CLASSIFICATION OF CONVEX ANCIENT FREE BOUNDARY MEAN CURVATURE FLOWS IN THE BALL.

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Abstract. We prove that there exists, in every dimension, a unique (modulo rotations about the origin and time translations) convex ancient mean curvature flow in the ball with free boundary on the sphere. This extends the main result of [4] to general dimensions.

Contents

1. Introduction

Mean curvature flow is the gradient flow of area for regular submanifolds. It models evolutionary processes governed by surface tension, such as grain boundaries in annealing metals [17, 20].

The systematic study of mean curvature flow was initiated by Brakke [6] (from the point of view of geometric measure theory) and later taken up by Huisken [15], who proved that closed convex hypersurfaces remain convex and shrink to "round" points in finite time. Different proofs of Huisken's theorem were discovered later by others [1, 2, 12].

Ancient solutions (that is, solutions defined on backwards-infinite time-intervals) are important from an analytical standpoint as they model singularity formation [13]. They also arise in quantum field theory, where they model the ultraviolet regime in certain Dirichlet sigma models [3]. They have generated a great deal of interest from a purely

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geometric standpoint due to their symmetry and rigidity properties (see, for example, [5] and the references therein).

The natural Neumann boundary value problem for mean curvature flow, called the *free boundary problem*, asks for a family of hypersurfaces whose boundary lies on (but is free to move on) a fixed barrier hypersurface which is met by the solution hypersurface orthogonally. Study of the free boundary problem was initiated by Huisken [16] and further developed by Stahl [18, 19] and others [8, 10, 21, 22]. In particular, Stahl proved that convex hypersurfaces with free boundary on a totally umbilic barrier remain convex and shrink to a "round" point on the barrier hypersurface. Hirsch and Li [14] proved that the same conclusion holds for "sufficiently convex" surfaces with free boundary on convex barriers in $\mathbb{R}^3$.

The analysis of ancient solutions to free boundary mean curvature flow remains in its infancy. We recently classified the convex ancient solutions to curve shortening flow in the disc [4] but, to our knowledge, the only other examples previously known seem to be those inherited from closed or complete examples (one may restrict the shrinking sphere, for example, to a halfspace).

We provide here a classification of convex[1] ancient free boundary mean curvature flows in the ball in all dimensions.

Theorem 1.1. *Modulo rotation about the origin and translation in time, there exists, for each $n \geq 1$, exactly one convex, locally uniformly convex ancient solution to free boundary mean curvature flow in the $(n+1)$-ball B^{n+1}. It converges to the point e_{n+1} as $t \to 0$ and smoothly to the horizontal bisector $B^n \times \{0\}$ as $t \to -\infty$. It is invariant under rotations about the e_{n+1}-axis. As a graph over B^n, it satisfies*

$$\mathrm{e}^{\mu_0 t} u(x,t) \to A\phi_0(x) \text{ uniformly in } x \text{ as } t \to -\infty$$

for some $A > 0$, where $\mu_0 < 0$ is the lowest eigenvalue and ϕ_0 the corresponding ground state of the "critical-Robin" Laplacian on B^n.

Theorem 1.1 is a consequence of Propositions 3.6, 4.4, and 4.5 proved below. Note that it is actually a classification of all convex ancient solutions, since the strong maximum principle and the Hopf boundary point lemma imply that any convex solution to the flow is either a stationary hyperball (and hence a bisector of the $(n+1)$-ball by the free boundary condition) or is locally uniformly convex at interior times.

[1]A free boundary hypersurface of the open ball B^{n+1} is *convex* if it bounds a convex region in B^2 and *locally uniformly convex* if it is of class C^2 and its second fundamental form is positive definite.

Our proof is strongly motivated by the one-dimensional case [4]. The main differences involve the use of the critical-Robin ground state in the construction of the solution and of a Liouville theorem for the critical-Robin heat equation (see §2) in proving its uniqueness.

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2. The critical-Robin heat equation

We will need[2] a Liouville theorem for ancient solutions to the "critical-Robin heat equation" on the unit n-ball B^n; that is, solutions u to the problem

$$\begin{cases} (\partial_t - \Delta)u = 0 \text{ in } B^n \times (-\infty, \infty) \\ \nabla u \cdot x = u \text{ on } \partial B^n \,. \end{cases} \tag{1}$$

Separation of variables leads us to consider eigenfunctions of the "critical-Robin Laplacian" on B^n; that is, solutions u to the problem

$$\begin{cases} -\Delta\phi = \mu\phi \text{ in } B^n \\ \nabla\phi \cdot x = \phi \text{ on } \partial B^n \,. \end{cases} \tag{2}$$

Observe that the lowest eigenvalue μ_0 is variationally characterized by

$$\mu_0 = \inf_{u \in H^1(B^n)} \frac{B(u,u)}{|u|^2_{L^2(B^n)}}\,,$$

where[3]

$$B(u,v) \doteqdot \int_{B^n} \nabla u \cdot \nabla v - \int_{\partial B^n} uv\,.$$

We note that standard methods (cf. [9, §6.5.1]) ensure that μ_0 is finite and simple, the spectrum forms a sequence $-\infty < \mu_0 < \mu_1 \leq \cdots \leq \mu_k \leq \ldots$ (with each eigenvalue appearing according to its multiplicity

[2] Most (if not all) of the results of this section appear to be well-known (they are simple consequences of, for example, the results of [11, §5]) but they are easy enough to obtain directly, so we include the details here.

[3] The second integral is interpreted according to the trace theorem.

and $\mu_k \to \infty$), and $L^2(B^n)$ admits an orthonormal basis consisting of corresponding eigenfunctions (which are necessarily smooth).

Separation of variables leads us to consider solutions of the form $u(r,z) = \phi(r)\Phi(z)$, where ϕ satisfies

$$-\left(\phi_{rr} + \frac{n-1}{r}\phi_r - \frac{\ell(\ell+n-2)}{r^2}\phi\right) = \mu\phi \text{ in } (0,1) \tag{3}$$

and Φ satisfies

$$-\Delta_{S^{n-1}}\Phi = \ell(\ell+n-2)\Phi$$

for some non-negative integer ℓ.

Lemma 2.1. *The negative eigenspace of (2) is one-dimensional.*

Proof. Assuming $\mu < 0$, consider the radial problem (3). We make the change of variable $r \mapsto \rho \doteqdot \lambda r$, where $\lambda \doteqdot \sqrt{-\mu}$, which results in the problem

$$\begin{cases} \phi'' + \dfrac{n-1}{\rho}\phi' - \dfrac{\ell(\ell+n-2)}{\rho^2}\phi = \phi \text{ in } (0,\lambda) \\ \lambda\phi'(\lambda) = \phi(\lambda)\,. \end{cases} \tag{4}$$

Consider a formal Taylor series solution

$$\phi(\rho) = \sum_{j=0}^{\infty} a_j \rho^j$$

to (4). Observe that

$$\phi'(\rho) = \sum_{j=1}^{\infty} j a_j \rho^{j-1} \text{ and } \phi''(\rho) = \sum_{j=2}^{\infty} j(j-1)a_j\rho^{j-2}$$

and hence ϕ satisfies (4) if and only if

$$\begin{aligned} 0 &= \phi'' + \frac{n-1}{\rho}\phi' - \frac{\ell(\ell+n-2)}{\rho^2}\phi - \phi \\ &= \frac{(n-1)a_1 - \ell(\ell+n-2)a_1}{\rho} - \frac{\ell(\ell+n-2)a_0}{\rho^2} \\ &\quad + \sum_{j=2}^{\infty}\big(j(n+j-2)a_j - \ell(n+\ell-2)a_j - a_{j-2}\big)\rho^{j-2}\,. \end{aligned}$$

Since $\phi \in L^2(0,1)$, the first two terms must vanish. Lest $\phi \equiv 0$, we conclude that either 1. $a_0 = 0$ and $\ell = 1$ (and hence Φ is linear), or 2. $a_1 = 0$ and $\ell = 0$ (and hence Φ is constant).

In the first case,

$$(j-1)(j+n-1)a_j = a_{j-2} \text{ for } j \geq 2\,,$$

so that

$$\phi(\rho) = \sum_{j=0}^{\infty} b_j \rho^{2j+1}\,, \quad b_j \doteqdot a_{2j+1}\,,$$

where the coefficients are given by

$$b_j = b_0 \prod_{\ell=1}^{j} \frac{1}{2\ell(n+2\ell)}\,.$$

We may assume that $b_0 = 1$. Since

$$\phi'(\rho) = 1 + \sum_{j=1}^{\infty}(2j+1) b_j \rho^{2j}\,,$$

the boundary condition yields the equation

$$\sum_{j=1}^{\infty} 2j b_j \lambda^{2j} = 0$$

for the frequency λ, which is impossible since $\lambda > 0$.

In the second case,

$$j(n+j-2)a_j = a_{j-2} \ \text{ for } \ j \geq 2\,,$$

so that

$$\phi(\rho) = \sum_{j=0}^{\infty} b_j \rho^{2j}\,, \quad b_j \doteqdot a_{2j}\,,$$

where the coefficients are given by

$$b_j = b_0 \prod_{\ell=1}^{j} \frac{1}{2\ell(n+2(\ell-1))}\,.$$

We may assume that $b_0 = 1$.

Since

$$\phi'(\rho) = \sum_{j=1}^{\infty} 2j b_j \rho^{2j-1}\,,$$

the boundary condition then yields the equation

$$\sum_{j=1}^{\infty}(2j-1) b_j \lambda^{2j} = 1$$

for the frequency λ. Since the coefficients are all positive, the expression on the left is monotone in λ. Since it goes to zero when $\lambda \to 0$ and to infinity when $\lambda \to \infty$, there must be exactly one solution, and hence exactly one negative eigenvalue. Since ϕ was determined by λ up to choosing b_0, the claim follows. $\square$

Denote by ϕ_0 the unique solution to (3) with $\mu = \mu_0$ and $\phi_0(0) = 1$ and set $\lambda_0 \doteqdot \sqrt{-\mu_0}$. Note that

$$\lambda_0 \frac{\phi'(\lambda_0)}{\phi(\lambda_0)} = 1 \tag{5}$$

and, since λ_0 is increasing in the dimension n, $\lambda_0 \geq \lambda_0|_{n=1} > 1$.

Lemma 2.2. *The null eigenspace of (2) consists of the linear functions.*

Proof. When $\mu = 0$, the equation (3) can be solved directly. Its admissible (i.e. square integrable) solutions are multiples of

$$\phi_\ell(r) \doteqdot r^\ell\,,$$

but the boundary condition rules out all but the linear ones, $\ell = 1$. The only admissible spherical harmonics Φ are then those of degree $\ell = 1$. □

Corollary 2.3. *Let u be a non-negative ancient solution to the critical-Robin heat equation on the unit ball. If $u(\cdot, t) = \mathrm{e}^{o(-t)}$ as $t \to -\infty$, then*

$$u(x,t) = A\mathrm{e}^{\lambda_0^2 t}\phi_0(\lambda_0|x|) \;\; \textit{for some} \;\; A \in \mathbb{R}\,.$$

Proof. We may represent u as

$$u(x,t) = \sum_{j=0}^{\infty} A_j \mathrm{e}^{-\mu_j t}\phi_j(x)\,,$$

where $\{\phi_j\}_{j=0}^\infty$ is an orthonormal basis for $L^2(B^n)$ consisting of eigenfunctions ϕ_j of the critical-Robin Laplacian (with eigenvalue μ_j). Since the null modes are linear and $\mu_j > 0$ for each $j \geq n+1$, the coefficients of these states must be zero. □

2.1. **Further properties of the ground state.** Recall that the radial ground state ϕ_0 is given by

$$\phi_0(\rho) = 1 + \sum_{j=1}^{\infty} b_j\rho^{2j}\,, \;\; \text{where} \;\; b_j = \prod_{\ell=1}^{j} \frac{1}{2\ell(n+2(\ell-1))}\,.$$

In particular, ϕ_0 well-defined and analytic on the whole real line, and satisfies

$$\phi_0'' + \frac{n-1}{\rho}\phi_0' = \phi_0$$

everywhere.[4] We shall need the following basic properties of ϕ_0.

[4]In the following, it is instructive to keep in mind that, when $n = 1$, $\phi_0(\rho) = \cosh\rho$.

Lemma 2.4. *The (odd) function ϕ_0'/ϕ_0 is monotone increasing and converges to one as $\rho \to \infty$.*

Proof. Setting $\Phi \doteqdot \phi_0'/\phi_0$, we find that

$$\Phi(\rho) = \frac{\sum_{j=0}^{\infty} 2(j+1)b_{j+1}\rho^{2j+1}}{\sum_{j=0}^{\infty} b_j \rho^{2j}} \sim \frac{\rho}{n} \text{ for } \rho \sim 0\,.$$

Observe also that

$$\rho\Phi' = \rho(1-\Phi^2) - (n-1)\Phi$$

and

$$\rho\Phi'' = 1 - \Phi^2 - n\Phi' - 2\rho\Phi\Phi'\,.$$

The first observation implies that $\Phi(\rho) > 0 = \Phi(0)$ for ρ close to (but above) zero. The second implies that Φ can never reach one, and the third then implies that Φ' remains positive for all $\rho > 0$. The second observation then implies that $\Phi(\rho) \to 1$ as $\rho \to \infty$. □

Lemma 2.5. *The (even) function $\phi_0'/\rho\phi_0$ is monotone decreasing for $\rho > 0$ and bounded from above by $1/n$.*

Proof. If we set $\Phi \doteqdot \phi_0'/\rho\phi_0$, then

$$\Phi(\rho) = \frac{\sum_{j=0}^{\infty} 2(j+1)b_{j+1}\rho^{2j}}{\sum_{j=0}^{\infty} b_j \rho^{2j}}$$

and

$$\rho\Phi'' = -(n+1)\Phi' - 2\rho\Phi^2 - 2\rho^2\Phi\Phi'\,.$$

The first observation implies that $\Phi(0) = 2b_1 = \frac{1}{n}$ and hence

$$\begin{aligned}\Phi(\rho) - \Phi(0) &= \frac{\sum_{j=0}^{\infty} 2(j+1)b_{j+1}\rho^{2j} - 2b_1 \sum_{j=0}^{\infty} b_j\rho^{2j}}{\sum_{j=0}^{\infty} b_j \rho^{2j}} \\ &= \frac{2\sum_{j=0}^{\infty}\big((j+1)b_{j+1} - b_1 b_j\big)\rho^{2j}}{\sum_{j=0}^{\infty} b_j\rho^{2j}} \\ &< 0\,.\end{aligned}$$

The second observation then implies that Φ remains decreasing for all $\rho > 0$. □

3. Existence

We are now ready to construct a non-trivial ancient free boundary mean curvature flow in the $(n+1)$-ball. It will be clear from the construction that the solution is rotationally symmetric about the vertical axis, emerges at time negative infinity from the horizontal n-ball, and converges at time zero to the point e_{n+1}.

We shall also prove an estimate for the height of the constructed solution (which will be needed to prove its uniqueness).

3.1. **Barriers.** Given $\theta \in (0, \frac{\pi}{2})$, denote by S_θ the n-sphere centred on the e_{n+1}-axis which meets ∂B^{n+1} orthogonally at $\{\cos\theta e + \sin\theta e_{n+1} : e \in S^{n-1} \times \{0\}\}$. That is,

$$\mathrm{S}_\theta \doteqdot \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 + (\csc\theta - y)^2 = \cot^2\theta\right\} . \tag{6}$$

If we set

$$\theta^-(t) \doteqdot \arcsin \mathrm{e}^{nt} \;\text{ and }\; \theta^+(t) \doteqdot \arcsin \mathrm{e}^{2nt} ,$$

then we find that the inward normal speed of $\mathrm{S}_{\theta^-(t)}$ is no greater than its mean curvature H^- and the inward normal speed of $\mathrm{S}_{\theta^+(t)}$ is no less than its mean curvature H^+. The maximum principle and the Hopf boundary point lemma then imply that

Proposition 3.1. *A solution to free boundary mean curvature flow in B^{n+1} which lies below (resp. above) the sphere S_{θ_0} at time t_0 lies below $\mathrm{S}_{\theta^+(t_0^++t-t_0)}$ (resp. above $\mathrm{S}_{\theta^-(t_0^-+t-t_0)}$) for all $t > t_0$, where $2nt_0^+ = \log\sin\theta_0$ (resp. $nt_0^- = \log\sin\theta_0$).*

Next, given $\lambda > 0$, consider the family $\{\Sigma_t^\lambda\}_{t\in(-\infty,0)}$ of hypersurfaces Σ_t^λ defined by

$$\Sigma_t^\lambda \doteqdot \left\{(x,y) \in \mathbb{R}^n \times (0, \tfrac{\pi}{2\lambda}) : \sin(\lambda y) = \mathrm{e}^{\lambda^2 t}\phi\big(\lambda|x|\big)\right\} ,$$

where $\phi = \phi_0$ is the unique solution to (4) with $\mu = \mu_0$ and $\phi(0) = 1$.

Differentiating the defining equation with respect to an arclength parameter s along the profile curve $\Lambda_t^\lambda \doteqdot \Sigma_t^\lambda \cap e_1 \wedge e_{n+1} \cap \{x_1 > 0\}$ yields (with the abuse of notation $x = x_1$)

$$\cos(\lambda y) y_s = \mathrm{e}^{\lambda^2 t}\phi'(\lambda x) x_s . \tag{7}$$

From this we find that the downwards pointing normal to the profile curve is given by

$$\nu = \left(\frac{\phi'(\lambda x)}{\phi(\lambda x)}, -\frac{\cos(\lambda y)}{\sin(\lambda y)}\right) \Bigg/ \sqrt{\left(\frac{\phi'(\lambda x)}{\phi(\lambda x)}\right)^2 + \left(\frac{\cos(\lambda y)}{\sin(\lambda y)}\right)^2} .$$

Differentiating (7) yields

$$\cos(\lambda y)y_{ss} - \mathrm{e}^{\lambda^2 t}\phi'(\lambda x)x_{ss} = \lambda\left(\sin(\lambda y)y_s^2 + \mathrm{e}^{\lambda^2 t}\phi''(\lambda x)x_s^2\right)$$
$$= \lambda\left(\sin(\lambda y)y_s^2 + \mathrm{e}^{\lambda^2 t}\left[\phi(\lambda x) - \frac{n-1}{\lambda x}\phi'(\lambda x)\right]x_s^2\right).$$

From this we find that the curvature of the profile curve is given by

$$\begin{aligned}\kappa = &-\gamma_{ss}\cdot\nu\\ &= \lambda\tan(\lambda y)x_s\left(1 - \frac{n-1}{\lambda x}\frac{\phi'(\lambda x)}{\phi(\lambda x)}x_s^2\right).\end{aligned}$$

On the other hand, the rotational curvature $\hat\kappa$ is given by

$$\hat\kappa = \frac{\nu\cdot e_1}{x} = \tan(\lambda y)\frac{x_s}{x}\frac{\phi'(\lambda x)}{\phi(\lambda x)}.$$

If we choose the arclength parameter so that $s = 0$ corresponds to $x = 0$, then we may write $\tau \doteqdot \gamma_s = (\cos\theta, \sin\theta)$ with $\theta \in [0, \frac{\pi}{2})$ increasing in s. The mean curvature of Σ_t^λ is then given, along the profile curve Λ_t^λ, by

$$H = \lambda\tan(\lambda y)\cos\theta\left(1 + (n-1)\sin^2\theta\frac{\phi'(\lambda x)}{\lambda x\phi(\lambda x)}\right).$$

On the other hand,

$$\cos(\lambda y)y_t = \mathrm{e}^{\lambda^2 t}\phi'(\lambda x)x_t + \lambda\mathrm{e}^{\lambda^2 t}\phi(\lambda x)$$

so that

$$-\gamma_t\cdot\nu = \lambda\tan(\lambda y)\cos\theta \leq H.$$

That is, $\{\Sigma_t^\lambda\}_{t\in(-\infty,0)}$ is a subsolution to mean curvature flow.

Proposition 3.2. *Given $\theta \in (0, \frac{\pi}{2})$, there exists a unique solution $\lambda(\theta)$ to*

$$\frac{\phi'(\lambda\cos\theta)}{\phi(\lambda\cos\theta)}\cos\theta = \frac{\cos(\lambda\sin\theta)}{\sin(\lambda\sin\theta)}\sin\theta.$$

Let $\{M_t\}_{t\in[\alpha,\omega)}$ be a rotationally symmetric solution to free boundary mean curvature flow in B^{n+1}. Define $\theta_\alpha \in (0, \frac{\pi}{2})$ by

$$\sin\theta_\alpha = \{X\cdot e_{n+1} : X \in \partial M_\alpha\}.$$

If $\lambda \leq \lambda(\theta_\alpha)$ and M_α lies above Σ_s^λ, then M_t lies above $\Sigma_{s+t-\alpha}^\lambda$ for all $t \in (\alpha,\omega)\cap(-\infty, \alpha - s)$.

Proof. Due to Lemma 2.4, we may proceed as in [4, Lemma 2.2 and Proposition 2.3]. □

Note also that, by the defining property (5) of λ_0, we have the inequality $\lambda(\theta) < \lambda_0$.

We will also require the following technical properties of Σ^λ.

Lemma 3.3. *The mean curvature H of the hypersurface Σ^λ satisfies*

$$\nabla H \cdot X \geq 0 \quad \textit{and} \quad \limsup_{\lambda\to\lambda_0} \max_{\overline{\Sigma}^\lambda} |\nabla \log H| \leq 1 \,.$$

Proof. If we set $\Phi(\rho) \doteqdot \phi'(\rho)/\rho\phi(\rho)$, then

$$\begin{aligned}\hat{\kappa}_s &= \frac{\cos\theta}{x}(\kappa - \hat{\kappa})\\ &= \frac{\cos\theta}{x}\left(\lambda\tan(\lambda y)\cos\theta\left[1 - n\Phi(\lambda x) + (n-1)\sin^2\theta\Phi(\lambda x)\right]\right)\end{aligned}$$

and

$$\begin{aligned}\kappa_s = {}& \big(\lambda^2\sec^2(\lambda y)\cos\theta\sin\theta - \lambda\tan(\lambda y)\sin\theta\kappa\big)(1-(n-1)\cos^2\theta\Phi(\lambda x))\\ & + (n-1)\lambda\tan(\lambda y)\cos\theta\big(2\cos\theta\sin\theta\Phi(\lambda x)\kappa - \lambda\cos^3\theta\Phi'(\lambda x)\big)\\ = {}& \big(\lambda^2\cos\theta\sin\theta + (n-1)\lambda^2\tan^2(\lambda y)\sin\theta\cos^3\theta\Phi(\lambda x)\big)\\ & \qquad\qquad\qquad\qquad \cdot(1-(n-1)\cos^2\theta\Phi(\lambda x))\\ & + (n-1)\lambda\tan(\lambda y)\cos\theta\big(2\cos\theta\sin\theta\Phi(\lambda x)\kappa - \lambda\cos^3\theta\Phi'(\lambda x)\big)\,.\end{aligned}$$

These are both positive for $\theta > 0$ by Lemma 2.5, which implies the first claim.

Next, estimating (in the *second* term below)

$$\kappa \leq H = \lambda\tan(\lambda y)\cos\theta(1+(n-1)\sin^2\theta\Phi(\lambda x))$$

and $\Phi'(\rho) \leq 0$ for $\rho \geq 0$, we observe that

$$\begin{aligned}(\log H)_s = {}& \log\big(\lambda\tan(\lambda y)\cos\theta\big)_s + \log\big(1+(n-1)\sin^2\theta\Phi(\lambda x)\big)_s\\ = {}& \frac{\lambda^2(1+\tan^2(\lambda y))\cos\theta\sin\theta - \lambda\tan(\lambda y)\sin\theta\kappa}{\lambda\tan(\lambda y)\cos\theta}\\ & + (n-1)\frac{2\sin\theta\cos\theta\kappa\Phi(\lambda x) + \lambda\sin^2\theta\cos\theta\Phi'(\lambda x)}{1+(n-1)\sin^2\theta\Phi(\lambda x)}\\ \leq {}& \frac{\lambda\sin\theta}{\tan(\lambda y)} + (n-1)\lambda\tan(\lambda y)\cos^2\theta\sin\theta\Phi(\lambda x)\\ & + 2(n-1)\lambda\tan(\lambda y)\sin\theta\cos^2\theta\Phi(\lambda x)\,.\end{aligned}$$

The second and third terms approach zero uniformly as $\lambda \to \lambda_0$ since θ does. The first may be rewritten as

$$\frac{\lambda\sin\theta}{\tan(\lambda y)} = \frac{\phi'(\lambda x)}{\phi(\lambda x)}\lambda\cos\theta\,.$$

The second claim now follows from Lemma 2.4 and the identity (5). $\square$

3.2. **Old-but-not-ancient solutions.** Given $\rho > 0$, choose a hypersurface M^ρ in B^{n+1} which satisfies the following properties:

- M^ρ is rotationally symmetric about the e_{n+1}-axis,
- M^ρ meets ∂B^{n+1} orthogonally at $\{\cos\rho e + \sin\rho e_{n+1} : e \in S^{n-1} \times \{0\}\}$,
- $M^\rho \cap B^{n+1}$ is the relative boundary of a convex region $\Omega^\rho \subset B^{n+1}$, and
- $\nabla H^\rho \cdot X \geq 0$, where X denotes the position vector.

For example, we could take $M^\rho \doteqdot \Sigma^{\lambda_\rho}_{t_\rho}$, where $\lambda_\rho \doteqdot \lambda(\rho)$ and

$$-t_\rho = \lambda_\rho^{-2} \log\left(\frac{\phi(\lambda_\rho \cos\rho)}{\sin(\lambda_\rho \sin\rho)}\right).$$

Recall ing the notation from (6), observe that the sphere S_{θ_ρ} defined by

$$\sin\theta_\rho = \frac{2\sin\rho}{1+\sin^2\rho}$$

is tangent to the plane $\{X \in \mathbb{R}^{n+1} : X \cdot e_{n+1} = \sin\rho\}$, and hence lies above M^ρ.

Work of Stahl [18, 19] now yields the following *old-but-not-ancient solutions*.

Lemma 3.4. *For each $\rho \in (0, \frac{\pi}{2})$, there exists a smooth solution*[5] *$\{M^\rho_t\}_{t\in[\alpha_\rho,0)}$ to free boundary mean curvature flow in B^{n+1} which satisfies the following properties:*

- $M^\rho_{\alpha_\rho} = M^\rho$,
- *M^ρ_t is convex and locally uniformly convex for each $t \in (\alpha_\rho, 0)$,*
- *M^ρ_t is rotationally symmetric about the e_{n+1}-axis for each $t \in (\alpha_\rho, 0)$,*
- *$M^\rho_t \to e_{n+1}$ uniformly as $t \to 0$,*
- *$\nabla H^\rho \cdot X \geq 0$, and*
- *$\alpha_\rho < \frac{1}{2n}\log\left(\frac{2\sin\rho}{1+\sin^2\rho}\right) \to -\infty$ as $\rho \to 0$.*

Proof. Existence of a maximal solution to mean curvature flow out of M^ρ which meets ∂B^{n+1} orthogonally was proved by Stahl [19, Theorem 2.1]. Stahl also proved that this solution remains convex and locally uniformly convex and shrinks to a point on the boundary of B^{n+1} at the final time (which is finite) [18, Proposition 1.4]. We obtain $\{M^\rho_t\}_{t\in[\alpha_\rho,0)}$ by time-translating Stahl's solution.

[5] Given by a one parameter family of immersions $X : \overline{M} \times [\alpha_\rho, 0) \to \overline{B}^{n+1}$ satisfying $X \in C^\infty(\overline{M} \times (\alpha_\rho, 0)) \cap C^{2+\beta, 1+\frac{\beta}{2}}(\overline{M} \times [\alpha_\rho, 0))$ for some $\beta \in (0,1)$.

By uniqueness of solutions (or the Alexandrov reflection principle) M_t^ρ remains rotationally symmetric about the e_{n+1}-axis for $t \in (\alpha_\rho, 0)$, so the final point is e_{n+1}.

The rotational symmetry also implies that $\nabla H = 0$ at the point $p_t \doteqdot M_t^\rho \cap \mathbb{R}e_{n+1}$ for all $t \in [\alpha_\rho, 0)$. By [18, Proposition 2.1], $\nabla H \cdot X = H > 0$ at the boundary for all $t \in (\alpha_\rho, 0)$. The maximum principle now implies that $\nabla H \cdot X \geq 0$. To see this, recall that

$$(\partial_t - \Delta)|\nabla H| \leq c|A|^2|\nabla H|$$

where c is a constant that depends only on n, and, given any $\sigma \in (\alpha_\rho, 0)$ and any $\varepsilon > 0$, consider the function

$$v_{\sigma,\varepsilon} \doteqdot \begin{cases} \langle \nabla H, \vec{r}\rangle + \varepsilon \mathrm{e}^{(C_\sigma c+1)(t-\alpha_\rho)} & \text{on } B^{n+1} \setminus \mathbb{R}e_{n+1} \\ \varepsilon \mathrm{e}^{(C_\sigma c+1)(t-\alpha_\rho)} & \text{on } \mathbb{R}e_{n+1}\,, \end{cases}$$

where $C_\sigma \doteqdot \max_{t\in[\alpha_\rho,\sigma]} \max_{\overline{M}_t^\rho} |A|^2$ and $\vec{r} \doteqdot X^\top/|X^\top|$. Note that $v_{\sigma,\varepsilon}$ is continuous. Observe that $v_{\sigma,\varepsilon}$ is no less than ε on ∂M_t^ρ for all t, on the axis off rotation for all t, and everywhere at the initial time. Thus, if $v_{\sigma,\varepsilon}$ is not positive everywhere in $\overline{M}^\rho \times [\alpha_\rho, \sigma]$, then there must be a first time $t_0 \in (0, \sigma]$ and an off-axis interior point x_0 such that $v_{\sigma,\varepsilon}(x_0, t_0) = 0$ and $v_{\sigma,\varepsilon} \geq 0$ in a small parabolic neighbourhood of (x_0, t_0). Since $\langle \nabla H, \vec{r}\rangle|_{(x_0,t_0)} < 0$, we find that $\langle \nabla H, \vec{r}\rangle = -|\nabla H|$ in a small spacetime neighborhood of (x_0, t_0). But then, at (x_0, t_0),

$$\begin{aligned} 0 &\geq (\partial_t - \Delta)v_{\sigma,\varepsilon} \\ &\geq -C_\sigma c|\nabla H| + \varepsilon(C_\sigma c + 1)\mathrm{e}^{(C_\sigma c+1)(t-\alpha_\rho)} \\ &= \varepsilon \mathrm{e}^{(C_\sigma c+1)(t-\alpha_\rho)} \\ &> 0\,, \end{aligned}$$

which is absurd. So $v_{\varepsilon,\sigma}$ is indeed positive in $\overline{M}_t^\rho$ for all $t \in [\alpha_\rho, \sigma]$. Taking $\varepsilon \to 0$ and $\sigma \to 0$ then implies that $\nabla H \cdot X \geq 0$.

Since $\mathrm{S}_{\theta_\rho} \subset \Omega^\rho$, the final property follows from Proposition 3.1. □

We now fix $\rho > 0$ and drop the super/subscript ρ. Denote by $\Gamma_t = M_t \cap e_1 \wedge e_{n+1} \cap \{x_1 > 0\}$ the profile curve of M_t and set

$$p_t \doteqdot M_t \cap \mathbb{R}e_{n+1}\,, \quad q_t \doteqdot \partial B^{n+1} \cap \Gamma_t\,,$$

$$\underline{H}(t) \doteqdot \min_{M_t} H = H(p_t) \quad\text{and}\quad \overline{H}(t) \doteqdot \max_{M_t} H = H(q_t)\,,$$

and define $\underline{y}(t)$, $\overline{y}(t)$ and $\overline{\theta}(t)$ by

$$p_t = \underline{y}(t)e_{n+1}\,, \quad q_t = \cos\overline{\theta}(t)e_1 + \sin\overline{\theta}(t)e_{n+1}\,, \quad\text{and}\quad \overline{y}(t) = \sin\overline{\theta}(t)\,.$$

Lemma 3.5. *Each old-but-not-ancient solution satisfies*

$$\underline{H} \leq n \tan \overline{\theta} \leq \overline{H}\,, \tag{8}$$

$$\sin \overline{\theta} \leq e^{nt}\,, \tag{9}$$

and

$$\frac{\sin \overline{\theta}}{1 + \cos \overline{\theta}} \leq \underline{y} \leq \sin \overline{\theta}\,. \tag{10}$$

Proof. To prove the lower bound for $\overline{H}$, it suffices to show that the sphere $\mathrm{S}_{\overline{\theta}(t)}$ (see (6)) lies locally below M_t near q_t. If this is not the case, then, locally around q_t, M_t lies below $\mathrm{S}_{\overline{\theta}(t)}$ and hence $H(q_t) \leq n \tan \overline{\theta}(t)$. But then we can translate $\mathrm{S}_{\overline{\theta}(t)}$ downwards until it touches M_t from below in an interior point at which the curvature must satisfy $H \geq n \tan \overline{\theta}(t)$. This contradicts the maximization of the mean curvature at q_t (unless M_t coincides with $\mathrm{S}_{\overline{\theta}(t)}$ in a neighbourhood of q_t, which by itself implies the claim).

The estimate (9) now follows by integrating the inequality

$$\frac{d}{dt} \sin \overline{\theta} = \cos \overline{\theta}\, \overline{H} \geq n \sin \overline{\theta}$$

between any initial time t and the final time 0 (at which $\overline{\theta} = \frac{\pi}{2}$ since the solution contracts to the point e_{n+1}).

The upper bound for $\underline{y}$ follows from convexity and the boundary condition $\overline{y} = \sin \overline{\theta}$. To prove the lower bound, we will show that the sphere $\mathrm{S}_{\overline{\theta}(t)}$ lies nowhere above M_t. Suppose that this is not the case. Then, since $\mathrm{S}_{\overline{\theta}(t)}$ lies locally below M_t near q_t, we can move $\mathrm{S}_{\overline{\theta}(t)}$ downwards until it is tangent from below to a point p'_t on M_t, at which we must have $H \geq n \tan \overline{\theta}(t)$. But then, since $\nabla H \cdot X \geq 0$, we find that $H \geq n \tan \overline{\theta}(t)$ for all points on the radial curve between p'_t and the nearest boundary point. But this implies that this whole arc (including p'_t) lies above $\mathrm{S}_{\overline{\theta}(t)}$, a contradiction.

To prove the upper bound for $\underline{H}$, fix t and consider the sphere S centred on the e_{n+1}-axis through the points p_t and q_t. Its radius is $r(t)$, where

$$r \doteqdot \frac{\cos^2 \overline{\theta} + (\sin \overline{\theta} - \underline{y})^2}{2(\sin \overline{\theta} - \underline{y})}\,.$$

We claim that M_t lies locally below S near p_t. Suppose that this is not the case. Then, by the rotational symmetry of M_t and S about the e_{n+1}-axis, M_t lies locally above S near p_t. This implies two things: first, that

$$H(p_t) \geq n r^{-1},$$

and second, that, by moving S vertically upwards, we can find a point p_t' (the final point of contact) which satisfies

$$H(p_t') \leq nr^{-1}\,.$$

These two inequalities contradict the (unique) minimization of H at p_t. We conclude that

$$\underline{H} \leq \frac{2n(\sin\overline{\theta} - \underline{y})}{\cos^2\overline{\theta} + (\sin\overline{\theta} - \underline{y})^2} \leq n\tan\overline{\theta}$$

due to the lower bound for $\underline{y}$. □

3.3. **Taking the limit.**

Proposition 3.6. *There exists a convex, locally uniformly convex ancient mean curvature flow in B^{n+1} with free boundary on ∂B^{n+1}.*

Proof. For each $\rho > 0$, consider the old-but-not-ancient solution $\{M_t^\rho\}_{t\in[\alpha_\rho,0)}$, $M_t^\rho = \partial\Omega_t^\rho$, constructed in Lemma 3.4. By (9), Ω_t^ρ contains $\mathrm{S}_{\omega(t)} \cap B^{n+1}$, where $\omega(t) \in (0, \frac{\pi}{2})$ is uniquely defined by

$$\frac{1-\cos\omega(t)}{\sin\omega(t)} = \mathrm{e}^{nt}\,.$$

If we represent M_t^ρ as a graph $x \mapsto y^\rho(x,t)$ over the horizontal n-ball, then convexity and the boundary condition imply that $|Dy^\rho| \leq \tan\omega$. Since $\omega(t)$ is independent of ρ, Stahl's (global in space, interior in time) Ecker–Huisken type estimates [19] imply uniform-in-ρ bounds for the curvature and its derivatives. So the limit

$$\{M_t^\rho\}_{t\in[\alpha_\rho,0)} \to \{M_t\}_{t\in(-\infty,0)}$$

exists in C^∞ (globally in space on compact subsets of time) and the limit $\{M_t\}_{t\in(-\infty,0)}$ satisfies mean curvature flow with free boundary in B^{n+1}. On the other hand, since $\{M_t^\rho\}_{t\in(\alpha_\rho,0)}$ contracts to e_{n+1} as $t \to 0$, (the contrapositive of) Proposition 3.1 implies that M_t^ρ must intersect the closed region enclosed by $\mathrm{S}_{\theta^+(t)}$ for all $t < 0$. It follows that M_t converges to e_{n+1} as $t \to 0$. By [18, Proposition 4.5], M_t is locally uniformly convex for each t. Since each M_t is the limit of convex hypersurfaces, each is convex. □

3.4. **Asymptotics for the height.** For the purposes of this section, we fix an ancient solution $\{M_t\}_{(-\infty,0)}$ obtained as in Proposition 3.6 by taking a sublimit as $\lambda \searrow \lambda_0$ of the specific old-but-not ancient solutions $\{M_t^\lambda\}_{t\in[\alpha_\lambda,0)}$ corresponding to $M_{\alpha_\lambda}^\lambda = \Sigma_{t_\lambda}^\lambda \cap B^{n+1}$, t_λ being the time at which $\{\Sigma_t^\lambda\}_{t\in(-\infty,0)}$ meets ∂B^{n+1} orthogonally. The asymptotics we obtain for this solution will be used to prove its uniqueness.

We will need to prove that the limit $\lim_{t\to-\infty} \mathrm{e}^{-\lambda_0^2 t}\underline{y}(t)$ exists in $(0,\infty)$. The following speed bound will imply that it exists in $[0,\infty)$.

Lemma 3.7. *The ancient solution* $\{M_t\}_{(-\infty,0)}$ *satisfies*

$$\underline{H} \geq \lambda_0^2 \underline{y}\,. \tag{11}$$

Proof. It suffices to prove that

$$\min_{M_t^\lambda} \frac{H}{y} \geq \min_{t=\alpha_\lambda} \frac{H}{y} \tag{12}$$

on each of the old-but-not-ancient solutions $\{M_t^\lambda\}_{t\in[\alpha_\lambda,0)}$ since

$$\begin{aligned}
\min_{t=\alpha_\lambda} \frac{H}{y} &= \min_{\Sigma^\lambda_{t_\lambda}} \frac{H}{y}\\
&= \lambda^2 \min_{\Sigma^\lambda_{t_\lambda}} \frac{\tan(\lambda y)}{\lambda y}\cos\theta\left(1+(n-1)\sin^2\theta\frac{\phi'(\lambda x)}{\lambda x\phi(\lambda x)}\right)\\
&\geq \lambda^2\cos\theta_\lambda\\
&\to \lambda_0^2 \;\text{ as }\; \lambda\to\lambda_0\,, \qquad (13)
\end{aligned}$$

where θ_λ is the angle that Λ_t^λ meets the boundary. But this is an easy consequence of the maximum principle, since

$$\begin{aligned}
(\partial_t-\Delta)\frac{H}{y} &= |A|^2\frac{H}{y} + 2\left\langle \nabla\frac{H}{y}, \frac{\nabla y}{y}\right\rangle\\
&\geq 2\left\langle \nabla\frac{H}{y}, \frac{\nabla y}{y}\right\rangle
\end{aligned}$$

in the interior and

$$\nabla\frac{H}{y} = 0$$

at the boundary. □

It follows that

$$\left(\mathrm{e}^{-\lambda_0^2 t}\underline{y}\right)_t \geq 0\,. \tag{14}$$

In particular, the limit

$$A \doteqdot \lim_{t\to-\infty} \mathrm{e}^{-\lambda_0^2 t}\underline{y}(t)$$

exists in $[0,\infty)$ as claimed.

We want next to prove that the above limit is positive. We will achieve this through a suitable upper bound for the speed.

Recall that

$$(\partial_t-\Delta)|\nabla H| \leq c|A|^2|\nabla H| \;\text{ and }\; (\partial_t-\Delta)\langle X,\nu\rangle = |A|^2\langle X,\nu\rangle - 2H, \tag{15}$$

where c depends only on n and X denotes the position. The good $-2H$ term in the second equation may be exploited to obtain the following crude speed bound.

Lemma 3.8. *There exist $T > -\infty$ and $C < \infty$ such that*

$$\overline{H} \leq Ce^{nt} \text{ for all } t < T\,. \tag{16}$$

Proof. We will prove the estimate on the (sufficiently) old-but-not-ancient solutions $\{M_t^\lambda\}_{t\in(\alpha_\lambda,0)}$. We first prove a crude gradient estimate of the form

$$|\nabla H| \leq 4H \tag{17}$$

for t sufficiently negative. It will suffice to prove that

$$|\nabla H| - 2H + 2\langle X, \nu\rangle \leq 0\,, \tag{18}$$

where X denotes the position. Indeed, since

$$\langle \nabla \langle X, \nu\rangle, X\rangle = A(X^\top, X^\top) > 0\,,$$

we may estimate, by (12) and (13),

$$|\langle X, \nu\rangle| \leq |\langle X, \nu\rangle|_{x=0} = y|_{x=0} \leq H|_{x=0} = \min_{M_t^\lambda} H \leq H \tag{19}$$

for λ sufficiently close to λ_0.

For λ sufficiently close to λ_0, we have $H|_{t=\alpha_\lambda} < 1/\sqrt{c}$, where c is the constant in (15). Denote by T^λ the first time at which H reaches $1/\sqrt{c}$. Since H is continuous up to the initial time α_λ, we have $T^\lambda > \alpha_\lambda$. We claim that (18) holds for $t < T^\lambda$ so long as λ is sufficiently large. It is satisfied on the initial timeslice $M^\lambda_{\alpha_\lambda} = \Sigma^\lambda_{t_\lambda}$ by Lemma 3.3. We will show that the function

$$f_\varepsilon \doteqdot |\nabla H| - 2H + 2\langle X, \nu\rangle - \varepsilon e^{t-\alpha_\lambda}$$

remains negative up to time T^λ. Suppose, to the contrary, that f_ε reaches zero at some time $t < T^\lambda$ at some point $p \in \overline{M}_t^\lambda$. Since $|\nabla H| - 2H + 2\langle X, \nu\rangle$ is negative at the boundary, p must be an interior point. At such a point, using the evolution equations (15), we have

$$\begin{aligned} 0 \leq (\partial_t - \Delta) f_\varepsilon &\leq |A|^2(c|\nabla H| - 2H + 2\langle X, \nu\rangle) - 4H - \varepsilon e^{t-\alpha_\lambda} \\ &= |A|^2\big(2(c-1)[H - \langle X, \nu\rangle] + c\varepsilon e^{t-\alpha_\lambda}\big) - 4H - \varepsilon e^{t-\alpha_\lambda}\,. \end{aligned}$$

Recalling (19) and estimating $|A| \leq H$ and $H \leq \frac{1}{\sqrt{c}}$ yields

$$0 \leq 4(c-1)H^3 - 4H + (cH^2 - 1)\varepsilon e^{t-\alpha_\lambda} < 0\,,$$

which is absurd. So f_ε does indeed remain negative, and taking $\varepsilon \to 0$ yields (17) for $t < T^\lambda$.

Since the length of the profile curve Γ_t^λ is bounded by 1, integrating (17) from the axis to the boundary yields

$$\overline{H} \leq \mathrm{e}^4 \underline{H} \;\text{ for }\; t < T^\lambda .$$

Recalling (8) and (9), this implies that

$$\overline{H} \leq \mathrm{e}^4 \frac{\mathrm{e}^{nt}}{\sqrt{1-\mathrm{e}^{2nt}}} \;\text{ for }\; t < T^\lambda .$$

Taking $t = T^\lambda$ we find that $T^\lambda \geq T$, where T is independent of λ, so we conclude that

$$\overline{H} \leq C\mathrm{e}^{nt} \;\text{ for }\; t < T ,$$

where C and T do not depend on λ. □

We now exploit (16) to obtain the desired speed bound.

Lemma 3.9. *There exist $C < \infty$ and $T > -\infty$ such that*

$$\frac{H}{y} \leq \lambda_0^2 + C\mathrm{e}^{2nt} \;\textit{ for }\; t < T .$$

Proof. Consider the old-but-not-ancient solution $\{M_t^\lambda\}_{t\in(-\infty,0)}$. By (16), we can find $C < \infty$ and $T > -\infty$ such that

$$\begin{aligned}(\partial_t - \Delta)\frac{H}{y} &= |A|^2\frac{H}{y} + 2\left\langle \nabla\frac{H}{y}, \frac{\nabla y}{y}\right\rangle \\ &\leq C\mathrm{e}^{2nt}\frac{H}{y} + 2\left\langle \nabla\frac{H}{y}, \frac{\nabla y}{y}\right\rangle \;\text{ for }\; t < T .\end{aligned}$$

Since, at a boundary point,

$$\nabla\frac{H}{y} = \frac{\nabla H}{y} - \frac{H}{y}\frac{\nabla y}{y} = 0 ,$$

the Hopf boundary point lemma and the ODE comparison principle yield

$$\max_{M_t^\lambda}\frac{H}{y} \leq C\max_{M_{\alpha_\lambda}^\lambda}\frac{H}{y} \;\text{ for }\; t \in (\alpha_\lambda, T) .$$

But now

$$(\partial_t - \Delta)\frac{H}{y} \leq C\mathrm{e}^{2nt}\max_{M_{\alpha_\lambda}^\lambda}\frac{H}{y} + 2\left\langle \nabla\frac{H}{y}, \frac{\nabla y}{y}\right\rangle \;\text{ for }\; t < T ,$$

and hence, by ODE comparison,

$$\max_{M_t^\lambda}\frac{H}{y} \leq \max_{M_{\alpha_\lambda}^\lambda}\frac{H}{y}\left(1 + C\mathrm{e}^{2nt}\right) \;\text{ for }\; t \in (\alpha_\lambda, T) .$$

Since, on the initial timeslice $M^\lambda_{\alpha_\lambda} = \Sigma^\lambda_{t_\lambda}$,

$$\frac{H}{y} = \frac{\lambda\tan(\lambda y)}{y}\cos\theta\left(1+(n-1)\sin^2\theta\frac{\phi'(\lambda x)}{\lambda\phi(\lambda x)}\right),$$

the claim follows upon taking $\lambda \to \lambda_0$. ☐

It follows that

$$\left(\log \underline{y}(t) - \lambda_0^2 t\right)_t \le C\mathrm{e}^{2nt} \ \text{ for } \ t < T$$

and hence, integrating from time t up to time T,

$$\log \underline{y}(t) - \lambda_0^2 t \ge \log \underline{y}(T) - \lambda_0^2 T - C \ \text{ for } \ t < T\,.$$

So we indeed find that

Lemma 3.10. *the limit*

$$A \doteqdot \lim_{t\to-\infty} \mathrm{e}^{-\lambda_0^2 t}\underline{y}(t) \tag{20}$$

exists in $(0,\infty)$ *on the particular ancient solution* $\{M_t\}_{(-\infty,0)}$.

4. Uniqueness

Now let $\{M_t\}_{t\in(-\infty,0)}$, $M_t = \partial_{\mathrm{rel}}\Omega_t$, be *any* convex, locally uniformly convex ancient free boundary mean curvature flow in the ball. By Stahl's theorem [18], we may assume that M_t contracts to a point on the boundary as $t \to 0$.

4.1. Backwards convergence.

We first show that $\overline{M}_t$ converges to a bisector as $t \to -\infty$.

Lemma 4.1. *Up to an ambient rotation,*

$$\overline{M}_t \underset{C^\infty}{\longrightarrow} \overline{B}^n \times \{0\} \ \text{ as } \ t \to -\infty\,.$$

Proof. Define $\Omega \doteqdot \cup_{t\in(-\infty,0)}\Omega_t$, where $\Omega_t \subset B^{n+1}$ is the convex region relatively bounded by M_t. Given $s \in \mathbb{R}$, define the free boundary mean curvature flow $\{M^s_t\}_{t\in(-\infty,-s)}$ by $M^s_t \doteqdot M_{t+s} = \partial\Omega^s_t$, where $\Omega^s_t \doteqdot \Omega_{t+s}$. Since the flow is monotone, the flows $\{M^s_t\}_{t\in(-\infty,-s)}$ converge to the stationary limit $\{\partial\Omega\}_{t\in(-\infty,\infty)}$ as $s \to -\infty$ uniformly in the Hausdorff topology on compact subsets of time. In fact, the convergence is smooth due to the Ecker–Huisken type estimates of Stahl [18]. Now, since Ω is convex and its boundary intersects ∂B^n orthogonally, it lies in some half-ball. But it cannot lie strictly in this half-ball due to Proposition 3.1. The strong maximum principle and Hopf boundary point lemma then imply that Ω is a half-ball. ☐

We henceforth assume, without loss of generality, that the backwards limit is the horizontal n-ball.

4.2. **Reflection symmetry.** We can now prove that the solution is rotationally symmetric using Alexandrov reflection across planes through the origin (see Chow and Gulliver [7]).

Lemma 4.2. *M_t is rotationally symmetric about the e_{n+1}-axis for all t.*

Proof. Given any $z \in S^n_+$ in the upper half sphere $S^n_+ \doteqdot \{z \in S^n : z \cdot e_{n+1} > 0\}$, we define the open halfspace

$$H_z \doteqdot \{X \in \mathbb{R}^{n+1} : X \cdot z > 0\}$$

and denote by R_z the reflection about ∂H_z. We first claim that, for every $z \in S^n_+$, there exists $t = t_z$ such that

$$(R_z \cdot M_t) \cap (M_t \cap H_z) = \emptyset \tag{21}$$

for all $t < t_z$. Assume, to the contrary, that there exists $z \in S^n_+$, a sequence of times $t_i \to -\infty$, and a sequence of pairs of points $p_i, q_i \in M_{t_i}$ such that $R_z(p_i) = q_i$. This implies that the line passing through p_i and q_i is parallel to the vector z, so the mean value theorem yields for each i a point r_i on M_{t_i} where the normal is orthogonal to z. But this contradicts Lemma 4.1.

The Alexandrov reflection principle [7] now implies that (21) holds for all $t < 0$ (note that $R_z \cdot M_t$ also intersects ∂B^{n+1} orthogonally). In fact, it is clear that $(R_z \cdot M_t) \cap H_z$ lies *above* $M_t \cap H_z$ for all $t < 0$ and all $z \in S^n_+$. The claim follows. □

4.3. **Asymptotics for the height.** We begin with a lemma.

Lemma 4.3. *For every $t < 0$,*

$$\nabla H \cdot X \geq 0 \ \ \textit{in} \ \ M_t$$

and hence

$$\frac{\sin \overline{\theta}}{1 + \cos \overline{\theta}} \leq \underline{y}\,. \tag{22}$$

Proof. Choose $T > -\infty$ so that $H < \frac{2}{2c+1}$ for $t < T$, where $c = c(n) \geq 2$ is the constant in the evolution inequality for $|\nabla H|$, and, given $\varepsilon > 0$, define

$$v_\varepsilon \doteqdot \langle \nabla H, \vec{r}\rangle + \varepsilon(1 - \langle X, \nu\rangle)$$

on $M_t \setminus \mathbb{R}e_{n+1}$, where $\vec{r} \doteqdot X^\top/|X^\top|$. Note that $v_\varepsilon \to \varepsilon(1 - \langle X, \nu\rangle)$ as $X \to \mathbb{R}e_{n+1}$.

We claim that $v_\varepsilon \geq 0$ for $t \in (-\infty, T)$. Suppose that this is not the case. Since $v_\varepsilon(\cdot, t) > \varepsilon$ at ∂M_t, $v_\varepsilon(x, t) > \varepsilon$ as $x \to M_t \cap \mathbb{R}e_{n+1}$, and $v_\varepsilon \to \varepsilon$ as $t \to -\infty$, there must exist a first time $t \in (-\infty, T)$ and an interior point $x \in M_t \setminus \mathbb{R}e_{n+1}$ at which $v_\varepsilon = 0$. But at such a point

$\langle \nabla H, \vec{r}\rangle = -\varepsilon(1-\langle X,\nu\rangle) < 0$, which means that $\langle \nabla H, \vec{r}\rangle = -|\nabla H|$ in a small spacetime neighbourhood of (p,t), and hence, at (p,t),

$$\begin{aligned} 0 \geq\ & (\partial_t - \Delta)\, v_\varepsilon \\ \geq\ & |A|^2\big(\langle \nabla H, \vec{r}\rangle - \varepsilon\langle X,\nu\rangle\big) + (c-1)|A|^2\,\langle \nabla H, \vec{r}\rangle + 2\varepsilon H \\ =\ & -\varepsilon|A|^2 - (c-1)\varepsilon|A|^2(1-\langle X,\nu\rangle) + 2\varepsilon H \\ \geq\ & \varepsilon(2-(2c+1)H)H \\ >\ & 0 \end{aligned}$$

which is absurd. Now take $\varepsilon \to 0$ to obtain $\langle \nabla H, X\rangle \geq 0$ in M_t for $t \in (-\infty, T]$. Applying the maximum principle as in the proof of Lemma 3.4, implies that $\langle \nabla H, X\rangle$ remains non-negative up to time 0.

Having established the first claim, the second follows as in Lemma 3.5. ☐

Proposition 4.4. *If we define $A \in (0,\infty)$ as in (20), then*

$$\mathrm{e}^{-\lambda_0^2 t} y(x,t) \to A\phi_0(\lambda_0|x|) \ \textit{uniformly as}\ t \to -\infty\,.$$

Proof. Given $\tau < 0$, consider the rescaled height function

$$y^\tau(x,t) \doteqdot \mathrm{e}^{-\lambda_0^2\tau} y(x,t+\tau)\,,$$

which is defined on the time-translated flow $\{M_t^\tau\}_{t\in(-\infty,-\tau)}$, where $M_t^\tau \doteqdot M_{t+\tau}$. Note that

$$\begin{cases} (\partial_t - \Delta^\tau) y^\tau = 0 \ \text{ in }\ \{M_t^\tau\}_{t\in(-\infty,-\tau)} \\ \ \langle \nabla^\tau y^\tau, N\rangle = y \ \text{ on }\ \{\partial M_t^\tau\}_{t\in(-\infty,-\tau)}\,, \end{cases} \tag{23}$$

where ∇^τ and Δ^τ are the gradient and Laplacian on $\{M_t^\tau\}_{t\in(-\infty,-\tau)}$, respectively, and N is the outward unit normal to ∂B^{n+1}.

Since $\{M_t\}_{t\in(-\infty,0)}$ reaches the origin at time zero, it must intersect the constructed solution for all $t < 0$. In particular, the value of $\underline{y}$ on the former can at no time exceed the value of $\overline{y}$ on the latter. But then (20) and (22) yield

$$\limsup_{t\to-\infty} \mathrm{e}^{-\lambda_0^2 t}\overline{y} < \infty\,. \tag{24}$$

This implies a uniform bound for y^τ on $\{M_t^\tau\}_{t\in(-\infty,T]}$ for any $T \in \mathbb{R}$. So Alaoglu's theorem yields a sequence of times $\tau_j \to -\infty$ such that y^{τ_j} converges in the weak* topology as $j \to \infty$ to some $y^\infty \in L^2_{\mathrm{loc}}(\overline{B}^n \times (-\infty,\infty))$. Since convexity and the boundary condition imply a uniform bound for $\nabla^\tau y^\tau$ on any time interval of the form $(-\infty, T]$, we may also arrange that the convergence is uniform in space at time zero, say.

We conclude from Corollary 2.3 that

$$y^\infty(x,t) = A\mathrm{e}^{-\lambda_0^2 t}\phi_0(\lambda_0|x|)$$

for some $A \geq 0$. In particular,

$$\mathrm{e}^{-\lambda_0^2 \tau_j} y(x, \tau_j) = y^{\tau_j}(x, 0) \to A\phi_0(\lambda_0 |x|) \;\text{ uniformly as }\; j \to \infty\,.$$

Now, if A is not equal to the corresponding value on the constructed solution (note that the full limit exists for the latter), then one of the two solutions must lie above the other at time τ_j for j sufficiently large. But this violates the avoidance principle. ☐

4.4. **Uniqueness.** Uniqueness of the constructed ancient solution now follows directly from the avoidance principle (cf. [4]).

Proposition 4.5. *Modulo time translation and rotation about the origin, there is only one convex, locally uniformly convex ancient solution to free boundary mean curvature flow in the ball.*

Proof. Denote by $\{M_t\}_{t\in(-\infty,0)}$ the constructed ancient solution and let $\{M'_t\}_{t\in(-\infty,0)}$ be a second ancient solution which, without loss of generality, contracts to the point e_{n+1} at time 0. Given any $\tau > 0$, consider the time-translated solution $\{M^\tau_t\}_{t\in(-\infty,-\tau)}$ defined by $M^\tau_t = M'_{t+\tau}$. By Proposition 4.4,

$$\mathrm{e}^{-\lambda_0^2 t} y^\tau(x,t) \to A\mathrm{e}^{\lambda_0^2 \tau}\phi_0(\lambda_0|x|) \;\text{ as }\; t \to -\infty$$

uniformly in x. So M^τ_t lies above M_t for $-t$ sufficiently large. The avoidance principle then ensures that M^τ_t lies above M_t for all $t \in (-\infty, 0)$. Taking $\tau \to 0$, we find that M'_t lies above M_t for all $t < 0$. Since the two solutions both reach the point e_{n+1} at time zero, they intersect for all $t < 0$ by the avoidance principle. The strong maximum principle then implies that the two solutions coincide for all t. ☐

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