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Uniform approximations and effective boundary conditions for a high-contrast elastic interface

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A difficulty in the theory of a thin elastic interface is that series expansions in its thickness become disordered in the high contrast limit, i.e. when the interface is much softer or much stiffer than the media on either side. We provide a mathematical analysis of such series for an annular coating around a cylindrical fibre embedded in an elastic matrix subject to biaxial forcing. We determine the order of magnitude of successive terms in the series, and hence the terms which need to be retained in order to ensure that every neglected term is smaller in order of magnitude than at at least one retained term. In this way, we obtain uniform approximations for quantities such as the jump in the displacement and stress across the coating, and explain some peculiarities which have been observed in numerical work. A key finding is that it is essential to distinguish three types of boundary-value problem, corresponding to 'distant forcing', 'localised forcing', and 'the homogeneous problem', since they give different patterns of disorder in the corresponding series expansions. This provides a meaningful correspondence between physical principles and our mathematical results.

1. Introduction

A widely used method in the theory of a thin elastic interface is that of effective boundary conditions. The idea is that within an elastic layer one may place an imaginary hypothetical surface and determine boundary conditions at this surface to reproduce as closely as possible the fields outside the layer. The surface may be anywhere in the layer, e.g. at its centre or coincident with

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one of its boundaries, and the fields in the adjacent media are envisaged as being extended by analytic continuation up to the hypothetical surface, so that the original three-phase problem is replaced by a simpler two-phase problem. A variant of the approach is not to introduce a hypothetical surface at all, but simply to determine jump conditions across the layer, and then work only with the surrounding media and the jump in displacement and traction between them. This likewise gives a two-phase problem. A basic ingredient of either approach is the construction of series expansions in the thickness of the layer, truncated appropriately for the geometry and parameter regime of interest.

The method has a long history and an extensive literature. Some early papers are [1] on thin-film planar layers for the guiding of surface waves, and [2,3] on spherical and cylindrical annular layers in the theory of composites. Series expansions have been obtained for spheroidal inclusions [4], layers of general shape [5–7], and anisotropic planar layers [8–10]. An asymptotic approach which includes variable curvature and rational scalings for soft or stiff layers is given in [11], the relation to the theory of a Steigmann-Ogden interface is given in [12], and surface operators are constructed in [13]. An early treatment related to a Galerkin boundary integral method is presented in [14]. Recent analytical and numerical results for a coated circular inhomogeneity are in [15–17], and for a coated spherical inhomogeneity in [18]. As an indication of the enormous scope of the underlying physical problem we are addressing, we may cite the comprehensive review paper [19], with over 700 references, and also mention specific physical effects such as thermoelasticity [20] and the adhesion properties of joints and interphases (e.g. [21,22]) as representative of another enormous literature. The subject spans mathematics, physics, and engineering to a high degree.

The aim of the present paper is to resolve a difficulty encountered directly in [17] (but also known about from much earlier, e.g. [7,21]). This is the loss of accuracy of layer models in certain parameter regimes, which is found empirically to be strongly associated with a highly irregular dependence of truncation error on the number of terms retained in an expansion. Our approach in this paper is to determine the analytic structure of the exact solution of a carefully chosen canonical problem when the solution is expressed as a series expansion in the dimensionless thickness ϵ of the thin layer, with coefficients which depend on the stiffnesses of the different phases and their Poisson's ratios. The problem we have chosen is that of a coated cylindrical fibre perfectly bonded to a matrix under biaxial forcing. Thus the material parameters are the shear moduli $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$ together with Poisson's ratios $(\nu_{\mathrm{f}}, \nu_{\mathrm{c}}, \nu_{\mathrm{m}})$, and all field variables are proportional to $\cos 2\theta$ or $\sin 2\theta$, where θ is the angular variable in cylindrical coordinates. Here and throughout, subscripts or superscripts $(\mathrm{f}, \mathrm{c}, \mathrm{m})$ denote (fibre, coat, matrix). We have found that this apparently simple and well-explored problem displays a quite remarkably complex and irregular behaviour when analysed from the point of view of what can go wrong in a truncated Taylor series expansion in the layer thickness. As problems can arise unexpectedly in late terms, after a sequence of tame early terms, it is necessary to present a large number of series in some detail, and we have done this. These details are needed if a sound judgement is to be formed of what is likely to occur in more complicated problems.

The two decisive quantities for our purposes are the softness parameter α and the stiffness parameter β, both dimensionless, defined by

$$\alpha = \frac{\mu_{\mathrm{c}}}{\min(\mu_{\mathrm{f}}, \mu_{\mathrm{m}})}, \qquad \beta = \frac{\mu_{\mathrm{c}}}{\max(\mu_{\mathrm{f}}, \mu_{\mathrm{m}})}. \tag{1.1}$$

We shall refer to a coat as soft if $\alpha \ll 1$, i.e. if it is much softer than the surrounding fibre and matrix; similarly we refer to it as stiff if $\beta \gg 1$, i.e. much stiffer than its surrounding media. These parameters make it easy to state the mathematical idea underlying the paper. Let us suppose that for a soft coat we encounter a series expansion in which the terms have successive orders of magnitude

$$(O(\alpha^{n_0}),\, O(\alpha^{n_1}\epsilon),\, O(\alpha^{n_2}\epsilon^2),\, \ldots), \tag{1.2}$$

where $(n_0, n_1, n_2, \ldots)$ is a sequence of integers (with no restriction on their sign). Then if for a particular $k \geq 1$ it should happen that $n_k < n_{k-1}$, the series is *disordered at position* k, because if α is small enough at fixed ϵ, then $\alpha^{n_k}\epsilon^k \gg \alpha^{n_{k-1}}\epsilon^{k-1}$. The degree of smallness for this to happen is easily quantified as $\alpha \ll \epsilon^{1/(n_{k-1}-n_k)}$. This occurs for many series, and moreover can occur for more than one k in a given series. Similarly, for a stiff coat a series expansion of the form

$$(O(\beta^{n_0}),\ O(\beta^{n_1}\epsilon),\ O(\beta^{n_2}\epsilon^2),\ \ldots) \tag{1.3}$$

is *disordered at any position* k *for which* $n_k > n_{k-1}$, because then $\beta^{n_k}\epsilon^k \gg \beta^{n_{k-1}}\epsilon^{k-1}$ if β is large enough. Quantitatively, this occurs when $\beta \gg 1/\epsilon^{1/(n_k-n_{k-1})}$.

Thus to obtain a uniform approximation to either of the two types of disordered series, one must go at least as far as the latest disordered term. If the disorder continues indefinitely, because the sequence $(n_0, n_1, n_2, \ldots)$ has no minimum in the first case, or no maximum in the second case, then a uniform approximation in the form of a truncated series does not exist. Of course, a series may not be disordered anywhere; it is then well-ordered, but perhaps surprisingly there appear to be few well-ordered series in the theory of elastic interfaces, or more generally in the theory of multi-phase media. We believe therefore that the type of analysis presented in this paper, with its sharp focus on exactly quantified orders of magnitude, is of wide generality.

One might ask whether it is possible to say in advance what the powers $(n_0, n_1, n_2, \ldots)$ will be for the quantities of interest in an interface problem, most notably the displacements and stresses in or near the interface itself. We have found that this is not possible. The powers vary unpredictably (as noted, a late disordered term may unexpectedly appear, and often does). Moreover, in calculating the jump between two accurately calculated quantities, a sequence of consecutive early terms in the two series may cancel, including among them disordered terms, so that the disorder in a jump is usually different from that in the quantities used in defining the jump, again unpredictably. Thus the fact that a disordered term may cancel out at a later stage in a calculation needs to be constantly borne in mind.

For the above reasons, we have thought it worthwhile to revisit the classical problem referred to above, of a coated fibre under uniform load. We calculate series expansions of the important physical quantities and jumps, and determine their degree of disorder, i.e. the powers $(n_0, n_1, n_2, \ldots)$ in the high-contrast limits $\alpha \ll 1$ and $\beta \gg 1$. Thereby we construct 'minimal uniform approximations', containing the necessary powers of ϵ but no more. For any other form of approximation, e.g. up to a lower power of ϵ, our method determines where in parameter space it is expected to be accurate, and where it will fail. Our approximations determine which terms are needed in effective boundary conditions for them to be uniform in the high-contrast limit. These approximations are consistent with the asymptotic theory in [11], which we thus confirm. In §9 we give illustrative examples of the theory. The reader may find it helpful to look ahead to this section on occasion, to inspect some uniform approximations in which the numerical coefficients are given explicitly.

Physically, the origin of the disorder is that a thin elastic layer may deform by both stretching and bending, the former described by low powers of ϵ, the latter by high powers of ϵ. Thus a deformation dominated by bending requires high powers of ϵ to be retained, whereas these powers are negligible in a deformation dominated by stretching [11]. The present paper provides a complete mathematical analysis of these cases in a canonical problem.

All our formulae are exact, given the starting-point of linear elasticity. We use Mathematica to calculate series expansions analytically, without approximation. The code with its output is in the supplementary material.

2. Boundary-value problems for a coated fibre in a matrix

(a) Geometry and boundary conditions

The fibre problem in §1 reduces to one of plane-strain elasticity. We take a cross-section of the fibre to be a disc of radius a, shear modulus μ_{f}, and Poisson's ratio ν_{f}, occupying the region $0 \leq r < a$

in a cylindrical coordinate system (r, θ). The fibre has an annular coat with parameters $(\mu_{\rm c}, \nu_{\rm c})$ occupying $a < r < b$, and this is perfectly bonded to a matrix with parameters $(\mu_{\rm m}, \nu_{\rm m})$ occupying $r > b$. Thus the fibre-coat interface is $r = a$ and the coat-matrix interface is $r = b$; in formulae, it is somewhat easier to use instead of the Poisson's ratios themselves, the Kolosov constants [23, p. 43] defined by

$$(\kappa_{\rm f},\, \kappa_{\rm c},\, \kappa_{\rm m}) = (3 - 4\nu_{\rm f},\, 3 - 4\nu_{\rm c},\, 3 - 4\nu_{\rm m}). \tag{2.1}$$

The displacement and stress components are $(\widehat{u}_r, \widehat{u}_\theta, \widehat{\sigma}_{rr}, \widehat{\sigma}_{r\theta})$, representing real functions of r and θ (we shall not need $\widehat{\sigma}_{\theta\theta}$). For biaxial forcing, we write these in the form

$$(\widehat{u}_r,\, \widehat{u}_\theta,\, \widehat{\sigma}_{rr},\, \widehat{\sigma}_{r\theta}) = \mathrm{Re}\{(u_r,\, \mathrm{i}u_\theta,\, \sigma_{rr},\, \mathrm{i}\sigma_{r\theta})\mathrm{e}^{2\mathrm{i}\theta}\}, \tag{2.2}$$

where $(u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta}) \equiv (u_r(r), u_\theta(r), \sigma_{rr}(r), \sigma_{r\theta}(r))$ are real functions of r; the argument r is omitted where this is clear. For convenience, we call these functions of r the displacement and stress components, leaving understood a factor of $\cos 2\theta$ or $\sin 2\theta$ determined by (2.2). Phases will be indicated by superscripts $(\mathrm{f}, \mathrm{c}, \mathrm{m})$ and interfaces by subscripts (a, b), so that, for example, the value of the radial displacement $u_r^{\rm f}(r)$ at the interface $r = a$ is denoted $u_{ra}^{\rm f}$, this being a shorthand for $u_r^{\rm f}(a)$.

We define a state vector $\mathbf{u} \equiv \mathbf{u}(r) = (u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta})^T$, and indicate phases and interfaces by, for example,

$$\mathbf{u}^{\rm f} \equiv \mathbf{u}^{\rm f}(r) = (u_r^{\rm f},\, u_\theta^{\rm f},\, \sigma_{rr}^{\rm f},\, \sigma_{r\theta}^{\rm f})^T, \quad \mathbf{u}_a^{\rm f} \equiv \mathbf{u}^{\rm f}(a) = (u_{ra}^{\rm f},\, u_{\theta a}^{\rm f},\, \sigma_{rra}^{\rm f},\, \sigma_{r\theta a}^{\rm f})^T, \tag{2.3}$$

and so on correspondingly. In this way, all the required properties throughout a phase or at an interface can be represented by a single vector symbol. It is convenient to use both a and b as reference lengths, and define dimensionless state vectors by

$$\tilde{\mathbf{v}}_a^{\rm f} \equiv \tilde{\mathbf{v}}_a^{\rm f}(r) = (2u_r^{\rm f}(r)/a,\, 2u_\theta^{\rm f}(r)/a,\, \sigma_{rr}^{\rm f}(r)/(2\mu_{\rm f}),\, \sigma_{r\theta}^{\rm f}(r)/(2\mu_{\rm f}))^T \tag{2.4}$$

and similarly for $\tilde{\mathbf{v}}_a^{\rm c}$, $\tilde{\mathbf{v}}_b^{\rm c}$, and $\tilde{\mathbf{v}}_b^{\rm m}$, in which the subscripts denote the reference length used in the first two components. Thus in the coat we may use either $\tilde{\mathbf{v}}_a^{\rm c}$ or $\tilde{\mathbf{v}}_b^{\rm c}$, depending on which reference length gives simpler formulae. The reason for the tildes here is that the $\tilde{\mathbf{v}}$ quantities are mostly needed at the values $r = a$ or $r = b$, and the corresponding values $(\tilde{\mathbf{v}}_a^{\rm f}(a),\, \tilde{\mathbf{v}}_a^{\rm c}(a),\, \tilde{\mathbf{v}}_b^{\rm c}(b),\, \tilde{\mathbf{v}}_b^{\rm m}(b))$ are written without tildes in the compact form

$$(\mathbf{v}_a^{\rm f},\, \mathbf{v}_a^{\rm c},\, \mathbf{v}_b^{\rm c},\, \mathbf{v}_b^{\rm m}) \equiv (\tilde{\mathbf{v}}_a^{\rm f}(a),\, \tilde{\mathbf{v}}_a^{\rm c}(a),\, \tilde{\mathbf{v}}_b^{\rm c}(b),\, \tilde{\mathbf{v}}_b^{\rm m}(b)). \tag{2.5}$$

The boundary conditions are (i) continuity of displacement and traction at interfaces, i.e. $\mathbf{u}_a^{\rm f} = \mathbf{u}_a^{\rm c}$ and $\mathbf{u}_b^{\rm c} = \mathbf{u}_b^{\rm m}$; (ii) continuity of displacement at the origin, i.e. $(u_r^{\rm f}, u_\theta^{\rm f}) \to (0, 0)$ as $r \to 0$; (iii) bounded stress at infinity; and (iv) a forcing condition. Here (i)–(iii) are homogeneous boundary conditions, and (iv) is an inhomogeneous boundary condition. As we see below, (iii) and (iv) are scalar expressions, thus making the number of boundary conditions twelve in total.

In the interests of a unified treatment of different boundary-value problems, we shall introduce the notion of a forcing amplitude F in all cases. One option is to leave F unspecified and arbitrary, so that F is simply a multiplying factor for each field variable, because the problem is linear,. We shall call this the homogeneous problem, and refer to it as problem (a). Alternatively, the forcing may be specified precisely, in which case we need to say whether the forcing is 'distant', i.e. applied at infinity, or is 'localised', i.e. applied to the fibre and/or coating. These options give the distant forcing problem and the localised forcing problem, respectively, and we refer to them as problems (b) and (c). It is then necessary to determine F in relation to the given data of the problem. Our approach does this in a straightforward way, and in general F turns out to be a function of ϵ, which depends on the boundary-value problem being solved.

In this paper, we concentrate on problems (a) and (b), and in case (b) specify the distant forcing by

$$(\widehat{\sigma}_{rr},\, \widehat{\sigma}_{r\theta}) \to (\sigma_\infty \cos 2\theta,\, -\sigma_\infty \sin 2\theta) \qquad (r \to \infty), \tag{2.6}$$

or equivalently

$$(\sigma_{rr}^{\rm m},\, \sigma_{r\theta}^{\rm m}) \to (\sigma_\infty,\, \sigma_\infty) \qquad (r \to \infty), \tag{2.7}$$

(see (2.2) for the notation), where σ_∞ is a prescribed constant. This forcing corresponds to uniform far-field strain. Note that it is the choice of F which ensures that σ_∞ in (2.6) and (2.7) is a constant independent of ϵ. For problem (c), localised forcing, the required choice of F would give a far-field stress of a similar form, but with σ_∞ now a function of ϵ, fully determined by this different forcing.

The literature contains examples of all three problems (a), (b), and (c). For example, [11] is largely concerned with (a), but includes (c) in the final discussion, where external loadings on the coating are introduced. References [15–17] are concerned with case (b).

(b) Form of solution

The boundary-value problems just specified have a known solution, given implicitly by inversion of an 8×8 matrix [3], or explicitly by evaluation of the Kolosov-Muskhelishvili potentials [14–17]. Its functional form is that of Michell's solution [23, p. 118] in each phase, conveniently expressed in terms of matrices defined by

$$\tilde{M}_a^{\mathrm{f}} = \tilde{M}(a, \kappa_{\mathrm{f}}, r) = \begin{pmatrix} 2(a/r)^3 & -2r/a & (\kappa_{\mathrm{f}}+1)a/r & (\kappa_{\mathrm{f}}-3)(r/a)^3 \\ -2(a/r)^3 & -2r/a & (\kappa_{\mathrm{f}}-1)a/r & -(\kappa_{\mathrm{f}}+3)(r/a)^3 \\ -3(a/r)^4 & -1 & -2(a/r)^2 & 0 \\ 3(a/r)^4 & -1 & (a/r)^2 & -3(r/a)^2 \end{pmatrix} \tag{2.8}$$

in the fibre, and similarly for $\tilde{M}_a^{\mathrm{c}}$, $\tilde{M}_b^{\mathrm{c}}$, and $\tilde{M}_b^{\mathrm{m}}$ in the coat and matrix. We maintain our convention that a tilde indicates a function of r. The two different forms $\tilde{M}_a^{\mathrm{c}}$ and $\tilde{M}_b^{\mathrm{c}}$ for Michell's solution in the coat correspond to the use of r/a and r/b respectively as the non-dimensional version of the radial variable r.

The dimensionless field in the fibre is $\tilde{\mathbf{v}}_a^{\mathrm{f}} = \tilde{M}_a^{\mathrm{f}}\mathbf{a}^{\mathrm{f}}$, with $\tilde{\mathbf{v}}_a^{\mathrm{f}} \equiv \tilde{\mathbf{v}}_a^{\mathrm{f}}(r)$ as defined in (2.4) and

$$\mathbf{a}^{\mathrm{f}} \equiv (a_1^{\mathrm{f}},\, a_2^{\mathrm{f}},\, a_3^{\mathrm{f}},\, a_4^{\mathrm{f}})^T \tag{2.9}$$

is a vector of dimensionless coefficients. These coefficients, and any other quantity we describe as a coefficient, does not depend on r. Here $a_1^{\mathrm{f}} = 0$ and $a_3^{\mathrm{f}} = 0$, corresponding to the absence of terms in $((a/r)^3, a/r)$ in both u_r^{f} and u_θ^{f}, as required by continuity of displacement at the origin.

In the coat we may write either $\tilde{\mathbf{v}}_a^{\mathrm{c}} = \tilde{M}_a^{\mathrm{c}}\mathbf{a}^{\mathrm{c}}$ or $\tilde{\mathbf{v}}_b^{\mathrm{c}} = \tilde{M}_b^{\mathrm{c}}\mathbf{b}^{\mathrm{c}}$, depending on which dimensionless variable $\tilde{\mathbf{v}}_a^{\mathrm{c}} \equiv \tilde{\mathbf{v}}_a^{\mathrm{c}}(r)$ or $\tilde{\mathbf{v}}_b^{\mathrm{c}} \equiv \tilde{\mathbf{v}}_b^{\mathrm{c}}(r)$ is used. Here the coefficient vectors are

$$\mathbf{a}^{\mathrm{c}} = (a_1^{\mathrm{c}},\, a_2^{\mathrm{c}},\, a_3^{\mathrm{c}}, a_4^{\mathrm{c}})^T, \qquad \mathbf{b}^{\mathrm{c}} = (b_1^{\mathrm{c}},\, b_2^{\mathrm{c}}, b_3^{\mathrm{c}}, b_4^{\mathrm{c}})^T, \tag{2.10}$$

and a check of definitions shows that $\mathbf{b}^{\mathrm{c}} = D\mathbf{a}^{\mathrm{c}}$ where

$$D = \mathrm{diag}((a/b)^4,\, 1,\, (a/b)^2,\, (b/a)^2). \tag{2.11}$$

In the matrix phase, we have $\tilde{\mathbf{v}}_b^{\mathrm{m}} = \tilde{M}_b^{\mathrm{m}}\mathbf{b}^{\mathrm{m}}$, with $\tilde{\mathbf{v}}_b^{\mathrm{m}} \equiv \tilde{\mathbf{v}}_b^{\mathrm{m}}(r)$, and the vector of dimensionless coefficients is now

$$\mathbf{b}^{\mathrm{m}} = (b_1^{\mathrm{m}},\, b_2^{\mathrm{m}},\, b_3^{\mathrm{m}},\, b_4^{\mathrm{m}})^T. \tag{2.12}$$

Here $b_4^{\mathrm{m}} = 0$, which corresponds to the absence of a term in $(r/b)^2$ in $\sigma_{r\theta}^{\mathrm{m}}$, as required by bounded stress at infinity.

(c) The Michell matrix

Define the Michell matrix by

$$M = M(\kappa) = \begin{pmatrix} 2 & -2 & \kappa+1 & \kappa-3 \\ -2 & -2 & \kappa-1 & -(\kappa+3) \\ -3 & -1 & -2 & 0 \\ 3 & -1 & 1 & -3 \end{pmatrix}. \tag{2.13}$$

This is a function of an arbitrary Kolosov constant κ, but of no other quantity, and we denote its values in the fibre, coat, and matrix by

$$(M_{\mathrm{f}},\, M_{\mathrm{c}},\, M_{\mathrm{m}}) = (M(\kappa_{\mathrm{f}}),\, M(\kappa_{\mathrm{c}}),\, M(\kappa_{\mathrm{m}})). \tag{2.14}$$

Thus at $r = a$ in the fibre, the relation $\tilde{\mathbf{v}}_a^{\mathrm{f}} = \tilde{M}_a^{\mathrm{f}} \mathbf{a}^{\mathrm{f}}$ given after (2.8) may be written as $\mathbf{v}_a^{\mathrm{f}} = M_{\mathrm{f}} \mathbf{a}^{\mathrm{f}}$, where $\mathbf{v}_a^{\mathrm{f}}$ is the first vector defined in (2.5). At $r = a$ on the coat side, the corresponding relation is $\mathbf{v}_a^{\mathrm{c}} = M_{\mathrm{c}} \mathbf{a}^{\mathrm{c}}$. Likewise, we have the two relations $\mathbf{v}_b^{\mathrm{c}} = M_{\mathrm{c}} \mathbf{b}^{\mathrm{c}}$ and $\mathbf{v}_b^{\mathrm{m}} = M_{\mathrm{m}} \mathbf{b}^{\mathrm{m}}$, corresponding to the two sides of $r = b$. Here the vectors $\mathbf{v}_a^{\mathrm{c}}$, $\mathbf{v}_b^{\mathrm{c}}$, and $\mathbf{v}_b^{\mathrm{m}}$ are the last three quantities in (2.5).

The Michell matrix has a non-trivial property which simplifies the subsequent algebra. This is that although $\det(M) = -12(\kappa+1)^2$, nevertheless a factor $\kappa + 1$ is common to every term in the adjugate of M, and so cancels out everywhere in the inverse of M, to leave

$$M^{-1} = \frac{1}{6(\kappa+1)} \begin{pmatrix} 0 & -3 & -(\kappa-3) & \kappa+3 \\ -6 & 3 & -3(\kappa+1) & -3(\kappa-1) \\ 3 & 3 & -6 & -6 \\ 3 & -3 & 2 & -2 \end{pmatrix}. \tag{2.15}$$

Thus although one might have expected the displayed terms in (2.15) to be quadratic expressions in κ, they are in fact only linear. The underlying reason for this is that M is only of rank 2 when $\kappa = -1$.

With the aid of (2.15), the coefficient vectors $(\mathbf{a}^{\mathrm{f}}, \mathbf{a}^{\mathrm{c}}, \mathbf{b}^{\mathrm{c}}, \mathbf{b}^{\mathrm{m}})$ are expressible in terms of the dimensionless boundary values $(\mathbf{v}_a^{\mathrm{f}}, \mathbf{v}_a^{\mathrm{c}}, \mathbf{v}_b^{\mathrm{c}}, \mathbf{v}_b^{\mathrm{m}})$ through the relations $\mathbf{a}^{\mathrm{f}} = M_{\mathrm{f}}^{-1} \mathbf{v}_a^{\mathrm{f}}$, etc. Note that the interface conditions $\mathbf{u}_a^{\mathrm{f}} = \mathbf{u}_a^{\mathrm{c}}$ and $\mathbf{u}_b^{\mathrm{c}} = \mathbf{u}_b^{\mathrm{m}}$ given in §2(*a*) are for dimensional quantities. To find their dimensionless versions, we use the relations

$$(\mathbf{u}_a^{\mathrm{f}},\, \mathbf{u}_a^{\mathrm{c}},\, \mathbf{u}_b^{\mathrm{c}},\, \mathbf{u}_b^{\mathrm{m}}) = (D_a^{\mathrm{f}} \mathbf{v}_a^{\mathrm{f}},\, D_a^{\mathrm{c}} \mathbf{v}_a^{\mathrm{c}},\, D_b^{\mathrm{c}} \mathbf{v}_b^{\mathrm{c}},\, D_b^{\mathrm{m}} \mathbf{v}_b^{\mathrm{m}}) \tag{2.16}$$

involving four diagonal matrices defined by, for example,

$$D_a^{\mathrm{f}} = \mathrm{diag}(a/2, a/2, 2\mu_{\mathrm{f}}, 2\mu_{\mathrm{f}}), \tag{2.17}$$

and so on. Then the interface conditions become

$$\mathbf{v}_a^{\mathrm{f}} = D_{\mathrm{f}}^{\mathrm{c}} \mathbf{v}_a^{\mathrm{c}}, \quad \mathbf{v}_b^{\mathrm{c}} = D_{\mathrm{c}}^{\mathrm{m}} \mathbf{v}_b^{\mathrm{m}}, \tag{2.18}$$

or equivalently

$$\mathbf{v}_a^{\mathrm{c}} = D_{\mathrm{c}}^{\mathrm{f}} \mathbf{v}_a^{\mathrm{f}}, \quad \mathbf{v}_b^{\mathrm{m}} = D_{\mathrm{m}}^{\mathrm{c}} \mathbf{v}_b^{\mathrm{c}}, \tag{2.19}$$

involving a second family of four diagonal matrices defined according to the pattern

$$D_{\mathrm{f}}^{\mathrm{c}} = \mathrm{diag}(1,\, 1,\, \mu_{\mathrm{c}}/\mu_{\mathrm{f}},\, \mu_{\mathrm{c}}/\mu_{\mathrm{f}}) \tag{2.20}$$

and so on. Matrices of the form (2.17) are defined with respect to a phase and a boundary, whereas those of the form (2.20) are defined with respect to two phases.

3. Method of solution

With the above definitions and relations, we have available a method of solution of the boundary-value problems defined in §2(*a*). The method involves two ideas. The first is that given any one of the twelve four-vectors we have defined, i.e. any coefficient vector selected from $(\mathbf{a}^{\mathrm{f}}, \mathbf{a}^{\mathrm{c}}, \mathbf{b}^{\mathrm{c}}, \mathbf{b}^{\mathrm{m}})$, or any boundary vector selected from either $(\mathbf{u}_a^{\mathrm{f}}, \mathbf{u}_a^{\mathrm{c}}, \mathbf{u}_b^{\mathrm{c}}, \mathbf{u}_b^{\mathrm{m}})$ or $(\mathbf{v}_a^{\mathrm{f}}, \mathbf{v}_a^{\mathrm{c}}, \mathbf{v}_b^{\mathrm{c}}, \mathbf{v}_b^{\mathrm{m}})$, the value of any other quantity may be written down as a product of this vector by a sequence of matrices with known entries. These matrices are those we have defined, and is the reason we have given prominence to the Michell matrix and its inverse. The problem thus resolves itself into finding a single four-vector, and this may be any one of the twelve available. The second idea is that of 'propagation of linear relations', which now follows.

(a) Propagation of linear relations

Let us suppose that we are given or have calculated a linear relation between the components of any of the vectors above. For example, if this vector is $\mathbf{b}^{\mathrm{m}}$, a linear relation between its components is of the form $\boldsymbol{\ell}^T\mathbf{b}^{\mathrm{m}}=0$ where $\boldsymbol{\ell}=(l_1,l_2,l_3,l_4)^T$ specifies the relation, and is taken as given. Since $\mathbf{b}^{\mathrm{m}}=M_{\mathrm{m}}^{-1}\mathbf{v}_b^{\mathrm{m}}$, we may write $(\boldsymbol{\ell}^T M_{\mathrm{m}}^{-1})\mathbf{v}_b^{\mathrm{m}}=0$, which is a linear relation between the components of $\mathbf{v}_b^{\mathrm{m}}$; then $\mathbf{v}_b^{\mathrm{m}}=D_{\mathrm{m}}^{\mathrm{c}}\mathbf{v}_b^{\mathrm{c}}$ gives $(\boldsymbol{\ell}^T M_{\mathrm{m}}^{-1}D_{\mathrm{m}}^{\mathrm{c}})\mathbf{v}_b^{\mathrm{c}}=0$, a linear relation between the components of $\mathbf{v}_b^{\mathrm{c}}$, and proceeding in this way we obtain also

$$(\boldsymbol{\ell}^T M_{\mathrm{m}}^{-1}D_{\mathrm{m}}^{\mathrm{c}}M_{\mathrm{c}})\mathbf{b}^{\mathrm{c}}=0 \quad \text{and} \quad (\boldsymbol{\ell}^T M_{\mathrm{m}}^{-1}D_{\mathrm{m}}^{\mathrm{c}}M_{\mathrm{c}}D)\mathbf{a}^{\mathrm{c}}=0. \tag{3.1}$$

Further applications give a similar but longer expression in $\mathbf{a}^{\mathrm{f}}$. Thus the original linear relation between the components of $\mathbf{b}^{\mathrm{m}}$ (which relates to the matrix phase) has propagated first to the coat, in the form of a relation between the components of $\mathbf{b}^{\mathrm{c}}$ or $\mathbf{a}^{\mathrm{c}}$, and then to the fibre, as a relation between the components of $\mathbf{a}^{\mathrm{f}}$. Similarly, whichever vector one starts with, in any phase, an arbitrary linear relation between its components may be propagated anywhere by means of a product of the matrices defined in §2.

(b) Effect of remote boundary conditions

We refer to the boundary conditions as $r\to 0$ and $r\to\infty$ as 'remote', in that they apply at a distance from the coat. These boundary conditions, given in §2(*a*), are equivalent to $(a_1^{\mathrm{f}},a_3^{\mathrm{f}})=(0,0)$ and $b_4^{\mathrm{m}}=0$, as required to remove terms of order $((a/r)^3,a/r)$ in the displacement near the origin, and terms of order $(r/b)^2$ in the distant stress. Thus the remote boundary conditions are equivalent to three linear relations which may be written

$$(\boldsymbol{\ell}^{(1)T}\mathbf{a}^{\mathrm{f}},\ \boldsymbol{\ell}^{(3)T}\mathbf{a}^{\mathrm{f}},\ \boldsymbol{\ell}^{(4)T}\mathbf{b}^{\mathrm{m}})=(0,0,0), \tag{3.2}$$

where

$$\boldsymbol{\ell}^{(1)}=(1,0,0,0)^T,\quad \boldsymbol{\ell}^{(3)}=(0,0,1,0)^T,\quad \boldsymbol{\ell}^{(4)}=(0,0,0,1)^T. \tag{3.3}$$

Let us now propagate these relations to the coat. For definiteness, we propagate to $\mathbf{a}^{\mathrm{c}}$, though an alternative would be to propagate to $\mathbf{b}^{\mathrm{c}}$, which differs only in being based on the reference length b instead of a, and satisfies $\mathbf{b}^{\mathrm{c}}=D\mathbf{a}^{\mathrm{c}}$ with D as defined in (2.11). Thus we obtain three linear relations satisfied by the components of $\mathbf{a}^{\mathrm{c}}$, of the form

$$(\mathbf{n}^{(1)T}\mathbf{a}^{\mathrm{c}},\ \mathbf{n}^{(3)T}\mathbf{a}^{\mathrm{c}},\ \mathbf{n}^{(4)T}\mathbf{a}^{\mathrm{c}})=(0,0,0) \tag{3.4}$$

in which, apart from arbitrary constants of proportionality, $(\mathbf{n}^{(1)},\mathbf{n}^{(3)},\mathbf{n}^{(4)})$ are products of matrices defined in §2. For example, we have in effect calculated $\mathbf{n}^{(4)}$ in $(3.1)_2$, since $\boldsymbol{\ell}=\boldsymbol{\ell}^{(4)}$ gives

$$\mathbf{n}^{(4)}\propto(\boldsymbol{\ell}^{(4)T}M_{\mathrm{m}}^{-1}D_{\mathrm{m}}^{\mathrm{c}}M_{\mathrm{c}}D)^T. \tag{3.5}$$

The expressions for $(\mathbf{n}^{(1)},\mathbf{n}^{(3)})$ are similar, but involve matrices defined in the fibre rather than the matrix.

Geometrically, the above shows that $\mathbf{a}^{\mathrm{c}}$ is perpendicular to three vectors in four-space, namely $(\mathbf{n}^{(1)},\mathbf{n}^{(3)},\mathbf{n}^{(4)})$. These vectors are linearly independent, because the boundary conditions are independent, and so $\mathbf{a}^{\mathrm{c}}$ is perpendicular to the hyperplane they generate. Hence $\mathbf{a}^{\mathrm{c}}$ is proportional to the cross-product of $(\mathbf{n}^{(1)},\mathbf{n}^{(3)},\mathbf{n}^{(4)})$, and we may write

$$\mathbf{a}^{\mathrm{c}}=F\mathbf{n}^{(1)}\times\mathbf{n}^{(3)}\times\mathbf{n}^{(4)}, \tag{3.6}$$

which is non-zero. Here F is the forcing amplitude, as discussed in §2(*a*). For the homogeneous problem (a), as there defined, F is arbitrary, whereas for a distant or localised forcing problem it is determined by the remaining boundary condition. In (3.6), the constants of proportionality used in defining $(\mathbf{n}^{(1)},\mathbf{n}^{(3)},\mathbf{n}^{(4)})$ may be chosen arbitrarily, but should then not be varied, so that F is well defined.

Thus the solution of the problem is reduced to evaluation of a triple product in four-space. The reason $\mathbf{a}^{\mathrm{c}}$ gives the full solution is that we may 'propagate back' from $\mathbf{a}^{\mathrm{c}}$ to any other quantity, in any phase, on multiplying $\mathbf{a}^{\mathrm{c}}$ by a sequence of known small matrices. The quantities obtained are coefficient vectors $(\mathbf{a}^{\mathrm{f}}, \mathbf{a}^{\mathrm{c}}, \mathbf{b}^{\mathrm{c}}, \mathbf{b}^{\mathrm{m}})$, boundary vectors $(\mathbf{u}^{\mathrm{f}}_a, \mathbf{u}^{\mathrm{c}}_a, \mathbf{u}^{\mathrm{c}}_b, \mathbf{u}^{\mathrm{m}}_b)$ and $(\mathbf{v}^{\mathrm{f}}_a, \mathbf{v}^{\mathrm{c}}_a, \mathbf{v}^{\mathrm{c}}_b, \mathbf{v}^{\mathrm{m}}_b)$, and field vectors $(\mathbf{u}^{\mathrm{f}}, \mathbf{u}^{\mathrm{c}}, \mathbf{u}^{\mathrm{m}})$ and $(\tilde{\mathbf{v}}^{\mathrm{f}}_a, \tilde{\mathbf{v}}^{\mathrm{c}}_a, \tilde{\mathbf{v}}^{\mathrm{c}}_b, \tilde{\mathbf{v}}^{\mathrm{m}}_b)$. The field vectors are functions of r (the dependence on θ is implicit, but may be recovered using (2.2)), and quantities containing the symbol $\mathbf{v}$ or $\tilde{\mathbf{v}}$ are dimensionless. Identities in the above are $\mathbf{u}^{\mathrm{f}}_a = \mathbf{u}^{\mathrm{c}}_a$ and $\mathbf{u}^{\mathrm{c}}_b = \mathbf{u}^{\mathrm{m}}_b$, by the interface boundary conditions. The parameters appearing in the solution are the shear moduli $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$, the Poisson's ratios $(\nu_{\mathrm{f}}, \nu_{\mathrm{c}}, \nu_{\mathrm{m}})$ defined via the Kolosov constants $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$, and the interface radii (a, b).

4. Calculation of the coefficient vectors

In §3(b), we showed how to determine linearly independent vectors $\mathbf{n}^{(1)}$, $\mathbf{n}^{(3)}$, and $\mathbf{n}^{(4)}$ which are perpendicular to the coefficient vector $\mathbf{a}^{\mathrm{c}}$. A suitable choice is

$$\mathbf{n}^{(1)} = \begin{pmatrix} 2(\mu_{\mathrm{f}} + \kappa_{\mathrm{f}}\mu_{\mathrm{c}}) \\ 2(\mu_{\mathrm{f}} - \mu_{\mathrm{c}}) \\ -(\kappa_{\mathrm{c}} - 1)\mu_{\mathrm{f}} + (\kappa_{\mathrm{f}} - 1)\mu_{\mathrm{c}} \\ -(\kappa_{\mathrm{c}} + 3)\mu_{\mathrm{f}} - (\kappa_{\mathrm{f}} + 3)\mu_{\mathrm{c}} \end{pmatrix}, \ \mathbf{n}^{(3)} = \begin{pmatrix} 0 \\ -(\mu_{\mathrm{f}} - \mu_{\mathrm{c}}) \\ (\kappa_{\mathrm{c}}\mu_{\mathrm{f}} + \mu_{\mathrm{c}})/2 \\ -3(\mu_{\mathrm{f}} - \mu_{\mathrm{c}})/2 \end{pmatrix}, \ \mathbf{n}^{(4)} = \begin{pmatrix} 2(\mu_{\mathrm{c}} - \mu_{\mathrm{m}})a^6 b^2 \\ 0 \\ (\mu_{\mathrm{c}} - \mu_{\mathrm{m}})a^4 b^4 \\ -(\mu_{\mathrm{c}} + \kappa_{\mathrm{c}}\mu_{\mathrm{m}})b^8 \end{pmatrix}. \tag{4.1}$$

Then with $\mathbf{a}^{\mathrm{c}}$ as in (3.6), the other coefficient vectors are given by the propagation formulae of §3. The result takes the form

$$\mathbf{a}^{\mathrm{f}} = F(a^6 b^2 \mathbf{a}^{\mathrm{f}}_{62} + a^4 b^4 \mathbf{a}^{\mathrm{f}}_{44} + b^8 \mathbf{a}^{\mathrm{f}}_{08}), \tag{4.2}$$

$$\mathbf{a}^{\mathrm{c}} = F(a^6 b^2 \mathbf{a}^{\mathrm{c}}_{62} + a^4 b^4 \mathbf{a}^{\mathrm{c}}_{44} + b^8 \mathbf{a}^{\mathrm{c}}_{08}), \tag{4.3}$$

$$\mathbf{b}^{\mathrm{c}} = F(a^8 \mathbf{b}^{\mathrm{c}}_{80} + a^6 b^2 \mathbf{b}^{\mathrm{c}}_{62} + a^4 b^4 \mathbf{b}^{\mathrm{c}}_{44} + a^2 b^6 \mathbf{b}^{\mathrm{c}}_{26} + b^8 \mathbf{b}^{\mathrm{c}}_{08}), \tag{4.4}$$

$$\mathbf{b}^{\mathrm{m}} = F\gamma(a^8 \mathbf{b}^{\mathrm{m}}_{80} + a^6 b^2 \mathbf{b}^{\mathrm{m}}_{62} + a^4 b^4 \mathbf{b}^{\mathrm{m}}_{44} + a^2 b^6 \mathbf{b}^{\mathrm{m}}_{26} + b^8 \mathbf{b}^{\mathrm{m}}_{08}), \tag{4.5}$$

where $\gamma = 1/\{(\kappa_{\mathrm{m}} + 1)\mu_{\mathrm{m}}\}$. Here the vector terms in parentheses in (4.2)–(4.4) have components which are cubics in $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$ with coefficients which depend on $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$; (4.5) is similar, but with quartics. In (4.2) and (4.5), a check of the algebra is that we must have

$$(a^{\mathrm{f}}_1,\ a^{\mathrm{f}}_3,\ b^{\mathrm{m}}_4) = (0,\ 0,\ 0), \tag{4.6}$$

because these are among the boundary conditions we started with.

So far, a and b are arbitrary, subject only to $a < b$. For a thin coat, we define its dimensionless thickness ϵ and mean radius a_0 by

$$\epsilon = (b - a)/a_0, \qquad a_0 = (a + b)/2, \tag{4.7}$$

so that

$$a = a_0(1 - \epsilon/2), \quad b = a_0(1 + \epsilon/2) \qquad (\epsilon \ll 1). \tag{4.8}$$

Then the coefficient vectors $(\mathbf{a}^{\mathrm{f}}, \mathbf{a}^{\mathrm{c}})$ are sextics in ϵ, with expansions of the form

$$\mathbf{a}^{\mathrm{f}} = F\sum_{i=0}^{6} \mathbf{a}^{\mathrm{f}}_i \epsilon^i, \qquad \mathbf{a}^{\mathrm{c}} = F\sum_{i=0}^{6} \mathbf{a}^{\mathrm{c}}_i \epsilon^i, \tag{4.9}$$

and the coefficient vectors $(\mathbf{b}^{\mathrm{c}}, \mathbf{b}^{\mathrm{m}})$ are octics, with expansions

$$\mathbf{b}^{\mathrm{c}} = F\sum_{i=0}^{8} \mathbf{b}^{\mathrm{c}}_i \epsilon^i, \qquad \mathbf{b}^{\mathrm{m}} = F\gamma\sum_{i=0}^{8} \mathbf{b}^{\mathrm{m}}_i \epsilon^i. \tag{4.10}$$

Here the coefficients of the powers of ϵ inherit their polynomial form from (4.2)–(4.5); i.e. cubic in $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$ for $(\mathbf{a}^{\mathrm{f}}, \mathbf{a}^{\mathrm{c}}, \mathbf{b}^{\mathrm{c}})$ and quartic for $\mathbf{b}^{\mathrm{m}}$. The same type of polynomial expansion is obtained if ϵ is defined by $b = a(1+\epsilon)$, except that the coefficients depend differently on $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$.

(a) Evaluation of the forcing amplitude F

We now evaluate F for the boundary-value problem of type (b) as specified in §2(*a*), i.e. distant forcing. This corresponds to a specified value of σ_∞ in the relation $(\sigma_{rr}^{\mathrm{m}}, \sigma_{r\theta}^{\mathrm{m}}) \to (\sigma_\infty, \sigma_\infty)$ as $r \to \infty$, as given in (2.7). On returning to the definitions in §2, it is found that the relation $\tilde{\mathbf{v}}_b^{\mathrm{m}} = \tilde{M}_b^{\mathrm{m}} \mathbf{b}^{\mathrm{m}}$ given after (2.11) yields

$$(\sigma_{rr}^{\mathrm{m}}, \sigma_{r\theta}^{\mathrm{m}}) \to -2b_2^{\mathrm{m}}(\mu_{\mathrm{m}}, \mu_{\mathrm{m}}) \qquad (r \to \infty). \tag{4.11}$$

Hence σ_∞ and b_2^{m} are related by

$$\sigma_\infty = -2\mu_{\mathrm{m}} b_2^{\mathrm{m}}, \tag{4.12}$$

so that b_2^{m} must be a constant. But in (4.10) we have an expression for $\mathbf{b}^{\mathrm{m}}$ in terms of F, and its 2-component is

$$b_2^{\mathrm{m}} = F\gamma \sum_{i=0}^{8} (\mathbf{b}_i^{\mathrm{m}})_2 \, \epsilon^i. \tag{4.13}$$

Hence substitution of (4.13) into (4.12) gives

$$F = \frac{-\sigma_\infty/\mu_{\mathrm{m}}}{2\gamma \sum_{i=0}^{8} (\mathbf{b}_i^{\mathrm{m}})_2 \, \epsilon^i}. \tag{4.14}$$

With this value of F, which depends on ϵ, all field values may be determined completely, in a form proportional to the specified σ_∞. A crucial feature is the occurrence of denominators which are octics in ϵ, and numerators in which the maximum power of ϵ is usually in the range 6–8, but in no case exceeds 9. Instead of powers of ϵ, one may equivalently use powers of a and b, by means of (4.2)–(4.5) instead of (4.9)–(4.10).

For a boundary-value problem of type (c), i.e. localised forcing, the analysis is similar, but typically involves field values specified in the fibre or coat, rather than the matrix, and leads to a different expression for F. As discussed after (2.7), the limiting values of $(\sigma_{rr}^{\mathrm{m}}, \sigma_{r\theta}^{\mathrm{m}})$ as $r \to \infty$ will now depend on ϵ, in contrast to the above. One may also leave F unspecified, i.e. solve the homogeneous problem (a). Although in principle this covers all cases, it should be remembered that many choices of F do not correspond to boundary-value problems which would arise in practice. For example, if F were taken to be independent of ϵ, then all distant and localised boundary values would depend on ϵ in a complicated way, and it is not easy to imagine what simple set of specified boundary conditions could lead to this.

(b) Non-uniformity in denominators for a stiff coat

Henceforth, we shall concentrate on the distant-forcing problem, and take the forcing amplitude F to be as given by (4.14). Let us define the *denominator polynomial* of F by

$$D^{\mathrm{m}} = D^{\mathrm{m}}(\epsilon) = \sum_{i=0}^{8} (\mathbf{b}_i^{\mathrm{m}})_2 \, \epsilon^i, \tag{4.15}$$

and examine its coefficients for a stiff coat, i.e. $\mu_{\mathrm{c}} \gg \max(\mu_{\mathrm{f}}, \mu_{\mathrm{m}})$. This is $\beta \gg 1$ in the notation of (1.1). Then in each coefficient in D^{m}, the highest-degree power of μ_{c} dominates the lower powers. On evaluating these coefficients, i.e. the quantities$(\mathbf{b}_i^{\mathrm{m}})_2$ for $i = 0, 1, \ldots, 8$, and picking out the highest powers, we find that the orders of magnitude of successive terms when $\beta \gg 1$ are in the ratio

$$(1, \, \beta\epsilon, \, \beta\epsilon^2, \, \beta\epsilon^3, \, \beta^2\epsilon^4, \, \beta\epsilon^5, \, \beta\epsilon^6, \, \beta\epsilon^7, \, \epsilon^8). \tag{4.16}$$

Here the power of β increases between the first and second terms, and between the fourth and fifth terms; hence for an expansion to be uniform with respect to β when $\beta \gg 1$, it is necessary to keep the first, second, and fifth terms, i.e.

$$(1,\ \beta\epsilon,\ \beta^2\epsilon^4), \tag{4.17}$$

because any one of these could be the largest depending on the value of β relative to $(1/\epsilon,\ 1/\epsilon^2,\ 1/\epsilon^3)$. For example, the term in $\beta^2\epsilon^4$ is largest if $\beta \gg 1/\epsilon^3$. The third and fourth terms in (4.16) are not needed for $\beta \gg 1$ because in these terms the power of β is no higher than in any preceding term: in progressing from the third to the fourth term, only the power of ϵ has increased, and for a thin coat we have $\epsilon \ll 1$, by definition. The same argument shows that terms beyond the fifth are not needed.

Without a theory such as that in the present paper, a numerical approach could be misleading. For example, if the first four terms were retained, i.e. up to order ϵ^3, it might appear numerically that convergence has occurred, because the the third and fourth terms are so much smaller than the first two; but in fact, the fifth term can be larger than any of these first four. Conversely, how one would know when to stop, if it has been found numerically that the fifth term is the largest? The above analysis provides an important conclusion for practical computation: however large the value of β, it is not necessary to go beyond the fifth term in approximation of $D^{\mathrm{m}}(\epsilon)$ for $\beta \gg 1$; but *to cover all cases, it is necessary that this fifth term be retained*.

(c) Non-uniformity in denominators for a soft coat

When the coat is soft, i.e. $\alpha \ll 1$ in the notation of (1.1), it is the lowest-degree powers of μ_{c} which dominate in the coefficients in D^{m}, and successive terms have orders of magnitude in the ratio

$$(\alpha^2,\ \alpha\epsilon,\ \epsilon^2,\ \alpha\epsilon^3,\ \epsilon^4,\ \alpha\epsilon^5,\ \epsilon^6,\ \alpha\epsilon^7,\ \alpha^2\epsilon^8). \tag{4.18}$$

Here the powers of α decrease between the first and second terms, and between the second and third terms. Therefore when $\alpha \ll 1$, *it is necessary to keep the first three terms*, i.e.

$$(\alpha^2,\ \alpha\epsilon,\ \epsilon^2), \tag{4.19}$$

but *the remaining six terms may be discarded*, because they are small compared with at least one of these first three terms when $\epsilon \ll 1$.

(d) Non-uniformity in numerators

We have performed a similar analysis to the above for the *numerator polynomials* of the vectors listed in (4.9)–(4.10) for both $\alpha \ll 1$ and $\beta \gg 1$, amounting to an inspection of 32 series in all. These polynomials are defined by the summations in (4.9)–(4.10), i.e. the right-hand sides without F and γ. One might have expected a simple pattern to emerge in the powers of α and β which occur, but none appears to be present. Thus to find out which powers of ϵ are important in a coefficient vector, direct calculation along the above lines appears to be called for. In general it is not necessary to go beyond the first five terms, i.e. beyond terms in ϵ^4; however, for particular values of material parameters, some coefficients turn out to be zero, and then further terms are needed. We give examples of this in §9. It seems fair to say that in the series we are studying, the degree of disorder is severe and unpredictable.

5. Jump in the field values across the coat

Since the state vector $\mathbf{u}$ is continuous at interfaces, as represented by the boundary conditions $\mathbf{u}^{\mathrm{f}}_a = \mathbf{u}^{\mathrm{c}}_a$ and $\mathbf{u}^{\mathrm{c}}_b = \mathbf{u}^{\mathrm{m}}_b$, we may omit the superscripts and write the interface values as $\mathbf{u}_a$ and $\mathbf{u}_b$, or in component form

$$\mathbf{u}_a = (u_{ra},\, u_{\theta a},\, \sigma_{rra},\, \sigma_{r\theta a}), \qquad \mathbf{u}_b = (u_{rb},\, u_{\theta b},\, \sigma_{rrb},\, \sigma_{r\theta b}). \tag{5.1}$$

The jump across the coat is $[\mathbf{u}] = [\mathbf{u}]_a^b = \mathbf{u}_b - \mathbf{u}_a$, or equivalently

$$[\mathbf{u}] = ([u_r],\, [u_\theta],\, [\sigma_{rr}],\, [\sigma_{r\theta}]) = ([u_r]_a^b,\, [u_\theta]_a^b,\, [\sigma_{rr}]_a^b,\, [\sigma_{r\theta}]_a^b). \tag{5.2}$$

There are various ways to write the dimensionless jump, and we select

$$[\mathbf{v}] = [\mathbf{v}]_a^b = ([u_r]/b,\, [u_\theta]/b,\, [\sigma_{rr}]/\mu_{\mathrm{c}},\, [\sigma_{r\theta}]/\mu_{\mathrm{c}}) \tag{5.3}$$

as being the most convenient for later formulae.

From (4.2)–(4.5) and the functional relations in §2(c), the dimensionless interface vectors $\mathbf{v}_a^{\mathrm{f}}$ and $\mathbf{v}_b^{\mathrm{m}}$ and jump vector $[\mathbf{v}]$ are found to have the form

$$\mathbf{v}_a^{\mathrm{f}} = F(a^6 b^2 \mathbf{v}_{62}^{\mathrm{f}} + a^4 b^4 \mathbf{v}_{44}^{\mathrm{f}} + b^8 \mathbf{v}_{08}^{\mathrm{f}}), \tag{5.4}$$

$$\mathbf{v}_b^{\mathrm{m}} = F\gamma(a^8 \mathbf{v}_{80}^{\mathrm{m}} + a^6 b^2 \mathbf{v}_{62}^{\mathrm{m}} + a^4 b^4 \mathbf{v}_{44}^{\mathrm{m}} + a^2 b^6 \mathbf{v}_{26}^{\mathrm{m}} + b^8 \mathbf{v}_{08}^{\mathrm{m}}), \tag{5.5}$$

$$[\mathbf{v}] = F(a^8 [\mathbf{v}]_{80} + a^7 b [\mathbf{v}]_{71} + \cdots + a b^7 [\mathbf{v}]_{17} + b^8 [\mathbf{v}]_{08}), \tag{5.6}$$

where γ is as defined after (4.5). On the right-hand sides here, the components of (5.4) and (5.6) are cubic in the shear moduli $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$ with coefficients which depend on $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$; the components of (5.5) are similar, but are quartic in $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$.

Some components of the vectors on the right of (5.4)–(5.6) are zero, and hence certain powers of a and b are absent from components of the corresponding vectors. For example, the last two components of $[\mathbf{v}]$, namely $([\sigma_{rr}]/\mu_{\mathrm{c}}, [\sigma_{r\theta}]/\mu_{\mathrm{c}})$, contain only even powers of a and b, and the first two components, $([u_r]/b, [u_\theta]/b)$, lack a term in $a^3 b^5$. One aspect of $[\mathbf{v}]$ is that it must be zero when $a = b$, because of the continuity of the state vector. A check reveals that this is so, and hence that $[\mathbf{v}]$ is divisible by $b - a$. On removing this factor, we could instead of (5.6) write $[\mathbf{v}]$ as proportional to an expression in

$$(a^7,\, a^6 b,\, \ldots,\, a b^6,\, b^7), \tag{5.7}$$

but there is no advantage in doing so, and the resulting expressions are longer.

As in §4, the above expressions in a and b may be written as polynomials in the dimensionless thickness ϵ. Thus $(\mathbf{v}_a^{\mathrm{f}}, \mathbf{v}_b^{\mathrm{m}}, [\mathbf{v}])$ are octics in ϵ, with expansions of the form

$$\mathbf{v}_a^{\mathrm{f}} = F \sum_{i=0}^{8} \mathbf{v}_{ai}^{\mathrm{f}} \epsilon^i, \qquad \mathbf{v}_b^{\mathrm{m}} = F\gamma \sum_{i=0}^{8} \mathbf{v}_{bi}^{\mathrm{m}} \epsilon^i, \qquad [\mathbf{v}] = F \sum_{i=1}^{8} [\mathbf{v}]_i \epsilon^i. \tag{5.8}$$

The vector coefficients $\mathbf{v}_{ai}^{\mathrm{f}}$ and $[\mathbf{v}]_i$ have components which are cubic in $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$, while the components of $\mathbf{v}_{bi}^{\mathrm{m}}$ are quartic. The expansion of $[\mathbf{v}]$ has no term with $i = 0$, because $[\mathbf{v}] = 0$ when $\epsilon = 0$. On converting these dimensionless formulae to their dimensional counterparts, using the definitions at the start of this section, and examining components, we find for example that $[u_r]$ and $[u_\theta]$ include terms up to ϵ^9. Since the stress components in $[\mathbf{v}]$ are made dimensionless by a factor μ_{c}, it follows that $[\sigma_{rr}]$ and $[\sigma_{r\theta}]$ are quartic in $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$. Here we are defining ϵ so that $a = a_0(1 - \epsilon/2)$ and $b = a_0(1 + \epsilon/2)$, from (4.7)–(4.8). If ϵ is defined by $b = a(1 + \epsilon)$, the results are similar, except that the coefficients have a different dependence on $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$.

(a) Numerators of the jump across a stiff coat

To determine the jump across the coat, we need the field values

$$(u_{ra},\, u_{\theta a},\, \sigma_{rra},\, \sigma_{r\theta a}) \quad \text{and} \quad (u_{rb},\, u_{\theta b},\, \sigma_{rrb},\, \sigma_{r\theta b}), \tag{5.9}$$

at a and b respectively, from which the jump

$$([u_r],\, [u_\theta],\, [\sigma_{rr}],\, [\sigma_{r\theta}]) \tag{5.10}$$

is obtained by subtraction. These quantities are proportional to F, from (5.8), and so have the denominator $D^{\mathrm{m}}(\epsilon)$ defined in (4.15). By omitting the factor F, we obtain the numerator polynomials, and as in §4(*b*) we may determine which powers of ϵ are needed for an approximation to be uniformly valid. On carrying this out for a stiff coat, i.e. $\beta \gg 1$, the result is that for the numerators of

$$(u_{ra},\ u_{\theta a},\ \sigma_{rra},\ \sigma_{r\theta a},\ u_{rb},\ u_{\theta b}) \tag{5.11}$$

two terms must be kept, with orders of magnitude in the ratio $(1, \beta\epsilon)$, whereas for each of

$$(\sigma_{rrb},\ \sigma_{r\theta b}) \tag{5.12}$$

a further term proportional to ϵ^4 is required, and terms in the ratio $(1, \beta\epsilon, \beta^2\epsilon^4)$ must be kept. For the jump (5.10), the numerators require two terms, but which two depends on the component: they are in the ratio $(\epsilon, \beta\epsilon^4)$ for $([u_r], [\sigma_{rr}], [\sigma_{r\theta}])$, and $(\epsilon, \beta\epsilon^2)$ for $[u_\theta]$.

The significance of these results is that for an approximation to cover the entire range of β for $\beta \gg 1$, the above terms are needed, but no others. In particular regimes, for example $\beta \sim 1/\epsilon$, or $\beta \sim 1/\epsilon^3$, individual terms can be neglected, but no term can be neglected for the entire range $\beta \gg 1$. The irregularity of the retained terms is noteworthy. Once all these terms are known, it it natural to inspect them in search of a pattern which would determine the terms to be retained; but none is evident.

(b) Cancellation and promotion

A noteworthy point in determining the jumps from the field values at a and b is that some terms cancel out exactly. In consequence, a knowledge of the dominant terms in u_{ra} and u_{rb}, for example, is not sufficient to determine the dominant terms in $[u_r]$, because any or all of these terms in u_{ra} and u_{rb} might cancel, and then later terms are 'promoted'. In general, one may use the term *promotion* to refer to the process in which *a term which is negligible at one stage of a calculation becomes dominant at a later stage*. To determine when this occurs, there is no escape from accurate calculation of higher order terms in field values. In general, the coefficients of these terms are complicated functions of $(\mu_{\mathrm{f}}, \mu_{\mathrm{c}}, \mu_{\mathrm{m}})$ and $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$, and advance warning is not available about when two of these coefficients, one for the field at a and one for the field at b, will turn out to be exactly equal. Thus although the final result, namely the terms to be retained in the jumps, is easy enough to state as above, the amount of analytical work required to obtain it is considerable, and is beyond the powers of hand calculation if one insists, as we do in this paper, that all results are to be obtained rigorously from the full elastostatic equations.

(c) Numerators of the jump across a soft coat

A soft coat is defined by $\alpha \ll 1$. We now find that two terms must be kept in the numerators of

$$(u_{ra},\ u_{\theta a},\ \sigma_{rra},\ \sigma_{r\theta a},\ \sigma_{rrb},\ \sigma_{r\theta b}), \tag{5.13}$$

the orders of magnitude being in the ratio (α, ϵ), whereas three terms must be kept for

$$(u_{rb},\ u_{\theta b}), \tag{5.14}$$

the orders being in the ratio $(\alpha^2, \alpha\epsilon, \epsilon^2)$. For the jump, two terms are required in the numerators, as for a stiff coat; for $([u_r], [u_\theta], [\sigma_{r\theta}])$ the orders of these terms are in the ratio $(\alpha\epsilon, \epsilon^2)$, whereas for $[\sigma_{rr}]$ the orders are in the ratio $(\alpha\epsilon, \epsilon^4)$. Our earlier remarks about the irregularity of these terms, and the importance of accounting for cancellation and promotion, apply here just as for a stiff coat.

(d) Disorder in cross-coat Taylor series

Our approach above has been to analyse numerators and denominators separately, giving approximations by means of rational functions of ϵ. An alternative is to calculate Taylor series approximations, which requires division by D^{m}. Now we saw in (4.17) and (4.19) that for problem (b), distant forcing, the dominant terms in D^{m} are in the ratio $(1, \beta\epsilon, \beta^2\epsilon^4)$ for a stiff coat, and in the ratio $(\alpha^2, \alpha\epsilon, \epsilon^4)$ for a soft coat. Hence on dividing by D^{m}, and expanding in powers of ϵ by means of the binomial theorem, the resulting series are disordered or divergent whenever $\beta \gg 1/\epsilon$ or $\alpha \ll \epsilon$. Moreover, the terms in the series may decrease only slowly (and therefore be of limited use) under the much less restrictive conditions $\beta \geq O(1/\epsilon)$ or $\alpha \leq O(\epsilon)$. Thus the conditions for the series to be useful may be quite onerous, for example when $\epsilon \sim 1/10$.

We have seen that for the fully specified distant-forcing boundary-value problem, it is necessary to divide all field quantities by D^{m}. Hence these onerous conditions cannot be evaded when the Taylor series is constructed. On the other hand, for the homogeneous problem, in which F is regarded as 'arbitrary constant', it is natural not to expand F, and then the series expansions are of the numerators only, which are polynomials. Questions of convergence would not then arise. However, such expansions raise difficulties of interpretation, particularly in relation to the question 'What is being held constant when ϵ is varied?'. It seems safer to consider always a fully specified boundary-value problem, in which case a denominator which is a function of ϵ will always be present.

(e) Normalised cross-coat jumps

Let us define normalised jumps by

$$([u_r]/u_{rb},\ [u_\theta]/u_{\theta b},\ [\sigma_{rr}]/\sigma_{rrb},\ [\sigma_{r\theta}]/\sigma_{r\theta b}). \tag{5.15}$$

These do not depend on F, because it cancels out, and in particular they do not depend on which of the boundary-value problems (a), (b), or (c) is being solved. Their orders of magnitude, in a compact notation, are

$$\left(\frac{[u_r]}{u_{rb}}, \frac{[u_\theta]}{u_{\theta b}}, \frac{[\sigma_{rr}]}{\sigma_{rrb}}, \frac{[\sigma_{r\theta}]}{\sigma_{r\theta b}}\right) \sim \begin{cases} \left(\dfrac{\epsilon+\beta\epsilon^4}{1+\beta\epsilon}, \dfrac{\epsilon+\beta\epsilon^2}{1+\beta\epsilon}, \dfrac{\beta\epsilon+\beta^2\epsilon^4}{1+\beta\epsilon+\beta^2\epsilon^4}, \dfrac{\beta\epsilon+\beta^2\epsilon^4}{1+\beta\epsilon+\beta^2\epsilon^4}\right) & (\beta \gg 1) \\ \left(\dfrac{\alpha\epsilon+\epsilon^2}{\alpha^2+\alpha\epsilon+\epsilon^2}, \dfrac{\alpha\epsilon+\epsilon^2}{\alpha^2+\alpha\epsilon+\epsilon^2}, \dfrac{\alpha\epsilon+\epsilon^4}{\alpha+\epsilon}, \dfrac{\alpha\epsilon+\epsilon^2}{\alpha+\epsilon}\right) & (\alpha \ll 1) \end{cases} \tag{5.16}$$

for the two cases of a stiff coat and soft coat, respectively. All order-one coefficients here have been replaced by 1, so that, for example, the first displayed term on the right indicates that $[u_r]/u_{rb}$ may be approximated for $\beta \gg 1$ by the ratio of the sum of two terms of orders $(\epsilon, \beta\epsilon^4)$ to the sum of two terms of orders $(1, \beta\epsilon)$. The exact coefficients are functions of $(\kappa_{\mathrm{f}}, \kappa_{\mathrm{c}}, \kappa_{\mathrm{m}})$. Alternative normalisations can be based on field values at a, or the mean of the field values at a and b, giving results of the same type as (5.16).

The expressions in (5.16) simplify if α or β are scaled with appropriate powers of ϵ, because individual terms in a numerator or denominator can then be ignored. For example, consider the relation

$$\frac{[\sigma_{rr}]}{\sigma_{rrb}} \sim \frac{\alpha\epsilon+\epsilon^4}{\alpha+\epsilon} \tag{5.17}$$

for a soft coat. Here the significant regimes, obtained by balancing orders of magnitude, are $\alpha \sim \epsilon^3$ and $\alpha \sim \epsilon$. Since $\epsilon^3 \ll \epsilon$, we can use the fact that if $\alpha \ll \epsilon^3$ then $\alpha \ll \epsilon$, and likewise if $\alpha \gg \epsilon$ then $\alpha \gg \epsilon^3$. Hence five different regimes may be identified, defined by the relations

$$\alpha \ll \epsilon^3, \quad \alpha \sim \epsilon^3, \quad \epsilon^3 \ll \alpha \ll \epsilon, \quad \alpha \sim \epsilon, \quad \alpha \gg \epsilon, \tag{5.18}$$

and correspondingly the dimensionless jump, written in our compact notation, takes the five different forms

$$\epsilon^3, \quad \alpha+\epsilon^3, \quad \alpha, \quad \frac{\alpha\epsilon}{\alpha+\epsilon}, \quad \epsilon. \tag{5.19}$$

Here again, order-one factors have been replaced by 1, to emphasize functional forms without extraneous detail. The same type of scaling analysis can be applied to all the terms on the right of (5.16), giving a large number of identifiable regimes.

Because F cancels out in the normalised jumps, the following phenomenon may occur, which could be very trying numerically in certain parameter regimes: individual field values and jumps, such as (u_{ra}, u_{rb}) and $[u_r]$, may have badly disordered or even divergent series expansions in ϵ, but the normalised jump $[u_r]/u_{rb}$ (or any alternative form such as $[u_r]/u_{ra}$) may nevertheless be well-ordered and convergent. This will happen when the disorder arises from F rather than from the field values and jumps without this factor, i.e. the numerators. It is another example of the great variability of the types of series which can be encountered.

6. Analytic continuation to the coat mid-surface

In deriving effective boundary conditions for an interface, it is often convenient to replace the original interface, of non-zero thickness, by a surface of zero thickness, and determine the jumps across this surface. To do this accurately, it is necessary to extend the field values by analytic continuation from outside the interface to a hypothetical surface inside the interface. We now carry this out by extending the field in the fibre and matrix to the mid-surface of the coat, defined as a cylinder of radius $a_0 = (a+b)/2$. Recall our definition in (4.7) that the dimensionless thickness of the coat is $\epsilon = (b-a)/a_0$, so that $a = a_0(1-\epsilon/2)$ and $b = a_0(1+\epsilon/2)$. Therefore the field values in the fibre are to be extended from their original domain $r \le a$ up to $r \le a_0$, and the values in the matrix from $r \ge b$ down to $r \ge a_0$. This gives effective values of fibre and matrix quantities at the mid-surface, from which the jumps in field values across it are obtained by subtraction. These will be compared with the jumps calculated in §6, which are across the entire thickness of the coat.

The analytic continuation is immediate, since all quantities are polynomials or rational functions of r, and so we may use the formulae derived already, but evaluate them at arbitrary r. The notation we shall use is that a subscript 0 denotes evaluation at the mid-surface, so that the analytically continued field values on its two sides are $\mathbf{u}_0^{\mathrm{f}}$ and $\mathbf{u}_0^{\mathrm{m}}$, or in component form

$$(u_{r0}^{\mathrm{f}}, u_{\theta 0}^{\mathrm{f}}, \sigma_{rr0}^{\mathrm{f}}, \sigma_{r\theta 0}^{\mathrm{f}}) \quad \text{and} \quad (u_{r0}^{\mathrm{m}}, u_{\theta 0}^{\mathrm{m}}, \sigma_{rr0}^{\mathrm{m}}, \sigma_{r\theta 0}^{\mathrm{m}}). \tag{6.1}$$

The jump across the mid-surface is $[\mathbf{u}_0] = [\mathbf{u}_0]_{\mathrm{f}}^{\mathrm{m}} = \mathbf{u}_0^{\mathrm{m}} - \mathbf{u}_0^{\mathrm{f}}$, or equivalently

$$[\mathbf{u}_0] = ([u_{r0}], [u_{\theta 0}], [\sigma_{rr0}], [\sigma_{r\theta 0}]). \tag{6.2}$$

As in §5(*a*), these quantities are proportional to F, and so have the denominator $D^{\mathrm{m}}(\epsilon)$. By omitting F, we obtain the numerator polynomials, and our aim is to calculate their series expansions in ϵ, keeping only those terms needed for approximations to be uniformly valid for an arbitrarily stiff or soft coat. As the method is similar to that of §5, we simply give the main results, indicating where these differ from the cross-coat jumps.

(a) Numerators of the mid-surface jump for a stiff coat

For a stiff coat, we have $\beta \gg 1$. When the numerators of the eight components listed in (6.1) are expanded in powers of ϵ, it is found that they fall into three groups. The first group consists of the four components of $\mathbf{u}_0^{\mathrm{f}}$, each requiring two terms in the ϵ-series of the numerator to be kept, these being in the ratio $(1, \beta\epsilon)$. The second group consists of the two displacement components u_{r0}^{m} and

$u^{\mathrm{m}}_{\theta 0}$, each requiring three terms in the numerator, in the ratio

$$(1,\ \beta\epsilon,\ \beta^2\epsilon^5). \tag{6.3}$$

The third group consists of the stress components $\sigma^{\mathrm{m}}_{rr0}$ and $\sigma^{\mathrm{m}}_{r\theta 0}$, which also require three terms in the ϵ-series of the numerators, but in the different ratio

$$(1,\ \beta\epsilon,\ \beta^2\epsilon^4). \tag{6.4}$$

Turning now to the jump components (6.2), we find that they fall into two groups: the displacement jumps $[u_{r0}]$ and $[u_{\theta 0}]$, which require three terms in the numerators, in the ratio

$$(\epsilon,\ \beta\epsilon^2,\ \beta^2\epsilon^5), \tag{6.5}$$

and the stress jumps $[\sigma_{rr0}]$ and $[\sigma_{r\theta 0}]$, which require only two terms, in the ratio

$$(\beta\epsilon,\ \beta^2\epsilon^4). \tag{6.6}$$

In (6.3) and (6.5), the occurrence of terms proportional to ϵ^5 is surprising, being without counterpart in the cross-coat series. These terms are needed if the mid-surface series for a stiff coat are to be uniformly valid for arbitrarily large β. We discuss this further in §8, under the heading of possible anomalies introduced by analytic continuation. Just as for the cross-coat series, there is no obvious pattern in the mid-surface series found here.

(b) Numerators of the mid-surface jump for a soft coat

For a soft coat, $\alpha \ll 1$, the three groups of components are the the same as in (a) above. In the first group, namely the four components of $\mathbf{u}^{\mathrm{f}}_0$, each component requires two terms in the ϵ-series of the numerator, in the ratio (α, ϵ). The other two groups each require three terms, but these differ between the two groups: the displacement components u^{m}_{r0}, and $u^{\mathrm{m}}_{\theta 0}$ require numerator terms in the ratio

$$(\alpha^2,\ \alpha\epsilon,\ \epsilon^2), \tag{6.7}$$

whereas for the stress components $\sigma^{\mathrm{m}}_{rr0}$ and $\sigma^{\mathrm{m}}_{r\theta 0}$ the numerator terms are in the ratio

$$(\alpha^2,\ \alpha\epsilon,\ \epsilon^3). \tag{6.8}$$

The displacement jumps $[u_{r0}]$ and $[u_{\theta 0}]$ require only two terms in their numerators, namely

$$(\alpha\epsilon,\ \epsilon^2), \tag{6.9}$$

whereas the stress jumps $[\sigma_{rr0}]$ and $[\sigma_{r\theta 0}]$ require

$$(\alpha^2\epsilon,\ \alpha\epsilon^2,\ \epsilon^3). \tag{6.10}$$

The terms in ϵ^3 in (6.8) and (6.10) are surprising, and are discussed in §8.

(c) Disorder in mid-surface Taylor series

If the Taylor series of the mid-surface jumps are required, they are found by the method of §5(*d*), involving division by $D^{\mathrm{m}}(\epsilon)$. The details are as before, because $D^{\mathrm{m}}(\epsilon)$ is the same in each case. As previously, the conditions for the resulting series to be neither disordered nor divergent can be quite onerous.

(d) Normalised mid-surface jumps

In the same notation as §5(*e*), the normalised jumps at the mid-surface, for a stiff coat and soft coat, are

$$\left(\frac{[u_{r0}]}{u^{\mathrm{m}}_{r0}},\frac{[u_{\theta 0}]}{u^{\mathrm{m}}_{\theta 0}},\frac{[\sigma_{rr0}]}{\sigma^{\mathrm{m}}_{rr0}},\frac{[\sigma_{r\theta 0}]}{\sigma^{\mathrm{m}}_{r\theta 0}}\right)$$
$$\sim\begin{cases}\left(\dfrac{\epsilon+\beta\epsilon^2+\beta^2\epsilon^5}{1+\beta\epsilon+\beta^2\epsilon^5},\dfrac{\epsilon+\beta\epsilon^2+\beta^2\epsilon^5}{1+\beta\epsilon+\beta^2\epsilon^5},\dfrac{\beta\epsilon+\beta^2\epsilon^4}{1+\beta\epsilon+\beta^2\epsilon^4},\dfrac{\beta\epsilon+\beta^2\epsilon^4}{1+\beta\epsilon+\beta^2\epsilon^4}\right) & (\beta\gg 1)\\[2ex] \left(\dfrac{\alpha\epsilon+\epsilon^2}{\alpha^2+\alpha\epsilon+\epsilon^2},\dfrac{\alpha\epsilon+\epsilon^2}{\alpha^2+\alpha\epsilon+\epsilon^2},\dfrac{\alpha^2\epsilon+\alpha\epsilon^2+\epsilon^3}{\alpha^2+\alpha\epsilon+\epsilon^3}\dfrac{\alpha^2\epsilon+\alpha\epsilon^2+\epsilon^3}{\alpha^2+\alpha\epsilon+\epsilon^3}\right) & (\alpha\ll 1).\end{cases} \tag{6.11}$$

The symbol $\sim$ indicates that order-one multiplying factors on the right-hand side have been replaced by 1. Individual regimes can be identified by scaling α or β with appropriate powers of ϵ; each scaling gives a reduced form. Since several regimes exist for each jump on the right of (6.11), the number of reduced forms is large. As in §5(*e*), a factor F is not present in normalised jumps, even when it is present earlier. Hence normalised jumps may be well-ordered despite the fact that field values and jumps are disordered.

7. Analytic continuation to arbitrary radius in the coat

Let us now extend the field in the fibre and matrix not just to a cylinder of radius $a_0=(a+b)/2$ as above, but to a cylinder of arbitrary radius dividing the interval $[a,b]$ in the ratio $k:1-k$, where $0\le k\le 1$. This radius is

$$a_0^{(k)}=(1-k)a+kb, \tag{7.1}$$

or equivalently

$$a_0^{(k)}=a_0\Big(1+\Big(k-\frac{1}{2}\Big)\epsilon\Big), \tag{7.2}$$

since $a=a_0(1-\epsilon/2)$ and $b=a_0(1+\epsilon/2)$. Thus our previous results are for $k=1/2$, with $a_0^{(1/2)}=a_0$. Although the effect of placing an interface at an arbitrary radius within the coat has been analysed previously (see the review article [19] and citations therein for a variety of approaches), we have not found an approach similar to ours in the literature, and the following analysis and results appear to be new.

The question of interest is whether different sets of dominant terms could arise through the dependence of coefficients on k; this would happen if previously non-zero coefficients for $k=1/2$ become zero when $k\neq 1/2$, or conversely if new terms with non-zero coefficients arise. The functional dependence of all coefficients on k is readily calculated, and it shows that the formulae just given still apply for any fixed value of k in the range $0\le k<1$. (We deal separately with the case $k=1$ below.) For example, consider the value $k=0$, corresponding to analytical continuation of the field in the matrix down to $r=a$. Few coefficients are then zero, and none of these is in a dominant term; hence no change arises in the terms to be included or excluded in series expansions in ϵ, though of course the numerical values of the coefficients are different. Note that placing the hypothetical surface at $r=a$, and extending the field in the matrix down to this interface with the fibre, is different from solving a three-phase problem with a coat of vanishingly small thickness at $r=a$: in this latter case, the field is continuous at $r=a$, but in the former case, there is a jump between the field values on the two sides of $r=a$.

Matters are different for $k=1$, i.e. analytic continuation up to $r=b$ from the fibre. Explicit formulae, in the form of functions of k, reveal that many coefficients contain $k-1$ as a factor, and so vanish when $k=1$. For example, in §6(*a*) we saw that for a stiff coat, the numerators of the mid-coat jumps $[u_{r0}]$ and $[u_{\theta 0}]$ require terms in the ratio $(\epsilon,\beta\epsilon^2,\beta^2\epsilon^5)$. However, the coefficient of $\beta^2\epsilon^5$ contains $k-1$ as a factor, so that the term is absent when $k=1$. In this case, it is necessary to inspect later terms in the series, for example terms of order

$$\beta^2\epsilon^6,\ \beta^2\epsilon^7,\dots, \tag{7.3}$$

as they could be promoted; that is, although they are negligible compared with the term of order $\beta^2\epsilon^5$ when $k \neq 1$, one of them could become dominant when $k = 1$. On inspecting these later terms, we find that every term of order $\beta^2\epsilon^n$ for $n \geq 5$ has a factor $k - 1$ in its coefficient. In conjunction with the fact that there are no terms of order $\beta^m\epsilon^n$ for $m > 2$, this implies that the numerators of the mid-coat jumps $[u_{r0}]$ and $[u_{\theta 0}]$ each require only two terms, in the ratio

$$(\epsilon,\ \beta\epsilon^2). \tag{7.4}$$

The same phenomenon occurs in the numerators of the field values u^{m}_{r0} and $u^{\mathrm{m}}_{\theta 0}$, where the terms needed are now only

$$(1,\ \beta\epsilon). \tag{7.5}$$

For the other field values and jumps for a stiff coat, the required dominant terms for $k = 1$ are the same as for $k < 1$.

Turning now to a soft coat for, we find that in the jumps $[\sigma_{rr0}]$ and $[\sigma_{r\theta 0}]$ the coefficients of $(\alpha\epsilon^2, \epsilon^3)$ in the numerator each contain a factor $k - 1$, but a later term of order $\alpha\epsilon^4$ does not, and hence the terms to be retained when $k = 1$ are in the ratio

$$(\alpha^2\epsilon,\ \alpha\epsilon^4). \tag{7.6}$$

Here we use the fact that although the coefficients contain a term of order $\alpha\epsilon^3$, nevertheless this term contains a factor $k - 1$; without this factor, the term $\alpha\epsilon^3$ would be retained in preference to $\alpha\epsilon^4$, since $\alpha\epsilon^3 \gg \alpha\epsilon^4$ when $\epsilon \ll 1$. Similarly the numerators of the field values $\sigma^{\mathrm{m}}_{rr0}$ and $\sigma^{\mathrm{m}}_{r\theta 0}$ require terms in the ratio

$$(\alpha^2,\ \alpha\epsilon) \tag{7.7}$$

when $k = 1$; this requires the fact that every term of order ϵ^n for $n \geq 3$ has a factor $k - 1$ in its coefficient, as otherwise another term would be needed. There is no term of order ϵ or ϵ^2 for any k. For other field values and jumps for a soft coat, the dominant terms for $k = 1$ are as for $k < 1$. It will be appreciated that the type of inference we are making in this and the previous paragraph requires careful attention to logic.

8. Anomalous terms introduced by analytic continuation

To explain the behaviour of the series in §8 when the parameter k is varied, a suitable starting-point is an examination of all the series presented in this paper so far. Such an examination reveals that a small number of terms appear to be anomalous, in that they do not fit the pattern of the results taken as a whole. These are the cross-surface terms in $\beta^2\epsilon^5$ for a stiff coat and the cross-surface terms in ϵ^3 for a soft coat, present when $k < 1$. The question therefore arises as to whether they are artefacts introduced by analytic continuation.

A check reveals that such terms are produced by continuation from the matrix phase in the direction of decreasing r, but not by continuation in the opposite direction, i.e. from the fibre phase while increasing r. Let us therefore make the hypothesis that the cause of the anomalous terms is analytic continuation in the direction of decreasing r.

This hypothesis is tentative, but it fits the facts. In particular, it explains two features of our results. First, it is consistent with the observation that there are no anomalous terms in the cross-layer jumps: these jumps do not involve analytic continuation at all. Second, it explains why there are no anomalous terms in the cross-surface jumps for $k = 1$: these jumps involve only analytic continuation in the direction of increasing r, because when $k = 1$ we have $a_0^{(k)} = b$. For $k < 1$ there is some analytic continuation in the direction of decreasing r, from $r = b$ down to the lesser value $r = (1 - k)a + kb$.

Moreover, the explanation is robust in the following sense. One might argue that for the surface jumps as a function of k, the small parameter $\epsilon = (b-a)/a_0$ is artificial, and a more natural choice would be

$$\epsilon^{(k)} = (b-a)/a_0^{(k)}, \tag{8.1}$$

based on the actual position of the surface at $a_0^{(k)}$ rather than the mid-coat position a_0. Therefore we repeated the calculations to obtain all quantities as series expansions in $\epsilon^{(k)}$ instead of ϵ. The coefficients in the resulting series become different functions of k from those obtained before; but in every case the presence or absence of a factor $k-1$ in a coefficient is unaltered, even when the dominant terms to be retained in an expansion depend on this factor being present in a large number of terms. Thus the the order-of-magnitude behaviour of the series expansions presented in the paper does not depend on the particular form of the small parameter used to represent the thickness of the coat.

The field in the fibre contains only positive powers of r (because it must be finite and continuous at the origin), whereas the field in the matrix includes negative powers of r. Perhaps this explains why analytic continuation outwards from the fibre to larger r does not produce anomalous terms, but continuation inwards from the matrix does produce them. In advance of further investigation this is speculative; however, *the occurrence of anomalous terms which can be dominant in certain parameter regimes for a stiff or soft coat after analytic continuation is definitely established by our results, and our demonstration of their existence appears to be a new result.*

9. Illustrative examples

To illustrate the above theory, we now present detailed formulae for the jump in the radial displacement for a stiff coat, i.e. $\beta \gg 1$. We do this for both the mid-surface jump and the cross-coat jump, and by selecting particular numerical values for shear moduli and Poisson's ratios, we give examples of promotion of terms, as described in §5(*b*). One example included in this scheme is that of incompressible media, for which the Kolosov constants all take the value 2, and we show that in this case the radial jump is of a smaller order of magnitude than for other values; this is in accord with physical intuition, and is another confirmation of the theory.

For definiteness, we take $\mu_{\rm f} = \mu_{\rm m}$ in what follows, and chose a system of units in which $\mu_{\rm m} = 1$ and $a_0 = 1$. Since all quantities may be expressed in terms of $\mu_{\rm m}$, a_0, and dimensionless quantities, this imposes no restriction; it may be checked that $(D^{\rm m}, \gamma, F)$ are proportional to $(\mu_{\rm m}^4 a_0^8,\ 1/\mu_{\rm m}, 1/(\mu_{\rm m}^3 a_0^8))$, consistent with $[u_{r0}]$ having the dimension of a_0. Recall that F is defined by (3.6), and $(\mathbf{n}^{(1)}, \mathbf{n}^{(3)}, \mathbf{n}^{(4)})$ by (4.1); other definitions are possible, but the advantage of those used here is that most quantities are then polynomials, and so can be evaluated exactly without convergence theory.

(a) Mid-surface jump in radial displacement

We saw in (6.5) that for a stiff coat, the mid-surface jump $[u_{r0}]$ requires, in general, three terms for a uniform approximation to its numerator, in the ratio (ϵ, $\beta\epsilon^2$, $\beta^2\epsilon^5$). In more detail, this jump is

$$\begin{aligned}[u_{r0}] = \frac{F\beta^2}{\kappa_{\rm m}+1}\big\{ &-2(\kappa_{\rm f}+1)(\kappa_{\rm c}^2-1)(\kappa_{\rm m}+1)\epsilon \;-\; 4(2\kappa_{\rm f}\kappa_{\rm m}+\kappa_{\rm f}-\kappa_{\rm m}-2)(\kappa_{\rm c}+1)\beta\epsilon^2 \\ &- 16\kappa_{\rm f}(\kappa_{\rm m}-1)\beta^2\epsilon^5 \;+\; O(\epsilon/\beta, \epsilon^2, \beta\epsilon^3, \beta^2\epsilon^6, \ldots)\big\}.\end{aligned} \tag{9.1}$$

For example, when $(\kappa_{\rm f}, \kappa_{\rm c}, \kappa_{\rm m}) = (1, 2, 2)$, we obtain

$$[u_{r0}] \simeq F\beta^2\big(-12\epsilon - 4\beta\epsilon^2 - \frac{16}{3}\beta^2\epsilon^5\big). \tag{9.2}$$

However, it may happen that some or all of the displayed coefficients are zero. For example, suppose that all three media are incompressible, i.e. $(\kappa_{\rm f}, \kappa_{\rm c}, \kappa_{\rm m}) = (1, 1, 1)$, corresponding to the three Poisson's ratios $(\nu_{\rm f}, \nu_{\rm c}, \nu_{\rm m})$ all taking the value $1/2$. In this special case, all three displayed coefficients in (9.1) are zero, which means that a number of previously neglected terms must now be promoted. This requires inspection of the coefficients of the terms indicated by $O(\epsilon/\beta, \epsilon^2, \beta\epsilon^3 \dots)$ indicated in (9.1). We find that the dominant terms then arise from the expression

$$[u_{r0}] = \frac{F\beta^2}{\kappa_m + 1}\{ -(11\kappa_f\kappa_m - 28\kappa_f - 4\kappa_m + 5)(\kappa_c + 1)\beta\epsilon^3 + 4\kappa_f(5 - \kappa_m)\beta^2\epsilon^6 + O(\epsilon^3, \beta\epsilon^4, \beta^2\epsilon^7, \dots)\}, \tag{9.3}$$

evaluated at $(\kappa_f, \kappa_c, \kappa_m) = (1, 1, 1)$, which gives

$$[u_{r0}] \simeq F\beta^2(16\beta\epsilon^3 + 8\beta^2\epsilon^6). \tag{9.4}$$

Thus in the incompressible case, the expansion begins with a term in ϵ^3 rather than merely ϵ.

If the three media have identical parameters, then in effect there is only one medium, and all mid-surface jumps must be zero when calculated without approximation. We have checked that all our general formulae satisfy this condition when $\mu_f = \mu_c = \mu_m$ and $\kappa_f = \kappa_c = \kappa_m$. As a simple illustration, the full series for which (9.4) is a uniform approximation is

$$[u_{r0}] = F\{(16\beta^3 - 28\beta^2 + 12\beta)\epsilon^3 + (8\beta^3 + 16\beta^2 - 40\beta + 16)\epsilon^4 + (19\beta^3 - 10\beta^2 - 9\beta)\epsilon^5 + (8\beta^4 - 2\beta^3 + 2\beta - 8)\epsilon^6 + \cdots\}. \tag{9.5}$$

The value $\beta = 1$ here corresponds to $\mu_f = \mu_c = \mu_m$ (recall that we are taking $\mu_f = \mu_m$), and then the coefficients in (9.5), together with those not displayed, become zero, giving $[u_{r0}] = 0$, as expected. Similarly, we also obtain $[u_{\theta 0}] = 0$, $[\sigma_{rr0}] = 0$, and $[\sigma_{r\theta 0}] = 0$ in this case.

In the above expressions, the forcing term F contains the denominator D^m, defined in (4.15). We saw in (4.17) that D^m requires three terms for its uniform approximation, in the ratio $(1,\ \beta\epsilon,\ \beta^2\epsilon^4)$. In more detail, we have

$$D^m = \beta^2\{ -(\kappa_f + 1)(\kappa_c + 1)^2(\kappa_m + 1) - 2(4\kappa_f\kappa_m + 3\kappa_f + \kappa_m)(\kappa_c + 1)\beta\epsilon - 16\kappa_f\kappa_m\beta^2\epsilon^4 + O(\epsilon, \beta\epsilon^2, \beta\epsilon^3, \dots)\}. \tag{9.6}$$

Here the displayed coefficients cannot be zero, and so promotion of later terms does not take place for any parameter values. However, some of the neglected terms can be identically zero, rather than merely smaller in order of magnitude than the retained terms. For example, the $O(\epsilon)$ term has coefficient

$$-2(2\kappa_f\kappa_m + \kappa_f - \kappa_m - 2)(\kappa_c^2 - 1), \tag{9.7}$$

and the $O(\beta\epsilon^2)$ term has coefficient

$$-6(2\kappa_f\kappa_m - \kappa_f - \kappa_m)(\kappa_c + 1), \tag{9.8}$$

and these can be zero for a variety of values of $(\kappa_f, \kappa_c, \kappa_m)$.

(b) Cross-coat jump in radial displacement

As noted in §5(*a*), the cross-coat jump $[u_r]$ for $\beta \gg 1$ requires two terms for uniform approximation to its numerator, in the ratio $(\epsilon, \beta\epsilon^4)$. In more detail,

$$[u_r] = F\beta^2\{(\kappa_f + 1)(\kappa_c + 1)(3 - \kappa_c)\epsilon - 2\kappa_f(3\kappa_c - 5)\beta\epsilon^4 + O(\epsilon/\beta, \epsilon^2, \beta\epsilon^5, \dots)\}. \tag{9.9}$$

When $(\kappa_f, \kappa_c, \kappa_m) = (1, 2, 2)$, this gives

$$[u_r] \simeq F\beta^2(6\epsilon - 2\beta\epsilon^4). \tag{9.10}$$

The coefficient of $\beta\epsilon^4$ in (9.9) is zero when $\kappa_f = 5/3$, i.e. $\nu_c = 1/3$, and in this case the term of order $\beta\epsilon^5$ within parentheses is promoted. Its coefficient is $8\kappa_f$. When $(\kappa_f, \kappa_c, \kappa_m) = (1, 5/3, 2)$, this gives

$$[u_r] = \frac{8}{9}F\beta^2\{8\epsilon + 9\beta\epsilon^5 + O(\epsilon/\beta, \epsilon^2, \beta\epsilon^6, \dots)\}. \qquad (9.11)$$

In these expressions, the denominator D^{m} in F is the same as for the mid-surface jump, i.e. is given by (9.6).

10. Conclusion

The problem addressed in this paper, namely that of determining the elastic displacements and stresses in a three-phase circular cylindrical configuration subject to biaxial forcing, is classical and appears to be simple. But in fact, we have found great complexity in its solution as soon as one proceeds to take the limit of a thin coating, $\epsilon \ll 1$, while taking account of other parameters in the problem. The underlying physical reason for this is that a coating supports bending deformation as well as dilatation and shear, and the relative importance of the different types of deformation introduces different scaling regimes. This was the point of view adopted in [11], in which asymptotic regimes were identified at the outset, and used to construct the leading terms in series expansions. Our work may be regarded as an extension and confirmation of that work, in which *we do not posit regimes but deduce them*. This is achieved by starting with the exact solution of boundary-value problems formulated and solved within linear elasticity theory, and calculating their series expansions in ϵ. The detailed results in [11] for our case (a), the homogeneous problem, and in [16,17] for our case (b), the distant forcing problem, have proved invaluable, because they give explicit expressions for many coefficients. We have compared these with our results, and found agreement in all cases. This provides a powerful check, since the methods used are so different.

Our work provides underpinning theory for assessing the likely accuracy of different types of effective boundary conditions for a thin coating in the high-contrast limit. For the three-phase configuration investigated, we have pinpointed which of the terms in the various series expansions must be kept in any given parameter regime of shear moduli and coat thickness, and which may be discarded without introducing significant error. Importantly, we have shown that the forcing amplitude F must be calculated explicitly as function of the coat thickness ϵ in any particular boundary-value problem, since the form of its Taylor series expansion in ϵ provides the most onerous restrictions on the range of validity of any proposed approximation. In effect, F provides the denominator of the solution of a problem, indicated in our notation by the function $D^{\mathrm{m}}(\epsilon)$, and as commonly occurs in mathematical physics, the denominator is the crucial function for determining analytic structure and convergence behaviour.

We have identified parameter regimes in which terms up to high order must be kept. No doubt in more complicated examples than analysed here, the details and scalings encountered will be different, but our results give an indication of what might be expected in general. Especially, they suggest that *irregular and unpredictable disorder in the magnitude of early and middle terms of series expansions in the thickness may be the usual behaviour rather than the exception, and to understand this disorder it is essential to consider a fully specified boundary-value problem*.

Authors' Contributions. CJC performed the mathematical analysis, instigated the disordered-series and convergence approach, created the algorithms for computer coding, and wrote the first draft of the paper. SGM identified the problem and its physical background (conceptualisation), prepared the literature review and its discussion, and edited/reviewed the final draft.

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