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Correlated scalar perturbations and gravitational waves from axion inflation

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Abstract. The scalar and tensor fluctuations produced during inflation can be correlated, if arising from the same underlying mechanism. In this paper we investigate such correlation in the model of axion inflation, where the rolling inflaton produces quanta of a $U(1)$ gauge field which, in turn, source scalar and tensor fluctuations. We compute the primordial correlator of the curvature perturbation, ζ, with the amplitude of the gravitational waves squared, $h_{ij}h_{ij}$, at frequencies probed by gravitational wave detectors. This two-point function receives two contributions: one arising from the correlation of gravitational waves with the scalar perturbations generated by the standard mechanism of amplification of vacuum fluctuations, and the other coming from the correlation of gravitational waves with the scalar perturbations sourced by the gauge field. Our analysis shows that the latter effect is generally dominant. The correlator, normalized by the amplitude of ζ and of $h_{ij}h_{ij}$, turns out to be of the order of $10^{-2} \times (f_{\rm NL}^{\rm equil})^{1/3}$, where $f_{\rm NL}^{\rm equil}$ measures the scalar bispectrum sourced by the gauge modes.

Contents

1 Introduction

The theory of inflation constitutes the dominant paradigm of primordial cosmology. Besides solving the most important problems of the standard Hot Big Bang model, it is able to provide an explanation, in excellent agreement with observations, for the origin of the temperature anisotropies present in the Cosmic Microwave Background (CMB) radiation and of the density fluctuations that characterize the large scale structure of the Universe. Among the many different inflationary scenarios, axion inflation is one of those giving a satisfying solution to the problem of UV sensitivity of the inflaton potential. In this model, proposed for the first time in 1990 as natural inflation [1], the inflaton is a pseudo-Nambu–Goldstone Boson that enjoys a (softly broken) shift symmetry, i.e., a symmetry under the transformation $\phi \to \phi + \text{const}$, which protects its potential against large radiative corrections.

The axionic inflaton is naturally coupled to gauge fields through the operator $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}/f$, where f is the axion decay constant [2]. In the presence of such coupling, the rolling zero mode of the inflaton acts as a source for the modes of the gauge field. As a result, quanta of the gauge field are amplified into classical modes, which in turn source, through a process of inverse decay, both scalar and tensor fluctuations. Since, due to the pseudoscalar nature of the inflaton, only one of the two helicities of the gauge field experiences a tachyonic instability, the spectra of the tensor modes of different helicities have different amplitudes. This scenario has multiple phenomenological predictions, including nongaussianities [3], deviations from scale invariance [4], formation of a population of primordial black holes [5], generation of primordial chiral gravitational waves at CMB [6] or interferometer [7] frequencies, baryogenesis [8], as well as the possible generation of cosmologically relevant magnetic fields [9] - see [10] for a review.

By comparing these phenomenological predictions with observations we can constrain the relevant parameters characterizing the models of axion inflation. More specifically, there are two significant observational lengthscales to examine. At large scales, probed by CMB measurements, the primary constraint arises from the non-observation of primordial nongaussianities for the scalar fluctuations. In axion inflation the sourced scalar fluctuations are highly nongaussian. Consequently, for a viable model, we must require that the sourced component of scalar modes is subdominant compared to that generated by the standard amplification of vacuum fluctuations. This is equivalent to stating that the amplitude of the gauge field, which sources the scalar and tensor fluctuations, must be relatively small. Therefore, the sourced component of tensor fluctuations is also small at this stage.

At smaller scales, corresponding to modes that left the horizon closer to the end of inflation, the situation becomes more interesting. For simple inflationary potentials, the inflaton's velocity increases as inflation progresses and therefore the population of gauge quanta, whose amplitude depends exponentially on the inflaton's velocity, becomes more sizable towards the end of inflation. As a consequence, sourced gravitational waves of shorter wavelengths, which are remarkably those probed by gravitational wave experiments, can have a much larger amplitude and might even be directly detectable [7] by a variety of observatories. Also in this regime we need the scalar fluctuations to remain bounded to avoid an overproduction of primordial black holes [10, 11].

A natural follow-up to the recent observational evidence [12–14] of a stochastic gravitational wave background (SGWB) is the search for anisotropies, in analogy to the scalar anisotropies observed in the CMB (see, e.g., [15] for a recent analysis of LIGO/Virgo/KAGRA and [16] for LISA's reach in this respect). Study of these anisotropies can allow us to distinguish between the astrophysical and cosmological origin of the SGWB. Furthermore, cosmological tensor anisotropies may be correlated with the scalar anisotropies of the CMB if they arise from the same underlying mechanisms [17]. Exploring such correlations can give important information about the cosmological background of gravitational waves, thus providing insights about the physics of the Early Universe. Reference [18] performed a study of the statistics of these anisotropies while [19] studied the consequences of a non-trivial primordial scalar-tensor-tensor nongaussianity on the energy density of gravitational waves.

In this work we compute the correlation between the curvature perturbation $\zeta(\mathbf{x})$ and the squared amplitude of the tensor modes $h_{ij}(\mathbf{x})\, h_{ij}(\mathbf{x})$ within the framework of axion inflation. The computation is conducted at frequencies tested by gravitational detectors, and the correlator is normalized by both the square root of the scalar power spectrum and the tensor power spectrum. The two point function receives two contributions, reflecting the fact that scalar fluctuations are generated both from the vacuum, through the standard amplification process, and by modes of the gauge field, through the inverse decay process. More specifically, we will study the two following situations:

- the rolling inflaton has fluctuations that are generated by the standard mechanism of amplification of vacuum fluctuations in an expanding Universe. The rolling inflaton then sources quanta of the gauge field, which in turn source gravitational waves. The fluctuations in the inflaton are thus imprinted in the fluctuations in the gravitational waves. We study this correlator in Section 3.1;
- the rolling inflaton sources quanta of the gauge field, which in turn source *both* scalar fluctuations and gravitational waves. Since these modes are produced by the same population of gauge modes, they are correlated. We study this correlator in Section 3.2.

As we will see, the second effect is generally dominant over the first one, and can lead to a primordial cross-correlation of the order of $(f_{\rm NL}^{\rm equil}/10^6)^{1/3}$, where $f_{\rm NL}^{\rm equil}$ measures the scalar bispectrum at CMB scales in the equilateral configuration generated by the gauge field dynamics.

The correlator studied in this work is the one between scalar perturbations at CMB scales, corresponding to modes that left the horizon early during inflation and gravitational waves at interferometer scales, which correspond to modes that left the horizon later during inflation. Even though these gravitational waves have relatively short (i.e., non cosmological) wavelengths, their *anisotropies* are at large, cosmological scales.

During the last stages of axion inflation the large amplitude acquired by the gauge modes implies that they can have strong backreaction effects on the inflating background. The nonperturbative inflaton-gauge field dynamics, studied in numerous papers including [20–29], is rich, complicated, and not yet fully understood. The production of gravitational waves, although generated during the phase of strong backreaction, is treated at the perturbative level. Reference [28] derived spectra of gravitational waves produced during this stage keeping into account the nonperturbative dynamics of the inflaton-gauge field system, even if it ignored inflaton inhomogeneities. Reference [30] performed an analogous study for the case of an $SU(2)$ gauge sector. The results of [28] suggest that, even though strong backreaction effects complicate significantly the dynamics of the inflaton and of the gauge quanta, if the inflaton evolution $\phi(t)$ is known, then the resulting gravitational wave spectra reflect quite accurately the shape of the function $\dot\phi(t)$. For the scope of our calculation, since we will formulate our results in terms of $\dot\phi(t)$ without referring to the specific dynamics that led to that expression, our results should be valid even in the strong backreaction regime, at least as long as the inflaton inhomogeneities are ignored. Moreover, there are reasons to expect that our results will not change even once inflaton gradients are accounted for, since causality will prevent the late strong dynamics from affecting physics at scales that have left the horizon at much earlier times.

This paper is organized as follows. Section 2 contains a review of the amplification process that quanta of gauge field undergo as the inflaton rolls down its potential, together with the generation of curvature perturbations and of gravitational waves. Then, in Section 3, we calculate the two contributions to the correlator between scalar fluctuations and the squared amplitude of the gravitational waves: in Subsection 3.1 we study the correlation of gravitational waves with the amplified vacuum scalar fluctuations and in Subsection 3.2 the correlation of gravitational waves with sourced scalar fluctuations. Finally, in Section 4 we discuss our results and we conclude.

2 Review of scalar and tensor perturbations from axion inflation

Our system consists of a pseudoscalar inflaton ϕ and a $U(1)$ gauge field A_μ in interaction with each other and with gravity through the action

$$\mathcal{S}=\int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-V(\phi)-\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu}-\frac{\phi}{8\,f}\frac{\epsilon^{\mu\nu\rho\lambda}}{\sqrt{-g}}F_{\mu\nu}\,F_{\rho\lambda}\right]\,,\tag{2.1}$$

where $g=\det(g_{\mu\nu})$, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, f is a constant with dimensions of mass, R is the Ricci scalar, and $\epsilon^{\mu\nu\rho\lambda}$ is the totally antisymmetric tensor defined by $\epsilon^{0123}=+1$. We will not make any assumption about the shape of the potential $V(\phi)$, other than it is flat enough to be able to support inflation.

Concerning the metric, we will assume that it is of the form of de Sitter space in flat slicing plus tensor perturbations (repeated latin indices are understood to be summed upon)

$$\begin{aligned}ds^2&=a^2(\tau)\left[-d\tau^2+\left(\delta_{ij}+h_{ij}(\mathbf{x},\,\tau)\right)dx^i\,dx^j\right]\,,\\ a(\tau)&=-\frac{1}{H\,\tau}\,,\qquad h_{ii}=\partial_i h_{ij}=0\,.\end{aligned}\tag{2.2}$$

We perturb the inflaton as

$$\phi(\mathbf{x},\,\tau)\equiv\phi_0(\tau)+\delta\phi(\mathbf{x},\,\tau)\,,\tag{2.3}$$

so that the curvature perturbation is given by $\zeta \equiv -\frac{H}{\dot\phi_0}\delta\phi$. We will denote the derivative with respect to conformal time τ by a prime and that with respect to the cosmic time t, defined through $dt = a(\tau)\, d\tau$, by an overdot. We set the scale factor to be equal to unity at the end of inflation, i.e., inflation will end at $\tau_{\rm end} = -1/H$.

We treat the homogeneous inflaton $\phi_0(\tau)$ and the scale factor $a(\tau)$ as background quantities, and we work with the following canonically normalized perturbations

$$
\begin{aligned}
& A_\mu(\mathbf{x},\,\tau) \qquad \text{with} \qquad A_0(\mathbf{x},\,\tau)=0\,, \quad \partial_i A_i(\mathbf{x},\,\tau)=0\,,\\
& \Phi(\mathbf{x},\,\tau) \equiv a(\tau)\,\delta\phi(\mathbf{x},\,\tau)\,,\\
& H_{ij}(\mathbf{x},\,\tau) \equiv \frac{M_P}{2}\,a(\tau)\,h_{ij}(\mathbf{x},\,\tau)\,.
\end{aligned}
\tag{2.4}
$$

Neglecting the mass of the inflaton, our perturbed Lagrangian takes the form

$$
\begin{aligned}
\mathcal{L} = & \left(\frac{1}{2}\Phi'^2 - \frac{1}{2}\partial_k\Phi\,\partial_k\Phi + \frac{a''}{2\,a}\Phi^2\right) + \left(\frac{1}{2}H'_{ij}\,H'_{ij} - \frac{1}{2}\partial_k H_{ij}\,\partial_k H_{ij} + \frac{a''}{2\,a}H_{ij}\,H_{ij}\right)\\
& + \left(\frac{1}{2}A'_i\,A'_i - \frac{1}{2}\partial_k A_i\,\partial_k A_i - \frac{\phi_0}{f}\epsilon^{ijk}\,A'_i\,\partial_j A_k\right)\\
& - \frac{H_{ij}}{a\,M_P}\,\left[A'_i\,A'_j - (\partial_i A_k - \partial_k A_i)\,(\partial_j A_k - \partial_k A_j)\right] - \frac{\Phi}{f\,a}\epsilon^{ijk}\,A'_i\,\partial_j A_k\,,
\end{aligned}
\tag{2.5}
$$

where the first line describes the free scalar and free tensor perturbations, the second line describes the free gauge field modes, and the last line contains the interactions that lead to processes of the form $A_iA_j \to H_{ij}$ and $A_iA_j \to \Phi$.

By varying the Lagrangian (2.5) with respect to Φ, H_{ij} and A_i, we obtain the equations of motion

$$
\Phi'' - \frac{a''}{a}\Phi - \nabla^2\Phi + \frac{1}{f\,a}\epsilon^{ijk}\,A'_i\,\partial_j A_k = 0\,,
\tag{2.6}
$$

$$
H''_{ij} - \frac{a''}{a}H_{ij} - \nabla^2 H_{ij} + \frac{1}{a\,M_P}\,\left[A'_i\,A'_j - (\partial_i A_k - \partial_k A_i)\,(\partial_j A_k - \partial_k A_j)\right] = 0\,,
\tag{2.7}
$$

$$
A''_i - \nabla^2 A_i - \frac{\phi'_0}{f}\epsilon^{ijk}\,\partial_j A_k = 0\,.
\tag{2.8}
$$

The solution of eq. (2.6) splits into two parts: the solution of the homogeneous equation, denoted as $\Phi_{\rm V}$, and the particular solution, denoted as $\Phi_{\rm S}$. The solution of the homogeneous equation represents the usual vacuum fluctuations generated during inflation due to the accelerated expansion of the background, while the particular solution is induced by the inverse decay of the gauge fields. The homogeneous solution can be quantized through the standard quantization of the free Lagrangian, using the first line of eq. (2.5), as

$$
\begin{aligned}
& \Phi_{\rm V}(\mathbf{x},\,\tau) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}}\,\left[\Phi_{\rm V}(k,\,\tau)\,\hat a(\mathbf{k}) + \Phi^*_{\rm V}(k,\,\tau)\,\hat a^\dagger(-\mathbf{k})\right]\,,\\
& \Phi_{\rm V}(k,\,\tau) \equiv \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)\,e^{-ik\tau}\,,
\end{aligned}
\tag{2.9}
$$

where the creation/annihilation operators $\hat a^\dagger(\mathbf{k})/\hat a(\mathbf{k})$ satisfy the usual commutation relations $\left[\hat a(\mathbf{k}),\,\hat a^\dagger(\mathbf{q})\right] = \delta(\mathbf{k}-\mathbf{q})$, $\left[\hat a(\mathbf{k}),\,\hat a(\mathbf{q})\right] = \left[\hat a^\dagger(\mathbf{k}),\,\hat a^\dagger(\mathbf{q})\right] = 0$.

The power spectrum of the curvature perturbation, $\mathcal{P}_\zeta$, defined through the two point function

$$\langle\zeta(\mathbf{k})\,\zeta(\mathbf{q})\rangle \equiv \frac{2\pi^2}{k^3}\,\mathcal{P}_\zeta(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{q})\,, \tag{2.10}$$

results in the sum of the power spectra corresponding to the homogeneous and the particular solutions, denoted as $\mathcal{P}_{\zeta,\,\mathrm{V}}$ and $\mathcal{P}_{\zeta,\,\mathrm{S}}$, respectively.

Specifically, the homogeneous solution, corresponding to the scalar perturbations associated to the mode functions (2.9), yields, at the end of inflation and for large scales,

$$\mathcal{P}_{\zeta,\,\mathrm{V}} = \frac{k^3}{2\pi^2}\,\frac{H^2}{\dot\phi_0^2}\,|\Phi_{\mathrm{V}}(k,\,\tau_{\mathrm{end}})|^2 \xrightarrow[k\ll H]{} \frac{H^4}{4\pi^2\,\dot\phi_0^2}\,. \tag{2.11}$$

An analogous discussion holds also for the tensor perturbations $H_{ij}(\mathbf{x},\,\tau)$, whose vacuum component gives rise to $\mathcal{P}_{h,\,\mathrm{V}} = \frac{2\,H^2}{\pi^2\,M_P^2}$.

In order to find the sourced components of the scalar and tensor power spectra we need to take into account the generation of the electromagnetic field by the rolling pseudoscalar. In order to do that, we start with the quantization of the vector field $A_i(\mathbf{x},\,\tau)$:

$$A_i(\mathbf{x},\,\tau) = \int\frac{d\mathbf{k}}{(2\pi)^{3/2}}\sum_{\lambda=\pm} e_i^\lambda(\widehat{\mathbf{k}})\,e^{i\mathbf{k}\mathbf{x}}\left[A_\lambda(k,\,\tau)\,\hat a_\lambda(\mathbf{k}) + A_\lambda^*(k,\,\tau)\,\hat a_\lambda^\dagger(-\mathbf{k})\right]\,, \tag{2.12}$$

where the helicity projectors $e_i^\pm(\widehat{\mathbf{k}})$ satisfy the relations

$$\begin{aligned}
&k_i\,e_i^\lambda(\widehat{\mathbf{k}}) = 0\,, && e_i^\lambda(\widehat{\mathbf{k}})^* = e_i^{-\lambda}(\widehat{\mathbf{k}}) = e_i^\lambda(-\widehat{\mathbf{k}})\,,\\
&i\epsilon_{ijk}k_j e_k^\lambda(\widehat{\mathbf{k}}) = \lambda\,k\,e_i^\lambda(\widehat{\mathbf{k}})\,, && e_i^\lambda(\widehat{\mathbf{k}})e_i^{\lambda'}(\widehat{\mathbf{k}}) = \delta_{\lambda,\,-\lambda'}\,.
\end{aligned} \tag{2.13}$$

Inserting the decomposition (2.12) into eq. (2.8) we obtain the equation of motion for the mode functions $A_\lambda(k,\,\tau)$,

$$A_\lambda''(k,\,\tau) + \left(k^2 - \lambda\frac{\phi_0'}{f}k\right)\,A_\lambda(k,\,\tau) = 0\,, \tag{2.14}$$

which can be solved explicitly in terms of special functions if $\dot\phi_0 = \text{constant}$. However, we do not need the exact solution. Defining

$$\xi \equiv \frac{\dot\phi_0}{2\,f\,H}\,, \tag{2.15}$$

we can rewrite eq. (2.14) as

$$\frac{d^2\,A_\lambda}{d(k\,\tau)^2} + \left(1 + 2\,\lambda\,\frac{\xi}{k\,\tau}\right)\,A_\lambda = 0\,, \tag{2.16}$$

so that, assuming $\xi > 0$, the helicity $\lambda = -1$ in eq. (2.16) has always real frequencies that are adiabatically evolving (remember that $\tau < 0$). As a consequence, the mode A_- stays in its vacuum and we will neglect it from now on. On the other hand, the positive helicity mode A_+ has imaginary frequencies for a range of values of $k\tau$ and is therefore exponentially amplified.

In the WKB approximation, the leading term in the solution of the tachyonic modes of A_+ reads [2]

$$A_+(k,\,\tau) \simeq \frac{1}{\sqrt{2\,k}}\,\left(-\frac{k\,\tau}{2\,\xi}\right)^{1/4}\,e^{-2\,\sqrt{-2\xi k\tau}+\pi\,\xi}\,, \tag{2.17}$$

which is strictly speaking valid only in the range [3] $\frac{1}{8\,\xi} \lesssim |k\,\tau| \lesssim 2\,\xi$ (we will assume $\xi \gtrsim O(1)$ throughout this paper). However, since the momenta in this range dominate the contributions to the observables we will be interested in, we will apply the expression (2.17) to the entire range $0 < |k\,\tau| < \infty$. Eq. (2.17) shows that the $\lambda = +$ helicity of the gauge field is amplified by a factor $e^{\pi\xi}$, which can be very large even for moderate values of ξ.

We are now in position to compute the leading order contribution of the amplified gauge field to the curvature perturbation ζ. Taking the Fourier of eq. (2.6), we obtain the equation

$$\Phi''(\mathbf{q},\,\tau)+q^2\Phi(\mathbf{q},\,\tau)-\frac{2}{\tau^2}\Phi(\mathbf{q},\,\tau)-i\,\frac{H\tau}{f}\,\epsilon^{ijk}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A_i'(\mathbf{p},\,\tau)\,(\mathbf{q}-\mathbf{p})_jA_k(\mathbf{q}-\mathbf{p},\,\tau)=0\,. \tag{2.18}$$

The particular solution of this equation, $\Phi_{\rm S}$, which corresponds to the sourced component of scalar fluctuations, can be found using the retarded propagator

$$\Phi_{\rm S}(\mathbf{q},\,\tau)\equiv i\int d\tau'\,G_q(\tau,\,\tau')\frac{H\tau'}{f}\,\epsilon^{ijk}\int\frac{d\mathbf{p}}{(2\pi)^{3/2}}\,A_i'(\mathbf{p},\,\tau')\,(\mathbf{q}-\mathbf{p})_jA_k(\mathbf{q}-\mathbf{p},\,\tau')\,. \tag{2.19}$$

Given that we are assuming an exact de Sitter background, the retarded propagator can be written explicitly as

$$G_k(\tau,\tau')=\frac{(1+k^2\,\tau\,\tau')\,\sin(k\,(\tau-\tau'))+k\,(\tau'-\tau)\,\cos(k\,(\tau-\tau'))}{k^3\,\tau\,\tau'}\,\Theta(\tau-\tau')\,, \tag{2.20}$$

where Θ denotes the Heaviside step function.

The sourced component of the scalar fluctuations induces an additional contribution to the power spectrum of the curvature perturbation, that for $\xi \gtrsim 3$, is well approximated by the formula [3]

$$\mathcal{P}_{\zeta,\,\rm S}=\frac{k^3}{2\pi^2}\frac{H^2}{\dot\phi_0^2}\,|\Phi_{\rm S}(k,\,\tau_{\rm end})|^2\xrightarrow[k\ll H]{}4.8\times10^{-8}\,\frac{H^8}{\dot\phi_0^4}\,\frac{e^{4\pi\xi}}{\xi^6}\,. \tag{2.21}$$

A commonly used measure of nongaussianity is the parameter $f_{\rm NL}$, which measures the amplitude of the bispectrum of the curvature perturbation and is defined via

$$\langle\zeta(\mathbf{k}_1)\,\zeta(\mathbf{k}_2)\,\zeta(\mathbf{k}_3)\rangle=\frac{3}{10}\,(2\pi)^{5/2}\,f_{\rm NL}(k_1,\,k_2,\,k_3)\,\mathcal{P}_\zeta^2\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,\frac{k_1^3+k_2^3+k_3^3}{k_1^3\,k_2^3\,k_3^3}\,. \tag{2.22}$$

For single field, slow-roll inflation, the bispectrum has a small amplitude, and $f_{\rm NL}$ is of the order of the slow-roll parameters [31]. On the other hand, the sourced component of the curvature perturbation, since it results from a $2\to1$ process, obeys an intrinsically nongaussian statistics. Since such nongaussianities originate from some sub-horizon dynamics, the bispectrum is peaked on equilateral configurations, i.e., for $k_1=k_2=k_3$, with [3]

$$f_{\rm NL}^{\rm equil}\simeq7.1\times10^5\,\frac{H^{12}}{\dot\phi^6}\,\frac{e^{6\pi\xi}}{\xi^9}\,, \tag{2.23}$$

for $\xi \gtrsim 3$ and in the regime $\mathcal{P}_{\zeta,\,\mathrm{S}} \ll \mathcal{P}_{\zeta,\,\mathrm{V}}$. In the regime of large ξ, where $\mathcal{P}_{\zeta,\,\mathrm{S}} \gg \mathcal{P}_{\zeta,\,\mathrm{V}}$, $f_{\rm NL}^{\rm equil}$ converges to a value of the order of 10^4, which exceeds by a $O(10^3)$ factor the constraints from Planck. This limits severely the value $\xi_{\rm CMB}$ taken by ξ when Cosmic Microwave Background scales are leaving the horizon, leading to [32] $\xi_{\rm CMB} \lesssim 2.5$.

The excited modes of the vector field are also a source of gravitational waves. To leading order, production of gravitational waves via this process is described by the equation

$$H_{ij}''(\mathbf{q},\,\tau) + q^2 H_{ij}(\mathbf{q},\,\tau) - \frac{2}{\tau^2} H_{ij}(\mathbf{q},\,\tau)$$
$$= \frac{H\,\tau}{M_P} \int \frac{d\mathbf{p}}{(2\pi)^{3/2}} \left(A_i'(\mathbf{p},\,\tau)\,A_j'(\mathbf{q}-\mathbf{p},\,\tau) - F_{ik}(\mathbf{p},\,\tau) F_{jk}(\mathbf{q}-\mathbf{p},\,\tau)\right)\,, \qquad (2.24)$$

where $F_{ij}(\mathbf{p},\,\tau) \equiv i\,\mathbf{p}_i A_j(\mathbf{p},\tau) - i\,\mathbf{p}_j A_i(\mathbf{p},\tau)$. As a consequence of the functional dependence of A_+ on $k\,\tau$ and on ξ, the electric field is stronger than the magnetic field by a factor $\sim \xi \gtrsim 1$. For this reason we will neglect the term $F_{ik}(\mathbf{p},\,\tau)F_{jk}(\mathbf{q}-\mathbf{p},\,\tau)$ in eq. (2.24). Using again the Green's function (2.20) we eventually obtain

$$H_{ij,\,\mathrm{S}}(\mathbf{q},\,\tau) \equiv \int d\tau'\, G_q(\tau,\tau')\,\frac{H\,\tau'}{M_P} \int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\, A_i'(\mathbf{p},\,\tau')\, A_j'(\mathbf{q}-\mathbf{p},\,\tau')\,. \qquad (2.25)$$

The resulting power spectrum for the tensor modes reads [6]

$$\mathcal{P}_h = \mathcal{P}_{h,\,\mathrm{V}} + \mathcal{P}_{h,\,\mathrm{S}} \simeq \frac{2\,H^2}{\pi^2\,M_P^2} + 8.7\times 10^{-8}\,\frac{H^4}{M_P^4}\,\frac{e^{4\pi\xi}}{\xi^6}\,. \qquad (2.26)$$

It is worth stressing that the sourced component of the gravitational waves is almost fully chiral, as a consequence of the fact that only the + helicity of the gauge field is excited. While this fact can lead to a rich and interesting phenomenology, we will not be concerned with it here.

The constraint on the parameter ξ coming from the limits on nongaussianities implies that $\mathcal{P}_{h,\,\mathrm{V}} \gg \mathcal{P}_{h,\,\mathrm{S}}$. This constraint, however, holds only for the value $\xi_{\rm CMB}$ taken by ξ when CMB scales left the horizon. The quantity ξ is slowly evolving, typically increasing, during inflation. Since the sourced component of the gravitational wave spectrum has an exponential dependence on ξ, it is possible that at later times $\mathcal{P}_{h,\,\mathrm{V}}$ is actually overwhelmed by $\mathcal{P}_{h,\,\mathrm{S}}$. We will denote by $\xi_{\rm LATE} > \xi_{\rm CMB}$ the value taken by ξ at this later stage. In particular, this leads to the possibility that gravitational waves sourced by the vector field have such large amplitude to be directly detectable by current or future gravitational detectors [7].

In the next section we will describe two mechanisms that induce correlation between the curvature perturbation and the gravitational waves produced in axion inflation.

3 The correlator between scalar fluctuations and gravitational waves

We define the normalized correlator of scalar fluctuations and gravitational waves as

$$\mathcal{C}_{h\zeta}(\mathbf{k},\,\tau) = \frac{1}{\mathcal{P}_h\,\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int \frac{d\mathbf{y}}{(2\pi)^{3/2}} e^{-i\mathbf{k}\mathbf{y}}\,\langle h_{ij}(\mathbf{x}+\mathbf{y},\,\tau)\,h_{ij}(\mathbf{x}+\mathbf{y},\,\tau)\,\zeta(\mathbf{x},\,\tau)\rangle$$
$$= \frac{1}{\mathcal{P}_h\,\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int \frac{d\mathbf{p}}{(2\pi)^3}\,\langle h_{ij}(\mathbf{k}-\mathbf{p},\,\tau)\,h_{ij}(\mathbf{p},\,\tau)\,\zeta(-\mathbf{k},\,\tau)\rangle'\,, \qquad (3.1)$$

where $\langle \ldots \rangle'$ denotes the correlator stripped of the Dirac delta associated to momentum conservation, i.e.,

$$\langle h_{ij}(\mathbf{k}_1,\tau)\, h_{ij}(\mathbf{k}_2,\tau)\, \zeta(\mathbf{k}_3,\tau)\rangle \equiv \langle h_{ij}(\mathbf{k}_1,\tau)\, h_{ij}(\mathbf{k}_2,\tau)\, \zeta(\mathbf{k}_3,\tau)\rangle' \,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,. \tag{3.2}$$

The correlator (3.1) receives two different contributions: the first is the result of the correlation of gravitational waves with the amplified vacuum scalar fluctuations; the second is due to the correlation of gravitational waves with the sourced scalar fluctuations. In the following we will examine the two cases separately.

3.1 Correlation with amplified vacuum scalar fluctuations

The spectrum $\mathcal{P}_{h,\mathrm{S}}$ of gravitational waves sourced by the gauge field depends on the values of ϕ and $\dot\phi$ evaluated approximately at the time when the tensor modes under consideration left the horizon, and where, in slow-roll approximation, $\dot\phi$ is a function of ϕ. As a consequence, long wavelength perturbations in the values of ϕ will lead to correlated long wavelength perturbations in the spectrum of gravitational waves. Schematically, in the long wavelength approximation we can write

$$\begin{aligned}&\left\langle \left[h_{ij}h_{ij}\right]_{\mathrm{S}}(\mathbf{x}+\mathbf{y})\zeta_{\mathrm{V}}(\mathbf{x})\right\rangle \simeq \left\langle \left[h_{ij}h_{ij}(\phi(\mathbf{x}+\mathbf{y}))\right]_{\mathrm{S}}\zeta_{\mathrm{V}}(\mathbf{x})\right\rangle \\ &\simeq \left\langle \left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}\zeta_{\mathrm{V}}(\mathbf{x})\right\rangle + \left\langle \frac{d\,\left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}}{d\phi_0}\,\delta\phi_{\mathrm{V}}(\mathbf{x}+\mathbf{y})\zeta_{\mathrm{V}}(\mathbf{x})\right\rangle\,,\end{aligned} \tag{3.3}$$

where the first correlator in the second line vanishes since it is equal to $\left\langle \left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}\right\rangle \langle \zeta_{\mathrm{V}}(\mathbf{x})\rangle$ with $\langle\zeta_{\mathrm{V}}(\mathbf{x})\rangle = 0$. Since $\left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}$ is proportional to $e^{4\pi\,\xi_{\mathrm{LATE}}}$, the dominant contribution to its variation, in the large-ξ limit, is given by

$$\frac{d\,\left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}}{d\phi_0} \simeq 4\pi\,\left[h_{ij}h_{ij}(\phi_0)\right]_{\mathrm{S}}\,\frac{d\xi_{\mathrm{LATE}}}{d\phi_0}\,, \tag{3.4}$$

where in slow roll, assuming $\dot\phi_0 > 0$, $V' < 0$,

$$\xi \equiv \frac{\dot\phi_0}{2\,fH} \simeq -\frac{V'}{6\,fH^2} = -\frac{M_P^2}{2\,f}\frac{V'}{V}\,. \tag{3.5}$$

As a consequence,

$$\frac{d\xi_{\mathrm{LATE}}}{d\phi_0} = -\frac{M_P^2}{2\,f}\left(\frac{V''}{V}-\frac{V'^2}{V^2}\right) = \left(\epsilon_{\mathrm{LATE}}-\frac{\eta_{\mathrm{LATE}}}{2}\right)\frac{1}{f}\,, \tag{3.6}$$

where we have defined as usual the slow-roll parameters as

$$\epsilon = \frac{M_P^2}{2}\frac{V'^2}{V^2}\,, \qquad \eta = M_P^2\frac{V''}{V}\,. \tag{3.7}$$

The above results allow us to write the contribution to $\mathcal{C}_{h\zeta}$ from the amplified vacuum fluctuations of the scalar spectrum, which we denote as $(\mathcal{C}_{h\zeta})_{\mathrm{V}}$, as

$$\begin{aligned}(\mathcal{C}_{h\zeta})_{\mathrm{V}} &= \frac{1}{\mathcal{P}_h\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{y}}{(2\pi)^{3/2}}e^{-i\mathbf{k}\mathbf{y}}\langle h_{ij,\mathrm{S}}(\mathbf{x}+\mathbf{y},\,\tau)\,h_{ij,\mathrm{S}}(\mathbf{x}+\mathbf{y},\,\tau)\,\zeta_{\mathrm{V}}(\mathbf{x},\,\tau)\rangle \\ &\simeq \frac{1}{\mathcal{P}_h\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{y}}{(2\pi)^{3/2}}\,e^{-i\mathbf{k}\mathbf{y}}\frac{4\pi}{f}\left(\epsilon_{\mathrm{LATE}}-\frac{\eta_{\mathrm{LATE}}}{2}\right)\,\langle h_{ij}h_{ij}\rangle_{\mathrm{S}}\,\langle\delta\phi_{\mathrm{V}}(\mathbf{x}+\mathbf{y})\,\zeta_{\mathrm{V}}(\mathbf{x})\rangle\,.\end{aligned} \tag{3.8}$$

Then using $\delta\phi = -\dot{\phi}_0\,\zeta/H$, where ζ, being evaluated at superhorizon scales, is constant, and writing $\langle h_{ij}h_{ij}\rangle_{\rm S} \propto \mathcal{P}_{h,\,\rm S} \simeq \mathcal{P}_h$, we obtain

$$(\mathcal{C}_{h\zeta})_{\rm V} = -\frac{1}{\sqrt{\mathcal{P}_\zeta}}\frac{k^3}{2\pi^2}\int\frac{d\mathbf{y}}{(2\pi)^{3/2}}\,e^{-i\mathbf{k}\mathbf{y}}\frac{4\pi}{f}\left(\epsilon_{\rm LATE}-\frac{\eta_{\rm LATE}}{2}\right)\frac{\dot{\phi}_{0,\,\rm LATE}}{H_{\rm LATE}}\left\langle\zeta_{\rm V}(\mathbf{x}+\mathbf{y})\,\zeta_{\rm V}(\mathbf{x})\right\rangle\,, \tag{3.9}$$

with

$$\int\frac{d\mathbf{x}}{(2\pi)^{3/2}}\,e^{-i\mathbf{k}\mathbf{x}}\left\langle\zeta_{\rm V}(\mathbf{x}+\mathbf{y})\zeta_{\rm V}(\mathbf{y})\right\rangle = \frac{1}{(2\pi)^{3/2}}\left\langle\zeta_{\rm V}(\mathbf{k})\zeta_{\rm V}(-\mathbf{k})\right\rangle' = \frac{1}{(2\pi)^{3/2}}\frac{2\pi^2}{k^3}\mathcal{P}_{\zeta,\,\rm V}\,, \tag{3.10}$$

so that we finally get

$$(\mathcal{C}_{h\zeta})_{\rm V} = 8\pi\,\xi_{\rm LATE}\frac{\sqrt{\mathcal{P}_{\zeta,\,\rm V}}}{(2\pi)^{3/2}}\left(\frac{\eta_{\rm LATE}}{2}-\epsilon_{\rm LATE}\right)\,. \tag{3.11}$$

To estimate our result for $(\mathcal{C}_{h\zeta})_{\rm V}$ we note that both $\epsilon_{\rm LATE}$ and $\eta_{\rm LATE}$ must be smaller than 1. Moreover, $\xi_{\rm LATE}$ is typically of the order of 10 or so. Using $\mathcal{P}_{\zeta,\,\rm V} \simeq \mathcal{P}_\zeta \simeq 2\times 10^{-9}$ (consistency with observations requires the sourced contribution to $\mathcal{P}_\zeta$ to be negligible), we finally obtain the upper bound

$$\left|(\mathcal{C}_{h\zeta})_{\rm V}\right| \lesssim O(10^{-3})\,, \tag{3.12}$$

which is very small and, as we will see, subdominant with respect to the correlation between sourced tensors and sourced scalar perturbations, that we will study now.

3.2 Correlation with sourced scalar fluctuations

In order to calculate the correlator between the sourced scalar and tensor fluctuations, that we denote as $(\mathcal{C}_{h\zeta})_{\rm S}$, we use eqs. (2.4), (2.19) and (2.25) to find $\langle h_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,h_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\zeta_{\rm S}(\mathbf{k}_3,\tau)\rangle$ in terms of the canonically normalized perturbations as

$$\begin{aligned}
&\langle h_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,h_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\zeta_{\rm S}(\mathbf{k}_3,\tau)\rangle = -\frac{4\,H(\tau)}{M_P^2\,\dot{\phi}_0(\tau)\,a^3(\tau)}\langle H_{ab,\,\rm S}(\mathbf{k}_1,\tau)\,H_{ab,\,\rm S}(\mathbf{k}_2,\tau)\,\Phi_{\rm S}(\mathbf{k}_3,\tau)\rangle\\
&= \frac{4\,H(\tau)}{M_P^4\,\dot{\phi}_0(\tau)\,a^3(\tau)\,f}\int_{-\infty}^{\tau}\frac{d\tau_1}{a(\tau_1)}\frac{d\tau_2}{a(\tau_2)}\frac{d\tau_3}{a(\tau_3)}\,G_{k_1}(\tau,\tau_1)\,G_{k_2}(\tau,\tau_2)\,G_{k_3}(\tau,\tau_3)\\
&\times\int\frac{d\mathbf{q}_1\,d\mathbf{q}_2\,d\mathbf{q}_3}{(2\pi)^{9/2}}\,e_a^+(\widehat{\mathbf{q}_1})\,e_b^+(\widehat{\mathbf{k}_1-\mathbf{q}_1})\,e_a^+(\widehat{\mathbf{q}_2})\,e_b^+(\widehat{\mathbf{k}_2-\mathbf{q}_2})\,e_i^+(\widehat{\mathbf{q}_3})\,e_i^+(\widehat{\mathbf{k}_3-\mathbf{q}_3})\,|\mathbf{k}_3-\mathbf{q}_3|\\
&\times\langle A'_+(q_1,\tau_1)\,A'_+(|\mathbf{k}_1-\mathbf{q}_1|,\tau_1)\,A'_+(q_2,\tau_2)\,A'_+(|\mathbf{k}_2-\mathbf{q}_2|,\tau_2)\,A'_+(q_3,\tau_3)\,A_+(|\mathbf{k}_3-\mathbf{q}_3|,\tau_3)\rangle\,,
\end{aligned} \tag{3.13}$$

where we have assumed that only the positive helicity photons contribute because, from eq. (2.14), A_+ is the only helicity that is amplified, and therefore

$$\begin{aligned}
&i\,\epsilon^{ijk}A'_i(\mathbf{q}_3,\tau_3)\,(\mathbf{k}_3-\mathbf{q}_3)_j\,A_k(\mathbf{k}_3-\mathbf{q}_3,\tau_3) =\\
&|\mathbf{k}_3-\mathbf{q}_3|\,e_i^+(\widehat{\mathbf{q}_3})\,e_i^+(\widehat{\mathbf{k}_3-\mathbf{q}_3})\,A'_+(q_3,\tau_3)\,A_+(|\mathbf{k}_3-\mathbf{q}_3|,\tau_3)\,,
\end{aligned} \tag{3.14}$$

and

$$A'_a(\mathbf{q}_1,\tau_1)\,A'_b(\mathbf{k}_1-\mathbf{q}_1,\tau_1) = e_a^+(\widehat{\mathbf{q}_1})\,e_b^+(\widehat{\mathbf{k}_1-\mathbf{q}_1})\,A'_+(q_1,\tau_1)\,A'_+(|\mathbf{k}_1-\mathbf{q}_1|,\tau_1)\,. \tag{3.15}$$

Using Wick's theorem to decompose the last line of eq. (3.13) and inserting it back into (3.1) we obtain

$$
\begin{aligned}
\left(\mathcal{C}_{h\zeta}\right)_{\rm S}(\mathbf{k},\,\tau) &= \frac{2\,k^3\,H(\tau)}{\pi^2\,M_P^4\,\dot\phi_0(\tau)\,a^3(\tau)\,f\,\mathcal{P}_h\,\sqrt{\mathcal{P}_\zeta}}\int\frac{d\mathbf{p}}{(2\pi)^3}\Bigg[\int_{-\infty}^{\tau}\frac{d\tau_1}{a(\tau_1)}\,\frac{d\tau_2}{a(\tau_2)}\,\frac{d\tau_3}{a(\tau_3)}\\
& G_{k_1}(\tau,\tau_1)\,G_{k_2}(\tau,\tau_2)\,G_{k_3}(\tau,\tau_3)\int\frac{d\mathbf{q}\,\mathcal{A}(\mathbf{q},\mathbf{k}_1-\mathbf{q},\mathbf{k}_2+\mathbf{q})}{(2\pi)^{9/2}}\Big(|\mathbf{k}_2+\mathbf{q}|\,A_+'(q,\tau_1)\,A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_1)\\
& A_+'(q,\tau_2)\,A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_3)\,A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_2)\,A_+(|\mathbf{k}_2+\mathbf{q}|,\tau_3)+|\mathbf{k}_1-\mathbf{q}|\,A_+'(q,\tau_1)\,A_+'(q,\tau_2)\\
& A_+'(|\mathbf{k}_1-\mathbf{q}|,\tau_1)\,A_+(|\mathbf{k}_1-\mathbf{q}|,\tau_3)\,A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_2)\,A_+'(|\mathbf{k}_2+\mathbf{q}|,\tau_3)\Big)\Bigg]_{\substack{\mathbf{k}_1=\mathbf{k}-\mathbf{p}\\ \mathbf{k}_2=\mathbf{p}\\ \mathbf{k}_3=-\mathbf{k}}}\,,
\end{aligned}\tag{3.16}
$$

where we have collected the angular part inside the expression $\mathcal{A}$:

$$
\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\delta_{ac}\,\delta_{bd}\left(\left(e_a^+(\widehat{\mathbf{k}_1})\,e_c^+(\widehat{-\mathbf{k}_1})\,e_b^+(\widehat{\mathbf{k}_2})\,e_i^+(\widehat{-\mathbf{k}_2})\,e_d^+(\widehat{\mathbf{k}_3})\,e_i^+(\widehat{-\mathbf{k}_3})+(a\leftrightarrow b)\right)+(c\leftrightarrow d)\right).
$$

Using the explicit form of the gauge field (2.17), the expression (3.16) becomes

$$
\begin{aligned}
\left(\mathcal{C}_{h\zeta}\right)_{\rm S}(\mathbf{k},\,\tau) &= -\frac{k^3\,H^4(\tau)\,e^{2\pi\xi_1}\,e^{2\pi\xi_2}\,e^{2\pi\xi_3}}{2\,\pi^2\,(2\pi)^{9/2}\,M_P^4\,\dot\phi_0(\tau)\,f\,a^3(\tau)\,\mathcal{P}_h\,\sqrt{\mathcal{P}_\zeta}}\\
&\times\int\frac{d\mathbf{p}}{(2\pi)^3}\int_{-\infty}^{\tau}d\tau_1\,d\tau_2\,d\tau_3\,\xi_1^{1/2}\,\xi_2^{1/2}\,\sqrt{\tau_1\,\tau_2}\,\tau_3\,G_p(\tau,\tau_2)\,G_{|\mathbf{k}-\mathbf{p}|}(\tau,\tau_1)\,G_k(\tau,\tau_3)\\
&\times\int d\mathbf{q}\,\mathcal{A}(\mathbf{q},\mathbf{k}-\mathbf{p}-\mathbf{q},\mathbf{p}+\mathbf{q})q^{1/2}\,|\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}\,|\mathbf{p}+\mathbf{q}|^{1/2}(|\mathbf{k}-\mathbf{p}-\mathbf{q}|^{1/2}+|\mathbf{p}+\mathbf{q}|^{1/2})\\
&\times e^{-2\sqrt{-2\,\xi_1\,q\,\tau_1}-2\sqrt{-2\,\xi_2\,q\,\tau_2}-2\sqrt{-2\,\xi_1\,|\mathbf{k}-\mathbf{p}-\mathbf{q}|\,\tau_1}-2\sqrt{-2\,\xi_2\,|\mathbf{p}+\mathbf{q}|\,\tau_2}-2\sqrt{-2\,\xi_3\,|\mathbf{p}+\mathbf{q}|\,\tau_3}-2\sqrt{-2\,\xi_3\,|\mathbf{k}-\mathbf{p}-\mathbf{q}|\,\tau_3}}
\end{aligned}\tag{3.17}
$$

where we have also allowed for the parameter ξ to be time-dependent, albeit adiabatically, and we have denoted $\xi_i\equiv\xi(\tau_i)$. In order to perform the calculation we set the time at the end of inflation to be $\tau_{\rm end}=-1/H$ and we measure all the integration variables in units of k, i.e. $\mathbf{q}'=\mathbf{q}/k$, $\mathbf{p}'=\mathbf{p}/k$, $\hat{\mathbf{k}}=\mathbf{k}/k$, $x_i=-k\,\tau_i$. Then we can further simplify the expression by performing the change of variables $\tilde{\mathbf{q}}=\mathbf{q}'+\mathbf{p}'$ and by taking the limit $k/H\to 0$. Subsequently we use the expressions

$$
\begin{aligned}
\int_0^\infty dx\,x^3\,e^{-a\sqrt{x}} &= \frac{2}{a^8}\Gamma(8)\,,\\
\int_0^\infty dx\,x^{5/2}\,e^{-a\sqrt{x}} &= \frac{2}{a^7}\Gamma(7)\,,
\end{aligned}\tag{3.18}
$$

where the object a depends on ξ and is in principle time-dependent, but we assume it to be constant as the integral gets contributions only from a relatively narrow range of times. Since the integration variables τ_1 and τ_2 are associate to the sourced tensor modes, whereas τ_3 is associated to the scalar modes, we can set $\xi_1=\xi_2=\xi_{\rm LATE}$ and $\xi_3=\xi_{\rm CMB}$. We thus finally obtain

$$
\left(\mathcal{C}_{h\zeta}\right)_{\rm S}=\frac{4\,H^7\,e^{4\pi\,\xi_{\rm LATE}}\,e^{2\pi\,\xi_{\rm CMB}}\,\Gamma(8)\,\Gamma(7)^2}{27\,\pi^2\,(2\pi)^{15/2}\,f\,(2\sqrt{2})^{22}\,\xi_{\rm LATE}^6\,\xi_{\rm CMB}^4\,M_P^4\,\dot\phi_0\,\mathcal{P}_h\,\sqrt{\mathcal{P}_\zeta}}\times\mathcal{I}\,,\tag{3.19}
$$

where

$$
\mathcal{I}=\int d\mathbf{p}'\int d\tilde{\mathbf{q}}\,\frac{\mathcal{A}(\tilde{\mathbf{q}}-\mathbf{p}',\hat{\mathbf{k}}-\tilde{\mathbf{q}},\tilde{\mathbf{q}})\,|\tilde{\mathbf{q}}-\mathbf{p}'|^{1/2}\,|\hat{\mathbf{k}}-\tilde{\mathbf{q}}|^{1/2}\,\tilde{q}^{1/2}}{(|\hat{\mathbf{k}}-\tilde{\mathbf{q}}|^{1/2}+\tilde{q}^{1/2})^7\,(|\tilde{\mathbf{q}}-\mathbf{p}'|^{1/2}+\tilde{q}^{1/2})^7\,(|\tilde{\mathbf{q}}-\mathbf{p}'|^{1/2}+|\hat{\mathbf{k}}-\tilde{\mathbf{q}}|^{1/2})^7}\,,\tag{3.20}
$$

with

$$
\begin{aligned}
\mathcal{A}(\mathbf{k}_1,\,\mathbf{k}_2,\,\mathbf{k}_3) = \frac{1}{4}\Big(& 2 + 3\,(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})^2 - 5\,\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3} + (\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})^2 + (\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})^2 - \widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3} + \widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2} \\
& - (\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2}) - (\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2}) + (\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3}) \\
& - (\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2})(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3})(\widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3}) - i\,\epsilon_{dil}\,\widehat{\mathbf{k}}_{1d}\,\widehat{\mathbf{k}}_{2i}\,\widehat{\mathbf{k}}_{3l}\,(\widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_2} - \widehat{\mathbf{k}_1}\cdot\widehat{\mathbf{k}_3} - \widehat{\mathbf{k}_2}\cdot\widehat{\mathbf{k}_3} + 1)\Big)\,. \qquad (3.21)
\end{aligned}
$$

The integral can be computed numerically and evaluates to $\mathcal{I} \simeq 6\times 10^{-4}$. After having substituted $\sqrt{\mathcal{P}_\zeta} \simeq \sqrt{\mathcal{P}_{\zeta,\,\mathrm{V}}} = H^2/(2\pi\,\dot\phi_0)$ and $\mathcal{P}_h \simeq \mathcal{P}_{h,\mathrm{S}}$ from (2.26), the expression (3.19) takes the simple form

$$
(\mathcal{C}_{h\zeta})_{\mathrm{S}} \simeq 2\times 10^{-4}\,\frac{e^{2\pi\,\xi_{\mathrm{CMB}}}\,H}{\xi_{\mathrm{CMB}}^4\,f}\,. \qquad (3.22)
$$

Finally, we use eq. (2.23) together with the measured amplitude of the scalar perturbations $\mathcal{P}_{\zeta,\,\mathrm{V}} \simeq 2\times 10^{-9}$ to obtain

$$
(\mathcal{C}_{h\zeta})_{\mathrm{S}} \simeq 1.5\times 10^{-2}\,(f_{\mathrm{NL}}^{\mathrm{equil}})^{1/3}\,. \qquad (3.23)
$$

4 Discussion and conclusions

An important component of current and future gravitational wave research is the detection and characterization of the stochastic gravitational wave background. This background may originate from astrophysical sources or have a cosmological origin. Specifically, identifying a cosmological gravitational wave background will provide important information about the very early universe.

A powerful approach to distinguish between astrophysical and cosmological backgrounds involves studying their anisotropies. Notably, it has been shown that these anisotropies are correlated with the anisotropies in the CMB [33, 34]. The exploration of such correlations can significantly contribute to the interpretation of the CMB and SGWB measurements.

In the present paper we have investigated the correlator between the curvature perturbation and the amplitude squared of the tensor modes, computed at the end of inflation, within the axion inflation model. In this model, scalar fluctuations are generated through two distinct mechanisms: first, from the vacuum via the standard amplification process, and second, as a consequence of the production of gauge fields through a process of inverse decay. Consequently, the correlator exhibits two distinct components.

Our analysis reveals that the correlation of gravitational waves with sourced scalar fluctuations is approximately of the order of $\sim 1.5\times 10^{-2}\,(f_{\mathrm{NL}}^{\mathrm{equil}})^{1/3}$, where $f_{\mathrm{NL}}^{\mathrm{equil}}$ measures the amplitude of the bispectrum, in the equilateral configuration, of the scalar perturbations induced at CMB scales by the gauge field population. Observations [35] constrain $f_{\mathrm{NL}}^{\mathrm{equil}} \lesssim O(50)$. Notably, this correlation turns out to be larger than the correlation with the vacuum scalar fluctuations. The correlator saturates to $\mathcal{C}_{h\zeta} = O(1)$ when $f_{\mathrm{NL}}^{\mathrm{equil}} = O(10^5)$, which is indeed the maximum value attainable by $f_{\mathrm{NL}}^{\mathrm{equil}}$ in the perturbative regime.

Our main result, provided by eq. (3.23), corresponds to the initial correlation between the tensor and scalar fluctuations in the model of axion inflation. The formalism of [36–38] can then be applied to derive potentially observable quantities. The actual observability of such correlators, subject to instrumental noise as well as to the intrinsic variance of the isotropic component [39, 40], will depend on the amplitude of the anisotropies in the gravitational wave spectra. Such an amplitude is encoded in the correlator $\langle h_{ij}(\mathbf{x})\, h_{ij}(\mathbf{x})\, h_{ab}(\mathbf{y}) h_{ab}(\mathbf{y})\rangle$, whose calculation, in the model of axion inflation, requires the evaluation of the gauge field's eight-point function – a calculation that we leave to future work. However, we expect anisotropies to have a potentially detectable amplitude. For instance, the lattice study of [41] showed that the spectrum of gravitational waves induced by preheating at the end of inflation display anisotropies with an amplitude of the order of $\sim 10^{-2}$.

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References

[1] K. Freese, J. A. Frieman and A. V. Olinto, "Natural inflation with pseudo - Nambu-Goldstone bosons," Phys. Rev. Lett. **65** (1990), 3233-3236 doi:10.1103/PhysRevLett.65.3233

[2] M. M. Anber and L. Sorbo, "N-flationary magnetic fields," JCAP **10**, 018 (2006) doi:10.1088/1475-7516/2006/10/018 [arXiv:astro-ph/0606534 [astro-ph]].

[3] N. Barnaby and M. Peloso, "Large Nongaussianity in Axion Inflation," Phys. Rev. Lett. **106**, 181301 (2011) doi:10.1103/PhysRevLett.106.181301 [arXiv:1011.1500 [hep-ph]].

[4] R. Namba, M. Peloso, M. Shiraishi, L. Sorbo and C. Unal, "Scale-dependent gravitational waves from a rolling axion," JCAP **01**, 041 (2016) doi:10.1088/1475-7516/2016/01/041 [arXiv:1509.07521 [astro-ph.CO]].

[5] A. Linde, S. Mooij and E. Pajer, "Gauge field production in supergravity inflation: Local non-Gaussianity and primordial black holes," Phys. Rev. D **87** (2013) no.10, 103506 doi:10.1103/PhysRevD.87.103506 [arXiv:1212.1693 [hep-th]].

[6] L. Sorbo, "Parity violation in the Cosmic Microwave Background from a pseudoscalar inflaton," JCAP **06**, 003 (2011) doi:10.1088/1475-7516/2011/06/003 [arXiv:1101.1525 [astro-ph.CO]].

[7] J. L. Cook and L. Sorbo, "Particle production during inflation and gravitational waves detectable by ground-based interferometers," Phys. Rev. D **85**, 023534 (2012) [erratum: Phys. Rev. D **86**, 069901 (2012)] doi:10.1103/PhysRevD.85.023534 [arXiv:1109.0022 [astro-ph.CO]].

[8] M. M. Anber and E. Sabancilar, "Hypermagnetic Fields and Baryon Asymmetry from Pseudoscalar Inflation," Phys. Rev. D **92**, no.10, 101501 (2015) doi:10.1103/PhysRevD.92.101501 [arXiv:1507.00744 [hep-th]].

[9] W. D. Garretson, G. B. Field and S. M. Carroll, "Primordial magnetic fields from pseudoGoldstone bosons," Phys. Rev. D **46**, 5346-5351 (1992) doi:10.1103/PhysRevD.46.5346 [arXiv:hep-ph/9209238 [hep-ph]].

[10] E. Pajer and M. Peloso, "A review of Axion Inflation in the era of Planck," Class. Quant. Grav. **30**, 214002 (2013) doi:10.1088/0264-9381/30/21/214002 [arXiv:1305.3557 [hep-th]].

[11] J. Garcia-Bellido, M. Peloso and C. Unal, "Gravitational waves at interferometer scales and primordial black holes in axion inflation," JCAP **12**, 031 (2016) doi:10.1088/1475-7516/2016/12/031 [arXiv:1610.03763 [astro-ph.CO]].

[12] G. Agazie *et al.* [NANOGrav], Astrophys. J. Lett. **951**, no.1, L8 (2023) doi:10.3847/2041-8213/acdac6 [arXiv:2306.16213 [astro-ph.HE]].

[13] J. Antoniadis *et al.* [EPTA and InPTA:], "The second data release from the European Pulsar Timing Array - III. Search for gravitational wave signals," Astron. Astrophys. **678**, A50 (2023) doi:10.1051/0004-6361/202346844 [arXiv:2306.16214 [astro-ph.HE]].

[14] D. J. Reardon, A. Zic, R. M. Shannon, G. B. Hobbs, M. Bailes, V. Di Marco, A. Kapur, A. F. Rogers, E. Thrane and J. Askew, *et al.* "Search for an Isotropic Gravitational-wave Background with the Parkes Pulsar Timing Array," Astrophys. J. Lett. **951**, no.1, L6 (2023) doi:10.3847/2041-8213/acdd02 [arXiv:2306.16215 [astro-ph.HE]].

[15] R. Abbott *et al.* [KAGRA, Virgo and LIGO Scientific], "Search for anisotropic gravitational-wave backgrounds using data from Advanced LIGO and Advanced Virgo's first three observing runs," Phys. Rev. D **104**, no.2, 022005 (2021) doi:10.1103/PhysRevD.104.022005 [arXiv:2103.08520 [gr-qc]].

[16] N. Bartolo *et al.* [LISA Cosmology Working Group], "Probing anisotropies of the Stochastic Gravitational Wave Background with LISA," JCAP **11**, 009 (2022) doi:10.1088/1475-7516/2022/11/009 [arXiv:2201.08782 [astro-ph.CO]].

[17] M. Geller, A. Hook, R. Sundrum and Y. Tsai, "Primordial Anisotropies in the Gravitational Wave Background from Cosmological Phase Transitions," Phys. Rev. Lett. **121** (2018) no.20, 201303 doi:10.1103/PhysRevLett.121.201303 [arXiv:1803.10780 [hep-ph]].

[18] F. Schulze, L. Valbusa Dall'Armi, J. Lesgourgues, A. Ricciardone, N. Bartolo, D. Bertacca, C. Fidler and S. Matarrese, "GW_CLASS: Cosmological Gravitational Wave Background in the Cosmic Linear Anisotropy Solving System," [arXiv:2305.01602 [gr-qc]].

[19] P. Adshead, N. Afshordi, E. Dimastrogiovanni, M. Fasiello, E. A. Lim and G. Tasinato, "Multimessenger cosmology: Correlating cosmic microwave background and stochastic gravitational wave background measurements," Phys. Rev. D **103**, no.2, 023532 (2021) doi:10.1103/PhysRevD.103.023532 [arXiv:2004.06619 [astro-ph.CO]].

[20] S. L. Cheng, W. Lee and K. W. Ng, "Numerical study of pseudoscalar inflation with an axion-gauge field coupling," Phys. Rev. D **93**, no. 6, 063510 (2016) [arXiv:1508.00251 [astro-ph.CO]].

[21] A. Notari and K. Tywoniuk,"Dissipative Axial Inflation," JCAP **1612**, 038 (2016) [arXiv:1608.06223 [hep-th]].

[22] O. O. Sobol, E. V. Gorbar and S. I. Vilchinskii, "Backreaction of electromagnetic fields and the Schwinger effect in pseudoscalar inflation magnetogenesis," Phys. Rev. D **100**, no.6, 063523 (2019) [arXiv:1907.10443 [astro-ph.CO]].

[23] G. Dall'Agata, S. González-Martín, A. Papageorgiou and M. Peloso, "Warm dark energy," JCAP **08**, 032 (2020) [arXiv:1912.09950 [hep-th]].

[24] V. Domcke, V. Guidetti, Y. Welling and A. Westphal, "Resonant backreaction in axion inflation," JCAP **09**, 009 (2020) [arXiv:2002.02952 [astro-ph.CO]].

[25] A. Caravano, E. Komatsu, K. D. Lozanov and J. Weller, "Lattice simulations of axion-U(1) inflation," Phys. Rev. D **108**, no.4, 043504 (2023) doi:10.1103/PhysRevD.108.043504 [arXiv:2204.12874 [astro-ph.CO]].

[26] M. Peloso and L. Sorbo, "Instability in axion inflation with strong backreaction from gauge modes," JCAP **01**, 038 (2023) doi:10.1088/1475-7516/2023/01/038 [arXiv:2209.08131 [astro-ph.CO]].

[27] D. G. Figueroa, J. Lizarraga, A. Urio and J. Urrestilla, "Strong Backreaction Regime in Axion Inflation," Phys. Rev. Lett. **131**, no.15, 151003 (2023) doi:10.1103/PhysRevLett.131.151003 [arXiv:2303.17436 [astro-ph.CO]].

[28] J. Garcia-Bellido, A. Papageorgiou, M. Peloso and L. Sorbo, "A flashing beacon in axion inflation: recurring bursts of gravitational waves in the strong backreaction regime," [arXiv:2303.13425 [astro-ph.CO]].

[29] R. von Eckardstein, M. Peloso, K. Schmitz, O. Sobol and L. Sorbo, "Axion inflation in the

strong-backreaction regime: decay of the Anber-Sorbo solution," JHEP **11**, 183 (2023) doi:10.1007/JHEP11(2023)183 [arXiv:2309.04254 [hep-ph]].

[30] O. Iarygina, E. I. Sfakianakis, R. Sharma and A. Brandenburg, "Backreaction of axion-SU(2) dynamics during inflation," [arXiv:2311.07557 [astro-ph.CO]].

[31] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," JHEP **05**, 013 (2003) doi:10.1088/1126-6708/2003/05/013 [arXiv:astro-ph/0210603 [astro-ph]].

[32] N. Barnaby, R. Namba and M. Peloso, "Phenomenology of a Pseudo-Scalar Inflaton: Naturally Large Nongaussianity," JCAP **04**, 009 (2011) doi:10.1088/1475-7516/2011/04/009 [arXiv:1102.4333 [astro-ph.CO]].

[33] A. Ricciardone, L. V. Dall'Armi, N. Bartolo, D. Bertacca, M. Liguori and S. Matarrese, "Cross-Correlating Astrophysical and Cosmological Gravitational Wave Backgrounds with the Cosmic Microwave Background," Phys. Rev. Lett. **127** (2021) no.27, 271301 doi:10.1103/PhysRevLett.127.271301 [arXiv:2106.02591 [astro-ph.CO]].

[34] M. Braglia and S. Kuroyanagi, "Probing prerecombination physics by the cross-correlation of stochastic gravitational waves and CMB anisotropies," Phys. Rev. D **104**, no.12, 123547 (2021) doi:10.1103/PhysRevD.104.123547 [arXiv:2106.03786 [astro-ph.CO]].

[35] Y. Akrami *et al.* [Planck], "Planck 2018 results. IX. Constraints on primordial non-Gaussianity," Astron. Astrophys. **641**, A9 (2020) doi:10.1051/0004-6361/201935891 [arXiv:1905.05697 [astro-ph.CO]].

[36] V. Alba and J. Maldacena, "Primordial gravity wave background anisotropies," JHEP **03**, 115 (2016) doi:10.1007/JHEP03(2016)115 [arXiv:1512.01531 [hep-th]].

[37] C. R. Contaldi, "Anisotropies of Gravitational Wave Backgrounds: A Line Of Sight Approach," Phys. Lett. B **771**, 9-12 (2017) doi:10.1016/j.physletb.2017.05.020 [arXiv:1609.08168 [astro-ph.CO]].

[38] N. Bartolo, D. Bertacca, S. Matarrese, M. Peloso, A. Ricciardone, A. Riotto and G. Tasinato, "Characterizing the cosmological gravitational wave background: Anisotropies and non-Gaussianity," Phys. Rev. D **102**, no.2, 023527 (2020) doi:10.1103/PhysRevD.102.023527 [arXiv:1912.09433 [astro-ph.CO]].

[39] G. Mentasti, C. R. Contaldi and M. Peloso, "Intrinsic Limits on the Detection of the Anisotropies of the Stochastic Gravitational Wave Background," Phys. Rev. Lett. **131**, no.22, 221403 (2023) doi:10.1103/PhysRevLett.131.221403 [arXiv:2301.08074 [gr-qc]].

[40] Y. Cui, S. Kumar, R. Sundrum and Y. Tsai, "Unraveling cosmological anisotropies within stochastic gravitational wave backgrounds," JCAP **10**, 064 (2023) doi:10.1088/1475-7516/2023/10/064 [arXiv:2307.10360 [astro-ph.CO]].

[41] L. Bethke, D. G. Figueroa and A. Rajantie, "Anisotropies in the Gravitational Wave Background from Preheating," Phys. Rev. Lett. **111**, no.1, 011301 (2013) doi:10.1103/PhysRevLett.111.011301 [arXiv:1304.2657 [astro-ph.CO]].