2 < \frac{\pi}{2}\right\}$. This establishes that $\widehat{\mathscr{D}}_{\mathrm{A}} = \{\phi : \|\phi - \phi^\star\|_2 < \frac{\pi}{2}\}$ is an estimate of the domain of attraction of the equilibrium $\phi = \phi^\star$.

On the other hand, if $\phi^\star \in \overline{\mathscr{M}}$, $\dot{V} = W_1 + W_2$ or $\dot{V} = W_2$. In either case, $\dot{V} < 0$ if $|2z_i| < \beta\pi$ for all i, and $|z_i - z_j| < \pi$ for all $(i,j) \in \mathscr{I}_2$ and $\beta \in (0,1)$ is such that $Q^*(Q^0, Q^\pi)$ is negative definite, which is implied by $\phi \in \mathscr{D}_{\mathrm{A}} = \left\{\phi : \|\phi - \phi^\star\|_2 < \frac{\beta\pi}{2}\right\}$, with $\beta \in (0,1)$ being such that $Q^*(Q^0, Q^\pi)$ is negative definite. This establishes that $\widehat{\mathscr{D}}_{\mathrm{A}} = \left\{\phi : \|\phi - \phi^\star\|_2 < \frac{\beta\pi}{2}\right\}$ is an estimate of the domain of attraction of the equilibrium $\phi = \phi^\star$ and completes the proof.

Theorem 8: *The domain of attraction of an asymptotically stable Type III equilibrium point $\phi = \phi^\star$ is given by*

$$
\widehat{\mathscr{D}}_{\mathrm{A}} = \left\{\phi : \|\phi - \phi^\star\|_2 \leq \frac{-\lambda_N(A(\phi^\star))}{2N\big(K(N-1) + 2K_{\mathrm{s}}\big)}\right\}.
$$

Proof: *Let $z = \phi - \phi^\star$. Then, we have*

$$
\dot{z} = A(\phi^\star)z + f(z + \phi^\star) - A(\phi^\star)z, \tag{19}
$$

where $f(\phi) = \mathrm{col}\{f_1(\phi), f_2(\phi), \dots, f_N(\phi)\}$, *with*

$$f_i(\phi) = -K \sum_{j=1, j\neq i}^{N} J_{ij} \sin(\phi_i - \phi_j) - K_s \sin(2\phi_i),$$

and

$$A(\phi^\star) = \left.\frac{\partial f}{\partial \phi}\right|_{\phi=\phi^\star}.$$

We recall that $A(\phi^\star)$*, an expression of which is given in the proof of Theorem 4, is symmetric with all its eigenvalues negative and, hence, is negative definite.*

By the mean value theorem, and noting that $f(\phi^\star) = 0$*, we have*

$$f_i(z+\phi^\star) = f_i(\phi^\star) + \frac{\partial f_i(z+\phi^\star)}{\partial z}(x_i) z = \frac{\partial f_i(z+\phi^\star)}{\partial z}(x_i) z,$$

where $x_i = \mathrm{col}\{x_{i1}, x_{i2}, \dots, x_{iN}\}$ *is a point on the line segment connecting* z *to the origin. Consequently,* $g(z) = f(z+\phi^\star) - A(\phi^\star) z$ *can be written as*

$$g(z) = \tilde{A}(x) z,$$

where $x = \mathrm{col}\{x_1, x_2, \dots, x_N\}$ *and*

$$\tilde{A}(x) = \begin{bmatrix} \frac{\partial f_1(z+\phi^\star)}{\partial z}(x_1) - \frac{\partial f_1(z+\phi^\star)}{\partial z}(0) \\ \frac{\partial f_2(z+\phi^\star)}{\partial z}(x_2) - \frac{\partial f_2(z+\phi^\star)}{\partial z}(0) \\ \vdots \\ \frac{\partial f_N(z+\phi^\star)}{\partial z}(x_N) - \frac{\partial f_N(z+\phi^\star)}{\partial z}(0) \end{bmatrix} = \tilde{A}_1(x) + \tilde{A}_2(x),$$

with $\tilde{A}_1(x)$ *being a diagonal matrix whose* i^{th} *diagonal entry is*

$$\begin{aligned}[\tilde{A}_1(x)]_{ii} = &-K \sum_{j=1, j\neq i}^{N} J_{ij} \cos(x_{ii} - x_{ij} + \phi_i^\star - \phi_j^\star) \\ &+ K \sum_{j=1, j\neq i}^{N} J_{ij} \cos(\phi_i^\star - \phi_j^\star) \\ &- 2K_s \cos(2x_{ii} + 2\phi_i^\star) + 2K_s \cos(2\phi_i^\star),\end{aligned}$$

and $\tilde{A}_2(x)$ *being a matrix whose diagonal entries are all zeros and* (i, j) *entry is given by*

$$[\tilde{A}_2(x)]_{ij} = K J_{ij}\big(\cos(x_{ii} - x_{ij} + \phi_i^\star - \phi_j^\star) - \cos(\phi_i^\star - \phi_j^\star)\big).$$

To obtain an upper bound of $\|\tilde{A}(x)\|_2$*, we first derive*

$$\begin{aligned}\|\tilde{A}(x)\|_\infty = \max_i \Bigg\{ \Bigg| &-K \sum_{j=1, j\neq i}^{N} J_{ij} \cos(x_{ii} - x_{ij} + \phi_i^\star - \phi_j^\star) \\ &+ K \sum_{j=1, j\neq i}^{N} J_{ij} \cos(\phi_i^\star - \phi_j^\star)) \\ &- 2K_s \cos(2x_{ii} + 2\phi_i^\star) + 2K_s \cos(2\phi_i^\star) \Bigg| \\ &+ K \sum_{j=1, j\neq i}^{N} \left| J_{ij} \cos(x_{ii} - x_{ij} + \phi_i^\star - \phi_j^\star) - J_{ij}\cos(\phi_i^\star - \phi_j^\star) \right| \Bigg\} \\ \le \max_i \Bigg\{ 2K &\sum_{j=1, j\neq i}^{N} \Big| J_{ij} \cos(x_{ii} - x_{ij} + \phi_i^\star - \phi_j^\star) \\ &- J_{ij} \cos(\phi_i^\star - \phi_j^\star) \Big| \\ &+ 2K_s \left| -\cos(2x_{ii} + 2\phi_i^\star) + \cos(2\phi_i^\star) \right| \Bigg\} \\ \le \max_i \Bigg\{ 2K &\sum_{j=1, j\neq i}^{N} \left| J_{ij}(x_{ii} - x_{ij}) \right| + 2K_s \left| 2x_{ii} \right| \Bigg\} \\ \le \max_i 2 &\Big\{ \big(K(N-1) + 2K_s\big) \|x_i\|_1 \Big\} \\ \le 2\big(&K(N-1) + 2K_s\big)\|z\|_1, \end{aligned} \tag{20}$$

where we have used Lemma 3 twice with $x = \phi_i^\star - \phi_j^\star$ *and* $\delta = x_{ii} - x_{ij}$*, and* $x = 2\phi_i^\star$ *and* $\delta = 2x_{ii}$*, respectively. In view of (20) and the fact that* $\|\tilde{A}(x)\|_2 \le \sqrt{N}\|\tilde{A}(x)\|_\infty$*, an upper bound of* $\|g(z)\|_2$ *can be obtained as*

$$\begin{aligned}\|g(z)\|_2 \le \|\tilde{A}\|_2 \|z\|_2 &\le 2\sqrt{N}\big(K(N-1) + 2K_s\big)\|z\|_1 \|z\|_2 \\ &\le 2N\big(K(N-1) + 2K_s\big)\|z\|_2^2,\end{aligned}$$

where we have used the fact that $\|z\|_1 \le \sqrt{N}\|z\|_2$.

Consider a Lyapunov function candidate $V(z) = \frac{1}{2} z^{\mathrm{T}} z$*. The derivative of* V *along the trajectory of (19) can be evaluated as*

$$\begin{aligned}\dot{V} &= z^{\mathrm{T}} A(\phi^\star) z + z^{\mathrm{T}} g(z) \\ &\le \lambda_N\big(A(\phi^\star)\big)\|z\|_2^2 + 2N\big(K(N-1) + 2K_s\big)\|z\|_2^3,\end{aligned}$$

where $\lambda_N\big(A(\phi^\star)\big) < 0$ *is the maximum eigenvalue of* $A(\phi^\star)$*. It is clear that* $\dot{V} < 0$ *for any* $z \neq 0$ *and*

$$\|z\|_2 \le \frac{-\lambda_N\big(A(\phi^\star)\big)}{2N\big(K(N-1) + 2K_s\big)}.$$

This shows that

$$\widehat{\mathscr{D}}_A = \left\{ \phi : \|\phi - \phi^\star\|_2 \le \frac{-\lambda_N\big(A(\phi^\star)\big)}{2N\big(K(N-1) + 2K_s\big)} \right\}$$

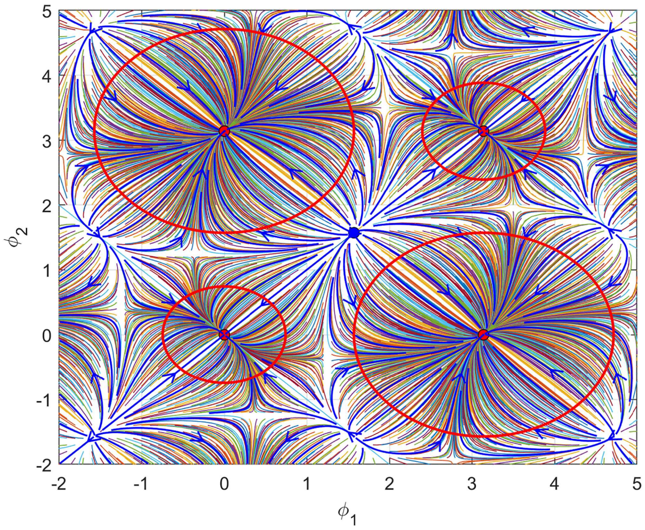

FIG. 6. A phase portrait when $K = 1$ and $K_s = 1.5$. The red points indicate the asymptotically stable equilibrium points $\phi^\star = \text{col}\{0, 0\}$, $\phi^\star = \text{col}\{0, \pi\}$, $\phi^\star = \text{col}\{\pi, 0\}$, and $\phi^\star = \text{col}\{\pi, \pi\}$. The blue point indicates the unstable equilibrium point $\phi^\star = \text{col}\left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}$. The red circles indicate the estimates of the domains of attraction.

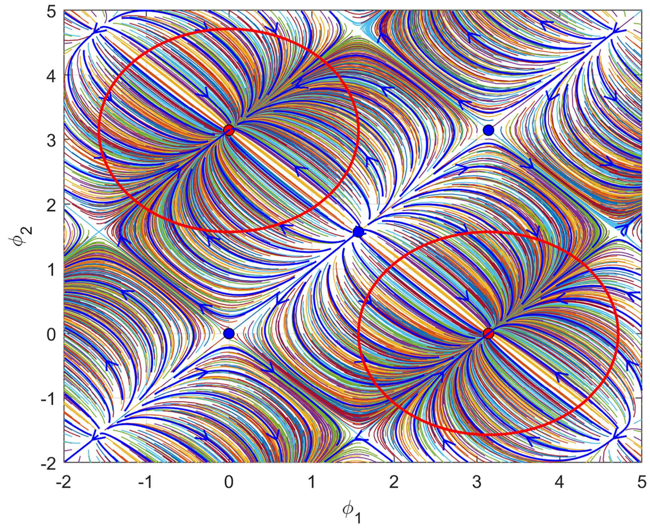

FIG. 7. A phase portrait when $K = 1$ and $K_s = 0.5$. The red points indicate the asymptotically stable equilibrium points $\phi^\star = \text{col}\{0, \pi\}$ and $\phi^\star = \text{col}\{\pi, 0\}$. The blue points indicate unstable equilibrium points $\phi^\star = \text{col}\{0, 0\}$, $\phi^\star = \text{col}\{\pi, \pi\}$, and $\phi^\star = \text{col}\left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}$. The red circles indicate the estimates of the domains of attraction.

is an estimate of the domain of attraction of the equilibrium point and the proof is completed.

IV. NUMERICAL EXPERIMENTS

In this section, we carry out numerical experiments to illustrate the analytical results obtained in this paper.

A. Stability analysis

Experiment 1: *Consider an Ising machine of two coupled spins. In this case, matrix $D(\phi^\star)$ is*

$$D(\phi^\star) = \begin{bmatrix} \cos(\phi_1^\star - \phi_2^\star) & -\cos(\phi_1^\star - \phi_2^\star) \\ -\cos(\phi_2^\star - \phi_1^\star) & \cos(\phi_2^\star - \phi_1^\star) \end{bmatrix}.$$

There are infinitely many equilibrium points. We will consider the stability property of five Type I equilibrium points $\phi^\star = \text{col}\{0, 0\}$, $\phi^\star = \text{col}\{0, \pi\}$, $\phi^\star = \text{col}\{\pi, 0\}$, $\phi^\star = \text{col}\{\pi, \pi\}$, $\phi^\star = \text{col}\{\frac{\pi}{2}, \frac{\pi}{2}\}$. By Theorem 2, $\phi^\star = \text{col}\{\frac{\pi}{2}, \frac{\pi}{2}\}$ is unstable for any values of the parameters K and K_s.

For $\phi^\star = \text{col}\{0, \pi\}$ and $\phi^\star = \text{col}\{\pi, 0\}$,

$$D = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

with $\lambda(D) = \{-2, 0\}$. By Theorem 2, $\phi^\star = \text{col}\{0, \pi\}$ and $\phi^\star = \text{col}\{\pi, 0\}$ are asymptotically stable if $K_s > 0$. Two phase portraits around these two points are shown in Fig. 6 ($K = 1$, $K_s = 1.5$) and Fig. 7 ($K = 1, K_s = 0.5$). Also shown in these figures are the estimates of the domains of attractions, obtained according to Theorem 7.

For $\phi^\star = \text{col}\{0, 0\}$ and $\phi^\star = \text{col}\{\pi, \pi\}$,

$$D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

with $\lambda(D) = \{0, 2\}$. By Theorem 2, $\phi^\star = \text{col}\{0, 0\}$ and $\phi^\star = \text{col}\{\pi, \pi\}$ are asymptotically stable if $K_s > K$, and unstable if $K_s < K$. Two phase portraits around these two equilibrium points are

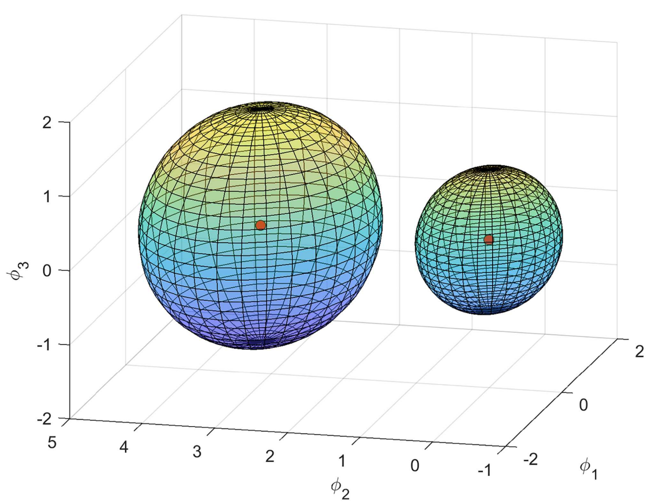

FIG. 8. The estimates of the domains of attraction of $\phi^\star = \text{col}\{0, \pi, 0\}$ (left) and $\phi^\star = \text{col}\{0, 0, 0\}$ (right).

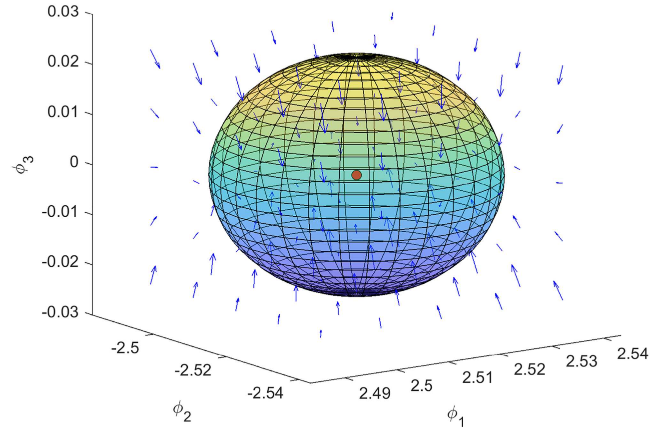

FIG. 9. The estimate of the domain of attraction of $\phi^\star = \mathrm{col}\{0.8\pi, -0.8\pi, 0\}$.

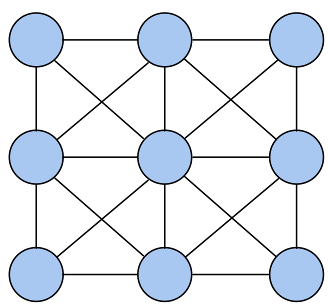

FIG. 10. The 3×3 King graph.

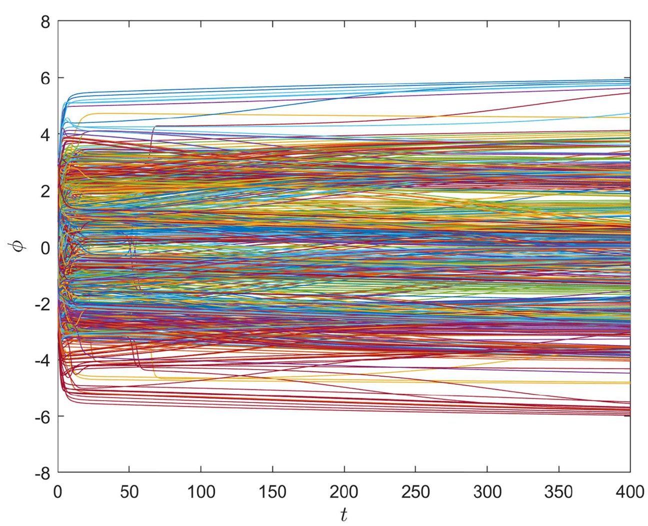

FIG. 11. Evolution of the phases of the nine spins in the King graph originating from multiple sets of initial phases: $K_s = 0.0100$ that is substantially below the threshold of $K_s = 0.2764$. The binarization does not occur and the spins converge to more than two non-equivalent phases.

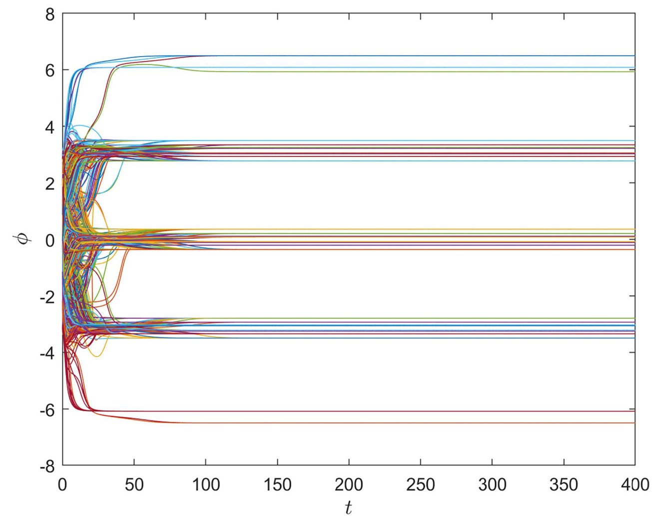

FIG. 12. Evolution of the phases of the nine spins in the King graph originating from multiple sets of initial phases: $K_s = 0.2400$ that is slightly below the threshold of $K_s = 0.2764$. The binarization does not occur and the spins converge to more than two non-equivalent phases.

shown in Fig. 6 ($K = 1, K_s = 1.5$) and Fig. 7 ($K = 1, K_s = 0.5$). The matrix Q^*, Q^0 for $\phi^\star = \mathrm{col}\{0, 0\}$ and Q^π for $\phi^\star = \mathrm{col}\{\pi, \pi\}$, at these two points is the same and is given by

$$Q^* = \begin{bmatrix} -4\dfrac{\sin(\beta\pi)}{\beta\pi}K_s + 2K & -2K \\ -2K & -4\dfrac{\sin(\beta\pi)}{\beta\pi}K_s + 2K \end{bmatrix}.$$

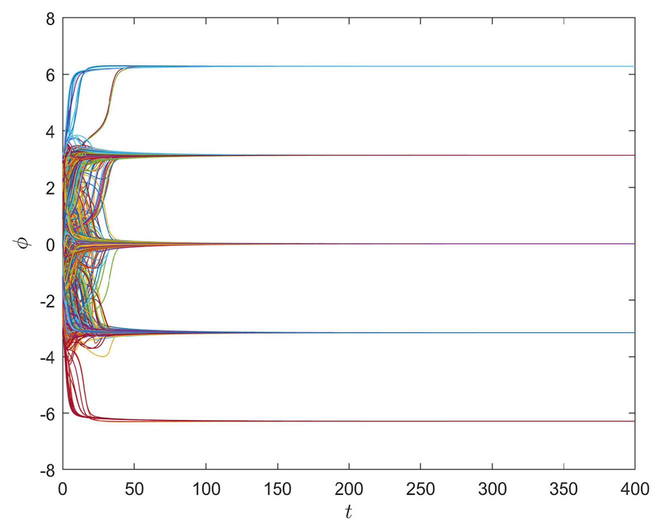

FIG. 13. Evolution of the phases of the nine spins in the King graph originating from multiple sets of initial phases: $K_s = 0.2770$ that is slightly above the threshold of $K_s = 0.2764$. The Ising machine is binarized, with their phases converging to two non-equivalent phases 0 and π.

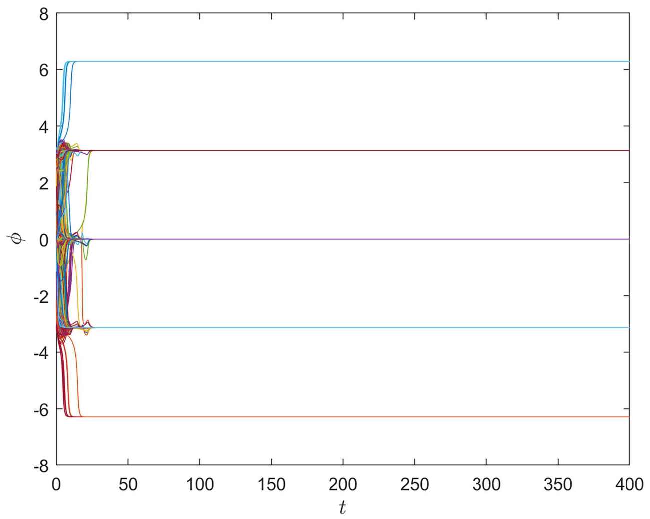

FIG. 14. Evolution of the phases of the nine spins in the King graph originating from multiple sets of initial phases: $K_s = 0.7900$ that is substantially above the threshold of $K_s = 0.2764$. The Ising machine is binarized, with their phases converging to two non-equivalent phases 0 and π.

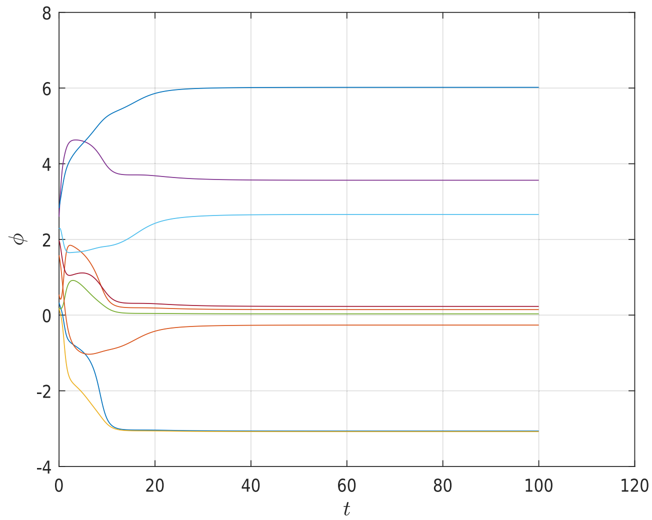

FIG. 16. Evolution of the phases of the nine spins in the spanning tree shown in Fig. 15: $K_s = 0.43$ without noise. The Ising machine is not binarized.

For $K = 1$ and $K_s = 1.5$, $\beta = 0.4743$ is such that $Q^* < 0$. An estimate of the domain of attraction can be obtained for these two equilibrium points according to Theorem 7 and shown in Fig. 6.

Experiment 2: *Consider an Ising machine of three coupled spins, with the coupling topology shown in Fig. 3. In this experiment, we estimate the domains of attraction of two Type I equilibrium points*

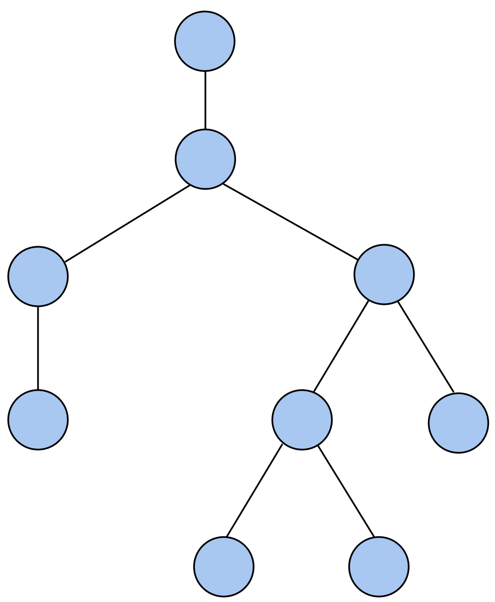

FIG. 15. A spanning tree graph.

$\phi^\star = \mathrm{col}\{0,0,0\}$ and $\phi^\star = \mathrm{col}\{0,\pi,0\}$. According to Theorem 2, we choose $K = 1$ and $K_s = 3$ such that these two points are asymptotically stable. By Theorem 7, the radius of the estimate of the domain of attraction for $\phi^\star = \mathrm{col}\{0,\pi,0\}$ is $\frac{\pi}{2}$.

For $\phi^\star = \mathrm{col}\{0,0,0\}$,

$$Q^0 = \begin{bmatrix} -12\dfrac{\sin(\beta\pi)}{\beta\pi}+2 & -2 & 0 \\ -2 & -12\dfrac{\sin(\beta\pi)}{\beta\pi}+4 & -2 \\ 0 & -2 & -12\dfrac{\sin(\beta\pi)}{\beta\pi}+2 \end{bmatrix}.$$

We calculate that $\beta = 0.6033$ is such that $Q^0 < 0$. Thus, the radius of the estimate of the domain of attraction of $\phi^\star = \mathrm{col}\{0,0,0\}$ is obtained according to Theorem 7 as $\frac{0.6033\pi}{2}$. The estimate of domains of attraction of $\phi^\star = \mathrm{col}\{0,\pi,0\}$ and $\phi^\star = \mathrm{col}\{0,0,0\}$ is shown in Fig. 8.

Experiment 3: *Consider an Ising machine of three coupled spins where spins are all-to-all connected. In this experiment, we estimate the domain of attraction of a Type III equilibrium point. For $K = 1$ and $K_s = \big(\sin(1.6\pi) + \sin(0.8\pi)\big)/\sin(1.6\pi) \approx 0.381\,966$, $\phi^\star = \mathrm{col}\{0.8\pi, -0.8\pi, 0\}$ is a Type III equilibrium point. By Theorem 4, $\phi^\star$ is asymptotically stable since $K_s > \frac{\lambda_N(\tilde{D}(\phi^\star))}{2} = 0.1878$. By Theorem 8, the radius of the estimate of the domain of attraction of $\phi^\star$ is $\frac{0.3884}{6\left(2+2\times K_s\right)} = 0.023\,421$. This estimate of the domain of attraction of $\phi^\star$ is shown in Fig. 9.*

B. Binarization

We will examine the threshold of SYNC that binarizes the Ising machine. We consider a 3×3 King graph, as shown in Fig. 10. We denote the maximum eigenvalue of the matrix $D(\phi^\star)$ at Type I equilibrium point $\phi^\star$ as $\lambda_9(D(\phi^\star))$. It can be verified that

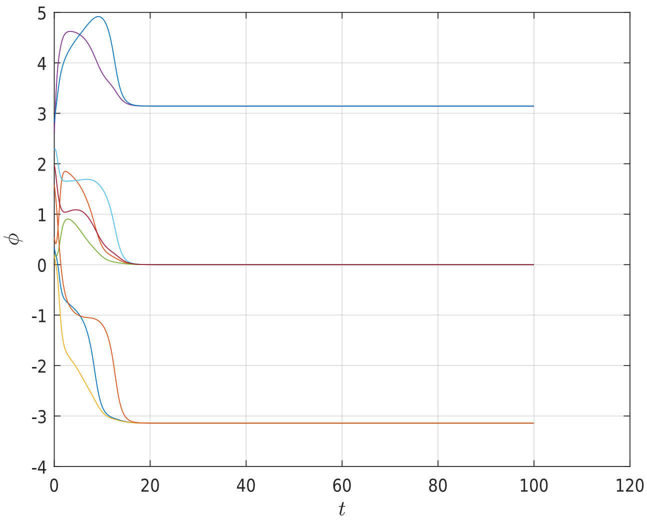

FIG. 17. Evolution of the phases of the nine spins in the spanning tree as shown in Fig. 15: $K_s = 0.43 + 0.02\xi(t)$, where $\xi(t) \in [0, 1]$ is a random noise. The Ising machine is binarized.

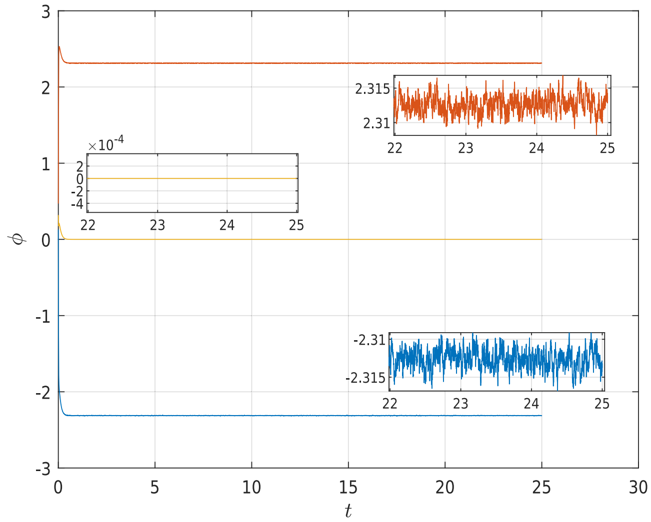

FIG. 19. Evolution of the phases of the three spins: $K_s = 25 + 2\xi(t)$, where $\xi(t) \in [0, 1]$ is a random noise. The spins' states do not converge to any equilibrium points but oscillate.

$\min_{\phi^\star}\{\lambda_9(D(\phi^\star))\} = 0.552\,799$. Thus, by Remark 7, the threshold of SYNC is $K_s = \frac{0.552799K}{2}$. Let $K = 1$, resulting in a threshold of SYNC $K_s = 0.2764$. The evolutions of the spins for $K_s = 0.0100$, $K_s = 0.2400$, $K_s = 0.2770$, and $K_s = 0.7900$ under the same initial phase values are shown in Figs. 11–14, respectively. We observe that the phase configuration in Fig. 12 is very close to the binarized phase configuration as $K_s = 0.2400$ is only slightly below the threshold of $K_s = 0.2764$. The Hamiltonian values of the phase configurations corresponding to $K_s = 0.2400$ (Fig. 12) and $K_s = 0.277$ (Fig.13) are, respectively, -8.085 and -8.

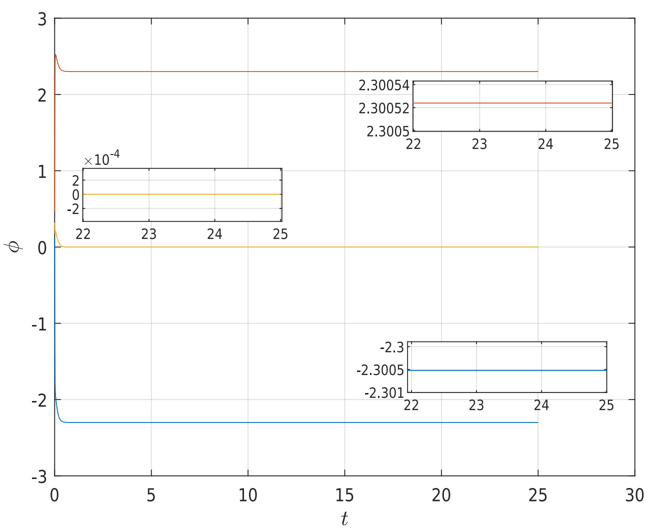

FIG. 18. Evolution of the phases of the three spins: $K_s = 25$ without noise. The spins' states converge to a Type III equilibrium point.

C. Effects of noise in the value of K_s

In this subsection, we provide two experiments to examine the effects of noise in the value of K_s. Both experiments show that noise in the value of K_s indeed eliminates Type III equilibrium points, leaving only binarized phase configurations.

Experiment 4: *Consider a spanning tree, as shown in Fig. 15. It can be verified that* $\min_{\phi^\star}\{\lambda_9(D(\phi^\star))\} = 0$. *We fix* $K = 1$ *and choose* $K_s = 0.43 > 0$ *such that at least one Type I equilibrium point is asymptotically stable. The initial phase configuration is* $\mathrm{col}\{0.33, 0.54, 0.18, 2.6, 0.19, 2.28, 1.97, 2.83, 1.58\}$. *The spins' states are not binarized in this case, shown in the Fig. 16. However, the spins' states are binarized after adding the random noise* $\xi(t)$ *on* K_s*, shown in Fig. 17.*

Experiment 5: *Consider three all-to-all connected spins. Choose* $K = 100$ *and* $K_s = 25$. *The initial phase configuration is* $\mathrm{col}\{0.13, 0.47, 0.32\}$. *It can be verified that* $\min_{\phi^\star}\{\lambda_3(D(\phi^\star))\} = 1$. *By Theorem 2, all Type I equilibrium points are unstable. The spins' states converge to a Type III equilibrium point without noise on* K_s*, while they oscillate after noise is introduced on* K_s*. The trajectories are shown in Figs. 18 and 19.*

V. CONCLUSIONS

This paper has studied the oscillator Ising machines from a control theoretic perspective. All equilibrium points of the spin-based dynamic system are classified into three types. We have conducted stability analysis on each type of equilibrium point. The analysis shows that the ratio of the coupling strength among the oscillators (K) to the coupling strength from SYNC (K_s) determines the stability properties of these equilibrium points. As a side result, the threshold of $\frac{K_s}{K}$ was obtained that determines whether binarization emerges in the machine. Additionally, an estimate of the domain of attraction for each asymptotically stable equilibrium point was obtained. It has been known that an Ising machine may potentially get trapped in a sub-optimal solution. The stability analysis carried out in this paper motivates the possibility that, for a COP with a certain special structure, the Ising machine with appropriate values of the parameters K and K_s could accurately find the global optimal solution. What such special structures are and how to tune the parameters are topics of our future study.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yi Cheng: Conceptualization (equal); Formal analysis (equal); Software (equal); Writing – original draft (equal); Writing – review & editing (equal). **Mohammad Khairul Bashar:** Conceptualization (equal); Validation (equal); Writing – review & editing (equal). **Nikhil Shukla:** Conceptualization (equal); Funding acquisition (equal); Supervision (equal); Validation (equal); Writing – review & editing (equal). **Zongli Lin:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Supervision (equal); Validation (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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