

COMPACT CURVE SHORTENING FLOW SOLUTIONS OUT OF NON COMPACT CURVES

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ABSTRACT. We construct a slingshot, that is a compact, embedded solution to curve shortening flow that comes out of a non compact curve and exists for a finite time.

1. INTRODUCTION

A smooth one-parameter family $\{\Gamma_t\}_{t \in I}$ of immersed planar curves $\Gamma_t \subset \mathbb{R}^2$ evolves by curve shortening flow if

$$(1) \quad \frac{\partial \gamma}{\partial t}(u, t) = \vec{\kappa}(u, t), \quad \forall (u, t) \in \Gamma \times I,$$

for some smooth family $\gamma : \Gamma \times I \rightarrow \mathbb{R}^2$ of immersions $\gamma(\cdot, t) : \Gamma \rightarrow \mathbb{R}^2$ of Γ_t , and where $\vec{\kappa}(u, t)$ is the curvature vector of Γ_t at the point $\gamma(u, t)$.

When Γ_0 is a smooth embedded compact curve, then by a famous theorem of Grayson [6], the solution of the curve shortening flow starting from Γ_0 exists on a maximal time interval $[0, T)$ and as $t \rightarrow T$ the solution converges to a round point. In the case when Γ_0 is additionally convex, this theorem was previously proved by Gage and Hamilton [5]. Contrary to the compact case, when Γ_0 is not compact solutions to curve shortening flow starting from Γ_0 are not that well understood in general. The particular case of graphical solutions has been extensively studied in the work of Ecker and Huisken [3, 4], who, among other things, showed that the flow of entire graphs exists for all times. Another case for which the behavior is known, is that of a curve asymptotic to two distinct halflines, for which Polden [9] showed that a solution exists for all $t > 0$ and the solution asymptotically approaches the selfsimilar solution associated with the two halflines. In [2], K-S Chou and X-P Zhu, showed that that if the initial curve divides the plane into two regions of infinite area, then a solution exists for all

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time. For the case that one of the regions of the plane defined by the curve has finite area, they showed that, if additionally the curve has finite total absolute curvature, then a solution exists for a finite time equal to that area divided by π . Moreover, they showed uniqueness of solutions when the initial curve has ends that are representable as graphs over two semi-infinite lines.

In the present paper we want to construct compact solutions emanating from a non compact initial curve. More precisely, given Γ_0 a smooth embedded curve in \mathbb{R}^2 , we want to construct a smooth family of compact embeddings

$$\gamma : S^1 \times (0, T) \rightarrow \mathbb{R}^2$$

that satisfy the curve shortening flow equation (1), and such that the curves $\Gamma_t = \gamma(S^1, t)$ converge to Γ_0 as $t \rightarrow 0$, in the sense that for any $\varepsilon > 0$, there exists t_ε such that Γ_t is in an ε -neighborhood of Γ_0 for all $t \in (0, t_\varepsilon)$. Note that such a solution is different from the one constructed in [2], as in [2] the family of solutions satisfying curve shortening flow is non-compact, that is the parameter space Γ in (1) is homeomorphic to \mathbb{R} .

We will consider a curve Γ_0 that satisfies the following:

- (i) Γ_0 is a smooth embedded 1-manifold diffeomorphic to $(0, 1)$ and it separates \mathbb{R}^2 into two regions, one of which has finite area, which we denote by $A_0 \in (0, \infty)$.
- (ii) $a + 1 < b$ and $c > 0$ are real numbers such that $\Gamma_0 \subset (a, \infty) \times (-c, c)$ and $\Gamma_0 \cap ([b, \infty) \times (-c, c))$ is the union of two smooth graphs, $u^\pm \in [b, \infty) \rightarrow \mathbb{R}$ with u^+ positive and decreasing to zero at infinity and u^- negative and increasing to zero at infinity, and with the derivatives of u^\pm converging to zero at infinity, as in Figure 1.

Moreover, we will denote by $B(\Gamma_0, \varepsilon)$ the ε neighborhood of Γ_0 , that is

$$B(\Gamma_0, \varepsilon) := \{p \in \mathbb{R}^2 : \text{dist}(p, \Gamma_0) < \varepsilon\}.$$

Our main theorem is the following

Theorem 1. *Let Γ_0 be a curve satisfying the above hypotheses (i)-(ii). There exists a smooth solution $\gamma : S^1 \times (0, \frac{A_0}{2\pi}) \rightarrow \mathbb{R}^2$ to the curve*

shortening flow [\[1\]](#) such that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\Gamma_t \subset B(\Gamma_0, \varepsilon)$ for $0 < t < t_\varepsilon$.

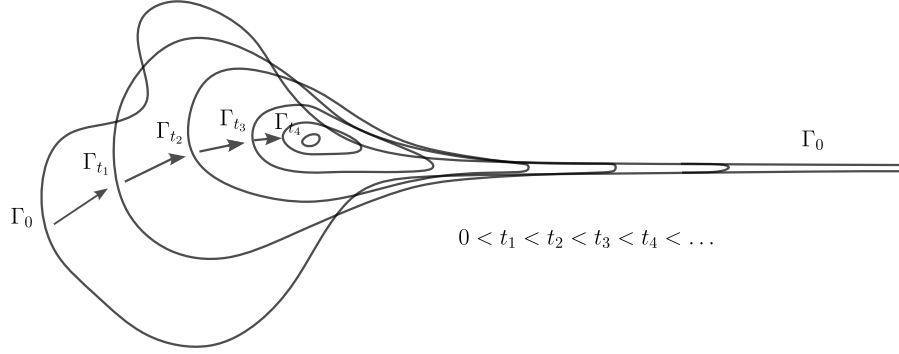


FIGURE 1. Schematic figure of the evolution.

The construction of the solution described in Theorem [\[1\]](#) is roughly as follows. We start with a sequence of compact curves Γ_0^i that approximate Γ_0 . Then, we define a sequence of curve shortening flows, using the curves Γ_0^i as initial conditions, which we refer to as slingshots. The idea, then, is to show that one can extract a limit of these slingshots. To do this, we establish uniform curvature bounds for the slingshots away from the initial time 0. This argument, which is the most novel part of this construction, is a direct argument, based on repeated applications of the avoidance principle, and in particular the fact that the number of intersections between two solutions of curve shortening flow (at least one of which is compact) cannot increase in time [\[1\]](#), together with the curvature estimates of Ecker and Huisken [\[4\]](#). The curvature bounds that we obtain when restricted to compact sets of \mathbb{R}^2 are in fact uniform up to time $t = 0$, a fact which allows us to conclude the convergence of Γ_t to Γ_0 as claimed in Theorem [\[1\]](#). We note here that one indeed obtains higher order convergence in compact sets, as long as the initial curve is smooth enough.

We remark that a construction of a compact solution coming out of a non compact curve has been discussed in [\[7\]](#) in order to show non uniqueness of the level set flow for non compact sets. In particular a compact solution is constructed with “initial curve” the boundary of a starshaped domain with tails the graph of $\pm|x|^{-1-\beta}$ $\beta > 0$. In [\[7\]](#) a sketch of the proof of this construction is given, which is different than

ours, as it uses global barriers and a foliation argument to control the degree to which the solutions are starshaped.

After completing this paper we were informed of a similar construction in the PhD Thesis of Peachey, (see also [8]). In this paper, a smooth solution $\gamma : \mathbb{R} \rightarrow (0, \infty) \rightarrow \mathbb{R}^2$ coming out of a cusp asymptotic to $\pm(1 + cx)^{-2}$ is constructed. The solution is constructed by an approximation method, as is done here, but the tools to extract a limit follow the ideas in [7] and as such differ from ours.

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2. CONSTRUCTION

We first show that if a curve is locally, in some rectangle, a graph, then under curve shortening flow and in a smaller rectangle it remains a graph. Moreover, we obtain estimates on the gradient. We remark that such estimates are known in more general contexts but as the proof of the version we need here is relatively simple we do include it for the convenience of the reader.

Proposition 2. *Let $\gamma_0 : S^1 \rightarrow \mathbb{R}^2$ be a smooth embedding and suppose that for $D > 0$, $R > 0$ and $r < \frac{D}{2}$, the following holds:*

- (1) *for any $|x_1| \leq R$ and $|x_2| \leq R$, the segment joining $(x_1, 0)$ to (x_2, D) intersects $\Gamma_0 = \gamma_0(S^1)$ transversely and at just one point.*
- (2) *for any $|x| \leq R$, the balls $B_r((x, 0))$ and $B_r((x, D))$ are disjoint from Γ_0 .*

Then, the curve shortening flow solution $\gamma : S^1 \times [0, T)$ starting at $\gamma(\cdot, 0) = \gamma_0(\cdot)$ satisfies $T \geq \frac{r^2}{2}$, and for all $t \in [0, \frac{r^2}{2}]$ the timeslices Γ_t satisfy the following: $\Gamma_t \cap ([-R, R] \times [0, D])$ can be represented as the

graph of a smooth function $g(\cdot, t) : [-R, R] \rightarrow \mathbb{R}$, with

$$\sup_{x \in [-\frac{R}{2}, \frac{R}{2}]} \left| \frac{\partial g}{\partial x}(x, t) \right| \leq \frac{2D}{R}, \text{ and}$$

$$\sqrt{r^2 - 2t} < g(x, t) < D - \sqrt{r^2 - 2t}, \quad \forall x \in [-R, R].$$

Proof. Note first that by hypothesis (2) of the proposition and the avoidance principle we obtain that

$$(2) \quad ([-R, R] \times \{0, D\}) \cap \Gamma_t = \emptyset, \quad \forall t \in [0, \frac{r^2}{2}],$$

and note that a simple linking argument, along with Grayson's theorem [6], shows that the curve shortening flow solution starting at Γ_0 does indeed have a lifespan of time at least $\frac{r^2}{2}$. Recall that the number of intersections between two compact solutions of curve shortening flow cannot increase, so long as the intersections do not develop at a boundary point, [1]. Therefore, hypothesis (1) of the proposition applied to segments with endpoints $(x, 0)$ and (x, D) , $x \in [-R, R]$, along with (2), imply that $\Gamma_t \cap ([-\frac{R}{2}, \frac{R}{2}] \times [0, D])$ can be represented as a graph of a smooth function $g(\cdot, t) : [-R, R] \rightarrow \mathbb{R}$. To prove the gradient bound, consider a point on the graph $p = (x, g(x, t))$ with $x \in [-\frac{R}{2}, \frac{R}{2}]$ and suppose that $g(x, t) \geq \frac{D}{2}$. Consider the two line segments joining $(x \pm \frac{R}{2}, 0)$ to p and extending them pass p we note that they intersect the segment $[-R, R] \times \{D\}$. Thus, by hypothesis (1), these segments lie below the graph of g_t and we obtain that $|\frac{\partial g}{\partial x}(x, t)| \leq \frac{g(x, t)}{R/2} \leq \frac{2D}{R}$. If the point p satisfies $g(x, t) \leq \frac{D}{2}$, we obtain the same estimate by considering the segments joining $(x \pm \frac{R}{2}, D)$ to p and extending them pass p . Finally, the height bounds are a cosequence of the avoidance principle and hypothesis (2). \square

Proposition [2] and the curvature estimates of Ecker-Huisken [4] yield the following

Corollary 3. *Under the hypotheses of Proposition [2], for every integer $m \geq 1$, there is a constant $c_m = c(m, R, D, \Gamma_0)$ such that*

$$(3) \quad \sup_{p \in \Gamma_t \cap ([-\frac{R}{4}, \frac{R}{4}] \times [0, D])} |\partial_s^m \kappa(p, t)| \leq c_m, \quad \forall t \in [0, \frac{r^2}{2}],$$

where $\kappa(p, t)$ denotes the curvature of Γ_t at the point p .

Proof. The proof is evident from the estimates in [4] by removing the time dependence from the bounds. Nonetheless, we include a sketch here for the convenience of the reader.

We first prove the case $m = 0$. Consider a point $p_0 = (x, y)$, with $|x| < \frac{R}{4}$ and $y \in (0, D)$, and let $v = v(p, t) = \langle \nu, e_2 \rangle^{-2}$, where $\nu = \nu(p, t)$ is a choice of the unit normal to Γ_t at p . Consider now G_t to be the connected component of $\Gamma_t \cap B_{\frac{R}{4}}(p_0)$ that is the graph of $g(\cdot, t)$ as in Proposition [2]. Then, by Proposition [2] we have that

$$v(p, t) \leq 1 + \frac{4D^2}{R^2}, \quad \forall p \in G_t, \quad \forall t \in [0, \frac{r^2}{2}].$$

Define the function $f(p, t) = \kappa(p, t)^2 \frac{v^2}{1-k^2v^2} ((\frac{R}{4})^2 - |p - p_0|^2)^2$, where $k = \frac{1}{2} + \frac{2D^2}{R^2}$. Note that $f(p, 0) \leq CR^2$, where $C = \sup_{G_0} \kappa^2$, a constant that depends only on γ_0 . If f has a maximum at a point $(p, t) \in G_t \times (0, \frac{r^2}{2}]$, then, by computing the heat operator of g (see [4], proof of Theorem 3.1]), we obtain

$$f(p, t) \leq c(n, k)R^2.$$

We therefore conclude the estimate for $m = 0$. The higher derivative bounds can be computed similarly by considering $\psi = 1$ in [4], proof of Theorem 3.4]. \square

Definition 4. A basic rectangle $\mathcal{F}(R, D, r)$ for an embedded curve Γ consists of a number $r > 0$ and a rectangle isometric to $[-R, R] \times [0, D]$ by an isometry T , such that:

- (1) for any $|x_1| \leq R$ and $|x_2| \leq R$, the segment joining $T((x_1, 0))$ to $T((x_2, D))$ intersects Γ transversely and at just one point.
- (2) for any $|x| \leq R$ the balls $B_r(T(x, 0))$ and $B_r(T(x, D))$ are disjoint from Γ .

T as above, will be referred to the isometry associated to $\mathcal{F}(R, D, r)$.

If $\mathcal{F}(R, D, r)$ is a basic rectangle for Γ and T is its associated isometry, then $T([- \frac{R}{4}, \frac{R}{4}] \times [0, D])$ together with r , form also a basic rectangle for Γ , which will be denoted by $\mathcal{F}_*(R, D, r)$.

It is clear that the estimates in the statement of Corollary [3] work exactly the same when we replace the basic rectangle $[-R, R] \times [0, D]$ by basic rectangles $\mathcal{F}(R, D, r)$ for the curve Γ_0 . More precisely, Proposition [2] and Corollary [3] yield the following:

Proposition 5. *Assume that $\mathcal{F}(R, D, r)$ is a basic rectangle for an embedded smooth curve Γ_0 . Then the curve shortening flow solution starting from Γ_0 exists for time at least $\frac{r^2}{2}$ and the timeslices Γ_t satisfy the following curvature estimate. For every integer $m \geq 1$, there is a constant $c_m = c(m, R, D, \Gamma_0)$, such that*

$$\sup_{p \in \Gamma_t \cap \mathcal{F}_*(R, D, r)} |\partial_s^m \kappa(p, t)| \leq c_m, \quad \forall t \in [0, \frac{r^2}{2}],$$

where $\kappa(p, t)$ denotes the curvature of Γ_t at the point p .

Definition 6. *For every integer $i \geq b + 3$, consider the connected part of Γ_0 between $(i, u^+(i))$ and $(i, u^-(i))$ and cup it off with an embedded piece joining these two end points and lying inside the rectangle $[i, i + 1] \times [u^-(i), u^+(i)]$, so that we obtain a smooth embedded and compact curve which we denote by Γ_0^i . Let $\gamma_0^i : S^1 \rightarrow \mathbb{R}^2$ be a parametrization of Γ_0^i . The solutions to the curve shortening flow starting from Γ_0^i are denoted by Γ_t^i and are called slingshots. Moreover, for each i , we will use $\gamma^i(\cdot, t)$ to denote any parametrization of the flow, which, as such, satisfies [\(1\)](#).*

The following lemma says essentially that the slingshots enter compact regions in arbitrarily small times uniformly in i .

Lemma 7. *For any decreasing sequence of times $t_j \downarrow 0$, there exists a sequence of numbers x_j , such that the slingshots, after passing to a subsequence Γ_t^j , satisfy*

$$\Gamma_t^k \subset [a, x_j] \times [-c, c], \quad \forall k \geq j, \text{ and } t \geq t_j.$$

Proof. Consider a sequence $t_j \downarrow 0$. Then, by the assumptions on the initial curve Γ_0 and by construction of the approximating sequence Γ_0^i , the slingshots, after passing to a subsequence Γ_t^j , satisfy the following. For any j , we can pick x_j such that the following hold.

- (i) Let $\mathcal{F}(R, 2c, \sqrt{2t_j}) := [-R + x_j, R + x_j] \times [-c, c]$, with $R = \frac{16c}{\pi}$. Then, for all $k \geq j$, $\Gamma_0^k \cap \mathcal{F}(R, 2c, \sqrt{2t_j})$ has two connected components, and for each of them $\mathcal{F}(R, 2c, \sqrt{2t_j})$ is a basic rectangle in the sense that on both components (i) and (ii) of Definition [4](#) are satisfied.

- (ii) For all $k \geq j$, the area of the compact region bounded by Γ_0^k in the halfplane $\{x \geq x_j - R\}$ is at most $\frac{\pi t_j}{2}$.

To prove the lemma, we will show that for all j and $t \geq t_j$ we have $\Gamma_t^k \subset [a, R + x_j] \times [-c, c]$, for all $k \geq j$, for which it suffices to prove that $\Gamma_{t_j}^k \subset [a, R + x_j] \times [-c, c]$, for all $k \geq j$. Assume on the contrary that for some j and $k \geq j$ we have $\Gamma_{t_j}^k \cap ((R + x_j, \infty) \times [-c, c]) \neq \emptyset$. First note that, by considering a small ball inside Γ_0 and by (i), the avoidance principle implies that

$$\Gamma_t^k \cap \mathcal{F}(R, 2c, \sqrt{2t_j}) \text{ has two connected components, } \forall t \in [0, t_j].$$

Let now $A_+^k(t)$ be the area of the compact region bounded by Γ_t^k in the halfplane $\{x \geq x_j\}$. Since $\Gamma_t^k \cap \mathcal{F}(R, 2c, \sqrt{2t_j})$ has two connected components, for all $t \in [0, t_j]$, Proposition 2 implies that

$$-\frac{d}{dt}A_+^k(t) \geq \pi - \frac{8c}{R}$$

and integration yields

$$A_+^k(t_j) \leq A_+^k(0) - t_j \left(\pi - \frac{8c}{R} \right) \leq -\frac{\pi t_j}{2} + \frac{8c}{R} t_j < 0$$

which contradicts the hypothesis that $A_+^k(t_j)$ is positive, which is implied since we assumed that $\Gamma_{t_j}^k \cap ((x_j + R, \infty) \times [-c, c]) \neq \emptyset$. \square

The following lemma, which is the central lemma for our constructions, says that there is a decreasing sequence $t_j \downarrow 0$ such that the slingshots, after passing to a subsequence Γ_t^j , for all j and $t_j \leq t \leq t_0$ (where t_0 is some fixed positive time), are covered by a fixed and finite set of basic rectangles and are therefore globally subject to the estimates of Corollary 3.

Lemma 8. *There exists a decreasing sequence of times $t_j \downarrow 0$, $j \geq 0$, such that the slingshots, after passing to a subsequence Γ_t^j satisfy the following. For every $j \geq 0$ there is a finite set of rectangles,*

$$(4) \quad \mathcal{F}(R_{j,1}, D_{j,1}, r_{j,1}), \dots, \mathcal{F}(R_{j,n_j}, D_{j,n_j}, r_{j,n_j}),$$

with $r_{j,k} \geq \sqrt{2t_0}$, $k = 1, \dots, n_j$, that are basic for Γ_t^j for any $t \in [0, t_0]$, and moreover,

$$(5) \quad \Gamma_t^j \subset \bigcup_{k=1}^{k=n_j} \mathcal{F}_*(R_{j,k}, D_{j,k}, r_{j,k}), \forall t \in [t_j, t_0].$$

Proof. We first construct basic rectangles that will cover the slingshots in a compact set, where all the initial curves Γ_0^i coincide.

Let $r_0 > 0$ be such that $[b, b+2] \times [0, c]$ and $[b, b+2] \times [-c, 0]$ together with r_0 form basic rectangles for Γ_0 , and we denote these by \mathcal{F}^\pm , respectively. Then, let

$$(6) \quad \mathcal{F}^1 = \mathcal{F}(R_1, D_1, r_1), \dots, \mathcal{F}^l = \mathcal{F}(R_l, D_l, r_l),$$

be a collection of basic rectangles for Γ_0 with associated isometries T_m and such that:

- (i) $\mathcal{F}^m \subset \{x < b+2\}$, for $m = 1, \dots, l$,
- (ii) $\mathcal{F}^1 \subset \text{Int}(\mathcal{F}_*^+)$ and $\mathcal{F}^l \subset \text{Int}(\mathcal{F}_*^-)$,
- (iii) $T_m(\{\frac{R_m}{4}\} \times [0, D_m]) \subset \text{Int}(\mathcal{F}_*^{m-1})$, for $m = 2, \dots, l$.

Note that the rectangles \mathcal{F}^\pm and \mathcal{F}^m , for $m = 1, \dots, l$, are also basic rectangles for Γ_0^i , for all $i \in \mathbb{N}$. This is because they are contained in the half plane $\{x \leq b+3\}$, where Γ_0^i and Γ_0 coincide. Define,

$$(7) \quad \bar{t} := \frac{1}{2} \min\{r_0^2, r_1^2, \dots, r_l^2\}$$

and also

$$\mathcal{F}(0) := \{\mathcal{F}^+, \mathcal{F}^-, \mathcal{F}^1, \dots, \mathcal{F}^l\}.$$

We claim that for any i and $0 \leq t \leq \bar{t}$ we have,

$$(8) \quad \Gamma_t^i \cap \{x \leq b+5/4\} \subset \bigcup_{\mathcal{F} \in \mathcal{F}(0)} \mathcal{F}_*.$$

To see this, let $\mathcal{F} \in \mathcal{F}(0)$. Then, by Proposition [2](#), we have that, for any i and any $0 \leq t \leq \bar{t}$, $\Gamma_t^i \cap \mathcal{F}_*$ is a connected 1-manifold with two boundary points lying in two opposite sides of the corresponding rectangle: $T_m(\{\pm \frac{R_m}{4}\} \times [0, D_m])$ if $\mathcal{F} = \mathcal{F}^m$, $m = 1, \dots, l$, and accordingly if $\mathcal{F} = \mathcal{F}^\pm$. By conditions (ii) and (iii) above the claim follows.

The next step is to construct basic rectangles that cover the entirety of the slingshots for times $t > t_j$. An essential tool to do that is Lemma [7](#), which allows us to deduce that after time t_j all slingshots have entered a compact set.

For any integer $k > b+1$, we let $y_k := \min\{u^+(2k), -u^-(2k)\}$ and set $q_k := (b, -y_k)$. We then define s_k^1 and s_k^2 be the two rays starting from q_k and passing through $(2k, 0)$ and $(b+1, c)$ respectively. Note that both rays intersect Γ_0 transversely and only once at a point with positive y -coordinate. Define also the rectangle $\mathcal{R}_k^+ := [b+1, k] \times [\frac{-ky_k}{2k-b}, c]$ and

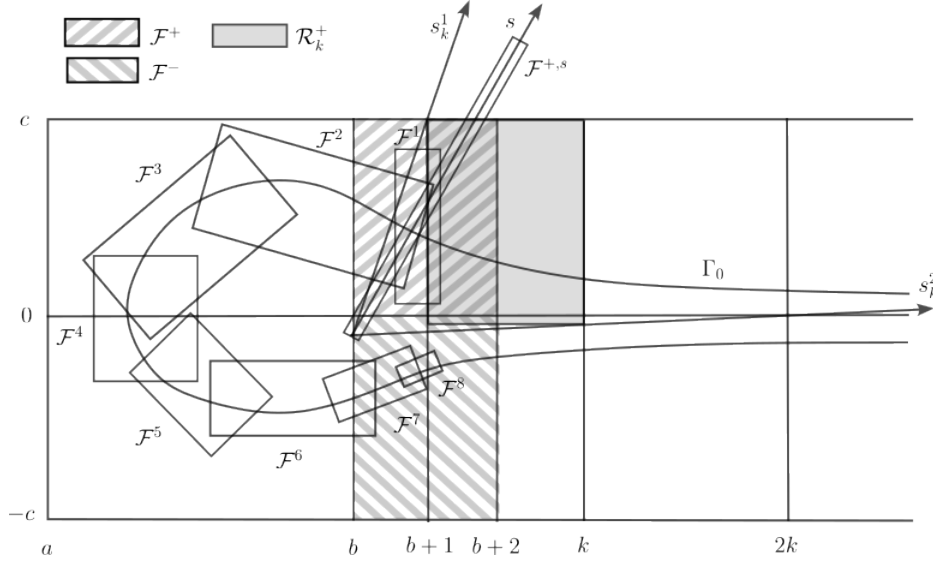


FIGURE 2. Schematic figure of the rectangles \mathcal{F}^\pm , the \mathcal{F}^m and the $\mathcal{F}^{+,s}$. The rectangle \mathcal{R}_k^+ is also shown.

note that it lies in the region between the two rays and has one vertex on each of them. Hence, any infinite ray from q_k and passing through any point in \mathcal{R}_k^+ intersects Γ_0 transversely and only once. We will use this fact to cover the slingshots by basic rectangles in \mathcal{R}_k^+ .

Let $\hat{r} \in (0, 1)$ be such that $\overline{B}_{\hat{r}}((b, 0))$ is contained in the open region of finite area enclosed by Γ_0 . Since, $y_k \downarrow 0$ as $k \rightarrow \infty$, we can choose \hat{k} such that $y_k \leq \frac{\hat{r}}{4}$, for all $k \geq \hat{k}$, and from now on we consider such a $k \geq \hat{k}$. Consider s to be a ray starting from q_k and passing through a point in \mathcal{R}_k^+ . Since every such ray has positive slope and intersects Γ_0 transversely and only once, for each such s , we can find a rectangle $T_s([-R_s, R_s] \times [0, D_s])$, for some isometry T_s , with the following properties:

- (1) $T_s(\{0\} \times [0, D_s]) \subset s$ and $T_s((0, 0)) = q_k$,
- (2) $R_s \leq \frac{\hat{r}}{4}$ and D_s is large enough so that $\langle T((0, D_s)), e_2 \rangle \geq c + \hat{r}$,
- (3) $\Gamma_0 \cap T_s([-R_s, R_s] \times [0, D_s])$ is a graph over $T([-R_s, R_s] \times \{0\})$.

Since $T_s([-R_s, R_s] \times \{0\}) \subset B_{\frac{\hat{r}}{2}}((b, 0))$ and by properties (2) and (3) above we conclude that $T_s([-R_s, R_s] \times [0, D_s])$ together with $r = \frac{\hat{r}}{4}$ is a basic rectangle for Γ_0 , which we denote as $\mathcal{F}^{+,s}$. By compactness, we can find a collection of rays $s_{k,1}, \dots, s_{k,l_k}$ such that $\mathcal{F}_*^{+,s_{k,1}}, \dots, \mathcal{F}_*^{+,s_{k,l_k}}$

cover \mathcal{R}_k^+ . From now on and to simplify notation we write $\mathcal{F}^{+,k,j}$ instead of $\mathcal{F}^{+,s_k,j}$. An identical reasoning shows that we can find a collection of basic rectangles,

$$(9) \quad \mathcal{F}^{-,k,1}, \dots, \mathcal{F}^{-,k,h_k},$$

for Γ_0 , all with $r = \frac{\hat{r}}{4}$ and such that $\mathcal{F}_*^{-,k,1}, \dots, \mathcal{F}_*^{-,k,h_k}$ are covering the rectangle $\mathcal{R}_k^- := [b+1, k] \times [-c, \frac{ky_k}{2k-b}]$. We will denote by $\mathcal{F}(k)$ all these rectangles

$$(10) \quad \mathcal{F}(k) = \{\mathcal{F}^{+,k,1}, \dots, \mathcal{F}^{+,k,l_k}, \mathcal{F}^{-,k,1}, \dots, \mathcal{F}^{-,k,h_k}\}.$$

Note that $\mathcal{R}_k^+ \cup \mathcal{R}_k^- = [b+1, k] \times [-c, c]$ and therefore

$$(11) \quad [b+1, k] \times [-c, c] \subset \bigcup_{\mathcal{F} \in \mathcal{F}(k)} \mathcal{F}_*.$$

Given $k \geq \hat{k}$ let $\hat{i}_k > 0$ be large enough so that none of the basic rectangles $\mathcal{F} \in \mathcal{F}(k)$ intersects the region $[\hat{i}_k, \infty] \times [-c, c]$. Note that this is possible, since all these rectangles have non zero slope and width bounded by $\frac{\hat{r}}{4}$. Recalling the definition of Γ_0^i , we deduce that these basic rectangles for Γ_0 are also basic rectangles for Γ_0^i when $i \geq \hat{i}_k$. Let $t_j \downarrow 0$ and x_j be the sequences of Lemma 7, for which, after dropping some initial terms if necessary, we will assume that $t_1 < t_0 := \min\{\bar{t}, \frac{\hat{r}^2}{32}\}$. Let k_1 be any integer such that $k_1 \geq \max\{\hat{k}, x_1\}$. By Lemma 7, we have that the slingshots, after passing to a subsequence Γ_t^j , satisfy, for any j and $t \geq t_1$,

$$(12) \quad \begin{aligned} \Gamma_t^j &\subset [a, x_1] \times [-c, c] \subset ([a, b+1] \times [-c, c]) \cup \mathcal{R}_{k_1}^+ \cup \mathcal{R}_{k_1}^- \\ &\subset \bigcup_{\mathcal{F} \in \mathcal{F}(0) \cup \mathcal{F}(k_1)} \mathcal{F}_*, \end{aligned}$$

with the second inclusion following by (8) and (11), and where $\mathcal{F}(k)$ is as constructed in (10). Finally note that for any $t \leq t_0$ and for $i \geq \hat{i}_k$, \mathcal{F} is a basic rectangle for Γ_t^i for all $\mathcal{F} \in \mathcal{F}(0) \cup \mathcal{F}(k_1)$ and $t \in [0, t_0]$. Hence, the slingshots, after passing to a further subsequence, still denoted by Γ_t^j , satisfy, for any j ,

$$\Gamma_t^j \subset \bigcup_{\mathcal{F} \in \mathcal{F}(0) \cup \mathcal{F}(k_1)} \mathcal{F}_*, \quad \forall t \in [t_1, t_0],$$

where $\mathcal{F}(0) \cup \mathcal{F}(k_1)$ is a finite family of rectangles that are basic for Γ_t^j , for all j and $t \leq t_0$ and moreover these rectangles are of the form

$\mathcal{F}(R, D, r)$ with $r \geq \sqrt{2t_0}$. We can now finish the proof of the proposition, by constructing the rest of the sequence as follows. For each t_j as above (from Lemma 7), with $j \geq 2$, we choose $k_j \geq \max\{k_{j-1}, x_j\}$. Then we construct the family of basic rectangles $\mathcal{F}(k_j)$ as in (10). We then note that there exists \hat{i}_{k_j} large enough, so that none of the basic rectangles $\mathcal{F} \in \mathcal{F}(k_j)$ intersects the region $[\hat{i}_{k_j}, \infty] \times [-c, c]$, therefore for all $i \geq \hat{i}_{k_j}$ and $t \in [0, t_0]$, \mathcal{F} is a basic rectangle for Γ_t^i for all $\mathcal{F} \in \mathcal{F}(0) \cup \mathcal{F}(k_j)$. Hence, the slingshots, after passing to a further subsequence, still denoted by Γ_t^j , satisfy, for any j ,

$$\Gamma_t^j \subset \bigcup_{\mathcal{F} \in \mathcal{F}(0) \cup \mathcal{F}(k_j)} \mathcal{F}_*, \quad \forall t \in [t_j, t_0],$$

where $\mathcal{F}(0) \cup \mathcal{F}(k_j)$ is a finite family of rectangles that are basic for Γ_t^j , for $j \geq 1$ and $t \leq t_0$ and moreover these rectangles are of the form $\mathcal{F}(R, D, r)$ with $r \geq \sqrt{2t_0}$. □

Proof of Theorem 1. Consider $t_0 > 0$ as in Lemma 8. Lemma 8 and Proposition 5 imply that we can apply a compactness argument (which amounts to the Arzela–Ascoli theorem) to the sequence of embeddings $\gamma_{t_0}^j : S^1 \rightarrow \mathbb{R}^2$. This yields that there exists a smooth embedding $\gamma_{t_0}^\infty : S^1 \rightarrow \mathbb{R}^2$ and a sequence of diffeomorphisms of S^1 , ϕ_j , such that after passing to a subsequence, $\gamma_{t_0}^j \circ \phi_j$ converges smoothly to $\gamma_{t_0}^\infty$. Let $t_j \downarrow 0$ be as in Lemma 8 and define the diffeomorphisms

$$\begin{aligned} \psi_j : S^1 \times [t_j, t_0] &\rightarrow S^1 \times [t_j, t_0] \\ (x, t) &\mapsto \psi_j(x, t) = (\phi_j(x), t). \end{aligned}$$

Note that Lemma 8 and Proposition 5, along with the evolution equation of the curvature and its derivatives (which yield time derivative bounds on the curvature and its derivatives), imply uniform bounds on the curvature and its derivatives for the sequence $\gamma^j \circ \psi_j$ (locally in $S^1 \times (0, t_0]$). Therefore, the Arzela–Ascoli theorem and a diagonal argument yield that there exists a smooth map $\gamma^\infty : S^1 \times (0, t_0] \rightarrow \mathbb{R}^2$, with $\gamma^\infty(\cdot, t) : S^1 \rightarrow \mathbb{R}^2$ a smooth embedding for each $t \in (0, t_0]$ and $\gamma^\infty(\cdot, t_0) = \gamma_{t_0}^\infty(\cdot)$, and such that, after passing to a further subsequence, $\gamma^j \circ \psi_j$ converges to γ^∞ smoothly on compact sets of $S^1 \times (0, t_0]$. The smooth convergence does imply that γ^∞ satisfies curve shortening flow

(1). Also, since $\gamma^\infty(\cdot, t) : S^1 \rightarrow \mathbb{R}^2$ a smooth embedding for each $t \in (0, t_0]$, by Grayson's theorem [6], we can extend the flow until it disappears to a round point. We have created thus a smooth flow $\gamma^\infty : S^1 \times (0, T) \rightarrow \mathbb{R}^2$, which agrees with the above defined γ^∞ in $(0, t_0)$ and such that it converges to a round point as $t \rightarrow T$.

Finally, to finish the proof we need to show that

- (i) $T = \frac{A_0}{2\pi}$ and
- (ii) $\forall \varepsilon > 0, \exists t_\varepsilon > 0: \Gamma_t := \gamma^\infty(S^1, t) \subset B(\Gamma_0, \varepsilon), \forall 0 < t < t_\varepsilon$.

To see (i), let $A^\infty(t)$ denote the (finite) area enclosed by Γ_t and $A^j(t)$ that of the approximating curves $\Gamma_t^j = \gamma^j(S^1, t)$. By the convergence for $t \in (0, t_0]$, we have

$$A^\infty(t_0) = \lim_j A^j(t_0) = \lim_j A^j(0) - 2\pi t_0 = A_0 - 2\pi t_0.$$

Since $0 = \lim_{t \rightarrow T} A^\infty(t) = A^\infty(t_0) - 2\pi(T - t_0)$, we obtain (i).

In order to see (ii), we let $\varepsilon > 0$. It suffices to show that there exists t_ε such that for all j large enough $\Gamma_t^j \subset B(\Gamma_0, \varepsilon)$, for all $t \in (0, t_\varepsilon)$. Assume that this is not the case, but instead, there exists a sequence of times $t_k \downarrow 0$ and a sequence of points of the slingshots $x_k \in \Gamma_{t_k}^{j_k}$, with $j_k \rightarrow \infty$, such that $\text{dist}(x_k, \Gamma_0) > \varepsilon$. Note first, that by the assumption on Γ_0 and the approximating sequence Γ_0^i , a simple argument using grim reapers, parallel to the x-axis, as barriers implies that eventually the points x_k must be in a compact set, that is, there exists k_0 and a compact set K , such that for all $k \geq k_0$, $x_k \in K$.

Finally, the proof of Lemma 8, yields a uniform curvature bound for the slingshots in compact sets, which amounts to a uniform bound in the velocity. This implies that the distance traveled goes uniformly to zero, that is $\text{dist}(\Gamma_{t_k}^{j_k} \cap K, \Gamma_0) \rightarrow 0$, as $k \rightarrow \infty$, and thus we obtain a contradiction.

□

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