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Singularly Perturbed Averaging with Application to Bio-Inspired 3D Source Seeking

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***Abstract*— We propose a novel 3D source seeking algorithm for rigid bodies with a non-collocated sensor inspired by the chemotactic navigation strategy of sea urchin sperm known as helical klinotaxis. We work directly with the rotation group SO(3) without parameterization in representing the attitude of a rigid body. As a consequence, the proposed algorithm does not require attitude feedback for implementation as opposed to all previous work on 3D source seeking. The stability of the proposed algorithm is proven using an intricate combination of singular perturbation and second order averaging.**

I. Introduction

Source seeking is the problem of locating a target that emits a scalar measurable signal, typically without any global positioning information. Extensive work has been done on the source seeking problem using extremum seeking control, of which we mention a few. For velocity-actuated point-like kinematics (i.e. single integrator dynamics), the source seeking problem can be easily solved using several non-resonating extremum seeking controllers if we assume full control authority on the velocity in all degrees of freedom. For double integrator dynamics, the problem is more difficult and requires careful considerations [1]. The situation is also difficult for kinematic models of planar rigid bodies under nonholonomic constraints, yet solutions have been proposed [2], [3]. Further difficulties arise when one considers the 2D underactuated dynamics with acceleration, rather than velocity, control [4]. However, most of the work in the literature focuses on the 2D case where the rigid body kinematics is described by a unicycle model. The pioneering work on the 3D case by Cochran et al [5] considers the 3D source seeking problem for an under-actuated vehicle with a rigid body kinematic model. However, in [5], a parameterization of the rotation group SO(3) through Euler angles is employed. In particular, the control law proposed in [5] assumes direct control authority on the Euler angle rates, which is not convenient for practical implementation; actuating the Euler angle rates requires measurement of the angles themselves. The situation is similar with all subsequent algorithms proposed for the 3D source seeking problem [6], [7], [8], [9], [10]. To elaborate, consider Euler's rigid body equations of motion:

$$\dot{\mathbf{q}} = \mathbf{R}\mathbf{v}, \qquad m\,\dot{\mathbf{v}} + m\,\boldsymbol{\omega}\times\mathbf{v} = \mathbf{f} \tag{1a}$$

$$\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\omega}}, \qquad \mathbf{J}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{J}\,\boldsymbol{\omega} = \boldsymbol{\tau} \tag{1b}$$

written in the body frame of reference, where $\mathbf{q}$ is the position of the center of the mass m, $\mathbf{R}$ is the rotation matrix relating the body axes to the axes of an inertial frame of reference, $\mathbf{v}$ and $\boldsymbol{\omega}$ are the linear and angular velocities in body coordinates, $\widehat{\boldsymbol{\omega}}$ is the skew-symmetric matrix associated with the vector $\boldsymbol{\omega}$, $\mathbf{J}$ is the moment of inertia matrix around the center of mass represented in the body frame, and the $\mathbf{f}$ and $\boldsymbol{\tau}$ are the body net forces and torques.

Control over free rigid body motion is achieved through the body forces and torques $\mathbf{f}$ and $\boldsymbol{\tau}$. All kinematic models are approximations of (1) when $\mathbf{f}$ and $\boldsymbol{\tau}$ are such that:

$$\mathbf{f} = \mathbf{C}_{\mathbf{v}}\,(\mathbf{v}_d - \mathbf{v}), \qquad \boldsymbol{\tau} = \mathbf{C}_{\boldsymbol{\omega}}\,(\boldsymbol{\omega}_d - \boldsymbol{\omega}) \tag{2}$$

where $\mathbf{v}_d$ and $\boldsymbol{\omega}_d$ and are considered as inputs, and the matrices $\mathbf{C}_{\mathbf{v}}$ and $\mathbf{C}_{\boldsymbol{\omega}}$ are positive definite with minimum eigenvalues that are much greater than the mass m and the maximum eigenvalue of the inertia tensor $\mathbf{J}$. Such an assumption is satisfied in the locomotion of microorganisms that swim in a low Reynolds number where viscous forces dominate inertia, or when high-gain velocity feedback is employed to enforce sufficient damping. In that setting, a singular perturbation argument leads to the quasi-steady approximation:

$$\dot{\mathbf{q}} = \mathbf{R}\mathbf{v}, \qquad \mathbf{v} = \mathbf{v}_d \tag{3a}$$

$$\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\omega}}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_d \tag{3b}$$

Notably, the control inputs in the quasi-steady approximation (3) are the linear and angular velocities in body coordinates.

If we are to design a source seeking controller for the kinematic model (3) that does not employ any global information, we must not use any information on either the position $\mathbf{q}$ or the orientation $\mathbf{R}$. However, a parameterization of the rotation matrix $\mathbf{R}$ through an Euler angle triplet $\boldsymbol{\theta}$ leads to relations of the form:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Phi}(\boldsymbol{\theta})\boldsymbol{\omega} \tag{4}$$

where $\boldsymbol{\Phi}$ involves the jacobian of the parameterization. As such, even if the motion stays within the range of attitudes in which the map $\boldsymbol{\Phi}$ is non-singular, any control authority over the Euler angles rates $\dot{\boldsymbol{\theta}}$ is achieved by inverting the map $\boldsymbol{\Phi}$ since one must only assume control over the angular velocities $\boldsymbol{\omega}$ in body coordinates when working with kinematic models of free rigid body motion. Hence, assuming

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direct control authority on the Euler angle rates $\dot{\boldsymbol{\theta}}$ implicitly requires measurement of at least some of the angles $\boldsymbol{\theta}$.

In this manuscript, we propose a novel 3D source seeking algorithm that only assumes control over the body angular velocities without any parameterization. As such, an implementation of the proposed algorithm using forces and torques would only require sufficiently strong damping either through an intrinsic damping mechanism in the system or an induced damping through velocity feedback, but it does not require attitude feedback. To the best of our knowledge, the proposed algorithm is the first 3D source seeking algorithm for rigid bodies with a non-collocated sensor based on extremum seeking that does not require any attitude information.

We remark that our proposed algorithm is inspired by the behavior of microorganisms that are routinely faced with the source seeking problem. For example, sea urchin sperm cells evolved to swim up the gradient of the concentration field of a chemical secreted by the egg [11], [12], thereby solving the source seeking problem, under direct evolutionary pressure. The sperm cells do so by swimming in helical paths that dynamically align with the gradient. The proposed algorithm is inspired by this strategy, which is also known as helical klinotaxis [13].

The remainder of this manuscript is organized as follows. In section II, we introduce the notation we use throughout the paper. In section III, we state without proof a trajectory approximation and stability result that is used in the computations and stability analysis. The proofs of the results in section III will appear in an extended version of the current manuscript. In section IV, we state the proposed 3D source seeking algorithm and analyze its stability. Numerical simulations are given in section V, followed by a conclusion in section VI.

II. Notations

The set $\mathcal{C}^k$ denotes the class of maps that are k-times differentiable with locally Lipschitz continuous derivatives. The notation $\partial_i \mathbf{f}$ denotes the derivative of the map $\mathbf{f}$ with respect to its i^{th} input. The Lie bracket between two vector fields $\mathbf{f}_i, \mathbf{f}_j$ on a domain $\mathcal{D}$ is denoted by $[\mathbf{f}_i, \mathbf{f}_j]$ and can be computed in coordinates as $[\mathbf{f}_i, \mathbf{f}_j] = \partial\mathbf{f}_j\, \mathbf{f}_i - \partial\mathbf{f}_i\, \mathbf{f}_j$, where $\partial\mathbf{f}_i$ is the jacobian of the map defining the vector field $\mathbf{f}_i$. The matrix exponential of a square matrix $\mathbf{F}$ is denoted as $\exp(\mathbf{F})$. The map $\widehat{\bullet} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ takes a vector $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^\intercal \in \mathbb{R}^3$ to the corresponding skew symmetric matrix, and it has the property that $\widehat{\mathbf{R}\boldsymbol{\omega}} = \mathbf{R}\widehat{\boldsymbol{\omega}}\mathbf{R}^\intercal$ when $\mathbf{R}$ is a rotation matrix, and the property that for two vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{R}^3$, the product $\widehat{\boldsymbol{\omega}}_1\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$, where $\times$ denotes the cross product between two vectors in $\mathbb{R}^3$. The vectors $\mathbf{e}_i \in \mathbb{R}^3$ for $i \in \{1, 2, 3\}$ denote the standard basis on $\mathbb{R}^3$. The vector $\mathbb{1}$ denotes the vector with all entries equal to unity. The notation $\mathcal{U}^{\mathcal{S}}_{\delta,\mathcal{D}}$ denotes the set $\{\mathbf{x} \in \mathcal{D} : \inf_{\mathbf{x}'\in\mathcal{S}} \|\mathbf{x} - \mathbf{x}'\| < \delta\}$ where $\mathcal{S} \subset \mathcal{D}$ is a bounded subset, and $\mathcal{D}$ is some domain of interest. The notation $\mathcal{U}^{\mathcal{S}}_{\delta}$ is defined similarly when the domain $\mathcal{D}$ is the entire space (i.e. $\mathbb{R}^n$ for some $n \in \mathbb{N}$).

III. Singularly Perturbed Second Order Averaging

In this section, we state, without proof, a trajectory approximation and stability result that will be used in analyzing the behavior of the proposed source seeking algorithm. The proofs of the results in this section will appear in a journal version of the current manuscript.

Consider the interconnected system:

$$\dot{\mathbf{x}}_1 = \sum_{k=1}^{2} \varepsilon^{k-2}\mathbf{f}_{1,k}(\mathbf{x}_1, \mathbf{x}_2, t, \varepsilon^{-2}t), \quad \mathbf{x}_1(t_0) = \mathbf{x}_{1,0} \tag{5a}$$

$$\dot{\mathbf{x}}_2 = \sum_{k=0}^{2} \varepsilon^{k-2}\mathbf{f}_{2,k}(\mathbf{x}_1, \mathbf{x}_2, t, \varepsilon^{-2}t), \quad \mathbf{x}_2(t_0) = \mathbf{x}_{2,0} \tag{5b}$$

where $\mathbf{x}_1, \mathbf{x}_{1,0} \in \mathbb{R}^{n_1}$, $\mathbf{x}_2, \mathbf{x}_{2,0} \in \mathbb{R}^{n_2}$, $t, t_0 \in \mathbb{R}$, and $\varepsilon \in (0, \infty)$ is a small parameter.

We adopt the following assumptions on the regularity of the right-hand side of the equations (5):

Assumption 3.1: Suppose that:

1) $\mathbf{f}_{j,k}(\cdot, \cdot, \cdot, \tau) \in \mathcal{C}^{3-k}\ \forall\tau \in \mathbb{R}$, and $\mathbf{f}_{j,k} \in \mathcal{C}^0$,
2) $\exists T > 0$ s.t. $\mathbf{f}_{j,k}$ are T-periodic in the last argument,
3) $\int_0^T \mathbf{f}_{j,1}(\cdot, \cdot, \cdot, \nu)d\nu = 0$
4) $\mathbf{f}_{2,0}(\mathbf{x}_1, \mathbf{x}_2, t, \tau) = \mathbf{F}\left(\mathbf{x}_2 - \boldsymbol{\varphi}_0(\mathbf{x}_1)\right)$, $\boldsymbol{\varphi}_0(\cdot) \in \mathcal{C}^3$, and $\mathbf{F}$ is Hurwitz,

Definition 3.1: A compact set $\mathcal{S}$ is said to be ***singularly semi-globally practically uniformly asymptotically stable*** (**sSPUAS**) for the system (5) if the following is satisfied:

1) $\forall\epsilon_1, \epsilon_2 \in (0, \infty)$, $\exists\delta_1, \delta_2, \varepsilon^* \in (0, \infty)$ such that $\forall\varepsilon \in (0, \varepsilon^*)$, $\forall t_0 \in \mathbb{R}$, and $\forall t \in [t_0, \infty)$ we have:

$$\left.\begin{matrix}\mathbf{x}_{1,0} \in \mathcal{U}^{\mathcal{S}}_{\delta_1}\\ \mathbf{x}_{2,0} \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_{1,0})}_{\delta_2}\end{matrix}\right\} \implies \left\{\begin{matrix}\mathbf{x}_1(t) \in \mathcal{U}^{\mathcal{S}}_{\epsilon_1}\\ \mathbf{x}_2(t) \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_1(t))}_{\epsilon_2}\end{matrix}\right.$$

2) $\forall\epsilon_1, \epsilon_2 \in (0, \infty)$ and all $\delta_1, \delta_2 \in (0, \infty)$, $\exists T_f, \varepsilon^* \in (0, \infty)$ such that $\forall\varepsilon \in (0, \varepsilon^*)$, $\forall t_0 \in \mathbb{R}$, $\forall t_1 \in [t_0 + T_f, \infty)$, and $\forall t_2 \in [t_0 + T_f\varepsilon^2, \infty)$ we have:

$$\left.\begin{matrix}\mathbf{x}_{1,0} \in \mathcal{U}^{\mathcal{S}}_{\delta_1}\\ \mathbf{x}_{2,0} \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_{1,0})}_{\delta_2}\end{matrix}\right\} \implies \left\{\begin{matrix}\mathbf{x}_1(t_1) \in \mathcal{U}^{\mathcal{S}}_{\epsilon_1}\\ \mathbf{x}_2(t_2) \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_1(t_2))}_{\epsilon_2}\end{matrix}\right.$$

3) $\forall\delta_1, \delta_2 \in (0, \infty)$, $\exists\epsilon_1, \epsilon_2, \varepsilon^* \in (0, \infty)$ such that $\forall\varepsilon \in (0, \varepsilon^*)$, $\forall t_0 \in \mathbb{R}$, and $\forall t \in [t_0, \infty)$ we have:

$$\left.\begin{matrix}\mathbf{x}_{1,0} \in \mathcal{U}^{\mathcal{S}}_{\delta_1}\\ \mathbf{x}_{2,0} \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_{1,0})}_{\delta_2}\end{matrix}\right\} \implies \left\{\begin{matrix}\mathbf{x}_1(t) \in \mathcal{U}^{\mathcal{S}}_{\epsilon_1}\\ \mathbf{x}_2(t) \in \mathcal{U}^{\boldsymbol{\varphi}_0(\mathbf{x}_1(t))}_{\epsilon_2}\end{matrix}\right.$$

We refer the reader to [14] for the definition of ***global uniform asymptotic stability*** (**GUAS**) for a compact set. In addition, consider the reduced order system:

$$\dot{\tilde{\mathbf{x}}}_1 = \sum_{k=1}^{2} \varepsilon^{k-2}\,\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}}_1, t, \varepsilon^{-2}t), \quad \tilde{\mathbf{x}}_1(t_0) = \mathbf{x}_{1,0}, \tag{6a}$$

$$\tilde{\mathbf{x}}_2 = \boldsymbol{\varphi}_0(\tilde{\mathbf{x}}_1) \tag{6b}$$

where the time-varying vector fields $\tilde{\mathbf{f}}_k$ are defined by:

$$\tilde{\mathbf{f}}_1(\mathbf{x}_1, t, \tau) = \mathbf{f}_{1,1}(\mathbf{x}_1, \boldsymbol{\varphi}_0(\mathbf{x}_1), t, \tau) \tag{7a}$$

$$\begin{aligned}\tilde{\mathbf{f}}_2(\mathbf{x}_1, t, \tau) = \mathbf{f}_{1,2}(\mathbf{x}_1, \boldsymbol{\varphi}_0(\mathbf{x}_1), t, \tau) \\ + \partial_2 \mathbf{f}_{1,1}(\mathbf{x}_1, \boldsymbol{\varphi}_0(\mathbf{x}_1), t, \tau)\boldsymbol{\varphi}_1(\mathbf{x}_1, t, \tau)\end{aligned} \tag{7b}$$

$$\boldsymbol{\varphi}_1(\mathbf{x}_1, t, \tau) = \left(\mathbf{I} - \mathrm{e}^{T\mathbf{F}}\right)^{-1} \int_0^T \mathrm{e}^{(T-\nu)\mathbf{F}}\, \mathbf{b}_1(\mathbf{x}_1, t, \nu + \tau) d\nu \tag{7c}$$

$$\begin{aligned}\mathbf{b}_1(\mathbf{x}_1, t, \tau) = \mathbf{f}_{2,1}(\mathbf{x}_1, \boldsymbol{\varphi}_0(\mathbf{x}_1), t, \tau) \\ - \partial\boldsymbol{\varphi}_0(\mathbf{x}_1)\, \mathbf{f}_{1,1}(\mathbf{x}_1, \boldsymbol{\varphi}_0(\mathbf{x}_1), t, \tau)\end{aligned} \tag{7d}$$

and $\mathbf{I}$ is the identity matrix of appropriate dimensions. Through second-order averaging [15], [16], we obtain the reduced order averaged system:

$$\dot{\bar{\mathbf{x}}}_1 = \bar{\mathbf{f}}(\bar{\mathbf{x}}_1, t), \quad \bar{\mathbf{x}}_1(t_0) = \mathbf{x}_{1,0}, \tag{8}$$

where the vector field $\bar{\mathbf{f}}$ is given by:

$$\bar{\mathbf{f}}(\mathbf{x}, t) = \frac{1}{T}\int_0^T \left(\tilde{\mathbf{f}}_2(\mathbf{x}, t, \tau) + \frac{1}{2}\Big[\int_0^\tau \tilde{\mathbf{f}}_1(\mathbf{x}_1, t, \nu) d\nu, \tilde{\mathbf{f}}_1(\mathbf{x}_1, t, \tau)\Big]\right) d\tau \tag{9}$$

Under Assumption 3.1, we have the following theorems characterizing the relation between the trajectories and the stability of the system (5) and the reduced order averaged system (8):

Theorem 3.1: Let a bounded subset $\mathcal{B}_1 \times \mathcal{B}_2 \subset \mathbb{R}^{n_1+n_2}$ and $t_f \in (0, \infty)$ be such that unique trajectories $\bar{\mathbf{x}}_1(t)$ for the system (8) exist $\forall t_0 \in \mathbb{R}$, $\forall t \in [t_0, t_0 + t_f]$, $\forall \mathbf{x}_{1,0} \in \mathcal{K}$. Then, $\exists \gamma, \lambda, \varepsilon^* \in (0, \infty)$ such that unique trajectories $(\mathbf{x}_1(t), \mathbf{x}_2(t))$ for the system (5) exist $\forall t_0 \in \mathbb{R}$, $\forall t \in [t_0, t_0 + t_f]$, $\forall (\mathbf{x}_{1,0}, \boldsymbol{\zeta}_{2,0}) \in \mathcal{B}_1 \times \mathcal{B}_2$, $\forall \varepsilon \in (0, \varepsilon^*)$, and satisfy:

$$\|\mathbf{x}_1(t) - \bar{\mathbf{x}}_1(t)\| \leq O(\eta(\varepsilon)) \tag{10}$$

$$\|\mathbf{y}_2(t)\| \leq \gamma\, \|\mathbf{y}_2(t_0)\|\, \mathrm{e}^{-\lambda\,\varepsilon^{-2}(t-t_0)} + O(\eta(\varepsilon)) \tag{11}$$

where $\mathbf{y}_2(t) = \mathbf{x}_2(t) - \boldsymbol{\varphi}_0(\mathbf{x}_1(t))$, and the order function $\eta(\varepsilon)$ satisfies the limit $\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0$.

Theorem 3.2: Let a compact subset $\mathcal{S} \subset \mathbb{R}^n$ be GUAS for the reduced order averaged system (8). Then, $\mathcal{S}$ is sSPUAS for the original system (5).

IV. Source Seeking for Rigid Bodies

We now turn our attention to the source seeking problem.

A. 3D Gradient Alignment

Recall that the kinematics of a rigid body in 3D space are given by:

$$\dot{\mathbf{q}} = \mathbf{R}\mathbf{v}, \qquad \dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\omega}} \tag{12}$$

First, we consider the problem of orienting a 3D rigid body with a fixed position such that one of the body axes points along the gradient of a signal strength field at the initial fixed position. We assume a model of the vehicle such that the origin of the body frame is fixed (i.e. $\mathbf{v} = 0$), and two of the angular velocities in body coordinates are the control inputs:

$$\boldsymbol{\omega} = \omega_{\parallel}\mathbf{e}_1 + \omega_{\perp}\mathbf{e}_3 \tag{13}$$

In addition, we assume that a non-collocated sensor is attached at the location $\mathbf{q}_s$ where:

$$\mathbf{q}_s = \mathbf{q} + r\mathbf{R}\mathbf{e}_2 \tag{14}$$

We would like to design a control law that aligns the body axis $\mathbf{R}\mathbf{e}_1$ with the gradient $\nabla\mathbf{c}(\mathbf{q})$ of a smooth signal strength field $c(\mathbf{q})$ at the fixed position $\mathbf{q}$, assuming that $\nabla\mathbf{c}(\mathbf{q}) \neq 0$. That is, we would like to stabilize the compact subset:

$$\mathcal{S}^+_{\mathbf{q}} = \{\mathbf{R} \in \mathrm{SO}(3) : \nabla\mathbf{c}(\mathbf{q})^\intercal\mathbf{R}\,\mathbf{e}_1 = \|\nabla\mathbf{c}(\mathbf{q})\|\} \tag{15}$$

We propose the linear dynamic feedback law:

$$\dot{\boldsymbol{\zeta}} = \varepsilon^{-2}\,\mathbf{F}\,\boldsymbol{\zeta} + \varepsilon^{-2}\,\mathbf{B}\,c(\mathbf{q}_s) \tag{16a}$$

$$\omega_\perp = \alpha_2\varepsilon^{-1}\,\mathbf{H}\,\boldsymbol{\zeta}, \qquad \omega_\parallel = \varepsilon^{-2} \tag{16b}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^2$, $\alpha_2, \varepsilon \in (0, \infty)$, and the matrices $\mathbf{F}, \mathbf{B}, \mathbf{H}$ are given by:

$$\mathbf{F} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} -4 & 4 \end{bmatrix} \tag{17}$$

We remark that the feedback law does not contain any information on the attitude of the vehicle and only employs the measured cost function $c(\mathbf{q}_s)$.

In the 'distinguished limit' [17] where the offset r of the sensor from the center of the vehicle is small: $r = O(\varepsilon) = r_0\varepsilon$ as $\varepsilon \to 0$ for some $r_0 > 0$, we have the following proposition:

Proposition 4.1: For a fixed $\mathbf{q}$ such that $\nabla\mathbf{c}(\mathbf{q}) \neq 0$, the compact subset $\mathcal{S}^+_{\mathbf{q}}$ is (almost) sSPUAS for the system defined by the equations (12)-(16).

Remark 4.1: Before we proceed with the proof of this proposition, we remark on the word 'almost' inside the parenthesis in its statement. Topological considerations [18] prohibit asymptotic stability via continuous feedback on the group SO(3). As such, it is impossible to conclude global practical stability results, since the reduced order averaged system cannot have any GUAS subsets. Instead, we can prove 'almost' sSPUAS which is a straightforward extension of Definition 3.1.

Proof: Let $\mathbf{R}_0 = \exp\left(\varepsilon^{-2}t\,\widehat{\mathbf{e}}_1\right)$, $\mathbf{P} = \mathbf{R}\mathbf{R}_0^\intercal$, and compute:

$$\dot{\mathbf{P}} = \dot{\mathbf{R}}\mathbf{R}_0^\intercal + \mathbf{R}\dot{\mathbf{R}}_0^\intercal = \omega_\perp\mathbf{R}\widehat{\mathbf{e}}_3\mathbf{R}_0^\intercal = \omega_\perp\mathbf{P}\mathbf{R}_0\widehat{\mathbf{e}}_3\mathbf{R}_0^\intercal \tag{18}$$

Let $\boldsymbol{\chi}(\boldsymbol{\zeta}, \varepsilon^{-2}t) = \alpha_2\mathbf{H}\,\boldsymbol{\zeta}\,\mathbf{R}_0\mathbf{e}_3$ and observe that:

$$\dot{\mathbf{P}} = \varepsilon^{-1}\,\mathbf{P}\widehat{\boldsymbol{\chi}}(\boldsymbol{\zeta}, \varepsilon^{-2}t) \tag{19}$$

We embed SO(3) into $\mathbb{R}^9$ by partitioning the matrix $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]$, defining the state vector $\mathbf{p} = [\mathbf{p}_1^\intercal, \mathbf{p}_2^\intercal, \mathbf{p}_3^\intercal]^\intercal$, and restricting the initial conditions for $\mathbf{p}$ to lie on the compact submanifold $\mathcal{M} = \left\{\mathbf{p}_i \in \mathbb{R}^3 : \mathbf{p}_i^\intercal\mathbf{p}_j = \delta_{ij},\ \mathbf{p}_i \times \mathbf{p}_j = \epsilon_{ijk}\mathbf{p}_k\right\}$, where δ_{ij} is the Kronecker symbol and ϵ_{ijk} is the Levi-Civita symbol. On $\mathcal{M} \times \mathbb{R}^2$, the system is governed by:

$$\dot{\mathbf{p}}_i = \varepsilon^{-1}\sum_{j,k=1}^{3} \chi_j(\boldsymbol{\zeta}, \varepsilon^{-2}t)\epsilon_{ijk}\mathbf{p}_k \tag{20a}$$

$$\dot{\boldsymbol{\zeta}} = \varepsilon^{-2}\left(\mathbf{F}\,\boldsymbol{\zeta} + \mathbf{B}\,c(\mathbf{q}_s)\right) \tag{20b}$$

where $\mathbf{q}_s$ is now given by:

$$\mathbf{q}_s = \mathbf{q} + r_0\varepsilon\,(\cos(\varepsilon^{-2}t)\,\mathbf{p}_2 + \sin(\varepsilon^{-2}t)\,\mathbf{p}_3) \qquad (21)$$

The signal strength field at $\mathbf{q}_s$ can be expanded as a series in $r = r_0\varepsilon$ using Taylor's theorem:

$$c(\mathbf{q}_s) = c(\mathbf{q}) + r_0\varepsilon\nabla c(\mathbf{q})^\intercal(\cos(\varepsilon^{-2}t)\mathbf{p}_2 + \sin(\varepsilon^{-2}t)\mathbf{p}_3) + \varepsilon^2\,\boldsymbol{\rho}(\mathbf{q},\mathbf{q}_s,\varepsilon^{-2}t,\varepsilon) \qquad (22)$$

where the remainder $\boldsymbol{\rho}$ is $\mathcal{C}^1$ in all of its arguments. Now, observe that the system defined by the equations (20) and (22) belongs to the class of systems described by (5). Hence, we may employ Theorem 3.1 and Theorem 3.2 in analyzing the stability of the system. In order to proceed, the reduced order averaged system must be computed. First, we observe that: $\boldsymbol{\varphi}_0 = \mathbb{1}c(\mathbf{q})$ is the zeroth order quasi-steady state for the singularly perturbed part of the system. We proceed to compute the first order periodic correction $\boldsymbol{\varphi}_1$ as given in equations (7). The vector $\mathbf{b}_1$ in this case is given by:

$$\mathbf{b}_1 = r_0\mathbf{B}\,\nabla c(\mathbf{q})^\intercal(\cos(\tau)\mathbf{p}_2 + \sin(\tau)\mathbf{p}_3) \qquad (23)$$

Direct computation using equations (7c) and (7d) shows that:

$$\boldsymbol{\varphi}_1 = \frac{r_0}{2}\begin{bmatrix} (\sin(\tau)\mathbf{p}_2 - \cos(\tau)\mathbf{p}_3)^\intercal\nabla c(\mathbf{q}) \\ (\cos(\tau)(\mathbf{p}_2-\mathbf{p}_3) + \sin(\tau)(\mathbf{p}_2+\mathbf{p}_3))^\intercal\nabla c(\mathbf{q}) \end{bmatrix} \qquad (24)$$

By applying the formulas in equations (7) and (9), we arrive at the reduced order averaged system:

$$\begin{bmatrix} \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \\ \dot{\bar{\mathbf{p}}}_3 \end{bmatrix} = \begin{bmatrix} \alpha_2 r_0\,(\bar{\mathbf{p}}_2\bar{\mathbf{p}}_2^\intercal + \bar{\mathbf{p}}_3\bar{\mathbf{p}}_3^\intercal)\,\nabla c(\mathbf{q}) \\ -\alpha_2 r_0\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2^\intercal\nabla c(\mathbf{q}) \\ -\alpha_2 r_0\bar{\mathbf{p}}_1\bar{\mathbf{p}}_3^\intercal\nabla c(\mathbf{q}) \end{bmatrix} \qquad (25a)$$

We claim that the subset $\mathcal{S}_{\mathbf{q}}^+$ is (almost) globally uniformly asymptotically stable for the system (25). To prove this statement, we define the candidate Lyapunov function:

$$V = \|\nabla c(\mathbf{q})\| - \nabla c(\mathbf{q})^\intercal\bar{\mathbf{p}}_1 \qquad (26)$$

which is clearly positive definite on $\mathcal{M}\backslash\mathcal{S}_{\mathbf{q}}^+$, and $V = 0$ if and only if $\mathbf{p} \in \mathcal{S}_{\mathbf{q}}^+$. We compute the derivative of V along the trajectories of the system (25):

$$\dot{V} = \alpha_2 r_0\nabla c(\mathbf{q})^\intercal\,(\bar{\mathbf{p}}_2\bar{\mathbf{p}}_2^\intercal + \bar{\mathbf{p}}_3\bar{\mathbf{p}}_3^\intercal)\,\nabla c(\mathbf{q}) \qquad (27)$$

However, it is not difficult to see that:

$$\bar{\mathbf{p}}_2\bar{\mathbf{p}}_2^\intercal + \bar{\mathbf{p}}_3\bar{\mathbf{p}}_3^\intercal = \mathbf{I} - \bar{\mathbf{p}}_1\bar{\mathbf{p}}_1^\intercal \qquad (28)$$

Hence, we have that:

$$\dot{V} = \alpha_2 r_0\|\nabla c(\mathbf{q})\|^2 - \alpha_2 r_0\,(\nabla c(\mathbf{q})^\intercal\bar{\mathbf{p}}_1)^2 \leq 0 \qquad (29)$$

Observe that $\dot{V} = 0$ if and only if $\mathbf{p} \in \mathcal{S}_{\mathbf{q}}^+$ or $\mathbf{p} \in \mathcal{S}_{\mathbf{q}}^-$ where:

$$\mathcal{S}_{\mathbf{q}}^- = \{\mathbf{R} \in \mathrm{SO}(3) : \nabla\mathbf{c}(\mathbf{q})^\intercal\mathbf{p}_1 = -\|\nabla\mathbf{c}(\mathbf{q})\|\} \qquad (30)$$

Thus, if we restrict our attention to the subset $\mathcal{M}\backslash\mathcal{S}_{\mathbf{q}}^-$, we have that $\dot{V} \leq 0\ \forall\mathbf{p} \in \mathrm{SO}(3)\backslash\mathcal{S}_{\mathbf{q}}^-$ and $\dot{V} = 0$ if and only if $\mathbf{p} \in \mathcal{S}_{\mathbf{q}}^+ \subset \mathrm{SO}(3)\backslash\mathcal{S}_{\mathbf{q}}^-$. We conclude that $\mathcal{S}_{\mathbf{q}}^+$ is almost globally uniformly asymptotically stable for the system (25). Hence, a modification of the statement of Theorem 3.2 to account for the zero measure unstable invariant set $\mathcal{S}_{\mathbf{q}}^-$ leads us to conclude that the subset $\mathcal{S}_{\mathbf{q}}^+$ is singularly (almost) semi-globally practically uniformly asymptotically stable. ■

B. 3D Source Seeking

Next, we use the gradient alignment algorithm to design a 3D source seeking algorithm for a rigid body with a non-collocated sensor. We assume a vehicle model in which the linear and angular velocity vectors are given in body coordinates by:

$$\mathbf{v} = v\mathbf{e}_1, \qquad \boldsymbol{\omega} = \omega_\parallel\mathbf{e}_1 + \omega_\perp\mathbf{e}_3 \qquad (31)$$

This model is a natural extension of the unicycle model to the 3D setting. It is well-known that this model is controllable using first order Lie brackets [19]. Let $c : \mathbb{R}^3 \to \mathbb{R}$ be the signal strength field emitted by the source, and consider the case of a non-collocated signal strength sensor that is mounted at the location $\mathbf{q}_s$ as defined by equation (14).

Assumption 4.1: Let the signal strength field $c \in \mathcal{C}^3$ be radially unbounded such that $\exists!\mathbf{q}^* \in \mathbb{R}^3$ such that $\nabla c(\mathbf{q}) = 0$ if and only if $\mathbf{q} = \mathbf{q}^*$, and satisfies $c(\mathbf{q}^*) - c(\mathbf{q}) \leq \kappa\|\nabla c(\mathbf{q})\|^2$, for $\kappa > 0$.

Assumption 4.1 is a sufficient condition for a gradient flow to have the extremum point $\mathbf{q}^*$ (i.e. the source location) as a GUAS point. The gradient alignment algorithm from the previous section can orient a seeking agent in the direction of the gradient. However, if the velocity v is taken as a positive constant, the agent always overshoots the source. Even more disturbingly, when the signal strength field is rotationally symmetric, the source is not even stable in the sense of Lyapunov; as long as $\alpha_1 \neq 0$, there will be an unbounded zero-measure invariant set containing the source along which the agent can escape to infinity. To elaborate, suppose that the control law (16) is applied, and the velocity v is taken to be a positive constant $v = \alpha_1$. Calculations similar to the proof of Proposition 4.1 produce the reduced order averaged system:

$$\begin{bmatrix} \dot{\bar{\mathbf{q}}} \\ \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \\ \dot{\bar{\mathbf{p}}}_3 \end{bmatrix} = \begin{bmatrix} \alpha_1\,\bar{\mathbf{p}}_1 \\ \alpha_2 r_0\,(\bar{\mathbf{p}}_2\bar{\mathbf{p}}_2^\intercal + \bar{\mathbf{p}}_3\bar{\mathbf{p}}_3^\intercal)\,\nabla c(\bar{\mathbf{q}}) \\ -\alpha_2 r_0\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2^\intercal\nabla c(\bar{\mathbf{q}}) \\ -\alpha_2 r_0\bar{\mathbf{p}}_1\bar{\mathbf{p}}_3^\intercal\nabla c(\bar{\mathbf{q}}) \end{bmatrix} \qquad (32)$$

which can be rewritten as a constrained mechanical system:

$$\ddot{\bar{\mathbf{q}}} = \alpha_1\alpha_2\,(\mathbf{I} - \alpha_1^{-2}\dot{\bar{\mathbf{q}}}\dot{\bar{\mathbf{q}}}^\intercal)\nabla c(\bar{\mathbf{q}}), \qquad \|\dot{\bar{\mathbf{q}}}\| = \alpha_1 \qquad (33)$$

Suppose that the signal strength field is given by $c(\mathbf{q}) = 2/(2 + \mathbf{q}^\intercal\mathbf{q})$, and therefore the system has the form:

$$\ddot{\bar{\mathbf{q}}} = -\frac{4\alpha_1\alpha_2\,(\mathbf{I} - \alpha_1^{-2}\dot{\bar{\mathbf{q}}}\dot{\bar{\mathbf{q}}}^\intercal)\bar{\mathbf{q}}}{(2 + \mathbf{q}^\intercal\mathbf{q})^2}, \qquad \|\dot{\bar{\mathbf{q}}}\| = \alpha_1 \qquad (34)$$

where $\alpha_1 > 0$. Now observe that the subset $\mathcal{T} = \{(\mathbf{q},\dot{\mathbf{q}}) : (\mathbf{q}^\intercal\dot{\mathbf{q}})^2 = \alpha_1^2\|\mathbf{q}\|^2\}$ is invariant for the system. Moreover, $\ddot{\bar{\mathbf{q}}} = 0$ in $\mathcal{T}$, and therefore $\dot{\bar{\mathbf{q}}}(t) = \dot{\bar{\mathbf{q}}}(0)$, $\bar{\mathbf{q}}(t) = \bar{\mathbf{q}}(0) + t\,\dot{\bar{\mathbf{q}}}(0)$, and $\lim_{t\to 0}\|\mathbf{q}(t)\| = \infty$ irrespective of $\bar{\mathbf{q}}(0)$. That is, if $(\bar{\mathbf{q}}(0), \dot{\bar{\mathbf{q}}}(0)) \in \mathcal{T}$, the seeking agent is guaranteed to eventually diverge from the source whenever $\alpha_1 \neq 0$.

To eliminate this undesired behavior, we tune the forward velocity of the vehicle. Consider the following dynamic control law:

$$v = 2\alpha_1 \varepsilon^{-1} \cos(2\varepsilon^{-2} t - c(\mathbf{q}_s)) \tag{35a}$$

$$\dot{\boldsymbol{\zeta}} = \varepsilon^{-2}\, \mathbf{F}\, \boldsymbol{\zeta} + \varepsilon^{-2}\, \mathbf{B}\, c(\mathbf{q}_s) \tag{35b}$$

$$\omega_\perp = \alpha_2 \varepsilon^{-1}\, \mathbf{H}\, \boldsymbol{\zeta}, \qquad \omega_\parallel = \varepsilon^{-2} \tag{35c}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^2$, $\mathbf{F}, \mathbf{B}, \mathbf{H}$ are as defined in equations (17), and α_1 and α_2 are positive tuning parameters. The proposed control law consists of two parts. The first part (35a) is a 1D extremum seeking control law [3]. The second part (35b)-(35c) is the gradient alignment algorithm proposed in the previous section. The first part behaves as a 1D line search along the direction of $\mathbf{R}\mathbf{e}_1$, while the second part is responsible for adjusting the attitude of the vehicle so that the body axis $\mathbf{R}\mathbf{e}_1$ is aligned with the gradient.

Proposition 4.2: Let Assumption 4.1 be satisfied. Then, the compact subset $\mathcal{S} = \{\mathbf{q}^*\} \times \mathrm{SO}(3)$ is sSPUAS for the system defined by equations (12) and (31) under the feedback law (35).

Proof: Following similar steps to the proof of Proposition 4.1 leads to the system:

$$\dot{\mathbf{q}} = 2\alpha_1 \varepsilon^{-1} \cos(2\varepsilon^{-2} t - c(\mathbf{q}_s))\, \mathbf{p}_1 \tag{36}$$

$$\dot{\mathbf{p}}_i = \alpha_2 \varepsilon^{-1} \sum_{j,k=1}^{3} \chi_j(\boldsymbol{\zeta}, \varepsilon^{-2} t) \epsilon_{ijk} \mathbf{p}_k \tag{37}$$

$$\dot{\boldsymbol{\zeta}} = \varepsilon^{-2} \left(\mathbf{F}\, \boldsymbol{\zeta} + \mathbf{B}\, c(\mathbf{q}_s) \right) \tag{38}$$

which also belongs to the class of systems described by (5) when the measured signal strength $c(\mathbf{q}_s)$ is expanded as in (22). Hence, we may employ Theorem 3.1 and Theorem 3.2 in analyzing the system. In order to proceed, the reduced order averaged system must be computed. By applying the formulas in equations (7) and (9), we arrive at the reduced order averaged system:

$$\begin{bmatrix} \dot{\bar{\mathbf{q}}} \\ \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \\ \dot{\bar{\mathbf{p}}}_3 \end{bmatrix} = \begin{bmatrix} \alpha_1^2\, \bar{\mathbf{p}}_1 \bar{\mathbf{p}}_1^\intercal \nabla c(\bar{\mathbf{q}}) \\ \alpha_2 r_0 \left(\bar{\mathbf{p}}_2 \bar{\mathbf{p}}_2^\intercal + \bar{\mathbf{p}}_3 \bar{\mathbf{p}}_3^\intercal \right) \nabla c(\bar{\mathbf{q}}) \\ -\alpha_2 r_0 \bar{\mathbf{p}}_1 \bar{\mathbf{p}}_2^\intercal \nabla c(\bar{\mathbf{q}}) \\ -\alpha_2 r_0 \bar{\mathbf{p}}_1 \bar{\mathbf{p}}_3^\intercal \nabla c(\bar{\mathbf{q}}) \end{bmatrix} \tag{39}$$

which can be equivalently written as:

$$\dot{\bar{\mathbf{q}}} = \alpha_1^2\, \bar{\mathbf{P}} \mathbf{e}_1 \mathbf{e}_1^\intercal \bar{\mathbf{P}}^\intercal \nabla c(\bar{\mathbf{q}}), \quad \dot{\bar{\mathbf{P}}} = \alpha_2 r_0 \bar{\mathbf{P}} \widehat{\bar{\chi}}(\bar{\mathbf{q}}, \bar{\mathbf{P}}) \tag{40}$$

where $\bar{\mathbf{P}} = [\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_3]$ is a rotation matrix, and the angular velocity vector $\bar{\chi}$ is given by:

$$\bar{\chi}(\bar{\mathbf{q}}, \bar{\mathbf{P}}) = -\mathbf{e}_1 \times \bar{\mathbf{P}}^\intercal \nabla c(\bar{\mathbf{q}}) \tag{41}$$

We claim that the compact subset $\mathcal{S}$ is GUAS for the reduced order averaged system (40). To prove this claim, we use the negative of the signal strength field as a candidate Lyapunov function $V_c(\mathbf{q}) = c(\mathbf{q}^*) - c(\mathbf{q})$. Observe that the system (40) is autonomous, and so the function V_c is indeed a candidate Lyapunov function for the compact subset $\mathcal{S}$ due to Assumption 4.1 [20]. We proceed to compute the derivative of V_c along the trajectories of the system (40):

$$\dot{V}_c = -\alpha_1^2\, \nabla c(\bar{\mathbf{q}})^\intercal \bar{\mathbf{P}} \mathbf{e}_1 \mathbf{e}_1^\intercal \bar{\mathbf{P}}^\intercal \nabla c(\bar{\mathbf{q}}) \leq 0 \tag{42}$$

Now, consider the subset $\mathcal{N} = \{(\mathbf{q}, \mathbf{P}) \in \mathbb{R}^3 \times \mathrm{SO}(3) : \dot{V}_c = 0\}$, and observe that $\mathcal{S} \subset \mathcal{N}$, and that $\mathcal{S}$ is an invariant subset of the reduced order averaged system (40). Suppose that a trajectory $(\bar{\mathbf{q}}(t), \bar{\mathbf{P}}(t))$ of the system (40) exists such that $(\bar{\mathbf{q}}(t), \bar{\mathbf{P}}(t)) \in \mathcal{N} \backslash \mathcal{S}$, $\forall t \in I$, where I is the maximal interval of existence and uniqueness of the trajectory. Such a trajectory must satisfy:

$$\nabla c(\bar{\mathbf{q}}(t))^\intercal \bar{\mathbf{P}}(t) \mathbf{e}_1 = 0, \quad \forall t \in I \tag{43}$$

The differentiability of the trajectories allows us to compute the derivative of this identity and obtain that:

$$\frac{d}{dt} \left(\nabla c(\bar{\mathbf{q}}(t))^\intercal \bar{\mathbf{P}}(t) \mathbf{e}_1 \right) = 0, \quad \forall t \in I \tag{44}$$

which simplifies to:

$$\nabla c(\bar{\mathbf{q}}(t))^\intercal \bar{\mathbf{P}}(t) (\bar{\chi}(\bar{\mathbf{q}}(t), \bar{\mathbf{P}}(t)) \times \mathbf{e}_1) = 0 \tag{45}$$

Recalling equation (41), we see that:

$$\bar{\chi}(\bar{\mathbf{q}}(t), \bar{\mathbf{P}}(t)) \times \mathbf{e}_1 = (\mathbf{I} - \mathbf{e}_1 \mathbf{e}_1^\intercal) \bar{\mathbf{P}}(t)^\intercal \nabla c(\bar{\mathbf{q}}(t)) \tag{46}$$

$$= \bar{\mathbf{P}}(t)^\intercal \nabla c(\bar{\mathbf{q}}(t)) \tag{47}$$

Hence, the equation (45) necessitates that:

$$\|\nabla c(\bar{\mathbf{q}}(t))\|^2 = 0, \quad \forall t \in I \tag{48}$$

which is clearly in contradiction with Assumption 4.1. Accordingly, it follows from LaSalle's Invariance principle [20, Corollary 4.2 to Theorem 4.4] that the compact subset $\mathcal{S}$ is GUAS for the system (40). Hence, the conclusion of the proposition follows by invoking Theorem 3.2. ■

Remark 4.2: Here, we comment on the roles of the parameters α_1 and α_2. The parameter α_2 is the rate of alignment of the direction of motion with the gradient, and α_1^2 multiplies the speed of the seeking agent along the direction of motion. The effect of these parameters is demonstrated through numerical examples in the next section.

V. Numerical Simulations

We illustrate the performance of our proposed algorithm through numerical simulations. Consider the signal strength field given by:

$$c(\mathbf{q}) = \frac{10}{1 + 0.025 \mathbf{q}^\intercal \mathbf{q}} \tag{49}$$

We simulate two cases. In both cases, we take the initial conditions as $\mathbf{q}(0) = [-3, -3, 3]^\intercal$, $\boldsymbol{\zeta}(0) = c(\mathbf{q}(0))\mathbb{1}$, $\varepsilon = 1/\sqrt{12\pi}$, $r_0 = 1$, and the columns of the matrix $\mathbf{R}(0)$ are:

$$\mathbf{r}_1(0) = \frac{-\mathbf{q}(0)}{\|\mathbf{q}(0)\|}, \quad \mathbf{r}_2(0) = \frac{\mathbf{r}_1(0) \times (\mathbf{e}_1 + \mathbf{e}_2)}{\|\mathbf{r}_1(0) \times (\mathbf{e}_1 + \mathbf{e}_2)\|}, \tag{50}$$

and $\mathbf{r}_3(0) = \mathbf{r}_1(0) \times \mathbf{r}_2(0)$. The numerical simulations are shown in Fig.1. Observe that the behavior near the source is complex, i.e. there is a nontrivial attractor. However, the size of this attractor is of the order $O(\alpha_1^2 \alpha_2^{-1} r_0^{-1} \varepsilon)$, and therefore it vanishes in the limit $\varepsilon \to 0$. Studying the nature of this attractor requires third-order calculations and so it will be addressed in future work.

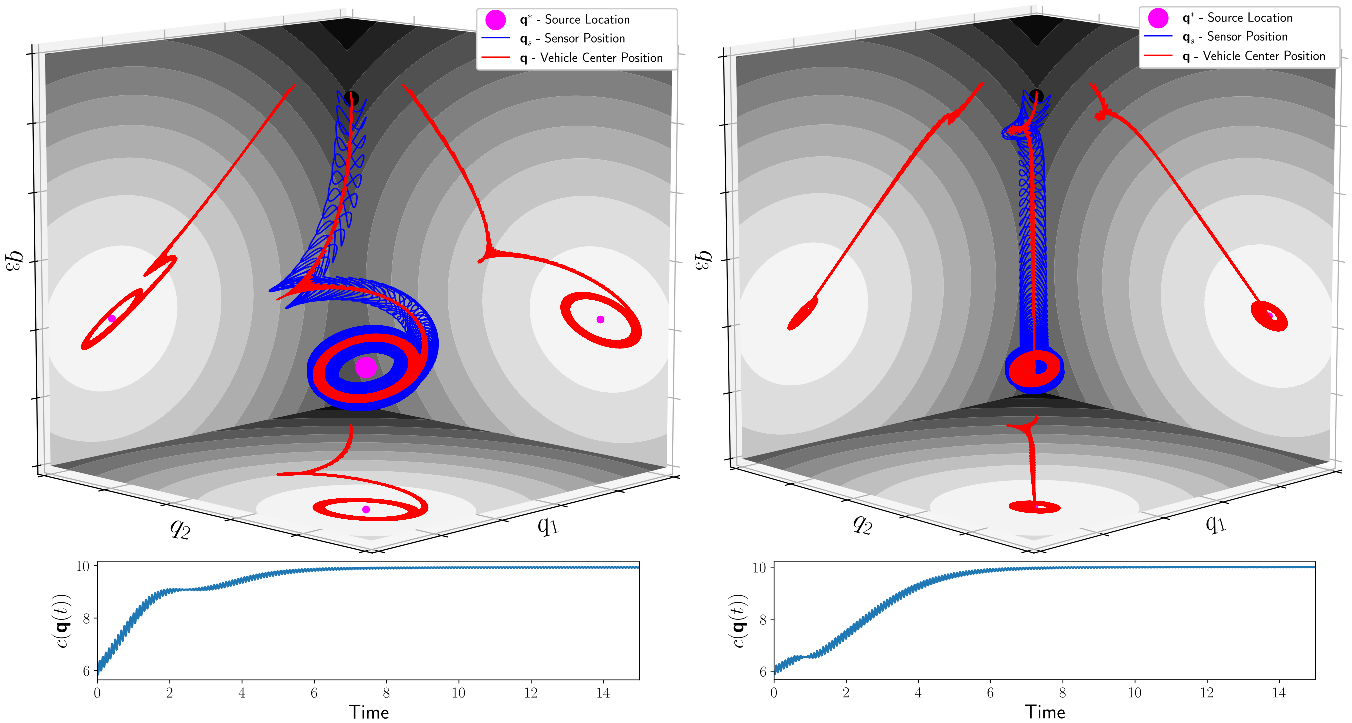

Fig. 1. The 3D spatial trajectory (top), and the history of $c(\mathbf{q}(t))$ (bottom) for the cases $\alpha_1 = \sqrt{2}$, $\alpha_2 = 1$ (left) and $\alpha_1 = 1$, $\alpha_2 = 3$ (right)

VI. Conclusion

In this manuscript, we proposed and analyzed the behavior of a novel 3D source seeking algorithm for rigid bodies with a non-collocated sensor inspired by the chemotaxis of sea urchin sperm cells. The proposed algorithm is the first 3D source seeking algorithm that does not require any attitude information. Future work will address the nonholonomic case where the velocity is constrained to be strictly forward.

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