

RIEMANN-ROCH THEOREMS IN MONOIDAL 2-CATEGORIES

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ABSTRACT. Smooth and proper dg-algebras have an Euler class valued in the Hochschild homology of the algebra. This Euler class is worthy of this name since it satisfies many familiar properties including compatibility with the familiar pairing on the Hochschild homology of the algebra and that of its opposite. This compatibility is the Riemann-Roch theorems of [Shk13, Pet13].

In this paper we prove a broad generalization of these Riemann-Roch theorems. We generalize from the bicategory of dg-algebras and their bimodules to symmetric monoidal bicategories and from Euler class to traces of non identity maps. Our generalization also implies spectral Riemann-Roch theorems.

We regard this result as an instantiation of a 2-dimensional generalized cobordism hypothesis. This perspective draws the result close to many others that generalize results about Euler characteristics and classes to bicategorical traces.

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1. INTRODUCTION

The trace of a matrix is the first example of an additive invariant that is usually encountered in mathematics. Less familiar than additivity, though no less fundamental, is the observation that trace is multiplicative:

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \mathrm{tr}(B)$$

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for square matrices A, B of the same size. Euler characteristics and Euler classes are generalizations of the notion of trace to more general categories. In this paper we prove a vast generalization to Riemann-Roch theorems that describe the compatibility between Euler classes and pairings, themselves generalizations of the multiplicativity of trace.

For a case of our theorem that indicates the true generality but minimizes unfamiliar terminology, we consider dg-algebras. If A is a smooth and proper dg- k -algebra there are homomorphisms

$$k \rightarrow \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \quad \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \rightarrow k$$

that display $\mathrm{HH}(A^{\mathrm{op}})$ as the dual (§2) of $\mathrm{HH}(A)$. Let \overline{A} and \overline{A} denote A as an $(A^{\mathrm{op}} \otimes A, k)$ - and $(k, A \otimes A^{\mathrm{op}})$ -bimodule, respectively. Tensoring with the modules \overline{A} and \overline{B} defines an isomorphism between the categories of (A, B) -bimodules and $(B^{\mathrm{op}}, A^{\mathrm{op}})$ -bimodules. For an (A, B) -bimodule M , let $\square M \square$ be the corresponding $(B^{\mathrm{op}}, A^{\mathrm{op}})$ -bimodule and use the same notation for bimodule homomorphisms.

Theorem 1.1 (Theorem 7.13i, compare to Theorem 8.8). *Suppose A and B are smooth and proper dg- k -algebras. If M is an (A, B) -bimodule that is finitely generated and projective as an A -module, N is a (B, A) -bimodule that is finitely generated and projective as a B -module, and $f: M \rightarrow M$ and $g: N \rightarrow N$ are homomorphisms, then the trace of*

$$\mathrm{HH}(A; M \otimes_B N) \xrightarrow{\mathrm{HH}(A; f \otimes_B g)} \mathrm{HH}(A; M \otimes_B N)$$

coincides with the map

$$(1.2) \quad k \rightarrow \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \xrightarrow{\mathrm{tr}(f) \otimes \mathrm{tr}(\square g \square)} \mathrm{HH}(B) \otimes \mathrm{HH}(B^{\mathrm{op}}) \rightarrow k.$$

The main results of [Shk13, Pet13] are special case of Theorem 1.1 where $A = k$ and f and g are identity maps.

Remark 1.3. As noted above, this is merely a special case of our main result. In Section 7 we show this theorem holds for symmetric monoidal bicategories. Because spectral categories play the role of algebras in a particular monoidal bicategory (see [CP22]), Theorem 1.1 also holds for spectra and topological Hochschild homology (THH).

Though this statement, its generalizations Theorems 7.13, 8.8 and 8.16, and predecessors [Shk13, Pet13] appear complicated, we show in Section 3 that when phrased in the correct generality, they become much more transparent. This understanding is motivated by graphical reasoning that interprets Theorems 1.1, 7.13, 8.8 and 8.16 as results about toroidal traces, in the sense of [CP22]. In the pictures in Figure 1.4, the colored regions correspond to the algebras, and the bimodules are to be read as living at the boundaries of the regions. The torus in Figure 1.4A is the trace of $\mathrm{HH}(f \otimes_B g)$ and the torus in Figure 1.4B is the composite in (1.2).

Indeed, one can see the same thing happening on both tori — the only reason that the theorem is not *immediately* obvious pictorially is the embedding of the torus in 3-space destroys the obvious symmetry between its two constituent copies of S^1 .

Remark 1.5. If one were willing to prove full coherence theorems for this kind of calculus and/or prove a kind of 2-dimensional cobordism hypothesis [SP09, Lur09] that took bimodules into account, this kind of graphical reasoning would actually *prove* the theorem. However, this kind of framed, bimodular cobordism hypothesis is overkill for the task at hand, so we content ourselves with translating the pictures into category theory, and verifying the category theory directly.

1.1. Organization. We begin in Section 2 with a review of duality and bicategories and the formalism of shadows. In Section 3 we outline the path to the main theorem — the proof is quite formal and superficially does not much resemble the statements in

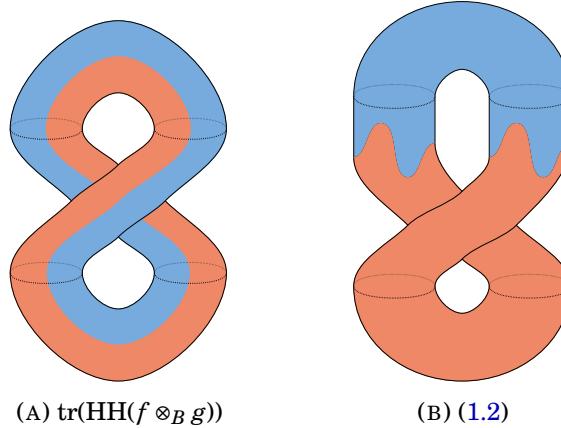


FIGURE 1.4. Toroidal traces without framing. Compare to Figure 4.12.

the introduction. Nevertheless, it is this relatively innocuous reformulation which will imply the results in the introduction.

Section 4 is a review of the details of monoidal bicategories that we will need.

Sections 5 and 6 are discussions of various flavors of duality that exist in monoidal bicategories, with the main content in Section 6.

In Section 7 we prove the main result in categorical generality and in Section 8 we relate the results in the literature to our statement of the theorem.

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2. DUALITY IN BICATEGORIES

Let V be a finite dimensional vector space over a field k . A dual vector space V^* can be defined without reference to the internal hom: a dual is specified by a unit $\epsilon : V^* \otimes V \rightarrow k$, a counit, $\eta : k \rightarrow V \otimes V^*$, and the fact that the following two composites are the identity map

$$V \xrightarrow{\eta \otimes \text{id}} V \otimes V^* \otimes V \xrightarrow{\text{id} \otimes \epsilon} V \quad V^* \xrightarrow{\text{id} \otimes \eta} V^* \otimes V \otimes V^* \xrightarrow{\epsilon \otimes \text{id}} V^*$$

Concretely, the maps η, ϵ can be defined in terms of a basis $\{e_i\}_{i=1}^n$ by $\eta(1) = \sum e_i \otimes e_i^*$, $\epsilon(e_j^* \otimes e_i) = \delta_{ij}$ and extending by linearity.

A dual immediately gives very explicit ways of extracting invariants of vector spaces. For instance, the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\epsilon} k$$

is multiplication by $\dim(V)$. For an endomorphism $f : V \rightarrow V$ the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\epsilon} k$$

is multiplication by $\text{tr}(f)$. This also has the benefit that dimension and trace can be seen as *morphisms*, and thus purely internal to the category.

Since the notion of dual, dimension, and trace can be expressed within a category with tensor product and symmetry, we can define all of these notions in a symmetric

monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ using the above definitions. The “dimension” of a dualizable object $C \in \mathcal{C}$ becomes the element $\chi(C) \in \text{hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ given by

$$\mathbf{1} \longrightarrow C \otimes C^* \xrightarrow{\cong} C^* \otimes C \longrightarrow \mathbf{1}$$

and similarly for the trace of an endomorphism $f : C \rightarrow C$.

While extracting invariants of objects required a symmetry isomorphism, the definition of duality did not. Thus, the definition of duality applies equally well in a monoidal category. Monoidal categories can be viewed as a degenerate version of a bicategory¹ — a monoidal category is a bicategory with one object, the objects of the monoidal category correspond to morphisms in a bicategory, and the tensor product structure corresponds to composition. With this context, we think of a bicategory as a monoidal category with many objects and the motivating example is the bicategory of rings, bimodules and bimodule maps. We find [Lei] to be a good source for definitions of bicategories.

Notation 2.1. We denote the bicategorical composition in a bicategory \mathcal{B} by \circ . If A is an object of \mathcal{B} we denote the identity 1-cell for A by U_A .

Example 2.2.

- i. In the category of bimodules, the unit 1-cell associated to a ring A is A regarded as an $A - A$ bimodule and \circ is the tensor product.
- ii. For a commutative ring R , there is a bicategory $\text{Alg}(R)$ whose objects are R -algebras, 1-morphisms are algebra bimodules, and 2-morphisms are bimodule homomorphisms. The bicategory composition \circ is tensor product over the shared algebra.
- iii. There is a bicategory of categories and profunctors (bimodules) enriched over a cocomplete closed symmetric monoidal category \mathcal{V} . The bicategory composition \circ is a coend.
- iv. There is a bicategory of dg-algebras, and the derived category of bimodules and homomorphisms. The bicategorical product is the tensor product.
- v. There is a bicategory of dg-categories, where 0-cells are dg-categories and 1- and 2-cells are the derived category of bimodules. We will abuse notation and also denote this category $\text{Mod}(\text{Cat}_{\text{dg}})$.
- vi. There is a bicategory of spectral categories where 1- and 2-cells are the homotopy category of bimodules. We will denote this category $\text{Mod}(\text{Cat}_{\text{Sp}})$.

See [CP22, Thm. 8.39] for an extended and explicit description of *iv* to *vi*.

Example *i* is a special case of example *ii* where $R = \mathbb{Z}$. Example *ii* is a subbicategory of example *iii* where we only consider the categories with a single object. (The same for examples *iv* and *v*.) Finally there is a spectral analog of example *iv* where the objects are replaced by ring spectra.

In this paper we will focus almost exclusively on bicategories that generalize the bicategory of bimodules. This reflects the particular objective of this paper and not the underlying theory. See [PS14a] for more topologically focused discussions.

The following definition first appeared in [MS06].

Definition 2.3. Let M be a 1-cell in a bicategory $\mathcal{B}(C, D)$. We say M is **right dualizable** if there is a 1-cell N together with 2-cells

$$\eta: U_C \rightarrow M \circ N \quad \epsilon: N \circ M \rightarrow U_D$$

¹The weakness of a bicategory is necessary for the examples of interest here.

such that the triangle identities (2.4) hold.

$$(2.4) \quad \begin{array}{ccccccc} M & \xrightarrow{\cong} & U_C \odot M & \xrightarrow{\eta \odot \text{id}} & M \odot N \odot M & \xrightarrow{\text{id} \odot \epsilon} & M \odot U_D & \xrightarrow{\cong} & M \\ N & \xrightarrow{\cong} & N \odot U_C & \xrightarrow{\text{id} \odot \eta} & N \odot M \odot N & \xrightarrow{\epsilon \odot \text{id}} & U_D \odot N & \xrightarrow{\cong} & N \end{array}$$

We say N is **right dual** to M , (M, N) is a **dual pair**, that N is **left dualizable**, and that M is its **left dual**.

Example 2.5. The examples in Example 2.2 all have many dualizable 1-cells and they are all variations on projectivity. Those in examples *iii*, *v*, and *vi*, go under the name of **smooth, proper** categories and are many object generalizations of finitely generated projective modules. For a beautiful introduction to this topic, see [Toë09]. See also [BGT13] for a careful discussion of dualizability in categories.

The following lemma is easy, but critical.

Lemma 2.6 (Compare to Lemma 5.13, [MS06, 16.5.1]). *If $M_1 \in \mathcal{B}(A, B)$ and $M_2 \in \mathcal{B}(B, C)$ are right dualizable, then so is $M_1 \odot M_2$.*

Definition 2.7. [Pon10] A **shadow** for a bicategory \mathcal{B} consists of functors

$$\langle\langle - \rangle\rangle: \mathcal{B}(R, R) \rightarrow \mathbf{T}$$

for each object R of \mathcal{B} and some fixed category \mathbf{T} , equipped with a natural isomorphism

$$\theta: \langle\langle M \odot N \rangle\rangle \rightarrow \langle\langle N \odot M \rangle\rangle$$

for $M \in \mathcal{B}(R, S)$ and $N \in \mathcal{B}(S, R)$ such that the following diagrams commute whenever they make sense:

$$\begin{array}{ccccc} \langle\langle (M \odot N) \odot P \rangle\rangle & \xrightarrow{\theta} & \langle\langle P \odot (M \odot N) \rangle\rangle & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle (P \odot M) \odot N \rangle\rangle & \langle\langle M \odot U_C \rangle\rangle & \xrightarrow{\theta} & \langle\langle U_C \odot M \rangle\rangle & \xrightarrow{\theta} & \langle\langle M \odot U_C \rangle\rangle \\ \langle\langle a \rangle\rangle \downarrow & & & & \uparrow \theta & & & & & & \downarrow \langle\langle l \rangle\rangle \\ \langle\langle M \odot (N \odot P) \rangle\rangle & \xrightarrow{\theta} & \langle\langle (N \odot P) \odot M \rangle\rangle & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle N \odot (P \odot M) \rangle\rangle & & & & & & \langle\langle r \rangle\rangle \nearrow & \downarrow \langle\langle r \rangle\rangle & \nearrow \langle\langle r \rangle\rangle \end{array}$$

Example 2.8. The bicategories in Example 2.2 all have shadows and they can be interpreted as the bicategorical product with the diagonal module associated to a 0-cell. In the bicategory of rings and modules this is Hochschild homology and in the spectral generalizations it is topological Hochschild homology.

Recent work of Hess and Rasekh [HR21] has shown there is an equivalence between functors out of THH of a bicategory and shadows on that bicategory so the examples in Example 2.8 capture those we need to consider.

Definition 2.9. [Pon10] Let $\phi: P \odot M \rightarrow M \odot Q$ be a 2-cell where M is right dualizable. The **twisted trace** of ϕ is the composite

$$\langle\langle P \rangle\rangle \cong \langle\langle P \odot U_A \rangle\rangle \rightarrow \langle\langle P \odot M \odot N \rangle\rangle \rightarrow \langle\langle M \odot Q \odot N \rangle\rangle \cong \langle\langle N \odot M \odot Q \rangle\rangle \rightarrow \langle\langle U_B \odot Q \rangle\rangle \cong \langle\langle Q \rangle\rangle$$

If ϕ is the identity map $M \rightarrow M$ we call the trace of ϕ the **Euler characteristic** of M . This is consistent with the Euler characteristic of spaces regarded as elements of the stable homotopy category.

Example 2.10. Let M be a right dualizable (A, B) -bimodule in the bicategory of rings and bimodules. Then the Euler characteristic is a map

$$\chi(A): \langle\langle U_A \rangle\rangle = \text{HH}(A) \longrightarrow \text{HH}(B) = \langle\langle U_B \rangle\rangle$$

As a more explicit example of this, let V be a finite dimensional complex representation of a finite group G . Then V is a $(\mathbb{C}[G], \mathbb{C})$ -bimodule which is right dualizable. The induced map

$$\mathrm{HH}_0(\mathbb{C}[G]) \longrightarrow \mathrm{HH}_0(\mathbb{C})$$

is the character of the representation.

We end this section with another easy, but critical result that builds on Lemma 2.6.

Theorem 2.11 (Compare to Proposition 5.15, [PS13, 7.5]). *Let $M_1 \in \mathcal{B}(A, B)$, $M_2 \in \mathcal{B}(B, C)$ be right dualizable and $Q_1 \in \mathcal{B}(A, A)$, $Q_2 \in \mathcal{B}(B, B)$ and $Q_3 \in \mathcal{B}(C, C)$. Let $f_1: Q_1 \odot M_1 \rightarrow M_1 \odot Q_2$ and $f_2: Q_2 \odot M_2 \rightarrow M_2 \odot Q_3$ be 2-cells. Then the trace of*

$$Q_1 \odot M_1 \odot M_2 \xrightarrow{f_1 \odot \mathrm{id}_{M_2}} M_1 \odot Q_2 \odot M_2 \xrightarrow{\mathrm{id}_{M_1} \odot f_2} M_1 \odot M_2 \odot Q_3$$

is

$$\langle\langle Q_1 \rangle\rangle \xrightarrow{\mathrm{tr}(f_1)} \langle\langle Q_2 \rangle\rangle \xrightarrow{\mathrm{tr}(f_2)} \langle\langle Q_3 \rangle\rangle.$$

3. PATH TO THE MAIN THEOREM

In this section we use the traces of Section 2 to state, but not yet prove, a bicategorical generalization of Theorem 1.1. This generalization requires hypotheses beyond a shadow on a bicategory and in this section we describe the additional conditions. When stating a theorem, there is often a choice between hypotheses that are easy to verify or state and those that are most compatible with the proof technique. The main theorem of this paper appears to be a case where there is a significant discrepancy between these two goals. We resolve that tension by focusing on precisely what our proof requires in this section. In Sections 5 through 7 we verify that symmetric monoidal bicategories with many dualizable 0-cells (Sections 5 and 6) and, in particular, subbicategories of Example 2.2 satisfy the conditions described in this section.

Examples 2.8 and 2.10 begin the process of generalizing Theorem 1.1 by recognizing the Hochschild homology groups as shadows and the Euler or Hochschild class as a trace. The next step is to appropriately interpret the pairing map. A non degenerate pairing of R -modules for a commutative ring R is a map

$$M \otimes_R N \rightarrow L$$

so that the adjoint $M \rightarrow \mathrm{Hom}(N, L)$ is an isomorphism. For pairings that take values in the ground ring R , this is a pair of modules (M, N) and an isomorphism $M \rightarrow \mathrm{Hom}(N, R)$.

We will replace this assumption by the assumption that (M, N) is a dual pair in a monoidal category $(\mathcal{C}, \otimes, I)$. If \mathcal{C} is closed and (M, N) is a dual pair, there is an isomorphism

$$N \rightarrow \mathrm{Hom}(M, I).$$

In the category of modules over a ring R , dualizability is not the same as a nondegenerate pairing, but working within the hypotheses of Theorem 1.1 we don't lose any generality. With this change, we gain an easy generalization of pairing to other monoidal categories and the coevaluations will be essential to our generalization of Theorem 1.1.

Then the first additional condition on a bicategory with shadow is the following:

Goal 3.1 (Section 5). *The shadow on \mathcal{B} takes values in a monoidal category.*

The Euler classes and Hochschild classes of Theorem 1.1 are traces in bicategories and so we think of them as maps $\langle\langle U_A \rangle\rangle \rightarrow \langle\langle U_B \rangle\rangle$ for 0-cells A and B of a bicategory \mathcal{B} . (In the motivating examples U_A and $\langle\langle U_A \rangle\rangle$ are the ground field.) Since $\langle\langle U_A \rangle\rangle$ and $\langle\langle U_B \rangle\rangle$ have to be compatible with Goal 3.1 we have the following condition:

Goal 3.2 (Section 6). *For each 0-cell A of \mathcal{B} , $\langle\langle U_A \rangle\rangle$ is dualizable.*

In Section 6 we will produce dualizability for shadows of more 1-cells, but this is an artifact of the process and not essential for the most immediate generalizations of the result in the introduction. It would be interesting to know if there are further generalizations that make use of the additional dualizability.

The final goal almost slides under the radar since it is such a fundamental and familiar fact about modules. It is the observation that an (A, B) -bimodule can be understood as an $(B^{\text{op}}, A^{\text{op}})$ -bimodule. In the generality here it is the following statement.

Goal 3.3 (Section 7). *There is a function $-\vee$ from 0-cells of \mathcal{B} to itself that so that the dual of $\langle\langle U_A \rangle\rangle$ is $\langle\langle U_{A^\vee} \rangle\rangle$ and this function extends to a functor*

$$\square_{-}: \mathcal{B}(A, B) \rightarrow \mathcal{B}(B^\vee, A^\vee)$$

that takes right dualizable 1-cells to right dualizable 1-cells.

Goals 3.1 through 3.3 don't together imply Goal 3.4, but they make it possible to understand the statement.

Goal 3.4 (Sections 6 and 7). *Suppose $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(B, A)$ are right dualizable 1-cells. For 2-cells $f: M \rightarrow M$ and $g: N \rightarrow N$ the following diagram commutes.*

$$\begin{array}{ccc} \langle\langle U_A \rangle\rangle \otimes \langle\langle U_{A^\vee} \rangle\rangle & \xrightarrow{\text{tr}(f) \otimes \text{tr}(\square_{-} g)} & \langle\langle U_B \rangle\rangle \otimes \langle\langle U_{B^\vee} \rangle\rangle \\ \uparrow & & \downarrow \\ I & \xrightarrow{\text{tr}(f \circ g)} & I \end{array}$$

The left vertical map is the coevaluation for the dual pair $(\langle\langle U_A \rangle\rangle, \langle\langle U_{A^\vee} \rangle\rangle)$ and the right vertical map is the evaluation for the dual pair $(\langle\langle U_B \rangle\rangle, \langle\langle U_{B^\vee} \rangle\rangle)$.

Example 3.5. Theorem 1.1 is a special case of this result under some simplifying assumptions. First suppose that $\langle\langle U_A \rangle\rangle \cong \langle\langle U_{A^\vee} \rangle\rangle \cong I$ and the coevaluation for I is the unit isomorphism. Then the diagram in Goal 3.4 becomes the following diagram.

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\text{tr}(f) \otimes \text{tr}(\square_{-} g)} & \langle\langle U_B \rangle\rangle \otimes \langle\langle U_{B^\vee} \rangle\rangle \\ \uparrow & & \downarrow \\ I & \xrightarrow{\text{tr}(f \circ g)} & I \end{array}$$

If the target of the shadow is the category of k -vector spaces for a field k then $I = k$. Taking Hochschild homology to be the shadow, the traces pick out Hochschild homology classes. Then the right vertical map is a nondegenerate pairing in the more conventional sense. The maps from $I = k$ pick out elements in their targets.

4. MONOIDAL BICATEGORIES

Our main theorem requires working in the generality of symmetric monoidal bicategories. In this section we review the necessary definitions and establish our graphical notation for manipulations. While the formal definition of a monoidal bicategory is unwieldy, the basic intuition, first examples and graphical descriptions are illuminating. We consider those first before recalling the formal definitions.

Intuitively, a monoidal bicategory is a bicategory \mathcal{B} equipped with a functor

$$\otimes: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$$

that is appropriately compatible with the bicategorical composition \odot and is unital and associative. A symmetric monoidal bicategory gains 1-cells $A \otimes B \rightarrow B \otimes A$ that satisfy the triangle identities for an adjoint pair. See Figure 4.9.

For motivating examples of monoidal bicategories we return to the motivating example of the previous section.

Example 4.1. The examples in Example 2.2 are (symmetric) monoidal bicategories. See [HS, Shu10, CP22].

- For a commutative ring R , the monoidal product \otimes in the bicategory $\text{Alg}(R)$ is tensor product over R .
- The monoidal product in the bicategory of categories and profunctors (bimodules) enriched over a cocomplete closed symmetric monoidal category \mathcal{V} is induced by the monoidal product on \mathcal{V} .

See [CP22, Thm. 8.39] for more details.

4.1. Circuit diagrams. Symmetric monoidal bicategories have an impressive amount of structure and keeping track of all it is burdensome. Fortunately, Gurski and Osorno [GO13] proved a coherence theorem for symmetric monoidal bicategories that massively simplifies bookkeeping challenges.

Theorem 4.2. [GO13, Theorem 1.23] *In the free symmetric monoidal bicategory on a single object, every diagram of 2-cells commutes. Equivalently, between every pair of parallel 1-cells there is either a unique invertible 2-cell or no 2-cells at all. Moreover, parallel 1-cells are isomorphic if and only if they have the same underlying permutation.*

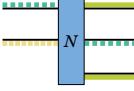
This alleviates the need to record associativity and unit maps, but leaves us with the challenge of keeping track of the bicategorical compositions \circ and the monoidal products \otimes of 1-cells. We will resolve this by writing compositions graphically using the conventions of [CP22]. They are closely related to those in [DSPS20, SP09, PS12].

We represent 0-cells by black horizontal lines with parallel colored shading.

(4.3) 

The default is that the shading is below the black lines, but in later sections we will flip the colored shading to the top to indicate the 1-dual of a 0-cell (see Definition 5.1). Vertically stacked shaded lines indicate the \otimes product.

One cells are represented by colored boxes with any number of shaded edges entering on the left and right sides. (Shaded edges do not enter on the top and bottom.)

(4.4) 

For example, a 1-cell from $A \otimes B$ to $C \otimes D \otimes E$ would have two edges entering on the left with the edge representing A below that for B and three edges on the right with that for C at the bottom and E at the top. If there are no edges attached to a side of a 1-cell then the 0-cell on that side is the monoidal unit 0-cell.

Vertical stacking of colored boxes represents \otimes product and horizontal concatenation is the \circ composition.

Two-cells are represented by arrows between groups of edges and colored boxes. In particular, we do not represent 2-cells with surfaces.

Example 4.5. As a limited first example, Figure 4.6 represents the maps and compatibility for a dual pair of 1-cells in a bicategory. As all 1-cells are assigned to a single 0-cell on left and right the colored boxes each have 1-edge entering on the left and one edge on the right. There is no vertical stacking since this is a definition in a bicategory rather than a monoidal bicategory.

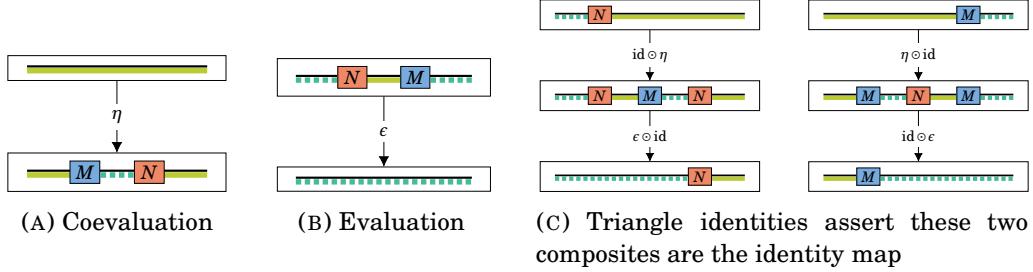


FIGURE 4.6. Circuit diagrams for dualizable 1-cells

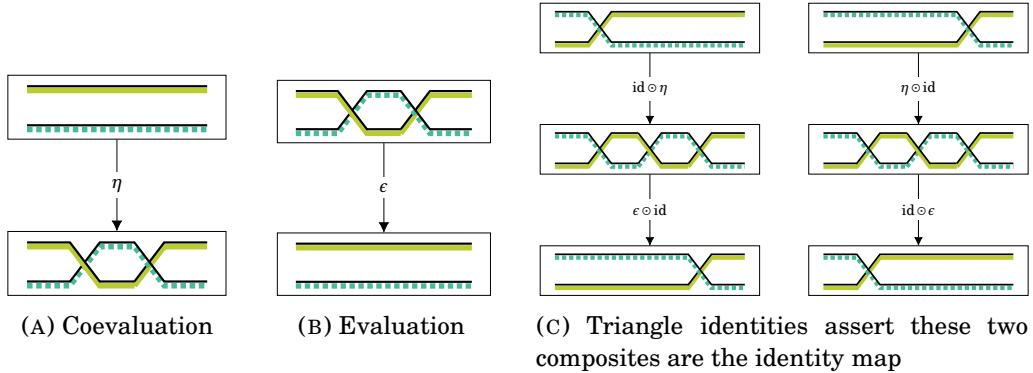


FIGURE 4.9. Circuit diagrams demonstrating dualizability for symmetry

Example 4.7. In a symmetric monoidal bicategory (see Section 4.2) for each pair of 0-cells A, B , there is a 1-cell Γ from $A \otimes B$ to $B \otimes A$. We represent this graphically as the following exchange of lines.



The 1-cell Γ is part of an adjoint equivalence. An adjoint equivalence is a dual pair where the coevaluation and evaluation 2-cells are isomorphisms. (In [CP19], motivated by the applications of the paper, we called this a Morita equivalence.) In particular, we have maps as in Figure 4.9A and Figure 4.9B and the composites in Figure 4.9C are the identity.

Remark 4.10. For the graphical representation of the 1-cell Γ in (4.8) the data of which line crosses over the other is an artifact of the representation that does not reflect the underlying mathematics. So the order of crossing should be disregarded. When read from left to right we will always have the top line cross over the bottom, but this is just to minimize distraction.

Example 4.11. Another illuminating example of the translation between circuit diagrams and more conventional expressions can be found in Definition 5.1. The map in (5.3) is written graphically in Figure 5.4A and the same for (5.2) and Figure 5.4B. These have both vertical stacking and horizontal composition and 1-cells have sides with no edges and sides with multiple edges. We find the graphical descriptions in Figures 5.4A and 5.4B significantly more clear than the descriptions in (5.3) and (5.2) and so we will default to the graphical representations.

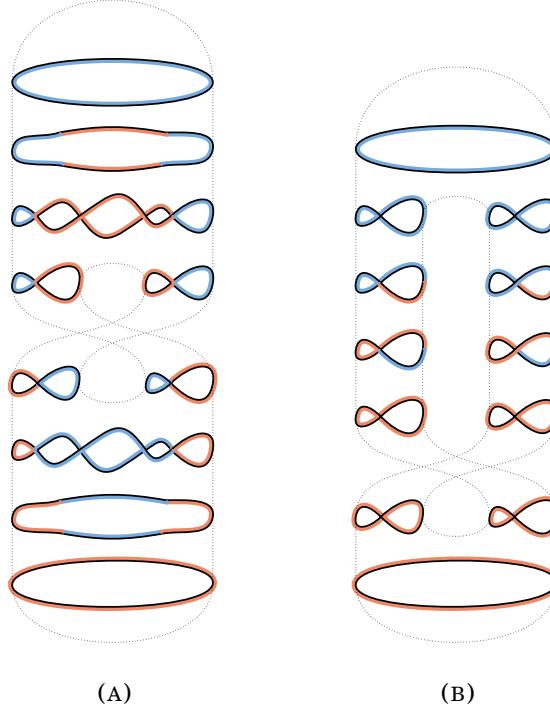


FIGURE 4.12. Toroidal traces with framing

Figure 4.12 is an interpretation of Figure 1.4 that is consistent with these graphical conventions.

4.2. Formal definitions. For the definition of a monoidal bicategory we follow [Sta16] and briefly recall the relevant definitions here.

Definition 4.13. [Sta16] A **monoidal bicategory** consists of:

- a bicategory \mathcal{B} .
- a functor $\otimes: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ and with an invertible 2-morphism

$$(M \otimes N) \circ (M' \otimes N') \Rightarrow (M \circ M') \otimes (N \circ N').$$

- An adjoint equivalence

$$a: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

that is pseudonatural in A, B, C .

- An invertible modification π relating the two different ways of moving parentheses from being clustered at the left of four objects to being clustered at the right.
- An equation of modifications relating the various ways of getting from the parentheses clustered at the left of five objects to clustered at the right.

- a 0-cell I
 - Adjoint equivalence 1-cells $L_A: I \otimes A \rightarrow A$ and $R_A: A \otimes I \rightarrow A$ that are pseudonatural in A .
 - Invertible modifications $\lambda: l \otimes B \Rightarrow l \circ a$, $\mu: r \otimes B \Rightarrow (A \otimes l) \circ a$, and $\rho: r \Rightarrow (A \otimes r) \circ a$.
 - Four equations of modifications relating the unit modifications.

Definition 4.14. [Sta16] A **symmetric monoidal bicategory** consists of the following:

- A monoidal bicategory \mathcal{B} .
- An adjoint equivalence

$$\Gamma : A \otimes B \rightarrow B \otimes A$$

pseudonatural in A and B .

- Invertible modifications R and S filling the hexagons

$$\begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{\Gamma} & (B \otimes C) \otimes A \\ \uparrow a & & \downarrow a \\ (A \otimes B) \otimes C & & B \otimes (C \otimes A) \\ \downarrow \Gamma \otimes \text{id}_C & & \uparrow \text{id}_B \otimes \Gamma \\ (B \otimes A) \otimes C & \xrightarrow{a} & B \otimes (A \otimes C) \\ & & \\ (A \otimes B) \otimes C & \xrightarrow{\Gamma} & C \otimes (A \otimes B) \\ \uparrow a & & \downarrow a \\ A \otimes (B \otimes C) & & \downarrow \text{id}_A \otimes \Gamma \\ \uparrow \text{id}_A \otimes \Gamma & & \downarrow \Gamma \otimes \text{id}_B \\ A \otimes (C \otimes B) & \xrightarrow{a} & (A \otimes C) \otimes B \end{array}$$

- Two equations shuffling one object into three objects.
- An equation shuffling two objects into two objects
- An equation relating multiple applications of Γ .
- An invertible modification

$$\nu : \Gamma \rightarrow \Gamma$$

satisfying two equations relating ν and the modifications R and S above and an equation relating ν applied to $A \otimes B$ and $B \otimes A$.

4.3. Pseudonaturality of Γ . We will make extensive use of the pseudonaturality of Γ to rearrange 1-cells to allow for the applications of 2-cells. (See, for example, Figure 6.11.) The pseudonaturality isomorphism is graphically represented as the following 2-cell.

$$(4.15) \quad \begin{array}{ccc} \text{Diagram 1} & \xrightarrow{\hspace{2cm}} & \text{Diagram 2} \end{array}$$

We will very rarely use such a simple form of this isomorphism, instead we will compose multiple instances or apply it to monoidal or bicategorical composites. For an example of the first, the map \mathfrak{s}_1 in Figure 4.17 is the composite

$$\begin{array}{ccc} \text{Diagram 1} & \xrightarrow{\hspace{2cm}} & \text{Diagram 2} \end{array}$$

where the first map is the pseudonaturality isomorphism where the role of M is played by the composite of \underline{B} , \underline{B} , and Q and N is an identity 1-cell. The second map is the inverse of the pseudonaturality isomorphism where the role of N is played by the composite of \underline{B} , Q , and \underline{A} . In this case M is an identity 1-cell. For an example of applying (4.15) to a monoidal composite see \mathfrak{s}_5 and \mathfrak{a}_5 in Figure 4.18.

To ensure that all such diagrams are related by unique 2-cells, we repeatedly invoke the following coherence theorem of Gurski–Osorno.

Theorem 4.16. [GO13, Theorem 1.25] *Let B be a bicategory that is free on an underlying 1-globular set. Then in the free symmetric monoidal bicategory on B , every diagram of 2-cells commutes.*

This is the “many objects and 1-morphisms” version of [GO13, Thm. 1.23]. In the case of interest for us, we consider the free symmetric monoidal bicategory generated by the 1-cells M and N in (4.15).

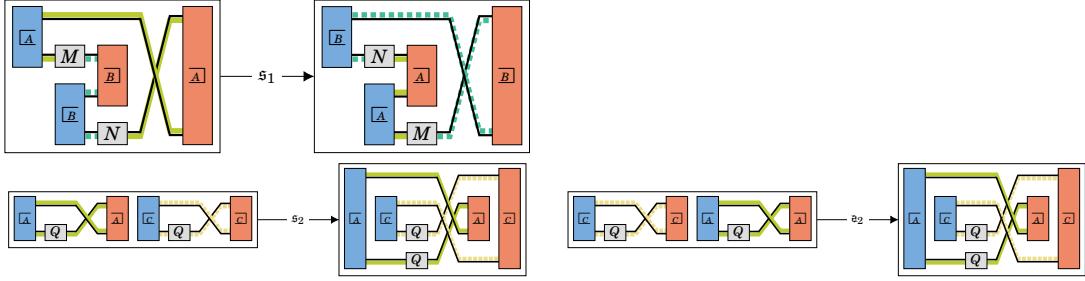


FIGURE 4.17. Pseudonaturality isomorphisms

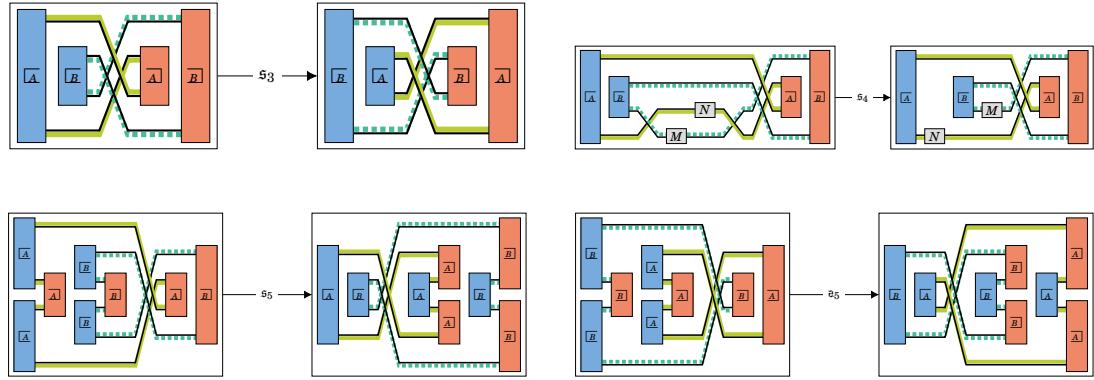


FIGURE 4.18. Pseudonaturality isomorphisms for Figure 5.19

The composites of (4.15) we use in this paper are displayed in Figures 4.17 through 4.20 and they are organized by where they are used. The maps in Figure 4.17 appear throughout this paper while those in Figures 4.18 through 4.20 appear only in very specific diagrams. There is a very valid argument that is strange to treat the maps in Figures 4.17 through 4.20 as distinct since they are all manifestations of (4.15). We choose to do so since identifying this list simplifies verifying the commutativity in later diagrams. For example, in Figures 5.19, 6.11 and 6.12 it makes it more clear which diagrams commute because of naturality.

In addition to the maps in Figures 4.17 through 4.20 there are shifts that only involve identity 0-cells. Examples of these include the maps in Figure 4.21 from Figure 6.11. These occur less commonly and so we will use s to refer to all instances of (4.15) involving only the monoidal until 0-cell except those for two separate components like s_{10} .

Together the maps in Figures 4.17 through 4.21 are all the ways we reorder 1-cells so we can regard each of these maps as a fixed composite of the map in (4.15) that we leave unchanged through the paper. Alternatively, we can cite [GO13, Theorem 1.25] again to see that a particular choice of composite is not necessary.

In the diagrams in Figures 5.12B, 5.19, 6.11 and 6.12 there are small regions entirely consisting of the maps in Figures 4.17 through 4.21. Again we can invoke [GO13, Theorem 1.25] to assert that all these regions commute.

5. 1-DUALIZABILITY

With the addition of a monoidal structure on a bicategory we gain more notions of duality. The first of these is duality for 0-cells. This duality and the duality in Definition 2.3

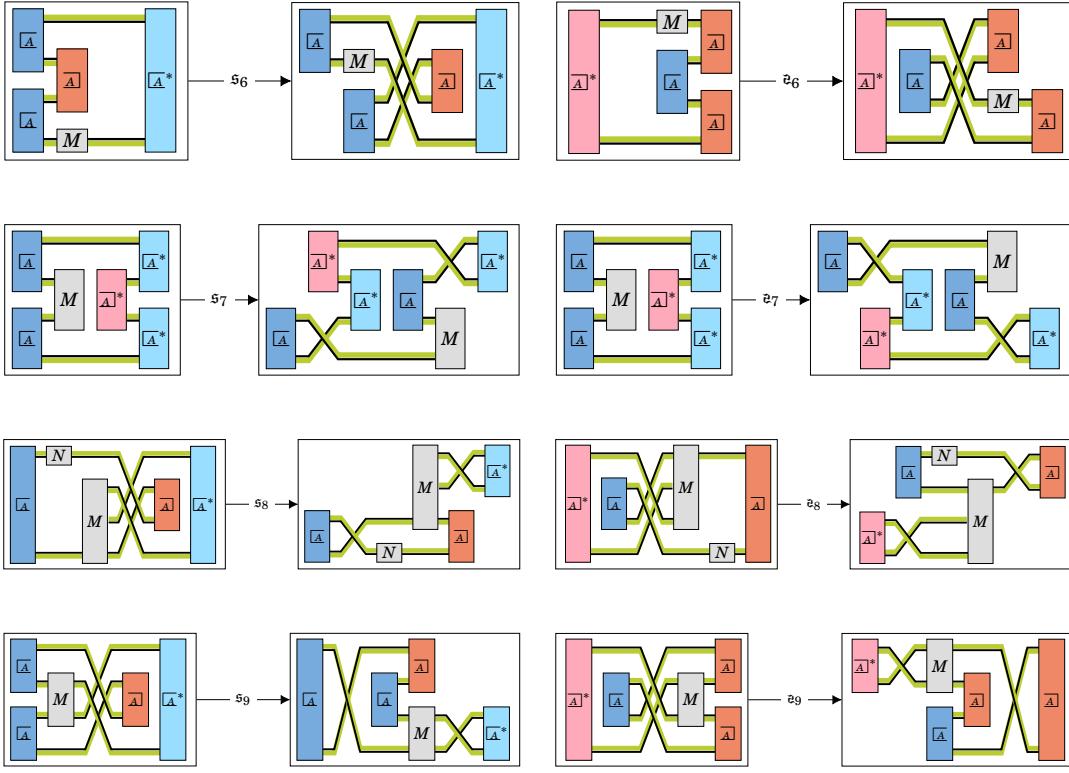


FIGURE 4.19. Pseudonaturality isomorphisms for Figures 6.11 and 6.12

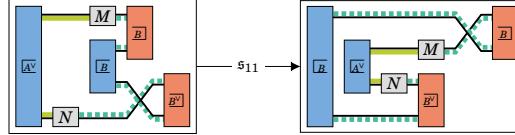


FIGURE 4.20

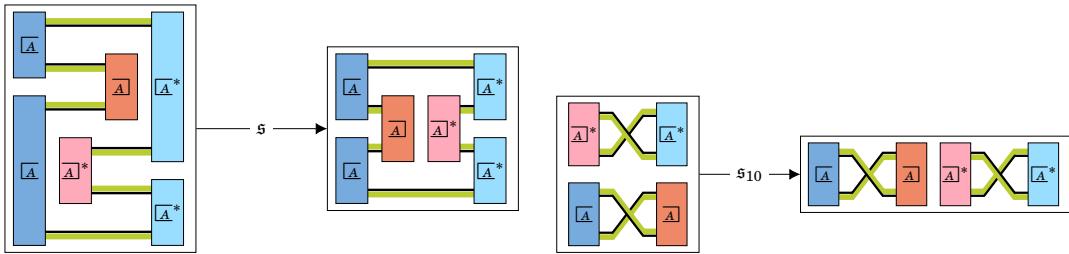


FIGURE 4.21

are distinct generalizations of dualizability for objects in a monoidal category [DP80]. In addition to its inherent interest, a crucial consequence is Proposition 5.7 below.

In a monoidal 1-category, a 0-cell (a.k.a. object) is **dualizable** A if there is a 0-cell A^\vee and maps $I \rightarrow A \otimes A^\vee$ and $A^\vee \otimes A \rightarrow I$ such that the two maps

$$A \cong I \otimes A \longrightarrow A \otimes A^\vee \otimes A \longrightarrow A \otimes I \cong A$$

and

$$A^\vee \cong A^\vee \otimes I \longrightarrow A^\vee \otimes A \otimes A^\vee \longrightarrow I \otimes A^\vee \cong A^\vee$$

are the identity. This generalizes the discussion for vector spaces in Section 2.

In a bicategory we cannot say that two 1-morphisms “are” identity maps, instead we must supply a 2-cell witnessing that statement. This accounts for the extra complexity of the definition below.

Definition 5.1. A 0-cell A in a monoidal bicategory \mathcal{B} is **1-dualizable** if there is

- a zero-cell A^\vee ,
- 1-cells $\underline{A} \in \mathcal{B}(I, A \otimes A^\vee)$ and $\overline{A} \in \mathcal{B}(A^\vee \otimes A, I)$ and
- invertible 2-cells (Figures 5.4A and 5.4B)

$$(5.2) \quad U_A \xrightarrow{\cdot} L_A^{-1} \circ (\underline{A} \otimes U_A) \circ (U_A \otimes \overline{A}) \circ R_A$$

$$(5.3) \quad U_{A^\vee} \xrightarrow{\cdot} R_{A^\vee}^{-1} \circ (U_{A^\vee} \otimes \underline{A}) \circ (\overline{A} \otimes U_{A^\vee}) \circ L_{A^\vee}$$

so that the diagrams in Figures 5.4C and 5.4D commute.

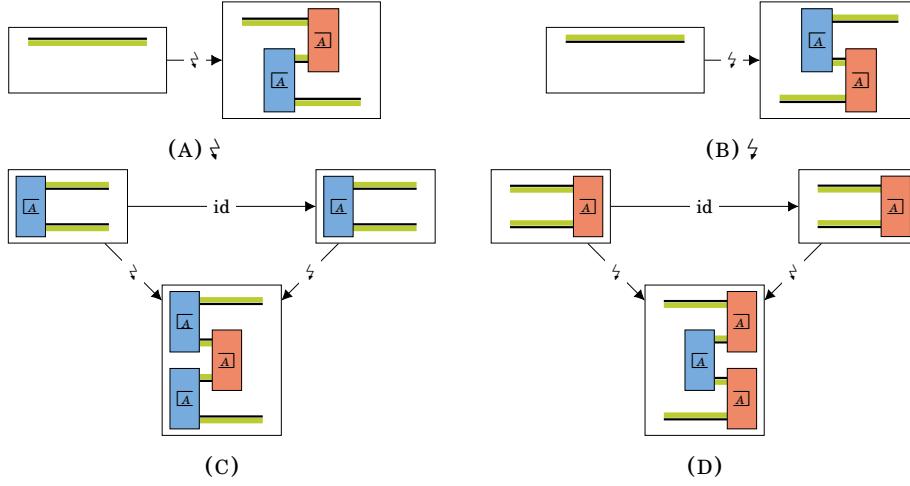


FIGURE 5.4. 2-cells and conditions for a 1-dualizable 0-cell A

Remark 5.5. Definition 5.1 is [CP22, Def. 6.4] with one substantive change and two notational changes.

In [CP22, Def. 6.4] the maps \cdot and \cdot were both denoted Δ . The 1-cell \underline{A} was denoted C and \overline{A} was denoted E . Unlike [CP22], the clarity provided by giving these distinct maps distinct names is necessary here.

The substantive difference is the condition that Figures 5.4C and 5.4D commute. This was not required for any of the results in [CP22], but will be required in Lemmas 5.18 and 6.6 (and Proposition 6.9 and Theorems 6.13 and 7.13).

Example 5.6.

- i. A ring A is 1-dualizable and A^\vee is A^{op} . The 1-cells \underline{A} and \overline{A} are A regarded as a $(\mathbb{Z}, A \otimes A^{\text{op}})$ -bimodule and $(A^{\text{op}} \otimes A, \mathbb{Z})$ -bimodule. The maps \cdot and \cdot are nearly tautological in this case.
- ii. If A is a category then A is 1-dualizable with dual A^{op} . The 1-cells \underline{A} and \overline{A} are the functors $* \times (A \times A^{\text{op}}) \rightarrow \text{Set}$ and $(A^{\text{op}} \times A) \times * \rightarrow \text{Set}$ given by the hom sets in the category A .

In this case the maps $\hat{\cdot}$ and $\check{\cdot}$ are slightly more complicated. Writing the objects in terms of standard tensors over categories, $\hat{\cdot}$ is the isomorphism

$$A(a, b) \cong A(a, -) \odot_A A(-, -) \odot_A A(-, -) \odot_A A(-, b).$$

and $\check{\cdot}$ is the same map for A^{op} .

iii. If \mathcal{C} is a dg-category or spectral category, entirely analogous statements hold.

A symmetric monoidal bicategory where all 0-cells are 1-dualizable endows the bicategory with a shadow. This shadow is the familiar one in many examples, including those in Example 2.2.

Proposition 5.7 (Goal 3.1, [BZN], [CP22, Prop. 6.11]). *If \mathcal{B} is a symmetric monoidal bicategory and all 0-cells of \mathcal{B} are 1-dualizable, the composition in Figure 5.8A defines a shadow on \mathcal{B} that takes values in the (symmetric monoidal) category $\mathcal{B}(I, I)$.*

If $M \in \mathcal{B}(A, A)$ and A is 1-dualizable, the shadow of M is the bicategorical composition in Figure 5.8A. The shadow isomorphism is defined in Figure 5.8B.

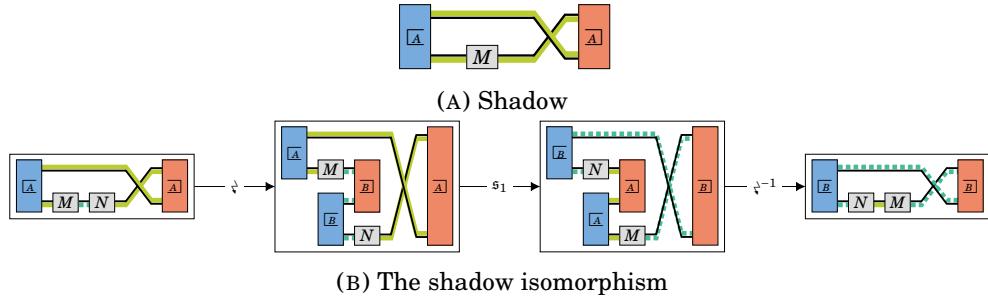


FIGURE 5.8. Shadows and shadow isomorphisms in symmetric monoidal bicategories with 1-dualizable 0-cells. (Proposition 5.7)

If $(\underline{A}, \overline{A})$ and $(\underline{A}', \overline{A}')$ are witnessing 1-cells for a 1-dualizable 0-cell A the maps $\hat{\cdot}$ and $\check{\cdot}$ for each pair define isomorphisms

$$\underline{A} \rightarrow \underline{A}' \quad \text{and} \quad \overline{A} \rightarrow \overline{A}'$$

These isomorphisms also show that the shadows defined by $(\underline{A}, \overline{A})$ and $(\underline{A}', \overline{A}')$ are isomorphic and the isomorphisms compatible with the trace construction.

Proposition 5.9. *If $(\underline{A}, \overline{A})$ and $(\underline{A}', \overline{A}')$ are witnessing 1-cells for a 1-dualizable 0-cell A , and $Q \in \mathcal{B}(A, A)$, there is an isomorphism*

$$(5.10) \quad \phi: \langle\langle Q \rangle\rangle \rightarrow \langle\langle Q' \rangle\rangle'$$

natural with respect to 2-cells in $\mathcal{B}(A, A)$.

If M is right dualizable and $f: Q \odot M \rightarrow M \odot P$, the diagram

$$(5.11) \quad \begin{array}{ccc} \langle\langle Q \rangle\rangle & \xrightarrow{\phi} & \langle\langle Q' \rangle\rangle' \\ \downarrow \text{tr}(f) & & \downarrow \text{tr}'(f) \\ \langle\langle P \rangle\rangle & \xrightarrow{\phi} & \langle\langle P' \rangle\rangle' \end{array}$$

commutes.

Proof. The isomorphism ϕ is a special case of the shadow isomorphism in Figure 5.8B. See Figure 5.12A for explicit definition.

To see the diagram in (5.11) commutes, we expand this diagram to that in Figure 5.12B. The top and bottom composites are the map in Figure 5.12A and the left and right are

the traces where the shadow isomorphisms have been expanded using the definition in Figure 5.8B. All small regions in Figure 5.12B notated with Nat. commute by naturality. This is the vast majority of the squares. The exceptions are the squares labeled with (1) and (2). These commute by the coherence theorem for symmetric monoidal bicategories [GO13, Thm. 1.25]. \square

Since there will not be a unique choice of witnessing 1-cells for a 1-dualizable 0-cell we will avoid introducing these isomorphisms in what follows by choosing a fixed but arbitrary dual and witnessing 1-cells for each 1-dualizable 0-cell. An exception to this is Proposition 6.9, where the choice of dual and 1-cells is essential to the way we have set up the proof.

5.1. Monoidal product and dualizability. Lemma 2.6 and Theorem 2.11 set up an expectation for compatibility between monoidal structures and duality. In this section we describe how 1-dualizability interacts with the monoidal product in a symmetric monoidal bicategory. The results here exactly parallel Lemma 2.6 and Theorem 2.11 in their statements and proofs.

Lemma 5.13 (Compare to Lemma 2.6). *If A and B are 1-dualizable then $A \otimes B$ is 1-dualizable.*

Proof. Using string diagram calculus for symmetric monoidal categories or bicategories, Lemma 2.6 is shown by “nesting” the evaluation and coevaluation maps for A and B . The proof of Lemma 5.13 is given by rotating that picture. (If we had chosen a different orientation for our circuit diagrams the proofs would be more similar.)

The coevaluation and evaluation are in Figures 5.14A and 5.14B. Note that the order is reversed between the coevaluation and evaluation. The invertible 2-cells are in Figures 5.14C and 5.14D.

The triangle diagrams from Figure 5.4C is in Figure 5.14E. The top two regions of that diagram commute by definition of $(\zeta_{B \otimes A})^{-1}$ and $\zeta_{B \otimes A}$. The regions labeled with Nat. commute by naturality. The remaining region is an example of Figure 5.4D. The left, bottom and right composite is the identity since it is also an example of Figure 5.4D.

The other triangle diagram is essentially a reflection of this diagram. \square

Proposition 5.15 (Compare to Theorem 2.11). *If A, B, C and D are dualizable 0-cells, $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(C, D)$ are dualizable 1-cells, and $f: Q_1 \circ M \rightarrow M \circ P_1$ and $g: Q_2 \circ N \rightarrow N \circ P_1$ are 2-cells, the diagram in Figure 5.16 commutes.*

As with Lemmas 2.6 and 5.13, this proof is a rotation of the proof of Theorem 2.11. The string diagram proof of Theorem 2.11 in either the bicategorical or symmetric monoidal version amounts to sliding disconnected strings past each other.

Proof. Using the evaluation and coevaluation from Lemma 5.13 (see Figures 5.14A and 5.14B), the right vertical composite in Figure 5.17 is the expansion of $\text{tr}(f \otimes g)$. This is also the right map in Figure 5.16. The left composite in Figure 5.17 is the expansion of the left map in Figure 5.16.

The square regions in Figure 5.16 commute by naturality of the symmetry isomorphism and the regions containing the shadow isomorphism θ commute by definition (Figure 5.8B).

Finally, for endomorphisms of the unit, composition can be identified with the monoidal product using the Eckman-Hilton argument. \square

The following is another simple consequence of Lemma 5.13.

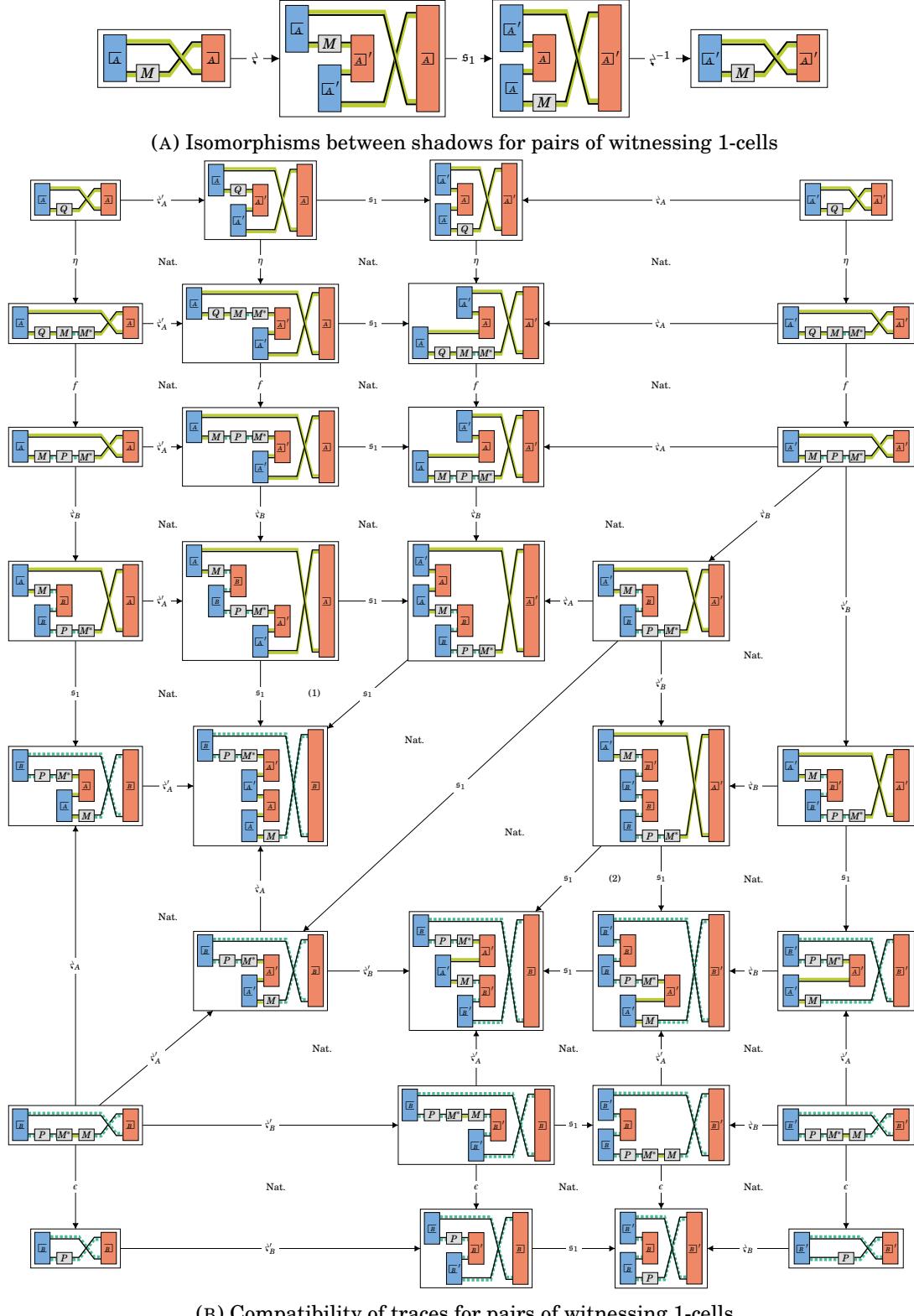


FIGURE 5.12. Compatibility of shadows and traces with different pairs of witnessing 1-cells. (Proposition 5.9)

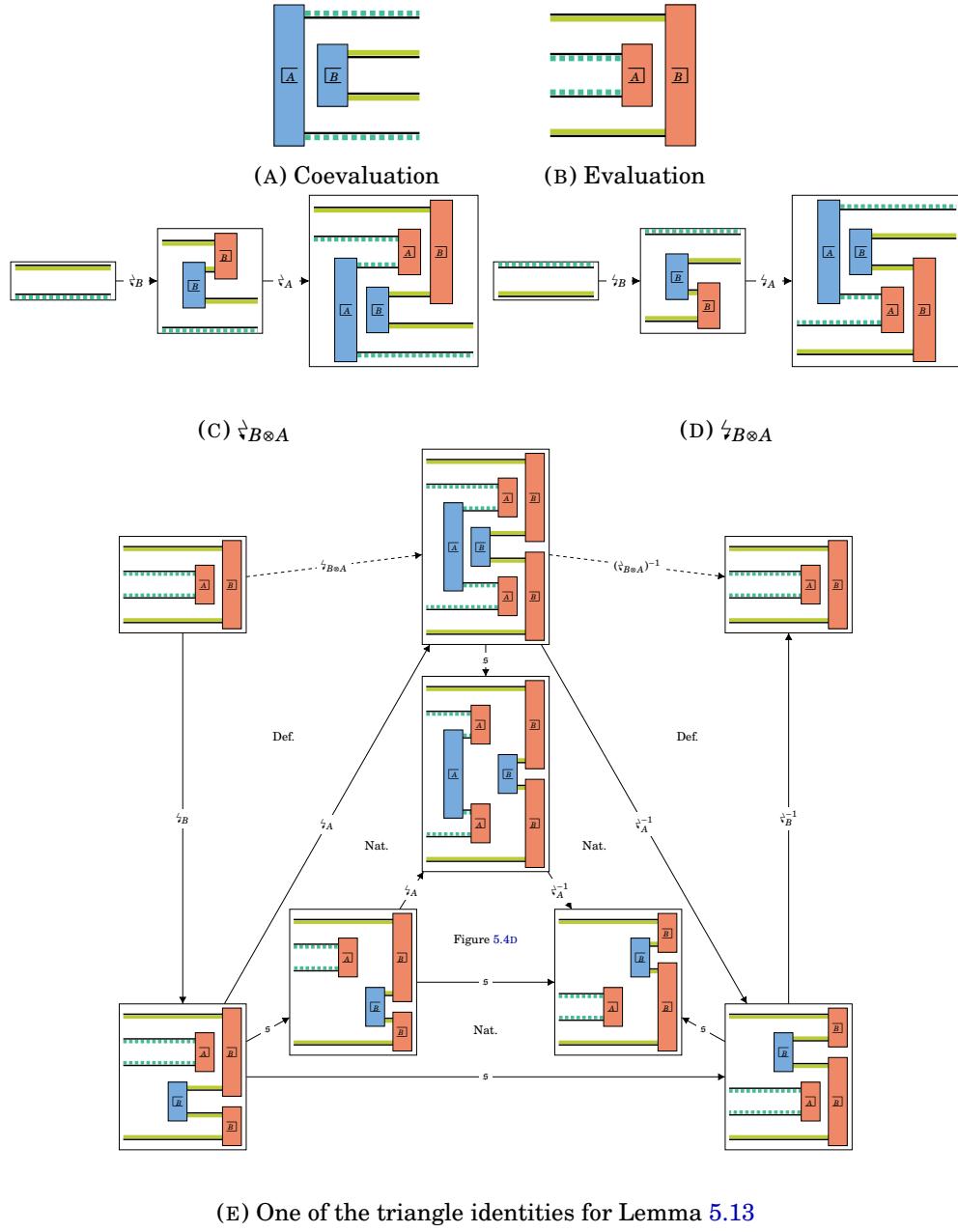


FIGURE 5.14. Compatibility of the monoidal product and 1-dualizability (Lemma 5.13)

Lemma 5.18. *If A and B are 1-dualizable 0-cells, the Euler characteristic of $\Gamma \in \mathcal{B}(A, B)$ (Example 4.7) is the isomorphism ς_3 from Figure 4.18.*

Proof. In the diagram in Figure 5.19 the Euler characteristic of Γ is the composite consisting of the left, bottom, and right maps. The top is the symmetry map s_3 . Five of the regions in the diagram in Figure 5.19 commute by naturality of the symmetry maps. These are all labeled by Nat. Two of the squares commute by the compatibility between \diamond and \circ from Figures 5.4C and 5.4D and are labeled as such. The remaining two regions, labeled (1) and (2), are examples of [GO13, Thm. 1.25]. \square

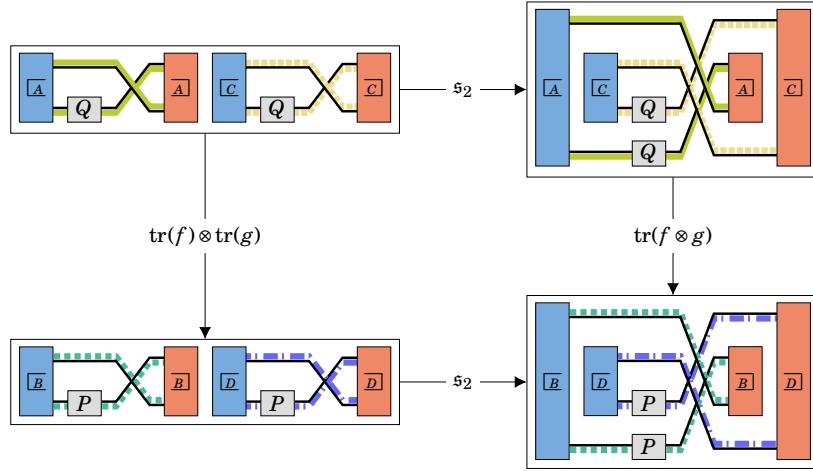


FIGURE 5.16. Compatibility of the trace and 1-dualizability (Proposition 5.15)

6. 2-DUALIZABILITY

As in [CP22] we are interested in examples where $\langle\langle N \rangle\rangle$ is dualizable for a dualizable 1-cell N . Given our motivating examples, the most relevant way to achieve this is to impose another condition on our 0-cells.

In the definition of 1-dualizability, we have coevaluation and evaluation 1-cells and the cells in (5.3) and (5.2) that witness the usual triangle equalities. For 2-dualizability, the extra input will be that the 1-cells $I \rightarrow A \otimes A^\vee$ and $A^\vee \otimes A \rightarrow I$ are *themselves* dualizable in the ambient bicategory.

A crucial consequence of 1-dualizability for every 0-cell in a symmetric monoidal bicategory is the existence of a shadow functor (Proposition 5.7). Similarly, a crucial consequence of 2-dualizability for 0-cells is the dualizability of $\langle\langle U_A \rangle\rangle$ in the monoidal category $\mathcal{B}(I, I)$ (Corollary 6.4) giving the pairing and “copairing” maps. In this section we also prove Goal 3.4 as Theorem 6.13. The only differences between these statements are in the notation used.

Definition 6.1. [CP22, 6.15, 6.16] A 1-dualizable 0-cell A is **2-dualizable** if the witnessing 1-cells $(\underline{A}, \overline{A})$ satisfy any of the following equivalent conditions:

- i. \underline{A} and \overline{A} are left dualizable
- ii. \underline{A} is left and right dualizable
- iii. \underline{A} and \overline{A} are right dualizable
- iv. \underline{A} is left and right dualizable

In addition to the 1-cells \underline{A} and \overline{A} and 2-cells \swarrow and \searrow , a 2-dualizable 0-cell has 1-cells \underline{A}^* and \overline{A}^* , the right duals of \underline{A} and \overline{A} , and four 2-cells illustrated in Figures 6.2A through 6.2D. These 2-cells also have to satisfy the triangle identities. In this case, the four composites in Figures 6.2E and 6.2F are identity 2-cells.

This is essentially a special case of *fully dualizable* objects in an n -category [SP09, Lur09].

Example 6.3. i. A dg- k -algebra B is **proper** if B is perfect as an object of $D(k)$ (i.e. B is perfect when regarded as a chain complex of k modules). It is **smooth** if B is perfect as an object of $D(B \otimes_k^L B^{\text{op}})$ [Toë09, 2.3].

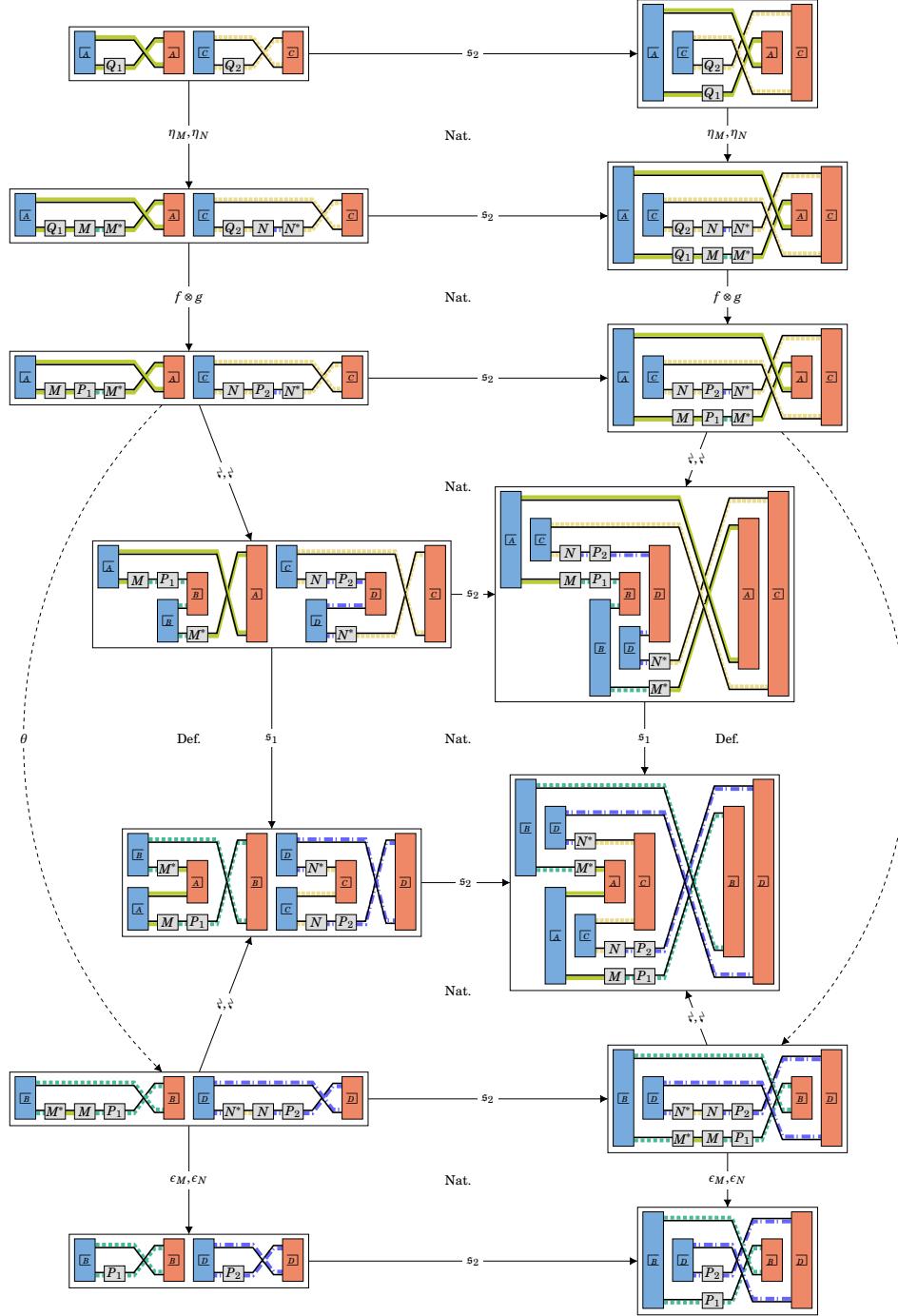
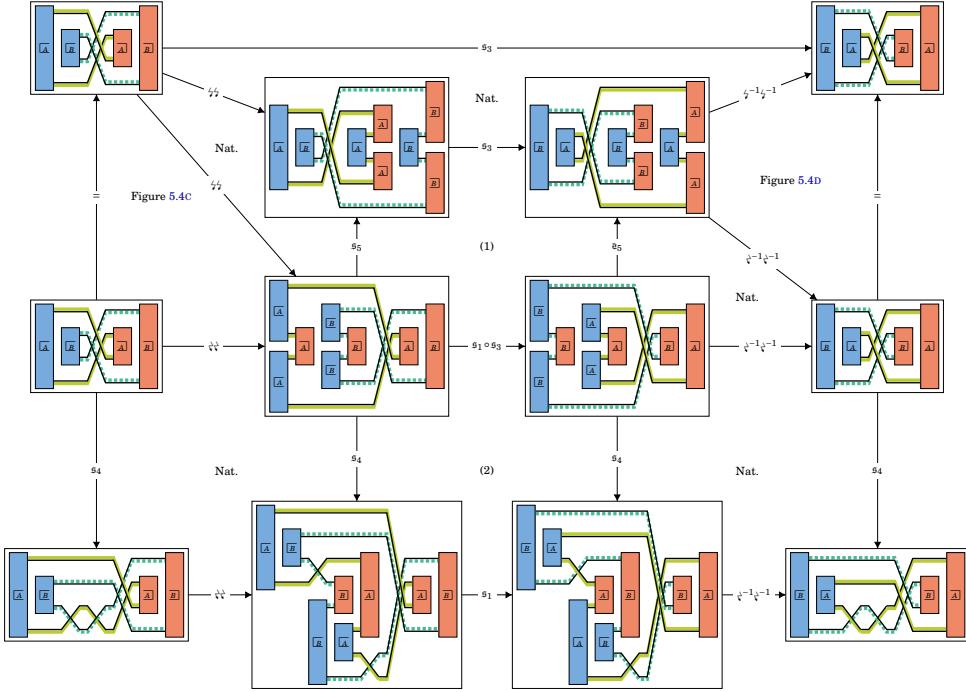
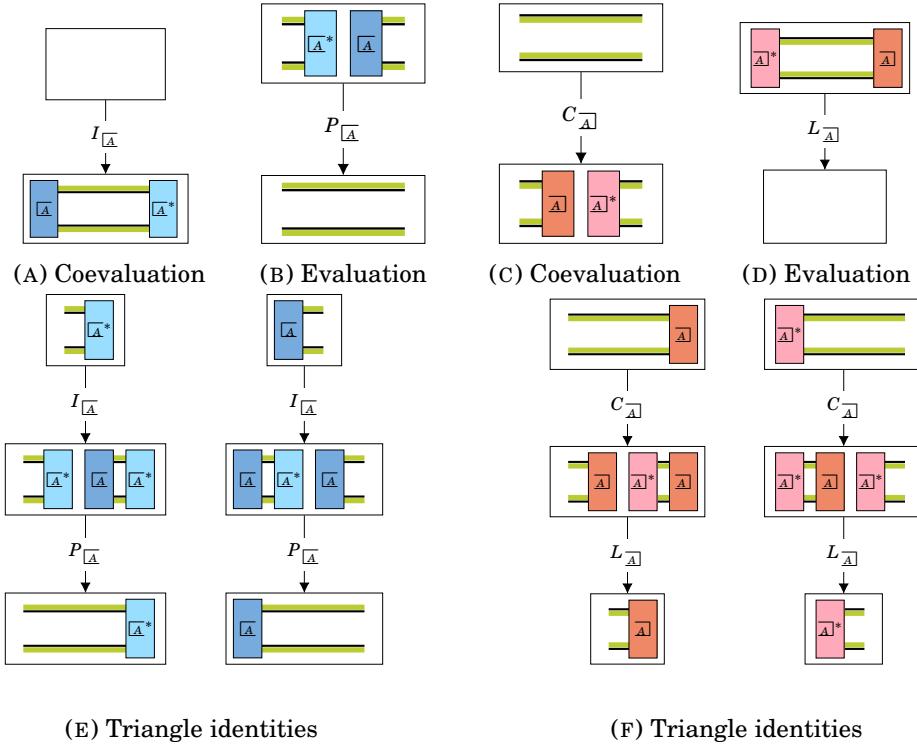


FIGURE 5.17. The diagram that expands Figure 5.16.

ii. If \mathcal{A} is a dg-category (respectively spectral category), the unit 1-cell $U_{\mathcal{A}}$ can be regarded as a $(k, \mathcal{A} \otimes \mathcal{A})$ -bimodule (respectively $(\mathbb{S}, \mathcal{A} \otimes \mathcal{A})$ -bimodule). Denote this module $\vec{U}_{\mathcal{A}}$.

A dg-category (respectively spectral category) \mathcal{A} is **proper** if $\vec{U}_{\mathcal{A}}$ is left dualizable. It is **smooth** if $\vec{U}_{\mathcal{A}}$ is right dualizable. For spectral or dg-categories, \mathcal{A} is 2-dualizable if and only if it is smooth and proper. See [CT12, Thm. 5.8].

FIGURE 5.19. Expanded Euler characteristic comparison for Γ FIGURE 6.2. Coevaluation, evaluation and triangle identities for right dualizability of 1-cells in $\mathcal{B}(1, A \otimes B)$ and $\mathcal{B}(A \otimes B, 1)$. Compare to Figure 4.6.

The following result is a consequence of Lemma 2.6 and the diagrams in Figure 4.9.

Corollary 6.4 (Goal 3.2, [CP22, 5.2 and 6.18]). *If A is a 2-dualizable 0-cell in a monoidal bicategory \mathcal{B} then $\langle\langle U_A \rangle\rangle$ is dualizable in the monoidal category $\mathcal{B}(I, I)$*

Proof. By definition, $\langle\langle U_A \rangle\rangle$ is the composite of \underline{A} , U_A , Γ , and \underline{A} in Figure 5.8A. In any bicategory, U_A is dualizable with dual U_A and the coevaluation and evaluation are unit isomorphisms. If A is 2-dualizable \underline{A} and \underline{A} are dualizable by definition. The 1-cell Γ is dualizable by Figure 4.9. Then the result follows from Lemma 2.6. \square

We can describe the coevaluation and evaluation for $\langle\langle U_A \rangle\rangle$ in terms of those for each of the pieces \underline{A} , U_A , Γ , and \underline{A} . The coevaluation is the composite in Figure 6.5A and the evaluation is the composite in Figure 6.5B. Note that we can omit the coevaluation and evaluation for U_A since it is the unit isomorphism. Following the discussion in Section 3 we will call the coevaluation a “copairing” and the evaluation a “pairing”.

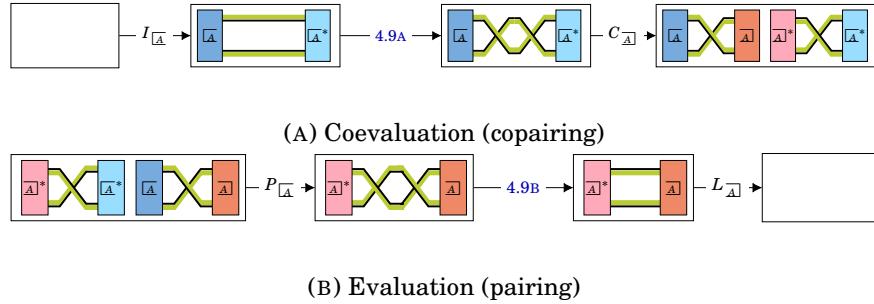


FIGURE 6.5. Copairing and pairing maps

Lemma 6.6 is a fundamental observation from [CP22]. Most of this result appeared in the proof of [CP22, Theorem 6.11], but since the definition of 1-dualizability is slightly different in this paper (Remark 5.5) we will formally state and prove the additional conditions required here.

Lemma 6.6. *If A is 2-dualizable then A^\vee is 1-dualizable with witnessing 1-cells $(\underline{A})^*$, $(\underline{A})^*$.*

Proof. The required 2-cells are defined in Figures 6.7A and 6.7C and their inverses are defined in Figures 6.7B and 6.7D. See [CP22, Figure 18] for the proof that these are the inverses.

The diagram in Figure 6.7 is an expansion of Figure 5.4C where \swarrow and \searrow are replaced by \nwarrow and \nearrow respectively. The regions labeled by Def. commute by definition (Figures 6.7A and 6.7D). The nine regions labeled by Nat. commute by naturality. The remaining two regions are triangle identities for the dualizability of \underline{A} and \underline{A} .

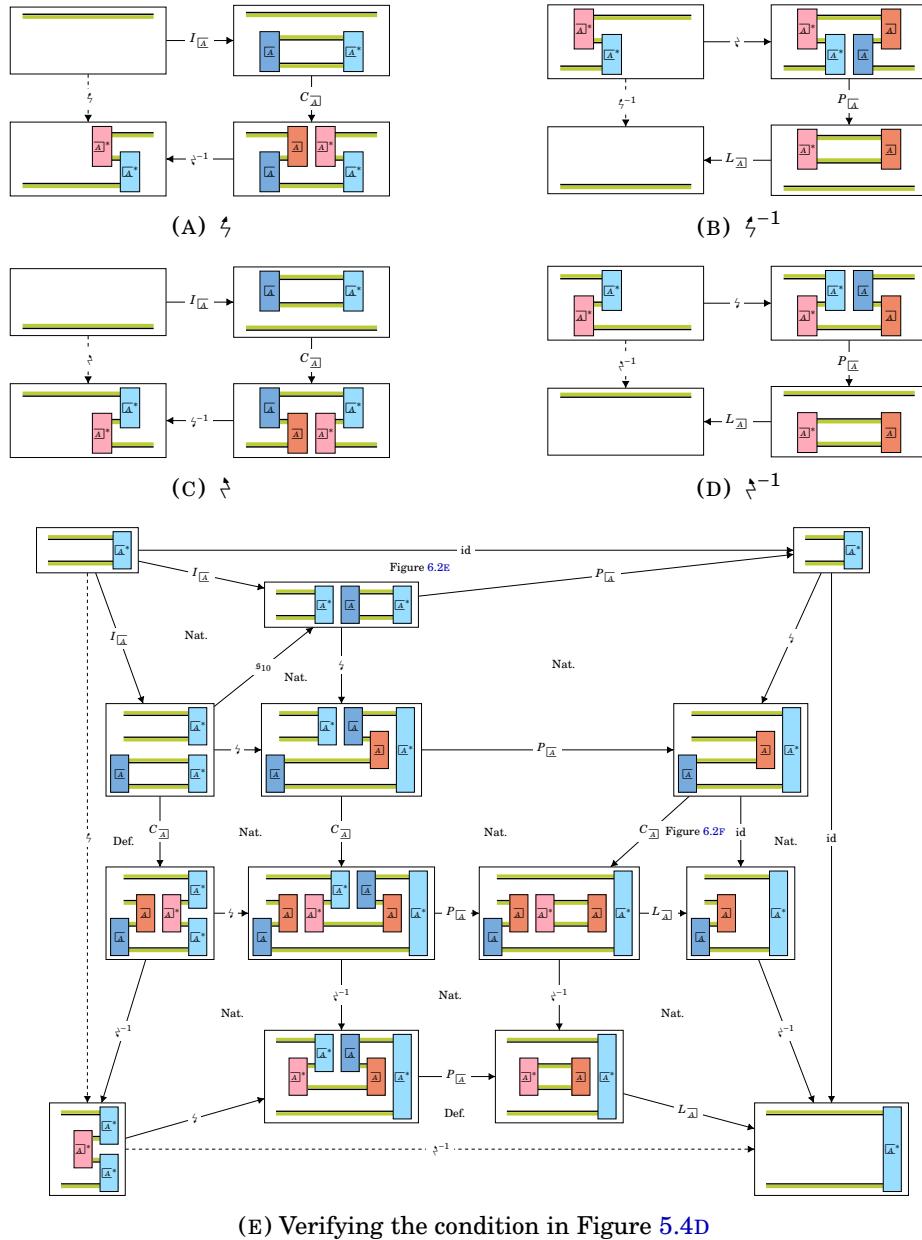
The diagram showing Figure 5.4D holds is similar. \square

Lemmas 5.13 and 6.6 give a preferred pair of 1-cells recognizing the 1-dualizability of $A^\vee \otimes A$ if A is 1-dualizable. See Figure 6.8. Whenever A is 2-dualizable we will use these 1-cells to define the shadow for a 1-cell in

$$\mathcal{B}(A^\vee \otimes A, A^\vee \otimes A).$$

Proposition 6.9. *If A is 2-dualizable then the Euler characteristics of \underline{A} and \underline{A} are defined and the triangles in Figure 6.10 commute.*

Proof. Using the 2-cells in Figure 6.8, the Euler characteristic of \underline{A} is the left composite in Figure 6.11. This may feel a little less complicated than the usual shadow isomorphism since there are unit 0-cells. The right composite is the copairing.

FIGURE 6.7. 1-dualizability of A^V (Lemma 6.6)FIGURE 6.8. 1-cells recognizing the 1-dualizability of $A^V \otimes A$

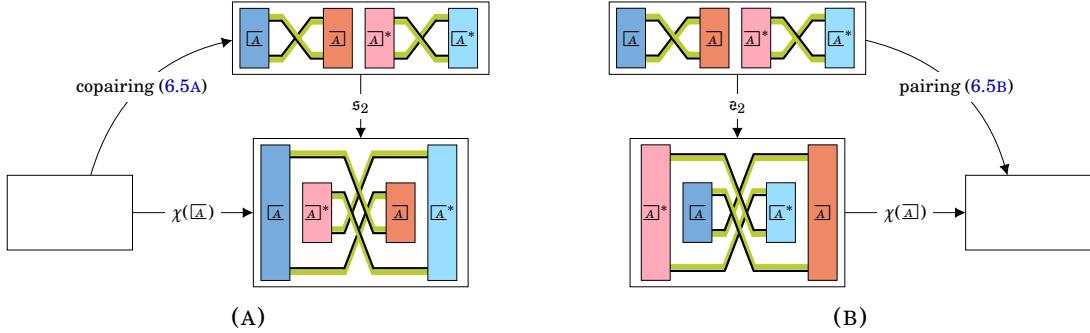


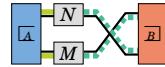
FIGURE 6.10. Compatibility between pairing/copairing and Euler characteristics (Proposition 6.9)

Note that the left composite contains the map $P_{\underline{A}}$ and the right composite contains the map $C_{\underline{A}}$. The primary goal of this diagram is to allow these maps to slide past each other. To allow these maps to be applied simultaneously we introduce a new pair of \underline{A} and \underline{A} using $\check{\wedge}$ and rearrange the 1-cells to obtain the diagram in the node in the middle of the diagram with the thicker border. To the right of this node is a commutative square where the orders of $P_{\underline{A}}$ and $C_{\underline{A}}$ are exchanged. This square is on a upper left to lower right diagonal of five naturality squares that pulls the map $C_{\underline{A}}$ across the diagram. It is also on a lower left to upper right diagonal of four naturality that pulls $P_{\underline{A}}$ across the diagram. These maps both vanish when they encounter the map $I_{\underline{A}}$. The map $C_{\underline{A}}$ can be replaced by instances of $\check{\wedge}$ and $\check{\wedge}$ when it encounters $I_{\underline{A}}$. The maps $P_{\underline{A}}$ and $I_{\underline{A}}$ are related by the triangle diagrams for \underline{A} and so the composite is the identity map.

Aside from many more naturality squares, there are five squares labeled (1)–(5) that commute by [GO13, Thm. 1.25]. The top commuting triangle labeled with $=$ is included to simplify the formating of the triangle directly below it. The vertical maps are the same. The remaining three regions labeled by Figures and Lemma commute by the relevant result.

The proof for the pairing is Eckman-Hilton dual to that for the pairing and the relevant diagram is a 180 degree rotation. Since it is pretty elaborate we include it in its entirety. See Figure 6.12. \square

For 2-cells $f: M \rightarrow M$ and $g: N \rightarrow N$ consider the endomorphism of



induced by f and g (and the identity maps of \underline{A} , \underline{B} , and Γ). This endomorphism will be written

$$\text{id}_{\underline{A}} \circ (f \otimes g) \circ \text{id}_{\Gamma} \circ \text{id}_{\underline{B}}$$

The following is the generalization of Theorem 1.1.

Theorem 6.13. *Suppose A and B are 2-dualizable 0-cells and $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(A^\vee, B^\vee)$ are dualizable 1-cells. If $f: M \rightarrow M$ and $g: N \rightarrow N$ are 2-cells the diagram 6.14 commutes.*

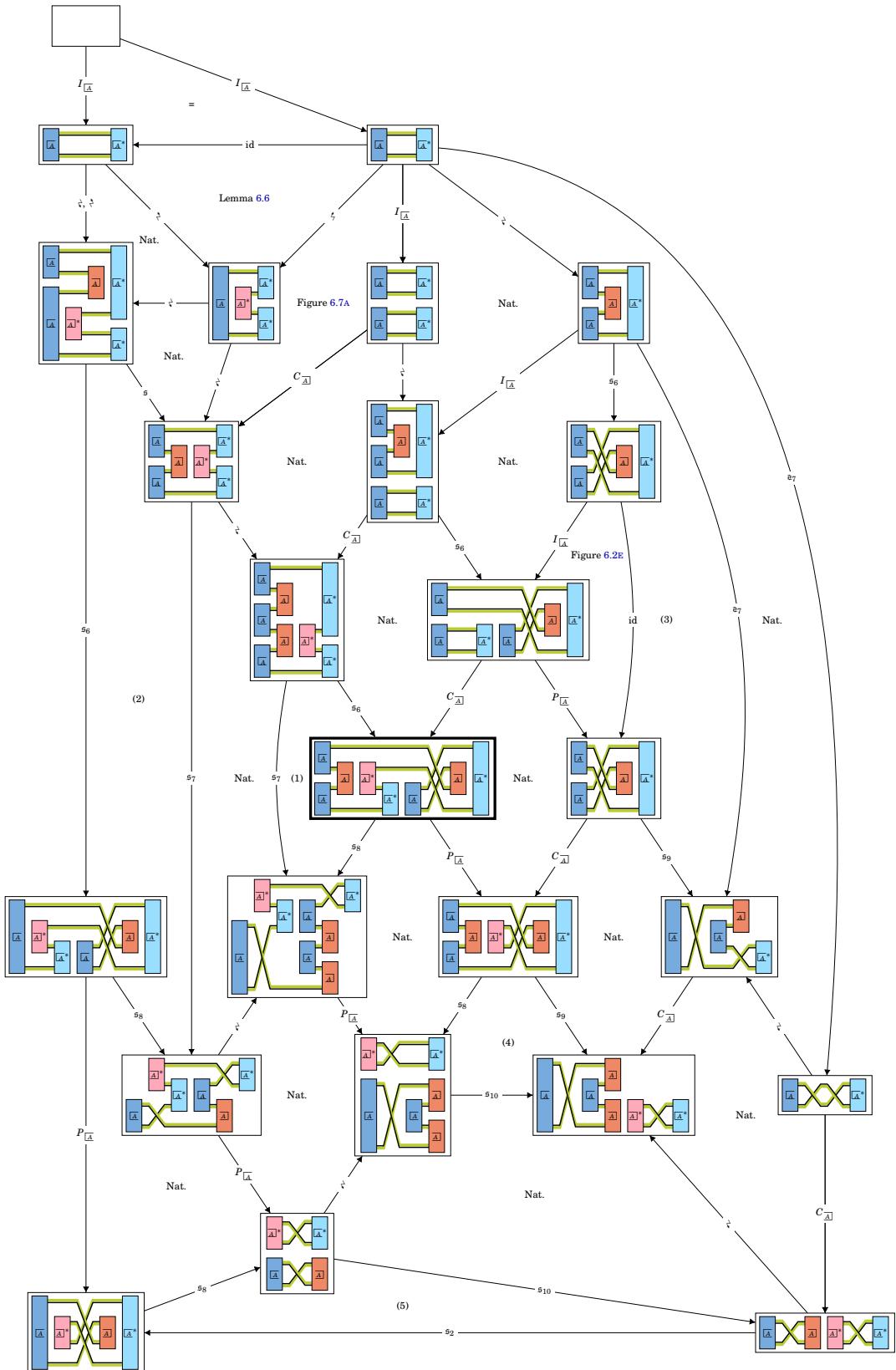


FIGURE 6.11. The expanded version of Figure 6.10A. The right composite is the copairing and the left composite is the Euler characteristic of C (Proposition 6.9)

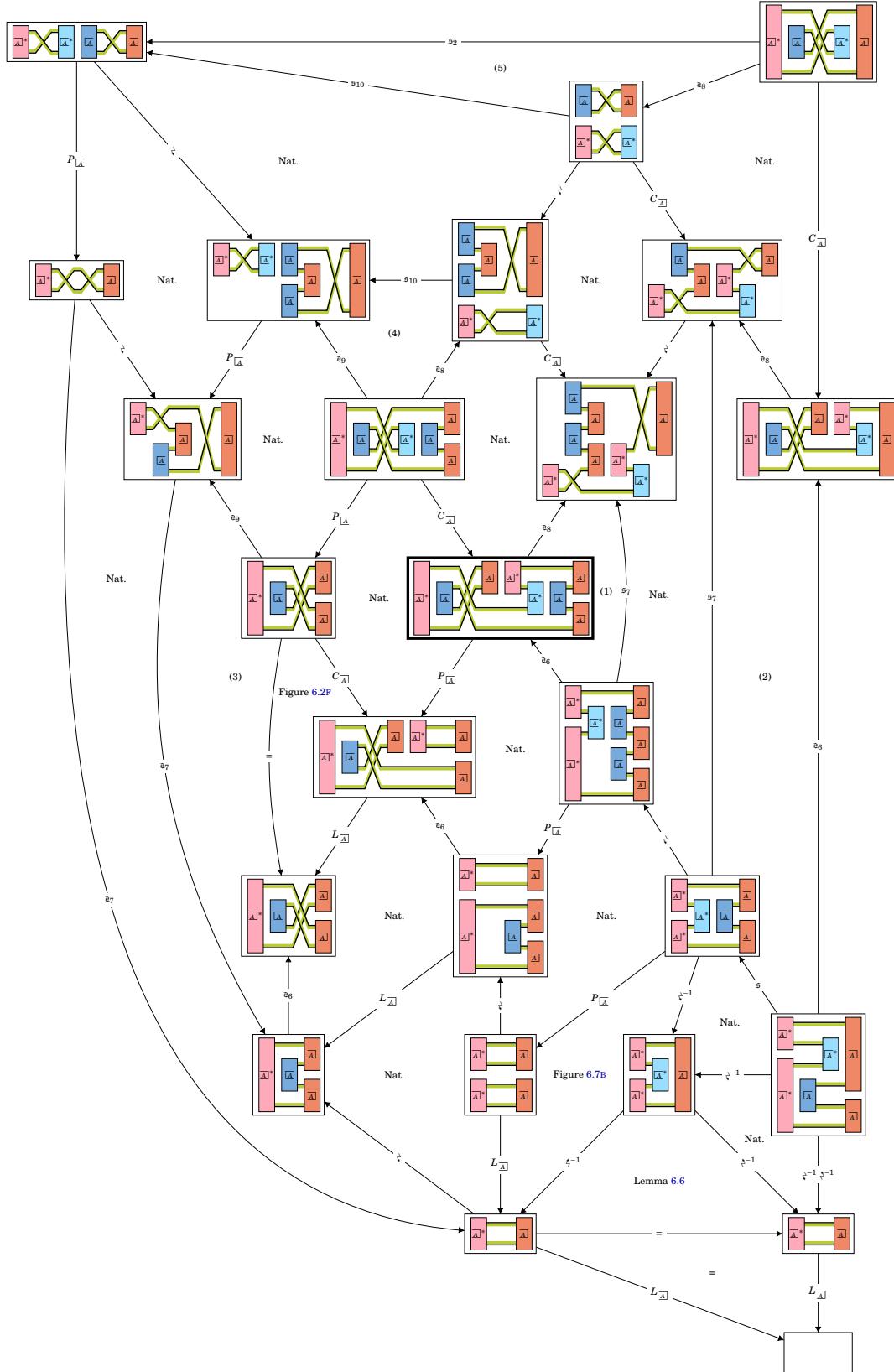


FIGURE 6.12. The expanded version of Figure 6.10B. The left map is the pairing and the right map is the Euler characteristic of E (Proposition 6.9)

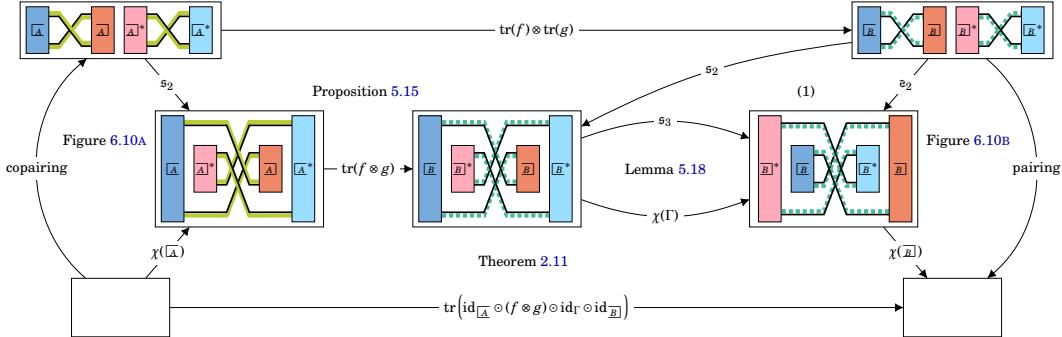
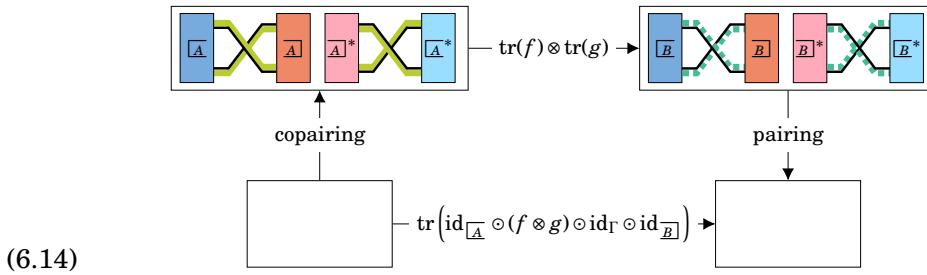


FIGURE 6.15. Expansion of Eq. (6.14) to verify the compatibility between trace, pairing and copairing (Theorem 6.13)



Proof. This theorem is the culmination of many of the preceding results. The first relevant result is Theorem 2.11 which we use to compute the trace of

$$\text{id}_{\boxed{A}} \circ (f \otimes g) \circ \text{id}_{\Gamma} \circ \text{id}_{\boxed{B}}$$

in terms of the trace of each piece. This is the bottom region in the expansion of the diagram in (6.14) to the diagram in Figure 6.15. The next result we use is Proposition 5.15 which relates the trace of $f \otimes g$ and the trace of f and g . This is the top left region in Figure 6.15. This is closely related to the oval region of Figure 6.15 that recognizes the Euler characteristic of Γ as a symmetry map (Lemma 5.18). Note that the region here differs from the diagram in Lemma 5.18 since \boxed{A}^* and \boxed{A}^* are replaced by \boxed{A}^* and \boxed{A}^* . This is permissible since Lemma 6.6 shows \boxed{A}^* is a choice of \boxed{A}^* .

The left and right regions are the conclusions of Proposition 6.9 that verifies the compatibility between the Euler characteristic of \boxed{A} and the copairing and the Euler characteristic of \boxed{A} and the pairing. The remaining region, labeled with (1), commutes by [GO13, Thm. 1.25]. \square

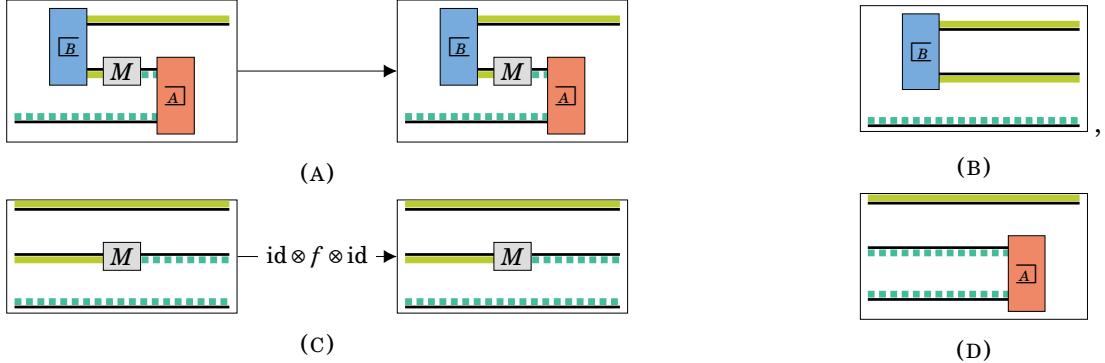
7. SERRE DUALITY AND SHKLYAROV'S THEOREM

It should now be clear that there are a considerable number of categorical operations in symmetric monoidal bicategories given by geometric operations. Thinking in a “cobordism hypothesis” way, this makes perfect sense. One more geometric move we can make in the presence of 1-dualizability is “putting a kink” in a bimodule, or turning a (A, B) -bimodule into a (B^V, A^V) -bimodule. More precisely, suppose A and B are 1-dualizable 0-cells. For a 1-cell $M \in \mathcal{B}(A, B)$, the Serre dual of M , denoted $\boxed{A} M \boxed{B} \in \mathcal{B}(B^V, A^V)$, is the composite in Figure 7.1B. For a 2-cell $f: M \rightarrow N$, let $\boxed{A} f \boxed{B}$ be the similar composite using the identity maps of \boxed{A} and \boxed{B} .

This Serre dual gives us a way of realizing one of our original goals:



FIGURE 7.1. Serre dual

FIGURE 7.4. Serre dual of a map $f: M \rightarrow M$

Lemma 7.2 (Goal 3.3). *If A and B are 2-dualizable and M is right dualizable then $\overline{A} M \overline{B}$ is right dualizable.*

Proof. By assumption M is right dualizable. The 1-cells \overline{A} and \overline{B} are right dualizable since A and B are 2-dualizable. Then Lemma 2.6 implies $\overline{A} M \overline{B}$ is right dualizable. \square

The following result is only used in the motivating result in Section 8.1, but it is an expected question following the lemma above so we address it here.

Corollary 7.3. *If $M \in \mathcal{B}(A, B)$ is right dualizable, the trace of the Serre dual of $f: M \rightarrow M$ is the composite*

$$\langle\langle U_{B^\vee} \rangle\rangle \xrightarrow{\text{id} \otimes \text{copairing}} \langle\langle U_{B^\vee} \rangle\rangle \otimes \langle\langle U_A \rangle\rangle \otimes \langle\langle U_{A^\vee} \rangle\rangle \xrightarrow{\text{id} \otimes \text{tr}(f) \otimes \text{id}} \langle\langle U_{B^\vee} \rangle\rangle \otimes \langle\langle U_B \rangle\rangle \otimes \langle\langle U_{A^\vee} \rangle\rangle \xrightarrow{\text{pairing} \otimes \text{id}} \langle\langle U_{A^\vee} \rangle\rangle$$

Proof. The Serre dual of f is a map of the form in Figure 7.4A. Using Theorem 2.11, its trace is the composite of three traces. The first is the trace of identity map of the 1-cell in Figure 7.4B. The second is the trace of the map $f: M \rightarrow M$ and identity maps as in Figure 7.4C. The third is the trace of the identity map of the 1-cell in Figure 7.4D. This identifies the trace of the Serre dual of f with the top composite in 7.5.

Proposition 5.15 allows us to expand each of these traces into a monoidal product of two traces. The trace of identity map of the 1-cell in Figure 7.4B becomes the traces of the identity maps of each of the 1-cells in the diagram in Figure 7.4B. The same simplification holds for the 1-cells in Figure 7.4D. For the trace of the map in Figure 7.4C we divide this into two traces, one for the identity map and f and one for an identity map of the unit 1-cell. Note that the trace of the identity map of the unit 1-cell is the identity. We repeat this simplification for the bottom center region in 7.5 to obtain the bottom arrow of the diagram. Both regions labeled by Proposition 6.9 are immediate consequences of that result and the observation that the Euler characteristic of a unit 1-cell is an identity map.

Finally, the region labeled (1) is another application of [GO13, Thm. 1.25]. \square

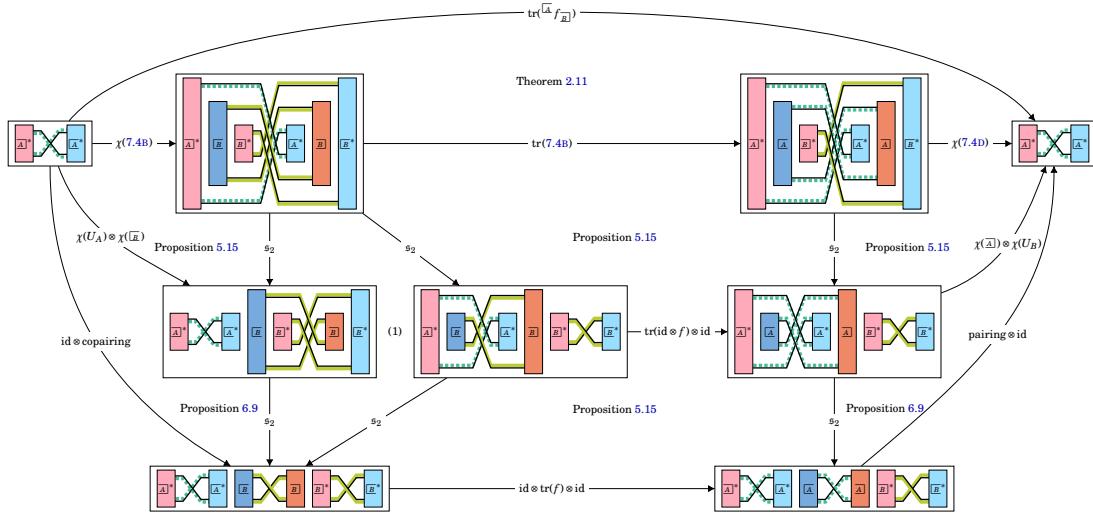


FIGURE 7.5. Trace of Serre dual

In this form, Serre duals arise naturally within the shadow isomorphisms.

Lemma 7.6. Suppose A and B are 1-dualizable.

i. If M is a 1-cell in $\mathcal{B}(A, B)$ and N is a 1-cell in $\mathcal{B}(B, A)$, there is a natural isomorphism

$$(7.7) \quad \langle\langle M \circ N \rangle\rangle \cong \overline{A} \odot (M \otimes \overline{B} N \overline{A}) \odot \Gamma \odot \overline{B}.$$

ii. If M is a 1-cell in $\mathcal{B}(A, B)$ and N is a 1-cell in $\mathcal{B}(A^\vee, B^\vee)$, there is a natural isomorphism

$$(7.8) \quad \langle\langle M \circ \overline{A^\vee} N \overline{B^\vee} \rangle\rangle \cong \overline{A^\vee} \odot (N \otimes M) \odot \Gamma \odot \overline{B^\vee}.$$

Proof. The composite in Figure 7.9A of the map $\check{\gamma}$ in (5.2) and one of the pseduonaturality maps for Γ defines the maps in (7.7). Since both maps in the composite are natural by assumption, the composite is natural.

The map in (7.8) is the composite in Figure 7.9B of $\check{\gamma}$ in (5.2) and one of the pseduonaturality maps for Γ . It is also natural by hypothesis. \square

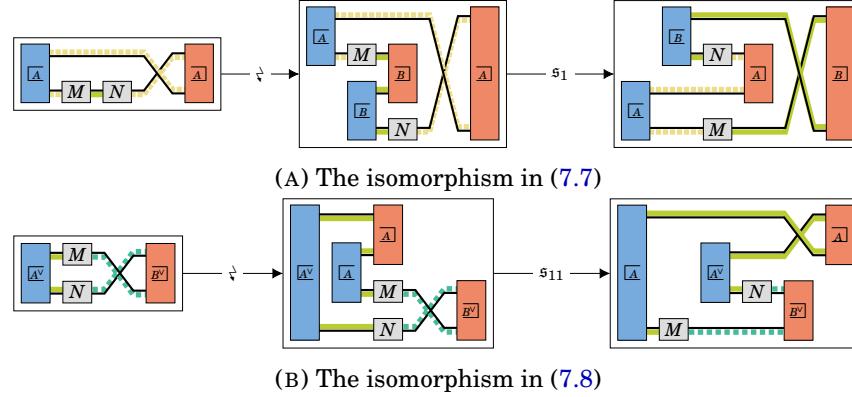


FIGURE 7.9. The isomorphisms relating compositions in Lemma 7.6.

Making some of the naturality statements in Lemma 7.6 explicit, if $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(B, A)$ and $f: M \rightarrow M$ and $g: N \rightarrow N$ the following diagram commutes.

$$(7.10) \quad \begin{array}{ccc} \langle\langle M \odot N \rangle\rangle & \xrightarrow{(7.7)} & \underline{A} \odot \left(M \otimes \underline{B} N \underline{A} \right) \odot \Gamma \odot \underline{B} \\ \downarrow \langle\langle f \circ g \rangle\rangle & & \downarrow \text{id}_{\underline{A}} \circ \left(f \otimes \underline{B} g \underline{A} \right) \circ \text{id}_{\Gamma} \circ \text{id}_{\underline{B}} \\ \langle\langle M \odot N \rangle\rangle & \xrightarrow{(7.7)} & \underline{A} \odot \left(M \otimes \underline{B} N \underline{A} \right) \odot \Gamma \odot \underline{B} \end{array}$$

If $M \in \mathcal{B}(A, B)$, $N \in \mathcal{B}(A^\vee, B^\vee)$, $f: M \rightarrow M$ and $g: N \rightarrow N$ the following diagram commutes.

$$(7.11) \quad \begin{array}{ccc} \langle\langle M \odot \underline{A}^\vee N \underline{B}^\vee \rangle\rangle & \xrightarrow{(7.8)} & \underline{A}^\vee \odot (N \otimes M) \odot \Gamma \odot \underline{B}^\vee \\ \downarrow \langle\langle f \circ \underline{A}^\vee g \underline{B}^\vee \rangle\rangle & & \downarrow \text{id}_{\underline{A}^\vee} \circ (g \otimes f) \circ \text{id}_{\Gamma} \circ \text{id}_{\underline{B}^\vee} \\ \langle\langle M \odot \underline{A}^\vee N \underline{B}^\vee \rangle\rangle & \xrightarrow{(7.8)} & \underline{A}^\vee \odot (N \otimes M) \odot \Gamma \odot \underline{B}^\vee \end{array}$$

These two commuting squares allow us to compare the traces of the vertical maps.

Corollary 7.12. *Suppose A and B are 2-dualizable 0-cells.*

i. *Let $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(B, A)$ be right dualizable 1-cells. If $f: M \rightarrow M$ and $g: N \rightarrow N$, then*

$$\text{tr}(\langle\langle f \circ g \rangle\rangle) = \text{tr} \left(\text{id}_{\underline{A}} \circ \left(f \otimes \underline{B} g \underline{A} \right) \circ \text{id}_{\Gamma} \circ \text{id}_{\underline{B}} \right)$$

ii. *Let $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(A^\vee, B^\vee)$ be right dualizable 1-cells. If $f: M \rightarrow M$ and $g: N \rightarrow N$, then*

$$\text{tr}(\langle\langle f \circ \underline{A}^\vee g \underline{B}^\vee \rangle\rangle) = \text{tr} \left(\text{id}_{\underline{A}^\vee} \circ (g \otimes f) \circ \text{id}_{\Gamma} \circ \text{id}_{\underline{B}^\vee} \right)$$

Proof. Note that the traces in the statement of this result are symmetric monoidal traces rather than bicategorical.

We first make an elementary observation about symmetric monoidal traces. Suppose X and Y are dualizable objects in a symmetric monoidal category, $h: X \rightarrow Y$ is an isomorphism, and the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{h} & Y \end{array}$$

Then

$$\text{tr}(f) = \text{tr}(h^{-1}hf) = \text{tr}(h^{-1}gh) = \text{tr}(ghh^{-1}) = \text{tr}(g)$$

The first and last equality replace $h^{-1}h$ and hh^{-1} with identity maps. The second equality from the left uses the commutativity of the square. The remaining equality is the invariance of trace under cyclic permutation.

Then the commutative diagram in (7.10) implies the traces of the vertical maps in (7.10) are the same and proves *i*. The identifications in *ii* follow from (7.11) in the same way. \square

We have now reached one of the main results of this paper. It is the generalization of Theorem 1.1 and the manifestation of Goal 3.4.

Theorem 7.13. *Suppose A and B are 2-dualizable 0-cells.*

- i. If $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(B, A)$ are right dualizable 1-cells and $f: M \rightarrow M$ and $g: N \rightarrow N$ then the diagram in Figure 7.14A commutes.
- ii. If $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(A^\vee, B^\vee)$ are right dualizable 1-cells and $f: M \rightarrow M$ and $g: N \rightarrow N$ then the diagram in Figure 7.14B commutes.

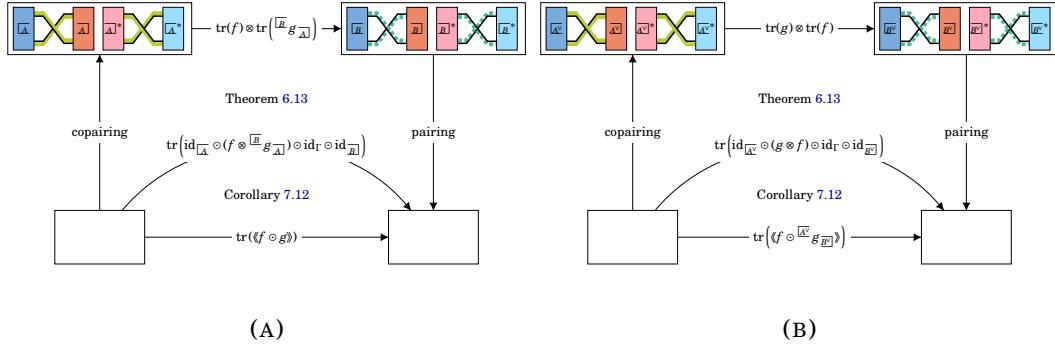


FIGURE 7.14. Compatibilities between trace, pairing, and copairing (Theorem 7.13)

Proof. This theorem is an immediate consequence of Corollary 7.12 and Theorem 6.13. Corollary 7.12 provides the equality between

$$\text{tr}(\langle\langle f \circ g \rangle\rangle) \quad \text{and} \quad \text{tr}\left(\text{id}_{\overline{A}} \circ (f \otimes \overline{B} g_{\overline{A}}) \circ \text{id}_\Gamma \circ \text{id}_{\overline{B}}\right).$$

Theorem 6.13 replaces this second trace by the composite of $\text{tr}(f)$ and $\text{tr}(\overline{B} g_{\overline{A}})$ with the copairing and pairing maps. The proof for *ii* is similar. \square

Theorem 7.13 deserves to be amplified and stated in more specific terms.

Example 7.15. We work in the familiar bicategory of rings and bimodules. In this category the shadow is Hochschild homology. For an (B, A) -bimodule N , the Serre dual in (A^\vee, B^\vee) -bimodules is denoted $\overline{B} N_{\overline{A}}$ as above (assuming A, B are suitably dualizable).

Let A be a k -algebra for some base ring k . The pairings defined in Figure 6.5 become maps

$$\text{HH}(A) \otimes \text{HH}(A^{\text{op}}) \rightarrow \text{HH}(k) \quad \text{HH}(B) \otimes \text{HH}(B^{\text{op}}) \rightarrow \text{HH}(k)$$

If we are given $M \in \mathcal{B}(A, B)$ and $N \in \mathcal{B}(B, A)$, we furthermore have traces induced by the identity maps of M and $\overline{B} N_{\overline{A}}$:

$$\text{HH}(A) \xrightarrow{\chi(M)} \text{HH}(B) \quad \text{HH}(A^\vee) \xrightarrow{\chi(\overline{B} N_{\overline{A}})} \text{HH}(B^\vee)$$

which we can tensor together to a map

$$\text{HH}(A) \otimes \text{HH}(A^\vee) \xrightarrow{\chi(M) \otimes \chi(\overline{B} N_{\overline{A}})} \text{HH}(B) \otimes \text{HH}(B^\vee).$$

In the presence of 2-duality, both $\text{HH}(A)$ and $\text{HH}(B)$ themselves become dualizable and we can compose with the copairing and pairing maps to obtain

$$\text{HH}(k) \rightarrow \text{HH}(A) \otimes \text{HH}(A^\vee) \xrightarrow{\chi(M) \otimes \chi(\overline{B} N_{\overline{A}})} \text{HH}(B) \otimes \text{HH}(B^\vee) \rightarrow \text{HH}(k).$$

The theorem says that this can be computed in one step: consider $M \odot_B N$ and take $\mathrm{HH}(A; M \odot_B N)$. This, itself, is dualizable as a k -algebra and so we can compute

$$\mathrm{HH}(k) \rightarrow \mathrm{HH}(A; M \odot_B N) \otimes \mathrm{HH}(A; M \odot_B N)^\vee \rightarrow \mathrm{HH}(k).$$

This corresponds to the toroidal trace diagrams in the introduction.

Remark 7.16. This also illustrates quite clearly why Serre duality needs to rear its head: if we have M an (A, B) -bimodule and N a (B, A) -bimodule, we can combine them by either looking at $M \odot_B N$ or $N \odot_A M$. In either case, in order to obtain pairings from their respective traces, we have to obtain (A^\vee, B^\vee) -bimodules.

8. THE MOTIVATING RESULTS

In this section we use our abstract categorical results to prove the results that motivated this work originally, Shklyarov's Hirzebruch-Riemann-Roch theorem for DG-algebras [Shk13] and Petit's Riemann-Roch theorem [Pet13]. Both theorems essentially follow from Theorem 7.13, however some work is needed to identify pairings defined in each paper with our pairing.

8.1. Shklyarov's Hirzebruch-Riemann-Roch theorem for DG-algebras. In this subsection we prove Shklyarov's Hirzebruch-Riemann-Roch theorem. It is an easy consequence of our theorems, but in order to state the theorem in his form, we restate his pairing and invariants. All of these are defined in terms of familiar properties of Hochschild homology and DG-categories: the shuffle isomorphism and the Morita invariance of Hochschild homology.

To define a pairing as Shklyarov does, we recall the following basic, but powerful fact about Hochschild homology: If A is a dg- k -algebra there is a canonical isomorphism

$$(8.1) \quad \mathrm{HH}(\mathrm{Perf} A) \simeq \mathrm{HH}(A).$$

given by Morita equivalence. The map $\mathrm{HH}(A) \rightarrow \mathrm{HH}(\mathrm{Perf} A)$ is induced by the inclusion. See [CP19] for (much) more detail.

Shklyarov defines a pairing $\cup: \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \rightarrow \mathrm{HH}(k)$ as follows.

Definition 8.2. [Shk13, (3.2)] We define

$$\cup: \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \rightarrow \mathrm{HH}(k)$$

as the composite

$$\mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \xrightarrow{\mathrm{sh}} \mathrm{HH}(A \otimes A^{\mathrm{op}}) \xrightarrow{\cong} \mathrm{HH}(\mathrm{Perf}_{A \otimes A^{\mathrm{op}}}) \xrightarrow{- \otimes_{A \otimes A^{\mathrm{op}}} A_k} \mathrm{HH}(\mathrm{Perf}_k) \xrightarrow{\cong} \mathrm{HH}(k)$$

where the first map is the shuffle map. The maps denoted with \cong are the isomorphism (or its inverse) in (8.1). The map denoted $- \otimes_{A \otimes A^{\mathrm{op}}} A_k$ tensors a module or homomorphism in $\mathrm{Perf}_{A \otimes A^{\mathrm{op}}}$ with $_{A \otimes A^{\mathrm{op}}} A_k$ over $A \otimes A^{\mathrm{op}}$.

Our work in [CP19] (in particular the Hochschild homology version of [CP19, Ex. 5.12]) immediately implies that our pairing agrees with Shklyarov's.

Lemma 8.3. *The pairing from Figure 6.5 agrees with the pairing in Definition 8.2.*

Proof. The proof is a diagram chase:

$$\begin{array}{ccc} \mathrm{HH}(\mathrm{Perf}_{A \otimes A^{\mathrm{op}}}) & \xrightarrow{\otimes_{A \otimes A^{\mathrm{op}}} A} & \mathrm{HH}(\mathrm{Perf}_k) \\ \cong \uparrow & & \cong \uparrow \\ \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) & \xrightarrow{\mathrm{sh}} & \mathrm{HH}(A \otimes A^{\mathrm{op}}) \xrightarrow{\chi(\boxed{A})} \mathrm{HH}(k) \end{array}$$

The commutativity of the square is a special case of [CP19, Lemma 5.8]. The fact that the composite along the bottom is the pairing from Figure 6.5 is Proposition 6.9. \square

Shklyarov defines invariants $\text{eu}(L)$ and $\text{Eu}(L)$ closely related to our trace invariants, but not *exactly* the same. Here are his definitions.

Definition 8.4. For a perfect dg A module L [Shk13, §1.3] defines $\text{Eu}(L) \in \text{HH}_0(\text{Perf } A)$ to be the image of L in $\text{HH}_0(\text{Perf } A)$ and $\text{eu}(L)$ to be the image of $\text{Eu}(L)$ under the isomorphism in (8.1).

We show agreement between these invariants and the Euler characteristics above.

Lemma 8.5. *The image of $1 \in k$ under the composite*

$$k \rightarrow \text{HH}_0(k) \xrightarrow{\chi(L)} \text{HH}(A)$$

is $\text{eu}(L)$.

The map $k \rightarrow \text{HH}_0(k)$ is the isomorphism that recognizes the quotient on $\text{HH}_0(k)$ is by the zero submodule.

Proof. Consider the diagram

$$(8.6) \quad \begin{array}{ccc} k & \longrightarrow & \text{HH}_0(\text{Perf}_A) \\ \downarrow & & \uparrow \cong \\ \text{HH}_0(k) & \xrightarrow{\chi(L)} & \text{HH}_0(A) \end{array}$$

where the top arrow sends 1 to the class of L in $\text{HH}_0(\text{Perf}_A)$. The diagram in (8.6) commutes by [CP19, Lemma 5.8]. \square

Assembling the results establishing the agreement between the various pairings and Euler characteristics, Shklyarov's HRR theorem follows easily from Theorem 7.13, but we need to say something about a subtle point. Following Shklyarov, we use the functor

$$(8.7) \quad \text{Perf}_A \rightarrow \text{Perf}_{A^{\text{op}}}$$

given by $M \mapsto \text{hom}_A(M, A)$. This functor hides some very common identifications.

- The functor $M \mapsto \text{hom}_A(M, A)$ takes an (A, k) -bimodule M to a (k, A) -bimodule: indeed, A acts on the right of $\text{hom}_A(M, A)$.

If one prefers, one could also note that $\text{hom}_A(M, A)$ being a (k, A) -bimodule is correct from the viewpoint of bicategorical duality.

- We then view an (k, A) -bimodule as a (A^{op}, k) -bimodule by declaring that an A^{op} -action on the left is the same as an A -action on the right (in this situation, we ignore k since it is commutative). This gives a functor

$$\text{Mod}_{(k, A)} \rightarrow \text{Mod}_{(A^{\text{op}}, k)}.$$

This transition is accomplished with a base-change object. Indeed, switching from a (k, A) -bimodule to an (A^{op}, k) -module amounts to tensoring with the base-change object $A \otimes A^{\text{op}} A_k$. So, the switch from (k, A) -bimodules to (A^{op}, k) -bimodules amounts to taking the Serre dual $\overline{\square}^k(\text{hom}_A(M, A)) \overline{\square}_A$.

When we very carefully keep track of sidedness, the functor (8.7), which we denote D , is the composite

$$\text{Perf}_{(A, k)} \xrightarrow{\text{hom}_A(-, A)} \text{Perf}_{(k, A)} \xrightarrow{\overline{\square}^k(-) \overline{\square}_A} \text{Perf}_{(A^{\text{op}}, k)}$$

Theorem 8.8. [Shk13, Thm. 3] *For any perfect DG A -modules M and N define*

$$\chi(M, N) := \chi(\text{hom}_{\text{Perf } A}(M, N)).$$

Then

$$\chi(M, N) = \text{eu}(N) \cup \text{eu}(DM)$$

Proof. This follows from Theorem 7.13 with the identifications in Lemmas 8.3 and 8.5 but we give more detail because of the subtleties noted above.

If M and N are (A, k) -bimodules and perfect as A -modules, the module $\hom_A(M, A)$ is a (k, A) -bimodule and we are in the situation of Theorem 7.13. That theorem gives the commutativity of the following (recalling that DM is the notation for the Serre dual of $\hom_A(M, A)$)

$$\begin{array}{ccc} \mathrm{HH}(k) \otimes \mathrm{HH}(k) & \xrightarrow{\chi(N) \otimes \chi(DM)} & \mathrm{HH}(A) \otimes \mathrm{HH}(A^{\mathrm{op}}) \\ \uparrow & & \downarrow \\ \mathrm{HH}(k) & \xrightarrow{\chi(\hom_A(M, A) \circ N)} & \mathrm{HH}(k) \end{array}$$

In our current situation, $\hom_A(M, N) \circ N = \hom_A(M, A) \otimes_A N$ which is $\hom_A(M, N)$ as a (k, k) -bimodule. We have already identified the long way around the diagram as the pairing of the Euler classes. \square

Remark 8.9. In fact, the above only requires that A be proper — only the right dualizability of \overline{A} is used. The pairing on Hochschild homology may not be non-degenerate in this situation, but the theorem remains agnostic about that.

Remark 8.10. Shklyarov also identifies $\mathrm{eu}(DM)$ with $\mathrm{eu}(M)^\vee$. This is the identification in Corollary 7.3.

Remark 8.11. In Shklyarov's formula an integral sign also appears. We viewed this as being rolled into the characters computed by Morita equivalence and so we omit it.

8.2. Petit's Riemann-Roch theorem for DG algebras. Like Theorem 1.1, the main result in [Pet13] describes the interaction between a Hochschild class and a pairing map. In [Pet13, Rmk. 4.11], Petit observes that his Hochschild class $hh_A(M, f)$ agrees with the bicategorical trace of the identity map in [PS14b, PS13] so no further identifications will be necessary for the comparison of invariants.

Petit gives several descriptions for his pairing. We will use the description in [Pet13, 5.2] since it allows the most direct comparison to Figure 6.5B. Unless explicitly stated, \hom and \otimes are derived so we will not separately notate this. Petit's pairing \cup is the following composite:

$$\begin{aligned} (8.12) \quad & \hom_{(B^{\mathrm{op}})^e}(\omega_{B^{\mathrm{op}}}^{-1}, B^{\mathrm{op}}) \otimes \hom_{B^e}(\omega_B^{-1}, B) \\ & \xrightarrow{(- \otimes \mathrm{id}_{\omega_{B^{\mathrm{op}}}}) \otimes \mathrm{id}} \hom_{(B^{\mathrm{op}})^e}(\omega_{B^{\mathrm{op}}}^{-1} \otimes_{B^{\mathrm{op}}} \omega_{B^{\mathrm{op}}}, B^{\mathrm{op}} \otimes_{B^{\mathrm{op}}} \omega_{B^{\mathrm{op}}}) \otimes \hom_{B^e}(\omega_B^{-1}, B) \\ & \xrightarrow{\hom(f, g) \otimes \mathrm{id}} \hom_{B^e}(B^{\mathrm{op}}, \omega_{B^{\mathrm{op}}}) \otimes \hom_{B^e}(\omega_B^{-1}, B) \\ & \xrightarrow{\otimes} \hom_k(B^{\mathrm{op}} \otimes_{B^e} \omega_B^{-1}, \omega_{B^{\mathrm{op}}} \otimes_{B^e} B) \xrightarrow{\hom(c, e)} \hom_k(k, k) \cong k \end{aligned}$$

where B is a DGA, $B^e := B \otimes B^{\mathrm{op}}$ and ${}^e B := B^{\mathrm{op}} \otimes B$ and the other objects and maps are defined in the left most column of Table 1. The translations of ω_B^{-1} and $\omega_{B^{\mathrm{op}}}^{-1}$ look different in Table 1 because of sidedness considerations. After converting to actions on the same side (as Petit does) there is no inconsistency.

The main theorem of [Pet13] follows from Theorem 7.13ii after we identify Petit's pairing (8.12) with ours. For the first step in the identification of the pairings the particular modules are not relevant, and in fact can be a distraction, so we replace the particular modules in (8.12) by the choices in Table 1 that have the minimum required compatibility for (8.12) to be defined.

The primary difference between (8.12) and our pairing is the presence of the internal \hom . The goal of the following result is to replace those internal homs by duals. In the

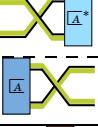
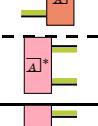
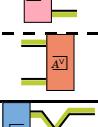
Petit [Pet13]	This paper	Lemma 8.13
$\omega_B^{-1} := \hom_{eB}(B^{\text{op}}, {}^eB)$		J^*
		J
		K
		K^*
$\omega_{B^{\text{op}}}^{-1} := \hom_{eB}(B, {}^eB)$		N
		N^*
		M
$\omega_{B^{\text{op}}} := \hom_k(B, k)$		L
$c: k \rightarrow \hom_{eB}(B^{\text{op}}, B^{\text{op}}) \cong B^{\text{op}} \otimes_{B^e} \omega_B^{-1}$	$P_{\overline{A}}$	η_J
$e: \omega_{B^{\text{op}}} \otimes_{B^e} B \rightarrow k$	$L_{\overline{A}}$	ϵ_K
$B^{\text{op}} \otimes_{B^{\text{op}}} \omega_{B^{\text{op}}} \rightarrow \omega_{B^{\text{op}}}$	Figure 8.15C	$g: M \otimes_R L \rightarrow K^*$
$B^{\text{op}} \rightarrow \omega_{B^{\text{op}}}^{-1} \otimes_{B^{\text{op}}} \omega_{B^{\text{op}}}$	Figure 8.15A	$f': N^* \otimes_k J \rightarrow L$,
	Figure 8.15B	$f: J \rightarrow N \otimes_R L$

TABLE 1. Notation translation

process we obtain a significant simplification of (8.12). To see this, compare the right and left hand sides of the displayed equation in Lemma 8.13. The right side is Petit's pairing and the left is the composite we will use in Theorem 8.16.

Lemma 8.13. *Let R be a k -algebra, J , M , and N be right R -modules, K be a left R -module, and L be an (R, R) -bimodule. Suppose J , N , and K are right dualizable. For module homomorphisms $g: M \otimes_R L \rightarrow K^*$ and $f': N^* \otimes_k J \rightarrow L$, the following diagram commutes.*

$$\begin{array}{ccc}
 (8.14) \quad M \otimes_R N^* \otimes_k J \otimes_R K & \xrightarrow{\sim} & \hom_R(N, M) \otimes \hom_R(J^*, K) \\
 \downarrow \text{id} \otimes f' \otimes \text{id} & & \downarrow \\
 M \otimes_R L \otimes_R K & & \hom_R(N \otimes_R L, M \otimes_R L) \otimes \hom_R(J^*, K) \\
 \downarrow g \otimes \text{id} & & \downarrow \hom((\text{id} \otimes f')(\eta_N \otimes \text{id}), g) \\
 K^* \otimes_R K & & \hom_R(J, K^*) \otimes \hom_R(J^*, K) \\
 \downarrow \epsilon_K & & \downarrow \otimes \\
 k & \xrightarrow{\hom(\eta_J, \epsilon_K)} & \hom(k, k)
 \end{array}$$

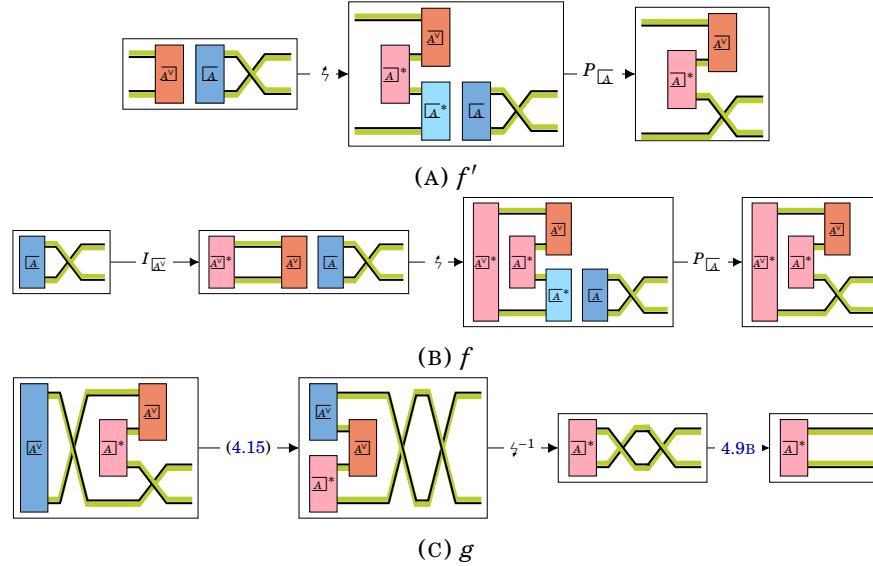


FIGURE 8.15. Maps

Note that $K^* = \text{hom}_k(K, k)$, $J^* = \text{hom}_R(J, R)$ and $N^* = \text{hom}_R(N, R)$. The map η_N is $k \rightarrow N \otimes_R N^*$ is the coevaluation for N . Similarly, η_J is the coevaluation for J and ϵ_K is the evaluation for K . The map f is $(\text{id} \otimes f')(\eta_N \otimes \text{id})$.

Proof. It is enough to prove the underived version of this result. So in this proof, all homs and tensors are underived. Since it seems simplest, we prove this by chasing elements, but this argument could be replaced by one using adjunctions.

The image of the tuple (m, δ, j, k) under the left composite is

$$g(m, f'(\delta, j))(k).$$

For $\alpha: N \rightarrow M$ and $\beta: J^* \rightarrow K$, the image under the right composite is the homomorphism

$$\begin{aligned} k &\xrightarrow{\eta_J} J \otimes_R J^* \xrightarrow{\eta_N \otimes \text{id} \otimes \text{id}} N \otimes_R N^* \otimes_k J \otimes_R J^* \xrightarrow{\text{id} \otimes f' \otimes \text{id}} N \otimes_R L \otimes_R J^* \\ &\xrightarrow{\alpha \otimes \text{id} \otimes \text{id}} M \otimes_R L \otimes_R J^* \xrightarrow{g \otimes \text{id}} K^* \otimes_R J^* \xrightarrow{\text{id} \otimes \beta} K^* \otimes_R K \xrightarrow{\epsilon_K} k \end{aligned}$$

We can take η_J to be given by linearly extending $1 \mapsto \sum_i (j_i, j_i^*)$ and η_N to be given by linearly extending $1 \mapsto \sum_\ell (n_\ell, n_\ell^*)$. Then the image of 1 under the composite above is

$$\begin{aligned} 1 &\mapsto \sum_i (j_i, j_i^*) \mapsto \sum_{i, \ell} (n_\ell, n_\ell^*, j_i, j_i^*) \mapsto \sum_{i, \ell} (n_\ell, f'(n_\ell^*, j_i), j_i^*) \\ &\mapsto \sum_{i, \ell} (\alpha(n_\ell), f'(n_\ell^*, j_i), j_i^*) \mapsto \sum_{i, \ell} (g(\alpha(n_\ell), f'(n_\ell^*, j_i)), j_i^*) \\ &\mapsto \sum_{i, \ell} (g(\alpha(n_\ell), f'(n_\ell^*, j_i)), \beta(j_i^*)) \mapsto \sum_{i, \ell} g(\alpha(n_\ell), f'(n_\ell^*, j_i))(\beta(j_i^*)) \end{aligned}$$

The image of (m, δ, j, k) under the top map is $(m\delta(-), j^*(-)k)$, so the image of (m, δ, j, k) under the top, right, and bottom composite is

$$\sum_{i, \ell} g(m\delta(n_\ell), f'(n_\ell^*, j_i))(j_i^*k) = \sum_{i, \ell} g(m, f'(\delta(n_\ell)n_\ell^*, j_i j^*(j_i^*k)))(k) = g(m, f'(\delta, j))(k)$$

□

The main theorem of [Pet13] is the following:

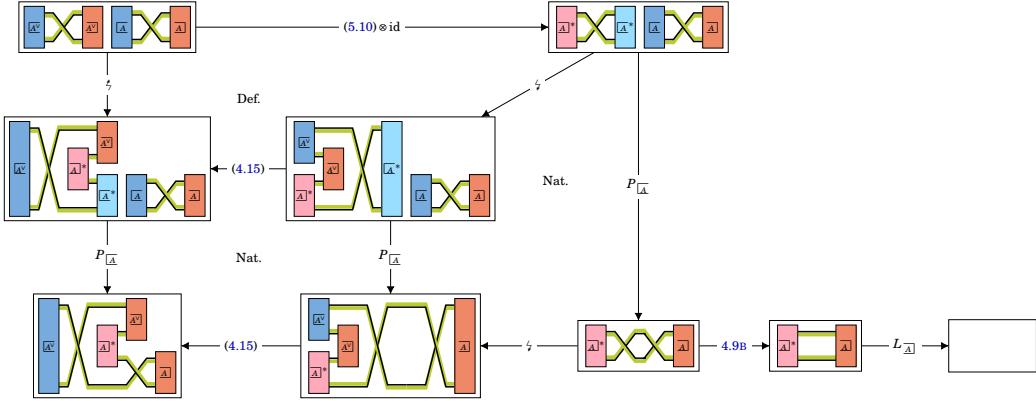


FIGURE 8.17. Comparison of pairings

Theorem 8.16. [Pet13, p. 4] Let A be a proper, smooth dg algebra, $M \in D_{\text{Perf}}(A)$, $f \in \text{hom}_A(M, M)$ and $N \in D_{\text{Perf}}(A^{\text{op}})$, $g \in \text{hom}_{A^{\text{op}}}(N, N)$. Then

$$hh_k(N \otimes_A M, g \otimes_A f) = hh_{A^{\text{op}}}(N, g) \cup hh_A(M, f).$$

Proof. First complete the translation of Petit's pairing (8.12) to a map without internal homs by replacing the arbitrary modules and maps in the left composite in (8.14) with those given by the middle column of Table 1 and the maps in Figures 8.15A and 8.15C. This composite is the left and bottom composite in Figure 8.17.

The two regions in Figure 8.17 labeled Nat. commute by naturality. The remaining region is labeled Def. and commutes by definition of the map in (5.10). The top and right composite is the pairing defined in this paper and so this completes the identification of Petit's pairing (8.12) and the pairing defined in this paper.

Then Theorem 8.16 follows from Theorem 7.13ii. \square

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