

TROPICALIZING THE GRAPH PROFILE OF SOME ALMOST-STARS*

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Abstract. Many important problems in extremal combinatorics can be stated as certifying polynomial inequalities in graph homomorphism numbers, and in particular, many ask to certify pure binomial inequalities. For a fixed collection of graphs $\mathcal{U}$, the *tropicalization of the graph profile of $\mathcal{U}$* essentially records all valid pure binomial inequalities involving graph homomorphism numbers for graphs in $\mathcal{U}$. Building upon ideas and techniques described by Blekherman and Raymond in 2022, we compute the tropicalization of the graph profile for the graph containing a single vertex as well as stars where one edge is subdivided. This allows pure binomial inequalities in homomorphism numbers (or densities) for these graphs to be verified through an explicit linear program where the number of variables is equal to the number of edges in the biggest graph involved.

Key words. graph profile, graph homomorphism, tropicalization, trees, stars

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1. Introduction. The *number of homomorphisms* from a graph H to a graph G , denoted by $\text{hom}(H; G)$, is the number of maps from $V(H)$ to $V(G)$ that yield a graph homomorphism, i.e., that map every edge of H to an edge of G . Many important problems and results in extremal graph theory can be framed as certifying the validity of polynomial inequalities in the number of graph homomorphisms which are valid on all graphs. In particular, in many cases, those polynomials are pure binomial inequalities. For example, the Sidorenko conjecture [Sid93] can be stated as $\text{hom}(P_0; G)^{2|E(H)| - |V(H)|} \cdot \text{hom}(H; G) \geq \text{hom}(P_1; G)^{|E(H)|}$ for any bipartite graph H , where P_i denotes the path with i edges. The Erdős–Simonovits conjecture [ES82], which was proven in [Saĝ18] and then generalized in [BR22], states that $\text{hom}(P_{2u}; G)^2 \cdot \text{hom}(P_{2v+1}; G)^{2v-1-2u} \geq \text{hom}(P_{2v-1}; G)^{2v+1-2u}$ for $u < v$ (the original conjecture was for $u = 0$). Another example is the Kruskal–Katona theorem [Kru63, Kat68], which states that $\text{hom}(K_p; G)^q \geq \text{hom}(K_q; G)^p$ for $2 \leq p < q$, where K_m is the complete graph on m vertices.

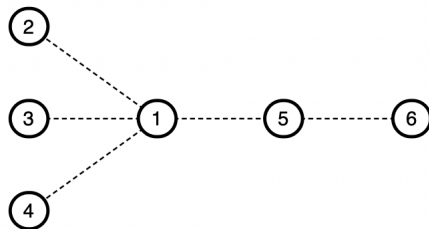
Graph profiles essentially record all valid polynomial inequalities in homomorphism numbers of some given graphs. In [BRST22, BR22], it was shown that the tropicalization of graph profiles is a way of capturing all true pure binomial inequalities. Graph profiles are complicated objects—for instance, they need not be semialgebraic. Very few two-dimensional density profiles are known (see, e.g., [Raz08, Nik11, Rei16]), and none are known in higher dimensions. Though it still retains interesting information, the tropicalization of a graph profile is a much simpler object: it is always a closed convex cone, and it has been computed for some arbitrarily large families of graphs [BRST22, BR22]. Moreover, all tropicalizations computed thus far have been rational polyhedral cones.

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FIG. 1. The $S_{2,13}$ tree.

In this paper, for $m \in \mathbb{N}$, we explicitly compute the rational polyhedral cone forming the tropicalization of the graph profile for the collection of graphs $\mathcal{U} = \{S_0, S_{2,1^0}, S_{2,1^1}, \dots, S_{2,1^{m-1}}\}$, where S_0 is a single vertex and $S_{2,1^k}$ is the tree where $V(S_{2,1^k}) = \{1, 2, \dots, k+3\}$ and $E(S_{2,1^k}) = \{\{1, j\} \mid j \in \{2, 3, \dots, k+2\}\} \cup \{\{k+2, k+3\}\}$, i.e., a $(k+1)$ -star where one edge is subdivided. The notation comes from thinking of the branch lengths of this subdivision of a star as a partition of the number of edges in the graph. (Note that $S_{2,1^0}$ and $S_{2,1^1}$ are respectively paths with two and three edges.) The tree $S_{2,1^3}$ is depicted in Figure 1. As a consequence, the validity of any pure binomial inequality in the graphs of $\mathcal{U}$ can be checked in a finite way from an explicit finite collection of binomial inequalities, and this can be done through a linear program.

We believe that understanding the tropicalization for all trees would be highly interesting but sadly out of reach at the moment. An ambitious intermediary goal might be to study the tropicalization of all possible subdivisions of a star up to a certain size, i.e., in our notation, S_λ for any partition λ of $n \leq N$ for some fixed $N \in \mathbb{N}$. Indeed, given that the tropicalization for stars is quite simple, whereas the tropicalization for paths is much more complicated [BR22], we expect this tropicalization to be highly nontrivial and to teach us some important ingredients necessary towards developing our understanding of the tropicalization of all trees. The family of graphs in this paper was chosen as a first step towards understanding the tropicalization for S_λ 's.

We now give a precise description of the aforementioned ideas.

1.1. Definitions, results, and proof strategies. The *number graph profile* of a collection of connected graphs $\mathcal{U} = \{H_1, \dots, H_s\}$, denoted as $\mathcal{N}_{\mathcal{U}}$, is the set of all vectors $(\text{hom}(H_1; G), \text{hom}(H_2; G), \dots, \text{hom}(H_s; G))$ as G varies over all graphs. For any $\mathcal{U}$, the number graph profile $\mathcal{N}_{\mathcal{U}}$ is contained in $\mathbb{N}^s$. Understanding all s -tuples that can occur as homomorphism numbers in $\mathcal{U}$ is essentially equivalent to understanding all polynomial inequalities in homomorphism numbers which are valid on all graphs. It is known that the problem of checking whether a polynomial expression in homomorphism numbers is nonnegative on all graphs is undecidable [IR95].

A *pure binomial inequality* has the form $x_1^{\alpha_1} \cdots x_s^{\alpha_s} \geq x_1^{\beta_1} \cdots x_s^{\beta_s}$, where $\mathbf{x} := (x_1, \dots, x_s)$, $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_s)$, $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s) \in \mathbb{R}_{\geq 0}^s$. As mentioned in the introduction, pure binomial inequalities in homomorphism numbers are of interest in extremal graph theory. To study such inequalities, let $\log : \mathbb{R}_{>0}^s \rightarrow \mathbb{R}^s$ be defined as $\log(\mathbf{v}) := (\log(v_1), \dots, \log(v_s))$. For a set $\mathcal{S} \subseteq \mathbb{R}_{\geq 0}^s$, we define $\log(\mathcal{S}) := \log(\mathcal{S} \cap \mathbb{R}_{>0}^s)$. The *tropicalization* of $\mathcal{S}$, which is also called the *logarithmic limit set* of $\mathcal{S}$, is defined to be

$$\text{trop}(\mathcal{S}) := \lim_{\tau \rightarrow \infty} \log_{\tau}(\mathcal{S}).$$

In [BRST22, Lemma 2.2], it was shown that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a closed convex cone equal to the closure of the conical hull of $\log(\mathcal{N}_{\mathcal{U}})$, i.e., $\text{trop}(\mathcal{N}_{\mathcal{U}}) = \text{cl}(\text{cone}(\log(\mathcal{N}_{\mathcal{U}})))$. Each pure binomial inequality in numbers for $\mathcal{N}_{\mathcal{U}}$ corresponds to a linear inequality in logarithms $\langle \alpha, \log \mathbf{x} \rangle \geq \langle \beta, \log \mathbf{x} \rangle$ for $\text{trop}(\mathcal{N}_{\mathcal{U}})$. An important fact we will use is that the extreme rays of the dual cone $\text{trop}(\mathcal{N}_{\mathcal{U}})^*$ generate all of the pure binomial inequalities valid on $\mathcal{N}_{\mathcal{U}}$. Note that $\text{trop}(\mathcal{N}_{\mathcal{U}}) \subseteq \mathbb{R}_{\geq 0}^s$ since $\mathcal{N}_{\mathcal{U}} \cap \mathbb{R}_{>0}^s$ contains only points where every coordinate is at least one. In [BR22, Proposition 2.4], it was shown that no spurious binomial inequalities are added by removing points with zero coordinates from $\mathcal{N}_{\mathcal{U}}$.

Blekherman and Raymond conjectured that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is in fact a rational polyhedral cone for any finite collection of graphs $\mathcal{U}$ [BR22, Conjecture 2.14]. They proved that this is the case when $\mathcal{U}$ consists of complete graphs, even cycles, odd cycles, or paths by fully characterizing $\text{trop}(\mathcal{N}_{\mathcal{U}})$ in those settings. They also showed that the conjecture holds for the following collections of graphs.

THEOREM 1.1 (see [BR22, Theorems 3.2 and 1.2]). *Let $\mathcal{U}$ be a finite collection of chordal series-parallel graphs. Then $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a rational polyhedral cone. Furthermore, there exists a finite collection of binomial inequalities such that any pure binomial inequality in the graphs of $\mathcal{U}$ can be deduced in a finite way from this finite collection.*

Note that this theorem already implies that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a rational polyhedral cone for the collection of graphs we are interested in, namely $\mathcal{U} = \{S_0, S_{2,1^0}, S_{2,1^1}, \dots, S_{2,1^{m-1}}\}$, since all of these graphs are chordal and series-parallel. The main contribution of this paper is finding an explicit description for $\text{trop}(\mathcal{N}_{\mathcal{U}})$.

THEOREM 5.1. *Let $\mathcal{U} = \{S_0, S_{2,1^0}, S_{2,1^1}, \dots, S_{2,1^{m-1}}\}$, and let*

$$Q = \left\{ \mathbf{y} \in \mathbb{R}^{m+1} \left| \begin{array}{l} -y_1 + y_2 \geq 0 \\ 4y_1 - 3y_2 \geq 0 \\ 3y_1 - 3y_3 + y_4 \geq 0 \\ y_1 + 2y_{m-1} - 2y_m \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_0 - 2y_1 + y_3 \geq 0 \\ y_{i-1} - 2y_i + y_{i+1} \geq 0 \quad \forall 2 \leq i \leq m-1 \\ m \cdot y_{m-1} - (m-1) \cdot y_m \geq 0 \end{array} \right. \right\}.$$

Then $\text{trop}(\mathcal{N}_{\mathcal{U}}) = Q$.

Having such an explicit description of $\text{trop}(\mathcal{N}_{\mathcal{U}})$ allows one to check the validity of any pure binomial inequality in the graphs of $\mathcal{U}$ through an explicit linear program. It is more convenient to check the validity of a binomial inequality for $\mathcal{N}_{\mathcal{U}}$ by checking the validity of the corresponding linear inequality for $\text{trop}(\mathcal{N}_{\mathcal{U}})$. There, we simply need to show that this linear inequality can be written as a conical combination of the defining inequalities for $\text{trop}(\mathcal{N}_{\mathcal{U}})$. If there is no such conical combination, then the original pure binomial inequality is not valid for $\mathcal{N}_{\mathcal{U}}$. Finding such a conical combination can be done via a linear program. Indeed, suppose one wants to check whether $S_0^{\alpha_0} \prod_{i=0}^{m-1} S_{2,1^i}^{\alpha_{i+1}} \geq S_0^{\beta_0} \prod_{i=0}^{m-1} S_{2,1^i}^{\beta_{i+1}}$ is a valid inequality for some α_i, β_i (some of which can be 0). This is equivalent to checking that $\alpha_0 y_0 + \sum_{i=1}^m \alpha_i y_i - \beta_0 y_0 - \sum_{i=1}^m \beta_i y_i \geq 0$ on $\text{trop}(\mathcal{N}_{\mathcal{U}})$ (since $y_0 = \log(\text{hom}(S_0; G))$ and $y_i = \log(\text{hom}(S_{2,1^{i-1}}; G))$ for $1 \leq i \leq m$). Thus one can simply minimize $\alpha_0 y_0 + \sum_{i=1}^m \alpha_i y_i - \beta_0 y_0 - \sum_{i=1}^m \beta_i y_i$

over the cone Q of Theorem 5.1. If the optimal value is 0, then the inequality is valid, and the dual solution gives the conical combination of inequalities of Q that yields $\alpha_0 y_0 + \sum_{i=1}^m \alpha_i y_i - \beta_0 y_0 - \sum_{i=1}^m \beta_i y_i \geq 0$, i.e., how one recovers this inequality from the pure binomial inequalities associated to the inequalities in Q . Otherwise, the inequality is not valid.

Note that these ideas carry on over to homomorphism densities. The *homomorphism density* from a graph H to a graph G , denoted as $t(H; G)$, is the probability that a map from $V(H)$ to $V(G)$ is a graph homomorphism, i.e., $t(H; G) = \frac{\text{hom}(H; G)}{|V(G)|^{|V(H)|}}$. The *density graph profile* $\mathcal{D}_{\mathcal{U}}$ of some finite collection of graphs $\mathcal{U}$ is the closure of the points $(t(H_1; G), t(H_2; G), \dots, t(H_s; G))$ as G varies over all graphs. It is more natural to write certain problems from extremal graph theory as certifying the validity of a polynomial inequality in densities. In this setting again, it is known that the problem of checking whether a polynomial expression in densities is nonnegative on all graphs is undecidable [HN11]. Note that any pure binomial inequality in homomorphism densities, say $t(H_1; G) - t(H_2; G) \geq 0$ (where H_1 and H_2 need not be connected graphs), can be rewritten as a pure binomial inequality in homomorphism numbers. Indeed, $t(H_1; G) - t(H_2; G) = \frac{\text{hom}(H_1; G)}{|V(G)|^{|V(H_1)|}} - \frac{\text{hom}(H_2; G)}{|V(G)|^{|V(H_2)|}}$, and since $|V(G)| = \text{hom}(K_1; G)$, the previous inequality can be rewritten as $\text{hom}(K_1; G)^{|V(H_2)| - |V(H_1)|} \text{hom}(H_1; G) - \text{hom}(H_2; G) \geq 0$ if $|V(H_2)| \geq |V(H_1)|$ or $\text{hom}(H_1; G) - \text{hom}(K_1; G)^{|V(H_1)| - |V(H_2)|} \text{hom}(H_2; G) \geq 0$ if $|V(H_1)| \geq |V(H_2)|$. Therefore, checking the validity of $t(H_1; G) - t(H_2; G) \geq 0$ over $\mathcal{D}_{\mathcal{U}}$ where $\mathcal{U}$ contains all connected components in H_1 and H_2 can be done by checking the corresponding pure binomial inequality in homomorphism numbers over $\mathcal{N}_{\mathcal{U}}$ if $\mathcal{U}$ contains K_1 or over $\mathcal{N}_{\mathcal{U}'}$ where $\mathcal{U}' = \mathcal{U} \cup \{K_1\}$ otherwise. In our setting, $\mathcal{U}$ contains $K_1 = S_0$, so by Theorem 5.1, the validity of any pure binomial inequality in densities in $\mathcal{U}$ can also be checked through the linear program described in the previous paragraph.

Our strategy to prove Theorem 5.1 is to first certify that all of the inequalities of Q come from valid pure binomial inequalities on the profile $\mathcal{N}_{\mathcal{U}}$, thus showing that $\text{trop}(\mathcal{N}_{\mathcal{U}}) \subseteq Q$. Certification is done using some standard techniques as well as tools presented in [KR11]. To show that $Q \subseteq \text{trop}(\mathcal{N}_{\mathcal{U}})$, we find all extreme rays of Q and show that these rays are realizable. We say a ray $\mathbf{r}$ is *realizable* if there exists a sequence of graphs G_n on n vertices such that $\alpha(\log \text{hom}(H_1; G_n), \dots, \log \text{hom}(H_s; G_n)) \rightarrow \mathbf{r}$ as $n \rightarrow \infty$ for some constant $\alpha \in \mathbb{R}_{\geq 0}$. These sequences of graphs often arise by constructing blow-up graphs and taking tensor powers and disjoint unions of those graphs.

The reader might notice that all inequalities in Q have a single negative term. This is no coincidence: Corollary 2.9 of [BR22] proves that each extreme ray of the dual cone $\text{trop}(\mathcal{N}_{\mathcal{U}})^*$ for any finite collection of connected graphs $\mathcal{U}$ is spanned by a vector with at most one negative coordinate.

Another nice property of $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is that it is *max-closed*: if $(x_1, \dots, x_s), (y_1, \dots, y_s) \in \text{trop}(\mathcal{N}_{\mathcal{U}})$, then $(x_1, \dots, x_s) \oplus (y_1, \dots, y_s) := (\max\{x_1, y_1\}, \dots, \max\{x_s, y_s\}) \in \text{trop}(\mathcal{N}_{\mathcal{U}})$. Given a cone $\mathcal{S} \subset \mathbb{R}_{\geq 0}^s$, the *double hull* of $\mathcal{S}$ is the smallest closed and max-closed convex cone containing $\mathcal{S}$. A point $\mathbf{p} \in \mathcal{S}$ is said to be *max-extreme* if whenever $\mathbf{p} = \mathbf{x} \oplus \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, then $\mathbf{p} = \mathbf{x}$ or $\mathbf{p} = \mathbf{y}$. Furthermore, $\mathbf{p} \in \mathcal{S}$ is said to span a *doubly extreme ray* of $\mathcal{S}$ if $\mathbf{p}$ spans an extreme ray of $\mathcal{S}$ and $\mathbf{p}$ is max-extreme. Theorem 2.13 in [BR22] states that a closed and max-closed convex cone in $\mathbb{R}_{\geq 0}^s$ is the double hull of its doubly extreme rays.

As a corollary of Theorem 5.1, we rewrite the tropicalization $\text{trop}(\mathcal{N}_{\mathcal{U}})$ as the double hull of five doubly extreme rays.

COROLLARY 5.2. *The set $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is the double hull of the following five doubly extreme rays in $\mathbb{R}^{m+1}$: $\mathbf{d}_{1,m} := (1, 0, 0, \dots, 0)$, $\mathbf{d}_{2,m} := \vec{1}$, $\mathbf{d}_{3,m} := (1, 2, 2, 3, 4, \dots, m)$, $\mathbf{d}_{4,m} := (1, 3, 4, 5, \dots, m+2)$, $\mathbf{d}_{5,m} := (2, 4, 5, 7, \dots, 2m+1)$.*

1.2. Outline. In section 3, we show that the defining inequalities of Q in Theorem 5.1 are valid for $\text{trop}(\mathcal{N}_{\mathcal{U}})$. In section 4, we show that the rays in Corollary 5.2 are present in $\text{trop}(\mathcal{N}_{\mathcal{U}})$, and we use those to construct a larger family of rays in $\text{trop}(\mathcal{N}_{\mathcal{U}})$. This larger family will turn out to be the extreme rays of $\text{trop}(\mathcal{N}_{\mathcal{U}})$, as will be shown in the proof of Theorem 5.1 in section 5. In section 2, we briefly recall helpful properties of graph homomorphisms and of tropicalizations of graph profiles, as well as some results of Kopparty and Rossman [KR11] which we use later on.

2. Background on tropicalizations, homomorphisms, and some results of Kopparty and Rossman. We go over different concepts and results which are used to prove Theorem 5.1 and Corollary 5.2 as well as the lemmas that precede them.

2.1. Useful results about tropicalizations. Recall from the introduction that for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^s$, we let $\mathbf{x} \oplus \mathbf{y}$ denote their *tropical sum*:

$$\mathbf{x} \oplus \mathbf{y} = (\max\{x_1, y_1\}, \dots, \max\{x_s, y_s\}).$$

A set $\mathcal{S} \subseteq \mathbb{R}^s$ is said to be *max-closed* if for any $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, we have $\mathbf{x} \oplus \mathbf{y} \in \mathcal{S}$. Corollary 2.3 in [BR22] proves that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a max-closed convex cone for any finite collection of connected graphs $\mathcal{U}$.

Recall also from the introduction that the *double hull* of a cone $\mathcal{S} \subset \mathbb{R}_{\geq 0}^s$ is defined to be the smallest max-closed convex cone containing $\mathcal{S}$ and is denoted by $\text{dh}(\mathcal{S})$. From Theorem 2.13 in [BR22], we know that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is equal to the double hull of its doubly extreme rays. Essential to the proof that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a max-closed convex cone (Lemma 2.2 of [BR22]) are the following two graph operations. It has long been known that $\text{hom}(H; G_1) + \text{hom}(H; G_2) = \text{hom}(H; G_1 G_2)$, where $G_1 G_2$ is the disjoint union of G_1 and G_2 , and that $\text{hom}(H; G_1) \cdot \text{hom}(H; G_2) = \text{hom}(H; G_1 \times G_2)$, where $G_1 \times G_2$ is the categorical product of G_1 and G_2 . From Lemma 2.2 in [BRST22], to show that a conical combination of two points $\log(\mathbf{v}), \log(\mathbf{w}) \in \log(\mathcal{N}_{\mathcal{U}})$ is in $\text{trop}(\mathcal{N}_{\mathcal{U}})$, it suffices to show that $a_1 \log(\mathbf{v}) + a_2 \log(\mathbf{w}) \in \text{trop}(\mathcal{N}_{\mathcal{U}})$ for $a_1, a_2 \in \mathbb{N}$. Since $a_1 \log(\mathbf{v}) + a_2 \log(\mathbf{w}) = \log(\mathbf{v}^{a_1} \cdot \mathbf{w}^{a_2})$, we see that this is true by using our graph operations. Similarly, from Lemma 2.2 in [BR22], to show that a tropical combination of these two points is in $\text{trop}(\mathcal{N}_{\mathcal{U}})$, it suffices to show that $a_1 \log(\mathbf{v}) \oplus a_2 \log(\mathbf{w}) \in \text{trop}(\mathcal{N}_{\mathcal{U}})$ for $a_1, a_2 \in \mathbb{N}$. It turns out that $a_1 \log(\mathbf{v}) \oplus a_2 \log(\mathbf{w}) = \frac{\log(\mathbf{v}^{a_1 l} + \mathbf{w}^{a_2 l})}{l}$ as $l \rightarrow \infty$, and so we see that the desired result holds since $\frac{\log(\mathbf{v}^{a_1 l} + \mathbf{w}^{a_2 l})}{l}$ is in $\text{trop}(\mathcal{N}_{\mathcal{U}})$ because of our graph operations. Therefore, taking the conical hull and the max-closure of any set of rays can also be understood in a graph theoretical way. Indeed, this tells us how to realize rays obtained by taking conical and tropical combinations of rays that are realizable. We will make use of this fact in section 4.

2.2. Gluing algebra. In the introduction, we defined $\text{hom}(H; G)$ when H is an unlabeled graph to be the number of homomorphisms from H to G . We now extend this definition for when H is a partially labeled graph. This will be useful to prove inequalities in the next section. A graph is *partially labeled* if a subset of its vertices are labeled with elements of $\mathbb{N} := \{1, 2, 3, \dots\}$ such that no vertex receives more than one label. If no vertices of H are labeled, then H is *unlabeled*. For a partially labeled graph H , $\text{hom}(H; G)$ is the number of homomorphisms from H to G that send the labeled vertices of H to the corresponding vertices in G (which one can think of as fully labeled in some fixed way). For example,

In the definition above, $A \cap B$ is said to separate $A \setminus B$ and $B \setminus A$ if there is no path in F_2 going from a vertex in $A \setminus B$ to a vertex in $B \setminus A$.

THEOREM 2.1 (Kopparty and Rossman, 2011). *Let F_1 be a chordal graph, and let F_2 be a series-parallel graph. Then*

$$\text{HDE}(F_1; F_2) = \min_{p \in \mathcal{P}(F_2)} \max_{\varphi \in \text{Hom}(F_1; F_2)} \sum_{S \subseteq \text{MaxCliques}(F_1)} -(-1)^{|S|} p(\varphi(\cap S)),$$

where $\text{MaxCliques}(F_1)$ is the set of maximal cliques of F_1 and $\cap S$ is the intersection of the maximal cliques in S .

3. Valid binomial inequalities for $S_{2,1^k}$ -trees. Recall that our goal is to show that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is equal to the cone Q defined in the introduction in Theorem 5.1. To show that $\text{trop}(\mathcal{N}_{\mathcal{U}}) \subseteq Q$, we need to show that all of the inequalities of Q come from valid pure binomial inequalities on the profile $\mathcal{N}_{\mathcal{U}}$. In this section, using some standard techniques as well as the result of Kopparty and Rossman presented in section 2, we derive some valid pure binomial inequalities for $\mathcal{N}_{\mathcal{U}}$ that yield the inequalities defining Q in Theorem 5.1 after taking the log, thus showing that these are valid inequalities for $\text{trop}(\mathcal{N}_{\mathcal{U}})$. Recall that $V(S_{2,1^k}) = \{1, 2, \dots, k+3\}$ and $E(S_{2,1^k}) = \{\{1, j\} \mid j \in \{2, 3, \dots, k+2\}\} \cup \{\{k+2, k+3\}\}$. Finally, for ease of reading, in this section, we will use H as shorthand for $\text{hom}(H; G)$ whenever there is no confusion.

First note that $S_{2,1^1} - S_{2,1^0} \geq 0$ and $S_{2,1^0}^4 - S_{2,1^1}^3 \geq 0$ are inequalities involving only paths with two and three edges and that these inequalities are already known (see [KR11], for example). The following inequality is also trivially true.

LEMMA 3.1. *We have that $S_0 S_{2,1^{m-2}} \geq S_{2,1^{m-1}}$ is a valid inequality for all graphs G .*

Proof. The inequality holds since there is a surjective homomorphism φ from $S_0 S_{2,1^{m-2}}$ to $S_{2,1^{m-1}}$. Indeed, labeling the vertex S_0 as 0, let $\varphi(i) = i$ for $1 \leq i \leq m-1$, $\varphi(0) = m$, $\varphi(m) = m+1$, $\varphi(m+1) = m+2$. $\square$

Recall the following consequence of the AM-GM inequality (see, for instance, Theorem 2.1 of [Sid94]). Let $\mathbf{v} = (H_1, \dots, H_s)$, where H_i 's are partially labelled graphs, and let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{N}^s$, $\alpha_1, \dots, \alpha_k \in \mathbb{R}_{>0}$, and $\mathbf{b} = \sum_{i=1}^k \alpha_i \mathbf{a}_i$ such that $\mathbf{b} \in \mathbb{N}^s$. Then

$$\llbracket \mathbf{v}^{\mathbf{a}_1} \rrbracket^{\alpha_1} \dots \llbracket \mathbf{v}^{\mathbf{a}_k} \rrbracket^{\alpha_k} \geq \llbracket \mathbf{v}^{\mathbf{b}} \rrbracket$$

is a valid inequality where $\mathbf{v}^{\mathbf{a}_i} = H_1^{a_{i1}} \dots H_s^{a_{is}}$. We now prove a few inequalities using this observation.

LEMMA 3.2. *We have that $S_{2,1^0}^3 S_{2,1^3} \geq S_{2,1^2}^3$.*

Proof. Applying AM-GM, we have that

$$\left[\begin{array}{c} 1 \quad 2 \\ \diagup \quad \diagdown \\ 4 \end{array} \right] \left[\left(\begin{array}{c} 1 \quad 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \right)^3 \right]^{\frac{1}{3}} \geq \left[\begin{array}{c} 1 \quad 2 \\ \diagup \quad \diagdown \\ 4 \end{array} \quad \begin{array}{c} 1 \quad 2 \\ \diagup \quad \diagdown \\ 3 \end{array} \right]$$

is a valid inequality, which proves the inequality above. $\square$

LEMMA 3.3. *We have that $S_{2,1^0} S_{2,1^{m-2}}^2 \geq S_{2,1^{m-1}}^2$.*

Proof. Applying AM-GM, we have that

$$\left[\left(\begin{array}{c} 2 \\ \vdots \\ 1 \end{array} \right)^2 \right]^{\frac{1}{2}} \left[\begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ m-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{m} \\ \vdots \\ \xrightarrow{m+1} \end{array} \right] \geq \left[\begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ m-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{m} \\ \vdots \\ \xrightarrow{m+1} \end{array} \right]$$

is a valid inequality, which proves the inequality above. $\square$

LEMMA 3.4. We have that $S_{2,1^{i-2}}S_{2,1^i} \geq S_{2,1^{i-1}}^2$ for $i \geq 2$.

Proof. Applying AM-GM, we have that

$$\left[\left(\begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ i-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{i} \\ \vdots \\ \xrightarrow{i+1} \end{array} \right)^2 \right]^{\frac{1}{2}} \left[\left(\begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ i-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{i} \\ \vdots \\ \xrightarrow{i+1} \end{array} \right)^2 \right]^{\frac{1}{2}} \geq \left[\begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ i-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{i} \\ \vdots \\ \xrightarrow{i+1} \end{array} \begin{array}{c} \begin{array}{c} 2 \\ \vdots \\ 3 \\ \vdots \\ i-1 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{i} \\ \vdots \\ \xrightarrow{i+1} \end{array} \end{array} \right]$$

is a valid inequality, which proves the inequality above. $\square$

Note that the inequality in Lemma 3.4 could also have been seen as an application of Hölder's inequality to a sequence of products of certain powers of degrees of endpoints of edges.

LEMMA 3.5. We have that $S_{2,1^{m-2}}^m \geq S_{2,1^{m-1}}^{m-1}$ for $m \geq 2$.

Proof. When $m = 2$, this is an inequality only involving paths and it is known to be true from previous work. For completeness, here is a short proof: applying AM-GM, we have that

$$\left[\begin{array}{c} 2 \\ \vdots \\ 3 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{3} \end{array} \right] \left[\begin{array}{c} 2 \\ \vdots \\ 3 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{3} \end{array} \right] \geq \left[\begin{array}{c} 2 \\ \vdots \\ 3 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{3} \end{array} \begin{array}{c} 2 \\ \vdots \\ 3 \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} 1 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{3} \end{array} \right]$$

is a valid inequality, which yields $\begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} \geq \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \end{array}$ as desired.

When $m \geq 3$, applying AM-GM, we have that

$$\left[\left(\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \right)^{m-1} \right]^{\frac{1}{m-1}} \left[\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \right] \geq \left[\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \begin{array}{c} \begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \end{array} \right]$$

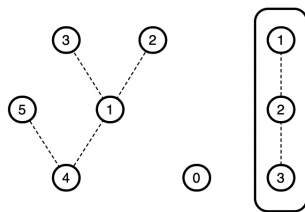
is a valid inequality which is equivalent to

$$\left[\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \right] \left[\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \right]^{m-1} \geq \left[\begin{array}{c} 4 \\ \vdots \\ 5 \\ \vdots \\ m \end{array} \begin{array}{c} \nearrow \\ \vdots \\ \nearrow \\ \vdots \\ \nearrow \end{array} 3 \begin{array}{c} \xrightarrow{2} \\ \vdots \\ \xrightarrow{1} \end{array} \right]^{m-1}$$

which proves the inequality above. $\square$

We now use Theorem 2.1 to prove the validity of one other important binomial inequality for $\mathcal{N}_{\mathcal{U}}$.

LEMMA 3.6. We have that $\text{HDE}(S_0S_{2,1^2}; S_{2,1^0}) = 2$ and thus that $S_0S_{2,1^2} \geq S_{2,1^0}^2$.

FIG. 2. Illustration of φ described in the proof of Lemma 3.6.

Proof. We first show that $\text{HDE}(S_0S_{2,12}; S_{2,1^0}) \leq 2$. For every $\emptyset \neq S \subseteq V(S_{2,1^0})$, let $p(S) = 1$, and let $p(\emptyset) = 0$. One can check that $p \in \mathcal{P}(S_{2,1^0})$.

For any homomorphism φ from $S_0S_{2,12}$ to $S_{2,1^0}$ (including the optimal one), since the maximal cliques in $S_0S_{2,12}$ are the vertex in S_0 and the edges in $S_{2,12}$, we have that

$$\sum_{S \subseteq \text{MaxCliques}(S_0S_{2,12})} -(-1)^{|S|} p(\varphi(\cap S)) = 1 + 4 - 3 = 2.$$

Indeed, the vertex S_0 contributes 1, the four edges of $S_{2,12}$ each contribute 1, the vertex of degree two in $S_{2,12}$ contributes -1 , as it is the intersection of two maximal cliques (edges), and the vertex of degree three in $S_{2,12}$ contributes -2 since it can be written in three different ways as the intersection of two maximal cliques and in one way as the intersection of three maximal cliques. As we are minimizing over all $p \in \mathcal{P}(S_{2,1^0})$, this yields that $\text{HDE}(S_0S_{2,12}; S_{2,1^0}) \leq 2$ as desired.

We next show that $\text{HDE}(S_0S_{2,12}; S_{2,1^0}) \geq 2$. Consider the homomorphism φ from $S_0S_{2,12}$ to $S_{2,1^0}$ such that $\varphi(0) = \varphi(4) = 3$, $\varphi(1) = \varphi(5) = 2$, and $\varphi(2) = \varphi(3) = 1$ (see Figure 2).

Every edge of $S_{2,1^0}$ is covered by the image of an edge of $S_{2,12}$ (each of which is a maximal clique, thus contributing a weight of 1) twice. The inner vertex of $S_{2,1^0}$ is covered by the image of the center vertex of $S_{2,12}$, which yields a multiplier -2 (because it can be written as the intersection of three maximal cliques and as three different intersections of two maximal cliques). One end vertex of $S_{2,1^0}$ is not covered by the image of any inner vertices, and leaves of $S_{2,12}$ contribute weights of 0 given that they cannot be written as intersections of maximal cliques. The other end vertex of $S_{2,1^0}$ is covered by S_0 (which contributes a weight of 1 since it is a maximal clique) and the inner vertex of the two-edge branch of $S_{2,1^0}$ (which contributes a weight of -1 since it is the intersection of two maximal cliques), yielding a multiplier of 0. Thus we have

$$\begin{aligned} \sum_{S \subseteq \text{MaxCliques}(S_0S_{2,12})} -(-1)^{|S|} p(\varphi(\cap S)) \\ &= 2p(\{1, 2\}) + 2p(\{2, 3\}) - 2p(\{2\}) \\ &= 2p(\{1, 2, 3\}) \\ &= 2, \end{aligned}$$

where the penultimate line follows from the last constraint of polymatroidal functions (letting $A = \{1, 2\}$ and $B = \{2, 3\}$, we have $p(\{2\}) + p(\{1, 2, 3\}) = p(\{1, 2\}) + p(\{2, 3\})$ since $\{2\}$ separates $\{1\}$ and $\{3\}$), and the last line follows from the fact that we are considering normalized polymatroidal functions, meaning that $p(V(F_2)) = 1$.

Therefore, for every $p \in \mathcal{P}(S_0 S_{2,1^2})$, there exists a homomorphism that yields 2, and since we are maximizing over all homomorphisms, we have $\text{HDE}(S_0, S_{2,1^2}; S_{2,1^0}) \geq 2$. This proves that $\text{HDE}(S_0 S_{2,1^2}; S_{2,1^0}) = 2$. $\square$

4. Rays of $\text{trop}(\mathcal{N}_{\mathcal{U}})$. Recall again that our goal is to show that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is equal to the cone Q defined in the introduction in Theorem 5.1. To show that $Q \subseteq \text{trop}(\mathcal{N}_{\mathcal{U}})$, we need to find all extreme rays of Q and show that these rays are realizable. In this section, we show that different rays are in $\text{trop}(\mathcal{N}_{\mathcal{U}})$. In the next section, we will see that the rays $\mathbf{d}_{j,m}$ are doubly extreme rays and the rays $\mathbf{s}_{j,m}$ and $\mathbf{r}_{j,m,i}$ are the remaining extreme rays of $\text{trop}(\mathcal{N}_{\mathcal{U}})$.

DEFINITION 4.1. Fix $m \in \mathbb{N}$, and let

- $\mathbf{d}_{1,m} := (1, 0, 0, \dots, 0)$,
- $\mathbf{d}_{2,m} := \vec{1} = (1, \dots, 1)$,
- $\mathbf{d}_{3,m} := (1, 2, 2, 3, 4, \dots, m)$,
- $\mathbf{d}_{4,m} := (1, 3, 4, 5, \dots, m+2)$, and
- $\mathbf{d}_{5,m} := (2, 4, 5, 7, \dots, 2m+1)$

be vectors in $\mathbb{R}^{m+1}$.

LEMMA 4.2. The rays $\mathbf{d}_{j,m}$ are in $\text{trop}(\mathcal{N}_{\mathcal{U}})$ for each $1 \leq j \leq 5$.

Proof. We show how to realize each ray as $\alpha(\log \text{hom}(S_0; G), \log \text{hom}(S_{2,1^0}; G), \dots, \log \text{hom}(S_{2,1^{m-1}}; G))$ for some graph G or as $\alpha(\log \text{hom}(S_0; G_n), \log \text{hom}(S_{2,1^0}; G_n), \dots, \log \text{hom}(S_{2,1^{m-1}}; G_n))$ for some sequence of graphs G_n on n vertices as $n \rightarrow \infty$ for some constant $\alpha \in \mathbb{R}_{\geq 0}$.

- (1) Consider the graph G_n consisting of $n-2$ isolated vertices and one edge. Then $\text{hom}(S_0; G_n) = n$ and $\text{hom}(S_{2,1^k}; G_n) = 2$ for $0 \leq k \leq m-1$. As $n \rightarrow \infty$, we have $\frac{\log(\text{hom}(S_0; G_n))}{\log(n)} = 1$ and $\frac{\log(\text{hom}(S_{2,1^k}; G_n))}{\log(n)} \rightarrow 0$ for $0 \leq k \leq m-1$, thus showing that $\mathbf{d}_{1,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$.
- (2) To realize $\mathbf{d}_{2,m} = \vec{1}$, consider the graph G consisting of a single edge. Note that $\text{hom}(S_0; G) = \text{hom}(S_{2,1^k}; G) = 2$ for $0 \leq k \leq m-1$. Thus $\frac{\log(\text{hom}(S_0; G))}{\log(2)} = \frac{\log(\text{hom}(S_{2,1^k}; G))}{\log(2)} = 1$ for $0 \leq k \leq m-1$, and so $\mathbf{d}_{2,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$ as desired.
- (3) To realize $\mathbf{d}_{3,m} = (1, 2, 2, 3, 4, \dots, m)$, consider S_n , the star with n branches. Then $\text{hom}(S_0; S_n) = n+1$ and $\text{hom}(S_{2,1^k}; S_n) = n^{k+1} + n^2$ for $0 \leq k \leq m-1$ (where the first term comes from sending the center of $S_{2,1^k}$ to the center of S_n , and the second term comes from sending it to one of the n leaves of S_n). As $n \rightarrow \infty$, we have $\frac{\log(\text{hom}(S_0; S_n))}{\log(n)} \rightarrow 1$, $\frac{\log(\text{hom}(S_{2,1^0}; S_n))}{\log(n)} \rightarrow 2$, and $\frac{\log(\text{hom}(S_{2,1^k}; S_n))}{\log(n)} \rightarrow k+1$ for $1 \leq k \leq m-1$, thus showing that $\mathbf{d}_{3,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$.
- (4) To realize $\mathbf{d}_{4,m} = (1, 3, 4, 5, \dots, m+2)$, consider the complete bipartite graph $K_{n,n}$. Then we have $\text{hom}(S_0; K_{n,n}) = 2n$ and $\text{hom}(S_{2,1^k}; K_{n,n}) = 2n \cdot n^{k+1} \cdot n$ for $0 \leq k \leq m-1$. As $n \rightarrow \infty$, we have $\frac{\log(\text{hom}(S_0; K_{n,n}))}{\log(n)} \rightarrow 1$ and $\frac{\log(\text{hom}(S_{2,1^k}; K_{n,n}))}{\log(n)} \rightarrow k+3$ for $0 \leq k \leq m-1$, thus showing that $\mathbf{d}_{4,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$.
- (5) To realize $\mathbf{d}_{5,m} = (2, 4, 5, 7, \dots, 2m+1)$, let G_{2n^2+1} be the tripartite graph with parts of size 1, n^2 , and n^2 and where there are n^2 edges between the first and second parts, n^3 edges between the second and third parts, and no edges between the first and third parts; i.e., this is the blow-up graph of a path of length two. Furthermore, the degree of every vertex in the second part is $n+1$ and the degree of every vertex in the third part is n . Then we have that $\text{hom}(S_0; G_{2n^2+1}) = 2n^2 + 1$, $\text{hom}(S_{2,1^0}; G_{2n^2+1}) = (n^2)^2 +$

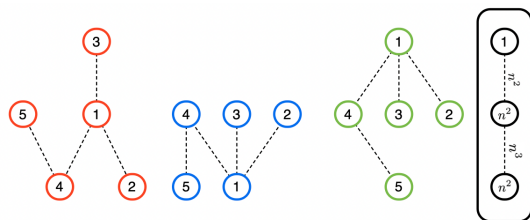


FIG. 3. Some possible ways of sending $S_{2,1^2}$ to G_{2n^2+1} for case (5) of Lemma 4.2.

$n^2 \cdot (n+1)^2 + n^2 \cdot n^2$, and $\text{hom}(S_{2,1^k}; G_{2n^2+1}) = (n^2)^{k+1}(n+1) + n^2(n+1)^{k+1}(n^2+n) + n^2n^{k+1}(n+1)$ for $1 \leq k \leq m-1$, where the different terms come from sending the center of $S_{2,1^k}$ to the different parts of G_{2n^2+1} (see Figure 3 for some examples). As $n \rightarrow \infty$, we have $\frac{\log(\text{hom}(S_0; G_{2n^2+1}))}{\log(n)} \rightarrow 2$, $\frac{\log(\text{hom}(S_{2,1^0}; G_{2n^2+1}))}{\log(n)} \rightarrow 4$, and $\frac{\log(\text{hom}(S_{2,1^k}; G_{2n^2+1}))}{\log(n)} \rightarrow 2k+3$ for $1 \leq k \leq m-1$, thus showing that $\mathbf{d}_{5,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$. $\square$

We now define additional families of rays. In Theorem 5.1, we will prove that these consist of the remaining extreme rays of $\text{trop}(\mathcal{N}_{\mathcal{U}})$.

DEFINITION 4.3. Fix $m \in \mathbb{N}$, and let

- $\mathbf{s}_{1,m} := (3, 6, 8, 10, 12, 15, \dots, 3(m-1), 3m)$,
- $\mathbf{s}_{2,m} := (6, 12, 16, 21, 27, 33, \dots, 6m-3, 6m+3)$,
- $\mathbf{s}_{3,m} := (2, 4, 5, 6, 8, 10, \dots, 2m)$,
- $\mathbf{s}_{4,m} := (3, 6, 8, 10, 13, 16, \dots, 3m+1)$,
- $\mathbf{r}_{1,m,i} := (6i-15, 12i-30, 16i-40, 21i-54, 27i-72, 33i-90, \dots, 6i^2-15i, 6i^2-9i-15, 6i^2-3i-30, \dots, (6i-15)m)$ for $5 \leq i \leq m-1$,
- $\mathbf{r}_{2,m,i} := (i+2, 3i, 4i, 5i, \dots, i(i+2), (i+1)(i+2), \dots, (m-1)(i+2), m(i+2))$ for $5 \leq i \leq m-1$,
- $\mathbf{r}_{3,m,i} := (3i-9, 6i-18, 8i-24, 10i-30, 13i-40, 16i-50, \dots, 3i^2-12i+10, 3i^2-9i, 3i^2-6i-9, \dots, (3i-9)m)$ for $5 \leq i \leq m-1$,
- $\mathbf{r}_{4,m,i} := (2i-5, 4i-10, 5i-12, 7i-18, 9i-24, \dots, 2i^2-7i+6, 2i^2-5i, 2i^2-3i-5, \dots, m(2i-5))$ for $5 \leq i \leq m-1$,
- $\mathbf{r}_{5,m,i} := (i, i, \dots, i, i+1, i+2, \dots, m)$ for $3 \leq i \leq m-1$, and
- $\mathbf{r}_{6,m,i} := (i+1, 3i-1, 4i-2, 5i-3, \dots, i^2+1, i^2+i, i^2+2i+1, \dots, (m-1)(i+1), m(i+1))$ for $4 \leq i \leq m-1$.

For now, we only show that all of these rays are also in $\text{trop}(\mathcal{N}_{\mathcal{U}})$.

LEMMA 4.4. The rays $\mathbf{s}_{j,m}$ and $\mathbf{r}_{l,m,i}$ are in $\text{trop}(\mathcal{N}_{\mathcal{U}})$ for $1 \leq j \leq 4$ and $1 \leq l \leq 6$.

Proof. Instead of finding explicit graphs to realize these rays as in the previous proof, we write them by taking tropical sums (i.e., by taking the maximum componentwise) of conical combinations of the rays $\mathbf{d}_{j,m}$ for $1 \leq j \leq 5$. Since we have already shown that $\mathbf{d}_{j,m} \in \text{trop}(\mathcal{N}_{\mathcal{U}})$ for $1 \leq j \leq 5$ in Lemma 4.2 and since $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is a max-closed closed convex cone, this shows that these rays are also in $\text{trop}(\mathcal{N}_{\mathcal{U}})$. Note that section 2.1 explains how one could use this to come up with explicit graphs to realize these rays.

- $\mathbf{s}_{1,m} = (3, 6, 8, 10, 12, 15, \dots, 3(m-1), 3m) = 3\mathbf{d}_{3,m} \oplus 2\mathbf{d}_{4,m}$.
- $\mathbf{s}_{2,m} = (6, 12, 16, 21, 27, 33, \dots, 6m-3, 6m+3) = 3\mathbf{d}_{5,m} \oplus 4\mathbf{d}_{4,m}$.
- $\mathbf{s}_{3,m} = (2, 4, 5, 6, 8, 10, \dots, 2m) = 2\mathbf{d}_{3,m} \oplus \mathbf{d}_{5,m}$.

- $\mathbf{s}_{4,m} = (3, 6, 8, 10, 13, 16, \dots, 3m+1) = (\mathbf{d}_{3,m} + \mathbf{d}_{5,m}) \oplus 2\mathbf{d}_{4,m}$.
- $\mathbf{r}_{1,m,i} = (6i-15, 12i-30, 16i-40, 21i-54, 27i-72, 33i-90, \dots, 6i^2-15i, 6i^2-9i-15, 6i^2-3i-30, \dots, (6i-15)m) = (4i-10)\mathbf{d}_{4,m} \oplus (6\mathbf{d}_{4,m} + 3(i-4)\mathbf{d}_{5,m}) \oplus (6i-15)\mathbf{d}_{3,m}$ for $5 \leq i \leq m-1$.
- $\mathbf{r}_{2,m,i} = (i+2, 3i, 4i, 5i, \dots, i(i+2), (i+1)(i+2), \dots, (m-1)(i+2), m(i+2)) = i\mathbf{d}_{4,m} \oplus (i+2)\mathbf{d}_{3,m}$ for $5 \leq i \leq m-1$.
- $\mathbf{r}_{3,m,i} = (3i-9, 6i-18, 8i-24, 10i-30, 13i-40, 16i-50, \dots, 3i^2-12i+10, 3i^2-9i, 3i^2-6i-9, \dots, (3i-9)m) = (\mathbf{d}_{2,m} + (i-5)\mathbf{d}_{3,m} + \mathbf{d}_{4,m} + (i-3)\mathbf{d}_{5,m}) \oplus 2(i-3)\mathbf{d}_{4,m} \oplus (3i-9)\mathbf{d}_{3,m}$ for $5 \leq i \leq m-1$.
- $\mathbf{r}_{4,m,i} = (2i-5, 4i-10, 5i-12, 7i-18, 9i-24, \dots, 2i^2-7i+6, 2i^2-5i, 2i^2-3i-5, \dots, m(2i-5)) = (2\mathbf{d}_{4,m} + (i-4)\mathbf{d}_{5,m}) \oplus (2i-5)\mathbf{d}_{3,m}$ for $5 \leq i \leq m-1$.
- $\mathbf{r}_{5,m,i} = (i, i, \dots, i, i+1, i+2, \dots, m) = i\mathbf{d}_{2,m} \oplus \mathbf{d}_{3,m}$ for $3 \leq i \leq m-1$.
- $\mathbf{r}_{6,m,i} = (i+1, 3i-1, 4i-2, 5i-3, \dots, i^2+1, i^2+i, i^2+2i+1, \dots, (m-1)(i+1), m(i+1)) = (2\mathbf{d}_{2,m} + (i-1)\mathbf{d}_{4,m}) \oplus (i+1)\mathbf{d}_{3,m}$ for $4 \leq i \leq m-1$. $\square$

5. Main results. In this section, we prove our main results. In Theorem 5.1, we show that $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is equal to the cone Q . We use the inequalities proven in section 3 to show that $\text{trop}(\mathcal{N}_{\mathcal{U}}) \subseteq Q$ and the rays realized in section 4 to show that $Q \subseteq \text{trop}(\mathcal{N}_{\mathcal{U}})$. Most of the proof consists of showing that the rays we realized are actually the extreme rays of Q . This requires some work, as the $(m+1)$ -dimensional cone Q is defined by $m+5$ inequalities. In Corollary 5.2, we restate the theorem in terms of doubly extreme rays.

THEOREM 5.1. Let $\mathcal{U} = \{S_0, S_{2,1^0}, S_{2,1^1}, \dots, S_{2,1^{m-1}}\}$, and let

$$Q = \left\{ \mathbf{y} \in \mathbb{R}^{m+1} \left| \begin{array}{l} -y_1 + y_2 \geq 0 \\ 4y_1 - 3y_2 \geq 0 \\ 3y_1 - 3y_3 + y_4 \geq 0 \\ y_1 + 2y_{m-1} - 2y_m \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_0 - 2y_1 + y_3 \geq 0 \\ y_{i-1} - 2y_i + y_{i+1} \geq 0 \quad \forall 2 \leq i \leq m-1 \\ m \cdot y_{m-1} - (m-1) \cdot y_m \geq 0 \end{array} \right. \right\}.$$

Then $\text{trop}(\mathcal{N}_{\mathcal{U}}) = Q$.

Proof. By taking the log of the inequalities proven to be valid for $\mathcal{N}_{\mathcal{U}}$ in section 3, we know that $\text{trop}(\mathcal{N}_{\mathcal{U}}) \subseteq Q$.

To show that $Q \subseteq \text{trop}(\mathcal{N}_{\mathcal{U}})$, we compute the extreme rays of Q and show that they are all present in $\text{trop}(\mathcal{N}_{\mathcal{U}})$. Any extreme ray must be tight with m linearly independent inequalities and respect all other inequalities. Note that there are $m-2$ inequalities of type $y_{i-1} - 2y_i + y_{i+1} \geq 0$ (which we call *gap inequalities*) for $2 \leq i \leq m-1$. In particular, observe that we will never discuss the gap between y_0 and y_1 . Moreover, there are seven other inequalities (which we call *non-gap inequalities*) for a total of $m+5$ inequalities. So there could potentially be $\binom{m+5}{m}$ extreme rays. However, many of these intersections are not feasible. To better understand which of these $\binom{m+5}{m}$ intersections we need to consider, it is useful to first look at the projection of $\text{trop}(\mathcal{N}_{\mathcal{U}})$ onto coordinates $y_0, y_1, y_2, y_3, y_4, y_{m-1}, y_m$ (i.e., the coordinates in the seven non-gap inequalities) which we call C . Computing C is nontrivial for a general m , so we end up giving a larger set C' that contains C (as the defining inequalities of C' are all valid for C) and which sufficiently restricts our search of extreme rays.

We first prove a few valid inequalities for C which we use to define C' . Observe that $y_i \leq (i+2)y_0$ for all $i \in [m]$ since $S_{2,1^{i-1}}$ has $i+2$ vertices. Moreover, from

the gap inequalities, we know the gap $y_{i+1} - y_i$ is greater than or equal to the gap $y_i - y_{i-1}$ for $2 \leq i \leq m-1$, i.e., gaps are nondecreasing, and so this implies that $y_{m-1} \geq y_3 + (m-4)(y_4 - y_3)$, $y_m \geq y_3 + (m-3)(y_4 - y_3)$, and $y_3 - y_4 - y_{m-1} + y_m \geq 0$. Also, the conical combination of gap inequalities $\sum_{i=5}^{m-1} (i-4)(y_{i-1} - 2y_i + y_{i+1} \geq 0)$ yields the inequality $y_4 - (m-4)y_{m-1} + (m-5)y_m \geq 0$. Finally, the conical combination $(-y_1 + y_2 \geq 0) + \sum_{i=2}^j (y_{i-1} - 2y_i + y_{i+1} \geq 0)$ yields $-y_j + y_{j+1} \geq 0$, which implies that y_1 through y_m are nondecreasing. Therefore, we let

$$C' := \left\{ \mathbf{y} \in \mathbb{R}^7 \mid \begin{array}{l} -y_1 + y_2 \geq 0 \\ 4y_1 - 3y_2 \geq 0 \\ 3y_1 - 3y_3 + y_4 \geq 0 \\ y_1 + 2y_{m-1} - 2y_m \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_0 - 2y_1 + y_3 \geq 0 \\ m \cdot y_{m-1} - (m-1) \cdot y_m \geq 0 \\ y_1 - 2y_2 + y_3 \geq 0 \\ y_2 - 2y_3 + y_4 \geq 0 \\ 3y_0 - y_1 \geq 0 \\ 4y_0 - y_2 \geq 0 \\ 5y_0 - y_3 \geq 0 \\ 6y_0 - y_4 \geq 0 \\ (m+1)y_0 - y_{m-1} \geq 0 \\ (m+2)y_0 - y_m \geq 0 \\ (m-5)y_3 - (m-4)y_4 + y_{m-1} \geq 0 \\ (m-4)y_3 - (m-3)y_4 + y_m \geq 0 \\ y_3 - y_4 - y_{m-1} + y_m \geq 0 \\ y_4 - (m-4)y_{m-1} + (m-5)y_m \geq 0 \\ -y_2 + y_3 \geq 0 \\ -y_3 + y_4 \geq 0 \\ -y_4 + y_{m-1} \geq 0 \\ -y_{m-1} + y_m \geq 0 \end{array} \right\} \supseteq C.$$

We can compute all the faces of C' (for example, in Sage or Polymake) to see which of the non-gap inequalities can be tight simultaneously. If the intersection of the hyperplanes corresponding to some of the non-gap inequalities with C' yields $\mathbf{0}$, then we know their intersection with C will also yield $\mathbf{0}$. Moreover, even if a particular set of k of the first seven inequalities (our original non-gap inequalities) does not yield $\mathbf{0}$ in C' , we know that extreme rays in C will require $m-k$ of the gap inequalities to be tight. In particular, if $k = 4$ (respectively, $k = 3$ and $k = 2$), then only two (respectively, one and zero) gap inequalities aren't tight, and so at least one (respectively, two and three) of $y_1 - 2y_2 + y_3 \geq 0$, $y_2 - 2y_3 + y_4 \geq 0$, and $(m-5)y_3 - (m-4)y_4 + y_{m-1} \geq 0$ must be tight in order for there to potentially be an extreme ray in C where these non-gap inequalities are tight. Taking all of this into consideration, Table 1 gives the set of non-gap inequalities that can potentially be tight with extreme rays of C . For each set, we find which extreme rays can be recovered.

S_1 : From the tight inequalities in this set, we have that $y_1 = y_2 = 0$ and $y_{m-1} = y_m$. Since gaps are nondecreasing, this means that all gaps are 0, and so this yields the extreme ray $\mathbf{d}_{1,m} = (1, 0, \dots, 0)$ which is tight with all $m-2$ gap inequalities.

S_2 : From the tight inequalities in this set, we have that $y_0 = a$, $y_1 = y_2 = 2a$, $y_3 = 3a$, $y_{m-1} = (m-1)a$, $y_m = ma$ for some $a \geq 0$. Since the gap between y_2 and y_3 is the same as the gap as between y_{m-1} and y_m and gaps are nondecreasing, this

TABLE 1
Cases of non-gap inequalities that can be tight simultaneously.

	$-y_1 + y_2 \geq 0$	$4y_1 - 3y_2 \geq 0$	$3y_1 - 3y_3 + y_1 + 2y_{m-1} - y_0 + y_{m-1} - y_0 - 2y_1 + y_3 \geq 0$	$y_4 \geq 0$	$2y_m \geq 0$	$y_m \geq 0$	$m \cdot y_{m-1} - (m-1) \cdot y_m \geq 0$
S_1	*	*	*	*	*	*	*
S_2	*			*	*	*	*
S_3		*	*	*	*		*
S_4		*	*	*	*		
S_5		*	*		*	*	
S_6		*	*		*		*
S_7		*		*	*		*
S_8			*	*	*		*
S_9				*	*	*	*
S_{10}	*				*	*	*
S_{11}		*		*	*		
S_{12}			*	*	*		
S_{13}					*	*	*
S_{14}	*					*	

yields the extreme ray $\mathbf{d}_{3,m} = (1, 2, 2, 3, 4, 5, \dots, m-1, m)$ which is tight with $m-3$ of the gap inequalities.

S_3 : From the tight inequalities in this set, we have that $y_0 = 3a$, $y_1 = 6a$, $y_2 = 8a$, $y_{m-1} = 3(m-1)a$, $y_m = 3ma$ for some $a \geq 0$. We know there are at most three gap inequalities that are not additionally tight. Two cases arise: either $y_1 - 2y_2 + y_3 \geq 0$ is tight or it is not.

In the first case, we then have $y_3 = 10a$, which in turn implies that $y_4 = 12a$ since $3y_1 - 3y_3 + y_4 = 0$. Note that the average gap between y_4 and y_m has size $\frac{3ma-12a}{m-4} = 3a$. Since the gap from y_{m-1} to y_m is $3a$ and gaps are nondecreasing, all gaps between y_4 and y_m must have size $3a$. This yields the extreme ray $\mathbf{s}_{1,m} = (3, 6, 8, 10, 12, 15, \dots, 3(m-1), 3m)$ which is tight with $m-3$ of the gap inequalities.

In the second case when $y_1 - 2y_2 + y_3 > 0$, let $y_3 = 10a + b$ for some $b > 0$. Then, since $3y_1 - 3y_3 + y_4 = 0$, $y_4 = 12a + 3b$, then $y_2 - 2y_3 + y_4 > 0$ as well. We know that gaps are nondecreasing, so the gap between y_3 and y_4 is at most the gap between y_{m-1} and y_m , which means that $2b \leq a$. Note that we cannot have $2b = a$ since then every gap from y_3 to y_m would be $3a$, and so $y_m = 10a + b + (m-3) \cdot 3a \neq 3ma$. So $2b < a$ and $y_4 - y_3 < 3a$, and since the gap between y_{m-1} and y_m is $3a$, there is one last gap inequality that isn't tight, say $y_{i-1} - 2y_i + y_{i+1} > 0$ for some $4 \leq i \leq m-1$. So $y_i = 10a + b + (i-3)(2a+2b)$ and $y_{i+1} = 10a + b + (i-3)(2a+2b) + 3a$, and we also know since all gaps thereafter are $3a$ and $y_m = 3ma$, $y_{i+1} = (i+1)3a$. Thus, $b = \frac{(i-4)a}{2i-5}$. This yields the family of rays $\mathbf{r}_{1,m,i} = (6i-15, 12i-30, 16i-40, 21i-54, 27i-72, 33i-90, \dots, 6i^2-15i, 6i^2-9i-15, 6i^2-3i-30, \dots, (6i-15)m)$ which are tight with $m-5$ of the gap inequalities. For example, for $m=7$, we have $\mathbf{r}_{1,7,5} = (15, 30, 40, 51, 63, 75, 90, 105)$ and $\mathbf{r}_{1,7,6} = (21, 42, 56, 72, 90, 108, 126, 147)$. Finally, observe that we do not include $i=4$ in this family, as this would make the inequality $y_1 - 2y_2 + y_3 = 0$ and bring us back to $\mathbf{s}_{1,m}$.

S_4 : From the tight inequalities in this set, we have that $y_0 = 6a$, $y_1 = 12a$, $y_2 = 16a$, $y_3 = c$, $y_4 = 3c - 36a$, $y_{m-1} = 2b$, and $y_m = 6a + 2b$ for some $a, b \geq 0$. Furthermore, we know that at most two gap inequalities aren't tight. Again, two cases arise: either $y_1 - 2y_2 + y_3 \geq 0$ is tight or it is not.

In the first case, we get that $c = 20a$, which implies that $y_4 = 24a$. The average gap between y_4 and y_m must thus be $\frac{6a+2b-24a}{m-4} = \frac{2b-18a}{m-4}$. However, we know that

$my_{m-1} - (m-1)y_m > 0$, which is equivalent to $b > 3a(m-1)$. Thus, the average gap is bigger than $6a$, which is a contradiction since the largest gap is $6a$.

In the second case when $y_1 - 2y_2 + y_3 > 0$, let $y_3 = c = 20a + d$ for some $d > 0$. Then $y_4 = 24a + 3d$. Further, note that $y_2 - y_1 = 4a$, $y_3 - y_2 = 4a + d$, $y_4 - y_3 = 4a + 2d$ (which are all distinct), and $y_m - y_{m-1} = 6a$ and since at most two gap inequalities aren't tight, we must have that $4a + 2d = 6a$ and thus that $d = a$. We thus get the ray $\mathbf{s}_{2,m} = (6, 12, 16, 21, 27, 33, \dots, 6m - 3, 6m + 3)$ which is tight with $m - 4$ of the gap inequalities.

$\mathcal{S}_5$: From the tight inequalities in this set, we have that $y_0 = b$, $y_1 = 3a$, $y_2 = 4a$, $y_3 = 6a - b$, $y_4 = 9a - 3b$, $y_{m-1} = c$, and $y_m = b + c$ for some $a, b, c \geq 0$. Since gaps are nondecreasing, we have that $a \leq 2a - b \leq 3a - 2b \leq b$, which implies that $a = b$ and that all gaps are a . This thus yields the ray $\mathbf{d}_{4,m} := (1, 3, 4, 5, 6, \dots, m + 2)$ which is tight with all gap inequalities.

$\mathcal{S}_6$: From the tight inequalities in this set, we have that $y_0 = b$, $y_1 = 3a$, $y_2 = 4a$, $y_3 = c$, $y_4 = 3c - 9a$, $y_{m-1} = (m-1)b$, $y_m = mb$ for some $a, b, c \geq 0$. Since gaps are nondecreasing, we have that $a \leq c - 4a \leq 2c - 9a \leq b$, but we know that at most two gap inequalities aren't tight, so not all of these can be distinct.

If $a = c - 4a$, then $c = 5a$ and $c - 4a = 2c - 9a$ as well. Suppose there are two gap inequalities that aren't tight, say $y_{i-1} - 2y_i + y_{i+1} > 0$ and $y_{j-1} - 2y_j + y_{j+1} > 0$ with $4 \leq i < j \leq m-1$ (since we have assumed the inequalities for $i \in \{2, 3\}$ are tight); then the $m - i + 1$ variables $y_0, y_{i+1}, y_{i+2}, \dots, y_m$ would be involved in $m - i - 2$ tight gap inequalities and the tight inequality $y_0 + y_{m-1} - y_m = 0$, and therefore this cannot result in a ray. So we know that in this case, we can assume that there is at most one gap inequality that isn't tight, say $y_{i-1} - 2y_i + y_{i+1} > 0$ for some $4 \leq i \leq m-1$. Then $y_i = (i+2)a = ib$, which implies that $b = \frac{i+2}{i}a$. This yields the family of rays $\mathbf{r}_{2,m,i} = (i+2, 3i, 4i, 5i, \dots, i(i+2), (i+1)(i+2), \dots, (m-1)(i+2), m(i+2))$ which are tight with $m-3$ of the gap inequalities. For example, for $m=7$, we have $\mathbf{r}_{2,7,5} = (7, 15, 20, 25, 30, 35, 42, 49)$, $\mathbf{r}_{2,7,6} = (8, 18, 24, 30, 36, 42, 48, 56)$. Finally, observe that we do not include $i=4$ in this family, as this would make the inequality $y_1 + 2y_{m-1} - 2y_m \geq 0$ tight.

Now let's consider the case when $a < c - 4a$, say $c = 5a + d$, which in turn implies that $c - 4a < 2c - 9a$, and since at most two gap inequalities aren't tight, we must have that $2c - 9a = b = a + 2d$. We thus have that $y_m = m(a + 2d) = y_3 + (m-3)(a + 2d) = 5a + d + (m-3)(a + 2d)$, and so $a = \frac{5d}{2}$. In turn, this gives that $y_1 = \frac{15d}{2}$, $y_m = m\frac{9d}{2}$, and $y_{m-1} = (m-1)\frac{9d}{2}$, and therefore $y_1 + 2y_{m-1} - 2y_m < 0$, a contradiction.

$\mathcal{S}_7$: From the tight inequalities in this set, we have that $y_0 = 3a$, $y_1 = 6a$, $y_2 = 8a$, $y_3 = b$, $y_4 = c$, $y_{m-1} = 3(m-1)a$, and $y_m = 3ma$ for some $a, b, c \geq 0$. Suppose that $y_1 - 2y_2 + 2y_3 = 0$. In that case, $b = 10a$, and since $3y_1 - 3y_3 + y_4 > 0$, let $c = 12a + d$ for some $d > 0$. We know there are at most two gap inequalities that aren't tight, and the three gaps present are $2a \leq 2a + d \leq 3a$, which are all distinct since $d > 0$ and since if $2a + d = 3a$, then y_m would be equal to $10a + (m-3)3a \neq 3ma$. We already know that $y_2 - 2y_3 + y_4 > 0$.

Suppose $y_{i-1} - 2y_i + y_{i+1} > 0$ as well for some $5 \leq i \leq m-1$. Then $y_i = 3ia = 10a + (i-3)(2a + d)$, which implies that $d = \frac{i-4}{i-3}a$. This thus yields the family of rays $\mathbf{r}_{3,m,i} = (3i-9, 6i-18, 8i-24, 10i-30, 13i-40, 16i-50, \dots, 3i^2-12i+10, 3i^2-9i, 3i^2-6i-9, \dots, (3i-9)m)$ which are tight with $m-4$ gap inequalities. For example, $\mathbf{r}_{3,7,5} = (6, 12, 16, 20, 25, 30, 36, 42)$ and $\mathbf{r}_{3,7,6} = (9, 18, 24, 30, 38, 46, 54, 63)$. Finally, observe that we do not include $i=4$ in this family, as this would make the inequality $3y_1 - 3y_3 + y_4 \geq 0$ tight.

Now suppose that $y_1 - 2y_2 + 2y_3 > 0$ and let $y_3 = b = 10a + d$. Since $3y_1 - 3y_3 + y_4 > 0$, let $y_4 = c = 12a + 3d + f$ for some $f > 0$. There are thus four gaps present, namely $2a < 2a + d < 2a + 2d + f \leq 3a$, and since at most two gap inequalities are not tight, we must have $2a + 2d + f = 3a$, which implies that $y_4 = y_m - (m - 4)3a = 12a \neq 12a + 3d + f$, a contradiction.

S_8 : From the tight inequalities in this set, we have that $y_0 = a$, $y_1 = 2a$, $y_2 = c$, $y_3 = b$, $y_4 = 3b - 6a$, $y_{m-1} = (m - 1)a$, $y_m = ma$ for some $a, b, c \geq 0$.

Suppose that $y_1 - 2y_2 + y_3 = 0$, and let $y_2 - y_1 = d$ for some $d > 0$ (since $y_2 - y_1 > 0$), which implies that $y_2 = 2a + d$, $y_3 = 2a + 2d$, $y_4 = 6d$. Since $4y_1 - 3y_2 > 0$, we have that $d < \frac{2a}{3}$. Moreover, since gaps are nondecreasing, we know that $y_4 - y_3 \geq d$, which implies that $d \geq \frac{2a}{3}$, a contradiction.

Now suppose that $y_1 - 2y_2 + y_3 > 0$, and let $y_2 = 2a + d$ and $y_3 = 2a + 2d + f$ for some $d, f > 0$, which implies that $y_4 = 6d + 3f$. There are thus four gaps present, namely $d < d + f \leq 4d + 2f - 2a \leq a$, and since at most two gap inequalities are not tight, we need to have either that $4d + 2f - 2a = a$ or $d + f = 4d + 2f - 2a$.

If $4d + 2f - 2a = a$, then $y_m = ma = 2a + 2d + f + a(m - 3)$, which implies that $6a = 12d + 6f$, contradicting $4d + 2f - 2a = a$.

If $d + f = 4d + 2f - 2a$ and the change from gaps of size $d + f = 4d + 2f - 2a$ to gaps of size a happens at position $4 \leq i \leq m - 1$, i.e., $y_{i-1} - 2y_i + y_{i+1} > 0$, then $y_i = (i + 2)d + (i - 1)f = i\frac{3d+f}{2}$, which implies that $f = \frac{i-4}{i-2}d$. This thus yields the family of rays $\mathbf{r}_{4,m,i} = (2i - 5, 4i - 10, 5i - 12, 7i - 18, 9i - 24, \dots, 2i^2 - 7i + 6, 2i^2 - 5i, 2i^2 - 3i - 5, \dots, m(2i - 5))$ which are tight with $m - 4$ gap inequalities. For example, $\mathbf{r}_{4,7,5} = (5, 10, 13, 17, 21, 25, 30, 35)$ and $\mathbf{r}_{4,7,6} = (7, 14, 18, 24, 30, 36, 42, 49)$. Finally, observe that we do not include $i = 4$ in this family, as this would make the inequality $4y_1 - 3y_2 \geq 0$ tight.

S_9 : From the tight inequalities in this set, we have that $y_0 = a$, $y_1 = 2a$, $y_2 = b$, $y_3 = 3a$, $y_4 = 4a$, $y_{m-1} = (m - 1)a$, $y_m = ma$ for some $a, b \geq 0$. Since we know that gaps are nondecreasing, we know that $y_i = ia$ for $3 \leq i \leq m$.

If $y_1 - 2y_2 + y_3 = 0$, then $y_2 = b = \frac{5}{2}a$, and so the only gap inequality that isn't tight is $y_2 - 2y_3 + y_4 \geq 0$. This yields the ray $\mathbf{s}_{3,m} = (2, 4, 5, 6, 8, 10, \dots, 2m)$.

If $y_2 - 2y_3 + y_4 = 0$, then $y_2 = b = 2a$, but then $-y_1 + y_2 = 0$, a contradiction.

Finally, note that we can't have an extreme ray for which both $y_1 - 2y_2 + y_3 > 0$ and $y_2 - 2y_3 + y_4 > 0$ since then no tight inequality would involve y_2 .

S_{10} : From the tight inequalities in this set, we have that $y_0 = b$, $y_1 = a$, $y_2 = a$, $y_3 = 2a - b$, $y_{m-1} = (m - 1)c$, $y_m = mc$ for some $a, b, c \geq 0$. Note that there is at most one gap inequality that isn't tight, and there are three gaps already present: $0 \leq a - b \leq c$, so at least two of those must be equal.

If $a - b = 0$ and the gap switches from 0 to c at position $2 \leq i \leq m - 1$, i.e., $y_{i-1} - 2y_i + y_{i+1} > 0$, this yields the family of rays $\mathbf{r}_{5,m,i} = (i, i, \dots, i, i + 1, i + 2, \dots, m)$. Finally, observe that we do not include $i = 2$ in this family, as this would make the inequality $y_1 + 2y_{m-1} - 2y_m \geq 0$ tight.

If $a - b = c$, then $y_2 = a = 2c$ since each gap after y_2 is c , but then $y_1 + 2y_{m-1} - 2y_m = 0$, a contradiction.

S_{11} : From the tight inequalities in this set, we have that $y_0 = \frac{3a}{2}$, $y_1 = 3a$, $y_2 = 4a$, $y_3 = c$, $y_{m-1} = b - \frac{3a}{2}$, $y_m = b$ for some $a, b \geq 0$. We know that there is at most one gap inequality that isn't tight, say $y_{i-1} - 2y_i + y_{i+1} > 0$.

If $i \geq 4$, then $y_3 = 5a$ and $y_4 = 6a$, which means that $3y_1 - 3y_3 + y_4 = 0$, a contradiction.

If $i = 3$, this yields the ray $\mathbf{s}_{4,m} = (3, 6, 8, 10, 13, 16, \dots, 3m + 1)$.

If $i = 2$, this implies that $y_3 = \frac{11a}{2}$ and $y_4 = 7a$, which implies that $3y_1 - 3y_3 + y_4 < 0$, a contradiction.

S_{12} : From the tight inequalities in this set, we have that $y_0 = a$, $y_1 = 2a$, $y_3 = c$, $y_4 = 3c - 6a$, $y_{m-1} = b - a$, $y_m = b$ for some $a, b, c \geq 0$. We know that there is at most one gap inequality that isn't tight, so the gap between y_3 and y_4 , $2c - 6a$ must be either equal to the gap between y_{m-1} and y_m , namely a , or to the gap between y_2 and y_3 .

If $y_4 - y_3 = a$, then $y_4 = c + a = 3c - 6a$, which implies that $a = \frac{2c}{7}$. Moreover, either $y_1 - 2y_2 + y_3 = 0$ or $y_2 - 2y_3 + y_4 = 0$. In the first case, this implies that $y_2 = \frac{11c}{14}$, but then $4y_1 - 3y_2 < 0$, a contradiction. In the second case, this implies that $y_2 = \frac{9c}{7}$ and yields the extreme ray $\mathbf{d}_{5,m} = (2, 4, 5, 7, \dots, 2m-1, 2m+1)$.

If $y_4 - y_3 = y_3 - y_2$ but $y_4 - y_3 < a$, then we must have that $y_2 - y_1 = y_3 - y_2$ as well since there is at most one gap inequality that isn't tight. So $y_0 = a$, $y_1 = 2a$, $y_2 = 2a + d$, $y_3 = 2a + 2d$, $y_4 = 2a + 3d$ for some $d \geq 0$. Since $3y_1 - 3y_3 + y_4 = 0$, we have that $2a = 3d$, but then $4y_1 - 3y_2 = 0$, a contradiction.

S_{13} : From the tight inequalities in this set, we have that $y_0 = a$, $y_1 = b$, $y_3 = 2b - a$, $y_{m-1} = (m-1)a$, and $y_m = ma$ for some $a, b \geq 0$. We know that at most one gap inequality isn't tight with any extreme ray in this setting, so $y_1 - 2y_2 + y_3 = 0$ or $y_2 - 2y_3 + y_4 = 0$.

If $y_1 - 2y_2 + y_3 = 0$, then $y_2 = \frac{3b-a}{2}$. We already have two gaps present: $\frac{b-a}{2} < a$ (note that they cannot be equal since $4y_1 - 3y_2 > 0$). Suppose that the switch from one gap to the other happens at position i , i.e., $y_{i-1} - 2y_i + y_{i+1} > 0$ for some $3 \leq i \leq m-1$. Since $y_i = 2ia = (i+1)b - (i-1)a$, we have that $b = \frac{(3i-1)a}{i+1}$, which yields the family of rays $\mathbf{r}_{6,m,i} = (i+1, 3i-1, 4i-2, 5i-3, \dots, i^2+1, i^2+i, i^2+2i+1, \dots, (m-1)(i+1), m(i+1))$. For example, we have $\mathbf{r}_{6,7,6} = (7, 17, 22, 27, 32, 37, 42, 49)$, $\mathbf{r}_{6,7,5} = (6, 14, 18, 22, 26, 30, 36, 42)$, and $\mathbf{r}_{6,7,4} = (5, 11, 14, 17, 20, 25, 30, 35)$. Note that we exclude $i = 3$ since in that case $y_1 + 2y_{m-1} - 2y_m = 0$, a contradiction.

If $y_1 - 2y_2 + y_3 > 0$, then $y_2 - 2y_3 + y_4 = 0$ and the gap between y_2 and y_3 must be equal to the gap between y_{m-1} and y_m , namely a , which implies that we have $y_i = ia$ for $2 \leq i \leq m$. Moreover, we also have $y_2 = 2b - 2a$ and $y_4 = 2b$. So $b = 2a$, but then $-y_1 + y_2 = 0$, a contradiction.

S_{14} : From the tight inequalities in this set, we have that $y_1 = a$ and $y_2 = a$ for some $a \geq 0$. Moreover, we know that all gap inequalities must be tight, so $y_i = a$ for $1 \leq i \leq m$. Finally, since $y_0 - 2y_1 + y_3 = 0$, we have that $y_0 = a$. This yields the ray $\mathbf{d}_{2,m} = (1, 1, \dots, 1)$.

Since all the extreme rays produced for Q were shown to be in $\text{trop}(\mathcal{N}_U)$ by Lemmas 4.2 and 4.4, we have that $Q \subseteq \text{trop}(\mathcal{N}_U)$ as desired. $\square$

COROLLARY 5.2. *The set $\text{trop}(\mathcal{N}_U)$ is the double hull of the following five doubly extreme rays in $\mathbb{R}^{m+1}$: $\mathbf{d}_{1,m} := (1, 0, 0, \dots, 0)$, $\mathbf{d}_{2,m} := \vec{1}$, $\mathbf{d}_{3,m} := (1, 2, 2, 3, 4, \dots, m)$, $\mathbf{d}_{4,m} := (1, 3, 4, 5, \dots, m+2)$, $\mathbf{d}_{5,m} := (2, 4, 5, 7, \dots, 2m+1)$.*

Proof. This follows from Lemma 4.4 and Theorem 5.1 and the fact that one can check that none of the doubly extreme rays is in the double hull of the others. $\square$

6. Final remarks. We first note that although we did not extensively use the results of Kopparty and Rossman [KR11] in how we presented our results, they were immensely useful in coming up with the tropicalization. Indeed, we used their linear program to compute valid inequalities and to check whether the doubly extreme rays of the cones generated by those inequalities were realizable. This computational work allowed us to guess what the tropicalization could potentially be before attempting to prove it. Furthermore, we could have written the proofs of the necessary inequalities

and realizations for the rays using the tools of Kopparty and Rossman; however, we elected to present more concise proofs instead. We also remark that our results show similarities to the tropicalization of stars described in [BR22, Theorem 2.17].

THEOREM 6.1. *Let $\mathcal{U} = \{S_0, S_1, \dots, S_m\}$, where S_i is the star graph with i branches. Then*

$$\text{trop}(\mathcal{N}_{\mathcal{U}}) = \left\{ \mathbf{y} \in \mathbb{R}^{m+1} \mid \begin{array}{l} -y_1 + y_2 \geq 0 \\ y_0 + y_{m-1} - y_m \geq 0 \\ y_{i-1} - 2y_i + y_{i+1} \geq 0 \\ m \cdot y_{m-1} - (m-1) \cdot y_m \geq 0 \end{array} \quad \forall 1 \leq i \leq m-1 \right\}.$$

The set $\text{trop}(\mathcal{N}_{\mathcal{U}})$ is the double hull of the following doubly extreme rays: $(1, 0, 0, \dots, 0)$, $(1, 1, \dots, 1)$, $(1, 1, 2, 3, \dots, m)$, and $(1, 2, 3, 4, \dots, m+1)$.

Furthermore, in this paper, we proved Theorem 5.1 by considering all extreme rays. However, it is possible to prove this theorem using only the doubly extreme rays (without finding all extreme rays) by doing some additional casework. This proof will be included in the thesis of the first-named author.

Finally, as mentioned in the introduction, computing the tropicalization for all trees would be highly interesting though probably out of reach at the moment. One family of trees of particular interest is a generalization of $S_{2,1^k}$'s that includes all subdivisions of a star, i.e., S_{λ} in our notation. Note that the tropicalization of paths in [BR22] is much more complicated than the tropicalization of stars. Therefore, given that S_{λ} can contain longer paths in general, computing their tropicalization might be highly nontrivial. Restricting ourselves to partitions of shape $(2, 1^k)$ was a first step in that direction.

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