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K-theoretic Tutte polynomials of morphisms of matroids

Rodica Dinu [a,b,c,*], Christopher Eur [d], Tim Seynnaeve [e]

[a] University of Bucharest, Faculty of Mathematics and Computer Science, Bucharest, Romania
[b] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania
[c] University of Konstanz, Fachbereich Mathematik und Statistik, Konstanz, Germany
[d] Stanford University, Stanford, CA, USA
[e] Universität Bern, Mathematisches Institut, Alpeneggstrasse 22, 3012 Bern, Switzerland

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We generalize the Tutte polynomial of a matroid to a morphism of matroids via the K-theory of flag varieties. We introduce two different generalizations, and demonstrate that each has its own merits, where the trade-off is between the ease of combinatorics and geometry. One generalization recovers the Las Vergnas Tutte polynomial of a morphism of matroids, which admits a corank-nullity formula and a deletion-contraction recursion. The other generalization does not, but better reflects the geometry of flag varieties.

1. Introduction

Matroids are combinatorial abstractions of hyperplane arrangements that have been fruitful grounds for interactions between algebraic geometry and combinatorics. One interaction concerns the Tutte polynomial of a matroid, an invariant first defined for graphs by Tutte [40] and then for matroids by Crapo [11].

* Corresponding author.
E-mail addresses: rdinu@fmi.unibuc.ro (R. Dinu), chriseur@stanford.edu (C. Eur), tim.seynnaeve@math.unibe.ch (T. Seynnaeve).

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Definition 1.1. Let M be a matroid of rank r on a finite set $[n] = \{1, 2, \dots, n\}$ with the rank function $\mathrm{rk}_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}$. Its **Tutte polynomial** $T_M(x, y)$ is a bivariate polynomial in x, y defined by

$$T_M(x,y) := \sum_{S \subseteq [n]} (x-1)^{r-\mathrm{rk}_M(S)} (y-1)^{|S|-\mathrm{rk}_M(S)}.$$

An algebro-geometric interpretation of the Tutte polynomial was given in [16] via the K-theory of the Grassmannian. Let $Gr(r; n)$ be the Grassmannian of r-dimensional linear subspaces in $\mathbb{C}^n$, and more generally let $Fl(\mathbf{r}; n)$ be the flag variety of flags of linear spaces of dimensions $\mathbf{r} = (r_1, \dots, r_k)$. The torus $T = (\mathbb{C}^*)^n$ acts on $Gr(r; n)$ and $Fl(\mathbf{r}; n)$ by its standard action on $\mathbb{C}^n$. A point $L \in Gr(r; n)$ on the Grassmannian corresponds to a realization of a matroid, and its torus-orbit closure defines a K-class $[\mathcal{O}_{\overline{T \cdot L}}] \in K^0(Gr(r; n))$ that depends only on the matroid. In general, a matroid M of rank r on $\{1, \dots, n\}$ defines a K-class $y(M) \in K^0(Gr(r; n))$. Fink and Speyer related $y(M)$ to the Tutte polynomial $T_M(x, y)$ via the diagram

$$\begin{tikzcd} & Fl(1,r,n-1;n) \arrow[dl, "\pi_r"'] \arrow[dr, "\pi_{(n-1)1}"] & & \\ Gr(r;n) & & Gr(n-1;n) \times Gr(1;n) \arrow[r, equal] & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}, \end{tikzcd} \tag{1}$$

where π_r and $\pi_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.

Theorem 1.2. *[16, Theorem 5.1] Let $\mathcal{O}(1)$ be the line bundle on $Gr(r; n)$ of the Plücker embedding $Gr(r; n) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$. With notations as above, we have*

$$T_M(\alpha, \beta) = (\pi_{(n-1)1})_* \pi_r^* \Big(y(M) \cdot [\mathcal{O}(1)] \Big) \in K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \simeq \mathbb{Q}[\alpha, \beta]/(\alpha^n, \beta^n),$$

where α, β are the K-classes of the structures sheaves of the hyperplanes of $(\mathbb{P}^{n-1})^\vee, \mathbb{P}^{n-1}$.

We extend this relation between matroids and the K-theory of Grassmannians to a relation between flag matroids and the K-theory of flag varieties. As a result, we show that there are (at least) *two* different generalizations of the Tutte polynomial to flag matroids, each with its own merits.

A **flag matroid** is a sequence of matroids $\mathbf{M} = (M_1, \dots, M_k)$ on a common ground set such that every circuit of M_i is a union of circuits of M_{i-1} for all $2 \leq i \leq k$. The **rank** of $\mathbf{M}$ is the sequence $(\mathrm{rk}(M_1), \dots, \mathrm{rk}(M_k))$. Most of this paper will concern the case of $k = 2$. In this case, the two-step flag matroids (M_1, M_2) are often called **matroid morphisms** or **matroid quotients**. They are combinatorial abstractions of graph homomorphisms, linear surjections, and embeddings of graphs on surfaces. See §2.1 for details on flag matroids and matroid quotients.

Many features of matroids naturally generalize to flag matroids. For instance, just as a point on a Grassmannian corresponds to a realization of a matroid, a point $\mathbf{L}$ on the flag variety $Fl(\mathbf{r};n)$ corresponds to a realization of a flag matroid. The torus-orbit closure of $\mathbf{L}$ defines a K-class $[\mathcal{O}_{\overline{T\cdot \mathbf{L}}}] \in K^0(Fl(\mathbf{r};n))$ that depends only on the flag matroid. In general, a flag matroid $\mathbf{M}$ of rank $\mathbf{r}$ on a ground set $\{1,\dots,n\}$ defines a K-class $y(\mathbf{M})$ of the flag variety $Fl(\mathbf{r};n)$. See §2.3 or [8, §8.5] for details.

At this point, however, extending the constructions on matroids to flag matroids splits into several strands, for there are (at least) two distinguished ways to generalize the diagram (1).

- The "flag-geometric" diagram:

$$\begin{tikzcd} & Fl(1,\mathbf{r},n-1;n) \arrow[dl, "\pi_{\mathbf{r}}"'] \arrow[dr, "\pi_{(n-1)1}"] & \\ Fl(\mathbf{r};n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1} \end{tikzcd} \tag{Fl}$$

  where $\pi_{\mathbf{r}}$ and $\pi_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.
- The "Las Vergnas" diagram:

$$\begin{tikzcd} & \widetilde{Fl}(1,\mathbf{r},n-1;n) \arrow[dl, "\widetilde{\pi}_{\mathbf{r}}"'] \arrow[dr, "\widetilde{\pi}_{(n-1)1}"] & \\ Fl(\mathbf{r};n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1} \end{tikzcd} \tag{$\widetilde{Fl}$}$$

  where $\widetilde{Fl}(1,\mathbf{r},n-1;n)$ is the variety defined as

$$\widetilde{Fl}(1,\mathbf{r},n-1;n) \\ := \left\{ \begin{array}{c} \text{linear subspaces} \\ (\ell, L_1,\dots,L_k,H) \end{array} \;\middle|\; \begin{array}{c} \dim \ell = 1, \dim H = n-1, (L_1,\dots,L_k) \in Fl(\mathbf{r};n), \\ \text{and } \ell \subseteq L_k \text{ and } L_1 \subseteq H \end{array} \right\},$$

  and $\widetilde{\pi}_{\mathbf{r}}$ and $\widetilde{\pi}_{(n-1)1}$ are maps that forget appropriate subspaces in the flag.

Let us first consider the construction ($\widetilde{Fl}$). While the construction ($\widetilde{Fl}$) may seem geometrically unnatural, since $\widetilde{Fl}(1,\mathbf{r},n-1;n)$ is not a flag variety, it leads to the previously established notion of Las Vergnas' Tutte polynomials of morphisms of matroids, defined as follows.

Definition 1.3. Let $\mathbf{M} = (M_1, M_2)$ be a two-step flag matroid on a ground set $[n] = \{1,\dots,n\}$. For $i = 1, 2$, let us write r_i for the rank of M_i, and $r_i(S)$ for the rank of $S \subseteq [n]$ in M_i. The **Las Vergnas Tutte polynomial** of (M_1, M_2) is a polynomial in three variables x, y, z defined by

$$LV\mathcal{T}_{\mathbf{M}}(x,y,z) := \sum_{S\subseteq[n]} (x-1)^{r_1-r_1(S)}(y-1)^{|S|-r_2(S)} z^{r_2-r_2(S)-(r_1-r_1(S))}.$$

Las Vergnas introduced this generalization of the Tutte polynomial in [27], and studied its properties in a series of subsequent works [28–30,14,31,32]. Our first main theorem is a K-theoretic interpretation of the Las Vergnas Tutte polynomial.

Theorem 5.2. *Let* $\mathbf{M} = (M_1, M_2)$ *be a flag matroid with* $r_1 = \mathrm{rk}(M_1)$*,* $r_2 = \mathrm{rk}(M_2)$*. Let* $\mathcal{O}(0,1)$ *be the line bundle on* $Fl(r_1, r_2; n)$ *of the map* $Fl(r_1, r_2; n) \to Gr(r_2; n) \hookrightarrow \mathbb{P}^{\binom{n}{r_2}-1}$*, and let* $\mathcal{S}_2/\mathcal{S}_1$ *be the vector bundle on* $Fl(r_1, r_2; n)$ *whose fiber over a point* $(L_1, L_2) \in Fl(r_1, r_2; n)$ *is* L_2/L_1*. Then,*

$$LV\mathcal{T}_{\mathbf{M}}(\alpha, \beta, w) = \sum_{m=0}^{r_2-r_1} (\widetilde{\pi}_{(n-1)1})_* \widetilde{\pi}_{\mathbf{r}}^* \Big(y(\mathbf{M})[\mathcal{O}(0,1)][\bigwedge^m(\mathcal{S}_2/\mathcal{S}_1)] \Big) w^m$$

as elements in $K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1})[w] \simeq \mathbb{Q}[\alpha, \beta, w]/(\alpha^n, \beta^n)$*.*

Let us now consider the construction (Fl). It leads to the following different generalization of the Tutte polynomial, first defined in the review [8].

Definition 6.1. Let $\mathbf{M}$ be a flag matroid of rank $\mathbf{r} = (r_1, \dots, r_k)$ on $\{1, \dots, n\}$, and let $\mathcal{O}(\mathbf{1})$ be the line bundle of the embedding $Fl(\mathbf{r}; n) \hookrightarrow Gr(r_1; n) \times \cdots \times Gr(r_k; n) \hookrightarrow \mathbb{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{r_k}-1}$. The **flag-geometric Tutte polynomial** of $\mathbf{M}$, denoted $K\mathcal{T}_{\mathbf{M}}(x, y)$, is the unique bivariate polynomial in x, y of bi-degree at most $(n-1, n-1)$ such that

$$K\mathcal{T}_{\mathbf{M}}(\alpha, \beta) = (\pi_{(n-1)1})_* \pi_r^* \Big(y(\mathbf{M}) \cdot [\mathcal{O}(\mathbf{1})] \Big) \in K((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \simeq \mathbb{Q}[\alpha, \beta]/(\alpha^n, \beta^n).$$

While the construction (Fl) may be more geometrically natural than the construction $(\widetilde{Fl})$, we show that the flag-geometric Tutte polynomial $K\mathcal{T}_{\mathbf{M}}$ fails to display the two characteristic combinatorial properties of the usual Tutte polynomial that the Las Vergnas Tutte polynomial satisfies.

The first property is the "corank-nullity formula": Both the usual Tutte polynomial and the Las Vergnas Tutte polynomial can be expressed as a summation over all subsets of the ground set, with terms involving coranks and nullities of subsets. As a result, for a matroid M or a flag matroid (M_1, M_2) on $\{1, \dots, n\}$, one has $T_M(2,2) = 2^n$ and $LV\mathcal{T}_{(M_1,M_2)}(2,2,1) = 2^n$. We show that the value of $K\mathcal{T}_{\mathbf{M}}(2,2)$ is more intricate.

Theorem 6.7. *Let* $\mathbf{M}$ *be a two-step flag matroid* $\mathbf{M} = (M_1, M_2)$ *on a ground set* $[n] = \{1, \dots, n\}$*. Let* $p\mathcal{B}(\mathbf{M})$ *be the set of subsets* $S \subseteq [n]$ *such that* S *is spanning in* M_1 *and independent in* M_2*. Then with* q *as a formal variable, we have*

$$K\mathcal{T}_{\mathbf{M}}(1+q^{-1}, 1+q) = q^{-r_2} \cdot (1+q)^n \cdot \Big(\sum_{S \in p\mathcal{B}(\mathbf{M})} q^{|S|} \Big),$$

$$\text{in particular,} \quad K\mathcal{T}_{\mathbf{M}}(2,2) = 2^n \cdot |p\mathcal{B}(\mathbf{M})|.$$

The second property is the deletion-contraction recursion that the Tutte polynomial and the Las Vergnas Tutte polynomial both satisfy. Unlike them, the flag-geometric Tutte polynomial $K\mathcal{T}_{\mathrm{M}}$ does not satisfy the usual deletion-contraction recursion. We instead show the following deletion-contraction-like relation.

Theorem 6.8. *Let M be a matroid on a ground set $\{0, 1, \dots, n\}$ such that the element 0 is neither a loop nor a coloop in M. Then we have*

$$K\mathcal{T}_{(M,M)}(x, y) = K\mathcal{T}_{(M/0,M/0)}(x, y) + K\mathcal{T}_{(M/0,M\setminus 0)}(x, y) + K\mathcal{T}_{(M\setminus 0,M\setminus 0)}(x, y).$$

The three main theorems Theorem 5.2, Theorem 6.7, and Theorem 6.8 are obtained by proving stronger torus-equivariant versions of the statements. The resulting torus-equivariant statements are then reduced to computing certain summations of lattice point generating functions, techniques for which we review, extend, and specialize in Section §4. Our main contribution here is Theorem 4.7, which serves as a key technical tool in this paper and may be of independent interest in the study of lattice polyhedra.

The reduction to lattice point generating functions is done via the method of equivariant localization, reviewed in Section §2.2, aided by certain push-pull computations in Section §3. Section §2 is largely a summary of a more detailed account [8, §8] on flag matroids and the torus-equivariant K-theory of flag varieties. We discuss some future directions in Section §7.

1.1. Computation

At https://github.com/chrisweur/kTutte, the reader can find a Macaulay2 code for computations with torus-equivariant K-classes and flag matroids. In particular, it computes the polynomials $LV\mathcal{T}_{\mathrm{M}}$ and $K\mathcal{T}_{\mathrm{M}}$ and their torus-equivariant versions.

1.2. Notation

Throughout we set $[n] := \{1, \dots, n\}$. For $i = 1, \dots, n$, we set $\mathbf{e}_i$ to be the standard coordinate vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and write $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$ for a subset $S \subseteq [n]$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$. Cardinality of a set S is denoted by $|S|$, and disjoint unions by $\sqcup$. A variety is a reduced and irreducible proper scheme over $\mathbb{C}$.

2. Preliminaries: flag matroids and their K-classes on flag varieties

Here we review flag matroids and their (torus-equivariant) K-classes on flag varieties. Most of the material in this section is described in more detail in the review [8].

2.1. Matroid quotients and flag matroids

We assume familiarity with the fundamentals of matroid theory, and point to [35,43,42] as references. We write $U_{r,n}$ for the uniform matroid of rank r on $[n]$. For a linear subspace $L \subseteq$

$\mathbb{C}^n$, let $M(L)$ denote the linear matroid whose ground set is the image of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ under the dual map $\mathbb{C}^n \twoheadrightarrow L^\vee$. For a matroid M on a ground set $[n]$ we set:

- $\mathrm{rk}_M : 2^{[n]} \to \mathbb{Z}$ to be the rank function of M, with $r(M) := \mathrm{rk}_M([n])$,
- $M|S$, $M\backslash S$ and M/S to be the restriction to, the deletion of, and the contraction by a subset $S \subseteq [n]$ (respectively),
- $\mathcal{B}(M)$ to be the set of bases of M, and
- $Q(M) \subset \mathbb{R}^n$ to be the base polytope of M, which is the convex hull of $\{\mathbf{e}_B \mid B \in \mathcal{B}(M)\}$.

In this paper, by **morphisms of matroids** we will mean matroid quotients, as defined below.[1] They are combinatorial abstractions of the graph homomorphisms, linear maps, and graphs embedded on surfaces; see [12] for illustrations of these examples.

Definition 2.1. Let M_1, M_2 be two matroids on a common ground set $[n]$. We say that M_1 is a **matroid quotient** of M_2, written $M_1 \twoheadleftarrow M_2$, if any of the following equivalent conditions are met [7, Proposition 7.4.7]:

(1) every circuit of M_2 is a union of circuits of M_1,
(2) $\mathrm{rk}_{M_2}(B) - \mathrm{rk}_{M_2}(A) \geq \mathrm{rk}_{M_1}(B) - \mathrm{rk}_{M_1}(A)$ for any $A \subseteq B \subseteq [n]$,
(3) there exists a matroid N on a ground set $[n] \sqcup S$ with $|S| = r(M_2) - r(M_1)$ such that $M_1 = N/S$ and $M_2 = N \setminus S$.

Example 2.2. Matroid quotients are combinatorial abstractions of linear maps of maximal rank. An inclusion of linear subspaces $L_1 \hookrightarrow L_2 \subseteq \mathbb{C}^n$, or equivalently a quotient $\mathbb{C}^n \twoheadrightarrow L_2^\vee \twoheadrightarrow L_1^\vee$, defines matroids $M(L_1)$ and $M(L_2)$, which form a matroid quotient $M(L_1) \twoheadleftarrow M(L_2)$.

Example 2.3 *(Canonical matroid quotients).* Just as any linear space L has two canonical linear maps, the identity $L \to L$ and the zero map $L \to 0$, any matroid M has two canonical matroid quotients, the identity $M \twoheadrightarrow M$ and the trivial quotient $M \twoheadrightarrow U_{0,n}$.

A matroid quotient $M_1 \twoheadleftarrow M_2$ is an **elementary quotient** if $r(M_2) - r(M_1) = 1$. Every matroid quotient $M_1 \twoheadleftarrow M_2$ can be realized as a composition of a series of elementary quotients. A canonical one is given by the **Higgs factorization** $M_1 = M^{(r_2-r_1)} \twoheadleftarrow \cdots \twoheadleftarrow M^{(1)} \twoheadleftarrow M^{(0)} = M_2$, defined by $\mathcal{B}(M^{(i)}) = \{S \subseteq [n] \mid |S| = r(M_2) - i,\ S$ spans M_1 and is independent in $M_2\}$. The subsets $S \subseteq [n]$ that span M_1 and are independent in M_2 are called **pseudo-bases** of (M_1, M_2). The set of pseudo-bases of $\mathbf{M} = (M_1, M_2)$ is denoted by $p\mathcal{B}(\mathbf{M})$. For a more on matroid quotients, we refer the reader to [7, §7.4] or [35, §7.3].

[1] The behavior of a morphism of matroids, in a more general sense of [12] or [22], is largely governed by an associated matroid quotient [12, Lemma 2.4].

Definition 2.4. A **flag matroid** is a sequence of matroids $\mathbf{M} = (M_1, \ldots, M_k)$[2] on a ground set $[n]$ such that $M_i \twoheadleftarrow M_{i+1}$ for all $i = 1, \ldots, k-1$. The matroids M_i are **constituents** of $\mathbf{M}$, and the **rank** of $\mathbf{M}$ is the sequence of ranks of its constituents $(r(M_1), \ldots, r(M_k))$. The set of **bases** of $\mathbf{M}$, denoted $\mathcal{B}(\mathbf{M})$, is the set of all k-flags of subsets $(B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k)$ such that $B_i \in \mathcal{B}(M_i)$.

Example 2.5 *(Linear flag matroids).* A sequence of matroids $(M(L_1), \ldots, M(L_k))$ defined by a flag $\mathbf{L}$ of linear subspaces $L_1 \subseteq \cdots \subseteq L_k \subseteq \mathbb{C}^n$ is a flag matroid. We denote this flag matroid by $M(\mathbf{L})$. Flag matroids arising in this way are called linear (or realizable) flag matroids.

For $\mathbf{S} = (S_1, \ldots, S_k)$ a flag of subsets of $[n]$, we write $\mathbf{e_S} = \mathbf{e}_{S_1} + \cdots + \mathbf{e}_{S_k}$. The **base polytope** $Q(\mathbf{M})$ of a flag matroid $\mathbf{M}$ is the convex hull of $\{\mathbf{e_B} \mid \mathbf{B} \in \mathcal{B}(\mathbf{M})\}$, whose vertices are in bijection with the bases of $\mathbf{M}$. The polytope $Q(\mathbf{M})$ is also the Minkowski sum of the base polytopes $Q(M_i)$ of the constituents of $\mathbf{M}$. The classical theorem of Gelfand, Goresky, MacPherson, and Serganova [18] a characterizes base polytopes of matroids. The analogue for flag matroids holds:

Theorem 2.6. *[3, Theorem 1.11.1] A lattice polytope $P \subset \mathbb{R}^n$ is the base polytope of a rank $(r_1, \ldots, r_k)$ flag matroid on $[n]$ if and only if the following two conditions hold:*

(1) *every vertex of P is a $\mathfrak{S}_n$-permutation of $\mathbf{e}_{\{1,2,\ldots,r_1\}} + \cdots + \mathbf{e}_{\{1,2,\ldots,r_k\}}$, and*
(2) *every edge of P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in [n]$.*

In particular, the normal fan of the base polytope $Q(\mathbf{M})$ of a flag matroid is a coarsening of the braid arrangement, which is the normal fan of the zonotope $\sum_{1 \le i < j \le n} \mathrm{Conv}(\mathbf{e}_i, \mathbf{e}_j)$.

Consequently, every face of a base polytope of a flag matroid is again a base polytope of a flag matroid. The faces can be described explicitly. For $u \in \mathbb{R}^n$ and a polytope $Q \subset \mathbb{R}^n$, let $Q^u := \{x \in Q \mid \langle x, u\rangle = \max_{y \in Q}\langle y, u\rangle\}$ be the face maximizing in the direction of u.

Proposition 2.7. *Let $\mathbf{M} = (M_1, \ldots, M_k)$ be a flag matroid on $[n]$ or rank $\mathbf{r} = (r_1, \ldots, r_k)$, and let $\mathbf{S} = S_1 \subseteq \cdots \subseteq S_m$ be a flag of subsets of $[n]$. Then $Q(\mathbf{M})^{\mathbf{e_S}}$ is the base polytope of a flag matroid whose i-th constituent (for $i = 1, \ldots, k$) is*

$$M_i|S_1 \oplus M_i|S_2/S_1 \oplus \cdots \oplus M_i|S_m/S_{m-1} \oplus M_i/S_m.$$

In other words, the bases of the flag matroid of $Q(\mathbf{M})^{\mathbf{e_S}}$ are bases $\mathbf{B} = (B_1, \ldots, B_k)$ of $\mathbf{M}$ such that $\mathrm{rk}_{M_i}(S_j) = |B_i \cap S_j|$ for all $1 \le i \le k$ and $1 \le j \le m$.

[2] We remark that, unlike [3] but in agreement with [12] and [8], we allow repetition of matroids in the sequence of matroids that constitute a flag matroid.

Proof. Note that if $Q = \sum_{i=1}^{k} Q_i$ is a Minkowski sum of polytopes, then for any $u \in \mathbb{R}^n$, the face Q^u is the Minkowski sum $\sum_{i=1}^{k} Q_i^u$ of faces. The proof of the proposition is thus reduced to the case of $\mathbf{M}$ being a matroid M. In this case, the statement is an immediate consequence of the greedy algorithm structure for matroids. $\square$

2.2. *Torus-equivariant K-theory of flag varieties*

We will study combinatorial properties of flag matroids through the geometry of (partial) flag varieties and their (torus-equivariant) K-theory. We point to [8, §8] or [15, §2] (and references therein) for a detailed exposition of equivariant K-theory of flag varieties.

We begin by describing torus-equivariant K-theory and the method of localization. Let $T = (\mathbb{C}^*)^n$, and write $\mathbb{Z}[\mathbf{t}^{\pm}] := \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}] = \mathbb{Z}[\mathbb{Z}^{[n]}]$ for the **character ring** of T. Let X be a smooth variety with a T-action, and let $\mathcal{E}$ be a (T-equivariant) vector bundle on X. We write:

- $K^0(X)$ for the Grothendieck ring of vector bundles on X, which is isomorphic to the Grothendieck group of coherent sheaves $K_0(X)$ since X is smooth,
- $K_T^0(X)$ for the T-equivariant Grothendieck ring,
- $[\mathcal{E}] \in K^0(X)$ for the K-class of $\mathcal{E}$ and $[\mathcal{E}]^T \in K_T^0(X)$ for its T-equivariant K-class,
- f_* for the (derived) pushforward map and f^* for the pullback map of K-classes along a proper map $f : X \to X'$ of smooth varieties,
- χ for the pushforward along the structure map $X \to \operatorname{Spec} \mathbb{C}$, and
- χ^T for the T-equivariant pushforward to $K_T^0(pt) = \mathbb{Z}[\mathbf{t}^{\pm}]$, the Lefschetz trace [34, §4].

We now assume that X has finitely many T-fixed points, denoted X^T, and finitely many 1-dimensional T-orbits. Moreover, we assume X to be equivariantly formal and contracting, the precise definitions of which can be found in [8, Remark 8.5 & Definition 8.6]. Examples of such X include flag varieties and smooth toric varieties. By definition, for each T-fixed point $x \in X^T$, there is a T-invariant affine neighborhood $U_x \simeq \mathbb{A}^{\dim X}$ whose characters $\{\lambda_1(x), \dots, \lambda_{\dim X}(x)\} \subset \mathbb{Z}^n$ generate a pointed semigroup. Fundamental results from the method of equivariant localization are collected in the following theorem.

Theorem 2.8. *Let X be a equivariantly formal and contracting smooth T-variety with finitely many T-fixed points and finitely many 1-dimensional T-orbits. Then:*

(1) *[41, Corollary 5.12], [25, Corollary A.5] (cf. [34, Theorem 3.2], [39, Theorem 2.7]) The restriction map*

$$K_T^0(X) \hookrightarrow K_T^0(X^T) \simeq (\mathbb{Z}[\mathbf{t}^{\pm}])^{X^T}, \qquad \epsilon \mapsto \epsilon(\cdot)$$

is injective. Moreover, an element $\epsilon(\cdot) \in (\mathbb{Z}[\mathbf{t}^{\pm}])^{X^T}$ is in the image if and only if for every one-dimensional T-orbit in X with boundary points $x, y \in X^T$ in the closure, the function $\epsilon(\cdot) : X^T \to \mathbb{Z}[\mathbf{t}^{\pm}]$ satisfies

$$\epsilon(x) \equiv \epsilon(y) \mod 1 - \mathbf{t}^\lambda$$

where λ is the character of the action of T on the one-dimensional orbit.

(2) *[15, Theorem 2.6], [33, Theorem 8.34] Let $\mathcal{E}$ be a T-equivariant coherent sheaf on X, and let $x \in X^T$. The image $[\mathcal{E}]^T(x)$ of $[\mathcal{E}]^T$ under the restriction $K_T^0(X) \to K_T^0(x) \simeq \mathbb{Z}[\mathbf{t}^\pm]$ is $\mathcal{K}(\mathcal{E}(U_x); \mathbf{t})$ where*

$$\mathrm{Hilb}(\mathcal{E}(U_x)) := \frac{\mathcal{K}(\mathcal{E}(U_x); \mathbf{t})}{\prod_{i=1}^{\dim X}(1 - \mathbf{t}^{-\lambda_i(x)})}$$

is the multigraded Hilbert series of the $\mathcal{O}_X(U_x)$-module $\mathcal{E}(U_x)$ [33, Theorem 8.20].

(3) *[9, Theorem 5.11.7] (cf. [34, §4]) Let $f : X \to Y$ be a proper T-equivariant map of equivariantly formal, contracting, and smooth T-varieties, and let $\alpha \in K_T^0(X)$, $\beta \in K_T^0(Y)$. Then we have*

$$(f^*\beta)(x) = \beta(f(x)) \text{ for every } x \in X^T, \text{ and}$$

$$(f_*\alpha)(y) = \left(\prod_{i=1}^{\dim Y}(1 - \mathbf{t}^{-\lambda_i(y)})\right)\left(\sum_{x \in X^T \cap f^{-1}(y)} \frac{\alpha(x)}{\prod_{i=1}^{\dim X}(1 - \mathbf{t}^{-\lambda_i(x)})}\right)$$

for every $y \in Y^T$.

We now specialize our discussion of K-theory to flag varieties. For a sequence of non-negative integers $\mathbf{r} = (r_1, \ldots, r_k)$ such that $0 < r_1 \leq \cdots \leq r_k < n$, denote by $Fl(\mathbf{r}; n)$ the **flag variety**

$$Fl(\mathbf{r}; n) := \{\mathbf{L} = (L_1 \subseteq \cdots \subseteq L_k \subseteq \mathbb{C}^n) \text{ linear subspaces with } \dim L_i = r_i \ \forall 1 \leq i \leq k\}.$$

For each $i = 1, \ldots, k$, we have the tautological sequence of vector bundles on $Fl(\mathbf{r}; n)$

$$0 \to \mathcal{S}_i \to \mathbb{C}^n \to \mathcal{Q}_i \to 0$$

where $\mathcal{S}_i$ is the (i-th) universal subbundle. It is a vector bundle whose fiber at a point $\mathbf{L} \in Fl(\mathbf{r}; n)$ is the subspace L_i. For $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we denote by $\mathcal{O}(\mathbf{a})$ the line bundle $\bigotimes_{i=1}^k (\det \mathcal{S}_i^\vee)^{\otimes a_i}$, and by $\mathcal{O}(\mathbf{1})$ the line bundle $\mathcal{O}(1, 1, \ldots, 1)$ on $Fl(\mathbf{r}; n)$. The torus $T := (\mathbb{C}^*)^n$ acts on $Fl(\mathbf{r}; n)$ by its action on $\mathbb{C}^n$ where $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1^{-1}x_1, \ldots, t_n^{-1}x_n)$. With this T-action, a flag variety is a equivariantly formal and contracting space with the following structure:

- The T-fixed points $x_\mathbf{S}$ of $Fl(\mathbf{r}; n)$ are flags of coordinate subspaces, which are in bijection with flags of subsets $S_1 \subseteq \cdots \subseteq S_k \subseteq [n]$ with $|S_i| = r_i$ for all $i = 1, \ldots, k$.
- For a flag $\mathbf{S}$, denote by $Ex(\mathbf{S})$ the set of $(i, j) \in [n] \times [n]$ such that $i \in S_\ell$ and $j \notin S_\ell$ for some $1 \leq \ell \leq k$. Then the set of characters of the T-neighborhood $U_\mathbf{S}$ of $x_\mathbf{S}$ is $\{\mathbf{e}_i - \mathbf{e}_j \mid (i, j) \in Ex(\mathbf{S})\}$.

The sign-convention we have adopted for the action of T ensures that T acts on the sections of $\mathcal{S}_i^\vee$ by positive characters. For instance, we have $[\mathcal{S}_i^\vee]^T(x_{\mathbf{S}}) = \sum_{j\in S_i} t_j$ and $[\mathcal{Q}_i^\vee]^T(x_{\mathbf{S}}) = \sum_{j\in[n]\setminus S_i} t_j$, and moreover $[\bigwedge^p \mathcal{S}_i^\vee]^T(x_{\mathbf{S}}) = \sum_{\substack{A\subseteq S_i\\ |A|=p}} \mathbf{t}^{\mathbf{e}_A}$ and $[\bigwedge^p \mathcal{S}_i]^T(x_{\mathbf{S}}) = \sum_{\substack{A\subseteq S_i\\ |A|=p}} \mathbf{t}^{-\mathbf{e}_A}$ (likewise for $\bigwedge^q \mathcal{Q}_i^\vee, \bigwedge^q \mathcal{Q}_i$).

2.3. K-class of a flag matroid

Flag matroids enter into the K-theory of flag varieties as T-equivariant K-classes as follows. Let $\mathbf{M}$ be a flag matroid of rank $\mathbf{r}$ on a ground set $[n]$. For $\mathbf{B}$ a basis of $\mathbf{M}$, define a polyhedral cone $\mathrm{Cone}_{\mathbf{B}}(\mathbf{M}) := \mathrm{Cone}(Q(\mathbf{M}) - \mathbf{e}_{\mathbf{B}}) \subset \mathbb{R}^n$, also known as the tangent cone of $Q(M)$ at the vertex $\mathbf{e}_{\mathbf{B}}$, and let $\mathrm{Hilb}_{\mathbf{B}}(\mathbf{M})$ be the multigraded Hilbert series of $\mathbb{C}[\mathbf{t}^\lambda \mid \lambda \in \mathrm{Cone}_{\mathbf{B}}(\mathbf{M}) \cap \mathbb{Z}^n]$ (see [33, Theorem 8.20]).

Definition 2.9. [8, Definition 8.19] Let $\mathbf{M}$ be a flag matroid of rank $\mathbf{r}$ on a ground set $[n]$. Then define $y(\mathbf{M})^T(\cdot) \in K_T^0(Fl(\mathbf{r};n)^T)$ by

$$y(\mathbf{M})^T(x_{\mathbf{S}}) := \begin{cases} \mathrm{Hilb}_{\mathbf{S}}(\mathbf{M}) \cdot \prod\limits_{(i,j)\in Ex(\mathbf{S})} (1 - t_i^{-1} t_j) & \text{if } \mathbf{S} \text{ a basis of } \mathbf{M} \\ 0 & \text{otherwise.} \end{cases}$$

By combining Theorem 2.8.(1) and Theorem 2.6, one observes that $y(\mathbf{M})^T$ can be considered as a class in $K_T^0(Fl(\mathbf{r};n))$ [8, Proposition 8.20]. We will write $y(\mathbf{M})$ for the underlying non-equivariant K-class. The geometric motivation for this K-class constitutes the remark below.

Remark 2.10. Recall from Example 2.5 that a point $\mathbf{L} \in Fl(\mathbf{r};n)$ defines a flag matroid $\mathbf{M} := M(\mathbf{L})$ of rank $\mathbf{r}$. One observes that the torus-orbit closure $\overline{T\cdot\mathbf{L}}$ is isomorphic to the toric variety of the base polytope $Q(\mathbf{M})$, and then by applying Theorem 2.8.(2) one shows that the class $[\mathcal{O}_{\overline{T\cdot\mathbf{L}}}]^T \in K_T^0(Fl(\mathbf{r};n))$ satisfies $[\mathcal{O}_{\overline{T\cdot\mathbf{L}}}]^T(\cdot) = y(\mathbf{M})^T(\cdot)$. See [8, §8.5] for details.

Remark 2.11. Let $\mathcal{P}(\mathsf{FMat}_{\mathbf{r};n})$ be a group generated by the indicator functions $\mathbb{1}(Q) : \mathbb{R}^n \to \mathbb{R}$ of base polytopes Q of rank $\mathbf{r}$ flag matroids on $[n]$. A function φ from the set of flag matroids of rank $\mathbf{r}$ on $[n]$ to an abelian group A is **(strongly) valuative** if it factors through $\mathcal{P}(\mathsf{FMat}_{\mathbf{r};\mathbf{n}})$. As taking tangent cones and taking Hilbert series are valuative, it follows easily from the definition that the assignment $\mathbf{M} \mapsto y(\mathbf{M})$ is valuative.

When $\mathbf{r} = (r)$ (that is, we are concerned with the Grassmannian $Gr(r;n)$ and hence matroids of rank r on $[n]$), invariants of a matroid M built from $y(M)$ were explored in [38] and [16] as follows. To avoid confusion we write $\mathbb{P}^{n-1}$ for $Gr(1;n)$ and $(\mathbb{P}^{n-1})^\vee$ for $Gr(n-1;n)$. Recall the diagram:

$$\begin{tikzcd}[column sep=small] & Fl(1,r,n-1;n) \arrow[dl, "\pi_{\mathbf{r}}"'] \arrow[dr, "\pi_{(n-1)1}"] & \\ Gr(r;n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}. \end{tikzcd} \tag{2}$$

Let α be the K-class of the structure sheaf of a hyperplane in $(\mathbb{P}^{n-1})^\vee$ and β the likewise K-class from $\mathbb{P}^{n-1}$. We remark that our notation of α, β is flipped from the notation in [16].[3] Recall that $K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}) \simeq \mathbb{Q}[\alpha,\beta]/(\alpha^n, \beta^n)$.

Theorem 2.12. *[16, Theorem 5.1] Let M be a matroid of rank r on $[n]$, and let $T_M(x,y)$ be its Tutte polynomial. Then we have*

$$T_M(\alpha,\beta) = (\pi_{(n-1)1})_* \pi_r^* \Big(y(M) \cdot [\mathcal{O}(1)] \Big).$$

We will generalize this K-theoretic formulation of Tutte polynomials of matroids to flag matroids in two different ways in subsequent sections. In both cases, similarly to Theorem 2.12, the Tutte polynomials of flag matroids are formulated via diagrams like (2), which we introduce in the next section.

3. Two diagrams and a fundamental computation

The main goal of this section is to prove Proposition 3.1, which relates a pushforward of a pullback of K-classes to Euler characteristics of certain associated sheaves. As this section is closely adapted from [16, §4], we only give sketches of proofs, save for the modified parts.

Let $\mathbf{r} = (r_1, \ldots, r_k)$ be a sequence of non-negative integers. For each $i = 1, \ldots, k$, recall that we have tautological bundles $\mathcal{S}_i$ and $\mathcal{Q}_i$ on $Fl(\mathbf{r};n)$ fitting into the short exact sequences

$$0 \to \mathcal{S}_i \to \mathbb{C}^n \to \mathcal{Q}_i \to 0. \tag{3}$$

For two vector bundles $\mathcal{E}, \mathcal{F}$ on $X = Fl(\mathbf{r};n)$, we write $\pi : \mathrm{BiProj}(\mathcal{E},\mathcal{F}) \to X$ for the bi-projectivization of the direct sum $\mathcal{E} \oplus \mathcal{F}$. That is, $\mathrm{BiProj}(\mathcal{E},\mathcal{F}) := \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}) \times_X \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{F})$, so that for each point $x \in X$, the fiber $\pi^{-1}(x)$ is $\mathbb{P}(\mathcal{E}_x^\vee) \times \mathbb{P}(\mathcal{F}_x^\vee)$. We consider the following two distinguished cases; note that the two cases are identical when $k = 1$ (i.e. when $Fl(\mathbf{r};n)$ is a Grassmannian $Gr(r;n)$).

[3] In [16], the authors consider $\pi_{1(n-1)} : Fl(1,r,n-1;n) \to \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$, and set α and β as the K-classes of the structure sheaves of hyperplanes from $\mathbb{P}^{n-1}$ and $(\mathbb{P}^{n-1})^\vee$ (respectively). Our flipped naming of α, β is to remedy a minor error in the proof of [16, Lemma 4.1] (bottom three lines on pg. 2709), which accidentally flips the correspondence of α, β to appropriate K-classes.

- $\mathrm{BiProj}(\mathcal{S}_1^\vee, \mathcal{Q}_k) \simeq Fl(1, \mathbf{r}, n-1; n)$. In this case, we have maps:

$$\begin{tikzcd} & Fl(1,\mathbf{r},n-1;n) \arrow[dl, "\pi_{\mathbf{r}}"'] \arrow[dr, "\pi_{(n-1)1}"] & \\ Fl(\mathbf{r};n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1} \end{tikzcd} \tag{Fl}$$

  where $\pi_{\mathbf{r}}$ and $\pi_{(n-1)1}$ are given by forgetting the linear spaces of appropriate dimensions.
- $\mathrm{BiProj}(\mathcal{S}_k^\vee, \mathcal{Q}_1) \simeq \widetilde{Fl}(1, \mathbf{r}, n-1; n)$ where $\widetilde{Fl}(1, \mathbf{r}, n-1; n)$ is a variety

$$\widetilde{Fl}(1,\mathbf{r},n-1;n) := \left\{ \begin{array}{c} \text{linear subspaces} \\ (\ell, L_1, \dots, L_k, H) \end{array} \middle| \begin{array}{c} \dim \ell = 1, \dim H = n-1, (L_1,\dots,L_k) \in Fl(\mathbf{r};n), \\ \text{and } \ell \subseteq L_k \text{ and } L_1 \subseteq H \end{array} \right\}.$$

  In this case, we also have maps:

$$\begin{tikzcd} & \widetilde{Fl}(1,\mathbf{r},n-1;n) \arrow[dl, "\widetilde{\pi}_{\mathbf{r}}"'] \arrow[dr, "\widetilde{\pi}_{(n-1)1}"] & \\ Fl(\mathbf{r};n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1} \end{tikzcd} \tag{$\widetilde{Fl}$}$$

  where $\widetilde{\pi}_{\mathbf{r}}$ and $\widetilde{\pi}_{(n-1)1}$ are given by forgetting the linear spaces of appropriate dimensions.

As before, let $\alpha = [\mathcal{O}_{H_1}]$ be the K-class of the structure sheaf of a hyperplane in $(\mathbb{P}^{n-1})^\vee$ and $\beta = [\mathcal{O}_{H_2}]$ the likewise K-class from $\mathbb{P}^{n-1}$. The main statement of this section is as follows.

Proposition 3.1. *Let $\epsilon \in K^0(Fl(\mathbf{r};n))$. With u and v as formal variables, define polynomials*

$$R_\epsilon(u,v) := \sum_{p,q} \chi\Big(\epsilon \cdot [\bigwedge^p \mathcal{S}_k][\bigwedge^q \mathcal{Q}_1^\vee]\Big) u^p v^q \quad \text{and}$$

$$\widetilde{R}_\epsilon(u,v) := \sum_{p,q} \chi\Big(\epsilon \cdot [\bigwedge^p \mathcal{S}_1][\bigwedge^q \mathcal{Q}_k^\vee]\Big) u^p v^q.$$

Then we have the following identities in $K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1})$.

$$R_\epsilon(\alpha-1, \beta-1) = (\pi_{(n-1)1})_* \pi_{\mathbf{r}}^*(\epsilon) \quad \text{and} \quad \widetilde{R}_\epsilon(\alpha-1,\beta-1) = (\widetilde{\pi}_{(n-1)1})_* \widetilde{\pi}_{\mathbf{r}}^*(\epsilon).$$

When $k = 1$, i.e. $Fl(\mathbf{r}; n)$ is a Grassmannian, Proposition 3.1 reduces to [16, Lemma 4.1]. We remark that, just as in [16], Proposition 3.1 is an identity in the *non*-equivariant K-theory. The proof of Proposition 3.1 is a minor modification of the proof of [16, Lemma 4.1]. Here, as a lemma, we separate out and also fix a minor error in the part of the proof in [16] that needs modification.

Lemma 3.2. *Denote* $\eta_1 := (1-\alpha)^{-1} = [\mathcal{O}(1,0)]$ *and* $\eta_2 := (1-\beta)^{-1} = [\mathcal{O}(0,1)]$, *and let* t *be a formal variable. Then the following identities hold in* $K^0(Fl(\mathbf{r};n))[[t]]$.

$$\sum_p [\bigwedge^p \mathcal{S}_k] t^p = (1+t)^n (\pi_{\mathbf{r}})_* \pi^*_{(n-1)1}\Big(\frac{1}{1+t\eta_1}\Big) \text{ and } \sum_q [\bigwedge^q \mathcal{Q}_1^\vee] t^q$$
$$= (1+t)^n (\pi_{\mathbf{r}})_* \pi^*_{(n-1)1}\Big(\frac{1}{1+t\eta_2}\Big).$$

And likewise,

$$\sum_p [\bigwedge^p \mathcal{S}_1] t^p = (1+t)^n (\widetilde{\pi}_{\mathbf{r}})_* \widetilde{\pi}^*_{(n-1)1}\Big(\frac{1}{1+t\eta_1}\Big) \text{ and}$$
$$\sum_q [\bigwedge^q \mathcal{Q}_k^\vee] t^q = (1+t)^n (\widetilde{\pi}_{\mathbf{r}})_* \widetilde{\pi}^*_{(n-1)1}\Big(\frac{1}{1+t\eta_2}\Big).$$

Proof. For each $i = 1, \ldots, k$ note that

$$\Big(\sum_\ell [\bigwedge^\ell \mathcal{S}_i] t^\ell\Big)\Big(\sum_m [\bigwedge^m \mathcal{Q}_i] t^m\Big) = (1+t)^n, \tag{4}$$

which follows from the short exact sequence (3) and [13, A2.2.(c)]. We also have an identity

$$\Big(\sum_\ell [\bigwedge^\ell \mathcal{S}_i] t^\ell\Big)\Big(\sum_m (-1)^m [\mathrm{Sym}^m \mathcal{S}_i] t^m\Big) = 1 \tag{5}$$

and likewise identities for $\mathcal{Q}_i$ and the duals $\mathcal{S}_i^\vee, \mathcal{Q}_i^\vee$, which follow from the exactness of the Koszul complex [13, A2.6.1]. Now, we note by [21, Exercise III.8.4] that

$$\begin{aligned} (\pi_{\mathbf{r}})_* \pi^*_{(n-1)1}(\eta_2^\ell \eta_1^m) &= [\mathrm{Sym}^\ell \mathcal{S}_1^\vee \otimes \mathrm{Sym}^m \mathcal{Q}_k] \text{ and} \\ (\widetilde{\pi}_{\mathbf{r}})_* \widetilde{\pi}^*_{(n-1)1}(\eta_2^\ell \eta_1^m) &= [\mathrm{Sym}^\ell \mathcal{S}_k^\vee \otimes \mathrm{Sym}^m \mathcal{Q}_1]. \end{aligned} \tag{6}$$

Combining (4), (5), and (6) then yields the desired identities. □

Sketch of proof of Proposition 3.1. One combines Lemma 3.2 with the projection formula for K-theory [17, §15.1]. Then by expanding the power series in u and v, which is in fact a finite sum, comparing coefficients yields the desired identity. See the proof in [16] for details. □

4. Summations of lattice point generating functions

The method of equivariant localization §2.2, aided by Proposition 3.1, will reduce our K-theoretic computations to summations of lattice point generating functions. Here we collect some useful results concerning summations of lattice point generating functions arising from polyhedra, along with variants that are suitable for our purposes. Our main novel contribution is Theorem 4.7, which is a useful variant of the method of flipping cones. The reader may see Example 4.12 for illustrations of main theorems here.

4.1. Brion's formula

Here we review the results in [6,23]. For a subset $S \subset \mathbb{R}^n$, denote by $\mathbb{1}(S) : \mathbb{Z}^n \to \mathbb{Q}$ its indicator function sending $x \mapsto 1$ if $x \in S$ and 0 otherwise. Let $\mathcal{P}_n$ be a vector space of $\mathbb{Q}$-valued functions on $\mathbb{Z}^n$ generated by $\{\mathbb{1}(P) \mid P \subset \mathbb{R}^n \text{ lattice polyhedra}\}$. It follows from the Brianchon-Gram formula [5,19,37] that $\mathcal{P}_n$ is generated by indicator functions of cones, and by triangulating one concludes that $\mathcal{P}_n$ is generated by indicator functions of smooth cones.

We will often consider elements of $\mathcal{P}_n$ as elements in the power series ring $\mathbb{Q}[[t_1^{\pm}, \dots, t_n^{\pm}]]$ by identifying $\mathbb{1}(P)$ with $\sum_{\lambda \in P \cap \mathbb{Z}^n} \mathbf{t}^{\lambda}$. The following fundamental theorem concerns convergence of these power series to a rational function.

Theorem 4.1. *[23, Theorem 1.2][4] Consider $\mathcal{P}_n$ as a $\mathbb{Q}[t_1^{\pm}, \dots, t_n^{\pm}]$-submodule of $\mathbb{Q}[[t_1^{\pm}, \dots, t_n^{\pm}]]$, and let $\mathbb{Q}(t_1, \dots, t_n)$ be the fraction field. There exists a unique $\mathbb{Q}[t_1^{\pm}, \dots, t_n^{\pm}]$-linear map*

$$\mathrm{Hilb} : \mathcal{P}_n \to \mathbb{Q}(t_1, \dots, t_n)$$

such that if $C = \mathrm{Cone}(v_1, \dots, v_k) \subset \mathbb{R}^n$ is a smooth cone with primitive ray generators $v_1, \dots, v_k \in \mathbb{Z}^n$ then $\mathrm{Hilb}(\mathbb{1}(C)) = \prod_{i=1}^{k} \frac{1}{1-\mathbf{t}^{v_i}}$.

Two remarks about the above linear map Hilb follow:

(1) The notation Hilb agrees with our previous notion of Hilbert series: when C is a pointed rational polyhedral cone, not necessarily smooth, $\mathrm{Hilb}(\mathbb{1}(C))$ equals the multigraded Hilbert series of $\mathbb{C}[\mathbf{t}^{\lambda} \mid \lambda \in C \cap \mathbb{Z}^n]$ in the sense of [33, Theorem 8.20].
(2) If P is a lattice polyhedron with a non-trivial lineality space, then $\mathrm{Hilb}(\mathbb{1}(P)) = 0$.

For P a lattice polyhedron, we will often by abuse of notation write $\mathrm{Hilb}(P)$ for $\mathrm{Hilb}(\mathbb{1}(P))$. An important result on rational generating functions for cones is the formula of Brion [6], which was slightly generalized in [23]. Here we will only need the following special case of [23, Theorem 2.3].

Theorem 4.2. *Let $P \subset \mathbb{R}^n$ be a lattice polyhedron with a nonempty set of vertices (so P has no lineality space), and let $C(P)$ be its recession cone. For every vertex v of P, write C_v for $\mathrm{Cone}(P - v)$. Then we have*

[4] Fink and Speyer in [16] and Postnikov in [36] cite [24], whereas Ishida in [23] writes that the theorem is originally due to Brion.

$$\mathrm{Hilb}(P) = \sum_{v\in \mathrm{Vert}(P)} \mathrm{Hilb}(C_v + v) \qquad \text{and} \qquad \mathrm{Hilb}(C(P)) = \sum_{v\in \mathrm{Vert}(P)} \mathrm{Hilb}(C_v).$$

4.2. Lawrence-Varchenko formula (flipping cones) and variants

Here we review the method of flipping cones [15, §6], [4, (11)]. Our contribution is a generalization Theorem 4.7, which will serve as a key technical tool in subsequent sections.

Let $\boldsymbol{\zeta} \in \mathbb{R}^n$. For every $a \in \mathbb{R}$, we will denote the hyperplane $\{x \in \mathbb{R}^n | \langle \boldsymbol{\zeta}, x\rangle = a\}$ by $H_{\boldsymbol{\zeta}=a}$ and the half-space $\{x \in \mathbb{R}^n | \langle \boldsymbol{\zeta}, x\rangle \geq a\}$ by $H_{\boldsymbol{\zeta}\geq a}$. For an element $f \in \mathcal{P}_n$, by considering f as an element of $\mathbb{Q}[[t_1^{\pm}, \dots, t_n^{\pm}]]$ we write $f|_{H_{\boldsymbol{\zeta}=a}}$ for the sum of terms ct^w in f such that $\langle w, \boldsymbol{\zeta}\rangle = a$.

Definition 4.3. A polyhedron $P \subset \mathbb{R}^n$ is **$\boldsymbol{\zeta}$-pointed** if $P \subseteq H_{\boldsymbol{\zeta}\geq a}$ for some $a \in \mathbb{R}$. Let $\mathcal{P}_n^{\boldsymbol{\zeta}}$ be the $\mathbb{Q}$-vector space generated by $\boldsymbol{\zeta}$-pointed elements in $\mathcal{P}_n$.

We note the following useful observation: Let $P \subset \mathbb{R}^n$ be a polyhedron with vertices $\mathrm{Vert}(P)$, and as before let $C_v := \mathrm{Cone}(P - v)$ for $v \in \mathrm{Vert}(P)$. Then, for $\boldsymbol{\zeta} \in \mathbb{R}^n$, the cone C_v is $\boldsymbol{\zeta}$-pointed if and only if v is a vertex of the face $P^{-\boldsymbol{\zeta}}$ of P minimizing in the $\boldsymbol{\zeta}$ direction.

If $f \in \mathcal{P}_n^{\boldsymbol{\zeta}}$, then one can compute $\mathrm{Hilb}(f)$ "slice-by-slice" in the following sense.

Lemma 4.4. *Let $f, g \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ and suppose that $\mathrm{Hilb}(f) = \mathrm{Hilb}(g)$. Then for every $a \in \mathbb{R}$, it holds that $\mathrm{Hilb}(f|_{H_{\boldsymbol{\zeta}=a}}) = \mathrm{Hilb}(g|_{H_{\boldsymbol{\zeta}=a}})$.*

Proof. Write $b = f - g$, and suppose by contradiction that there is an $a \in \mathbb{R}$ with $\mathrm{Hilb}(b|_{H_{\boldsymbol{\zeta}=a}}) \neq 0$. Since $b \in \mathcal{P}_n^{\boldsymbol{\zeta}}$, there is a minimal such a, which we denote by a_0. Writing $b = \sum_i p_i \mathbb{1}(C_i)$ with C_i smooth cones and p_i Laurent polynomials, we define a nonzero Laurent polynomial $q(\mathbf{t}) = \sum_{e\in\mathbb{Z}^n} \lambda_e \boldsymbol{t}^e$ by $q(\mathbf{t}) := \prod_i \prod_{j_i} (1 - \mathbf{t}^{j_i})$ where j_i ranges over the primitive rays of C_i. By construction $q \cdot b$ has finite support, i.e. is a Laurent polynomial, and $\mathrm{Hilb}(q \cdot b) = q\,\mathrm{Hilb}(b) = 0$. Hence, we have $q \cdot b = 0$. Let $c = \min\{\langle \boldsymbol{\zeta}, e\rangle | \lambda_e \neq 0\}$, and let $q_0 = \sum_{e:\langle\boldsymbol{\zeta},e\rangle=c} \lambda_e \boldsymbol{t}^e$. Then $0 = \mathrm{Hilb}((q \cdot b)|_{H_{\boldsymbol{\zeta}=a_0+c}}) = q_0\,\mathrm{Hilb}(b|_{H_{\boldsymbol{\zeta}=a_0}}) \neq 0$, a contradiction. $\square$

Suppose that $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ is chosen such that the ζ_i's are $\mathbb{Q}$-linearly independent, in which case we say "$\boldsymbol{\zeta}$ is **irrational**." Then for any $a \in \mathbb{R}$, the intersection $H_{\boldsymbol{\zeta}=a} \cap \mathbb{Z}^n$ consists of at most one point. In this case Lemma 4.4 reduces to saying $\mathrm{Hilb} : \mathcal{P}_n^{\boldsymbol{\zeta}} \to \mathbb{Q}(t_1, \dots, t_n)$ is injective, recovering [15, Lemma 6.3]. We next recall the notion of cone flips. We begin with a lemma for their existence.

Lemma 4.5. *[20, Lemma 6], [16, Lemma 2.1] Assume $\boldsymbol{\zeta}$ is irrational. For every $f \in \mathcal{P}_n$, there is a unique $f^{\boldsymbol{\zeta}} \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ such that $\mathrm{Hilb}(f) = \mathrm{Hilb}(f^{\boldsymbol{\zeta}})$. The map $f \mapsto f^{\boldsymbol{\zeta}}$ is linear.*

The map $(\cdot)^{\boldsymbol{\zeta}}$ in the lemma is described explicitly as follows. Let $C \subseteq \mathbb{R}^n$ be a rational simplicial cone

$$C = \{w + \textstyle\sum_{i=0}^{n-1} a_i v_i \mid a_i \geq 0 \text{ for all } i \in [n]\}.$$

Then the image $C^{\boldsymbol{\zeta}} \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ under the map of Lemma 4.5 is given by

$$C^{\boldsymbol{\zeta}} = (-1)^{\ell}\, \mathbb{1}\left(\left\{ w + \sum_{i=0}^{n-1} a_i v_i \;\middle|\; \begin{array}{c} a_i \geq 0 \text{ for all } i \text{ with } \langle \boldsymbol{\zeta}, v_i\rangle > 0, \\ \text{and } a_i < 0 \text{ for all } i \text{ with } \langle \boldsymbol{\zeta}, v_i\rangle < 0 \end{array} \right\}\right), \tag{7}$$

where ℓ is the number of rays v_i for which $\langle \boldsymbol{\zeta}, v_i\rangle < 0$. We will refer to $C^{\boldsymbol{\zeta}}$ as the *cone flip* of C in direction $\boldsymbol{\zeta}$. For a general pointed rational cone C, one defines the flipped cone $C^{\boldsymbol{\zeta}} \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ by triangulating the cone.[5]

Remark 4.6. The assumption that $\boldsymbol{\zeta}$ is irrational is essential for Lemma 4.5: if $\boldsymbol{\zeta}$ is not irrational then $\mathcal{P}_n^{\boldsymbol{\zeta}}$ contains some lattice polyhedron P with a non-trivial lineality space, and $\operatorname{Hilb}(P) = 0 = \operatorname{Hilb}(0)$, contradicting uniqueness.

Now, suppose we are given an expression over a finite index set Λ

$$\varphi = \sum_{\lambda \in \Lambda} a_\lambda \operatorname{Hilb}(C_\lambda) \in \mathbb{Q}(t_1, \ldots, t_n), \tag{8}$$

where the C_λ are pointed cones with vertices not necessarily at the origin and $a_\lambda \in \mathbb{Q}$ are scalars. Suppose we know that $\varphi \in \mathbb{Q}(t_1, \ldots, t_n)$ is in fact a Laurent polynomial, for example, because φ arose from a computation in T-equivariant K-theory. Then we can use cone-flipping to get partial information about the coefficients of φ. The following proposition is our "cone-flipping in slices" technique which will be used repeatedly in later sections.

Theorem 4.7. *Suppose $\varphi = \sum_\lambda a_\lambda \operatorname{Hilb}(C_\lambda)$ is a Laurent polynomial, i.e. $\varphi \in \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$, and let P be the convex hull of the vertices of the C_λ. For $\boldsymbol{\zeta} \in \mathbb{R}^n$, not necessarily irrational, and $b \in \mathbb{R}$, suppose that every cone C_λ whose vertex w_λ satisfies $\langle \boldsymbol{\zeta}, w_\lambda\rangle < b$ is $\boldsymbol{\zeta}$-pointed. Then*

$$\varphi|_{H_{\boldsymbol{\zeta}=b}} = \sum_{C_\lambda \in \mathcal{P}_n^{\boldsymbol{\zeta}}} a_\lambda \operatorname{Hilb}(C_\lambda \cap H_{\boldsymbol{\zeta}=b}).$$

In particular, if $P \cap H_{\boldsymbol{\zeta}=b}$ is the face $P^{-\boldsymbol{\zeta}}$ of P minimizing in the $\boldsymbol{\zeta}$ direction, then

$$\varphi|_{H_{\boldsymbol{\zeta}=b}} = \sum_{\substack{\boldsymbol{\zeta}\text{-pointed } C_\lambda \text{ whose} \\ \text{vertex } w_\lambda \text{ is on } P^{-\boldsymbol{\zeta}}}} a_\lambda \operatorname{Hilb}(C_\lambda \cap H_{\boldsymbol{\zeta}=b}).$$

[5] We remark that calling $C^{\boldsymbol{\zeta}}$ the "flipped cone" of C is a slight abuse of terminology when C is not simplicial, since $C^{\boldsymbol{\zeta}}$ is not necessarily the support function of a polyhedron. It can be a genuine linear combination of some of those; see [15, Remark 6.7].

We note two useful immediate consequences of Theorem 4.7 in the following corollary, of which the second statement appeared previously in [15].

Corollary 4.8. *With assumptions as in Theorem 4.7, one has the following:*

(a) *If $H_{\boldsymbol{\zeta}=b} \cap P = \{w\}$ is a vertex of P, the coefficient of $\mathbf{t}^w$ in φ is equal to $\sum a_\lambda$, where the sum is over all λ for which $C_\lambda \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ and the vertex of C_λ is at w.*
(b) *[15, Corollary 6.9] The Newton polytope* $\mathrm{Newt}(\varphi)$ *of φ is contained in P.*

Proof. The first statement is a special case of Theorem 4.7. For the second statement, observe that for any lattice point $v \in \mathrm{Newt}(\varphi)$ and any $\boldsymbol{\zeta} \in \mathbb{R}^n$, there must exist a cone C_λ such that its vertex w_λ satisfies $\langle \boldsymbol{\zeta}, w_\lambda \rangle \leq b$ where $b = \langle \boldsymbol{\zeta}, v \rangle$, since otherwise $\varphi|_{H_{\boldsymbol{\zeta}=b}} = 0$ by Theorem 4.7. $\square$

We prepare for the proof by noting a useful feature of the cone-flipping operation, starting with the following notion.

Definition 4.9. Let C be a pointed cone, and $\boldsymbol{\zeta} \in \mathbb{R}^n$. We say that an irrational $\boldsymbol{\zeta}' \in \mathbb{R}^n$ is an **irrational approximation** of $\boldsymbol{\zeta}$ with respect to C, if for every ray generator $v \in \mathbb{R}^n$ of C it holds that $\langle \boldsymbol{\zeta}, v \rangle > 0 \implies \langle \boldsymbol{\zeta}', v \rangle > 0$ and that $\langle \boldsymbol{\zeta}, v \rangle < 0 \implies \langle \boldsymbol{\zeta}', v \rangle < 0$.

Note that an irrational approximation of $\boldsymbol{\zeta}$ can always be obtained as a small perturbation of $\boldsymbol{\zeta}$. The following is a minor generalization of [16, Lemma 2.3], with almost identical proof, which we have included for completeness.

Lemma 4.10. *Let $\boldsymbol{\zeta} \in \mathbb{R}^n$, let C be a pointed cone with vertex at w, and let $\boldsymbol{\zeta}' \in \mathbb{R}^n$ be an irrational approximation of $\boldsymbol{\zeta}$. Then its cone flip $C^{\boldsymbol{\zeta}'}$ is supported in the half space $\{x \mid \langle \boldsymbol{\zeta}, x \rangle \geq \langle \boldsymbol{\zeta}, w \rangle\}$. Furthermore, if C is not contained in $\{x \mid \langle \boldsymbol{\zeta}, x \rangle \geq \langle \boldsymbol{\zeta}, w \rangle\}$, then $C^{\boldsymbol{\zeta}'}$ is supported in the open half space $\{x \mid \langle \boldsymbol{\zeta}, x \rangle > \langle \boldsymbol{\zeta}, w \rangle\}$; in particular $w \notin C^{\boldsymbol{\zeta}'}$.*

Proof. If C is simplicial, the result follows immediately from the construction of cone flips (7) and Definition 4.9. For general C, we can obtain the first statement by considering any triangulation of C. For the second one, choose a ray v of C such that $\langle \boldsymbol{\zeta}, v \rangle < 0$ and a triangulation of C such that every interior cone contains v. Such a triangulation can for instance be constructed by triangulating the faces of C that do not contain v, and then coning that triangulation from v. Now $C = \sum_F (-1)^{\dim C - \dim F}\, \mathbb{1}(F)$ and $C^{\boldsymbol{\zeta}'} = \sum_F (-1)^{\dim C - \dim F}\, \mathbb{1}(F)^{\boldsymbol{\zeta}'}$, where the sum is over all interior cones of the triangulation. The result now follows from the simplicial case. $\square$

Proof of Theorem 4.7. Since the summation defining φ is over a finite collection of cones $\{C_\lambda\}_{\lambda \in \Lambda}$, there exists a $\boldsymbol{\zeta}' \in \mathbb{R}$ which is an irrational approximation of $\boldsymbol{\zeta}$ with respect to every cone C_λ. By assumption $\varphi = \mathrm{Hilb}(f)$, where $f \in \mathcal{P}_n$ has finite support, in particular $f \in \mathcal{P}_n^{\boldsymbol{\zeta}}$. Hence, by Lemma 4.4, $\varphi|_{H_{\boldsymbol{\zeta}=b}} = \mathrm{Hilb}(\sum a_\lambda\, \mathbb{1}(C_\lambda^{\boldsymbol{\zeta}'} \cap H_{\boldsymbol{\zeta}=b}))$. If $C_\lambda \notin \mathcal{P}_n^{\boldsymbol{\zeta}}$, then by assumption the vertex w_λ of C_λ satisfies $\langle \boldsymbol{\zeta}, w_\lambda\rangle \geq b$, and by Lemma 4.10 $C_\lambda^{\boldsymbol{\zeta}'}$ is supported on the open half-space $\{x \mid \langle \boldsymbol{\zeta}, x\rangle > b\}$, in particular $C_\lambda^{\boldsymbol{\zeta}'} \cap H_{\boldsymbol{\zeta}=b} = \emptyset$. If $C_\lambda \in \mathcal{P}_n^{\boldsymbol{\zeta}}$, then since C_λ and $C_\lambda^{\boldsymbol{\zeta}'}$ are both in $\mathcal{P}_n^{\boldsymbol{\zeta}}$, it follows from Lemma 4.4 that $\mathrm{Hilb}(C_\lambda^{\boldsymbol{\zeta}'} \cap H_{\boldsymbol{\zeta}=b}) = \mathrm{Hilb}(C_\lambda \cap H_{\boldsymbol{\zeta}=b})$. $\square$

4.3. Flipping cones for base polytopes

Let us now specialize our discussion of summing lattice point generating functions to ones arising from flag matroids. For the rest of this section, let $\mathbf{M}$ be a flag matroid of rank $\mathbf{r} = (r_1, \ldots, r_k)$ on a ground set $[n]$, whose constituent matroids have rank functions $\mathrm{rk}_1, \ldots, \mathrm{rk}_k$. As before, for a basis $\mathbf{B}$ of $\mathbf{M}$ let us write $\mathrm{Cone}_{\mathbf{B}}(\mathbf{M}) := \mathrm{Cone}(Q(\mathbf{M}) - \mathbf{e}_{\mathbf{B}})$.

Consider the expression below, which is a finite summation

$$\varphi = \sum_{\lambda \in \Lambda} a_\lambda \mathbf{t}^{w_\lambda} \,\mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}_\lambda}(\mathbf{M})), \tag{9}$$

where $a_\lambda \in \mathbb{Q}$, $w_\lambda \in \mathbb{Z}^n$, and $\mathbf{B}_\lambda$ a basis of $\mathbf{M}$. We allow the same basis to occur several times in the sum. Note that $\mathbf{t}^{w_\lambda}\,\mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}_\lambda}(\mathbf{M})) = \mathrm{Hilb}(C_\lambda)$, where C_λ is a cone with vertex at w_λ, so (9) is a special case of (8). As before, we assume that $\varphi \in \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$, i.e. φ is a Laurent polynomial, and we write $P := \mathrm{Conv}(w_\lambda \mid \lambda \in \Lambda)$ for the convex hull of the w_λ. We will assume that all w_λ lie in $\mathbb{Z}^n_{\geq 0}$, and that there exists a $c \in \mathbb{Z}_{\geq 0}$ such that the sum of the entries of any w_λ is equal to c. Let $\widetilde{P} := \mathrm{Conv}(\sigma \cdot w_\lambda \mid \sigma \in S_n,\ \lambda \in \Lambda)$ be the convex hull of all points in $\mathbb{Z}^n_{\geq 0}$ that are equal to one of the w_λ up to permuting entries.

The following theorem will be repeatedly applied in the next sections.

Theorem 4.11. *Let φ, P, and $\widetilde{P}$ be as above, and let v be a vertex of $\widetilde{P}$. Write $v = \mathbf{e}_{S_1} + \cdots + \mathbf{e}_{S_m}$, with $S_1 \subseteq \ldots \subseteq S_m \subseteq [n]$. Fix a basis $\mathbf{B} = (B_1, \ldots, B_k)$ of $\mathbf{M}$ such that $\mathbf{e}_{\mathbf{B}}$ is a vertex of the face $Q(\mathbf{M})^v$ of $Q(\mathbf{M})$ maximizing the direction v, that is, a basis $\mathbf{B}$ satisfying $|S_i \cap B_j| = \mathrm{rk}_{M_j}(S_i)$ for all $1 \leq i \leq m$ and $1 \leq j \leq k$ (Proposition 2.7). Then the coefficient of $\mathbf{t}^v$ in $\varphi \in \mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$ is equal to the sum of all a_λ for which $w_\lambda = v$ and $\mathbf{B}_\lambda = \mathbf{B}$.*

Proof. Since the Newton polytope of φ is contained in P (Corollary 4.8. (b)), the result is true for $v \notin P$. So, we now consider the case $v \in P$. Let us write $v = (v_1, \ldots, v_n)$ and $\mathbf{e}_{\mathbf{B}} = (b_1, \ldots, b_n)$. By permuting the coordinates of $\mathbb{N}^n$, we may assume that $v_i \geq v_{i+1}$ for all $i \in [n]$, and that $b_i \geq b_{i+1}$ whenever $v_i = v_{i+1}$.

We first show that we may choose a $\boldsymbol{\zeta}' = (\zeta_1', \ldots, \zeta_n') \in \mathbb{R}^n$ that satisfies the following properties:

(i) The vertex $\{v\}$ is the face of $\widetilde{P}$ maximizing in the $\boldsymbol{\zeta}'$ direction, and hence is the vertex of P maximizing in the $\boldsymbol{\zeta}'$ direction.
(ii) $\zeta_1' > \zeta_2' > \ldots > \zeta_n'$.

To choose such a $\boldsymbol{\zeta}'$, start with any $\boldsymbol{\zeta}'$ satisfying (i), which by perturbing the entries, we may assume to have all distinct entries. For any pair $i, j \in [n]$ such that $v_i > v_j$, we have $\zeta_i' > \zeta_j'$,

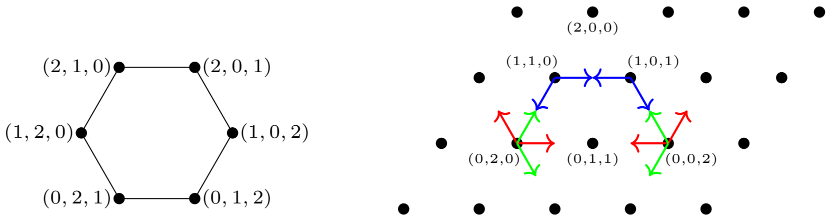

Fig. 1. Base polytope $Q(\mathbf{M})$ and translates of its vertex cones. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

since else we can swap v_i and v_j and obtain a vertex of $\widetilde{P}$ where $\boldsymbol{\zeta}'$ attains a larger value. For any collection $i, i+1, \dots, j \in [n]$ such that $v_i = \cdots = v_j$, we may reorder the corresponding entries of $\boldsymbol{\zeta}'$ in decreasing order, since such a reordering does not change the value of $\langle \boldsymbol{\zeta}', v \rangle$. This procedure produces the desired $\boldsymbol{\zeta}'$ since we had assumed $(v_1, \dots, v_n)$ to be weakly decreasing.

We then claim that the vertex face of $Q(\mathbf{M})$ maximizing in the $\boldsymbol{\zeta}'$ direction is $\{\mathbf{e}_{\mathbf{B}}\}$. Indeed, note that $\boldsymbol{\zeta}'$ is an interior point in the cone

$$\mathrm{Cone}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_1 + \cdots + \mathbf{e}_{n-1}) + \mathbb{R}\mathbf{e}_{[n]},$$

of which the cone

$$\mathrm{Cone}(\mathbf{e}_{S_1}, \mathbf{e}_{S_2}, \dots, \mathbf{e}_{S_m}) + \mathbb{R}\mathbf{e}_{[n]}$$

is a face. This face contains v in its relative interior. These two cones are cones in the braid arrangement, of which the normal fan of $Q(\mathbf{M})$ is a coarsening (Theorem 2.6). Thus, the vertex face of $Q(\mathbf{M})$ maximizing in the $\boldsymbol{\zeta}'$ direction is among the vertices of $Q(\mathbf{M})^v$, and our assumption $b_i \geq b_{i+1}$ for all $i = 1, \dots, n$ such that $v_i = v_{i+1}$ ensures that $\mathbf{e}_{\mathbf{B}}$ is indeed the one. Now, applying Corollary 4.8.(a) with $\boldsymbol{\zeta} = -\boldsymbol{\zeta}'$ gives the desired statement. $\square$

Example 4.12. We illustrate Theorem 4.2, Theorem 4.7, and Theorem 4.11 in an example. Let $\mathbf{M}$ be the flag matroid $(U_{1,3}, U_{2,3})$. Its base polytope $Q(\mathbf{M})$ is drawn on the left in Fig. 1. We arrange the six vertex cones of $Q(\mathbf{M})$ as on the right hand side of the figure, getting a summation

$$\begin{aligned} \varphi =& t_1 t_2 \,\mathrm{Hilb}(\mathrm{Cone}_{(2,1,0)}(\mathbf{M})) + t_1 t_3 \,\mathrm{Hilb}(\mathrm{Cone}_{(2,0,1)}(\mathbf{M})) && \text{(in blue)} \\ &+ t_2^2 \,\mathrm{Hilb}(\mathrm{Cone}_{(1,2,0)}(\mathbf{M})) + t_3^2 \,\mathrm{Hilb}(\mathrm{Cone}_{(1,0,2)}(\mathbf{M})) && \text{(in green)} \\ &+ t_2^2 \,\mathrm{Hilb}(\mathrm{Cone}_{(0,2,1)}(\mathbf{M})) + t_3^2 \,\mathrm{Hilb}(\mathrm{Cone}_{(0,1,2)}(\mathbf{M})) && \text{(in red)} \end{aligned}$$

By generalized Brion's theorem (second half of Theorem 4.2), one may replace the two cones at $(0,2,0)$, one red and one green, by a single cone $\mathrm{Cone}(\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_2)$, and likewise for the two cones at $(0,0,2)$. Then φ is now a summation of the four vertex cones of the trapezoid $\mathrm{Conv}(2\mathbf{e}_2, 2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3)$, whose value by Brion's theorem (first half of Theorem 4.2) is

$$\varphi = t_1 t_2 + t_1 t_3 + t_2^2 + t_2 t_3 + t_3^2.$$

If we apply Theorem 4.7 with $\boldsymbol{\zeta} = \mathbf{e}_1$ and $b = 0$, noting that the $\mathbf{e}_1$-pointed cones are $\mathrm{Cone}_{(0,2,1)}(\mathbf{M})$ and $\mathrm{Cone}_{(0,1,2)}(\mathbf{M})$, colored red in the figure, we find that, as expected,

$$\varphi|_{H_{\mathbf{e}_1=0}} = \frac{t_2^2}{1 - t_3 t_2^{-1}} + \frac{t_3^2}{1 - t_2 t_3^{-1}} = \frac{t_2^3 - t_3^3}{t^2 - t_3} = t_2^2 + t_2 t_3 + t_3^2.$$

To apply Theorem 4.11, we first note that the polytope $\widetilde{P}$ is the convex hull of $(2,0,0)$, $(0,2,0)$, $(0,0,2)$. Since there are no cones placed at $(2,0,0)$, Theorem 4.11 says that the coefficient of t_1^2 is equal to 0. For the coefficient of t_2^2, we can take either $\mathbf{B} = (1,2,0)$ or $\mathbf{B} = (0,2,1)$; in both cases Theorem 4.11 tells us that the coefficient of t_2^2 is equal to 1.

In Example 4.12, because we had only one translate of each vertex cone, arranged in a suitable manner, we could apply Brion's theorem to compute φ. In subsequent sections, we will typically have several parallel translates of each vertex cone, where Theorem 4.7 or Theorem 4.11 will better suit our needs.

5. The Las Vergnas Tutte polynomial of a matroid quotient

In [27], Las Vergnas introduced a Tutte polynomial of a matroid quotient as follows, and studied its properties in a series of subsequent works [28–30,14,31,32]. The reader may find the survey [28] particularly useful.

Definition 5.1. Let $\mathbf{M} = (M_1, M_2)$ be a two-step flag matroid on a ground set $[n]$. For $i = 1, 2$ write r_i for the rank of M_i and $r_i(S)$ for the rank of $S \subseteq [n]$ in M_i. The **Las Vergnas Tutte polynomial** of $\mathbf{M}$ is

$$LV\mathcal{T}_{\mathbf{M}}(x, y, z) := \sum_{S \subseteq [n]} (x-1)^{r_1 - r_1(S)} (y-1)^{|S| - r_2(S)} z^{r_2 - r_2(S) - (r_1 - r_1(S))} \tag{10}$$

For the remainder of this section, we let $\mathbf{M}$ be a two-step flag matroid, i.e. a matroid quotient $M_1 \twoheadleftarrow M_2$ on a ground set $[n]$. We show in this section that $LV\mathcal{T}_{\mathbf{M}}$ arises K-theoretically from $y(\mathbf{M})$. We start by recalling the construction $(\widetilde{Fl})$ of $\widetilde{Fl}(1, r_1, r_2, n-1; n)$ in §3 with the maps

$$\begin{tikzcd} & \widetilde{Fl}(1, r_1, r_2, n-1; n) \arrow[dl, "\widetilde{\pi}_{\mathbf{r}}"'] \arrow[dr, "\widetilde{\pi}_{(n-1)1}"] & \\ Fl(r_1, r_2; n) & & (\mathbb{P}^{n-1})^{\vee} \times \mathbb{P}^{n-1}. \end{tikzcd}$$

We have an inclusion of tautological vector bundles $0 \to \mathcal{S}_1 \to \mathcal{S}_2$ on the flag variety $Fl(r_1, r_2; n)$. Let $\mathcal{S}_2/\mathcal{S}_1$ be the quotient bundle.

Theorem 5.2. *With the notations as above, we have*

$$LV\mathcal{T}_{\mathbf{M}}(\alpha,\beta,w) = \sum_{m=0}^{r_2-r_1} (\widetilde{\pi}_{(n-1)1})_* \widetilde{\pi}_{\mathbf{r}}^* \Big(y(\mathbf{M})[\mathcal{O}(0,1)][\bigwedge^m (\mathcal{S}_2/\mathcal{S}_1)] \Big) w^m \tag{11}$$

as elements in $K^0((\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1})[w]$.

We will prove the stronger statement that the T-equivariant version of Theorem 5.2 holds. By Proposition 3.1, we have that the following equality implies Theorem 5.2:

$$LV\mathcal{T}_{\mathbf{M}}(u+1,v+1,w) = \sum_{p,q,m} \chi\Big(y(\mathbf{M})[\mathcal{O}(0,1)][\bigwedge^p \mathcal{S}_1][\bigwedge^q \mathcal{Q}_2^\vee][\bigwedge^m (\mathcal{S}_2/\mathcal{S}_1)] \Big) u^p v^q w^m. \tag{12}$$

We thus define the **T-equivariant Las Vergnas Tutte polynomial** of M by

$$\begin{aligned} &LV\mathcal{T}_{\mathbf{M}}^T(u+1,v+1,w) \\ &\quad := \sum_{p,q,m} \chi^T \Big(y(\mathbf{M})^T[\mathcal{O}(0,1)]^T[\bigwedge^p \mathcal{S}_1]^T[\bigwedge^q \mathcal{Q}_2^\vee]^T[\bigwedge^m (\mathcal{S}_2/\mathcal{S}_1)]^T \Big) u^p v^q w^m. \end{aligned}$$

Theorem 5.3. *With the notations as above, we have*

$$LV\mathcal{T}_{\mathbf{M}}^T(u+1,v+1,w) = \sum_{S\subseteq[n]} \mathbf{t}^{\mathbf{e}_S} u^{r_1-r_1(S)} v^{|S|-r_2(S)} w^{r_2-r_1-r_2(S)+r_1(S)}.$$

Proof. First, it follows from Theorem 2.8.(3) that

$$\begin{aligned} &\sum_{p,q,m} \chi^T \Big(y(\mathbf{M})^T[\mathcal{O}(0,1)]^T[\bigwedge^p \mathcal{S}_1]^T[\bigwedge^q \mathcal{Q}_2^\vee]^T[\bigwedge^m (\mathcal{S}_2/\mathcal{S}_1)]^T \Big) u^p v^q w^m \\ &= \sum_{\substack{\mathbf{B}=(B_1,B_2),\\ \mathbf{B}\in\mathcal{B}(\mathbf{M})}} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \cdot \mathbf{t}^{\mathbf{e}_{B_2}} \Big(\sum_{\mathfrak{p}'\subseteq B_1} \mathbf{t}^{-\mathbf{e}_{\mathfrak{p}'}} u^{|\mathfrak{p}'|} \Big) \Big(\sum_{\mathfrak{q}\subset[n]\setminus B_2} \mathbf{t}^{\mathbf{e}_{\mathfrak{q}}} v^{|\mathfrak{q}|} \Big) \\ &\quad \times \Big(\sum_{\mathfrak{m}'\subseteq B_2\setminus B_1} \mathbf{t}^{-\mathbf{e}_{\mathfrak{m}'}} w^{|\mathfrak{m}'|} \Big) \\ &= \sum_{\mathbf{B}\in\mathcal{B}(\mathbf{M})} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \cdot \Big(\mathbf{t}^{\mathbf{e}_{B_1}} \sum_{\mathfrak{p}'\subseteq B_1} \mathbf{t}^{-\mathbf{e}_{\mathfrak{p}'}} u^{|\mathfrak{p}'|} \Big) \Big(\sum_{\mathfrak{q}\subset[n]\setminus B_2} \mathbf{t}^{\mathbf{e}_{\mathfrak{q}}} v^{|\mathfrak{q}|} \Big) \\ &\quad \times \Big(\mathbf{t}^{\mathbf{e}_{B_2\setminus B_1}} \sum_{\mathfrak{m}'\subseteq B_2\setminus B_1} \mathbf{t}^{-\mathbf{e}_{\mathfrak{m}'}} w^{|\mathfrak{m}'|} \Big) \\ &= \sum_{\mathbf{B}\in\mathcal{B}(\mathbf{M})} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\substack{\mathfrak{p}\subseteq B_1\\ \mathfrak{m}\subseteq B_2\setminus B_1\\ \mathfrak{q}\subseteq[n]\setminus B_2}} \mathbf{t}^{\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{m}}+\mathbf{e}_{\mathfrak{q}}} u^{r_1-|\mathfrak{p}|} v^{|\mathfrak{q}|} w^{r_2-r_1-|\mathfrak{m}|}. \end{aligned} \tag{13}$$

We can now compute the sum

$$\sum_{\mathbf{B}\in\mathcal{B}(\mathbf{M})} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\substack{\mathfrak{p}\subseteq B_1,\, |\mathfrak{p}|=p \\ \mathfrak{m}\subseteq B_2\setminus B_1,\, |\mathfrak{m}|=m \\ \mathfrak{q}\subseteq [n]\setminus B_2,\, |\mathfrak{q}|=q}} \mathbf{t}^{\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{m}}+\mathbf{e}_{\mathfrak{q}}},$$

for fixed p, m, q.

To compute the coefficient of $\mathbf{t}^{\mathbf{e}_S}$, we apply Theorem 4.11. Pick a basis (B_1, B_2) such that $|S\cap B_1| = \mathrm{rk}_1(S)$ and $|S\cap B_2| = \mathrm{rk}_2(S)$. We need to compute the number of terms in the sum above for which $\mathbf{B} = (B_1, B_2)$ and $\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{m}} + \mathbf{e}_{\mathfrak{q}} = \mathbf{e}_S$. But such a term needs to satisfy $\mathfrak{p} = S\cap B_1$, $\mathfrak{p}\cup\mathfrak{m} = S\cap B_2$, and $\mathfrak{p}\cup\mathfrak{m}\cup\mathfrak{q} = S$. In particular $p = \mathrm{rk}_1(S)$, $p+m = \mathrm{rk}_2(S)$, and $p+m+q = |S|$. If these three equalities are satisfied, there is indeed exactly one such term. $\square$

Remark 5.4. We remark that the Las Vergnas polynomial, and our K-theoretic interpretation of it, generalizes the Tutte polynomial of a matroid in the following ways. Recall that any matroid M has two canonical matroid quotients, $M \twoheadrightarrow M$ and $M \twoheadrightarrow U_{0,n}$.

- When $\mathbf{M} = (M)$ (i.e. one constituent), the equation (11) reduces to the one in Theorem 2.12 [16, Theorem 5.1].
- When $\mathbf{M} = (M, M)$, one can observe from (10) or (11) that $LV\mathcal{T}_{\mathbf{M}}(x, y, z) = T_M(x, y)$.
- When $\mathbf{M} = (U_{0,n}, M)$, one can observe from (10) or (12) that $LV\mathcal{T}_{\mathbf{M}}(x, y, z) = T_M(z + 1, y)$.

Remark 5.5. The Las Vergnas Tutte polynomial satisfies a deletion-contraction relation similar to that of the Tutte polynomial [28, Proposition 5.1]. We remark that our "cone-flipping with slices" (Theorem 4.7) can be used to show deletion-contraction relation for $LV\mathcal{T}_{\mathbf{M}}$ and T_M. For example, if $i\in[n]$ is neither a loop nor a coloop of M_2,

$$LV\mathcal{T}_{M_1,M_2}(x, y, z) = LV\mathcal{T}_{M_1\setminus i, M_2\setminus i}(x, y, z) + LV\mathcal{T}_{M_1/i, M_2/i}(x, y, z).$$

This identity is obtained by applying Theorem 4.7 to (13) as follows. By considering $\boldsymbol{\zeta} = \mathbf{e}_i$, we find that the terms in (13) that are not divisible by t_i sum to $LV\mathcal{T}^{T'}_{M_1\setminus i, M_2\setminus i}(x, y, z)$, where $T' = (\mathbb{C}^*)^{n-1}$. By considering $\boldsymbol{\zeta} = -\mathbf{e}_i$, we find that the terms that are divisible by t_i sum to $t_i LV\mathcal{T}^{T'}_{M_1/i, M_2/i}(x, y, z)$. We leave the details to the reader.

Remark 5.6. Unlike the Tutte polynomials of matroids, the constant term of $LV\mathcal{T}_{\mathbf{M}}$ is no longer necessarily zero. This reflects the fact that for most $\mathbf{L}\in Fl(r_1, r_2; n)$, the map $\widetilde{\pi}_{(n-1)1} : \widetilde{\pi}_{\mathbf{r}}^{-1}(\overline{T\cdot\mathbf{L}}) \to (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}$ is surjective. If further $r_2 - r_1 = 1$, then this map is a finite morphism, and by a similar computation as in [38, Theorem 5.1], one can show that the degree of the map is the Crapo's beta invariant $\beta(N)$ where N is a matroid such that $M_1 = N/e$ and $M_2 = N\setminus e$.

Remark 5.7. It follows from Remark 2.11 and Theorem 5.2 that the assignment $\mathbf{M} \mapsto LV\mathcal{T}_{\mathbf{M}}$ is valuative.

6. Flag-geometric Tutte polynomial of a flag matroid

In this section, we explore the behavior of another notion of Tutte polynomials of flag matroids that differs from that of Las Vergnas in the previous section. Here, instead of the construction $(\widetilde{Fl})$, we consider the more geometrically natural construction (Fl) in §3 with the maps

$$\begin{tikzcd} & Fl(1, \mathbf{r}, n-1; n) \arrow[dl, "\pi_{\mathbf{r}}"'] \arrow[dr, "\pi_{(n-1)1}"] & \\ Fl(\mathbf{r}; n) & & (\mathbb{P}^{n-1})^\vee \times \mathbb{P}^{n-1}. \end{tikzcd}$$

Definition 6.1. [8, Definition 8.23] Let $\mathbf{M}$ be a flag matroid of rank $\mathbf{r} = (r_1, \ldots, r_k)$ on $[n]$. Then the **flag-geometric Tutte polynomial** of $\mathbf{M}$, denoted $K\mathcal{T}_{\mathbf{M}}(x, y) \in \mathbb{Z}[x, y]$, is the (unique) polynomial of bi-degree at most $(n-1, n-1)$ such that

$$K\mathcal{T}_{\mathbf{M}}(\alpha, \beta) = (\pi_{(n-1)1})_* \pi_r^* \Big(y(\mathbf{M}) \cdot [\mathcal{O}(\mathbf{1})] \Big). \tag{14}$$

While the construction (Fl) leading to $K\mathcal{T}_{\mathbf{M}}$ may be more geometrically natural than $(\widetilde{Fl})$, the combinatorial properties of $K\mathcal{T}_{\mathbf{M}}$ seem more mysterious than those of $LV\mathcal{T}_{\mathbf{M}}$. For example, in contrast to $LV\mathcal{T}_{\mathbf{M}}$, the polynomial $K\mathcal{T}_{\mathbf{M}}$ does not readily reduce to the Tutte polynomial of M when $\mathbf{M}$ is one of the two canonical matroid quotients of a matroid M (i.e. $M \twoheadleftarrow M$ and $U_{0,n} \twoheadleftarrow M$).

We illuminate some combinatorial structures of $K\mathcal{T}_{\mathbf{M}}$ as follows.

- There is no known (corank-nullity) combinatorial formula for $K\mathcal{T}_{\mathbf{M}}$ that is similar to (10) for $LV\mathcal{T}_{\mathbf{M}}$. Our result in §6.2, which in particular computes $K\mathcal{T}_{\mathbf{M}}(2, 2)$, can be considered as a first step in this direction.
- No deletion-contraction relation is known to hold for $K\mathcal{T}_{\mathbf{M}}$; one may construe this to be a consequence of the fact that the base polytope of a flag matroid generally has lattice points that are not vertices. In §6.3 we formulate and prove a deletion-contraction-like relation for elementary matroid quotients.

6.1. First properties of $K\mathcal{T}_{\mathbf{M}}$

Again, by Proposition 3.1, we have that

$$K\mathcal{T}_{\mathbf{M}}(u+1, v+1) = \sum_{p,q} \chi\Big(y(\mathbf{M})[\mathcal{O}(\mathbf{1})][\bigwedge^p \mathcal{S}_k][\bigwedge^q \mathcal{Q}_1^\vee] \Big) u^p v^q,$$

which leads us to the following T-equivariant version of $K\mathcal{T}_{\mathbf{M}}$.

Definition 6.2. The T-equivariant flag-geometric Tutte polynomial of a flag matroid $\mathbf{M}$ is

$$K\mathcal{T}^T_{\mathbf{M}}(u+1,v+1) := \sum_{p,q} \chi^T\Big(y(\mathbf{M})^T[\mathcal{O}(\mathbf{1})]^T[\bigwedge^p \mathcal{S}_k]^T[\bigwedge^q \mathcal{Q}_1^\vee]^T\Big)u^p v^q.$$

Theorem 2.8.(3) again yields $K\mathcal{T}^T_{\mathbf{M}}$ as a sum of rational functions as follows via a similar computation as one in the proof of Theorem 5.2.

Lemma 6.3. *For a flag matroid $\mathbf{M} = (M_1,\dots,M_k)$ on a ground set $[n]$, we have*

$$K\mathcal{T}^T_{\mathbf{M}}(u+1,v+1) = \sum_{\mathbf{B}\in\mathbf{M}} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\mathfrak{p}\subseteq B_k}\sum_{\mathfrak{q}\subseteq[n]\setminus B_1} \mathbf{t}^{\mathbf{e}_{B_1}+\cdots+\mathbf{e}_{B_{k-1}}+\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{q}}} u^{r_k-|\mathfrak{p}|} v^{|\mathfrak{q}|}. \tag{15}$$

Many of our results on $K\mathcal{T}_{\mathbf{M}}$ will be obtained by manipulation with the equation (15). We start with the following example.

Example 6.4. For any matroid M on $[n]$, we have $K\mathcal{T}_{U_{0,n},M}(x,y) = y^n T_M(x,1)$.[6] To verify this, we compute

$$\begin{aligned} K\mathcal{T}^T_{U_{0,n},M}(u+1,v+1) &= \sum_{B\in\mathcal{B}(M)} \mathrm{Hilb}(\mathrm{Cone}_B(M)) \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]} \mathbf{t}^{\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{q}}} u^{r-|\mathfrak{p}|} v^{|\mathfrak{q}|} \\ &= \Big(\prod_{i=1}^n (1+t_i v)\Big) \cdot \sum_{B\in\mathcal{B}(M)} \mathrm{Hilb}(\mathrm{Cone}_B(M)) \sum_{\mathfrak{p}\subseteq B} \mathbf{t}^{\mathbf{e}_{\mathfrak{p}}} u^{r-|\mathfrak{p}|} \qquad (16) \\ &= \Big(\prod_{i=1}^n (1+t_i v)\Big) \cdot K\mathcal{T}^T_M(u+1,1). \end{aligned}$$

Setting $t_i = 1$, $u = x-1$, and $v = y-1$ yields the desired claim. This example shows that we cannot recover T_M from $K\mathcal{T}_{U_{0,n},M}$ although $U_{0,n} \twoheadleftarrow M$ is a canonical matroid quotient of M.

Proposition 6.5. *Let $\mathbf{M} = (M_1,\dots,M_k)$ be a flag matroid on $[n]$. The following properties hold for the flag-geometric Tutte polynomial $K\mathcal{T}^T_{\mathbf{M}}$:*

(1) *(Direct sum) If $\mathbf{M}$ is a direct sum $\mathbf{M}'\oplus\mathbf{M}''$ of two flag matroids on ground sets A, B with $A\sqcup B = [n]$, then $K\mathcal{T}^T_{\mathbf{M}}(x,y) = K\mathcal{T}^{T'}_{\mathbf{M}'}(x,y)\cdot K\mathcal{T}^{T''}_{\mathbf{M}''}(x,y)$ (where $T' = (\mathbb{C}^*)^A$, $T'' = (\mathbb{C}^*)^B$).*

[6] The diagram (Fl) makes sense only when $r_1 \geq 1$, so $K\mathcal{T}_{U_{0,n},M}$ cannot be defined as a push-pull of a K-class. However, we define $K\mathcal{T}_{U_{0,n},M}$ by specializing $K\mathcal{T}^T_{U_{0,n},M}$ at $t_i = 1$.

(2) *(Loops & coloops) Let ℓ be the number of loops in M_1, and c the number of coloops in M_k. Then $x^c y^\ell$ divides $K\mathcal{T}_{\mathbf{M}}(x,y)$.*
(3) *(Duality) If $\mathbf{M}^\vee$ is the dual flag matroid of $\mathbf{M}$, whose constituents are matroid duals of the original, then $K\mathcal{T}_{\mathbf{M}}(y,x) = K\mathcal{T}_{\mathbf{M}^\vee}(x,y)$.*
(4) *(Base polytope) $K\mathcal{T}^T_{\mathbf{M}}(1,1) = \mathrm{Hilb}(Q(\mathbf{M}))$.*
(5) *(Valuativeness) The map $\mathbf{M} \mapsto K\mathcal{T}_{\mathbf{M}}$ is valuative.*

Proof. The first two statements follow from manipulating with the identity (15) in a similar way as the computation (16) in Example 6.4. For the third statement, we claim that the T-equivariant version of the statement is $t^{\mathbf{e}_{[n]}} K\mathcal{T}^{T^{-1}}_{\mathbf{M}}(y,x) = K\mathcal{T}_{\mathbf{M}^\vee}(x,y)$, where the T^{-1} superscript means that we have replaced t_i by t_i^{-1}. Verifying this identity is then another easy manipulation with (15). The fourth statement follows from Brion's formula (Theorem 4.2). The last statement follows from Remark 2.11. $\square$

We can use Theorem 4.11 to compute some of the terms in (15):

Theorem 6.6. *Let $\mathbf{M} = (M_1, M_2)$ be a 2-step flag matroid and let $\mathbf{t}^{\mathbf{k}} u^{r_2-i} v^j$ be a monomial occurring in (15). Then $\sum_{\ell=1}^n k_\ell = r_1 + i + j$. Let c denote the number of entries in $\mathbf{k}$ that are equal to 1. If $c \leq |r_1 + j - i|$, the coefficient of $\mathbf{t}^{\mathbf{k}} u^{r_2-i} v^j$ is equal to*

(1) 1*, if S_2 is spanning for M_1, S_1 is independent in M_2, and $c = |r_1 + j - i|$,*
(2) 0*, otherwise,*

where S_1 and S_2 are defined by $S_1 \subseteq S_2$ and $\mathbf{k} = \mathbf{e}_{S_1} + \mathbf{e}_{S_2}$.

Proof. The equality $\sum_{\ell=1}^n k_\ell = r_1 + i + j$ follows immediately from (15). The coefficient of $u^{r_2-i} v^j$ is equal to

$$\sum_{\substack{\mathbf{B} = (B_1, B_2), \\ \mathbf{B} \in \mathcal{B}(\mathbf{M})}} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\substack{\mathfrak{p} \subseteq B_2, \\ |\mathfrak{p}| = i}} \sum_{\substack{\mathfrak{q} \subseteq J_1, \\ |\mathfrak{q}| = j}} \mathbf{t}^{\mathbf{e}_{B_1} + \mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}}}, \tag{17}$$

where we have denoted $J_1 := [n] \setminus B_1$. The vertices of $\widetilde{P}$ have $|r_1 + j - i|$ entries equal to 1. This proves that the coefficient is 0 if $c < |r_1 + j - i|$. So from now on we assume $c = |r_1 + j - i|$.

Next, we apply Theorem 4.11. Writing $\mathbf{k} = \mathbf{e}_{S_1} + \mathbf{e}_{S_2}$, and noting that $|S_1| = \min(i, r_1 + j)$ and $|S_2| = \max(i, r_1 + j)$, we find a basis $\mathbf{B} = (B_1, B_2)$ of $\mathbf{M}$ for which $r_j(S_i) = |S_i \cap B_j|$. We now need to compute the number of ways $\mathbf{k}$ can be written as a sum $\mathbf{e}_{B_1} + \mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}}$. If S_2 is not spanning for M_1, or if S_1 is not independent in M_2, there are no ways to do this, and the coefficient is 0. Otherwise, if $i \leq r_1 + j$, we need to put $\mathfrak{p} = S_1$ and $\mathfrak{q} = S_2 \setminus S_1$. If $i \geq r_1 + j$, we need to put $\mathfrak{q} = S_1 \cap J_1$ and $\mathfrak{p} = S_1 \cup J_1$. In both cases, there is just one way, so the coefficient is 1. $\square$

6.2. Towards a corank-nullity formula

For a matroid M on $[n]$, the corank-nullity formula for the Tutte polynomial $T_M(x,y) = \sum_{S\subseteq[n]}(x-1)^{r-r(S)}(y-1)^{|S|-r(S)}$ expresses T_M as a sum over all subsets of $[n]$. In particular, we have $T_M(2,2) = 2^n$; in fact, $K\mathcal{T}_M^T(2,2) = \prod_{i=1}^n(1+t_i)$. As a first step towards a similar formula for $K\mathcal{T}_{\mathbf{M}}$, we show the following for a two-step flag matroid.

Theorem 6.7. *Let* $\mathbf{M}$ *be a two-step flag matroid* $\mathbf{M} = (M_1, M_2)$ *of rank* (r_1, r_2)*, and let* $p\mathcal{B}(\mathbf{M})$ *be the set of pseudo-bases of* $\mathbf{M}$*, i.e. subsets* $S \subseteq [n]$ *such that* S *is spanning in* M_1 *and independent in* M_2*. With* q *as a formal variable, we have*

$$K\mathcal{T}_{\mathbf{M}}^T(1+q^{-1}, 1+q) = q^{-r_2}\Big(\prod_{i=1}^n(1+t_iq)\Big)\Big(\sum_{S\in p\mathcal{B}(\mathbf{M})}\mathbf{t}^{\mathbf{e}_S}q^{|S|}\Big),$$

and in particular, we have

$$\begin{aligned}
&K\mathcal{T}_{\mathbf{M}}(1+q^{-1}, 1+q) = q^{-r_2}\cdot(1+q)^n\cdot\Big(\sum_{S\in p\mathcal{B}(\mathbf{M})}q^{|S|}\Big),\\
&K\mathcal{T}_{\mathbf{M}}^T(2,2) = \Big(\prod_{i=1}^n(1+t_i)\Big)\Big(\sum_{S\in p\mathcal{B}(\mathbf{M})}\mathbf{t}^{\mathbf{e}_S}\Big), \text{ and}\\
&K\mathcal{T}_{\mathbf{M}}(2,2) = 2^n|p\mathcal{B}(\mathbf{M})|.
\end{aligned}$$

Proof. Setting $u = q^{-1}$ and $v = q$ in (15) of Lemma 6.3 gives us

$$\begin{aligned}
&K\mathcal{T}_{\mathbf{M}}^T(1+q^{-1}, 1+q)\\
&\quad= \sum_{\substack{\mathbf{B}=(B_1,B_2),\\ \mathbf{B}\in\mathcal{B}(\mathbf{M})}}\operatorname{Hilb}(\operatorname{Cone}_{\mathbf{B}}(\mathbf{M}))\sum_{\mathfrak{p}\subseteq B_2}\sum_{\mathfrak{q}\subseteq[n]\setminus B_1}\mathbf{t}^{\mathbf{e}_{B_1}+\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{q}}}q^{|\mathfrak{p}|+|\mathfrak{q}|-r_2}\\
&\quad= \sum_{\substack{\mathbf{B}=(B_1,B_2),\\ \mathbf{B}\in\mathcal{B}(\mathbf{M})}}\operatorname{Hilb}(\operatorname{Cone}_{\mathbf{B}}(\mathbf{M}))\sum_{R\subseteq E}\sum_{S\subseteq B_2\setminus B_1}\mathbf{t}^{\mathbf{e}_{B_1}+\mathbf{e}_R+\mathbf{e}_S}q^{|R|+|S|-r_2}\\
&\quad= q^{-r_2}\prod_{i\in E}(1+t_iq)\sum_{\substack{\mathbf{B}=(B_1,B_2),\\ \mathbf{B}\in\mathcal{B}(\mathbf{M})}}\operatorname{Hilb}(\operatorname{Cone}_B(\mathbf{M}))\sum_{S\subseteq B_2\setminus B_1}\mathbf{t}^{\mathbf{e}_{B_1}+\mathbf{e}_S}q^{|S|}.
\end{aligned}$$

We now use Theorem 4.11 to compute the sum

$$\varphi_r := \sum_{\substack{\mathbf{B}=(B_1,B_2),\\ \mathbf{B}\in\mathcal{B}(\mathbf{M})}}\operatorname{Hilb}(\operatorname{Cone}_{\mathbf{B}}(\mathbf{M}))\sum_{\substack{B_1\subseteq\mathfrak{p}\subseteq B_2,\\ |\mathfrak{p}|=r}}\mathbf{t}^{\mathbf{e}_{\mathfrak{p}}}$$

for a fixed $r_1 \le r \le r_2$. First, we note that the polytope $\widetilde{P} = \operatorname{Conv}(\mathbf{e}_S \mid S\subseteq E,\ |S| = r)$, obtained as the convex hull of the S_n-orbit of $\{\mathbf{e}_{\mathfrak{p}} \mid B_1\subseteq\mathfrak{p}\subseteq B_2,\ |\mathfrak{p}| = r\}$, has no interior lattice points.

For $S \subseteq E$ with $|S| = r$, if S is not a pseudo-basis of $M_1 \twoheadleftarrow M_2$, then there is no basis $\mathbf{B}$ of $\mathbf{M}$ such that $B_1 \subseteq S \subseteq B_2$, and hence the coefficient of $\mathbf{t}^{\mathbf{e}_S}$ is 0 in this case. Now, suppose S is a pseudo-basis of $M_1 \twoheadleftarrow M_2$, which by definition implies that there exists basis $\mathbf{B} = (B_1, B_2)$ of $\mathbf{M}$ with $B_1 \subseteq S \subseteq B_2$. This basis $\mathbf{B}$ is a vertex of the face $Q(\mathbf{M})^{\mathbf{e}_S}$ by Proposition 2.7, and thus by Theorem 4.11 the coefficient of $\mathbf{t}^{\mathbf{e}_S}$ is equal to 1 in φ_r. $\square$

We do not know of analogues of Theorem 6.7 for flag matroids with more than two constituents.

6.3. A deletion-contraction-like relation

In this section, we consider $K\mathcal{T}_{\mathbf{M}}$ of an elementary quotient $\mathbf{M} = (M_1, M_2)$. By definition we have $r(M_2) - r(M_1) = 1$, and in this case there is a unique matroid M on a ground set $[\widetilde{n}] := \{0\} \sqcup [n]$ such that $M_1 = M/0$ and $M_2 = M \setminus 0$ [35, §7.3]. Our main theorem of this subsection is the following deletion-contraction-like relation.

Theorem 6.8. *Let M be a matroid of rank r on $[\widetilde{n}] := \{0\} \sqcup [n]$ such that the element 0 is neither a loop nor a coloop in M. Let $\widetilde{T} = \mathbb{C}^* \times T = (\mathbb{C}^*)^{n+1}$ be the torus with character ring $\mathbb{Z}[t_0^{\pm}, \dots, t_n^{\pm}]$. Then we have*

$$K\mathcal{T}^{\widetilde{T}}_{M,M}(x,y) = t_0^2 K\mathcal{T}^{T}_{M/0,M/0}(x,y) + t_0 K\mathcal{T}^{T}_{M/0,M\setminus 0}(x,y) + K\mathcal{T}^{T}_{M\setminus 0,M\setminus 0}(x,y). \qquad (18)$$

In particular, we have $K\mathcal{T}_{M,M}(x,y) = K\mathcal{T}_{M/0,M/0}(x,y) + K\mathcal{T}_{M/0,M\setminus 0}(x,y) + K\mathcal{T}_{M\setminus 0,M\setminus 0}(x, y)$.

We use $\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$ for the standard basis of $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$. For a polyhedron $P \subset \mathbb{R}^n$, we will often abuse the notation and write P also for $\{\mathbf{0}\} \times P \subset \mathbb{R} \oplus \mathbb{R}^n$. We prepare for the proof of Theorem 6.8 by an observation that motivated the theorem.

As the base polytope $Q(M)$ is a $(0,1)$-polytope (i.e. a lattice polytope contained in the Boolean cube $[0,1]^{n+1} \subset \mathbb{R}^{n+1}$), every lattice point is a vertex. Moreover, observe that the vertices of $Q(M)$ partition into two parts, the bases of $M/0$ and the bases of $M \setminus 0$. As a result, the lattice points of $Q(M,M) = Q(M) + Q(M)$ partition into the following three parts, with $Q_1 = \frac{1}{2}(Q_0 + Q_2)$:

- $Q_2 := Q(M,M) \cap H_{\mathbf{e}_0=2} = \{2\mathbf{e}_0\} \times Q(M/0, M/0)$,
- $Q_1 := Q(M,M) \cap H_{\mathbf{e}_0=1} = \{\mathbf{e}_0\} \times Q(M/0, M \setminus 0)$, and
- $Q_0 := Q(M,M) \cap H_{\mathbf{e}_0=0} = \{\mathbf{0}\} \times Q(M \setminus 0, M \setminus 0)$.

The case of setting $x = y = 1$ (cf. Proposition 6.5.(4)) in (18) of Theorem 6.8 witnesses this partition of the lattice points of $Q(M,M)$. The following lemma in preparation for the proof of Theorem 6.8 is a consequence of $Q_1 = \frac{1}{2}(Q_0 + Q_2)$.

Lemma 6.9. *Let the notations be as above. Then for $B \in \mathcal{B}(M)$ with $0 \notin B$, we have*

$$\mathrm{Hilb}(\mathrm{Cone}_B(Q(M,M)) \cap H_{\mathbf{e}_0=1}) = \sum_{\substack{I \in \mathcal{B}(M/0), \\ I \subset B}} t_0 t_{B\setminus I}^{-1} \mathrm{Hilb}(\mathrm{Cone}_{(I,B)}(Q(M/0, M\setminus 0))$$

and

$$\mathrm{Hilb}(\mathrm{Cone}_B(Q(M,M)) \cap H_{\mathbf{e}_0=0}) = \sum_{\substack{I \in \mathcal{B}(M/0), \\ I \subset B}} \mathrm{Hilb}(\mathrm{Cone}_{(I,B)}(Q(M/0, M\setminus 0)).$$

Proof. We have an equality of polyhedra

$$\mathrm{Cone}_B(Q(M,M)) \cap H_{\mathbf{e}_0=1} = \mathrm{Cone}_B(Q(M\setminus 0)) + Q_1 - 2\mathbf{e}_B.$$

We claim that $\mathrm{Cone}_B(Q(M\setminus 0)) + Q_1$ has vertices $\{\mathbf{e}_I + \mathbf{e}_B\}$ for $I \in \mathcal{B}(M/0)$ such that $I \subset B$. The two statements in the lemma then follow from Brion's formula Theorem 4.2.

For the claim, we start by noting that if $I \in \mathcal{B}(M/0)$ then there exists $B' \in \mathcal{B}(M \setminus 0)$ such that $I \subset B'$ (since $M/0 \twoheadleftarrow M\setminus 0$). Consequently, if $\mathbf{e}_B$ is the vertex of $Q(M\setminus 0)$ that minimizes $\langle v, \mathbf{e}_B\rangle$ for some $v \in \mathbb{R}^n$, then a vertex of $Q(M/0)$ that minimizes $\langle v, \cdot\rangle$ must be $\mathbf{e}_I$ satisfying $I \subset B$. Our claim now follows from $Q_1 = \frac{1}{2}(Q_0 + Q_2)$. $\square$

Proof of Theorem 6.8. Let us begin by noting that the equation (15) for $K\mathcal{T}^{\widetilde{T}}_{M,M}$ reads

$$K\mathcal{T}^{\widetilde{T}}_{M,M}(u+1, v+1) = \sum_{B\in\mathcal{B}(M)} \mathrm{Hilb}(\mathrm{Cone}_B(Q(M,M))) \sum_{\mathfrak{p}\subseteq B} \sum_{\mathfrak{q}\subseteq[\widetilde{n}]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|} v^{|\mathfrak{q}|}. \tag{19}$$

We apply Theorem 4.7 with $\boldsymbol{\zeta} = \mathbf{e}_0$ and L defined by $t_0 = 0$. Note that $\mathrm{Cone}_B(Q(M,M)) \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ if and only if $0 \notin B$. Hence all cones occurring in (19) with vertex on L are in $\mathcal{P}_n^{\boldsymbol{\zeta}}$, and we find that the terms in (19) not divisible by t_0 sum to

$$\begin{aligned}
&\sum_{\substack{B \in \mathcal{B}(M), \\ 0 \notin B}} \mathrm{Hilb}(\mathrm{Cone}_B(Q(M,M)) \cap H_{\mathbf{e}_0=0}) \sum_{\mathfrak{p}\subseteq B} \sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|} v^{|\mathfrak{q}|} \\
&\qquad = \sum_{B\in\mathcal{B}(M\setminus 0)} \mathrm{Hilb}(\mathrm{Cone}_B(M\setminus 0)) \sum_{\mathfrak{p}\subseteq B} \sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|} v^{|\mathfrak{q}|} \\
&\qquad = K\mathcal{T}^{T}_{M\setminus 0, M\setminus 0}(u+1, v+1).
\end{aligned}$$

A similar argument, with $\boldsymbol{\zeta} = -\mathbf{e}_0$, shows that the coefficient of t_0^2 in (19) is $K\mathcal{T}^{T}_{M/0,M/0}$.

Finally, we apply Theorem 4.7 once more, this time with $\boldsymbol{\zeta} = \mathbf{e}_0$ and $L = H_{\mathbf{e}_0=1}$. We find that the terms in (19) divisible by t_0 but not by t_0^2 sum to

$$
\begin{aligned}
&\Bigg(\sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \operatorname{Hilb}(\operatorname{Cone}_B(Q(M,M))) \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[\widetilde{n}]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|} \Bigg)\Bigg|_{H_{\mathbf{e}_0=1}}\\
&= \Bigg(\sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \operatorname{Hilb}(\operatorname{Cone}_B(Q(M,M))) \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} (1+t_0v) u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|} \Bigg)\Bigg|_{H_{\mathbf{e}_0=1}}\\
&= \sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \operatorname{Hilb}(\operatorname{Cone}_B(Q(M,M))\cap H_{\mathbf{e}_0=1}) \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|}\\
&\quad + t_0 \sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \operatorname{Hilb}(\operatorname{Cone}_B(Q(M,M))\cap H_{\mathbf{e}_0=0}) \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|+1},
\end{aligned}
$$

which by Lemma 6.9 is equal to

$$
\begin{aligned}
&\sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \sum_{\substack{I\,\in\,\mathcal{B}(M/0),\\ I\,\subset\,B}} t_0 t_{B\setminus I}^{-1} \operatorname{Hilb}(\operatorname{Cone}_{(I,B)}(Q(M/0, M\setminus 0)))\\
&\qquad \times \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|}\\
&\qquad + t_0 \sum_{\substack{B\,\in\,\mathcal{B}(M),\\ 0\,\notin\,B}} \sum_{\substack{I\,\in\,\mathcal{B}(M/0),\\ I\,\subset\,B}} \operatorname{Hilb}(\operatorname{Cone}_{(I,B)}(Q(M/0, M\setminus 0)))\\
&\qquad \times \sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_B+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|+1}\\
&= t_0 \sum_{(I,B)\in\mathcal{B}(M/0,M\setminus 0)} \operatorname{Hilb}(\operatorname{Cone}_{(I,B)}(Q(M/0, M\setminus 0)))\\
&\qquad \times \Bigg(\sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus B} \mathbf{t}^{\mathbf{e}_I+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} (1+t_{B\setminus I}v) \Bigg) u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|}\\
&= t_0 \sum_{(I,B)\in\mathcal{B}(M/0,M\setminus 0)} \operatorname{Hilb}(\operatorname{Cone}_{(I,B)}(Q(M/0, M\setminus 0)))\\
&\qquad \times \Bigg(\sum_{\mathfrak{p}\subseteq B}\sum_{\mathfrak{q}\subseteq[n]\setminus I} \mathbf{t}^{\mathbf{e}_I+\mathbf{e}_\mathfrak{p}+\mathbf{e}_\mathfrak{q}} \Bigg) u^{r-|\mathfrak{p}|}v^{|\mathfrak{q}|}\\
&= t_0 K\mathcal{T}^T_{M/0,M\setminus 0}(u+1, v+1),
\end{aligned}
$$

as desired. □

Remark 6.10. We remark that for a general flag matroid $\mathbf{M}$, the slices $\{Q(\mathbf{M}) \cap H_{\mathbf{e}_i=k}\}_{k\in\mathbb{Z}}$ need not be flag matroid base polytopes. Moreover, even when they are, we do not observe an identity like the one in Theorem 6.8 that expresses $K\mathcal{T}_\mathbf{M}$ in terms of the slices.

For example, consider $\mathbf{M} = (U_{1,3}, U_{2,3})$. We have $K\mathcal{T}_\mathbf{M}(x,y) = x^2y^2 + x^2y + xy^2 + x^2 + 2xy + y^2$. In any coordinate direction, its three slices are $(U_{0,2}, U_{1,2})$, $(U_{1,2}, U_{1,2})$, and $(U_{1,2}, U_{2,2})$, whose $K\mathcal{T}$ are (respectively), $xy^2 + y^2$, $xy + x + y$, and $x^2y + x^2$.

Remark 6.11. One can generalize Theorem 6.8 as follows. Denote by $M^\ell := (M, \dots, M)$, the flag matroid whose constituents are M repeated ℓ times. Then we have

$$K\mathcal{T}^{\widetilde{T}}_{M^\ell} = t_0^\ell K\mathcal{T}^T_{(M/0)^\ell} + t_0^{\ell-1} K\mathcal{T}^T_{(M/0)^{\ell-1}, M\setminus 0} + \cdots + K\mathcal{T}_{(M\setminus 0)^\ell}.$$

The proof is essentially identical to one given for Theorem 6.8.

7. Future directions

We list two future directions stemming from our work here.

7.1. g and h polynomials for flag matroids

For a matroid M, Speyer introduced in [38] a polynomial invariant $g_M(t) \in \mathbb{Q}[t]$ and a close cousin $h_M(t) \in \mathbb{Q}[t]$, which is related to $g_M(t)$ by $h_M(t) = (-1)^c g_M(-t)$ where c is the number of connected components of M. A K-theoretic interpretation of the polynomial h_M was given in [16].

Theorem 7.1. *[16, Theorem 6.1 & Theorem 6.5] Let M be a matroid of rank r on $[n]$ without loops or coloops. Let $\pi_r, \pi_{(n-1)1}, \alpha, \beta$ be as in §2.3. Then the polynomial h_M is the (unique) univariate polynomial of degree at most $n-1$ such that*

$$(\pi_{(n-1)1})_* \pi_r^* \Big(y(M) \Big) = h_M(\alpha + \beta - \alpha\beta).$$

For a flag matroid $\mathbf{M}$ on $[n]$, this motivates us to consider $(\pi_{(n-1)1})_* \pi_{\mathbf{r}}^* \Big(y(\mathbf{M}) \Big)$, where the maps are as in the flag-geometric construction (Fl). By Proposition 3.1, this is equal to

$$\sum_{p,q} \chi\Big(y(\mathbf{M}) [\textstyle\bigwedge^p \mathcal{S}_k] [\bigwedge^q \mathcal{Q}_1^\vee] \Big) (\alpha - 1)^p (\beta - 1)^q.$$

Let us consider its torus-equivariant version

$$\sum_{p,q} \chi^T \Big(y(\mathbf{M})^T [\textstyle\bigwedge^p \mathcal{S}_k]^T [\bigwedge^q \mathcal{Q}_1^\vee]^T \Big) u^p v^q$$

where u and v are formal variables. We show that this is a polynomial in uv, which thereby establishes that $(\pi_{(n-1)1})_* \pi_{\mathbf{r}}^* \Big(y(\mathbf{M}) \Big)$ is a polynomial in $\alpha + \beta - \alpha\beta$ (since the substitution $u = \alpha - 1, v = \beta - 1$ yields $1 - uv = \alpha + \beta - \alpha\beta$).

Lemma 7.2. *(cf. [16, Lemma 6.2]) Let $\mathbf{M} = (M_1, \dots, M_k)$ be a flag matroid on $[n]$, and suppose every constituent of $\mathbf{M}$ is both loopless and coloopless. Then*

$$\sum_{p,q} \chi^T\Big(y(\mathbf{M})^T[\bigwedge^p \mathcal{S}_k]^T[\bigwedge^q \mathcal{Q}_1^\vee]^T\Big)u^p v^q \in \mathbb{Q}[u,v]$$

is a polynomial in $\mathbb{Q}[uv]$.

We remark that the condition about a flag matroid $\mathbf{M} = (M_1, \ldots, M_k)$ being loopless or coloopless depends only on M_1 or M_k, respectively. First, note that by the condition (2) in Definition 2.1, if $\ell \in [n]$ is a loop in M_i then it is a loop in M_{i-1} also. By duality, if $\ell \in [n]$ is a coloop in M_i then it is a coloop in M_{i+1} also. Hence, the flag matroid $\mathbf{M}$ is loopless (coloopless) if and only if M_1 has no loops (M_k has no coloops).

Proof. Once more by Theorem 2.8.(3), we get

$$\begin{aligned}&\sum_{p,q} \chi^T\Big(y(\mathbf{M})^T[\bigwedge^p \mathcal{S}_k]^T[\bigwedge^q \mathcal{Q}_1^\vee]^T\Big)u^p v^q \\ &\quad = \sum_{\mathbf{B}\in\mathbf{M}} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\mathfrak{p}\subseteq B_k} \sum_{\mathfrak{q}\subseteq [n]\setminus B_1} \mathbf{t}^{-\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{q}}} u^{|\mathfrak{p}|} v^{|\mathfrak{q}|}.\end{aligned}$$

Fix $|\mathfrak{p}| = i$, $|\mathfrak{q}| = j$, and consider the sum

$$\varphi_{ij} = \sum_{\mathbf{B}\in\mathbf{M}} \mathrm{Hilb}(\mathrm{Cone}_{\mathbf{B}}(\mathbf{M})) \sum_{\substack{\mathfrak{p}\,\subseteq\, B_k,\\ |\mathfrak{p}|\,=\,i}} \sum_{\substack{\mathfrak{q}\,\subseteq\,[n]\,\setminus\, B_1,\\ |\mathfrak{q}|\,=\,j}} \mathbf{t}^{-\mathbf{e}_{\mathfrak{p}}+\mathbf{e}_{\mathfrak{q}}}. \tag{20}$$

We need show that φ_{ij} is zero if $i \neq j$. Let P be the convex hull of $\{-\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}}\}$ appearing in the summation (20). Note that P is contained in the intersection of $H_{\mathbf{e}_{[n]}=j-i}$ and the cube $\{x \in \mathbb{R}^n \mid -1 \leq x_\ell \leq 1 \ \forall \ell \in [n]\}$. By Corollary 4.8. (b), it thus suffices to show that $\varphi_{ij}|_{H_{\mathbf{e}_\ell=-1}} = 0$ and $\varphi_{ij}|_{H_{\mathbf{e}_\ell=1}} = 0$ for all $\ell \in [n]$.

Let us now fix any $\ell \in [n]$. As none of the constituents have coloops (and in particular ℓ is not a coloop in M_k), the intersection $Q(\mathbf{M}) \cap H_{\mathbf{e}_\ell=0}$ is a non-empty face of $Q(\mathbf{M})$ minimizing in the $\mathbf{e}_\ell$ direction, consisting of bases $\mathbf{B} = (B_1, \ldots B_k)$ such that $\ell \notin B_k$. Thus, we have that $\mathrm{Cone}_{\mathbf{B}}(\mathbf{M}) \in \mathcal{P}_n^{\mathbf{e}_\ell}$ if and only if $\ell \notin B_k$, and by Theorem 4.7 with $\boldsymbol{\zeta} = \mathbf{e}_\ell$ we have

$$\varphi_{ij}|_{H_{\mathbf{e}_\ell=-1}} = \sum_{\mathbf{B}:\ell\notin B_k} \sum_{\substack{\mathfrak{p}\,\subseteq\, B_k,\\ |\mathfrak{p}|\,=\,i}} \sum_{\substack{\mathfrak{q}\,\subseteq\,[n]\,\setminus\, B_1,\\ |\mathfrak{q}|\,=\,j}} \mathrm{Hilb}((-\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}} + \mathrm{Cone}_{\mathbf{B}}(\mathbf{M}))|_{H_{\mathbf{e}_\ell=-1}}).$$

But since $\ell \notin B_k$ implies $\ell \notin \mathfrak{p}$, every cone $-\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}} + \mathrm{Cone}_{\mathbf{B}}(\mathbf{M})$ occurring in the sum above will have vertex v with $v_\ell > -1$. Hence, noting that $\mathrm{Cone}_{\mathbf{B}}(\mathbf{M}) \in \mathcal{P}_n^{\mathbf{e}_\ell}$ for such cones, we have $\varphi_{ij}|_{H_{\mathbf{e}_\ell=-1}} = 0$. A similar argument with $\boldsymbol{\zeta} = -\mathbf{e}_\ell$, noting that ℓ is not a loop in M_1, shows that $\varphi_{ij}|_{H_{\mathbf{e}_\ell=1}} = 0$. $\square$

We thus make the following definition that generalizes the polynomial h_M of a matroid M to the setting of flag matroids. It is well-defined by Lemma 7.2.

Definition 7.3. Let $\mathbf{M} = (M_1, \ldots, M_k)$ be a flag matroid $[n]$ such that every constituent of $\mathbf{M}$ is both loopless and coloopless. Let $\pi_{(n-1)1}, \pi_{\mathbf{r}}, \alpha, \beta$ be as in §3. Then the polynomial $h_{\mathbf{M}}$ is defined as the (unique) univariate polynomial of degree at most $n-1$ such that

$$(\pi_{(n-1)1})_* \pi_{\mathbf{r}}^* \Big(y(\mathbf{M}) \Big) = h_{\mathbf{M}}(\alpha + \beta - \alpha\beta).$$

Remark 7.4. We have constructed the polynomial $h_{\mathbf{M}}$ via the flag-geometric diagram (Fl). Although one may also consider a similar construction via the "Las Vergnas" diagram $(\widetilde{Fl})$, a computer computation (§1.1) shows that the analogue of Lemma 7.2 fails in this case, for instance with $\mathbf{M} = (U_{2,4}, U_{3,4})$.

In the case of matroids realizable over $\mathbb{C}$, the behavior of the polynomial g_M of a matroid M, in particular the non-negativity of its coefficients, was used to establish a bound on the number of interior faces in a matroidal subdivision of a base polytope of a matroid [38]. Extending these results to arbitrary matroids is so far open, but an announcement of a relevant forthcoming work has been made in [26].

In another work [2], the authors study flag-matroidal subdivisions of base polytopes of flag matroids, and extend the tropical geometry of matroids used in [38] to the setting of flag matroids. We are thus led to ask the following.

Question 7.5. Does a suitable modification of our polynomial $h_{\mathbf{M}}$ give an analogue of the polynomial g_M for flag matroids, and does its behavior lead to a bound on the number of interior faces in a flag-matroidal subdivision of a base polytope of a flag matroid?

7.2. Characteristic polynomials of matroid morphisms

A recent breakthrough in matroid theory is the log-concavity of the coefficients of the characteristic polynomial of a matroid [1]. We consider here several candidates for characteristic polynomials of morphisms of matroids. We begin with the one coming from the flag-geometric Tutte polynomial.

Definition 7.6. For a flag matroid $\mathbf{M}$, define the **flag-geometric characteristic polynomial** $K\chi_{\mathbf{M}}(q)$ of $\mathbf{M}$ by

$$K\chi_{\mathbf{M}}(q) := (-1)^{r_k} K\mathcal{T}_{\mathbf{M}}(1-q, 0).$$

Like the usual characteristic polynomial, the polynomial $K\mathcal{T}_{\mathbf{M}}$ satisfies $K\chi_{\mathbf{M}}(q) = 0$ whenever the first constituent M_1 of $\mathbf{M}$ has a loop by Proposition 6.5. The following conjecture is supported by computer computations (§1.1). It suggests that the flag-geometric characteristic polynomial of a two-step flag matroid may contain little information about the flag matroid itself.

Conjecture 7.7. *Let M be a matroid of rank r with no loops, so that $U_{1,n} \twoheadleftarrow M$ is a valid matroid quotient. Then $K\chi_{(U_{1,n},M)}(q) = (q-1)^r$.*

Let us now turn to the Las Vergnas Tutte polynomial. The last two bullet points of Remark 5.4 suggest two different ways of generalizing the characteristic polynomial of a matroid. The case of $\mathbf{M} = (M, M)$ gives rise to the polynomial $p_{\mathbf{M}}(q, s) := (-1)^{r_2} LV\mathcal{T}_{\mathbf{M}}(1 - q, 0, -s)$, which was studied by Las Vergnas as the **Poincaré polynomial** of a matroid quotient [28, §4]. Here we introduce another generalization following the case of $\mathbf{M} = (U_{0,n}, M)$.

Definition 7.8. For a flag matroid $\mathbf{M} = (M_1, M_2)$ on $[n]$, define its **beta polynomial** $\beta_{\mathbf{M}}(q)$ by

$$\beta_{\mathbf{M}}(q) := (-1)^{r_2 - r_1} LV\mathcal{T}_{\mathbf{M}}(0, 0, -q).$$

When $\mathbf{M} = (U_{0,n}, M)$, it follows from $LV\mathcal{T}_{\mathbf{M}}(x, y, z) = T_M(z + 1, y)$ that $\beta_{\mathbf{M}}(q) = \chi_M(q)$, the characteristic polynomial of M. The terminology for $\beta_{\mathbf{M}}(q)$ is motivated by Proposition 7.9 below. First, let us recall that the **beta invariant** $\beta(M)$ of a matroid M of rank r is defined as

$$\beta(M) := (-1)^{r-1} \left(\frac{d}{dq} \chi_M(q) \Big|_{q=1} \right),$$

and that if e is an element that is neither a loop nor a coloop, then $\beta(M/e) + \beta(M \setminus e) = \beta(M)$ [10].

Proposition 7.9. *Let $M_1 = M^{(r_2-r_1)} \twoheadleftarrow \cdots \twoheadleftarrow M^{(1)} \twoheadleftarrow M^{(0)} = M_2$ be the Higgs factorization of a matroid quotient $M_1 \twoheadleftarrow M_2$ as described in §2.1. The beta polynomial $\beta_{M_1,M_2}(q)$ is divisible by $(q - 1)$, and the* ***reduced beta polynomial*** *of $M_1 \twoheadleftarrow M_2$, defined as $\overline{\beta}_{M_1,M_2}(q) := \beta_{M_1,M_2}(q)/(q - 1)$, satisfies*

$$\overline{\beta}_{M_1,M_2}(q) = \sum_{i=0}^{r_2-r_1-1} (-1)^{r_2-r_1-1-i} \big(\beta(M^{(i)}) + \beta(M^{(i+1)})\big) q^i.$$

If $\widetilde{M}^{(i)}$ is the (unique) matroid on $[n] \sqcup \{0\}$ such that $\widetilde{M}^{(i)}/0 = M^{(i+1)}$ and $\widetilde{M}^{(i)} \setminus 0 = M^{(i)}$, which exists by [35, §7.3], then $\beta(M^{(i)}) + \beta(M^{(i+1)}) = \beta(\widetilde{M}^{(i)})$. So, the proposition says that

$$\overline{\beta}_{M_1,M_2}(q) = \sum_{i=0}^{d-1} (-1)^{d-1-i} \beta(\widetilde{M}^{(i)}) \qquad \text{where } d = r_2 - r_1.$$

Proof. Let $\mathbf{M} = (M_1, M_2)$ and $d = r_2 - r_1$. [28, Theorem 3.1] states that

$$LV\mathcal{T}_{\mathbf{M}}(x, y, z) = \sum_{i=0}^{d} t_i(\mathbf{M}; x, y) z^i,$$

where

$$t_i(\mathbf{M};x,y) = \frac{1}{xy-x-y}\Big((y-1)T_{M^{(i-1)}}(x,y) + (-xy+x+y-2)T_{M^{(i)}}(x,y) \\ + (x-1)T_{M^{(i+1)}}(x,y)\Big)$$

for $i = 1, \ldots, d-1$, and

$$t_0(\mathbf{M};x,y) = \frac{1}{xy-x-y}\Big(-T_{M^{(0)}}(x,y) + (x-1)T_{M^{(1)}}(x,y)\Big),$$
$$t_d(\mathbf{M};x,y) = \frac{1}{xy-x-y}\Big((y-1)T_{M^{(d-1)}}(x,y) - T_{M^{(d)}}(x,y)\Big).$$

Let us express the beta invariant $\beta(M)$ of a matroid M of rank r equivalently as

$$\beta(M) = (-1)^{r-1}\left(\frac{d}{dq}\chi_M(q)\Big|_{q=1}\right),$$
$$= (-1)^{r-1}\lim_{q\to 1}\frac{(-1)^r T_M(1-q,0) - (-1)^r T_M(0,0)}{q-1} = -\lim_{q\to 1}\frac{T_M(1-q,0)}{q-1}.$$

As $LV\mathcal{T}_{\mathbf{M}}(x,y,z)$ is a polynomial, each $t_i(\mathbf{M};x,y)$ is also a polynomial. Hence, we have $t_i(\mathbf{M};0,0) = \lim_{q\to 1} t_i(\mathbf{M};1-q,0)$, and thus the above expressions for $t_i(\mathbf{M};x,y)$ give

$$t_i(\mathbf{M};0,0) = \beta(M^{(k-1)}) + 2\beta(M^{(k)}) + \beta(M^{(k+1)}) \quad \text{for } i = 1,\ldots,d-1, \text{ and}$$
$$t_0(\mathbf{M};0,0) = \beta(M^{(0)}) + \beta(M^{(1)}),$$
$$t_d(\mathbf{M};0,0) = \beta(M^{(d-1)}) + \beta(M^{(d)}).$$

As a result, we have

$$(-1)^d LV\mathcal{T}_{\mathbf{M}}(0,0,-q) = (q-1)\Big(\sum_{i=0}^{d-1}(-1)^{d-1-i}\big(\beta(M^{(i)}) + \beta(M^{(i+1)})\big)q^i\Big),$$

yielding the desired result for the reduced beta polynomial $\overline{\beta}_{\mathbf{M}}(q)$. □

Log-concavity of the coefficients of the reduced characteristic polynomial $\overline{\chi}_M(q) = \frac{\chi_M(q)}{q-1}$ was established in [1]. This motivates the following conjecture.

Conjecture 7.10. *The coefficients of $\overline{\beta}_{M_1,M_2}(q)$ form a log-concave sequence. Consequently, the coefficients of $LV\mathcal{T}_{\mathbf{M}}(0,0,q)$ form a log-concave sequence.*

The coefficients of $LV\mathcal{T}_{\mathbf{M}}(1,1,q)$ were shown to be (ultra) log-concave in [12].

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