Inventiones mathematicae
https://doi.org/10.1007/s00222-023-01194-5

Tautological classes of matroids

Andrew Berget[1] · Christopher Eur[2] · Hunter Spink[3] · Dennis Tseng[4]

Received: 28 September 2021 / Accepted: 6 April 2023

Abstract
We introduce certain torus-equivariant classes on permutohedral varieties which we call "tautological classes of matroids" as a new geometric framework for studying matroids. Using this framework, we unify and extend many recent developments in matroid theory arising from its interaction with algebraic geometry. We achieve this by establishing a Chow-theoretic description and a log-concavity property for a 4-variable transformation of the Tutte polynomial, and by establishing an exceptional Hirzebruch-Riemann-Roch-type formula for permutohedral varieties that translates between K-theory and Chow theory.

Contents

✉ A. Berget
andrew.berget@gmail.com

C. Eur
ceur@math.harvard.edu

H. Spink
hspink@stanford.edu

D. Tseng
dennisctseng@gmail.com

1 Western Washington University, Bellingham, WA, USA

2 Harvard University, Cambridge, MA, USA

3 Stanford University, Stanford, CA, USA

4 New York, NY, USA

Published online: 03 May 2023

1 Introduction

In Ardila's survey on the interaction between matroid theory and algebraic geometry [8], recent developments are classified according to three geometric models for matroids. However, developments from these different geometric models remained partially disjoint, as evidenced in Conjectures 1.1 below. We introduce a new unifying framework we call "tautological classes of matroids." An advantage of our approach is that we are able to exploit different techniques that were previously applicable in one model but not in others. Two prominent such techniques are localization methods in torus-equivariant geometry and positivity properties in tropical Hodge theory.

Let $E = \{0, 1, \dots, n\}$, and let T be the algebraic torus $(\mathbb{C}^*)^E$. The standard action of T on $\mathbb{C}^E$ is $(t_0, \dots, t_n) \cdot (x_0, \dots, x_n) = (t_0x_0, \dots, t_nx_n)$, which induces a T-action on the Grassmannian $\mathrm{Gr}(r; E)$ of r-dimensional subspaces of $\mathbb{C}^E$. Let X_E be the n-dimensional permutohedral variety, which is the projective toric variety associated to the permutohedron $\Pi(E) = \mathrm{Conv}\big(\sigma \cdot (0, \dots, n) \mid \sigma \text{ a permutation of } E\big) \subset \mathbb{R}^E$. We follow the conventions of [31, 52] for toric varieties and polyhedral geometry.

Let M be a matroid of rank r with ground set E, and let $\mathrm{rk}_M : 2^E \to \mathbb{Z}$ be its rank function. We refer to [109] or [94] for a general background on matroids. We always assume that a matroid has a nonempty ground set unless explicitly noted otherwise. For a set S and an element $i \in S$, as is customary in matroid theory, we often denote $S \setminus i = S \setminus \{i\}$. For $i \in E$, we denote by H_i the i-th coordinate hyperplane in $\mathbb{C}^E$. If M is realizable over $\mathbb{C}$, a realization of M is an r-dimensional linear subspace $L \subseteq \mathbb{C}^E$ such that the set of bases of M equals the sub-collection $\{B \in \binom{E}{r} \mid L \cap \bigcap_{i \in B} H_i = \{0\}\}$ of size r subsets of E.

1.1 Three previous geometric models of matroids

For the reader's convenience, we include an overview of the previous three geometric models of matroids. In each case, one begins with an algebro-geometric model defined for a realizable matroid, and then captures its combinatorial essence by a polyhedral model defined for an arbitrary (not necessarily realizable) matroid.

Model (1) (Base polytope and K-theory) In this model, one considers a realization $L \subseteq \mathbb{C}^E$ as a point on $\mathrm{Gr}(r; E)$. The resulting algebro-geometric model is the torus orbit closure $\overline{T \cdot L} \subset \mathrm{Gr}(r; E)$. The polyhedral model is the base polytope $P(M) = \mathrm{Conv}(\sum_{i \in B} \mathbf{e}_i \mid B \text{ a basis of } M) \subset \mathbb{R}^E$, whose associated projective toric variety is isomorphic to $\overline{T \cdot L}$ when L is a realization of M [58]. This geometric approach to matroids led to:

- a matroid invariant called the g-polynomial [103],
- a K-theoretic expression for the Tutte polynomial of M [50],
- an Ehrhart-style lattice point counting expression for the Tutte polynomial of M [25], and
- a generalization of the Tutte polynomial of a matroid to flag matroids [26, 39].

Model (2) (Bergman fan and Chow ring) In this model, one considers a realization $L \subseteq \mathbb{C}^E$ as a hyperplane arrangement $\{\mathcal{H}_i \subset L \mid \mathcal{H}_i = L \cap H_i\}_{i \in E}$ of the restrictions of $H_i \subset \mathbb{C}^E$ to L. The resulting algebro-geometric model is the wonderful compactification W_L of the projective hyperplane arrangement complement $\mathbb{P}L \setminus \bigcup \mathbb{P}\mathcal{H}_i$, introduced in [35]. It is a subvariety of the permutohedral variety X_E. The polyhedral model is the Bergman fan Σ_M, a subfan of the normal fan of the permutohedron, whose cones correspond to chains of flats of M [9]. When considered as a Minkowski weight, it defines a homology class Δ_M in the Chow ring $A^\bullet(X_E)$ that equals the class $[W_L]$ when L is a realization of M [78, 104]. This geometric approach to matroids led to:

- a notion of Chow rings of arbitrary (not necessarily realizable) matroids [49],
- a Chow-theoretic expression for the (reduced) characteristic polynomial of M [72], and
- the development of the Hodge theory of matroids, and in particular a proof of the log-concavity of coefficients of the characteristic polynomial of a matroid [1], settling the long-standing conjectures of Heron, Rota, Mason, and Welsh [64, 88, 98, 109]

Model (3) (Conormal fan and Chow ring) In this model, one considers a realization $L \subseteq \mathbb{C}^E$ as a Lagrangian subvariety $L \times L^\perp$ of the cotangent space $\mathbb{C}^E \times (\mathbb{C}^E)^\vee$ of $\mathbb{C}^E$, where $L^\perp = (\mathbb{C}^E/L)^\vee$. Projectivizing, one obtains the conormal space $\mathbb{P}L \times \mathbb{P}L^\perp$ of the linear subvariety $\mathbb{P}L \subseteq \mathbb{P}^n$. The resulting algebro-geometric model is the critical set variety $\mathfrak{X}_L$ [30, 93], a variety birational to $\mathbb{P}L \times \mathbb{P}L^\perp$. The polyhedral model is the conormal fan $\Sigma_{M,M^\perp}$, whose support equals the support of the product of two Bergman fans $\Sigma_M \times \Sigma_{M^\perp}$. Here $M^\perp$ denotes the dual matroid of M, a matroid that is realized by $L^\perp$ when M is realized by L. This geometric approach to matroids led to:

- a notion of Chern-Schwartz-MacPherson (CSM) classes of matroids [84], inspired by the related geometry of log-tangent sheaves [37, 68],
- a Chow-theoretic expression for the coefficients of $T_M(x, 0)$, i.e. the Tutte polynomial $T_M(x, y)$ of M evaluated at $y = 0$ [84], and
- a proof of the log-concavity of the coefficients of $T_M(x, 0)$ [13, 70], settling long-standing conjectures of Brylawski and Dawson [23, 34].

In these models, the Tutte polynomial of a matroid and its specializations manifested geometrically in many different ways, but the connection between them has until now remained unclear. The following collects conjectures about such connections from several groups of authors.

Conjectures 1.1

(*a*) (*within* (1)) *The authors of* [25] *conjectured there was a connection between their formulation of the Tutte polynomial and that of* [50] (*see the discussion below* [25, *Theorem* 3.4]).

(*b*) ((1) & (2)) *The authors of* [26] *asked how the* K*-theoretic computations in* [50] *relate to the Chow-theoretic computations made in* [72] (*see the discussion above* [26, §1.1]). *Moreover, they generalized the characteristic polynomial of a matroid to that of a flag matroid via* K*-theory, and conjectured that it also has log-concave coefficients* [26, *Conjecture* 9.4].

(*c*) ((1) & (3)) *The authors of* [84] *conjectured a Chow-theoretic formula for the* g*-polynomial of matroids, originally defined in* [103] *via* K*-theory* (*see* [84, §5.3]).

We affirm all parts of Conjectures 1.1 (see the discussion at the end of §1.6).

1.2 Tautological bundles and tautological classes

We now introduce our new framework. Let $\mathbb{C}^E_{\text{inv}}$ denote the vector space $\mathbb{C}^E$ with the *inverse* action of T where $(t_0, \dots, t_n) \in T$ acts on $(x_0, \dots, x_n) \in \mathbb{C}^E_{\text{inv}}$ by $(t_0^{-1}x_0, \dots, t_n^{-1}x_n)$. Denote by $\underline{\mathbb{C}}^E_{\text{inv}}$ the T-equivariant vector bundle $X_E \times \mathbb{C}^E_{\text{inv}}$. Finally, write $\mathbf{1}$ for the identity point of the open torus $T/\mathbb{C}^* \subset X_E$, where $\mathbb{C}^*$ acts diagonally on T.

Definition 1.2 For an r-dimensional linear subspace $L \subseteq \mathbb{C}^E$, the **tautological subbundle** $\mathcal{S}_L$ and the **tautological quotient bundle** $\mathcal{Q}_L$ of L are defined by

$$\mathcal{S}_L := \text{the unique } T\text{-equivariant rank } r \text{ subbundle of } \underline{\mathbb{C}}^E_{\text{inv}} \text{ whose fiber at } \mathbf{1} \text{ is } L, \quad \text{and}$$

$$\mathcal{Q}_L := \text{the unique } T\text{-equivariant rank } |E| - r \text{ quotient bundle of } \underline{\mathbb{C}}^E_{\text{inv}} \text{ whose fiber at } \mathbf{1} \text{ is } \mathbb{C}^E/L.$$

The uniqueness and existence of these bundles are verified in Proposition 3.6, where we induce them from the tautological sub and quotient bundles of the Grassmannian $\text{Gr}(r; E)$. By construction, one has a short exact sequence $0 \to \mathcal{S}_L \to \underline{\mathbb{C}}^E_{\text{inv}} \to \mathcal{Q}_L \to 0$ of vector bundles on X_E.

When $L \subseteq \mathbb{C}^E$ is a realization of a matroid M, the T-equivariant K-classes $[\mathcal{S}_L], [\mathcal{Q}_L] \in K_0^T(X_E)$ depend only on the matroid M (Proposition 3.7). From this, we construct classes $[\mathcal{S}_M], [\mathcal{Q}_M] \in K_0^T(X_E)$ on X_E satisfying $[\mathcal{S}_M]+[\mathcal{Q}_M] = [\underline{\mathbb{C}}^E_{\text{inv}}]$ for arbitrary (not necessarily realizable) matroids M with ground set E (Definition 3.9) which we call the **tautological *K*-classes** of M. We will write $[\mathcal{S}_M^\vee], [\mathcal{Q}_M^\vee]$ for their duals, and write $c_i(\mathcal{S}_M), c_i(\mathcal{Q}_M) \in A^i(X_E)$ for their i-th non-equivariant Chern classes, which we call the **tautological Chern classes** of M.

1.3 Two fundamental properties

The tautological K-classes and Chern classes of a matroid display useful features of both the K-theoretic approach to matroids via base polytopes (Model (1) in §1.1) and the Chow-theoretic approach via tropical geometry (Models (2) and (3) in §1.1). More precisely:

(A) The T-equivariant structure of tautological classes of matroids allows for the use of localization techniques in torus-equivariant geometry, reviewed in §2. Using these localization techniques, we show in §4 that tautological classes of matroids satisfy a deletion-contraction property, and we show in §5 that they display the following properties often shared by matroid invariants derived from base polytopes of matroids:
   - A Hopf-algebraic structure reflecting the fact that a face of the base polytope $P(M)$ of a matroid M is a product of the base polytopes of certain matroid minors of M [2, 40, 79].
   - Valuativity, which means that the invariant satisfies an inclusion-exclusion property with respect to any subdivision of $P(M)$ into smaller matroid base polytopes [10, 11, 38].
   - Well-behavedness under matroid duality and direct sums.

(B) Several long-standing conjectures in matroid theory concerning the log-concavity of sequences were resolved by adapting positivity properties of nef line bundles in algebraic geometry to a tropical geometry setting [1, 13]. In our case, if a matroid M has a realization L, then $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ are globally generated, and hence nef vector bundles. Equivalently, the relative hyperplane classes of the bi-projective bundle $\mathbb{P}(\mathcal{Q}_L^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_L)$ are nef divisor classes, implying that the Chern classes of $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ (as the Segre classes of $\mathcal{Q}_L^\vee$ and $\mathcal{S}_L$ respectively) have positivity and log-concavity properties. In §9, we show that the Chern classes of $\mathcal{S}_M^\vee$ and $\mathcal{Q}_M$ for arbitrary (not necessarily realizable) matroids M retain these same properties. An essential tool for establishing these results is the tropical Hodge theory of Lefschetz fans developed in [13, §5].

1.4 A unifying formula and log-concavity for the Tutte polynomial

An element $i \in E$ in a matroid M is a loop (resp. a coloop) if no basis of M contains i (resp. every basis of M contains i). When i is neither a loop nor a coloop in M, the deletion $M \setminus i$ and the contraction M/i are matroids on $E \setminus i$ defined by

$$\text{the set of bases of } M \setminus i = \{B \mid B \text{ a basis of } M \text{ such that } B \not\ni i\}, \quad \text{and}$$

$$\text{the set of bases of } M/i = \{B \setminus i \mid B \text{ a basis of } M \text{ such that } B \ni i\}.$$

When i is a loop or a coloop in M, one writes $M \setminus i = M/i$ for the matroid whose set of bases equal the nonempty one among the two sets of bases above. These notions give rise to the Tutte polynomial, the universal deletion-contraction invariant, defined for graphs in [106], and for matroids in [33].

Definition 1.3 Let M be a matroid with ground set E. The **Tutte polynomial** $T_M(x, y)$ is the unique bivariate polynomial determined by the following two properties.

- (Base case) If $|E| = 1$, then

$$T_M(x, y) = \begin{cases} x & \text{if } M \text{ has rank 1 (i.e. } M \text{ is a coloop)} \\ y & \text{if } M \text{ has rank 0 (i.e. } M \text{ is a loop).} \end{cases}$$

- (Deletion-contraction relation) If $|E| \geq 2$ and $i \in E$, then

$$T_M(x,y) = \begin{cases} x \cdot T_{M/i}(x,y) & \text{if } i \in E \text{ is a coloop in } M \\ y \cdot T_{M\setminus i}(x,y) & \text{if } i \in E \text{ is a loop in } M \\ T_{M\setminus i}(x,y) + T_{M/i}(x,y) & \text{if } i \in E \text{ is neither a loop nor a coloop} \\ & \text{in } M. \end{cases}$$

We use the fundamental properties (A) and (B) of tautological classes of matroids to prove the following two theorems about the Tutte polynomial. The first theorem is a Chow-theoretic expression for the Tutte polynomial that generalizes every previous expression for the Tutte polynomial and its specializations mentioned in §1.1. To state it, we recall two distinguished nef divisor classes on the permutohedral variety X_E from [1, 72]. Writing $U_{1,E}$ and $U_{n,E}$ for the uniform matroids of rank 1 and corank 1 on E (respectively), denote in the non-equivariant Chow group $A^1(X_E)$ the elements (see Example 3.10)

$$\alpha = c_1(\mathcal{Q}_{U_{n,E}}) \quad \text{and} \quad \beta = c_1(\mathcal{S}^\vee_{U_{1,E}}).$$

Equivalently, X_E resolves the Cremona map $\mathbb{P}^n \overset{\text{crem}}{\dashrightarrow} \mathbb{P}^n$ defined by $[x_0 : \ldots : x_n] \mapsto [x_0^{-1} : \ldots : x_n^{-1}]$, and α, β are the pullbacks of the hyperplane classes from the domain and target respectively (see §2.6).

Theorem A *Let $\int_{X_E} : A^\bullet(X_E) \to \mathbb{Z}$ be the degree map on X_E. For a matroid M of rank r with ground set E, define a polynomial*

$$t_M(x,y,z,w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}\,T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big).$$

Then, we have an equality

$$\sum_{i+j+k+\ell=n} \Big(\int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}_M^\vee) c_\ell(\mathcal{Q}_M)\Big) x^i y^j z^k w^\ell = t_M(x,y,z,w).$$

We prove Theorem A in §4 by showing that the tautological Chern classes of matroids satisfy a deletion-contraction relation (Theorem 4.8). We also present in Appendix I a different proof obtained by establishing a recursive convolution formula for both tautological Chern classes and Tutte polynomials of matroids, which may be of independent interest.

The second theorem is a log-concavity property for the Tutte polynomial expression in Theorem A, which generalizes every log-concavity result mentioned in §1.1. To state it, we recall that a nonnegative sequence $(a_0, a_1, \ldots, a_m)$ is **log-concave** if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq m-1$, and has **no internal zeros** if $a_i a_j > 0 \implies a_k > 0$ for all $0 \leq i \leq k \leq j \leq m$. For a homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_N]$ of degree d with nonnegative coefficients, we say that its coefficients form a **log-concave unbroken array** if, for any $1 \leq i < j \leq N$ and a monomial $x^{\mathbf{m}}$ of degree $d' \leq d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\mathbf{m}}\}_{0 \leq k \leq d-d'}$ in f form a log-concave sequence with no internal zeros.

Theorem B *For a matroid M of rank r with ground set E, the coefficients of the polynomial*

$$t_M(x, y, z, w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r} T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big)$$

form a log-concave unbroken array.

We prove Theorem B in §9. In fact, we establish a stronger statement (Theorem 9.13) implying that the polynomial $t_M(x, y, z, w)$ is a "denormalized Lorentzian polynomial" in the sense of [17].

By considering coefficients of $x^{r-1-i}y^i w^{|E|-r}$ in Theorem B (i.e. setting $z = 0$ and considering terms divisible by $w^{|E|-r}$), one recovers the log-concavity for the coefficients of the unsigned reduced characteristic polynomial $T_M(q+1, 0)/(q+1)$ from [1]. By considering coefficients of $x^{r-1-i}z^i w^{|E|-r}$ (i.e. setting $y = 0$ and considering terms divisible by $w^{|E|-r}$), one recovers the log-concavity for the coefficients of $T_M(q, 0)$, the h-vector of the broken circuit complex of M, from [13]. To deduce the log-concavity of the coefficients of $T_M(q, 1)$, the h-vector of the independence complex, [13] appeals to a combinatorial property of the free coextension matroid of M due to Brylawski [22]. Here, one can deduce this result directly from Theorem B applied to M by considering the coefficients of $x^{|E|-r-1+i}z^{r-i}$ (i.e. setting $y = w = 0$).[1] We note [34] showed that h-vector log-concavity always implies f-vector log-concavity.

1.5 Minkowski weights associated to matroids

We use Theorem A to relate the tautological Chern classes of a matroid to prior constructions such as the Bergman fan and the Chern-Schwartz-MacPherson classes of a matroid (Definition 7.5 and Definition 8.2, respectively). See §7.1 for the definition of, and notations concerning, Minkowski weights used in the statements below.

Theorem C *Let M be a matroid of rank r with ground set E. Let Δ_M be its Bergman class. For $0 \le k \le r-1$, let $\operatorname{csm}_k(M)$ be its k-dimensional Chern-Schwartz-MacPherson class. Then*

$$\Delta_M = c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E} \quad \textit{and} \quad \operatorname{csm}_k(M) = c_{r-1-k}(\mathcal{S}_M) c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E}.$$

The result for Δ_M is proved in Theorem 7.6 and the result for $\operatorname{csm}_k(M)$ is proved in Theorem 8.4. For realizable matroids, we establish stronger geometric statements in Theorem 7.10 and Theorem 8.8. Using these theorems, in §7 and §8 we recover the properties of the Bergman fans and the CSM classes of matroids previously established in [9, 48] and [84], respectively. In light of Theorem C, Theorem A generalizes [72, Lemma 6.1], which states that the intersection degrees

[1] A strengthening of the log-concavity of the f-vector of the independence complex to ultra-log-concavity, conjectured by Mason [88], was established in [7] and [17]. A strengthening of the log-concavity of the h-vector is given in [15]. Neither strengthening is implied by Theorem B.

$\alpha^i \beta^{r-1-i} \Delta_M$ equal the coefficients of the unsigned reduced characteristic polynomial $T_M(q+1, 0)/(q+1)$, and generalizes [84, Theorem 5.8], which states that the intersection degrees $\alpha^i \operatorname{csm}_i(M)$ equal the coefficients of $T_M(q, 0)$. These combinatorial interpretations of the tautological Chern classes, which are particular cases of Schur classes of $\mathcal{S}_M$ and $\mathcal{Q}_M$, motivate us to pose the following question.

Question 1.4 What combinatorial interpretations do products of Schur classes of $\mathcal{S}_M^\vee$ and $\mathcal{Q}_M$ admit? Do they similarly satisfy positivity and log-concavity properties?

For instance, [36, 54] establish positivity properties for Schur classes of nef vector bundles, which apply to the globally generated bundles $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ if L is a realization of M.

1.6 A K-theory to Chow theory bridge

We now turn to connecting Theorem A, a Chow-theoretic expression, to expressions for the Tutte polynomial obtained via K-theoretic tools. One could try using the Hirzebruch-Riemann-Roch (HRR) theorem, which states that the Euler characteristic $\chi([\mathcal{E}])$ of a K-class $[\mathcal{E}]$ on a smooth projective variety X satisfies

$$\chi\big([\mathcal{E}]\big) = \int_X \operatorname{Td}(X) \cdot \operatorname{ch}([\mathcal{E}]),$$

where $\operatorname{Td}(X) \in A^\bullet(X)_{\mathbb{Q}}$ is the Todd class of X and $\operatorname{ch}([\mathcal{E}])$ is the Chern character of $[\mathcal{E}]$ (by convention $\int_X \gamma = 0$ if $\gamma \in A^i(X)$ for $i < \dim(X)$). However, the Hirzebruch-Riemann-Roch theorem does not appear to be useful in our context (see Remark 10.3).

We construct an exceptional isomorphism $\zeta_{X_E} \colon K_0(X_E) \xrightarrow{\sim} A^\bullet(X_E)$, unrelated to the Chern character ch, that translates between K-theoretic and Chow-theoretic computations using $1+\alpha+\cdots+\alpha^n$ in place of the Todd class $\operatorname{Td}(X_E)$. This map behaves particularly well on a collection of T-equivariant K-classes that we say "have simple Chern roots" (Definition 10.4), which includes $[\mathcal{S}_M^\vee]$ and $[\mathcal{Q}_M^\vee]$ for any matroid M.

Theorem D *There exists a ring isomorphism $\zeta_{X_E} \colon K_0(X_E) \xrightarrow{\sim} A^\bullet(X_E)$ which satisfies*

$$\chi\big([\mathcal{E}]\big) = \int_{X_E} (1+\alpha+\cdots+\alpha^n) \cdot \zeta_E([\mathcal{E}])$$

for any $[\mathcal{E}] \in K_0(X_E)$. Denote by $\bigwedge^i$ for the i-th exterior power and $c(\mathcal{E}, u) := \sum_{i \geq 0} c_i(\mathcal{E}) u^i$ the Chern polynomial of $[\mathcal{E}]$. If $[\mathcal{E}]$ has simple Chern roots and rank $\operatorname{rk}(\mathcal{E})$ then we have

$$\sum_{i \geq 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}]\big) u^i = (u+1)^{\operatorname{rk}(\mathcal{E})} c(\mathcal{E}, \tfrac{u}{u+1}), \quad \textit{and}$$

$$\sum_{i \geq 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}^\vee]\big) u^i = (u+1)^{\operatorname{rk}(\mathcal{E})} c(\mathcal{E}, 1)^{-1} c(\mathcal{E}, \tfrac{1}{u+1}).$$

We prove the first part of Theorem D in Theorem 10.1, and the second part in Proposition 10.5. Applying Theorem D to Theorem A, we recover both the K-theoretic formula for the Tutte polynomial [50, Theorem 5.1] (see Theorem 10.9) and the lattice-point-counting formula for the Tutte polynomial [25, Theorem 3.2] (see Theorem 10.11), thereby answering Conjectures 1.1.(a) and the first part of Conjectures 1.1.(b). We also use Theorem D to give a Chow-theoretic formula for Speyer's g-polynomial of a matroid (Theorem 10.12) conjectured in [84, Conjecture 1], answering Conjectures 1.1.(c).

Finally, in §11 we show that our methods generalize well to flag matroids, answering two conjectures concerning the characteristic polynomials of flag matroids that were defined and studied in [26, 39]. In particular, we establish a log-concavity property answering the second part of Conjectures 1.1.(b).

Matroids and flag matroids are the "type A" examples of Coxeter matroids [16, 57]. They are also examples of polymatroids (Remark 10.7). Motivated by these, we pose the following question.

Question 1.5 How do the results here generalize to Coxeter matroids or polymatroids?

For instance, we show in §10.4 that Theorem D recovers Postnikov's result [95, Theorem 11.3] relating Ehrhart and volume polynomials of generalized permutohedra, of which polymatroids are a subfamily.

2 Equivariant geometry of permutohedral varieties

We set notations and collect results relevant to the torus-equivariant K-theory and Chow theory of permutohedral varieties. We also note some features that are special to permutohedral varieties not shared by arbitrary toric varieties.

2.1 Equivariant K-ring and equivariant Chow ring

Recall that $E = \{0, 1, \dots, n\}$, and $T = (\mathbb{C}^*)^E$. Let $\mathrm{Char}(T)$ be the character group of T. For a smooth T-variety X, let $K_T^0(X)$ denote the T-equivariant Grothendieck K-ring of vector bundles on X, as defined in for example [97, 108]. For $[\mathcal{E}] \in K_T^0(X)$, we write $[\mathcal{E}^\vee]$ for its dual class. Writing $T_0, \dots, T_n$ for the standard characters of T, we identify the character ring $\mathbb{Z}[\mathrm{Char}(T)]$ with the Laurent polynomial ring $\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$. In particular, the T-equivariant K-ring $K_T^0(\mathrm{pt})$ of a point is identified with $\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$, where a T-representation corresponds to the sum of its characters.

We let $A_T^\bullet(X)$ denote the T-equivariant Chow ring of X as defined in [41], noting the identification of the equivariant Chow homology with the equivariant Chow cohomology occurring in [41, Proposition 4] when X is smooth. For a T-equivariant K-class $[\mathcal{E}] \in K_T^0(X)$, we write $c_i(\mathcal{E})$ for its i-th *non-equivariant* Chern class, reserving the notation $c_i^T(\mathcal{E})$ for the i-th T*-equivariant* Chern class. We identify the symmetric algebra $\mathrm{Sym}^\bullet \mathrm{Char}(T)$, which is the T-equivariant Chow ring $A_T^\bullet(\mathrm{pt})$ of a point, with the polynomial ring $\mathbb{Z}[t_0, \dots, t_n]$. Here we have used the lowercase t for notational clarity: In the context of equivariant K-theory or Chow rings, the T_i variables will denote elements in the Laurent polynomial ring, whereas the t_i variables will denote elements in the polynomial ring.

2.2 Grassmannians

Let $\mathcal{S}$ and $\mathcal{Q}$ denote the tautological sub and quotient bundle on $\mathrm{Gr}(r; E)$, respectively. For an r-element subset I of E, let p_I be the T-fixed point of $\mathrm{Gr}(r; E)$ corresponding to $\mathrm{span}(\mathbf{e}_i \mid i \in I) \subseteq \mathbb{C}^E$. For a T-equivariant K-class $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$, write $[\mathcal{E}]_I$ for its image under the restriction map $K_T^0(\mathrm{Gr}(r; E)) \to K_T^0(p_I) = \mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$. Then, we have

$$[\mathcal{S}]_I = \sum_{i \in I} T_i \quad \text{and} \quad [\mathcal{Q}]_I = \sum_{j \in E \setminus I} T_j \quad \text{for any } I \in \tbinom{E}{r}.$$

In particular, the ample line bundle $\mathcal{O}(1)$ on $\mathrm{Gr}(r; E)$, whose global sections give the Plücker embedding into $\mathbb{P}(\mathbb{C}^{\binom{E}{r}})$, satisfies $[\mathcal{O}(1)]_I = [\det \mathcal{S}^\vee]_I = \prod_{i \in I} T_i^{-1}$ for every $I \in \binom{E}{r}$. If I and J are r-element subsets of E such that $J = I \setminus \{i\} \cup \{j\}$ for some $i \in I$ and $j \in J$, then every T-equivariant K-class $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$ satisfies $[\mathcal{E}]_I \equiv [\mathcal{E}]_J \mod (1 - T_i/T_j)$. Conversely a collection $([\mathcal{E}]_I)_{I \in \binom{E}{r}}$ satisfying this congruence for all such I, J determines a unique $[\mathcal{E}]$ (see for example the discussion in [50, §2.2]).

2.3 Conventions for permutations and cones

We now specialize to permutohedral varieties. We first set some notations and conventions. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^E$, and let $\mathfrak{S}_E$ be the set of permutations on E. The normal fan $\widetilde{\Sigma}_E$ of the permutohedron $\Pi(E) = \mathrm{Conv}(\sigma \cdot (0, 1, \dots, n) \mid \sigma \in \mathfrak{S}_E)$ is the fan in $\mathbb{R}^E$ induced by the type A_n hyperplane arrangement, which consists of the $\binom{n+1}{2}$ hyperplanes $\{x \in \mathbb{R}^E \mid \langle x, \mathbf{e}_i - \mathbf{e}_j \rangle = 0\}_{0 \le i < j \le n}$. Every cone of $\widetilde{\Sigma}_E$ has 1-dimensional linearity space $\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)$. Let Σ_E be the quotient fan in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$. It is a rational unimodular fan over the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{1}$, whose dual lattice is $\mathbf{1}^\perp = \{x \in \mathbb{Z}^E \mid \langle x, \mathbf{1} \rangle = 0\}$.

The cones of Σ_E correspond to flags of nonempty proper subsets of E. Such a flag $\mathscr{S}\colon \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ corresponds to $\mathrm{Cone}(\overline{\mathbf{e}}_{S_1}, \dots, \overline{\mathbf{e}}_{S_k})$, where $\overline{\mathbf{e}}_S$ denotes the image of $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$ under $\mathbb{R}^E \to \mathbb{R}^E/\mathbb{R}\mathbf{1}$. The permutohedral variety X_E is the toric variety of the fan Σ_E, whose dense open torus is $T/\mathbb{C}^*$ with $\mathbb{C}^*$ acting diagonally on T. We often consider X_E as a T-variety, where the diagonal $\mathbb{C}^*$ acts trivially on X_E. As the fan Σ_E is unimodular, the variety X_E is smooth.

We set a bijection between the set $\mathfrak{S}_E$ of permutations on E and the set of maximal cones of Σ_E by

$$\mathfrak{S}_E \ni \sigma \quad \longleftrightarrow \quad \begin{array}{c} \mathrm{Cone}(\overline{\mathbf{e}}_{S_1}, \dots, \overline{\mathbf{e}}_{S_n}) \text{ corresponding to the chain } \mathscr{S} \\ \text{where } S_i = \{\sigma(0), \dots, \sigma(i-1)\} \text{ for } 1 \le i \le n \end{array}.$$

The elements in the interior of the cone corresponding to σ are precisely those of the form $v_0\overline{\mathbf{e}}_{\sigma(0)} + v_1\overline{\mathbf{e}}_{\sigma(1)} + \cdots + v_n\overline{\mathbf{e}}_{\sigma(n)}$ for $v_0 > \cdots > v_n$. This bijection naturally induces a bijection between $\mathfrak{S}_E$ and the set X_E^T of T-fixed points of X_E. For a permutation σ, we write p_σ for the T-fixed point corresponding to σ, and write $U_\sigma \simeq \mathbb{C}^n$ for the T-invariant affine chart around p_σ. Since $\operatorname{Cone}(\mathbf{e}_{\sigma(0)} - \mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n-1)} - \mathbf{e}_{\sigma(n)}) \subset \mathbf{1}^\perp$ is the dual cone of the maximal cone corresponding to σ, the torus T acts on U_σ, and hence in particular the tangent space to p_σ, with characters $T_{\sigma(0)}T_{\sigma(1)}^{-1}, \dots, T_{\sigma(n-1)}T_{\sigma(n)}^{-1}$.

2.4 Localization theorems

By the bijection between permutations $\mathfrak{S}_E$ and T-fixed points X_E^T, we identify $K_T^0(X_E^T)$ with $\prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$, and identify $A_T^\bullet(X_E^T)$ with $\prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[t_0, \dots, t_n]$. For f in $\prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$ (resp. $\prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[t_0, \dots, t_n]$), we write f_σ for its projection to the copy of the factor $\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$ (resp. $\mathbb{Z}[t_0, \dots, t_n]$) indexed by the permutation σ.

Theorem 2.1 *Let X_E be the permutohedral variety as above.*

(*a*) *The restriction map $K_T^0(X_E) \to K_T^0(X_E^T)$ from the T-equivariant K-ring of X_E to that of its T-fixed points is injective, and its image is the subring of $K_T^0(X_E^T)$ given by*

$$\left\{ f \in \prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}] \;\middle|\; \begin{array}{c} f_\sigma \equiv f_{\sigma'} \mod (1 - \frac{T_{\sigma(i+1)}}{T_{\sigma(i)}}) \\ \text{whenever } \sigma' = \sigma \circ (i, i+1) \text{ for a transposition} \\ (i, i+1) \end{array} \right\}.$$

In particular, the ring $K_T^0(X_E)$ can be identified with the ring $PLaur(\widetilde{\Sigma}_E)$ of piecewise Laurent polynomials on the fan $\widetilde{\Sigma}_E$ in $\mathbb{R}^E$, and the non-equivariant K-ring $K^0(X_E)$ is isomorphic to the quotient of $PLaur(\widetilde{\Sigma}_E)$ by the ideal generated by $f(T_0, \dots, T_n) - f(1, \dots, 1)$ for each global Laurent polynomial f on $\mathbb{R}^E$.

(*b*) *The restriction map $A_T^\bullet(X_E) \to A_T^\bullet(X_E^T)$ from the T-equivariant Chow ring of X_E to that of its T-fixed points is injective, and its image is the subring of $A_T^\bullet(X_E^T)$ given by*

$$\left\{ \varphi \in \prod_{\sigma\in\mathfrak{S}_E}\mathbb{Z}[t_0, \dots, t_n] \;\middle|\; \begin{array}{c} \varphi_\sigma \equiv \varphi_{\sigma'} \mod (t_{\sigma(i)} - t_{\sigma(i+1)}) \\ \text{whenever } \sigma' = \sigma \circ (i, i+1) \text{ for a transposition} \\ (i, i+1) \end{array} \right\}.$$

In particular, the ring $A_T^\bullet(X_E)$ can be identified with the ring $PPoly(\widetilde{\Sigma}_E)$ of piecewise polynomial functions on the fan $\widetilde{\Sigma}_E$ in $\mathbb{R}^E$, and the non-equivariant Chow ring $A^\bullet(X_E)$ is isomorphic to the quotient of $PPoly(\widetilde{\Sigma}_E)$ by the ideal generated by $\varphi(t_0, \dots, t_n) - \varphi(0, \dots, 0)$ for each global polynomial φ on $\mathbb{R}^E$.

In light of the above theorem, for $[\mathcal{E}] \in K_T^0(X_E)$ we also write $[\mathcal{E}]$ for its image in $K_T^0(X_E^T)$, and $[\mathcal{E}]_\sigma \in \mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$ for the restriction of $[\mathcal{E}]$ to the T-fixed point p_σ. We notate similarly for $\xi \in A_T^\bullet(X_E)$.

With one exception, Theorem 2.1 collects standard results in equivariant geometry that hold, e.g., for smooth proper toric varieties. Theorem 2.1.(a) follows from [108, Corollary 5.12, Theorem 5.19] while (b) follows from [21, Corollary 2.3, Theorem 3.4]. The non-standard exception above is the identification of $K^0_T(X_E)$ with a ring of piecewise Laurent polynomials on a fan. This result is special to the permutohedral variety, and fails for arbitrary toric varieties. The validity of the identification follows from two straightforward observations: No codimension 1 cone of $\widetilde{\Sigma}_E$, whose linear span is the hyperplane normal to $\mathbf{e}_{\sigma(i)} - \mathbf{e}_{\sigma(i+1)}$ for some σ and i, is contained in a coordinate hyperplane of $\mathbb{R}^E$, and two Laurent polynomials f_σ and $f_{\sigma'}$ satisfy $f_\sigma \equiv f_{\sigma'} \mod (1 - \frac{T_{\sigma(i+1)}}{T_{\sigma(i)}})$ if and only if $f_\sigma \equiv f_{\sigma'} \mod (T_{\sigma(i)} - T_{\sigma(i+1)})$.

2.5 Duality, rank, exterior powers, and Chern classes

For $\mathbf{m} = (m_0, \ldots, m_n) \in \mathbb{Z}^E$, write $\mathbf{T}^{\mathbf{m}} = T_0^{m_0} \cdots T_n^{m_n}$, and write $\mathbf{m} \cdot \mathbf{t} = m_0 t_0 + \cdots + m_n t_n$. Let $[\mathcal{E}] \in K^0_T(X_E)$. Then, for each $\sigma \in \mathfrak{S}_E$ we have

$$[\mathcal{E}]_\sigma = \sum_{i=1}^{k_\sigma} a_{\sigma,i} \mathbf{T}^{\mathbf{m}_{\sigma,i}}$$

for some integer $k_\sigma \geq 0$, signs $a_1, \ldots, a_{k_\sigma} \in \{-1, 1\}$, and $\mathbf{m}_{\sigma,1}, \ldots, \mathbf{m}_{\sigma,k_\sigma} \in \mathbb{Z}^E$. The dual class $[\mathcal{E}^\vee]$ is defined by saying

$$[\mathcal{E}^\vee]_\sigma = \sum_{i=1}^{k_\sigma} a_{\sigma,i} \mathbf{T}^{-\mathbf{m}_{\sigma,i}}.$$

The map which takes a vector bundle $\mathcal{E}$ to its rank is additive, and hence extends to a map $\mathrm{rk}\colon K_0^T(X_E) \to \mathbb{Z}$. We have $\mathrm{rk}(\mathcal{E}) = a_{\sigma,1} + \cdots + a_{\sigma,k_\sigma}$ for any σ, since when $\mathcal{E}$ is a vector bundle the right hand side is the rank of the pullback of the bundle to the torus fixed point p_σ. The j-th exterior power, denoted $[\bigwedge^j \mathcal{E}]$, and the j-th T-equivariant Chern class, are given by equating

$$\sum_{j=0}^{\infty} [\textstyle\bigwedge^j \mathcal{E}]_\sigma u^j = \prod_{i=1}^{k_\sigma} (1 + \mathbf{T}^{\mathbf{m}_{\sigma,i}} u)^{a_{\sigma,i}} \quad \text{and}$$

$$c^T(\mathcal{E}, u) = \sum_{j=0}^{\infty} c_j^T(\mathcal{E})_\sigma u^j = \prod_{i=1}^{k_\sigma} (1 + \mathbf{m}_{\sigma,i} \cdot \mathbf{t} u)^{a_{\sigma,i}},$$

where u is a formal variable. We note that this implies $c^T(\mathcal{E}^\vee, u) = c^T(\mathcal{E}, -u)$. When $a_{\sigma,i} = 1$ for all σ and i, in which case $k_\sigma = \mathrm{rk}(\mathcal{E})$ for all permutations σ, we have for $0 \leq j \leq \mathrm{rk}(\mathcal{E})$ that

$$[\textstyle\bigwedge^j \mathcal{E}]_\sigma = \mathrm{Elem}_j(\mathbf{T}^{\mathbf{m}_{\sigma,1}}, \ldots, \mathbf{T}^{\mathbf{m}_{\sigma,\mathrm{rk}(\mathcal{E})}}) \quad \text{and}$$

$$c_j^T(\mathcal{E}) = \mathrm{Elem}_j(\mathbf{m}_{\sigma,1} \cdot \mathbf{t}, \ldots, \mathbf{m}_{\sigma,\mathrm{rk}(\mathcal{E})} \cdot \mathbf{t}),$$

where Elem_j denotes the j-th elementary symmetric polynomial. More generally, when $a_{\sigma,i} = 1$ for all σ and i, given an element $\lambda(x) \in \Lambda \subset \mathbb{Z}[[x_1, x_2, \ldots]]$ in the ring Λ of symmetric functions [85, Section I.2], we define the T-equivariant K-class $[\mathsf{S}^\lambda \mathcal{E}]$ by

$$[\mathsf{S}^\lambda \mathcal{E}]_\sigma = \lambda(\mathbf{T}^{\mathbf{m}_{\sigma,1}}, \ldots, \mathbf{T}^{\mathbf{m}_{\sigma,\mathrm{rk}(\mathcal{E})}}, 0, 0, \ldots) \quad \text{for all } \sigma \in \mathfrak{S}_E,$$

and define the T-equivariant Chow class $\mathsf{s}_\lambda^T(\mathcal{E})$ by

$$\mathsf{s}_\lambda^T(\mathcal{E})_\sigma = \lambda(\mathbf{m}_{\sigma,1} \cdot \mathbf{t}, \ldots, \mathbf{m}_{\sigma,\mathrm{rk}(\mathcal{E})} \cdot \mathbf{t}, 0, 0, \ldots) \quad \text{for all } \sigma \in \mathfrak{S}_E.$$

2.6 Cremona involution

The projective space $\mathbb{P}^n$ is a T-variety by the standard action of T on $\mathbb{C}^E$. The fan of $\mathbb{P}^n$ as a toric variety is then the coarsening of Σ_E with rays $\mathrm{Cone}(\overline{\mathbf{e}}_i)$ for each $i \in E$, and we have the naturally induced birational map $\pi_E\colon X_E \to \mathbb{P}^n$ that is the identity on the common dense open torus $T/\mathbb{C}^*$. Let $\mathrm{crem}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the Cremona transformation defined by $[t_0 : \cdots t_n] \mapsto [t_0^{-1} : \cdots : t_n^{-1}]$ on the open dense torus $T/\mathbb{C}^*$ of $\mathbb{P}^n$. Relatedly, the map $\mathbb{R}^E/\mathbb{R}\mathbf{1} \to \mathbb{R}^E/\mathbb{R}\mathbf{1}$ given by $x \mapsto -x$ defines an involution of the fan Σ_E, and hence induces the Cremona involution $\mathrm{crem}\colon X_E \to X_E$ of the permutohedral variety X_E, fitting into the diagram

$$\begin{tikzcd} X_E \arrow[r, "\mathrm{crem}"] \arrow[d, "\pi_E"'] & X_E \arrow[d, "\pi_E"] \\ \mathbb{P}^n \arrow[r, dashed, "\mathrm{crem}"] & \mathbb{P}^n. \end{tikzcd}$$

Because crem is an involution, we have $\mathrm{crem}_* = \mathrm{crem}^*$ on $K_0^T(X_E)$ and $A_T^\bullet(X_E)$, so we simply write crem when acting on K-classes or Chow classes.

Remark 2.2 The map $\mathrm{crem}\colon X_E \to X_E$ is not T-equivariant, but is a toric morphism, where the map of tori $T \to T$ is given by $t \mapsto t^{-1}$. For a permutation $\sigma \in \mathfrak{S}_E$, note that the T-fixed point p_σ maps under crem to the point $p_{\overline{\sigma}}$, where

$$\overline{\sigma} \in \mathfrak{S}_E \quad \text{is defined by} \quad \overline{\sigma}(i) = \sigma(n-i) \quad \text{for } i = 0, \ldots, n.$$

As a result, if $[\mathcal{E}] \in K_T^0(X_E)$ with $[\mathcal{E}]_\sigma = \sum_{i=1}^{k_\sigma} a_{\sigma,i} \mathbf{T}^{\mathbf{m}_{\sigma,i}}$ for $\sigma \in \mathfrak{S}_E$, then $\mathrm{crem}[\mathcal{E}] \in K_T^0(X_E)$ satisfies

$$(\mathrm{crem}[\mathcal{E}])_\sigma = [\mathcal{E}]_{\overline{\sigma}}(T_0^{-1}, \ldots, T_n^{-1}) = \sum_{i=1}^{k_{\overline{\sigma}}} a_{\overline{\sigma},i} \mathbf{T}^{-\mathbf{m}_{\overline{\sigma},i}}.$$

The Cremona involution gives rise to two distinguished divisor classes on X_E as follows.

Definition 2.3 Writing $h = c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}^n)$ for the hyperplane class in $\mathbb{P}^n$, define the divisor classes $\alpha_E = \pi_E^* h$ and $\beta_E = \mathrm{crem}\, \alpha_E = (\pi_E \circ \mathrm{crem})^* h$ in $A^1(X_E)$.

We omit the subscript E from α_E and β_E when the ground set E is clear. The line bundles of the divisor classes α and β are given T-linearizations in Remark 2.4.

2.7 Generalized permutohedra and T-linearized line bundles

Let $P \subset \mathbb{R}^E$ be a lattice polytope contained in a translate of the sublattice $\mathbf{1}^\perp = \{\mathbf{m} \in \mathbb{Z}^E \mid m_0 + \cdots + m_n = 0\}$. Let $h_P \colon \mathbb{R}^E \to \mathbb{R}$ be its support function defined by $h_P(x) = \max_{\mathbf{m}\in P}\langle \mathbf{m}, x\rangle$. It was shown in several places [43, §II], [89, Ch. 4], [2, Theorem 12.3 & references therein] that the following two statements are equivalent:

- The polytope P is a **generalized permutohedron**, i.e. its normal fan coarsens the fan $\widetilde{\Sigma}_E$.
- The function $\mathrm{rk}_P \colon 2^E \to \mathbb{R}$ defined by $\mathrm{rk}_P(S) = h_P(\mathbf{e}_S)$ for a subset $S \subseteq E$ is **submodular**, i.e.

$$\mathrm{rk}_P(S) + \mathrm{rk}_P(S') \geq \mathrm{rk}_P(S \cup S') + \mathrm{rk}_P(S \cap S') \quad \text{for all subsets } S, S' \subseteq E,$$

  and $P = \{x \in \mathbb{R}^E \mid \langle x, \mathbf{1}\rangle = \mathrm{rk}_P(E) \text{ and } \langle x, \mathbf{e}_S\rangle \leq \mathrm{rk}_P(S) \text{ for all nonempty } S \subseteq E\}$.

Let P now be a generalized permutohedron. The negated polytope $-P$ has support function $h_{-P}(v) = h_P(-v)$, and is also a generalized permutohedron as $\mathrm{rk}_{-P}(S) = \mathrm{rk}_P(E \setminus S) - \mathrm{rk}_P(E)$ is submodular. We associate to P a T-linearized line bundle on X_E as follows. Since $\mathbf{1}^\perp$ is the dual lattice of $\mathbb{Z}^E/\mathbb{Z}\mathbf{1}$, for every translate P' of P such that $P' \subset \mathbf{1}^\perp$, there is the associated $(T/\mathbb{C}^*)$-invariant divisor $D_{P'}$ on X_E (see [31, Theorem 4.2.12 & Proposition 4.2.14]) given by

$$D_{P'} = \sum_{\emptyset \subsetneq S \subsetneq E} -\min_{\mathbf{m}\in P'}\langle \mathbf{m}, \overline{\mathbf{e}}_S\rangle Z_S = \sum_{\emptyset \subsetneq S \subsetneq E} \max_{\mathbf{m}\in -P'}\langle \mathbf{m}, \overline{\mathbf{e}}_S\rangle Z_S = \sum_{\emptyset \subsetneq S \subsetneq E} \mathrm{rk}_{-P'}(S) Z_S$$

where Z_S denotes the $(T/\mathbb{C}^*)$-invariant divisor in X_E corresponding to the ray $\mathrm{Cone}(\mathbf{e}_S)$ in the fan Σ_E. We note here that the divisor class $[D_{P'}] \in A^1(X_E)$ in the non-equivariant Chow ring of X_E is independent of the translation P', so we may write $[D_P]$ for this divisor class. The resulting line bundle $\mathcal{O}(D_P)$ admits a T-linearization given by

$$[\mathcal{O}(D_P)]_\sigma = \mathbf{T}^{-\mathbf{m}_\sigma} \in \mathbb{Z}[T_0^\pm, \ldots, T_n^\pm] \quad \text{for any } \sigma \in \mathfrak{S}_E,$$

where $\mathbf{m}_\sigma$ is the vertex of P minimizing the pairing with any vector in the interior of the cone corresponding to the permutation σ. Concretely, the lattice point $\mathbf{m}_\sigma$ is the vertex of P achieving for any sequence $v_0 > \cdots > v_n$ the minimum in $\min_{\mathbf{m}\in P}\langle \mathbf{m}, v_0\mathbf{e}_{\sigma(0)} + v_1\mathbf{e}_{\sigma(1)} + \cdots + v_n\mathbf{e}_{\sigma(n)}\rangle$. In particular, we have

$$[\mathcal{O}(D_{-P})]_\sigma = \mathbf{T}^{\widetilde{\mathbf{m}}_\sigma} \in \mathbb{Z}[T_0^\pm, \ldots, T_n^\pm] \quad \text{for any } \sigma \in \mathfrak{S}_E,$$

where the lattice point $\widetilde{\mathbf{m}}_\sigma$ is the vertex of P achieving for any sequence $v_0 > \cdots > v_n$ the maximum in $\max_{\mathbf{m}\in P}\langle \mathbf{m}, v_0\mathbf{e}_{\sigma(0)} + v_1\mathbf{e}_{\sigma(1)} + \cdots + v_n\mathbf{e}_{\sigma(n)}\rangle$.

Notation For a generalized permutohedron P, we write $\mathcal{O}(D_P)$ for the corresponding line bundle on X_E with the T-linearization given as above.

Remark 2.4 Let $\Delta = \mathrm{Conv}(\mathbf{e}_i \mid i \in E)$ be the standard simplex in $\mathbb{R}^E$, and let $-\Delta = \mathrm{Conv}(-\mathbf{e}_i \mid i \in E)$ be the negative standard simplex. Since the fan of $\mathbb{P}^n$ as a toric variety is the normal fan of Δ, the T-equivariant line bundles $\mathcal{O}(D_\Delta)$ and $\mathcal{O}(D_{-\Delta})$ on X_E are non-equivariantly isomorphic to $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$, respectively. Moreover, by the discussion above they satisfy

$$[\mathcal{O}(D_\Delta)]_\sigma = T_{\sigma(n)}^{-1} \quad \text{and} \quad [\mathcal{O}(D_{-\Delta})]_\sigma = T_{\sigma(0)}$$

for every permutation $\sigma \in \mathfrak{S}_E$. Furthermore, fixing an element $i \in E$, we can consider the translate $\Delta - \mathbf{e}_i$, which is contained in the sublattice $\mathbf{1}^\perp$, so that we have the equality of divisor classes

$$\alpha = [D_\Delta] = \sum_{\emptyset \subsetneq S \subsetneq E} \mathrm{rk}_{\mathbf{e}_i - \Delta}(S)[Z_S] = \sum_{i \in S \subsetneq E} [Z_S]$$

in $A^1(X_E)$, and by a similar computation $\beta = \sum_{\emptyset \subsetneq S \subseteq E \setminus \{i\}} [Z_S]$. These last definitions of α and β are the same as in [1, 13, 72].

3 Tautological classes of matroids

In §3.1, we define the tautological bundles of a realizations of matroids, and define the tautological K-classes and Chern classes of matroids, which we collectively refer to as “tautological classes of matroids”. In §3.2, we provide a slight generalization that we will not need until §10.

3.1 Well-definedness

We prepare by recalling some properties of the base polytope of a matroid. Introduced in [58], the **base polytope** $P(M)$ of a matroid M with ground set E is defined as

$$P(M) = \mathrm{Conv}\Big(\mathbf{e}_B : B \text{ a basis of } M\Big) \subset \mathbb{R}^E,$$

where $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ for a subset $S \subseteq E$. We recall two well-known facts about the base polytope $P(M)$.

Proposition 3.1 *Let M be matroid of rank r with ground set E.*

(*a*) *The base polytope $P(M)$ is a generalized permutohedron. In other words, the normal fan of $P(M)$, as a fan in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$, coarsens the permutohedral fan Σ_E.*

(*b*) *Let $L \subseteq \mathbb{C}^E$ be a realization of M, considered as a point $L \in \mathrm{Gr}(r; E)$. Then, the torus-orbit-closure $\overline{T \cdot L} \subseteq \mathrm{Gr}(r; E)$ is torus-equivariantly isomorphic to the normal projective toric variety $X_{P(M)}$ associated to the polytope $P(M)$. For a basis $B \subseteq E$ of M, the T-fixed point in $X_{P(M)}$ corresponding to the vertex $\mathbf{e}_B$ of $P(M)$ maps under the isomorphism to the T-fixed point p_B in $\mathrm{Gr}(r; E)$, as denoted in §2.2.*

Part (a) is classical, tracing back to [43]; see [12, §4.4] for a proof and a generalization to Coxeter matroids. Part (b) follows from combining [58, Corollary 2.4] and [110]; see [26, §5] for a proof. Proposition 3.1.(a) allows us to make the following definition.

Definition 3.2 For a permutation $\sigma \in \mathfrak{S}_E$, the **lex-first-basis** of a matroid M with respect to σ, denoted by $B_\sigma(M)$, is the unique basis of M such that the vertex $\mathbf{e}_{B_\sigma(M)}$ of $P(M)$ achieves for any sequence $v_0 > \cdots > v_n$ the maximum in

$$\max_{\mathbf{m}\in P(M)} \langle \mathbf{m}, v_0\mathbf{e}_{\sigma(0)} + v_1\mathbf{e}_{\sigma(1)} + \cdots + v_n\mathbf{e}_{\sigma(n)}\rangle.$$

Writing $\overline{\sigma} \in \mathfrak{S}_E$ for the permutation defined by $\overline{\sigma}(i) = \sigma(n-i)$, define the basis $B_{\overline{\sigma}}(M)$ to be the **reverse-lex-basis** of M with respect to σ. Equivalently, the basis $B_{\overline{\sigma}}(M)$ is the basis of M such that the vertex $\mathbf{e}_{B_{\overline{\sigma}}(M)}$ of $P(M)$ achieves for any sequence $v_0 > \cdots > v_n$ the minimum in

$$\min_{\mathbf{m}\in P(M)} \langle \mathbf{m}, v_0\mathbf{e}_{\sigma(0)} + v_1\mathbf{e}_{\sigma(1)} + \cdots + v_n\mathbf{e}_{\sigma(n)}\rangle.$$

We simply write B_σ or $B_{\overline{\sigma}}$ if the matroid in question is clear.

That the respective maximum or minimum is achieved uniquely at a vertex of $P(M)$, independently of all $v_0 > \cdots > v_n$, follows from Proposition 3.1.(a). The equivalence of the two definitions for $B_{\overline{\sigma}}(M)$ follows from noting that if $w_i = -v_{n-i}$ then $v_0 > \cdots > v_n$ is equivalent to $w_0 > \cdots > w_n$, and $v_0\mathbf{e}_{\sigma(0)} + v_1\mathbf{e}_{\sigma(1)} + \cdots + v_n\mathbf{e}_{\sigma(n)} = -(w_0\mathbf{e}_{\overline{\sigma}(0)} + w_1\mathbf{e}_{\overline{\sigma}(1)} + \cdots + w_n\mathbf{e}_{\overline{\sigma}(n)})$.

Remark 3.3 Under the min-convention for polyhedral geometry, the cone in the normal fan of $P(M)$ corresponding to the vertex $\mathbf{e}_B$ consists of $x \in \mathbb{R}^E/\mathbb{R}\mathbf{1}$ such that $\mathbf{e}_B$ achieves the minimum in $\min_{\mathbf{m}\in P(M)}\langle \mathbf{m}, x\rangle$. In particular, for a permutation $\sigma \in \mathfrak{S}_E$, the definition of $\mathbf{e}_{B_{\overline{\sigma}}(M)}$ implies that the cone of Σ_E corresponding to σ is contained in the cone in the normal fan of $P(M)$ corresponding to the vertex $\mathbf{e}_{B_{\overline{\sigma}}(M)}$.

Remark 3.4 Choosing $v_0 \gg \cdots \gg v_n$ justifies the terminology "lex-first-basis" because it implies $B_\sigma(M)$ is the first basis in the lexicographic ordering when the ground set has the linear order $\sigma(0) \prec \cdots \prec \sigma(n)$.

Combining the two parts of Proposition 3.1, we have the following.

Lemma 3.5 *If $L \subseteq \mathbb{C}^E$ is a realization of a matroid M of rank r, then one has a T-equivariant map*

$$\varphi_L\colon X_E \to X_{P(M)} \xrightarrow{\sim} \overline{T\cdot L} \subset \mathrm{Gr}(r; E) \qquad (\dagger)$$

which sends the identity point $\mathbf{1}$ of the open torus $T/\mathbb{C}^*$ of X_E to the point L in $\mathrm{Gr}(r; E)$*. For each permutation $\sigma \in \mathfrak{S}_E$, the map φ_L sends the T-fixed point p_σ in X_E to the T-fixed point $p_{B_{\overline{\sigma}}(M)}$ in* $\mathrm{Gr}(r; E)$.

Proof Proposition 3.1.(a) implies that we have a T-equivariant map $X_E \to X_{P(M)}$ induced by a coarsening of fans. By Remark 3.3, for a permutation $\sigma \in \mathfrak{S}_E$, the cone of Σ_E corresponding to σ is contained in the cone in the normal fan of $P(M)$ corresponding to the vertex $\mathbf{e}_{B_{\overline{\sigma}}(M)}$. The rest of the lemma now follows from Proposition 3.1.(b). $\square$

We thus have the following proposition. Recall that $\underline{\mathbb{C}}^E_{\text{inv}} = X_E \times \mathbb{C}^E_{\text{inv}}$, where $\mathbb{C}^E_{\text{inv}}$ denotes the vector space $\mathbb{C}^E$ with the inverse action of T.

Proposition 3.6 *The assignment $L \mapsto \operatorname{crem} \varphi_L^* \mathcal{S}$ (resp. $L \mapsto \operatorname{crem} \varphi_L^* \mathcal{Q}$), where $\mathcal{S}$ (resp. $\mathcal{Q}$) is the tautological sub (resp. quotient) bundle of* $\mathrm{Gr}(\dim L; E)$, *is a bijection between subspaces $L \subseteq \mathbb{C}^E$ and T-equivariant sub (resp. quotient) bundles of $\underline{\mathbb{C}}^E_{\text{inv}}$ whose fiber at the identity $\mathbf{1} \in T/\mathbb{C}^*$ is L (resp. $\mathbb{C}^E/L$).*

Proof Since $\mathcal{S}$ (resp. $\mathcal{Q}$) is a sub (resp. quotient) bundle of the trivial bundle $\mathrm{Gr}(\dim L; E) \times \mathbb{C}^E$, where T acts on $\mathbb{C}^E$ by the standard action $t \cdot (x_0, \dots, x_n) = (t_0 x_0, \dots, t_n x_n)$, the pullback $\varphi_L^* \mathcal{S}$ (resp. $\varphi_L^* \mathcal{Q}$) is a T-equivariant sub (resp. quotient) bundle of $X_E \times \mathbb{C}^E$. Applying the Cremona involution, we thus have that $\operatorname{crem} \varphi_L^* \mathcal{S}$ (resp. $\operatorname{crem} \varphi_L^* \mathcal{Q}$) is a T-equivariant sub (resp. quotient) bundle of $\underline{\mathbb{C}}^E_{\text{inv}} = X_E \times \mathbb{C}^E_{\text{inv}}$ whose fiber over $\mathbf{1}$ is L (resp. $\mathbb{C}^E/L$). Since such a sub (resp. quotient) bundle is uniquely determined by its fibers over the dense torus of X_E, which by torus-equivariance is uniquely determined by its fiber at the identity, the assignment is a bijection. $\square$

We thus find that the notions in Definition 1.2, reproduced below, are well-defined.

Definition 1.2 For an r-dimensional linear subspace $L \subseteq \mathbb{C}^E$, the **tautological subbundle** $\mathcal{S}_L$ and the **tautological quotient bundle** $\mathcal{Q}_L$ of L are defined by

$$\begin{aligned} \mathcal{S}_L &:= \text{the unique } T\text{-equivariant rank } r \text{ subbundle of } \underline{\mathbb{C}}^E_{\text{inv}} \text{ whose fiber at } \mathbf{1} \text{ is } L, \quad \text{and} \\ \mathcal{Q}_L &:= \text{the unique } T\text{-equivariant rank } |E|-r \text{ quotient bundle of } \underline{\mathbb{C}}^E_{\text{inv}} \text{ whose fiber} \\ &\qquad \text{at } \mathbf{1} \text{ is } \mathbb{C}^E/L. \end{aligned}$$

Equivalently, $\mathcal{S}_L$ and $\mathcal{Q}_L$ are defined as

$$\mathcal{S}_L := \operatorname{crem} \varphi_L^* \mathcal{S} \qquad \text{and} \qquad \mathcal{Q}_L := \operatorname{crem} \varphi_L^* \mathcal{Q}.$$

In other words, the fiber of $\mathcal{S}_L$ over a point $\overline{t}$ in the open torus $T/\mathbb{C}^*$ of X_E is identified with the subspace $t^{-1}L$ of $\mathbb{C}^E$. Remark 7.12 explains the inverse t^{-1}. We now identify the localizations of $[\mathcal{S}_L]$ and $[\mathcal{Q}_L]$ at the torus fixed points of X_E, proving that the K-classes depend only on the matroid M of the realization L.

Proposition 3.7 *For any realization $L \subseteq \mathbb{C}^E$ of a matroid M with ground set E, the T-equivariant K-classes $[\mathcal{S}_L]$ and $[\mathcal{Q}_L]$ only depend on the matroid M, and satisfy*

$$[\mathcal{S}_L]_\sigma = \sum_{i \in B_\sigma(M)} T_i^{-1} \quad \textit{and}$$

$$[\mathcal{Q}_L]_\sigma = \sum_{i \in E \setminus B_\sigma(M)} T_i^{-1} \quad \textit{for every permutation } \sigma \in \mathfrak{S}_E.$$

Proof Let $r = \dim L$, and let $\varphi_L : X_E \to X_{P(M)} \xrightarrow{\sim} \overline{T \cdot L} \subset \mathrm{Gr}(r; E)$ be as above. For a permutation $\sigma \in \mathfrak{S}_E$, the map $\varphi_L : X_E \to \overline{T \cdot L}$ sends p_σ to the fixed point of X_E corresponding to $B_{\overline{\sigma}}(M)$. Therefore $\varphi_L(p_\sigma) = p_{B_{\overline{\sigma}}} \in \mathrm{Gr}(r; E)$, so we have

$$[\varphi_L^* \mathcal{S}]_\sigma = [\mathcal{S}]_{B_{\overline{\sigma}}(M)} = \sum_{i \in B_{\overline{\sigma}}(M)} T_i \quad \text{and} \quad [\varphi_L^* \mathcal{Q}]_\sigma = [\mathcal{Q}]_{B_{\overline{\sigma}}(M)} = \sum_{i \in E \setminus B_{\overline{\sigma}}(M)} T_i$$

by §2.2. Applying the Cremona involution (Remark 2.2) then yields the desired statement. □

This description of the T-equivariant K-classes of tautological bundles of realizations of matroids extend to arbitrary (not necessarily realizable) matroids.

Proposition 3.8 *For any matroid M (not necessarily realizable) on ground set E, the two $\mathfrak{S}_E$-tuples $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ of Laurent polynomials defined by*

$$[\mathcal{S}_M]_\sigma = \sum_{i \in B_\sigma} T_i^{-1} \quad \textit{and} \quad [\mathcal{Q}_M]_\sigma = \sum_{i \notin B_\sigma} T_i^{-1} \quad \textit{for } \sigma \in \mathfrak{S}_E$$

are well-defined T-equivariant K-classes on X_E satisfying $[\mathcal{S}_M] + [\mathcal{Q}_M] = [\underline{\mathbb{C}}^E_{\mathrm{inv}}]$.

Proof Let σ and σ' be permutations such that $\sigma' = \sigma \circ (i, i+1)$ for some $i \in E$. Note that $\overline{\sigma'} = \overline{\sigma} \circ (n-i, n-i-1)$. So, the two maximal cones in Σ_E corresponding to $\overline{\sigma}$ and $\overline{\sigma'}$ intersect in a codimension 1 cone whose linear span is the hyperplane normal to $\mathbf{e}_{\overline{\sigma}(n-i)} - \mathbf{e}_{\overline{\sigma}(n-i-1)}$. Since the normal fan of the base polytope $P(M)$ coarsens Σ_E, the two vertices $\mathbf{e}_{B_\sigma}$ and $\mathbf{e}_{B_{\sigma'}}$ are either identical, or their difference is equal to $\mathbf{e}_{\overline{\sigma}(n-i)} - \mathbf{e}_{\overline{\sigma}(n-i-1)} = \mathbf{e}_{\sigma(i)} - \mathbf{e}_{\sigma(i+1)}$. In other words, the lex-first-bases B_σ and $B_{\sigma'}$ are thus identical or have symmetric difference $\{\sigma(i), \sigma(i+1)\}$. Hence, the two $\mathfrak{S}_E$-tuples satisfy the condition in Theorem 2.1.(a). Their sum is the $\mathfrak{S}_E$-tuple such that we have $\sum_{i \in E} T_i^{-1}$ for all $\sigma \in \mathfrak{S}_E$, which defines the class of $[\underline{\mathbb{C}}^E_{\mathrm{inv}}]$. □

Definition 3.9 For a matroid M with ground set E, we define the **tautological sub** (resp. **quotient**) ***K*-class** of M to be the T-equivariant K-class $[\mathcal{S}_M]$ (resp. $[\mathcal{Q}_M]$) in $K_0^T(X_E)$ defined by

$$[\mathcal{S}_M]_\sigma = \sum_{i \in B_\sigma} T_i^{-1} \quad \text{resp.} \quad [\mathcal{Q}_M]_\sigma = \sum_{i \notin B_\sigma} T_i^{-1} \quad \text{for } \sigma \in \mathfrak{S}_E.$$

We define the **tautological sub** (resp. **quotient**) **equivariant Chern classes** of M to be associated equivariant Chow classes $c_j^T(\mathcal{S}_M)$ (resp. $c_j^T(\mathcal{Q}_M)$) in $A_T^\bullet(X_E)$, which by definition are given by

$$c_j^T(\mathcal{S}_M)_\sigma = \mathrm{Elem}_j(\{-t_i\}_{i \in B_\sigma}) \quad \text{resp.} \quad c_j^T(\mathcal{Q}_M)_\sigma = \mathrm{Elem}_j(\{-t_i\}_{i \notin B_\sigma}) \quad \text{for } \sigma \in \mathfrak{S}_E.$$

Example 3.10 Recall the two distinguished divisor classes α and β on X_E, and denote by $U_{r,E}$ the uniform matroid on E of rank r. Note that the base polytope $P(U_{1,E})$ is the standard simplex $\Delta = \mathrm{Conv}(\mathbf{e}_i \mid i \in E)$. Since $E \setminus B_\sigma(U_{n,E}) = \{\sigma(n)\}$ and $B_\sigma(U_{1,E}) = \{\sigma(0)\}$, by comparing the localizations at p_σ for $\sigma \in \mathfrak{S}_E$, it follows from Remark 2.4 that we have $[\mathcal{Q}_{U_{n,E}}] = [\mathcal{O}(D_{P(U_{1,E})})]$ and $[\mathcal{S}^\vee_{U_{1,E}}] = [\mathcal{O}(D_{-P(U_{1,E})})]$ as classes in $K^0_T(X_E)$, and thus

$$[\mathcal{Q}_{U_{n,E}}] = [\mathcal{O}(\alpha)] \quad \text{and} \quad [\mathcal{S}^\vee_{U_{1,E}}] = [\mathcal{O}(\beta)]$$

as non-equivariant K-classes. Moreover, since $[\mathcal{S}_M] + [\mathcal{Q}_M] = [\underline{\mathbb{C}}^E_{\mathrm{inv}}]$ for a matroid M, we have

$$c(\mathcal{S}^\vee_{U_{n,E}}) = c(\mathcal{Q}^\vee_{U_{n,E}})^{-1} = 1 + \alpha + \cdots + \alpha^n \quad \text{and}$$

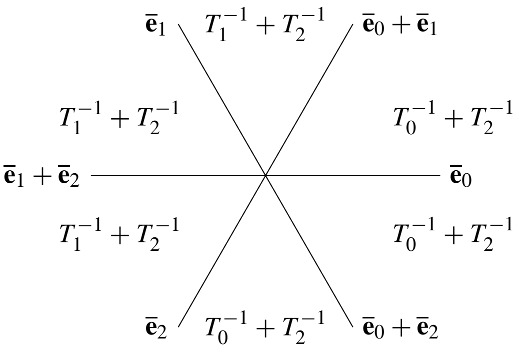
$$c(\mathcal{Q}_{U_{1,E}}) = c(\mathcal{S}_{U_{1,E}})^{-1} = 1 + \beta + \cdots + \beta^n$$

as non-equivariant Chow classes in $A^\bullet(X_E)$.

Example 3.11 We note $[\mathcal{O}(D_{-P(M)})] = [\det \mathcal{S}^\vee_M]$. Indeed, by the discussion in §2.7 above Remark 2.4,

$$[\mathcal{O}(D_{-P(M)})]_\sigma = \mathbf{T}^{\mathbf{e}_{B_\sigma(M)}} = \prod_{i \in B_\sigma(M)} T_i = [\det \mathcal{S}^\vee_M]_\sigma \quad \text{for any } \sigma \in \mathfrak{S}_E.$$

Example 3.12 Let $M = U_{1,\{0,1\}} \oplus U_{1,\{2\}}$. The following figure illustrates the fan $\Sigma_{\{0,1,2\}}$ and the class $[\mathcal{S}_M]$ represented by assignments of Laurent polynomials to the maximal cones.



This matroid M has a realization $L \subseteq \mathbb{C}^{\{0,1,2\}}$ where L is the row-span of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Consider a permutation σ defined by $(\sigma(0), \sigma(1), \sigma(2)) = (2, 0, 1)$. Note that $\bar{\mathbf{e}}_0 + 2\bar{\mathbf{e}}_2$ lies in the interior of $\mathrm{Cone}(\bar{\mathbf{e}}_{\sigma(0)}, \bar{\mathbf{e}}_{\sigma(0)} + \bar{\mathbf{e}}_{\sigma(1)})$, the cone corresponding to σ. Hence, the map $\lambda_\sigma : \mathbb{C}^* \to (\mathbb{C}^*)^{\{0,1,2\}}/\mathbb{C}^*$ defined by $s \mapsto [s : 1 : s^2]$ limits as $s \to 0$ to the T-fixed point p_σ in $X_{\{0,1,2\}}$. The limit as $s \to 0$ of $\lambda_\sigma(s)^{-1} L =$ row-span $\begin{bmatrix} s^{-1} & 1 & 0 \\ 0 & 0 & s^{-2} \end{bmatrix}$ in $\mathrm{Gr}(2; \{0,1,2\})$ is row-span $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This verifies as expected that $[\mathcal{S}_L]_\sigma = T_0^{-1} + T_2^{-1}$.

3.2 Matroid analogues of Grassmannian K-classes

Here we generalize the association of $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ from the vector bundles $\mathcal{S}$ and $\mathcal{Q}$ on $\mathrm{Gr}(r; E)$, to arbitrary K-classes on $\mathrm{Gr}(r; E)$. We will not need this until §10, where we relate our computations to the computations of [50]. One may consider it as a combinatorial abstraction for arbitrary matroids of a pullback map from the K-ring of $\mathrm{Gr}(r; E)$ to the K-ring of the permutohedral variety.

Proposition 3.13 *Let M be a matroid of rank r with ground set E. For any class $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$, an element $[\mathcal{E}_M]$ defined by*

$$[\mathcal{E}_M]_\sigma = [\mathcal{E}]_{B_\sigma(M)}(T_0^{-1}, \dots, T_n^{-1}) \in \mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}] \quad \textit{for each } \sigma \in \mathfrak{S}_E$$

is a well-defined element in $K_T^0(X_E)$ such that if $L \subseteq \mathbb{C}^E$ is any realization of M, then $[\mathcal{E}_M] = \mathrm{crem}\,\varphi_L^*[\mathcal{E}]$ where $\varphi_L \colon X_E \to \mathrm{Gr}(r; E)$ is the map (†) in Lemma 3.5. Moreover, the assignment $[\mathcal{E}] \mapsto [\mathcal{E}_M]$ is a ring homomorphism $K_T^0(\mathrm{Gr}(r; E)) \to K_T^0(X_E)$ that respects exterior powers.

The notation of Proposition 3.13 is consistent with our definition of the tautological K-classes $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ of a matroid M. In particular, Proposition 3.13 implies that the notation $[\bigwedge^i \mathcal{S}_M]$ is unambiguous, since $[(\bigwedge^i \mathcal{S})_M] = [\bigwedge^i (\mathcal{S}_M)]$, and likewise for exterior powers of $\mathcal{Q}_M$ and the duals $\mathcal{S}_M^\vee$, $\mathcal{Q}_M^\vee$.

Proof By the property of the Cremona involution (Remark 2.2), it suffices to show that an element $[\mathcal{E}'_M]$ defined by $[\mathcal{E}'_M]_\sigma = [\mathcal{E}]_{B_{\overline{\sigma}}(M)} \in \mathbb{Z}[T_0^{\pm}, \dots, T_n^{\pm}]$ for $\sigma \in \mathfrak{S}_E$ is a well-defined element in $K_T^0(X_E)$ such that $[\mathcal{E}'_M] = \varphi_L^*[\mathcal{E}]$ for a realization L. To see well-definedness, we check that $[\mathcal{E}'_M]$ satisfies the condition in Theorem 2.1.(a). Suppose that σ and σ' are maximal cones in Σ_E sharing a codimension 1 face, whose linear span is $\{x \in \mathbb{R}^E/\mathbb{R}\mathbf{1} \mid x_i = x_j\}$ for some $i \neq j \in E$. Since the subsets $B_{\overline{\sigma}}(M)$ and $B_{\overline{\sigma'}}(M)$ are either identical or have symmetric difference $\{i, j\}$, the condition for $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$ as noted in §2.2 implies that $[\mathcal{E}'_M]_{B_{\overline{\sigma}}(M)} \equiv [\mathcal{E}'_M]_{B_{\overline{\sigma'}}(M)}$ mod $(T_i - T_j)$, as desired. That $[\mathcal{E}'_M] = \varphi_L^*[\mathcal{E}]$ for a realization L follows from the fact that φ_L maps the point p_σ in X_E to $p_{B_{\overline{\sigma}}(M)}$ in $\mathrm{Gr}(r; E)$ for any permutation $\sigma \in \mathfrak{S}_E$ by Lemma 3.5. That the assignment $[\mathcal{E}] \mapsto [\mathcal{E}_M]$ is a ring homomorphism respecting exterior powers is straightforward to check from the defining formula $[\mathcal{E}_M]_\sigma = [\mathcal{E}]_{B_\sigma(M)}(T_0^{-1}, \dots, T_n^{-1})$ for each $\sigma \in \mathfrak{S}_E$. $\square$

4 A unifying Tutte polynomial formula

In this section, we prove Theorem A, reproduced below, by establishing a deletion-contraction relation for the tautological Chern classes of matroids.

Theorem A *Let $\int_{X_E} \colon A^\bullet(X_E) \to \mathbb{Z}$ be the degree map on X_E. For a matroid M of rank r with ground set E, define a polynomial*

$$t_M(x, y, z, w) = (x + y)^{-1}(y + z)^r (x + w)^{|E|-r}\, T_M\Big(\frac{x + y}{y + z}, \frac{x + y}{x + w}\Big).$$

Then, we have an equality

$$\sum_{i+j+k+\ell=n}\left(\int_{X_E}\alpha^i\beta^j c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M)\right)x^i y^j z^k w^\ell = t_M(x,y,z,w).$$

To do so, we will use the following property of Tutte polynomials. See [24, Sect. 6.2], specifically Exercise 6.5(b), for details and further references.

Proposition 4.1 *There is a unique 5-variable polynomial $G_M(u,v,a,b,\gamma)\in\mathbb{Z}[u,v,a,b,\gamma]$ associated to a matroid M, called a **generalized Tutte-Grothendieck invariant** of M, satisfying the following properties*:

(1) (*Base case*) *If $|E|=1$, then*

$$G_M(u,v,a,b,\gamma)=\begin{cases} u & \text{if } M \text{ has rank } 1 \text{ (i.e. } M \text{ is a coloop)}\\ v & \text{if } M \text{ has rank } 0 \text{ (i.e. } M \text{ is a loop)}.\end{cases}$$

(2) (*Deletion-contraction relation*) *If $|E|\geq 2$ and $i\in E$, then*

$$G_M(u,v,a,b,\gamma)=\begin{cases} \gamma u G_{M/i}(u,v,a,b,\gamma) & \text{if } i\in E \text{ is a coloop in } M\\ \gamma v G_{M\setminus i}(u,v,a,b,\gamma) & \text{if } i\in E \text{ is a loop in } M\\ aG_{M\setminus i}(u,v,a,b,\gamma)+bG_{M/i}(u,v,a,b,\gamma) & \\ \quad \text{if } i\in E \text{ is neither a loop nor a coloop in } M. & \end{cases}$$

For a matroid M with ground set E and of rank r, this polynomial is given by

$$G_M(u,v,a,b,\gamma)=\gamma^{-1}b^r a^{|E|-r}T_M\Big(\frac{u\gamma}{b},\frac{v\gamma}{a}\Big).$$

In particular, for the polynomial appearing in Theorem A we have

$$G_M(1,1,x+w,y+z,x+y)=(x+y)^{-1}(y+z)^r(x+w)^{|E|-r}T_M\Big(\frac{x+y}{y+z},\frac{x+y}{x+w}\Big).$$

Let us now restate Theorem A as follows. Denote an element $\xi_M\in A^\bullet(X_E)[x,y,z,w]$ by

$$\begin{aligned}\xi_M&=(1+\alpha x+\cdots+\alpha^n x^n)(1+\beta y+\cdots+\beta^n y^n)c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w)\\ &=c(\mathcal{S}_{U_{n,E}}^\vee,x)c(\mathcal{Q}_{U_{1,E}},y)c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w),\end{aligned}$$

where the second equality follows from Example 3.10. We show that $\int_{X_E}\xi_M$ is the generalized Tutte-Grothendieck invariant in Proposition 4.1 with $u=v=1$, $a=x+w$, $b=y+z$, and $\gamma=x+y$.

For the base case, note that if $|E|=1$ then $\xi_M=1$ because X_E is 0-dimensional. It follows that $\int_{X_E}\xi_M=1$.

If $|E|\geq 2$, we will show that $\int_{X_E}\xi_M$ satisfies the deletion-contraction relation in Proposition 4.1 in two steps (for concreteness we will take $i=n$ in our arguments).

First, in §4.1, we consider a surjective map $f\colon X_E \to X_{E\setminus n}$ of permutohedral varieties defined by deleting $n \in E$, and study the behavior of the tautological Chern classes of M under the pushforward map $f_*\colon A^\bullet(X_E) \to A^{\bullet-1}(X_{E\setminus n})$. Then, in §4.2, we use these observations to show Theorem 4.8, which states that ξ_M satisfies

$$f_*\xi_M = \begin{cases} (x+y)\xi_{M\setminus n} & \text{if } n \text{ is a loop} \\ (x+y)\xi_{M/n} & \text{if } n \text{ is a coloop} \\ (x+w)\xi_{M\setminus n} + (y+z)\xi_{M/n} & \text{if } n \text{ is neither a loop nor a coloop.} \end{cases}$$

Noting that $\int_{X_E} \xi_M = \int_{X_{E\setminus n}} f_*\xi_M$ by the functoriality of pushforward maps, we conclude that $\int_{X_E} \xi_M$ satisfies the deletion-contraction relation of Proposition 4.1, and Theorem A follows.

4.1 A projection map of permutohedral varieties

As before, we let $E = \{0, 1, \dots, n\}$, and assume throughout that $|E| \geq 2$. Denote $T' = (\mathbb{C}^*)^{E\setminus n}$. The projection map $\mathbb{R}^E/\mathbb{R}\mathbf{1} \to \mathbb{R}^{(E\setminus n)}/\mathbb{R}\mathbf{1}$ induces a map of fans $\Sigma_E \to \Sigma_{E\setminus n}$ because the cone of Σ_E corresponding to a permutation $\widetilde{\sigma} \in \mathfrak{S}_E$ maps to the cone of $\Sigma_{E\setminus n}$ corresponding to the permutation $\sigma \in \mathfrak{S}_{E\setminus n}$ where the sequence $(\sigma(0), \dots, \sigma(n-1))$ is obtained from $(\widetilde{\sigma}(0), \dots, \widetilde{\sigma}(n))$ by omitting the entry $n = \widetilde{\sigma}(\widetilde{\sigma}^{-1}(n))$.

Definition 4.2 Let $f\colon X_E \to X_{E\setminus n}$ be the toric morphism of permutohedral varieties induced by the projection map $\mathbb{R}^E/\mathbb{R}\mathbf{1} \to \mathbb{R}^{(E\setminus n)}/\mathbb{R}\mathbf{1}$, where X_E and $X_{E\setminus n}$ are considered as a T-variety and a T'-variety, respectively. The underlying map of tori $T/\mathbb{C}^\times \to T'/\mathbb{C}^\times$ corresponding to the toric morphism f is induced by the map $T \to T'$ given by projection onto the first n coordinates.

Given a permutation $\sigma \in \mathfrak{S}_{E\setminus n}$, define for each $i \in E$ a permutation $\sigma^i \in \mathfrak{S}_E$ by

$$\sigma^i(j) = \begin{cases} \sigma(j) & j < i \\ n & j = i \\ \sigma(j-1) & j > i \end{cases} \qquad \text{for } 0 \leq j \leq n,$$

so that the preimage $f^{-1}(p_\sigma)$ of the T'-fixed point p_σ of $X_{E\setminus n}$ consists of T-fixed points $\{p_{\sigma^i}\}_{i\in E}$ of X_E.

One may consider the permutation $\sigma^i \in \mathfrak{S}_E$ as the permutation obtained from $\sigma \in \mathfrak{S}_{E\setminus n}$ by inserting n right after $\sigma(i-1)$ in the linear order $\sigma(0) \prec \ldots \prec \sigma(n-1)$ of σ. In other words, the linear order defining the permutation σ^i is given by $\sigma(0) \prec \cdots \prec \sigma(i-1) \prec n \prec \sigma(i) \prec \cdots \prec \sigma(n-1)$.

In order to push forward ξ_M under $f_*\colon A^\bullet(X_E) \to A^{\bullet-1}(X_{E\setminus n})$, our strategy will be to take an equivariant version ξ_M^T of ξ_M and compute the image under the composite $A_T^\bullet(X_E) \xrightarrow{f_*} A_T^{\bullet-1}(X_{E\setminus\{n\}}) \to A_{T'}^{\bullet-1}(X_{E\setminus\{n\}})$, where the first map is given by equivariant pushforward and the second map is induced by the map $T' \to T$ given by inclusion into the first n coordinates. This would recover the pushforward $f_*(\xi_M)$ by the commutativity of the diagram below.

$$\begin{tikzcd}
\xi_M^T \in A_T^\bullet(X_E) \arrow[rr] \arrow[d, "f_*"] & & A^\bullet(X_E) \ni \xi_M \arrow[d, "f_*"] \\
A_T^{\bullet-1}(X_{E\setminus\{n\}}) \arrow[r] & A_{T'}^{\bullet-1}(X_{E\setminus\{n\}}) \arrow[r] & A^{\bullet-1}(X_{E\setminus\{n\}})
\end{tikzcd}$$

To compute the image of ξ_M^T under the composite $A_T^\bullet(X_E) \xrightarrow{f_*} A_T^{\bullet-1}(X_{E\setminus\{n\}}) \to A_{T'}^{\bullet-1}(X_{E\setminus\{n\}})$, we compute the localizations of the image at each of the torus-fixed points. Our basic tool for doing so is Lemma 4.3 below.

Lemma 4.3 *For a T-equivariant Chow class $\xi^T \in A_T^\bullet(X_E)$, the pushforward map $f_*\colon A_T^\bullet(X_E) \to A_T^{\bullet-1}(X_{E\setminus n})$ satisfies*

$$(f_*\xi^T)_\sigma|_{t_n=0} = \sum_{i=0}^{n-1} t_{\sigma(i)}^{-1}\big(\xi_{\sigma^{i+1}}^T\big|_{t_n=0} - \xi_{\sigma^i}^T|_{t_n=0}\big) \in A_{T'}^{\bullet-1}(\mathrm{pt})$$

$$\textit{for any permutation } \sigma \in \mathfrak{S}_{E\setminus n},$$

where the right-hand-side always simplifies to a polynomial in $A_{T'}^{\bullet-1}(\mathrm{pt}) = \mathbb{Z}[t_0, \ldots, t_{n-1}]$.

Proof We apply the localization formula [21, Corollary 4.2] with the identification of the torus action on the tangent spaces to the torus-fixed points of X_E at the end of §2.3, and write

$$\xi^T = \sum_{\tau\in\mathfrak{S}_E} \frac{\xi_\tau^T}{\prod_{\ell=1}^n (t_{\tau(\ell-1)} - t_{\tau(\ell)})}[p_\tau],$$

where $[p_\tau]$ is the class of the T-fixed point $p_\tau \in X_E$, and its coefficient is the equivariant multiplicity of ξ^T at p_τ by [21, Theorem 5.4]. Applying f_* to this equation and regrouping gives,

$$f_*\xi^T = \sum_{\sigma\in\mathfrak{S}_{E\setminus n}} \sum_{i=0}^n \frac{\xi_{\sigma^i}^T}{\prod_{\ell=1}^n (t_{\sigma^i(\ell-1)} - t_{\sigma^i(\ell)})}[p_\sigma],$$

Here, we are treating $X_{E\setminus n}$ as a T-variety via the map $T \to T'$, and the class $f_*\xi^T$ as a T-equivariant Chow class in $A_T^\bullet(X_{E\setminus n})$. Applying the equivariant multiplicity map of [21, Theorem 4.2] we obtain

$$\frac{(f_*\xi^T)_\sigma}{\prod_{j=1}^{n-1}(t_{\sigma(j-1)} - t_{\sigma(j)})} = \sum_{i=0}^n \frac{\xi_{\sigma^i}^T}{\prod_{\ell=1}^n (t_{\sigma^i(\ell-1)} - t_{\sigma^i(\ell)})},$$

which yields

$$(f_*\xi^T)_\sigma = \sum_{i=0}^n \frac{\prod_{j=1}^{n-1}(t_{\sigma(j-1)} - t_{\sigma(j)})}{\prod_{\ell=1}^n (t_{\sigma^i(\ell-1)} - t_{\sigma^i(\ell)})}\xi_{\sigma^i}^T$$

$$= -\frac{1}{t_{\sigma(0)} - t_n}\xi^T_{\sigma^0} + \frac{1}{t_{\sigma(n-1)} - t_n}\xi^T_{\sigma^n} + \sum_{i=1}^{n-1} \frac{t_{\sigma(i-1)} - t_{\sigma(i)}}{(t_{\sigma(i-1)} - t_n)(t_n - t_{\sigma(i)})}\xi^T_{\sigma^i}$$

$$= -\frac{1}{t_{\sigma(0)} - t_n}\xi^T_{\sigma^0} + \frac{1}{t_{\sigma(n-1)} - t_n}\xi^T_{\sigma^n} + \sum_{i=1}^{n-1} (\frac{1}{t_{\sigma(i-1)} - t_n} - \frac{1}{t_{\sigma(i)} - t_n})\xi^T_{\sigma^i}.$$

Reordering the terms and setting $t_n = 0$ then yields the desired result. □

We now consider how the T-equivariant tautological Chern classes of matroids behave with respect to the pushforward map f_*. Let us prepare with the following.

Lemma 4.4 *Let M be a matroid on ground set E, and let $\sigma \in \mathfrak{S}_{E\setminus n}$. For all $0 \le i \le n$, if n is a loop then $B_{\sigma^i}(M) = B_\sigma(M \setminus n)$, and if n is a coloop then $B_{\sigma^i}(M) = B_\sigma(M/n) \sqcup \{n\}$. If $n \in E$ is neither a loop nor a coloop, there is a $0 \le k \le n-1$ such that*

- $B_{\sigma^0}(M) = B_{\sigma^1}(M) = \cdots = B_{\sigma^k}(M) = B_\sigma(M/n) \sqcup \{n\}$,
- $B_{\sigma^{k+1}}(M) = \cdots = B_{\sigma^n}(M) = B_\sigma(M \setminus n)$, *and*
- $B_\sigma(M/n) \sqcup \{\sigma(k)\} = B_\sigma(M \setminus n)$.

Proof When n is not a coloop, the set of bases of $M \setminus n$ is $\{B \mid B$ a basis of M such that $B \not\ni n\}$, and when n is not a loop, the set of bases of M/n is $\{B \setminus n \mid B$ a basis of M such that $B \ni n\}$. Thus, for all $0 \le i \le n$, if n is a loop then $B_{\sigma^i}(M) = B_\sigma(M \setminus n)$, and if n is a coloop then $B_{\sigma^i}(M) = B_\sigma(M/n) \sqcup \{n\}$.

For all $0 \le i \le n$, the definition of $\sigma^i \in \mathfrak{S}_E$ and the "lex-first" property (Remark 3.4) imply that if $n \in B_{\sigma^i}(M)$ then $n \in B_{\sigma^j}(M)$ for all $0 \le j \le i$, and if $n \notin B_{\sigma^i}(M)$ then $n \notin B_{\sigma^j}(M)$ for all $i \le j \le n$. If n is neither a loop nor a coloop, then it is contained in some basis of M, and also avoids some other basis of M. Thus, the lex-first-basis $B_{\sigma^0}(M)$ of M with respect to the linear ordering $n \prec \sigma(0) \prec \cdots \prec \sigma(n-1)$ contains n, and the lex-first-basis $B_{\sigma^n}(M)$ of M with respect to the linear ordering $\sigma(0) \prec \cdots \prec \sigma(n-1) \prec n$ does not contain n. Hence, the maximum $\max\{0 \le i \le n-1 \mid n \in B_{\sigma^i}(M)\}$ is well-defined, and setting k to be this maximum, we see that the first two bullet points follow from the lex-first property.

For the third, we first note that the two cones in Σ_E corresponding to permutations σ^k and σ^{k+1} share the codimension 1 face whose linear span is the hyperplane normal to $\mathbf{e}_{\sigma(k)} - \mathbf{e}_n$. Hence, since the normal fan of $P(M)$ coarsens Σ_E (Proposition 3.1.(a)), and since $B_{\sigma^k}(M) \ne B_{\sigma^{k+1}}(M)$ by definition of k, then as in the proof of Proposition 3.8 we have the symmetric difference of $B_{\sigma^k}(M)$ and $B_{\sigma^{k+1}}(M)$ is $\{n, \sigma(k)\}$. □

Definition 4.5 Let M be a matroid on ground set E, and let $\sigma \in \mathfrak{S}_{E\setminus n}$. Then define $k_\sigma(M) \in \mathbb{Z}$ by

$$k_\sigma(M) = \begin{cases} -1 & \text{if } n \in E \text{ is loop in } M \\ n & \text{if } n \in E \text{ is a coloop in } M \\ \max\{0 \le i \le n-1 \mid n \in B_{\sigma^i}(M)\} & \text{if } n \in E \text{ is neither a loop} \\ & \text{nor a coloop.} \end{cases}$$

The following lemma records the key behavior of tautological Chern classes of matroids that we will need to establish a deletion-contraction relation (Theorem 4.8) in the next subsection.

Lemma 4.6 *Let M be a matroid on ground set E, and $\sigma \in \mathfrak{S}_{E\setminus n}$.*
(a) For any $0 \le i \le n$ we have

$$c^T(\mathcal{S}_M^\vee, u)_{\sigma^i}|_{t_n=0} = \begin{cases} c^{T'}(\mathcal{S}_{M/n}^\vee, u)_\sigma & \text{if } i \le k_\sigma(M) \\ c^{T'}(\mathcal{S}_{M\setminus n}^\vee, u)_\sigma & \text{if } i > k_\sigma(M), \end{cases} \quad \text{and}$$

$$c^T(\mathcal{Q}_M, u)_{\sigma^i}|_{t_n=0} = \begin{cases} c^{T'}(\mathcal{Q}_{M/n}, u)_\sigma & \text{if } i \le k_\sigma(M) \\ c^{T'}(\mathcal{Q}_{M\setminus n}, u)_\sigma & \text{if } i > k_\sigma(M). \end{cases}$$

(b) If $n \in E$ is neither a loop nor a coloop, then writing $k = k_{\sigma(M)}$, we have

$$(1 + t_{\sigma(k)}u)c^{T'}(\mathcal{S}_{M/n}^\vee, u)_\sigma = c^{T'}(\mathcal{S}_{M\setminus n}^\vee, u)_\sigma \qquad \text{and}$$
$$(1 - t_{\sigma(k)}u)c^{T'}(\mathcal{Q}_{M\setminus n}, u)_\sigma = c^{T'}(\mathcal{Q}_{M/n}, u)_\sigma.$$

Proof Recalling that for any permutation $\widetilde{\sigma} \in \mathfrak{S}_E$ we have

$$c^T(\mathcal{S}_M^\vee, u)_{\widetilde{\sigma}} = \prod_{j \in B_{\widetilde{\sigma}}(M)} (1 + t_j u) \quad \text{and} \quad c^T(\mathcal{Q}_M, u)_{\widetilde{\sigma}} = \prod_{j \notin B_{\widetilde{\sigma}}(M)} (1 - t_j u),$$

this is a direct reformulation of Lemma 4.4 after noting $(1 \pm t_n u)|_{t_n=0} = 1$. □

4.2 A deletion-contraction relation

With these preparations in place, we are now ready to prove the deletion-contraction relation. Throughout, assume that $|E| \ge 2$. As before, let $T' = (\mathbb{C}^*)^{E\setminus n}$. For notational clarity, let us define

$$a_E(x) = c(\mathcal{S}_{U_{n,E}}^\vee, x) \qquad \text{and} \qquad b_E(y) = c(\mathcal{Q}_{U_{1,E}}, y),$$

and their T-equivariant counterparts $a_E^T(x) = c^T(\mathcal{S}_{U_{n,E}}^\vee, x)$ and $b_E^T(y) = c^T(\mathcal{Q}_{U_{1,E}}, y)$. In particular, for a matroid M on ground set E, we have

$$\xi_M = a_E(x) b_E(y) c(\mathcal{S}_M^\vee, z) c(\mathcal{Q}_M, w).$$

The following corollary of Lemma 4.6 will be useful in our computations.

Corollary 4.7 *The T-equivariant classes $a_E^T(x) = c^T(\mathcal{S}_{U_{n,E}}^\vee, x)$ and $b_E^T(y) = c^T(\mathcal{Q}_{U_{1,E}}, y)$ satisfy*

$$a_E^T(x)_{\sigma^i}|_{t_n=0} = a_{E\setminus n}^{T'}(x)_\sigma \quad \text{for} \quad 0 \le i < n \quad \text{and}$$

$$a_E^T(x)_{\sigma^n}|_{t_n=0} = (1 + t_{\sigma(n-1)}x)a_{E\setminus n}^{T'}(x)_\sigma, \text{ and}$$

$$b_E^T(y)_{\sigma^i}|_{t_n=0} = b_{E\setminus n}^{T'}(y)_\sigma \quad \text{for}$$

$$0 < i \leq n \quad \text{and} \quad b_E^T(y)_{\sigma^0}|_{t_n=0} = (1 - t_{\sigma(0)}y)b_{E\setminus n}^{T'}(y)_\sigma$$

for any permutation $\sigma \in \mathfrak{S}_{E\setminus n}$.

Proof For the part concerning a_E^T, apply Lemma 4.6 to the matroid $U_{n,E}$, noting that $(U_{n,E})/n = U_{n-1,E\setminus n}$ and that $k_\sigma(U_{n,E}) = n - 1$ for any $\sigma \in \mathfrak{S}_{E\setminus n}$. Likewise, for b_E^T, apply Lemma 4.6 to the matroid $U_{1,E}$, noting that $(U_{1,E}) \setminus n = U_{1,E\setminus n}$ and that $k_\sigma(U_{1,E}) = 0$ for any $\sigma \in \mathfrak{S}_{E\setminus n}$. $\square$

Theorem 4.8 *Let M be a matroid on ground set E with $|E| \geq 2$. Let $f_*\colon A^\bullet(X_E) \to A^{\bullet-1}(X_{E\setminus n})$ be the pushforward map of the toric map $f\colon X_E \to X_{E\setminus n}$ in Definition 4.2. Then, we have*

$$f_*\xi_M = \begin{cases} (x+y)\xi_{M\setminus n} & \text{if } n \in E \text{ is a loop in } M \\ (x+y)\xi_{M/n} & \text{if } n \in E \text{ is a coloop in } M \\ (x+w)\xi_{M\setminus n} + (y+z)\xi_{M/n} & \text{otherwise.} \end{cases}$$

Proof We compute using the T-equivariant classes. That is, denote by

$$\xi_M^T = a_E^T(x)b_E^T(y)c^T(\mathcal{S}_M^\vee, z)c^T(\mathcal{Q}_M, w)$$

an element in $A_T^\bullet(X_E)[x, y, z, w]$, which maps to the non-equivariant class ξ_M. We wish to show that for any permutation $\sigma \in \mathfrak{S}_{E\setminus n}$

$$(f_*\xi_M^T)_\sigma|_{t_n=0} = \begin{cases} (x+y)(\xi_{M\setminus n}^{T'})_\sigma & \text{if } k_{\sigma(M)} = -1 \\ (x+w)(\xi_{M\setminus n}^{T'})_\sigma + (y+z)(\xi_{M/n}^{T'})_\sigma & \text{if } 0 \leq k_{\sigma(M)} \leq n-1 \\ (x+y)(\xi_{M/n}^{T'})_\sigma & \text{if } k_{\sigma(M)} = n. \end{cases} \quad (\#)$$

Let us fix an arbitrary permutation $\sigma \in \mathfrak{S}_{E\setminus n}$. By Lemma 4.3, we have

$$(f_*\xi_M^T)_\sigma|_{t_n=0} = \sum_{i=0}^{n-1} t_{\sigma(i)}^{-1}\big((\xi_M^T)_{\sigma^{i+1}}\big|_{t_n=0} - (\xi_M^T)_{\sigma^i}\big|_{t_n=0}\big).$$

Noting $k_{\sigma(M)} = -1$ if n is a loop, $k_{\sigma(M)} = n$ if n is a loop, and $0 \leq k_{\sigma(M)} \leq n-1$ otherwise, the desired equality (#) is implied by the following claim consisting of three cases.

Claim: For $i = 0$ we have

$$t_{\sigma(0)}^{-1}\big((\xi_M^T)_{\sigma^1}\big|_{t_n=0} - (\xi_M^T)_{\sigma^0}\big|_{t_n=0}\big) = \begin{cases} y(\xi_{M\setminus n}^{T'})_\sigma & \text{if } k_\sigma(M) = -1 \\ w(\xi_{M\setminus n}^{T'})_\sigma + (y+z)(\xi_{M/n}^{T'})_\sigma & \text{if } k_\sigma(M) = 0 \\ y(\xi_{M/n}^{T'})_\sigma & \text{if } k_\sigma(M) > 0. \end{cases}$$

For $0 < i < n-1$ we have

$$t_{\sigma(i)}^{-1}\big((\xi_M^T)_{\sigma^{i+1}}\big|_{t_n=0} - (\xi_M^T)_{\sigma^i}\big|_{t_n=0}\big) = \begin{cases} 0 & \text{if } k_\sigma(M) \neq i \\ w(\xi_{M\setminus n}^{T'})_\sigma + z(\xi_{M/n}^{T'})_\sigma & \text{if } k_\sigma(M) = i. \end{cases}$$

Finally, for $i = n-1$ we have

$$\begin{aligned} &t_{\sigma(n-1)}^{-1}\big((\xi_M^T)_{\sigma^n}\big|_{t_n=0} - (\xi_M^T)_{\sigma^{n-1}}\big|_{t_n=0}\big) \\ &= \begin{cases} x(\xi_{M\setminus n}^{T'})_\sigma & \text{if } k_\sigma(M) < n-1 \\ (x+w)(\xi_{M\setminus n}^{T'})_\sigma + z(\xi_{M/n}^{T'})_\sigma & \text{if } k_\sigma(M) = n-1. \\ x(\xi_{M/n}^{T'})_\sigma & \text{if } k_\sigma(M) = n \end{cases} \end{aligned}$$

The proofs for $i = 0$ and $i = n-1$ are nearly identical so we only show the former. Also, Lemma 4.6.(a) and Corollary 4.7 together imply that the difference $(\xi_M^T)_{\sigma^{i+1}}|_{t_n=0} - (\xi_M^T)_{\sigma^i}|_{t_n=0}$ is zero when $0 < i < n-1$ and $i = k_\sigma(M)$, so when $0 < i < n-1$ we only need to establish the case $k_{\sigma(M)} = i$.

Case $i = 0$ and $k_\sigma(M) \neq 0$.

Write $M' = M \setminus n$ if $k_{\sigma(M)} = -1$ and $M' = M/n$ if $k_{\sigma(M)} > 0$. Since $k_\sigma(M) \neq 0$, Lemma 4.6.(a) implies that

$$\begin{aligned} c^T(\mathcal{S}_M^\vee, z)_{\sigma^0}|_{t_n=0} &= c^T(\mathcal{S}_M^\vee, z)_{\sigma^1}|_{t_n=0} = c^{T'}(\mathcal{S}_{M'}^\vee, z)_\sigma, \text{ and} \\ c^T(\mathcal{Q}_M, w)_{\sigma^0}|_{t_n=0} &= c^T(\mathcal{Q}_M, w)_{\sigma^1}|_{t_n=0} = c^{T'}(\mathcal{Q}_{M'}, w)_\sigma. \end{aligned}$$

By Corollary 4.7, we also have

$$\begin{aligned} &a_E^T(x)_{\sigma^0}|_{t_n=0} = a_E^T(x)_{\sigma^1}|_{t_n=0} = a_{E\setminus n}^{T'}(x)_\sigma, \\ &b_E^T(y)_{\sigma^0}|_{t_n=0} = (1 - t_{\sigma(0)}y) b_{E\setminus n}^{T'}(y)_\sigma, \quad \text{and} \\ &b_E^T(y)_{\sigma^1}|_{t_n=0} = b_{E\setminus n}^{T'}(y)_\sigma. \end{aligned}$$

We conclude that

$$\begin{aligned} &t_{\sigma(0)}^{-1}\big((\xi_M^T)_{\sigma^1}\big|_{t_n=0} - (\xi_M^T)_{\sigma^0}\big|_{t_n=0}\big) \\ &= t_{\sigma(0)}^{-1} a_{E\setminus n}^{T'}(x)_\sigma \big(1 - (1 - t_{\sigma(0)}y)\big) b_{E\setminus n}^{T'}(y)_\sigma c^{T'}(\mathcal{S}_{M'}, z)_\sigma c^{T'}(\mathcal{Q}_{M'}, w)_\sigma \\ &= y(\xi_{M'}^{T'})_\sigma. \end{aligned}$$

Case $i = 0$ and $k_\sigma(M) = 0$.

Since $k_\sigma(M) = 0$, Lemma 4.6 implies that

$$\begin{aligned} &c^T(\mathcal{S}_M^\vee, z)_{\sigma^0}|_{t_n=0} = c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma, \\ &c^T(\mathcal{S}_M^\vee, z)_{\sigma^1}|_{t_n=0} = c^{T'}(\mathcal{S}_{M\setminus n}^\vee, z)_\sigma = (1 + t_{\sigma(0)}z) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma, \end{aligned}$$

$$c^T(\mathcal{Q}_M, w)_{\sigma^0}|_{t_n=0} = c^{T'}(\mathcal{Q}_{M/n}, w)_\sigma = (1 - t_{\sigma(0)} w) c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma, \quad \text{and}$$
$$c^T(\mathcal{Q}_M, w)_{\sigma^1}|_{t_n=0} = c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma.$$

Similarly to the previous case, by Corollary 4.7, we also have

$$a_E^T(x)_{\sigma^0}|_{t_n=0} = a_E^T(x)_{\sigma^1}|_{t_n=0} = a_{E\backslash n}^{T'}(x)_\sigma,$$
$$b_E^T(y)_{\sigma^0}|_{t_n=0} = (1 - t_{\sigma(0)} y) b_{E\backslash n}^{T'}(y)_\sigma, \quad \text{and}$$
$$b_E^T(y)_{\sigma^1}|_{t_n=0} = b_{E\backslash n}^{T'}(y)_\sigma.$$

Thus, we conclude that

$$\begin{aligned}
&t_{\sigma(0)}^{-1}\left((\xi_M^T)_{\sigma^1}\Big|_{t_n=0} - (\xi_M^T)_{\sigma^0}\Big|_{t_n=0}\right) \\
&= t_{\sigma(0)}^{-1} a_{E\backslash n}^{T'}(x)_\sigma \Big((1 + t_{\sigma(0)} z) - (1 - t_{\sigma(0)} y)(1 - t_{\sigma(0)} w)\Big) \\
&\quad \times b_{E\backslash n}^{T'}(y)_\sigma c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma \\
&= a_{E\backslash n}^{T'}(x)_\sigma b_{E\backslash n}^{T'}(y)_\sigma \Big(z + y + w - t_{\sigma(0)} y w\Big) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma \\
&= a_{E\backslash n}^{T'}(x)_\sigma b_{E\backslash n}^{T'}(x)_\sigma \Big((w(1 + t_{\sigma(0)} z) + (y + z)(1 - t_{\sigma(0)} w)\Big) \\
&\quad \times c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma \\
&= a_{E\backslash n}^{T'}(x)_\sigma b_{E\backslash n}^{T'}(x)_\sigma \Big(w c^{T'}(\mathcal{S}_{M\backslash n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\backslash n}, z)_\sigma \\
&\quad + (y + z) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M/n}, z)_\sigma\Big) \\
&= w(\xi_{M\backslash n}^{T'})_\sigma + (y + z)(\xi_{M/n}^{T'})_\sigma,
\end{aligned}$$

where the second last equality follows from Lemma 4.6.(b).

Case $0 < i < n - 1$ and $k_\sigma(M) = i$. Applying Lemma 4.6 to M with $i = k_\sigma(M)$ implies that

$$c^T(\mathcal{S}_M^\vee, z)_{\sigma^i}|_{t_n=0} = c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma, \quad \text{and}$$
$$c^T(\mathcal{S}_M^\vee, z)_{\sigma^{i+1}}|_{t_n=0} = c^{T'}(\mathcal{S}_{M\backslash n}^\vee, z)_\sigma = (1 + t_{\sigma(i)} z) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma,$$

and moreover that

$$c^T(\mathcal{Q}_M, w)_{\sigma^i}|_{t_n=0} = c^{T'}(\mathcal{Q}_{M/n}, w)_\sigma = (1 - t_{\sigma(i)} w) c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma, \quad \text{and}$$
$$c^T(\mathcal{Q}_M, w)_{\sigma^{i+1}}|_{t_n=0} = c^{T'}(\mathcal{Q}_{M\backslash n}, w)_\sigma.$$

Since $0 < i < n - 1$, by Corollary 4.7 we also have

$$a_E^T(x)_{\sigma^i}|_{t_n=0} = a_E^T(x)_{\sigma^{i+1}}|_{t_n=0} = a_{E\backslash n}^{T'}(x)_\sigma, \quad \text{and}$$

$$b_E^T(y)_{\sigma^i}|_{t_n=0} = b_E^T(y)_{\sigma^{i+1}}|_{t_n=0} = b_{E\setminus n}^{T'}(y)_\sigma.$$

Thus, we conclude that

$$\begin{aligned}
&t_{\sigma(i)}^{-1}\big((\xi_M^T)_{\sigma^{i+1}}\big|_{t_n=0} - (\xi_M^T)_{\sigma^i}\big|_{t_n=0}\big)\\
&= t_{\sigma(i)}^{-1} a_{E\setminus n}^{T'}(x)_\sigma b_{E\setminus n}^{T'}(y)_\sigma \Big((1+t_{\sigma(i)}z) - (1-t_{\sigma(i)}w)\Big) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\setminus n}, w)_\sigma\\
&= a_{E\setminus n}^{T'}(x)_\sigma b_{E\setminus n}^{T'}(y)_\sigma \Big(z+w\Big) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\setminus n}, w)_\sigma\\
&= a_{E\setminus n}^{T'}(x)_\sigma b_{E\setminus n}^{T'}(y)_\sigma \Big(w(1+t_{\sigma(i)}z) + z(1-t_{\sigma(i)}w)\Big) c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\setminus n}, w)_\sigma\\
&= a_{E\setminus n}^{T'}(x)_\sigma b_{E\setminus n}^{T'}(y)_\sigma \Big(w c^{T'}(\mathcal{S}_{M\setminus n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M\setminus n}, w)_\sigma\\
&\quad + z c^{T'}(\mathcal{S}_{M/n}^\vee, z)_\sigma c^{T'}(\mathcal{Q}_{M/n}, w)_\sigma\Big)\\
&= w(\xi_{M\setminus n}^{T'})_\sigma + z(\xi_{M/n}^{T'})_\sigma,
\end{aligned}$$

where the second last equality follows from Lemma 4.6.(b).

This concludes our proof of the claim, and thereby that of Theorem 4.8. □

From this, we conclude the proof of Theorem A.

Proof of Theorem A Theorem 4.8 shows that the pushforward $f_*\xi_M$ satisfies the same deletion-contraction relation as $t_M(x, y, z, w)$ does. Noting that $\int_{X_E} \xi_M = \int_{X_{E\setminus n}} f_*\xi_M$ by the functoriality of pushforward maps, we conclude Theorem A by induction on the cardinality of E. □

5 Base polytope properties

We establish the base polytope properties of tautological classes of matroids and their Chern classes listed in §1.3(A)—matroid minors decomposition, valuativity, and well-behavedness under duality and direct sum.

5.1 Matroid minors decomposition

For a matroid M on E and subset $S \subseteq E$, recall that the **restriction** $M|S$ is a matroid on S with rank function $\mathrm{rk}_{M|S}(\cdot) = \mathrm{rk}_M(\cdot)$, and that the **contraction** M/S is a matroid on $E \setminus S$ with rank function $\mathrm{rk}_{M/S}(\cdot) = \mathrm{rk}_M(\cdot \cup S) - \mathrm{rk}_M(S)$. A **minor** of M is a matroid $M|S/S'$ on $S \setminus S'$ for some $S' \subseteq S \subseteq E$. One can verify that $M|S/S' = (M/S')|S$.

Let $\mathscr{S}: \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ be a chain of nonempty proper subsets of E. We always denote by convention $S_0 = \emptyset$ and $S_{k+1} = E$ for such a chain. Faces of the base polytope of a matroid have the following decomposition property.

Proposition 5.1 [9, *Proposition* 2] *Let M be a matroid with ground set E. The face of the base polytope $P(M)$ maximizing the pairing $\langle \cdot, \mathbf{e}_{S_1} + \cdots + \mathbf{e}_{S_k}\rangle$ is equal to the product*

$$P(M|S_1) \times P(M|S_2/S_1) \times \cdots \times P(M|S_k/S_{k-1}) \times P(M/S_k).$$

Many combinatorial invariants on matroids respect this matroid minors decomposition behavior, which underlies the Hopf algebra structure on matroids studied in [2, 40, 74, 79, 99]. We show that the tautological classes of matroids also display such behavior. We prepare with the following, which is a geometric restatement of the fact that faces of permutohedra are products of smaller permutohedra (see for example [2, §4.1] and references therein).

Proposition 5.2 *Let $\mathscr{S} : \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ be a chain of nonempty proper subsets of E. Then, the torus-orbit closure $Z_{\mathscr{S}} \subset X_E$ corresponding to* $\mathrm{Cone}(\overline{\mathbf{e}}_{S_1}, \ldots, \overline{\mathbf{e}}_{S_k}) \in \Sigma_E$ *has a natural T-equivariant isomorphism*

$$Z_{\mathscr{S}} \simeq X_{S_1} \times X_{S_2 \setminus S_1} \times \cdots \times X_{E \setminus S_k},$$

where the torus $T = (\mathbb{C}^*)^E$ acts on the torus $T_{S_{i+1}\setminus S_i} = (\mathbb{C}^*)^{S_{i+1}\setminus S_i}$ via the obvious projection for each $i = 0, \ldots, k$. In particular, we have

$$A^\bullet_T(Z_{\mathscr{S}}) \simeq \bigotimes_{i=0}^{k} A^\bullet_{T_{S_{i+1}\setminus S_i}}(X_{S_{i+1}\setminus S_i}) \quad \textit{and} \quad K^0_T(Z_{\mathscr{S}}) \simeq \bigotimes_{i=0}^{k} K^0_{T_{S_{i+1}\setminus S_i}}(X_{S_{i+1}\setminus S_i}).$$

Under the isomorphism, for a $(k+1)$-tuple $(\sigma_0, \ldots, \sigma_k)$ of permutations where $\sigma_i \in \mathfrak{S}_{S_{i+1}\setminus S_i}$, the inclusion $Z_{\mathscr{S}} \hookrightarrow X_E$ maps the T-fixed point $p_{\sigma_0} \times \cdots \times p_{\sigma_k}$ of $Z_{\mathscr{S}}$ to the point p_σ of X_E where σ is the permutation $\sigma_0 \circ \cdots \circ \sigma_k$ on $E = \bigsqcup_{i=0}^k (S_{i+1} \setminus S_i)$.

Proposition 5.3 *Let $\mathscr{S} : \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ be a chain of nonempty proper subsets of E, and let M be a matroid on E. Then, under the isomorphism in Proposition 5.2, one has*

$$[\mathcal{S}_M]|_{Z_{\mathscr{S}}} = \sum_{i=0}^{k} 1^{\otimes i} \otimes [\mathcal{S}_{M|S_{i+1}/S_i}] \otimes 1^{\otimes(k-i)} \quad \textit{and}$$

$$[\mathcal{Q}_M]|_{Z_{\mathscr{S}}} = \sum_{i=0}^{k} 1^{\otimes i} \otimes [\mathcal{Q}_{M|S_{i+1}/S_i}] \otimes 1^{\otimes(k-i)}$$

as elements in $K^0_T(Z_{\mathscr{S}})$. In particular, with a formal variable u, one has

$$c(\mathcal{S}_M, u)|_{Z_{\mathscr{S}}} = \bigotimes_{i=0}^{k} c(\mathcal{S}_{M|S_{i+1}/S_i}, u) \quad \textit{and} \quad c(\mathcal{Q}_M, u)|_{Z_{\mathscr{S}}} = \bigotimes_{i=0}^{k} c(\mathcal{Q}_{M|S_{i+1}/S_i}, u)$$

as elements in $\bigotimes_{i=0}^k A^\bullet(X_{S_{i+1}\setminus S_i}) \simeq A^\bullet(Z_{\mathscr{S}})$.

Proof Let $\sigma \in \mathfrak{S}_E$ be the composition of permutations $\sigma_0, \ldots, \sigma_k$ where $\sigma_i \in \mathfrak{S}_{S_{i+1} \setminus S_i}$ for $i = 0, \ldots, k$. Since $B_\sigma(M) = \bigsqcup_{i=0}^k B_{\sigma_i}(M|S_{i+1}/S_i)$ by Proposition 5.1, the restrictions of the K-classes $[\mathcal{S}_M]|_{Z_{\mathscr{S}}}$ and $\sum_{i=0}^k 1^{\otimes i} \otimes [\mathcal{S}_{M|S_{i+1}/S_i}] \otimes 1^{\otimes(k-i)}$ to the point p_σ both give the same Laurent polynomial $\sum_{j \in B_\sigma(M)} T_j^{-1}$. The statement for $[\mathcal{Q}_M]$ is proved similarly. □

Corollary 5.4 *Let $\mathscr{S} : \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ be a chain of nonempty proper subsets of E, and let M be a matroid on E. The divisor classes α_E and β_E on X_E defined in Definition 2.3 satisfy*

$$\alpha|_{Z_{\mathscr{S}}} = 1^{\otimes k} \otimes \alpha_{E \setminus S_k} \quad \textit{and} \quad \beta|_{Z_{\mathscr{S}}} = \beta_{S_1} \otimes 1^{\otimes k}$$

as elements in $\bigotimes_{i=0}^k A^\bullet(X_{S_{i+1} \setminus S_i}) \simeq A^\bullet(Z_{\mathscr{S}})$.

Proof In Example 3.10, we showed that the divisor classes α_E and β_E on X_E satisfy $[\mathcal{Q}_{U_{n,E}}] = [\mathcal{O}(\alpha)]$ and $[\mathcal{S}^\vee_{U_{1,E}}] = [\mathcal{O}(\beta)]$ as non-equivariant K-classes. For any subsets $\emptyset \subseteq S' \subseteq S \subseteq E$, the matroid minor $U_{n,E}|S/S'$ is a uniform matroid of corank 1 if $S = E$ and $S' \subsetneq E$, and is corank 0 otherwise. Likewise, the matroid minor $U_{1,E}|S/S'$ is a uniform matroid of rank 1 if $\emptyset \subsetneq S$ and $S' = \emptyset$, and is rank 0 otherwise. Thus, applying Proposition 5.3 with $M = U_{n,E}$ and $M = U_{1,E}$ implies the desired result. □

5.2 Valuativity

For a subset $P \subseteq \mathbb{R}^E$, let $1_P : \mathbb{R}^E \to \mathbb{Z}$ denote its indicator function defined by $1_P(x) = 1$ if $x \in P$ and 0 otherwise. An important tool for extending algebraic constructions from realizable matroids to arbitrary matroids is the following notion of "valuativity".

Definition 5.5 [38, Definition 3.5] A function ϕ from the set of matroids with ground set E to an abelian group A is **valuative** if, for any matroids $M_1, \ldots, M_\ell$ on E and integers $a_1, \ldots, a_\ell$ such that $\sum_{i=1}^\ell a_i 1_{P(M_i)} = 0$, the function ϕ satisfies $\sum_{i=1}^\ell a_i \phi(M_i) = 0$.

In other words, the map ϕ is valuative if it factors through the map sending M to the indicator function $1_{P(M)}$ of its base polytope $P(M)$. Many invariants of matroids satisfy valuativity, including the Tutte polynomial and its specializations [11, 38]. For a more comprehensive list see [10], and see [47] for a study of valuativity for Coxeter matroids.

We show that a wide range of classes associated to the tautological K-classes of matroids are also valuative. These will include any polynomial expression in exterior powers or Chern classes of $[\mathcal{S}_M]$, $[\mathcal{Q}_M]$, and their dual K-classes $[\mathcal{S}_M^\vee]$ and $[\mathcal{Q}_M^\vee]$. For example, one may consider assigning to a matroid M the class $[\bigwedge^2 \mathcal{S}_M][\bigwedge^3 \mathcal{Q}_M] + 4[\bigwedge^5 \mathcal{S}_M^\vee]$, or the class $c_1(\mathcal{S}_M)^2 c_2(\mathcal{Q}_M) - c_4(\mathcal{S}_M)^3$.

More precisely, recall from Sect. 2.5 that for an element $\lambda(x) \in \Lambda \subset \mathbb{Z}[[x_1, x_2, \ldots]]$ in the ring of symmetric functions, we may construct the classes $[\mathsf{S}^\lambda \mathcal{E}] \in$

$K_T(X_E)$ and $\mathsf{s}^T_\lambda(\mathcal{E}) \in A^\bullet_T(X_E)$ when $\mathcal{E} = [\mathcal{S}_M]$, $[\mathcal{Q}_M]$, $[\mathcal{S}^\vee_M]$, or $[\mathcal{Q}^\vee_M]$. For a polynomial $f(a,b,c,d) \in \mathbb{Z}[a,b,c,d]$ and a sequence $\boldsymbol{\lambda} = (\lambda_a(x), \lambda_b(x), \lambda_c(x), \lambda_d(x))$ of symmetric functions, we define

$$\begin{aligned}
&\phi_{f,\boldsymbol{\lambda}}\colon \{\text{Matroids on } E\} \to K_T(X_E)\\
&\quad \text{by} \quad M \mapsto f([\mathsf{S}^{\lambda_a}\mathcal{S}_M], [\mathsf{S}^{\lambda_b}\mathcal{Q}_M], [\mathsf{S}^{\lambda_c}\mathcal{S}^\vee_M], [S^{\lambda_d}\mathcal{Q}^\vee_M]), \quad \text{and}\\
&\psi_{f,\boldsymbol{\lambda}}\colon \{\text{Matroids on } E\} \to A^\bullet_T(X_E)\\
&\quad \text{by} \quad M \mapsto f(\mathsf{s}^T_{\lambda_a}(\mathcal{S}_M), \mathsf{s}^T_{\lambda_b}(\mathcal{Q}_M), \mathsf{s}^T_{\lambda_c}(\mathcal{S}^\vee_M), \mathsf{s}^T_{\lambda_d}(\mathcal{Q}^\vee_M)).
\end{aligned}$$

For instance, by taking $\boldsymbol{\lambda}$ to be appropriate elementary symmetric functions and f certain polynomials, one obtains the two aforementioned examples.

Proposition 5.6 *The maps $\phi_{f,\boldsymbol{\lambda}}$ and $\psi_{f,\boldsymbol{\lambda}}$ defined above are valuative.*

We prepare the proof with the following lemma.

Lemma 5.7 *Let $\mathbb{Z}^{\binom{E}{r}}$ be the free abelian group on the set of r-subsets of E, with its standard basis denoted $\{\langle B\rangle \mid B \in \binom{E}{r}\}$. For any fixed permutation $\sigma \in \mathfrak{S}_E$, the assignment $M \mapsto \langle B_\sigma(M)\rangle \in \mathbb{Z}^{\binom{E}{r}}$ on matroids of rank r with ground set E is valuative.*

Proof For a total order $<$ on the ground set E, recall that an element i in a basis B is internally active if there is no $j \in E$ such that $j < i$ and $(B \setminus i) \cup j$ is a basis. When the total order $<$ is given by the permutation σ, the lex-first basis $B_\sigma(M)$ is the unique basis with r internally active elements (Remark 3.4). The lemma is now a special case of [11, Theorem 5.4]. □

Proof of Proposition 5.6 For a matroid M and a permutation σ, by the construction of $[\mathsf{S}^\lambda\mathcal{E}]$ in Sect. 2.5, the restriction of the equivariant K-class $\phi_{f,\boldsymbol{\lambda}}(M)$ to the T-fixed point p_σ is a Laurent polynomial $\phi_{f,\boldsymbol{\lambda}}(M)_\sigma \in \mathbb{Z}[T_0^\pm, \dots, T_n^\pm]$ determined completely by $B_\sigma(M)$. Similarly, the polynomial $\psi_{f,\boldsymbol{\lambda}}(M)_\sigma$ in $\mathbb{Z}[t_0, \dots, t_n]$ representing the restriction to p_σ of the equivariant Chow class $\psi_{f,\boldsymbol{\lambda}}(M)$ is completely determined by $B_\sigma(M)$. In other words, the map $M \mapsto \phi_{f,\boldsymbol{\lambda}}(M)_\sigma$ factors through the free abelian group $\mathbb{Z}^{\binom{E}{r}}$ by $M \mapsto \langle B_\sigma(M)\rangle \in \mathbb{Z}^{\binom{E}{r}}$, and similarly for $\psi_{f,\boldsymbol{\lambda}}$. Thus, the method of localization Theorem 2.1 implies that both $\phi_{f,\boldsymbol{\lambda}}$ and $\psi_{f,\boldsymbol{\lambda}}$ factor through the map $M \mapsto (\langle B_\sigma(M)\rangle)_{\sigma\in\mathfrak{S}_E} \in \bigoplus_{\sigma\in\mathfrak{S}_E} \mathbb{Z}^{\binom{E}{r}}$, which is valuative by Lemma 5.7, and the result follows. □

We note for future use in §10 the following generalization of Proposition 5.6 concerning the valuativity of classes defined in Proposition 3.13.

Proposition 5.8 *For a fixed set of classes $[\mathcal{E}^{(1)}], \dots, [\mathcal{E}^{(m)}] \in K^0_T(\mathrm{Gr}(r; E))$, the map ϕ that assigns to a matroid M of rank r on E a fixed polynomial expression in the classes $[\mathcal{E}^{(1)}_M], \dots, [\mathcal{E}^{(m)}_M] \in K^0_T(X_E)$ defined in Proposition 3.13 is valuative.*

Proof It follows from the definition of the classes $[\mathcal{E}_M^{(i)}]$ (Proposition 3.13) that, for each permutation $\sigma \in \mathfrak{S}_E$, the restriction of the assignment $\phi(M)$ to the T-fixed point $p_\sigma \in X_E^T$ is completely determined by $B_\sigma(M)$. Hence, the map ϕ factors through $M \mapsto (\langle B_\sigma(M)\rangle)_{\sigma\in\mathfrak{S}_E} \in \bigoplus_{\sigma\in\mathfrak{S}_E} \mathbb{Z}^{\binom{E}{r}}$. Thus, the assignment is valuative by Lemma 5.7. □

We conclude with a lemma that will be useful later for deducing results for arbitrary matroids from realizable matroids. Recall that an element $e \in E$ is a loop (resp. coloop) in a matroid M if no basis of M contains e (resp. every basis of M contains e).

Lemma 5.9 *Let M be a matroid on E. Then there exist matroids $M_1, \dots, M_\ell$, all realizable over $\mathbb{C}$, and integers $a_1, \dots, a_\ell$, such that $1_{P(M)} = \sum_{i=1}^{\ell} a_i 1_{P(M_i)}$. If M is loopless* (*resp. coloopless*), *then all M_i's can also be taken to be loopless* (*resp. coloopless*).

Proof [38, Theorem 5.4] states that the $\mathbb{Z}$-span of the indicator functions of base polytopes of matroids admits a basis consisting of matroids known as Schubert matroids, which are all realizable over $\mathbb{C}$. In particular, one can write $1_{P(M)} = \sum a_i 1_{P(M_i)}$ with M_i being Schubert matroids.

Suppose now that M is loopless; the coloopless case is proved similarly. We first claim that if we have polytopes $P_1, \dots, P_\ell$ and integers b_i such that $\sum b_i 1_{P_i} = 0$, and all P_i lie in a half-space H^+ bounded by a hyperplane H, then the sum is also zero if we restrict to just the polytopes completely contained in H. Assuming the claim, for $j \in E$ take the hyperplane $H_j = \{x_j = 0\}$ and half-space $H_j^+ = \{x_j \geq 0\}$ in $\mathbb{R}^E$. For all i we have $P(M_i) \subset H_j^+$, and we have $P(M_i) \subset H_j$ if and only if j is a loop in M_i. Therefore, by considering each $j \in E$ in turn, we can remove all polytopes with loops from the sum $\sum a_i 1_{P(M_i)}$ without affecting the equation $1_{P(M)} = \sum a_i 1_{P(M_i)}$, and the result follows.

For the proof of the claim, we note that the set of all polytopes contained in H^+, which we denote as $\mathscr{P}$, is intersection closed. [60, Theorem 1] states that a function f on such a set $\mathscr{P}$ of polytopes factors through the map $P \mapsto 1_P$ if it satisfies $f(\emptyset) = 0$ and $f(P \cup Q) + f(P \cap Q) = f(P) + f(Q)$ for any $P, Q \in \mathscr{P}$ with $P \cup Q \in \mathscr{P}$. We verify that the function h mapping $P \in \mathscr{P}$ to 1_P if $P \subset H$ and 0 otherwise satisfies this relation as follows. If $P \subseteq Q$ or $Q \subseteq P$, or if $P \cap Q \not\subset H$, or if $P \cup Q \subset H$, then the relation is easily verified for h. But the condition for P and Q leaves no other cases: Suppose otherwise. Then $(P \cup Q) \setminus H$ is convex and nonempty. The relatively closed decomposition $(P \cup Q) \setminus (P \cap Q) = (P \setminus Q) \cup (Q \setminus P)$ disconnects $(P \cup Q) \setminus (P \cap Q)$, so the connectedness of $(P \cup Q) \setminus H$ implies that $P \setminus Q \subset H$ or $Q \setminus P \subset H$. But then $P \subset H$ or $Q \subset H$, so the convexity of $P \cup Q$ together with $P \cup Q \not\subset H$ necessitates that $P \subseteq Q$ or $Q \subseteq P$, in contradiction to the assumption. □

Remark 5.10 Lemma 5.9 gives rise to the following recurring theme for matroid constructions motivated from geometry. Let $f : \mathrm{Gr}(r; E) \to A$ be a function from the Grassmannian, i.e. the space of realizations of matroids of rank r, to an abelian group A. Suppose f satisfies the property that

(i) the value $f(L)$ only depends on the matroid M that $L \in \mathrm{Gr}(r; E)$ realizes.

If the function f, now considered as a function on the set of realizable matroids, extends to a function $\widetilde{f}$ on the set of all (not necessarily realizable) matroids satisfying

(ii) the assignment $M \mapsto \widetilde{f}(M)$ is valuative,

then Lemma 5.9 implies that such an extension is unique. Many matroid constructions motivated from the geometry of realizations of matroids are characterized by the two properties (i) and (ii) above. These constructions include: tautological K and Chern classes (Propositions 3.7 & 5.6), Bergman classes (Corollary 7.11), Chern-Schwartz-MacPherson classes (Corollary 8.10), the combinatorial biprojective classes defined in Definition 9.1 (Proposition 9.2), the K-class denoted $y(M)$ for a matroid M in [50] (Definition 10.8), and the assignment of $[\mathcal{E}] \mapsto [\mathcal{E}_M]$ in Proposition 3.13 (Proposition 5.8).

5.3 Matroid duality and direct sum

For a matroid M of rank r with ground set E, the **dual matroid** $M^\perp$ is the matroid of rank $|E| - r$ with ground set E whose bases are $\{E \setminus B \mid B \text{ a basis of } M\}$. If $L \subseteq \mathbb{C}^E$ is a realization of M, then $L^\perp = (\mathbb{C}^E/L)^\vee \subseteq (\mathbb{C}^E)^\vee \simeq \mathbb{C}^E$ is a realization of $M^\perp$, where the isomorphism $(\mathbb{C}^E)^\vee \simeq \mathbb{C}^E$ is induced by the standard basis of $\mathbb{C}^E$. Recall from Sect. 2.6 the Cremona involution $\mathrm{crem}\colon X_E \to X_E$, induced by the map of tori $T/\mathbb{C}^* \to T/\mathbb{C}^*$ mapping $[t_0 : \cdots : t_n]$ to $[t_0^{-1} : \cdots : t_n^{-1}]$.

Proposition 5.11 *Let M be a matroid on E. Then, one has* $\mathrm{crem}[\mathcal{S}_M] = [\mathcal{Q}^\vee_{M^\perp}]$ *and* $\mathrm{crem}[\mathcal{Q}_M] = [\mathcal{S}^\vee_{M^\perp}]$. *In particular,* $\mathrm{crem}\, c(\mathcal{S}_M, u) = c(\mathcal{Q}^\vee_{M^\perp}, u)$ *and* $\mathrm{crem}\, c(\mathcal{Q}_M, u) = c(\mathcal{S}^\vee_{M^\perp}, u)$.

Proof For a permutation $\sigma \in \mathfrak{S}_E$, by Remark 2.2 we have $(\mathrm{crem}[\mathcal{S}_M])_\sigma = \sum_{i \in B_{\overline{\sigma}}(M)} T_i$. Because $P(M^\perp) = -P(M) + (1, 1, \dots, 1)$, for any $v_0 > v_1 > \cdots > v_n$, the basis $E \setminus B$ of $M^\perp$ which maximizes the pairing $\langle \mathbf{e}_{E \setminus B}, v_0\mathbf{e}_{\sigma(0)} + \cdots + v_n\mathbf{e}_{\sigma(n)} \rangle$ corresponds to the basis B of M which minimizes $\langle \mathbf{e}_B, v_0\mathbf{e}_{\sigma(0)} + \cdots + v_n\mathbf{e}_{\sigma(n)} \rangle$. But this was earlier shown to be a defining property of the reverse-lex-basis $B = B_{\overline{\sigma}}$. Hence $[\mathcal{Q}^\vee_{M^\perp}]_\sigma = \sum_{i \in B_{\overline{\sigma}}(M)} T_i$ as well. The proof for $\mathrm{crem}[\mathcal{Q}_M] = [\mathcal{S}^\vee_{M^\perp}]$ is similar. □

Let $E = E_1 \sqcup E_2$ be a disjoint union of nonempty subsets E_1 and E_2 of E. For matroids M_1 and M_2 on E_1 and E_2, respectively, their **direct sum** $M_1 \oplus M_2$ is a matroid whose base polytope is given by $P(M_1 \oplus M_2) = P(M_1) \times P(M_2)$, where we have identified $\mathbb{R}^E = \mathbb{R}^{E_1} \times \mathbb{R}^{E_2}$. A matroid M is **connected** if it is not a direct sum of two matroids on nonempty ground sets, and is **disconnected** otherwise.

Proposition 5.12 *Let M be a matroid on E. Then M can be uniquely written as a direct sum $M_1 \oplus \cdots \oplus M_k$ of connected matroids M_i on E_i satisfying $E = \bigsqcup_{i=1}^k E_i$, called the connected components of M. The number k of connected components of M satisfies* $\dim P(M) = |E| - k$.

Proof See [94, 4.2.9] and [48, Proposition 2.4]. □

Let us establish an analogous direct sum behavior for the tautological K-classes of matroids. We prepare by noting that, for a nonempty subset $E' \subseteq E$, the projection map $\mathbb{R}^E/\mathbb{R}\mathbf{1} \to \mathbb{R}^{E'}/\mathbb{R}\mathbf{1}$ induces a map of fans $\Sigma_E \to \Sigma_{E'}$. If we write $E' = \{j_0, \ldots, j_{n'}\} \subseteq E$ with $j_0 < \cdots < j_{n'}$, then the cone in Σ_E corresponding to a permutation $\sigma \in \mathfrak{S}_E$ maps to the cone in $\Sigma_{E'}$ corresponding to the permutation $\sigma_{E'} \in \mathfrak{S}_{E'}$ defined by

$$\sigma_{E'}(j_k) = \sigma(i_k) \text{ where } 0 \le i_0 < \cdots < i_{n'} \le n \text{ such that } \{\sigma(i_0), \ldots, \sigma(i_{n'})\} = E'.$$

In particular, considering $X_{E'}$ as a T-variety via the projection map $T = (\mathbb{C}^*)^E \to (\mathbb{C}^*)^{E'} = T'$, we have a T-equivariant map of T-varieties $X_E \to X_{E'}$, and under which the point $p_\sigma \in X_E^T$ corresponding to $\sigma \in \mathfrak{S}_E$ maps to the point $p_{\sigma_{E'}} \in X_{E'}^T$.

Proposition 5.13 *Let M be a matroid on E with connected components $M_1, \ldots, M_k$ on $E_1, \ldots, E_k \subseteq E$. For each $i = 1, \ldots, k$, let $f_i \colon X_E \to X_{E_i}$ be the toric morphism induced by the map of fans $\Sigma_E \to \Sigma_{E_i}$ arising from the projection $\mathbb{R}^E/\mathbb{R}\mathbf{1} \to \mathbb{R}^{E_i}/\mathbb{R}\mathbf{1}$. Then, we have*

$$[\mathcal{S}_M] = \sum_{i=1}^{k} f_i^*[\mathcal{S}_{M_i}] \quad \textit{and} \quad [\mathcal{Q}_M] = \sum_{i=1}^{k} f_i^*[\mathcal{Q}_{M_i}] \quad \textit{as elements in } K_T^0(X_E).$$

Proof Since $M = M_1 \oplus \cdots \oplus M_k$, any basis B of M can be uniquely written $B = B_1 \sqcup \cdots \sqcup B_k$ where B_i is a basis of M_i for each $i = 1, \ldots, k$. Thus, from the description of the permutation σ_{E_i} above for each $i = 1, \ldots, k$, it follows that $B_\sigma(M) = \bigsqcup_{i=1}^k B_{\sigma_{E_i}}(M_i)$. We also have $(f_i^*[\mathcal{S}_{M_i}])_\sigma = [\mathcal{S}_{M_i}]_{\sigma_{E_i}}$ because f_i maps the point $p_\sigma \in X_E^T$ to $p_{\sigma_{E_i}} \in X_{E_i}^T$, and the desired equality for $[\mathcal{S}_M]$ follows. The proof for $[\mathcal{Q}_M]$ is similar. $\square$

6 Beta invariants via tautological classes

In this section, we record an immediate corollary of Theorem A as Theorem 6.2, which is an expression for the beta-invariants of matroids in terms of the tautological Chern classes of matroids. We will use this specialization of Theorem A in the next two sections to study certain Chow classes associated to matroids.

Definition 6.1 For a matroid M, denote by $\beta(M)$ and $\beta(M^\perp)$ the coefficients of the linear terms x and y, respectively, in the Tutte polynomial $T_M(x, y)$. The quantity $\beta(M)$ is called the **beta invariant** of the matroid M.

The notation $\beta(M)$ and $\beta(M^\perp)$ is consistent with matroid duality since $T_M(x, y) = T_{M^\perp}(y, x)$. Beta invariants of matroids were introduced and studied in [32]. We express beta invariants in terms of tautological Chern classes of matroids as follows.

Theorem 6.2 *Let M be a matroid of rank r on ground set E. Then,*

$$\int_{X_E} c_{r-1}(\mathcal{S}_M^\vee)c_{|E|-r}(\mathcal{Q}_M) = \beta(M) \quad \textit{and} \quad \int_{X_E} c_r(\mathcal{S}_M^\vee)c_{|E|-r-1}(\mathcal{Q}_M) = \beta(M^\perp),$$

where we set by convention $c_{-1}(\mathcal{E}) = 0$ for a K-class $[\mathcal{E}]$.

Proof Note that Tutte polynomials have no constant terms, and that $c_k(\mathcal{S}_M^\vee) = 0$ for $k > r$ and $c_\ell(\mathcal{Q}_M) = 0$ for $\ell > |E| - r$. Thus, by setting $x = y = 0$ in Theorem A, we find

$$\sum_{k+\ell=n} \left(\int_{X_E} c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M) \right) z^k w^\ell = z^r w^{|E|-r} \left(\beta(M)\frac{1}{z} + \beta(M^\perp)\frac{1}{w} \right),$$

so we conclude the desired equalities. □

Remark 6.3 If M has a realization L, there are a number of known geometric manifestations of the beta invariant. [103, Theorem 5.1] showed that $\beta(M)$ equals the number of points in the intersection of $\overline{T \cdot L} \subset \mathrm{Gr}(r; E)$ with the Schubert subvariety of $\mathrm{Gr}(r; E)$ corresponding to r-dimensional subspaces that form a flag with a generic point-hyperplane pair. For an alternate proof of Theorem 6.2 using this geometry and valuativity (Proposition 5.6), see the end of Appendix I.

In the discussion at the end of [103, §5], it is noted that the beta invariant arose in [28, Theorem 28] as the top Chern class of the log-tangent sheaf on a certain compactification of the projectivized hyperplane arrangement complement $\mathbb{P}(L) \setminus \bigcup \mathbb{P}(\mathcal{H}_i)$ where $\mathcal{H}_i = \{x_i = 0\} \cap L$, which also appeared in [62, §2.2]. The top Chern class of the log-tangent sheaf is known by the logarithmic Poincaré-Hopf theorem to compute the Euler characteristic of $\mathbb{P}(L) \setminus \bigcup \mathbb{P}(\mathcal{H}_i)$, which is $(-1)^{\dim \mathbb{P}(L)}\beta(M)$ by [92, Theorem 5.2]. These computations were motivated by a problem that involves counting in a very-affine linear subvariety the number of critical points of a product of powers of linear forms, first considered by Varchenko over the real numbers [107], and established over the complex numbers in [93]. A generalization of these results on very-affine linear subvarieties to those on arbitrary very-affine smooth subvarieties is [68]. For us, this will manifest in the geometric constructions in Theorem 7.10 and Theorem 8.8, where we show that $c_{|E|-r}(\mathcal{Q}_M)$ is the Chow class of the "wonderful compactification" W_L of $\mathbb{P}(L) \setminus \bigcup \mathbb{P}(\mathcal{H}_i)$, and $\mathcal{S}_L|_{W_L}$ is a trivial extension of the log-tangent sheaf of the compactification $W_L \supset (\mathbb{P}(L) \setminus \bigcup \mathbb{P}(\mathcal{H}_i))$. See Remark 8.9.

Lastly, in an attempt to formulate a tropical Poincaré-Hopf formula, Rau in [96] showed that a certain tropical intersection also computes $\beta(M)$. We give a geometric interpretation of this in Appendix II.

7 Bergman classes via tautological classes

In this section and the next, we use the special case of Theorem A (Theorem 6.2) and the matroid minors decomposition property (Proposition 5.3) to express certain

Chow classes associated to matroids in terms of the tautological Chern classes of matroids. We recover along the way several known results about these Chow classes. The Chow classes of interest will be phrased in terms of Minkowski weights. We review Minkowski weights in §7.1, and then we consider the Bergman classes of matroids in §7.2 and §7.3. In the next section, we consider the Chern-Schwartz-MacPherson classes of matroids.

7.1 Minkowski weights

Let Σ be a polyhedral fan in $\mathbb{R}^m$ that is rational over the lattice $\mathbb{Z}^m$. Let $\Sigma(d)$ denote the set of d-dimensional cones of the fan Σ. Recall that Σ is said to be unimodular if for every cone τ in Σ, the primitive vectors of the rays of τ form a subset of a $\mathbb{Z}$-basis of $\mathbb{Z}^m$. The toric variety X_Σ is smooth if and only if Σ is unimodular. When Σ is unimodular and complete, the validity of the Poincaré duality for the Chow ring $A^\bullet(X_\Sigma)$ of X_Σ states that we have an isomorphism $A^\bullet(X_\Sigma) \cong \mathrm{Hom}(A^{m-\bullet}(X_\Sigma), \mathbb{Z})$ given by sending

$$\xi \in A^{m-d}(X_\Sigma) \text{ to the function } A^d(X_\Sigma) \to \mathbb{Z} \text{ defined by } [Z] \mapsto \int_{X_\Sigma} \xi \cdot [Z].$$

Because the Chow classes of torus-orbit-closures of a toric variety generate its Chow ring, the function $\int_{X_\Sigma} \xi \cdot (-)\colon A^d(X_\Sigma) \to \mathbb{Z}$ is determined by its values on the complementary dimensional torus-orbit-closures $[Z_\tau]$ corresponding to cones $\tau \in \Sigma(d)$. Because the classes $[Z_\tau]$ are in general not linearly independent, an assignment $\Sigma(d) \to \mathbb{Z}$ of integers to each $[Z_\tau]$ must satisfy certain "balancing conditions" to define a map $A^d(X_\Sigma) \to \mathbb{Z}$. These observations lead to the notion of a Minkowski weight.

Definition 7.1 A d-dimensional **Minkowski weight** on a unimodular fan Σ is a function $\Delta\colon \Sigma(d) \to \mathbb{Z}$ such that the following balancing condition is satisfied for every cone $\tau' \in \Sigma(d-1)$:

$$\sum_{\tau \succ \tau'} \Delta(\tau) u_{\tau' \setminus \tau} \in \mathrm{span}(\tau')$$

where the summation is over all cones $\tau \in \Sigma(d)$ containing τ', and $u_{\tau'\setminus\tau}$ denotes the primitive generator of the unique ray of τ that is not in τ'. Write $\mathrm{MW}_d(\Sigma)$ for the set of d-dimensional Minkowski weights on Σ.

Theorem 7.2 [55, *Theorem* 3.1] *Let Σ be a complete unimodular fan of dimension m, and let X_Σ be its toric variety. Then, for every $0 \le d \le m$, one has an isomorphism*

$$A^{m-d}(X_\Sigma) \xrightarrow{\sim} \mathrm{MW}_d(\Sigma) \quad \textit{defined by} \quad \xi \mapsto \left(\tau \mapsto \int_X \xi \cdot [Z_\tau]\right).$$

For a smooth complete toric variety X_Σ, when a Chow class $\xi \in A^\bullet(X_\Sigma)$ maps to a Minkowski weight $\Delta \in \mathrm{MW}_\bullet(\Sigma)$ by the isomorphism in Theorem 7.2, we say that

Δ and ξ are **Poincaré duals** of each other, which is notated by writing

$$\xi \cap \Delta_\Sigma = \Delta.$$

See [86, Chap. 6] for a further treatment of tropical intersection theory.

7.2 Minkowski weights of tautological classes

We compute the Poincaré duals of the tautological Chern classes of a matroid in Proposition 7.4. In the next subsection, we will use a special case of Proposition 7.4 concerning the top Chern classes to study the Bergman classes of matroids. We prepare with the following lemma. Recall that we say that a matroid is a **loop** (resp. **coloop**) if it has rank 0 (resp. rank 1) and its ground set has cardinality 1.

Lemma 7.3 *For a matroid M of rank r with ground set E, we have*

$$\int_{X_E} c(\mathcal{Q}_M) = \begin{cases} 1 & \text{if } M \text{ is a loop, or is the rank } 1 \text{ uniform matroid } U_{1,E} \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{X_E} c(\mathcal{S}_M^\vee) = \begin{cases} 1 & \text{if } M \text{ is a coloop, or is the corank } 1 \text{ uniform matroid } U_{n,E} \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let us prove the statement for $\mathcal{Q}_M$; the statement for $\mathcal{S}_M^\vee$ is proved similarly. First, because X_E is n-dimensional and $\mathcal{Q}_M$ has rank $n+1-r$, we have that the expression $\int_{X_E} c(\mathcal{Q}_M) = \int_{X_E}(1 + c_1(\mathcal{Q}_M) + \cdots + c_{n+1-r}(\mathcal{Q}_M))$ equals 0 unless $r \leq 1$. If $r = 0$, so that $M = U_{0,E}$, then setting $r = 0$ in the second identity of Theorem 6.2 gives

$$\int_{X_E} c(\mathcal{Q}_M) = \int_{X_E} c_{|E|-1}(\mathcal{Q}_M) = \beta(M^\perp) = \begin{cases} 1 & \text{if } |E| = 1 \\ 0 & \text{if } |E| \geq 2, \end{cases}$$

where for the last equality we note from the definition of Tutte polynomials that $T_{U_{0,E}}(x,y) = y^{|E|}$. If $r = 1$, so that $M = U_{0,E_\ell} \oplus U_{1,E\setminus E_\ell}$ for some subset $\emptyset \subseteq E_\ell \subsetneq E$, then setting $r = 1$ in the first identity of Theorem 6.2 gives

$$\int_{X_E} c(\mathcal{Q}_M) = \int_{X_E} c_{|E|-1}(\mathcal{Q}_M) = \beta(M) = \begin{cases} 1 & \text{if } E_\ell = \emptyset, \text{ i.e. } M = U_{1,E} \\ 0 & \text{otherwise,} \end{cases}$$

where for the last equality we note from the definition of Tutte polynomials that $T_{U_{0,E_\ell} \oplus U_{1,E\setminus E_\ell}}(x,y) = y^{|E_\ell|} T_{U_{1,E\setminus E_\ell}}(x,y) = y^{|E_\ell|}(x + y + \cdots + y^{|E\setminus E_\ell|-1})$. $\square$

Proposition 7.4 *Let M be a matroid of rank r with ground set E, and write* $\mathrm{crk}_M = |E| - r$ *for its corank. Let j and k be nonnegative integers such that $j + k = |E| - 1$. For every chain $\mathscr{S}: \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ of k nonempty proper subsets of E, we* *have*

$$\int_{X_E} c_j(\mathcal{Q}_M)\cdot[Z_{\mathscr{S}}] = \begin{cases} 1 & \textit{for } i=0,\dots,k, \textit{ exactly } \operatorname{crk}_M - j \textit{ minors } M|S_{i+1}/S_i \\ & \textit{are loops and the rest are rank } 1 \textit{ uniform matroids} \\ & U_{1,S_{i+1}\setminus S_i} \\ 0 & \textit{otherwise.} \end{cases}$$

$$\int_{X_E} c_j(\mathcal{S}_M^\vee)\cdot[Z_{\mathscr{S}}] = \begin{cases} 1 & \textit{for } i=0,\dots,k, \textit{ exactly } r - j \textit{ minors } M|S_{i+1}/S_i \\ & \textit{are coloops and the rest are corank } 1 \textit{ uniform matroids} \\ & U_{|S_{i+1}\setminus S_i|-1,S_{i+1}\setminus S_i} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof We prove the result for $\mathcal{Q}_M$, the result for $\mathcal{S}_M^\vee$ can be proved nearly identically, or by invoking the duality property (Proposition 5.11). By the matroid minors decomposition property (Proposition 5.3), with u a formal variable we have that

$$c(\mathcal{Q}_M, u)|_{Z_{\mathscr{S}}} = \bigotimes_{i=0}^{k} c(\mathcal{Q}_{M|S_{i+1}/S_i}, u) \in \bigotimes_{i=0}^{k} A^\bullet(X_{S_{i+1}\setminus S_i})[u].$$

Lemma 7.3 implies that, for each $i=0,\dots,k$, we have $\int_{X_{S_{i+1}\setminus S_i}} c(\mathcal{Q}_{M|S_{i+1}/S_i}, u) = u^{\operatorname{crk}_{M|S_{i+1}/S_i}-1}$ if the minor $M|S_{i+1}/S_i$ is a loop, $u^{\operatorname{crk}_{M|S_{i+1}/S_i}}$ for a rank 1 uniform matroid, and 0 otherwise, which yields the desired statement since $\sum_{i=0}^k \operatorname{crk}_{M|S_{i+1}/S_i} = \operatorname{crk}_M$. □

7.3 Bergman classes of matroids and wonderful compactifications of hyperplane arrangement complements

By considering the Minkowski weight for the top tautological Chern class $c_{|E|-r}(\mathcal{Q}_M)$ of a matroid M, we recover the notion of the Bergman class of a matroid and its relation to the geometry of the wonderful compactification of a hyperplane arrangement complement.

Definition 7.5 For a matroid M with ground set E, a subset $F \subseteq E$ is a **flat** of M if $\operatorname{rk}_M(F\cup\{e\}) > \operatorname{rk}_M(F)$ for all $e \in E\setminus F$. The **Bergman fan** Σ_M of a loopless matroid M of rank r is a subfan of Σ_E whose set of maximal cones consists of $\operatorname{Cone}(\overline{\mathbf{e}}_{F_1},\dots,\overline{\mathbf{e}}_{F_{r-1}})$ for each maximal chain of nonempty proper flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E$ of M. The **Bergman class** Δ_M of a matroid M of rank r with ground set E is the function $\Delta_M\colon \Sigma_E(r-1)\to\mathbb{Z}$ defined by

$$\operatorname{Cone}(\overline{\mathbf{e}}_{S_1},\dots,\overline{\mathbf{e}}_{S_{r-1}}) \mapsto \begin{cases} 1 & \text{if } M \text{ is loopless and } \operatorname{Cone}(\overline{\mathbf{e}}_{S_1},\dots,\overline{\mathbf{e}}_{S_{r-1}}) \in \Sigma_M \\ 0 & \text{otherwise.} \end{cases}$$

The Bergman fan of a matroid was previously studied in [9, 48, 102, 104] as a tropical linear space, and is a key object in the Hodge theory of matroids [1]. We caution that in these works, the terminology "Bergman fan" of a matroid M sometimes refers to a more coarsened fan structure on the support of Σ_M than defined here. Here, we show that Δ_M is a Minkowski weight, and identify it with $c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E}$, the Poincaré dual of the top Chern class of $\mathcal{Q}_M$.

Theorem 7.6 *Let M be a matroid of rank r with ground set E. For every chain $\mathscr{S} \colon \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_{r-1} \subsetneq E$ of $(r-1)$ nonempty proper subsets of E, we have*

$$\int_{X_E} c_{|E|-r}(\mathcal{Q}_M) \cdot [Z_{\mathscr{S}}] = \Delta_M(\mathrm{Cone}(\overline{\mathbf{e}}_{S_1}, \ldots, \overline{\mathbf{e}}_{S_{r-1}})).$$

In particular, the function Δ_M is a Minkowski weight, and we have the equality of Minkowski weights

$$c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E} = \Delta_M.$$

Proof By Proposition 7.4, we have that $\int_{X_E} c_{|E|-r}(\mathcal{Q}_M) \cdot [Z_{\mathscr{S}}]$ equals 1 if all minors $M|S_{i+1}/S_i$ are rank 1 uniform matroids and equals 0 otherwise. Thus, we need to show that the chain $\mathscr{S} \colon \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_{r-1} \subsetneq E$ satisfies $M|S_{i+1}/S_i = U_{1,S_{i+1}\setminus S_i}$ for all $i = 0, \ldots, r-1$ if and only if M is loopless and $\mathscr{S}$ is a maximal chain of nonempty proper flats of M.

We first show that if $M|S_{i+1}/S_i = U_{1,S_{i+1}\setminus S_i}$ for all $1 \leq i \leq r-1$, then S_i is a flat for all $1 \leq i \leq r-1$. If a subset $S \subsetneq E$ is not a flat, then by definition there exists $e \in E \setminus S$ such that $\mathrm{rk}_M(S \cup \{e\}) = \mathrm{rk}_M(S)$, and hence the element e is a loop in M/S. If S is a flat of corank 1 in M, then M/S is a matroid of rank 1 with no loops, and hence is the matroid $U_{1,E\setminus S}$. Moreover, note that if $S \subseteq E$ is a flat of M, then the flats of $M|S$ are the flats of M contained in S. Hence, by backwards induction on i starting with $i = r-1$, we conclude that $M|S_{i+1}/S_i = U_{1,S_{i+1}\setminus S_i}$ for $i = 1, \ldots, r-1$ if and only if S_i is a flat of corank 1 in $M|S_{i+1}$, and hence a flat in M, for all $i = 1, \ldots, r-1$. Lastly, if S_1 is a flat of M, then it contains the loops of M, so that $M|S_1$ is loopless if and only if M is loopless. □

The fact that Δ_M is a Minkowski weight recovers the computation from [71, Proposition 4.3] that Δ_M satisfies the balancing condition of Definition 7.1. Also, in Remark III.1, we show that a set of Chern roots of $\mathcal{Q}_M$ is given by $\{\alpha - \alpha_{\mathscr{F}_i}\}_{i=r,\ldots,n}$ for certain modular filters $\mathscr{F}_i$. In this light, Theorem 7.6 states that the Minkowski weight of the product $\prod_{i=r}^n (\alpha - \alpha_{\mathscr{F}_i})$ equals the Bergman class Δ_M. In [69, Remark 31], the same statement is deduced via a tropical intersection formula in [3].

Remark 7.7 By a similar computation, one can recover the computation of the Minkowski weight for the "truncated Bergman fan" $\Delta_{M,[d_1,d_2]}$ from [69, 77], which in our notation is equal to the Minkowski weight $\alpha^{r-1-d_2}\beta^{d_1-1}c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E}$ by Theorem 7.6. Indeed, along the lines of Proposition 7.4, we have by Proposition 5.3 and Corollary 5.4 that

$$\alpha^i \beta^j c(\mathcal{Q}_M)|_{Z_{\mathscr{S}}} = \beta^j c(\mathcal{Q}_{M|S_1}, u) \otimes \bigotimes_{i=1}^{k-1} c(\mathcal{Q}_{M|S_{i+1}/S_i}, u) \otimes \alpha^i c(\mathcal{Q}_{M/S_k}, u),$$

and one can similarly apply Theorem A to each term of the tensor product, noting that for any flat F, the expression $|\mu(\emptyset, F)|$ in [69, 77] is equal to $T_{M|F}(1, 0)$.

For M a realizable matroid, we now show how our expression of Δ_M as $c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E}$ recovers the relation between the class Δ_M and the geometry of wonderful compactifications of hyperplane arrangement complements introduced in [35].

Definition 7.8 Let $L \subseteq \mathbb{C}^E$ be a realization of a matroid M, and let $\mathbb{P}L \subset \mathbb{P}^n$ be the projectivization. The **wonderful compactification** W_L associated to the realization L is the closure of $\mathbb{P}L \cap (T/\mathbb{C}^*)$ in X_E.[2]

Remark 7.9 For $i \in E$, let H_i be the coordinate hyperplane of $\mathbb{C}^E$. A realization $L \subseteq \mathbb{C}^E$ of a matroid defines a projective hyperplane arrangement $\{\mathbb{P}\mathcal{H}_i \subset \mathbb{P}L\}_{i\in E}$ on $\mathbb{P}L$, where $\mathcal{H}_i = L \cap H_i$. In other words, the set $\mathbb{P}L \cap (T/\mathbb{C}^*)$ is the complement of the projective hyperplane arrangement. [35, Theorem 4.2] states that the wonderful compactification W_L is the compactification of $\mathbb{P}L \cap (T/\mathbb{C}^*)$ whose boundary $D_{W_L} = W_L \setminus (\mathbb{P}L \cap (T/\mathbb{C}^*))$ is a union of smooth irreducible divisors with simple normal crossing.

The following theorem relates the wonderful compactification to the tautological quotient bundle $\mathcal{Q}_L$ of a realization L of a matroid.

Theorem 7.10 *Let $L \subseteq \mathbb{C}^E$ be a subspace of dimension r, and for $\mathbf{v} \in \mathbb{C}^E$, let $s_{\mathbf{v}}$ be the global section of the tautological quotient bundle $\mathcal{Q}_L$ obtained as the image under $\underline{\mathbb{C}}^E_{\text{inv}} \to \mathcal{Q}_L$ of the constant $\mathbf{v} \in \mathbb{C}^E$ section of the bundle $\underline{\mathbb{C}}^E_{\text{inv}}$. Then, for any $\mathbf{a} \in (\mathbb{C}^*)^E$ we have*

$$W_{\mathbf{a}^{-1}L} = \textit{the vanishing loci } \{p \in X_E \mid s_{\mathbf{a}}(p) = 0\} \textit{ of the global section } s_{\mathbf{a}} \textit{ of } \mathcal{Q}_L.$$

In particular, since $\mathcal{Q}_L$ is globally generated by $\underline{\mathbb{C}}^E_{\text{inv}} \to \mathcal{Q}_L$, for any $\mathbf{a} \in (\mathbb{C}^*)^E$ we have

$$[W_{\mathbf{a}^{-1}L}] = c_{|E|-r}(\mathcal{Q}_L) \quad \textit{as elements in } A^\bullet(X_E).$$

Proof We prove the theorem for $\mathbf{a} = \mathbf{1} = (1, 1, \ldots, 1)$, since the general case follows by T-equivariance. By the short exact sequence $0 \to \mathcal{S}_L \to \underline{\mathbb{C}}^E_{\text{inv}} \to \mathcal{Q}_L \to 0$, for any $p \in X_E$ we have $s_{\mathbf{1}}(p) = 0$ precisely when $\mathbf{1}$ belongs to the fiber $\mathcal{S}_L|_p$ of $\mathcal{S}_L$ at p. Recall that for $\bar{t}$, the point in the dense open torus $T/\mathbb{C}^*$ corresponding to a point $t \in T$, the fiber $\mathcal{S}_L|_{\bar{t}}$ is by definition $t^{-1}L$.

[2]Our definition here may look different from the one in [35]. First, the wonderful compactification as defined here is the "maximal building set" wonderful compactification, whereas [35] more generally studies wonderful compactifications from arbitrary maximal building sets. Second, the wonderful compactification is originally constructed via blow-ups. From the fact that X_E can be constructed as a series of blow-ups from $\mathbb{P}^n$, one can deduce the equivalence between the description of W_L as a blow-up and the description here as a closure in X_E. See for example [71, Sect. 6] for an exposition of this equivalence.

We first treat the case where L is contained in a coordinate hyperplane $\{x_i = 0\}$, i.e. the corresponding matroid has a loop. In this case, by construction we have $W_L = \emptyset$. Also, the fiber of $\mathcal{S}_L$ over $\bar{t}$ in the dense open torus $T/\mathbb{C}^*$ is always contained in $\{x_i = 0\}$. Since this is a closed condition, the same true is for every fiber of $\mathcal{S}_L \to X_E$. Since $\mathbf{1} \notin \{x_i = 0\}$, this means $\mathbf{1}$ is not contained in any fiber of $\mathcal{S}_L$, even on the boundary of X_E. That is, the section $s_{\mathbf{1}}$ is nowhere vanishing on X_E.

We now treat with the case where L is not contained in a coordinate hyperplane. On the dense open torus $T/\mathbb{C}^*$ we have

$$s_{\mathbf{1}}(\bar{t}) = 0 \iff \mathbf{1} \in \mathcal{S}_L|_{\bar{t}} = t^{-1}L \iff t \in L \iff \bar{t} \in \mathbb{P}L.$$

Thus, the vanishing locus of the section $s_{\mathbf{1}}$ on the dense open torus $T/\mathbb{C}^*$ is $\mathbb{P}L \cap (T/\mathbb{C}^*)$. Since W_L is the closure in X_E of $\mathbb{P}L \cap (T/\mathbb{C}^*)$, we are done once we show that the vanishing loci of $s_{\mathbf{1}}$ is irreducible

To show irreducibility, we consider the map $\pi : \mathcal{S}_L \to \underline{\mathbb{C}}^E_{\text{inv}} = X_E \times \mathbb{C}^E_{\text{inv}} \to \mathbb{C}^E_{\text{inv}}$, which is dominant since L is not contained in a coordinate hyperplane. Note that for $\mathbf{a} \in \mathbb{C}^E$, the fiber $\pi^{-1}(\mathbf{a}) = \{(x, \mathbf{a}) \in X_E \times \mathbb{C}^E_{\text{inv}} : \mathbf{a} \in \mathcal{S}_L|_x\}$ is isomorphic to the vanishing loci of the section $s_{\mathbf{a}}$. The affine bundle $\mathcal{S}_L$ is irreducible, and since a general fiber of any dominant map of two varieties is pure-dimensional, we have that a general fiber of π is irreducible if a general fiber of the restriction of π to the open subset $\mathcal{S}_L|_{T/\mathbb{C}^*}$ is irreducible. By T-equivariance, the previous conclusion that $\{s_{\mathbf{1}} = 0\} \cap (T/\mathbb{C}^*) = \mathbb{P}L \cap (T/\mathbb{C}^*)$ thus implies that a general fiber of π is irreducible. Again by T-equivariance, the fiber $\pi^{-1}(\mathbf{1})$ is a general fiber and hence irreducible, as desired. □

Combining Theorem 7.6 with Theorem 7.10, we obtain the following properties of the Bergman class of a matroid.

Corollary 7.11 *Let M be a matroid with ground set E. Then,*

(*i*) *If $L \subseteq \mathbb{C}^E$ is a realization of M, then the Chow class $[W_L] \in A^\bullet(X_E)$ of the wonderful compactification is the Poincaré dual of Δ_M, and in particular is independent of the realization.*

(*ii*) *The assignment $M \mapsto \Delta_M$ is valuative.*

Part (i) recovers [78, Proposition 4.2]. Part (ii) follows by applying Proposition 5.6 to Theorem 7.6. Lemma 5.9 implies that the two properties in Corollary 7.11 characterize the assignment $M \mapsto \Delta_M$.

Remark 7.12 For a realization $L \subseteq \mathbb{C}^E$, we defined $\mathcal{S}_L$ to be the vector bundle on X_E such that the fiber over $\bar{t} \in T/\mathbb{C}^*$ is the subspace $t^{-1}L \subseteq \mathbb{C}^E$. If we defined $\mathcal{S}_L$ such that the fiber at $\bar{t}$ is $tL \subseteq \mathbb{C}^E$ instead, the proof of Theorem 7.10 implies that the vanishing loci of the section $s_{\mathbf{1}}$ on the open torus $T/\mathbb{C}^*$ is not the linear space $\mathbb{P}L \cap (T/\mathbb{C}^*)$, but instead its Cremona image $\text{crem}(\mathbb{P}L \cap (T/\mathbb{C}^*))$. In particular, such alternate definitions of $\mathcal{S}_L$ and $\mathcal{Q}_L$ result in $c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E} = \text{crem}\, \Delta_M$.

8 Chern-Schwartz-MacPherson classes via tautological classes

By considering the Poincaré duals of products of tautological Chern classes, we recover the notion of Chern-Schwartz-MacPherson (CSM) classes of a matroid studied in [84], and their relation to the geometry of hyperplane arrangement complements.

8.1 Minkowski weights of products of Chern classes

We compute the Poincaré duals of any product $c_i(\mathcal{S}_M^\vee)c_j(\mathcal{Q}_M)$ of tautological Chern classes of a matroid M. We will use this to express the CSM classes of matroids in terms of the tautological Chern classes.

Proposition 8.1 *Let M be a matroid with ground set E, and write rk_M and crk_M for its rank and corank, respectively. For a chain $\mathscr{S}: \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ of nonempty proper subsets of E, we have*

$$\int_{X_E} c(\mathcal{S}_M^\vee, z)c(\mathcal{Q}_M, w)\cdot[Z_{\mathscr{S}}]$$
$$= z^{\mathrm{rk}_M} w^{\mathrm{crk}_M} \prod_{i=0}^{k} \Big(\beta(M|S_{i+1}/S_i)\frac{1}{z} + \beta((M|S_{i+1}/S_i)^\perp)\frac{1}{w}\Big).$$

Proof By the matroid minor decomposition property Proposition 5.2 we have

$$\int_{X_E} c(\mathcal{S}_M^\vee, z)c(\mathcal{Q}_M, w)\cdot[Z_{\mathscr{S}}] = \int_{Z_{\mathscr{S}}} \big(c(\mathcal{S}_M^\vee, z)c(\mathcal{Q}_M, w)|_{Z_{\mathscr{S}}}\big)$$
$$= \prod_{i=0}^{k} \int_{X_{S_{i+1}\setminus S_i}} c(\mathcal{S}_{M|S_{i+1}/S_i}, z)c(\mathcal{Q}_{M|S_{i+1}/S_i}, w).$$

By Theorem 6.2, for a matroid M' on a ground set E' we have

$$\int_{X_{E'}} c(\mathcal{S}_{M'}^\vee, z)c(\mathcal{Q}_{M'}, w) = z^{\mathrm{rk}_{M'}} w^{\mathrm{crk}_{M'}} \big(\beta(M')\frac{1}{z} + \beta(M'^\perp)\frac{1}{w}\big).$$

Applying this with $M' = M|S_{i+1}/S_i$ for each $i = 0, \dots, k$, the result follows. $\square$

8.2 CSM classes of matroids and hyperplane arrangement complements

The following definition of CSM classes of a matroid was introduced in [84, Definition 5].

Definition 8.2 Let M be a matroid of rank r on ground set E. For $0 \le k \le r-1$, define the k**-dimensional CSM class** of M to be the function $\mathrm{csm}_k(M)\colon \Sigma_E(k) \to \mathbb{Z}$ defined by

$\mathrm{Cone}(\bar{\mathbf{e}}_{S_1}, \dots, \bar{\mathbf{e}}_{S_k})$

$$\mapsto \begin{cases} (-1)^{r-1-k} \prod_{i=0}^{k} \beta(M|S_{i+1}/S_i) & \text{if } S_i \text{ is a flat of } M \text{ for all } 0 \le i \le k \\ 0 & \text{otherwise} \end{cases}$$

for every chain $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ of k nonempty proper subsets of E.

Remark 8.3 One may alternately define the function $\mathrm{csm}_k(M)\colon \Sigma_E(k) \to \mathbb{Z}$ to be

$$\mathrm{Cone}(\overline{\mathbf{e}}_{S_1}, \ldots, \overline{\mathbf{e}}_{S_k}) \mapsto (-1)^{r-1-k} \prod_{i=0}^{k} \beta(M|S_{i+1}/S_i)$$

for the following reason. If a matroid M has a loop, say $e \in E$, then $M = U_{0,\{e\}} \oplus M \setminus \{e\}$, so that $\beta(M) = 0$ because $T_M(x, y)$ is divisible by $y = T_{U_{0,\{e\}}}(x, y)$. Now, arguing the same way as in the proof of Theorem 7.6, we have that the minors $M|S_{i+1}/S_i$ are loopless for all $i = 1, \ldots, k$ if and only if the subsets S_i are flats of M for all $i = 1, \ldots, k$, and $S_0 = \emptyset$ is a flat of M if and only if M is loopless.

We express the CSM classes of matroids in terms of products of tautological Chern classes of M as follows.

Theorem 8.4 *Let M be a matroid of rank r with ground set E. Then, for every $k = 0, 1, \ldots, r-1$, and for every chain $\mathscr{S}\colon \emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ of k nonempty proper subsets of E, we have*

$$\int_{X_E} c_{r-1-k}(\mathcal{S}_M) c_{|E|-r}(\mathcal{Q}_M) \cdot [Z_{\mathscr{S}}] = (-1)^{r-1-k} \prod_{i=0}^{k} \beta(M|S_{i+1}/S_i).$$

In particular, for every $k = 0, 1, \ldots, r-1$, the function $\mathrm{csm}_k(M)$ is a Minkowski weight, and we have the equality of Minkowski weights

$$c_{r-1-k}(\mathcal{S}_M) c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E} = \mathrm{csm}_k(M).$$

Proof Since $c_{r-1-k}(\mathcal{S}_M^\vee) = (-1)^{r-1-k} c_{r-1-k}(\mathcal{S}_M)$, we show that

$$\int_{X_E} c_{r-1-k}(\mathcal{S}_M^\vee) c_{|E|-r}(\mathcal{Q}_M) \cdot [Z_{\mathscr{S}}] = \prod_{i=0}^{k} \beta(M|S_{i+1}/S_i).$$

Since we are considering the $(|E| - r)$-th Chern class of $\mathcal{Q}_M$, we first extract the $w^{|E|-r} = w^{\mathrm{crk}_M}$ coefficient from Proposition 8.1. Because no terms involving a $\frac{1}{w}$ can thus contribute, the coefficient of w^{crk_M} in the expression in Proposition 8.1 is

$$z^{\mathrm{rk}_M} \prod_{i=0}^{k} \beta(S_{i+1}/S_i) \frac{1}{z} = z^{r-k-1} \prod_{i=0}^{k} \beta(S_{i+1}/S_i),$$

so that $\int_{X_E} c_{r-1-k}(\mathcal{S}_M^\vee) c_{|E|-r}(\mathcal{Q}_M) \cdot [Z_{\mathscr{S}}] = \prod_{i=0}^{k} \beta(M|S_{i+1}/S_i)$ as desired. $\square$

We note that setting $k = r - 1$ recovers Theorem 7.6. The fact that $\mathrm{csm}_k(M)$ are Minkowski weights recovers the computation [84, Theorem 2.3] that the functions $\mathrm{csm}_k(M)$ satisfy the balancing condition of Definition 7.1. Setting $x = y = 0$ in Theorem 4.8 and applying it to Theorem 8.4, one recovers [84, Proposition 5.2], which states that CSM classes of a matroid M on E satisfies a deletion-contraction relation

$$f_* \mathrm{csm}_k(M) = \begin{cases} 0 & \text{if } i \in E \text{ is a loop or a coloop} \\ \mathrm{csm}_k(M \setminus i) - \mathrm{csm}_k(M/i) & \text{otherwise} \end{cases}$$

for all k, where $f\colon X_E \to X_{E\setminus i}$ is the map in Definition 4.2.

For M a realizable matroid, we now show how our expression of $\mathrm{csm}_k(M)$ as $c_{r-1-k}(\mathcal{S}_M)c_{|E|-r}(\mathcal{Q}_M) \cap \Delta_{\Sigma_E}$ recovers the relation between $\mathrm{csm}_k(M)$ and the geometry of Chern-Schwartz-MacPherson (CSM) classes of hyperplane arrangement complements. We do this by first reviewing how CSM classes relate to log-tangent sheaves, and then by relating log-tangent sheaves to the tautological subbundles of realizations of matroids.

CSM classes, introduced in [87, 100], are generalizations of characteristic classes of smooth and complete algebraic varieties. See [5] for a survey and [6] for a construction. For a constructible subset Z of a complete complex algebraic variety X, its CSM class is an element $\mathrm{csm}(Z) \in A_\bullet(X)$ that equals the total Chern class $[Z] \cap c(\mathcal{T}_Z)$ of its tangent bundle $\mathcal{T}_Z$ when Z is a smooth complete variety. We consider CSM classes of hyperplane arrangement complements.

Definition 8.5 For a realization $L \subseteq \mathbb{C}^E$ of a loopless matroid M of rank r (i.e. an r-dimensional subspace not contained in a coordinate hyperplane), let $\mathcal{C}(L) = \mathbb{P}L \cap (T/\mathbb{C}^*)$ be the hyperplane arrangement complement. We decompose its CSM class $\mathrm{csm}(\mathcal{C}(L)) \in A^\bullet(X_E)$ as

$$\mathrm{csm}(\mathcal{C}(L)) = \sum_{k=0}^{r-1} \mathrm{csm}_k(\mathcal{C}(L)),$$

where $\mathrm{csm}_k(\mathcal{C}(L))$ is the graded piece lying in $A_k(X_E) \cong A^{n-k}(X_E)$. For a realization $L \subseteq \mathbb{C}^E$ of a matroid M containing loops, the intersection $\mathbb{P}L \cap (T/\mathbb{C}^*)$ is necessarily empty, and hence we define $\mathrm{csm}(\mathcal{C}(L))$ to be $0 \in A^\bullet(X_E)$.

Recall from Remark 7.9 that the wonderful compactification W_L is the compactification of $\mathbb{P}L \cap (T/\mathbb{C}^*)$ with the simple normal crossing boundary $D_{W_L} = W_L \setminus (\mathbb{P}L \cap (T/\mathbb{C}^*))$. CSM classes of realizable matroids are related to log tangent sheaves from the following fact.

Proposition 8.6 *Assume that $L \subset \mathbb{C}^E$ is not contained in a coordinate hyperplane. The CSM class $\mathrm{csm}(\mathcal{C}(L))$ is equal to the Chern class $c(\mathcal{T}_{W_L}(-\log D_{W_L})) \in A^\bullet(W_L)$ pushed forward to $A^\bullet(X_E)$.*

Proof The CSM class of $\mathcal{C}(L)$ inside of W_L is given by $c(\mathcal{T}_{W_L}(-\log D_{W_L})) \in A^\bullet(W_L)$ as it is the complement of a simple normal crossing divisor in a smooth variety [4, Theorem 1]. Since CSM classes behave compatibly with pushforward along proper maps, $\mathrm{csm}(\mathcal{C}(L)) \in A^\bullet(X_E)$ is given by the Chern classes of the log tangent sheaf $\mathcal{T}_{W_L}(-\log D_{W_L})$ pushed forward under $A^\bullet(W_L) \to A^\bullet(X_E)$. □

Remark 8.7 For D a divisor in a variety X, local sections of $T_X(-\log D)$ should be viewed as vector fields that are tangent to D. We will only consider the case when D is a simple normal crossing divisor and X is smooth. In this case, the intuition above is formalized in a short exact sequence

$$0 \to T_X(-\log D) \to T_X \to \bigoplus \mathcal{O}_{D_i}(D_i) \to 0,$$

where D_i are the components of D. See [4, Sect. 2] or [101, Sect. 3].

The following theorem relates the log tangent sheaf to tautological bundles.

Theorem 8.8 *For $L \subseteq \mathbb{C}^E$ a dimension r subspace of $\mathbb{C}^E$ not contained in a coordinate hyperplane, we have a short exact sequence*

$$0 \to \mathcal{O}_{W_L} \to \mathcal{S}_L|_{W_L} \to \mathcal{T}_{W_L}(-\log D_{W_L}) \to 0,$$

of sheaves on the wonderful compactification W_L. As a consequence, for any $L \subset \mathbb{C}^E$ and each $k = 0, \dots, r-1$, we have

$$\mathrm{csm}_k(\mathcal{C}(L)) = c_{r-1-k}(\mathcal{S}_L)c_{|E|-r}(\mathcal{Q}_L) \quad \textit{as elements in } A^\bullet(X_E).$$

Proof If L is contained in a coordinate hyperplane then its matroid must have at least one loop. It follows from Proposition 7.4 that $c_{|E|-r}(\mathcal{Q}_L) = 0$ and hence $\mathrm{csm}_k(\mathcal{C}(L)) = c_{r-1-k}(\mathcal{S}_L)c_{|E|-r}(\mathcal{Q}_L) = 0$ for all $0 \le k \le r-1$, in agreement with Definition 8.5.

For the rest of the argument we assume that L is not contained in a coordinate hyperplane. In this case, it is sufficient to prove the first claim of the theorem, since the second is an immediate consequence.

To begin, we claim there is a short exact sequence

$$0 \to T_{W_L}(-\log D_{W_L}) \to T_{X_E}(-\log D_{X_E})|_{W_L} \to N_{W_L/X_E} \to 0.$$

The components $D_{W_L,S}$ of D_{W_L} are in bijection with partial intersections of the hyperplanes $\mathcal{H}_i = L \cap \{x_i = 0\}$, which are the nonempty proper flats $S \subset E = \{0, \dots, n\}$ of the matroid M. The components $D_{X_E,S}$ of D_{X_E} are similarly indexed by all nonempty proper subsets $S \subset E$.

Consider the following diagram, where the dashed arrows are maps that we need to show exist. Here the left two vertical columns are the defining short exact sequence for log-tangent sheaves.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{W_L}(-\log D_{W_L}) \arrow[r, dashed] \arrow[d] & T_{X_E}(-\log D_{X_E})|_{W_L} \arrow[r] \arrow[d] & N_{W_L/X_E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{W_L} \arrow[r] \arrow[d] & T_{X_E}|_{W_L} \arrow[r] \arrow[d] & N_{W_L/X_E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{\substack{S \text{ proper} \\ \text{flat of } M}} \mathscr{O}_{D_{W_L,S}}(D_{W_L,S}) \arrow[r, dashed] \arrow[d] & \bigoplus_{\emptyset \subsetneq S \subsetneq \{0,\dots,n\}} \mathscr{O}_{D_{X_E,S}}(D_{X_E,S})|_{W_L} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Following [71, Proof of Theorem 6.3(1)], each divisor $D_{X_E,S}$ for S a proper subset of E intersects W_M if and only if S is a flat of M (a maximal collection of indices such that $\cap_{i\in S}\mathcal{H}_i$ intersects in a fixed subspace), and in this case $D_{X_E,S} \cap W_L = D_{W_L,S}$ scheme-theoretically. Thus, $\bigoplus_{S \text{ flat of } M} \mathscr{O}_{D_{W_L,S}}(D_{W_L,S})$ is isomorphic to $\bigoplus_{\emptyset\subsetneq S\subsetneq\{0,\dots,n\}} \mathscr{O}_{D_{X_E,S}}(D_{X_E,S})|_{W_L}$. This then implies the top map exists, and by the nine-lemma the top row is exact.

Next, the log-tangent sheaf $T_{X_E}(-\log D_{X_E})$ is trivial since X_E is a smooth complete toric variety, and fits into the short exact sequence

$$0 \to \underline{\mathbb{C}} \to \bigoplus_{i=0}^{n} \underline{\mathbb{C}} t_i \frac{\partial}{\partial t_i} \to T_{X_E}(-\log D_{X_E}) \to 0,$$

where the first inclusion takes $\mathbf{1} \mapsto \sum t_i \frac{\partial}{\partial t_i}$.

Pulling back the inclusion $T_{W_L}(-\log D_{W_L}) \hookrightarrow T_{X_E}(-\log D_{X_E})|_{W_L}$ under the surjective mapping $\bigoplus_{i=0}^n \underline{\mathbb{C}} t_i \frac{\partial}{\partial t_i} \to T_{X_E}(-\log D_{X_E})$ restricted to W_L, we get some subbundle $\mathcal{F} \subset \bigoplus_{i=0}^n \underline{\mathbb{C}}|_{W_L} t_i \frac{\partial}{\partial t_i}$. This yields the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{C}}|_{W_L} \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, hook] & T_{W_L}(-\log D_{W_L}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \underline{\mathbb{C}}|_{W_L} \arrow[r] & \bigoplus_{i=0}^n \underline{\mathbb{C}}|_{W_L} t_i \frac{\partial}{\partial t_i} \arrow[r] & T_{X_E}(-\log D_{X_E})|_{W_L} \arrow[r] & 0
\end{tikzcd}$$

Under the identification[3] $\bigoplus_{i=0}^n \underline{\mathbb{C}} t_i \frac{\partial}{\partial t_i} \cong \underline{\mathbb{C}}^{n+1}_{\text{inv}}$ with $t_i \frac{\partial}{\partial t_i} \mapsto e_i$, we need to check $\mathcal{F}$ agrees with $\mathcal{S}_L|_{W_L}$.

To check this, it suffices to restrict to $W_L \cap T$. Fix a point $t \in W_L \cap T$. In coordinates, t is specified by a point in $(\mathbb{C}^*)^{n+1}$ up to the diagonal scaling under $\mathbb{C}^*$. As a subbundle of $\underline{\mathbb{C}}^{n+1}_{\text{inv}}$, the fiber of $\mathcal{S}_L|_{W_L\cap T}$ over t is $t^{-1}L$. The fiber of

[3] This identification does not respect the natural T-equivariant structure, which is to act trivially on the left hand side.

$\mathcal{F}$ over t is all $v = (v_0, \dots, v_n)$ such that $\sum_{i=0}^{n} v_i t_i \frac{\partial}{\partial t_i}$ lies in the tangent space to t at $t \in L \cap (\mathbb{C}^\times)^{n+1} \subset \mathbb{C}^{n+1}$. This is equivalently described as the set of all $v = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$ such that $(t_0 v_0, \dots, t_n v_n) \in L \subset \mathbb{C}^{n+1}$, which is $t^{-1}L$ as well.

Since $\mathcal{F}$ agrees with $\mathcal{S}_L|_{W_L}$, the top row of the commutative diagram gives us our desired short exact sequence. □

Remark 8.9 Theorem 8.8 implies $c_{r-1}(\mathcal{S}_L)|_{W_L} = c_{r-1}(T_{W_L}(-\log D_{W_L}))$ and Theorem 7.10 implies $c_{r-1}(\mathcal{S}_L)|_{W_L} = (-1)^{r-1} c_{r-1}(\mathcal{S}_L^\vee) c_{|E|-r}(\mathcal{Q}_M)$. A logarithmic version of the Poincaré-Hopf theorem [101, Theorem 4.1] implies that $c_{r-1}(T_{W_L}(-\log D_{W_L}))$ equals the topological Euler characteristic of the projective hyperplane arrangement corresponding to L. The topological Euler characteristic is equal to $(-1)^{r-1}\beta(M)$ by [92, Theorem 5.2]. This yields an alternative proof of Theorem 6.2 for realizable matroids, which can be extended to all matroids via valuativity (Proposition 5.6 and Lemma 5.9) by arguing similarly as in the end of Appendix I.

Combining Theorem 8.4 with Theorem 8.8, we obtain the following properties of the CSM classes of a matroid.

Corollary 8.10 *Let M be a matroid or rank r with ground set E. Then, for each $k = 0, \dots, r-1$*

(*i*) *The CSM class $\mathrm{csm}_k(\mathcal{C}(L))$ of the hyperplane arrangement complement $\mathcal{C}(L)$ is the Poincaré dual of $\mathrm{csm}_k(M)$ for any realization $L \subseteq \mathbb{C}^E$ of M.*

(*ii*) *The assignment $M \mapsto \mathrm{csm}_k(M)$ is valuative.*

Part (i) recovers [84, Theorem 3.1]. Part (ii), which follows from applying Proposition 5.6 to Theorem 8.4, recovers [84, Theorem 4.1]. For each $k = 0, \dots, r-1$, Lemma 5.9 implies that the two properties in Corollary 8.10 characterize the assignment $M \mapsto \mathrm{csm}_k(M)$.

9 Positivity and log-concavity

When a matroid M has a realization $L \subset \mathbb{C}^E$, the vector bundles $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ are globally generated and hence nef, so that their relative $\mathcal{O}(1)$ classes satisfy positivity and log-concavity properties listed in [83, §1.6]. In this section, we show that these properties persist for the K-classes $[\mathcal{S}_M^\vee]$ and $[\mathcal{Q}_M]$ for an arbitrary (not necessarily realizable) matroid M. In particular, we prove Theorem B, which states that the unifying Tutte formula $t_M(x, y, z, w)$ of Theorem A satisfies a log-concavity property.

We will proceed in three steps. In §9.1, we define the Chow class $[\mathbb{P}(\mathcal{S}_M) \times_{X_E} \mathbb{P}(\mathcal{Q}_M^\vee)]$ on $X_E \times \mathbb{P}^n \times \mathbb{P}^n$ which equals the Chow class of the biprojectivation $\mathbb{P}(\mathcal{S}_L) \times_{X_E} \mathbb{P}(\mathcal{Q}_L^\vee)$ when L is a realization of M. In §9.2, we show that the class $[\mathbb{P}(\mathcal{S}_M) \times_{X_E} \mathbb{P}(\mathcal{Q}_M^\vee)]$ is a pushforward of the Poincaré dual of a Minkowski weight supported on a "Lefschetz fan" in the sense of [13, Definition 1.5]. In §9.3 and §9.4, we derive positivity and log-concavity properties via the tropical Hodge theory of Lefschetz fans developed in [13, §5]. Our approach allows us to avoid the intricate computations with the bipermutohedron carried out in [13, §2 & §4]; see Remark 9.9.

9.1 Bi-projectivizations of tautological classes

For a vector bundle $\mathcal{E}$ on a variety X, we denote by $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}_X \operatorname{Sym}^\bullet(\mathcal{E}^\vee) = (\mathcal{E} \setminus (X \times \{0\}))/\mathbb{C}^*$ its projectivization. Let $\mathbb{P}^n_{\text{inv}} = \mathbb{P}(\mathbb{C}^E_{\text{inv}})$. For a realization $L \subseteq \mathbb{C}^E$ of a matroid M, one has inclusions $\mathbb{P}(\mathcal{S}_L) \subseteq X_E \times \mathbb{P}^n_{\text{inv}}$ and $\mathbb{P}(\mathcal{Q}_L^\vee) \subseteq X_E \times (\mathbb{P}^n_{\text{inv}})^\vee$, and hence we can form the bi-projective bundle

$$\mathbb{P}(\mathcal{Q}_L^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_L) \subset X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}.$$

We now define a combinatorial abstraction of the Chow class of the bi-projectivization $\mathbb{P}(\mathcal{Q}_L^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_L)$ for arbitrary matroids. Let us denote the following elements in $A^1(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}})$ by

δ and ϵ = the pullbacks of the hyperplane classes of $(\mathbb{P}^n_{\text{inv}})^\vee$ and $\mathbb{P}^n_{\text{inv}}$, respectively.

Let $\mu\colon X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}} \to X_E$ be the projection map, and for a class $\xi \in A^\bullet(X_E)$, we often write ξ also for the pullback $\mu^*\xi$ when we trust that no confusion will arise.

Definition 9.1 For matroids M_1 and M_2 of ranks r_1 and r_2 on the common ground set E, we define a Chow class $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})] \in A^\bullet(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}})$ by

$$[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})] = \sum_{k=0}^{r_1} \sum_{\ell=0}^{n+1-r_2} c_k(\mathcal{S}_{M_1}^\vee) c_\ell(\mathcal{Q}_{M_2}) \delta^{r_1-k} \epsilon^{n+1-r_2-\ell}.$$

The Chow class defined in Definition 9.1 has the following characterizing properties.

Proposition 9.2 *Let notations be as in the above definition. The class* $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ *satisfies and is determined by the following two properties.*

- (*i*) *If* L_1 *and* $L_2 \subseteq \mathbb{C}^E$ *are realizations of* M_1 *and* M_2*, respectively, then* $[\mathbb{P}(\mathcal{Q}_{L_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{L_2})] = [\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ *as Chow classes in* $A^\bullet(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}})$*, and*
- (*ii*) *the assignment* $(M_1, M_2) \mapsto [\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ *is valuative in each factor* M_1 *and* M_2.

Proof The property (i) is immediate by the formula for the Chow class of the projectivization of a subbundle [44, Proposition 9.13], noting that $\mathcal{S}_{L_1}^\vee = (\underline{\mathbb{C}}^E_{\text{inv}})^\vee/(\mathcal{Q}_{L_1}^\vee)$ and $\mathcal{Q}_{L_2} = \underline{\mathbb{C}}^E_{\text{inv}}/\mathcal{S}_{L_2}$. The property (ii) follows from Proposition 5.6. That these two properties characterize $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ follows from Lemma 5.9. $\square$

The following proposition relates mixed intersections of certain nef divisors with $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ to the mixed intersections appearing in Theorem A.

Proposition 9.3 *Let* M_1 *and* M_2 *be matroids of rank* r_1 *and* r_2 *on the common ground set* E*. Then, the pushfoward map* $\mu_*\colon A^\bullet(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}) \to A^\bullet(X_E)$ *satisfies for all nonnegative integers* k *and* ℓ

$$\mu_*(\delta^{n-r_1+k} \epsilon^{r_2-1+\ell} [\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]) = c_k(\mathcal{S}_{M_1}^\vee) c_\ell(\mathcal{Q}_{M_2}).$$

In particular, for $i + j + k + \ell = n$ we have

$$\int_{X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}} \alpha^i \beta^j \delta^{n-r_1+k} \epsilon^{r_2-1+\ell} [\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$$
$$= \int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}^\vee_{M_1}) c_\ell(\mathcal{Q}_{M_2}).$$

Proof As each δ and ϵ is a hyperplane class pullback from a projective space $\mathbb{P}^n$, for any integers $i, j \geq 0$ we have $\mu_*(\delta^i \epsilon^j) = 1$ if $i = j = n$, and 0 otherwise. We conclude by the definition of $[\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ and the push-pull formula. □

Remark 9.4 We note at this point that we can conclude Theorem B when a matroid M of rank r has a realization $L \subseteq \mathbb{C}^E$ as follows. The classes $\alpha, \beta, \delta, \epsilon$ on $X_{A_E} \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}$ are nef divisors, and hence restrict to nef divisors on the variety $\mathbb{P}(\mathcal{Q}^\vee_L) \times_{X_E} \mathbb{P}(\mathcal{S}_L)$. By Proposition 9.3 and Theorem A, we have

$$\sum_{i+j+k+\ell=n} \left(\int_{\mathbb{P}(\mathcal{Q}^\vee_L) \times_{X_E} \mathbb{P}(\mathcal{S}_L)} \alpha^i \beta^j \delta^k \epsilon^\ell \right) x^i y^j z^k w^\ell$$
$$= \frac{1}{x+y} (y+z)^r (x+w)^{n+1-r} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right).$$

One concludes the desired log-concavity property by the classical Khovanskii-Teissier inequality for intersection multiplicities of nef divisors (see [65, 105] or [83, Corollary 1.6.3 (i)]).

In the next few subsections, we will show Theorem B for arbitrary matroids by relating the intersection in Proposition 9.3 with an equivalent intersection on a Lefschetz fan as defined in [13]. The log-concavity will then follow from the validity of mixed Hodge-Riemann relations on the Lefschetz fan.

We conclude here with an observation that allows us to assume matroids to be loopless or coloopless under certain contexts. For a matroid M, let $M^{coloop\to loop}$ be M with all coloops replaced by loops, and $M^{loop\to coloop}$ be M with all loops replaced by coloops.

Proposition 9.5 *Let M_1 and M_2 be matroids of rank r_1 and r_2 on the common ground set E. Then,*

$$[\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})] = [\mathbb{P}(\mathcal{Q}^\vee_{M_1^{coloop\to loop}}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2^{loop\to coloop}})]$$

as elements in $A^\bullet(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}})$.

Proof From the definition of $[\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$, it suffices to show for any matroid M that

$$c(\mathcal{S}_{M^{loop\to coloop}}) = c(\mathcal{S}_M), \text{ and } c(\mathcal{Q}_{M^{loop\to coloop}}) = c(\mathcal{Q}_M).$$

We only prove $c(\mathcal{S}_{M^{loop\to coloop}}) = c(\mathcal{S}_M)$, as $c(\mathcal{Q}_{M^{loop\to coloop}}) = c(\mathcal{Q}_M)$ is proved similarly. Let $E_\ell \subseteq E$ be the (possibly empty) set of loops in M. Then, we have $M = M|(E \setminus E_\ell) \oplus U_{0,E_\ell}$ and $M^{loop\to coloop} = M|(E \setminus E_\ell) \oplus U_{|E_\ell|,E_\ell}$. Hence, since $\mathcal{S}_{0,E_\ell} = 0$ and $\mathcal{S}_{|E_\ell|,E_\ell} = [\underline{\mathbb{C}}^{E_\ell}_{\text{inv}}]$, Proposition 5.13 implies that $[\mathcal{S}_{M^{loop\to coloop}}]$ equals the sum of $[\mathcal{S}_M]$ and a trivial bundle (the pullback of $[\underline{\mathbb{C}}^{E_\ell}_{\text{inv}}]$). Thus, their Chern classes coincide. $\square$

9.2 Minkowski weight of a birational model of the biprojectivization

Let M_1 and M_2 be matroids on the common ground set E. The goal of this subsection is to express the Chow class $[\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ as a pushforward of a Minkowski weight that has certain "Lefschetz properties" we'll exploit in the next subsection.

Let Σ' be a pure d-dimensional subfan of an m-dimensional complete unimodular fan Σ. We say that Σ' is a **balanced fan** if the function

$$\Delta\colon \Sigma(d) \to \mathbb{Z} \quad \text{defined by} \quad \Delta(\tau) = \begin{cases} 1 & \tau \subseteq \tau' \text{ for some } \tau' \in \Sigma'(d) \\ 0 & \text{otherwise} \end{cases}$$

is a Minkowski weight on Σ, in which case we say that $\Delta \in \mathrm{MW}_d(\Sigma)$ is the **constant-1 Minkowski weight** on Σ'. We write $[\Sigma'] \in A^{m-d}(X_\Sigma)$ for the Poincaré dual of Δ.

Example 9.6 The Bergman class Δ_M is the constant-1 Minkowski weight on the Bergman fan Σ_M of a loopless matroid M (see for example Theorem 7.6). More generally, the Minkowski weights of $c_k(\mathcal{S}^\vee_M)$ and $c_\ell(\mathcal{Q}_M)$ are constant-1 Minkowski weights (Proposition 7.4).

Theorem 9.7 *Consider $X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}$ as a toric variety with dense open torus $(T/\mathbb{C}^*)^3$. Then, there exists a smooth projective toric variety X_Σ associated to a unimodular fan Σ in $(\mathbb{R}^E/\mathbb{R}\mathbf{1})^3$, together with a birational toric morphism $\widetilde{\phi}\colon X_\Sigma \to X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}$, such that for M_1 a coloopless matroid and M_2 a loopless matroid we have*

(1) *the product of fans $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$ is a coarsening of a subfan $\Sigma_{E,M_1^\perp,M_2} \subset \Sigma$, and*
(2) *under the pushfoward map $\widetilde{\phi}_*\colon A^\bullet(X_\Sigma) \to A^\bullet(X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}})$ we have*

$$\widetilde{\phi}_*[\Sigma_{E,M_1^\perp,M_2}] = [\mathbb{P}(\mathcal{Q}^\vee_{M_1}) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})].$$

We prepare the proof by stating some facts from tropical intersection theory. For a d-dimensional very-affine subvariety Y in a torus $(\mathbb{C}^*)^m$, the **tropicalization** of Y, denoted $\mathrm{trop}(Y)$, is a pure d-dimensional polyhedral complex in $\mathbb{R}^m$, along with $\mathbb{Z}$-valued weight function w on the set of its d-dimensional polyhedral cells. The weight w has the property that, for every complete unimodular fan Σ in $\mathbb{R}^m$ containing a subfan whose support equals $\mathrm{trop}(Y)$, the assignment

$$\Delta_Y\colon \Sigma(d) \to \mathbb{Z} \quad \text{defined by}$$

$$\tau \mapsto \begin{cases} w(C) & \text{if there exists a polyhedral cell } C \text{ in } \operatorname{trop}(Y) \text{ containing } \tau \\ 0 & \text{otherwise} \end{cases}$$

is a Minkowski weight on Σ. One such unimodular fan Σ can be constructed from the Gröbner fan of Y, which is a fan in $\mathbb{R}^m$, not necessarily unimodular, that contains a subfan whose support equals $\operatorname{trop}(Y)$. See [86, Ch. 3] for an introduction to tropicalizations and Gröbner fans. The tropicalization of a product of very-affine subvarieties is the product of each tropicalization. We will need the following two facts about tropicalizations from the theory of tropical compactifications.

Lemma 9.8

(*a*) *Let $L \subset \mathbb{C}^E$ be a realization of a loopless matroid M. Then, the tropicalization of $\mathbb{P}L \cap (T/\mathbb{C}^*)$ is a polyhedral complex whose support equals the support of the Bergman fan Σ_M, along with the weight function w that is constantly* 1. *Moreover, the Bergman fan refines a subfan of the Gröbner fan of $\mathbb{P}L \cap (T/\mathbb{C}^*)$.*

(*b*) *Let $Y \subseteq (\mathbb{C}^*)^m$ be a very-affine subvariety, and Σ a complete unimodular fan in $\mathbb{R}^m$ that refines the Gröbner fan of Y and contains a subfan whose support equals* $\operatorname{trop}(Y)$. *Then, the Chow class $[\overline{Y}] \in A^\bullet(X_\Sigma)$ of the closure of Y in the toric variety X_Σ is equal to the Poincaré dual of the Minkowski weight Δ_Y defined by* $\operatorname{trop}(Y)$.

Proof The first part of (a) was first observed in [104, §9.3]; see [86, Theorem 4.1.11] for a proof. The second part that Σ_M refines the Gröbner fan is [86, Exercise 4.7.(7)], which was implicitly stated in [9, Theorem 1]. The statement (b) follows from combining [75, Proposition 9.4] with [61, Theorem 14.9], or from combining [86, Theorem 6.4.17] with [86, Theorem 6.7.7]. □

We are now ready to prove Theorem 9.7.

Proof of Theorem 9.7 First, we clarify how we are treating $X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}$ as a toric variety. The standard basis of $\mathbb{C}^E$ induces an isomorphism $\mathbb{C}^E \simeq (\mathbb{C}^E)^\vee$, and by forgetting the T-action on $\mathbb{P}^n_{\text{inv}}$ and $(\mathbb{P}^n_{\text{inv}})^\vee$, we identify $X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}} = X_E \times \mathbb{P}^n \times \mathbb{P}^n$, where the latter is a product of three toric varieties, each with open dense torus $(T/\mathbb{C}^*)$.

We now specify the map $\widetilde{\phi}$ first on the torus $(T/\mathbb{C}^*)^3$. Define $\phi_{\text{trop}}\colon (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1})^3 \to (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1})^3$ by $\phi(u_0, u_1, u_2) = (u_0, u_0 + u_1, -u_0 + u_2)$. This induces an invertible map of tori $\phi\colon (T/\mathbb{C}^*)^3 \to (T/\mathbb{C}^*)^3$ given by $\phi(t_0, t_1, t_2) = (t_0, t_0t_1, t_0^{-1}t_2)$. Let Σ be a unimodular fan in $(\mathbb{R}^E/\mathbb{R}\mathbf{1})^3$ that sufficiently refines the fan $\Sigma_E^3 = \Sigma_E \times \Sigma_E \times \Sigma_E$ such that ϕ_{trop} induces a map of fans $\Sigma \to \Sigma_E \times \Sigma_n \times \Sigma_n$, where Σ_n is the fan of the toric variety $\mathbb{P}^n$. Such a fan Σ can be constructed by noting that a collection of normal fans of polytopes admit a common refinement [31, Proposition 6.2.13.(b)] and by applying the toric resolution of singularities [31, Theorem 11.1.9]. Then, the invertible map ϕ of tori extends to a birational toric morphism $\widetilde{\phi}\colon X_\Sigma \to X_E \times \mathbb{P}^n \times \mathbb{P}^n$.

We now verify that Σ and $\widetilde{\phi}$ satisfies the desired properties. The property (1) is immediate from the construction. Indeed, $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$ is a subfan of Σ_E^3, and

Σ refines Σ_E^3, we can set $\Sigma_{M,M_1^\perp,M_2}$ to be the subfan of Σ whose support equals the support of $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$.

For the property (2), we first claim that both $[\Sigma_{E,M_1^\perp,M_2}]$ and $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ are valuative separately in coloopless matroids M_1 and loopless matroids M_2. For $[\Sigma_{E,M_1^\perp,M_2}]$, by considering the corresponding constant-1 Minkowski weight on Σ, the desired valuativity is equivalent to the valuativity of the indicator function for the support of the fan $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$. Corollary 7.11.(ii) implies that the assignment $M \mapsto$ (indicator function for the support of Σ_M) is valuative, and similarly for the assignment $M \mapsto$ (indicator function for the support of $\Sigma_{M^\perp}$) since $M \mapsto 1_{P(M^\perp)} = 1_{-P(M)+\mathbf{1}}$ is valuative. Thus, we conclude that $[\Sigma'_{E,M_1^\perp,M_2}]$ is valuative separately in coloopless matroids M_1 and loopless matroids M_2. The valuativity for $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ follows from its definition and Proposition 5.6. Applying Lemma 5.9 to these valuative properties, we conclude that it suffices to show $\widetilde{\phi}_*[\Sigma'_{E,M_1^\perp,M_2}] = [\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})]$ assuming that M_1 and M_2 both have realizations.

Now, let $L_1 \subseteq \mathbb{C}^E$ and $L_2 \subseteq \mathbb{C}^E$ be realizations of M_1 and M_2, respectively. Recall that under $\mathbb{C}^E \simeq (\mathbb{C}^E)^\vee$, the subspace $L_1^\perp = (\mathbb{C}^E/L_1)^\vee \subseteq (\mathbb{C}^E)^\vee \simeq \mathbb{C}^E$ realizes $M_1^\perp$. Let Y_{L_1,L_2} be the intersection of $X_E \times \mathbb{P}L_1^\perp \times \mathbb{P}L_2 \subseteq X_E \times \mathbb{P}^n \times \mathbb{P}^n = X_E \times (\mathbb{P}^n_{\text{inv}})^\vee \times \mathbb{P}^n_{\text{inv}}$ with the open dense torus, namely,

$$Y_{L_1,L_2} = (T/\mathbb{C}^*) \times (\mathbb{P}L_1^\perp \cap T/\mathbb{C}^*) \times (\mathbb{P}L_2 \cap T/\mathbb{C}^*).$$

Note that Y_{L_1,L_2} is nonempty because $M_1^\perp$ and M_2 are assumed to be loopless. The very-affine subvariety Y_{L_1,L_2} is an "untwisting" of $\mathbb{P}(\mathcal{Q}_{L_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{L_2})$ on the open dense torus $(T/\mathbb{C}^*)^3$ in the sense that ϕ maps Y_{L_1,L_2} isomorphically onto $(\mathbb{P}(\mathcal{Q}_{L_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{L_2})) \cap (T/\mathbb{C}^*)^3$. Indeed, for any $t_0 \in T/\mathbb{C}^*$, the fibers over $\{t_0\} \times (T/\mathbb{C}^*)^2$ are

$$\begin{aligned}(Y_{L_1,L_2})_{\{t_0\}\times(T/\mathbb{C}^*)^2} &= \{t_0\} \times (\mathbb{P}L_1^\perp \cap T/\mathbb{C}^*) \times (\mathbb{P}L_2 \cap T/\mathbb{C}^*) \quad \text{and} \\ (\mathbb{P}(\mathcal{Q}_{L_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{L_2}))_{\{t_0\}\times(T/\mathbb{C}^*)^2} &= \{t_0\} \times (t_0\mathbb{P}L_1^\perp \cap T/\mathbb{C}^*) \times (t_0^{-1}\mathbb{P}L_2 \cap T/\mathbb{C}^*),\end{aligned}$$

and ϕ was given by $\phi(t_0,t_1,t_2) = (t_0, t_0t_1, t_0^{-1}t_2)$. Thus, denoting $\overline{Y_{L_1,L_2}}$ for the closure of Y_{L_1,L_2} in X_Σ, we have that $\widetilde{\phi}_*[\overline{Y_{L_1,L_2}}] = [\mathbb{P}(\mathcal{Q}_{L_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{L_2})]$. It only remains to show that $[\overline{Y_{L_1,L_2}}] = [\Sigma_{E,M_1^\perp,M_2}]$. Lemma 9.8.(a) implies that $\operatorname{trop}(Y_{L_1,L_2})$ is a polyhedral complex whose support equals the support of $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$, and the weight function is constantly 1. Then, since Lemma 9.8.(b) implies that the Chow class $[\overline{Y_{L_1,L_2}}] \in A^\bullet(X_\Sigma)$ is Poincaré dual to the Minkowski weight on Σ defined by the tropicalization $\operatorname{trop}(Y_{L_1,L_2})$, we conclude that $[\overline{Y_{L_1,L_2}}] = [\Sigma_{E,M_1^\perp,M_2}]$, as desired. $\square$

Remark 9.9 The fan $\Sigma_{E,M_1^\perp,M_2}$, serving as our combinatorial model of a biprojective bundle, is valuative separately in M_1 and M_2, allowing us to reduce to the realizable case. However, $\Sigma_{E,M^\perp,M}$ is *not* valuative in M. Similarly, the "conormal fan" $\Sigma_{M,M^\perp}$, whose support coincides with $\Sigma_M \times \Sigma_{M^\perp}$, is not valuative in M. In both cases, the final degree computation yields a valuative answer, which for us gave $t_M(x, y, z, w)$ and for [13] gave $T_M(x, 0)$, despite the fans not being valuative. This prevents one from automatically extending the final degree computation from the realizable case to all matroids.

However, in both cases, the valuativity of the final expression can be explained by instead working with the degrees $\alpha^i \beta^j c_k(\mathcal{S}_M^\vee) c_\ell(\mathcal{Q}_M)$. Then, the valuativity follows from Proposition 5.6. In contrast to our equivariant methods, valuativity seems less accessible using non-equivariant methods, as evidenced by the intricate computations with the bipermutohedron in [13, §2 and §4] required to generalize the realizable case done in [70].

9.3 Log-concavity for the Tutte polynomial

In this subsection, we combine Proposition 9.3 and Theorem 9.7 with properties of "Lefschetz fans" established in [13, §5] to prove Theorem B, reproduced below. Recall that the coefficients of a homogeneous polynomial $f \in \mathbb{R}[x_0, \dots, x_N]$ of degree d form a **log-concave unbroken array** if, for any $0 \le i < j \le N$ and a monomial x^m of degree $d' \le d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^m\}_{0 \le k \le d-d'}$ in f form a nonnegative log-concave sequence with no internal zeros.

Theorem B *For a matroid M of rank r with ground set E, the coefficients of the polynomial*

$$t_M(x, y, z, w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}\, T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

form a log-concave unbroken array.

We prepare by stating the tools we need from the tropical Hodge theory developed in [13, §5]. A **Lefschetz fan** is a pure-dimensional unimodular fan Σ, not necessarily complete, that is a balanced and satisfies certain Lefschetz properties [13, Definition 1.5]. In our case, the properties of a Lefschetz fan we need are collected in the following proposition.

Lemma 9.10

(*a*) [13, *Theorem* 1.6] *If Σ is a Lefschetz fan, then any unimodular fan whose support equals that of Σ is a Lefschetz fan.*

(*b*) [13, *Lemma* 5.26] *A product of Lefschetz fans is a Lefschetz fan.*

(*c*) [1, *Theorem* 8.9] *The Bergman fan of a loopless matroid is a Lefschetz fan.*

(*d*) [13, *Theorem* 5.28] *Let Σ be an m-dimensional projective unimodular fan, and let Σ' be a d-dimensional subfan that is a Lefschetz fan, which as a balanced fan defines the Chow class $[\Sigma'] \in A^{m-d}(X_\Sigma)$. Then, for nef divisors $D_1, D_2, \dots, D_k \in A^1(X_\Sigma)$ on the projective toric variety X_Σ, the sequence $(a_0, \dots, a_{d-(k-2)})$ defined by*

$$a_i = \int_{X_\Sigma} D_1^i D_2^{d-(k-2)-i} D_3 \cdots D_k \cdot [\Sigma']$$

is a nonnegative sequence that is log-concave with no internal zeros.[4]

We can now prove a strengthening of Theorem B.

Theorem 9.11 *For matroids M_1 and M_2 on the common ground set E, the coefficients of the polynomial*

$$\sum_{i+j+k+\ell=n} \left(\int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}_{M_1}^\vee) c_\ell(\mathcal{Q}_{M_2}) \right) x^i y^j z^k w^\ell \tag{$\star$}$$

form a log-concave unbroken array.

Proof By Proposition 9.3, the polynomial ($\star$) is equal to

$$\sum_{i+j+k+\ell=n} \left(\int_{X_E \times (\mathbb{P}^n_{\mathrm{inv}})^\vee \times \mathbb{P}^n_{\mathrm{inv}}} \alpha^i \beta^j \delta^{n-r_1+k} \epsilon^{r_2-1+\ell} [\mathbb{P}(\mathcal{Q}_{M_1}^\vee) \times_{X_E} \mathbb{P}(\mathcal{S}_{M_2})] \right) x^i y^j z^k w^\ell.$$

In this expression, by Proposition 9.5 we may assume that M_1 is coloopless and M_2 is loopless. Then, by Theorem 9.7 and the push-pull formula we have that the above equals

$$\sum_{i+j+k+\ell=n} \left(\int_{X_\Sigma} \widetilde{\phi}^* \alpha^i \widetilde{\phi}^* \beta^j \widetilde{\phi}^* \delta^{k+\mathrm{crk}_{M_1}-1} \widetilde{\phi}^* \epsilon^{\ell+\mathrm{rk}_{M_2}-1} [\Sigma_{E, M_1^\perp, M_2}] \right) x^i y^j z^k w^\ell.$$

That the coefficients of this polynomial form a log-concave unbroken array is now a result of Theorem 9.7 and Lemma 9.10 as follows. Because Σ_E, $\Sigma_{M_1^\perp}$, and Σ_{M_2} are Lefschetz fans (Lemma 9.10.(c)), so is their product (Lemma 9.10.(b)). Since the product $\Sigma_E \times \Sigma_{M_1^\perp} \times \Sigma_{M_2}$ and the fan $\Sigma_{E, M_1^\perp, M_2}$ have the same support (Theorem 9.7), the fan $\Sigma_{E, M_1^\perp, M_2}$ is also a Lefschetz fan (Lemma 9.10.(a)). Because the divisors $\widetilde{\phi}^*\alpha, \widetilde{\phi}^*\beta, \widetilde{\phi}^*\delta, \widetilde{\phi}^*\epsilon$ are nef divisors on X_Σ, being pullbacks of nef divisors, that the coefficients of the polynomial form a log-concave unbroken array follows from Lemma 9.10.(d). □

Proof of Theorem B Set $M_1 = M_2 = M$ in Theorem 9.11, and apply Theorem A. □

9.4 Denormalized Lorentzian polynomials

Let us note a strengthening of Theorem 9.11 that we will only need in §11 when we consider flag matroids. The theorem is strengthened in two ways.

First, we use the language of Lorentzian polynomials developed in [17]. For a homogeneous degree d polynomial $f = \sum_{\mathbf{m} \in \mathbb{Z}^N_{\geq 0}} a_{\mathbf{m}} x^{\mathbf{m}} \in \mathbb{R}[x_1, \ldots, x_N]$, its **normalization** is $N(f) = \sum_{\mathbf{m} \in \mathbb{Z}^N_{\geq 0}} a_{\mathbf{m}} \frac{x^{\mathbf{m}}}{\mathbf{m}!}$ where $\mathbf{m}! = m_1! \cdots m_N!$. The polynomial f is

[4] [13, Theorem 5.28] does not state no internal zeros, but its proof implies that the sequence $(a_0, \ldots, a_{d-k-2})$ is a limit of log-concave positive sequences. A limit of such sequences is a log-concave sequence with no internal zeros; see [67, Lemma 34] for a proof.

said to be the denormalization of $N(f)$. The polynomial f is a **strictly Lorentzian polynomial** if every monomial of degree d has positive coefficient in f and every $(d-2)$-th coordinate partial derivative of f is a quadric form with signature $(+,-,-,\ldots,-)$. It is a **Lorentzian polynomial** if f is a limit of strictly Lorentzian polynomials. [17, Example 2.26] combined with [17, Theorem 2.10] implies that the coefficients of a denormalized Lorentzian polynomial form a log-concave unbroken array. A minor modification of the proof of [17, Theorem 4.6] applied to the mixed Hodge-Riemann relations for Lefschetz fans [13, Definition 5.6.(2)] implies the following strengthening of Lemma 9.10.(d).

Lemma 9.12 *Let Σ be a m-dimensional projective unimodular fan, and let Σ' be a d-dimensional subfan that is a Lefschetz fan, which as a balanced fan defines the Chow class $[\Sigma'] \in A^{m-d}(X_\Sigma)$. Then, for nef divisors $D_1, D_2, \ldots, D_N \in A^1(X_\Sigma)$ on the projective toric variety X_Σ, the polynomial $f \in \mathbb{R}[x_1,\ldots,x_N]$ defined by*

$$f = \sum_{i_1+\cdots+i_N=d} \left(\int_{X_\Sigma} D_1^{i_1}\cdots D_N^{i_N}\cdot[\Sigma'] \right) x_1^{i_1}\cdots x_N^{i_N}$$

is a denormalization of a Lorentzian polynomial.

Second, we note that one can define multi-projectivization analogues of the definition of the biprojectivization class $[\mathbb{P}(\mathcal{Q}_{M_1}^\vee)\times_{X_E}\mathbb{P}(\mathcal{S}_{M_2})]$ for any number of $\mathcal{Q}_{M_i}^\vee$'s and $\mathcal{S}_{M_j}$'s. Proofs of the multi-projectivization analogues of Proposition 9.3, Proposition 9.5, and Theorem 9.7 are essentially identical to the proofs given for the biprojectivization case here. As a result, we obtain the following.

Theorem 9.13 *Let $M_1,\ldots,M_m$ and $M_1',\ldots,M_{m'}'$ be matroids with the common ground set E. Then, the polynomial defined by*

$$\sum \left(\int \alpha^i\beta^j c_{k_1}(\mathcal{S}_{M_1}^\vee)\cdots c_{k_m}(\mathcal{S}_{M_m}^\vee)c_{\ell_1}(\mathcal{Q}_{M_1'})\cdots c_{\ell_{m'}}(\mathcal{Q}_{M_{m'}'}) \right) x^i y^j z_1^{k_1}\cdots z_m^{k_m} w_1^{\ell_1} \cdots w_{m'}^{\ell_{m'}}$$

is a denormalization of a Lorentzian polynomial, where $\int = \int_{X_E\times((\mathbb{P}^n_{\mathrm{inv}})^\vee)^m\times(\mathbb{P}^n_{\mathrm{inv}})^{m'}}$ and the sum is over all $i+j+k_1+\cdots+k_m+\ell_1+\cdots+\ell_{m'}=n$.

Proof Similarly as in the proof of Theorem 9.11, the multi-projectivization analogues of Proposition 9.3, Proposition 9.5, and Theorem 9.7 expresses the polynomial as a sum over intersection numbers of nef divisors multiplied to the Chow class of a Lefschetz fan $\Sigma_{E,M_1^\perp,\ldots,M_m^\perp,M_1',\ldots,M_{m'}'}$. Then, one concludes by applying Lemma 9.12. □

10 A K-theory to Chow theory bridge

In this section, we develop a method special to permutohedral varieties that allows us to translate K-theoretic computations to Chow-theoretic computations, as stated in

Theorem D. We apply this method to the tautological K-classes of matroids to bridge the K-theoretic and the Chow-theoretic approach to studying matroid invariants. Our method is a replacement, not a derivative, of the classical Hirzebruch-Riemann-Roch theorem, although the statement looks similar. As before, let $E = \{0, 1, \dots, n\}$.

10.1 A Hirzebruch-Riemann-Roch-type formula

Recall that $\widetilde{\Sigma}_E$ denotes the normal fan in $\mathbb{R}^E$ of the permutohedron, whose quotient fan in $\mathbb{R}^E/\mathbb{R}\mathbf{1}$ is the fan Σ_E of the permutohedral variety X_E. To state our GHRR-type formula for the permutohedral variety X_E, we recall from Theorem 2.1 that we had the identifications $K_0^T(X_E) \simeq PLaur(\widetilde{\Sigma}_E)$ and $A_T^\bullet(X_E) \simeq PPoly(\widetilde{\Sigma}_E)$ where

$$\begin{aligned} PLaur(\widetilde{\Sigma}_E) &= \text{the ring of piecewise Laurent polynomials in variables } T_0, \dots, T_n \\ &\quad\ \text{on the fan } \widetilde{\Sigma}_E, \\ PPoly(\widetilde{\Sigma}_E) &= \text{the ring of piecewise polynomials in variables } t_0, \dots, t_n \\ &\quad\ \text{on the fan } \widetilde{\Sigma}_E. \end{aligned}$$

Let $\chi^T : K_T^0(X_E) \to K_T^0(\mathrm{pt})$ be the T-equivariant pushforward map of K-rings (i.e. the T-equivariant sheaf Euler characteristic), and let $\int^T : A_T^\bullet(X_E) \to A_T^{\bullet-n}(\mathrm{pt})$ be the T-equivariant pushfoward map of Chow rings (i.e. the T-equivariant degree map). We relate these two pushforward maps as follows.

Theorem 10.1 *Denote by $A^\bullet(X_E)[\prod(1+t_i)^{-1}]$ the ring obtained from $A_T^\bullet(X_E)$ by adjoining the inverse of the polynomial $(1+t_0)(1+t_1)\cdots(1+t_n)$. Then, the map $\zeta_{X_E}^T : K_0^T(X_E) \to A_T^\bullet(X_E)[\prod(1+t_i)^{-1}]$ defined by sending*

$$f(T_0, \dots, T_n) \mapsto f(t_0+1, \dots, t_n+1)$$
$$\textit{for } f \textit{ a piecewise Laurent polynomial on } \widetilde{\Sigma}_E$$

is a ring isomorphism, which descends to a ring isomorphism $\zeta_{X_E} : K^0(X_E) \to A^\bullet(X_E)$. Moreover, the following diagrams commute

$$\begin{tikzcd} K_0^T(X_E) \arrow[r, "\zeta_{X_E}^T"] \arrow[d, "\chi^T"'] & A_T^\bullet(X_E)[\prod(1+t_i)^{-1}] \arrow[d, "\int^T (c^T(S^\vee_{U_{n,E}})\cdot -)"] \\ K_0^T(\mathrm{pt}) \arrow[r, "\zeta_{\mathrm{pt}}^T"] & A_T^\bullet(\mathrm{pt})[\prod(1+t_i)^{-1}] \end{tikzcd} \qquad \begin{tikzcd} K^0(X_E) \arrow[r, "\zeta_{X_E}"] \arrow[d, "\chi"'] & A^\bullet(X_E) \arrow[d, "\int((1+\alpha+\cdots+\alpha^n)\cdot -)"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z}. \end{tikzcd}$$

In particular, denoting by $\deg_\alpha : A^\bullet(X_E) \to \mathbb{Z}$ the map $\xi \mapsto \int_{X_E}(1+\alpha+\cdots+\alpha^n)\cdot\xi$, we have for a K-class $[\mathcal{E}] \in K^0(X_E)$ that

$$\chi_{X_E}([\mathcal{E}]) = \deg_\alpha(\zeta_{X_E}[\mathcal{E}]).$$

For the proof of the commutativity of the diagrams, we will need the Atiyah-Bott localization formulas for K-theory and Chow, which we specialize to permutohedral varieties using the identification of the torus action on the tangent spaces to the torus-fixed points of X_E at the end of §2.3.

Theorem 10.2 *Let X_E be the permutohedral variety.*
(*a*) [91, 4.7] *The T-equivariant Euler characteristic χ^T satisfies*

$$\chi^T([\mathcal{E}]) = \sum_{\sigma \in \mathfrak{S}_E} \frac{[\mathcal{E}]_\sigma}{\left(1 - \frac{T_{\sigma(1)}}{T_{\sigma(0)}}\right) \cdots \left(1 - \frac{T_{\sigma(n)}}{T_{\sigma(n-1)}}\right)} \in K_T^0(\mathrm{pt}) = \mathbb{Z}[T_0^{\pm}, \ldots, T_n^{\pm}].$$

(*b*) [42, *Corollary* 1] *The T-equivariant degree map $\int^T$ satisfies*

$$\int^T (\xi) = \sum_{\sigma \in \mathfrak{S}_E} \frac{\xi_\sigma}{(t_{\sigma(0)} - t_{\sigma(1)}) \ldots (t_{\sigma(n-1)} - t_{\sigma(n)})} \in A_T^\bullet(\mathrm{pt}) = \mathbb{Z}[t_0, \ldots, t_n].$$

To obtain the non-equivariant Euler characteristic $\chi([\mathcal{E}])$ (resp. the non-equivariant degree $\int \xi$), one evaluates the respective sum in Theorem 10.2 at $T_0 = \cdots T_n = 1$ (resp. $t_0 = \cdots = t_n = 0$) after simplifying the expression to a Laurent polynomial (resp. a polynomial). Implicit in the theorem is that such simplification always occurs, and that in the case of $\int^T$, if the sum is a rational function of degree less than 0, then the sum simplifies to zero.

Proof of Theorem 10.1 That $\zeta_{X_E}^T$ is a ring isomorphism is clear once we show that the map is well-defined. Let σ and σ' be any maximal cones in $\widetilde{\Sigma}_E$ sharing a codimension 1 face. The linear span of the face is $\{x \in \mathbb{R}^E \mid x_i = x_j\}$ for some $i \neq j \in E$. By definition, an element $f \in PLaur(\widetilde{\Sigma}_E) \simeq K_0^T(X_E)$ satisfies $f_\sigma \equiv f_{\sigma'} \mod (T_i - T_j)$. Since $(t_i + 1) - (t_j + 1) = t_i - t_j$, its image $\zeta_{X_E}^T(f)$ also satisfies $\zeta_{X_E}^T(f)_\sigma \equiv \zeta_{X_E}^T(f)_{\sigma'} \mod (t_i - t_j)$, and hence is a well-defined element in $PPoly(X_E)[\prod(1 + t_i)^{-1}] \simeq A_T^\bullet(X_E)[\prod(1 + t_i)^{-1}]$.

We now show that the isomorphism $\zeta_{X_E}^T$ descends to an isomorphism of the non-equivariant rings. We first recall from Theorem 2.1 that the kernels of the surjections to the non-equivariant rings are

$$\begin{aligned} I_K &= \ker(K_T^0(X_E) \twoheadrightarrow K^0(X_E)) = \begin{array}{c}\text{the ideal generated by } f(T_0, \ldots, T_n) - f(1, \ldots, 1)\\ \text{for } f \text{ a global Laurent polynomial,}\end{array} \\ I_A &= \ker(A_T^\bullet(X_E) \twoheadrightarrow A^\bullet(X_E)) = \begin{array}{c}\text{the ideal generated by } f(t_0, \ldots, t_n) - f(0, \ldots, 0)\\ \text{for } f \text{ a global polynomial.}\end{array} \end{aligned} \tag{1}$$

Since the image of the global polynomial $\prod_{i \in E}(1 + t_i) \in PPoly(\Sigma_E)$ under $A_T^\bullet(X_E) \twoheadrightarrow A_T^\bullet(X_E)/I_A \simeq A^\bullet(X_E)$ is 1, and in particular invertible, the universal property of localization naturally induces a map $A_T^\bullet(X_E)[\prod(1 + t_i)^{-1}] \to A^\bullet(X_E)$. As localization commutes with quotient, we have $A_T^\bullet(X_E)[\prod(1 + t_i)^{-1}]/I_A' = A^\bullet(X_E)$, where I_A' is the ideal $I_A[\prod(1 + t_i)^{-1}]$. We need to show that $\zeta_{X_E}^T(I_K) = I_A'$.

Given a generator of I'_A, say $f(t_0,\dots,t_n) - f(0,\dots 0)$ where $f \in \mathbb{Z}[t_0,\dots,t_n]$ is a polynomial, we have

$$(\zeta^T_{X_E})^{-1}(f(t_0,\dots,t_n) - f(0,\dots 0)) = f(T_0-1,\dots,T_n-1) - f(0,\dots,0) \in I_K$$

since $f(T_0-1,\dots,T_n-1)$ is a Laurent polynomial evaluating to $f(0,\dots,0)$ when $T_0 = \cdots T_n = 1$. Thus, we have $(\zeta^T_{X_E})^{-1}(I'_A) \subset I_K$. Conversely, given a generator of I_K, say $f(T_0,\dots,T_n) - f(1,\dots,1)$ where $f \in \mathbb{Z}[T_0^{\pm},\dots,T_n^{\pm}]$ is a Laurent polynomial, we can write $f(T_0,\dots,T_n) = (\prod_{i\in E} T_i^{-1})^m g(T_0,\dots,T_n)$ for a polynomial $g(T_0,\dots,T_n)$, so that

$$\begin{aligned}
&\zeta^T_{X_E}(f(t_0,\dots,t_n) - f(1,\dots,1)) \\
&\quad = (\textstyle\prod_{i\in E}(1+t_i)^{-1})^m (g(t_0+1,\dots,t_n+1) - g(1,\dots,1)) \in I'_A.
\end{aligned}$$

Since $\prod_{i\in E}(1+t_i)^{-1}$ is a unit and $g(t_0+1,\dots,t_n+1) - g(1,\dots,1)$ is a generator of I'_A, we have $\zeta^T_{X_E}(I_K) = I'_A$. We conclude that $\zeta^T_{X_E}(I_K) = I'_A$, and hence that the isomorphism $\zeta^T_{X_E}$ descends to an isomorphism of the respective quotients by the ideals, yielding $\zeta_{X_E} \colon K^0(X_E) \to A^\bullet(X_E)$.

Finally, we now show that the two diagrams commute. Let $f \in PLaur(\widetilde{\Sigma}_E) \simeq K_0^T(X_E)$. Using Theorem 10.2.(a), we compute that

$$\begin{aligned}
(\zeta^T_{\mathrm{pt}} \circ \chi^T)(f) &= \zeta^T_{\mathrm{pt}} \left(\sum_{\sigma\in\mathfrak{S}_E} \frac{f_\sigma(T_0,\dots,T_n)}{(1-\frac{T_{\sigma(1)}}{T_{\sigma(0)}})\cdots(1-\frac{T_{\sigma(n)}}{T_{\sigma(n-1)}})} \right) \\
&= \sum_{\sigma\in\mathfrak{S}_E} \frac{f_\sigma(t_0+1,\dots,t_n+1)}{\prod_{i=0}^{n-1}(1-\frac{1+t_{\sigma(i+1)}}{1+t_{\sigma(i)}})}.
\end{aligned}$$

Note that for any permutation $\sigma \in \mathfrak{S}_E$, we have

$$\frac{\prod_{i=0}^{n-1}(t_{\sigma(i)} - t_{\sigma(i+1)})}{\prod_{i=0}^{n-1}(1-\frac{1+t_{\sigma(i+1)}}{1+t_{\sigma(i)}})} = \prod_{i=0}^{n-1}(1+t_{\sigma(i)}) = c^T(\mathcal{S}^\vee_{U_{n,E}})_\sigma,$$

where the last equality follows from $[\mathcal{S}^\vee_{U_{n,E}}]_\sigma = \sum_{i=0}^{n-1} T_{\sigma(i)}$. We thus compute further that

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_E} \frac{f_\sigma(t_0+1,\dots,t_n+1)}{\prod_{i=0}^{n-1}(1-\frac{1+t_{\sigma(i+1)}}{1+t_{\sigma(i)}})} &= \sum_{\sigma\in\mathfrak{S}_E} \frac{f_\sigma(t_0+1,\dots,t_n+1)c^T(\mathcal{S}^\vee_{U_{n,E}})_\sigma}{\prod_{i=0}^{n-1}(t_{\sigma(i)} - t_{\sigma(i+1)})} \\
&= \sum_{\sigma\in\mathfrak{S}_E} \frac{(\zeta^T_{X_E} f)_\sigma \cdot c^T(\mathcal{S}^\vee_{U_{n,E}})_\sigma}{\prod_{i=0}^{n-1}(t_{\sigma(i)} - t_{\sigma(i+1)})},
\end{aligned}$$

and by Theorem 10.2.(b) we have

$$\sum_{\sigma\in\mathfrak{S}_E} \frac{(\zeta^T_{X_E} f)_\sigma \cdot c^T(\mathcal{S}^\vee_{U_{n,E}})_\sigma}{\prod_{i=0}^{n-1}(t_{\sigma(i)} - t_{\sigma(i+1)})} = \int^T c^T(\mathcal{S}^\vee_{U_{n,E}}) \cdot \zeta^T_{X_E}(f),$$

showing that the left diagram of the theorem commutes. The commutativity of the right diagram follows since non-equivariantly $c(\mathcal{S}_{U_{n,E}}^{\vee}) = 1 + \alpha + \alpha^2 + \cdots + \alpha^n \in A^{\bullet}(X_E)$, as noted in Example 3.10. □

Remark 10.3 For a smooth projective variety X, the Hirzebruch-Riemann-Roch (HRR) theorem states that there exists an isomorphism ch called the Chern character map

$$\mathrm{ch}\colon K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} A^{\bullet}(X) \otimes \mathbb{Q},$$

and that there exists a Chow class $\mathrm{Td}(X) \in A^{\bullet}(X)_{\mathbb{Q}}$ called the Todd class of X such that the diagram

$$\begin{tikzcd} K_0(X) \otimes \mathbb{Q} \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\chi"'] & A^{\bullet}(X) \otimes \mathbb{Q} \arrow[d, "\int_X (\mathrm{Td}(X) \cdot -)"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

commutes, where χ denotes the Euler characteristic. See [44, Chap. 14] for an account of this. We note that for the permutohedral variety X_E, the map ζ_{X_E} in Theorem 10.1 is different from ch, and the class $1 + \alpha + \cdots + \alpha^n \in A^{\bullet}(X_E)$ is not equal to the Todd class of X_E. While the HRR theorem is a standard tool for translating between K-theory computations and Chow ring computations, sufficiently explicit descriptions of Todd classes, even in the case of permutohedral varieties, are often not available. See [27] for a study of Todd classes of permutohedral varieties in low dimensions.

The map ζ_{X_E} of Theorem 10.1 behaves particularly well for a family of K-classes defined as follows.

Definition 10.4 We say that a T-equivariant K-class $[\mathcal{E}] \in K_T^0(X_E)$ **has simple Chern roots** if for each permutation σ there is a sequence of non-negative integers $\mathbf{m}_\sigma = (m_{\sigma,0}, \ldots, m_{\sigma,n}) \in \mathbb{Z}_{\geq 0}^E$ such that

$$[\mathcal{E}]_\sigma = \mathbf{m}_\sigma \cdot \mathbf{T} = \sum_{i=1}^{n} m_{\sigma,i} T_i \in \mathbb{Z}[T_0^{\pm}, \ldots, T_n^{\pm}].$$

For instance, for any matroid M the K-classes $[\mathcal{S}_M^{\vee}]$ and $[\mathcal{Q}_M^{\vee}]$ have simple Chern roots. One can verify that the property of having simple Chern roots is closed under virtual extensions, subbundles, and quotients of T-equivariant K-classes, but we will not need this fact. We now show how ζ_{X_E} behaves for K-classes with simple Chern roots.

Proposition 10.5 *Let $[\mathcal{E}] \in K_T^0(X_E)$ have simple Chern roots. Then, with u a formal variable, we have*

$$\sum_{i \geq 0} \zeta_{X_E}([\textstyle\bigwedge^i \mathcal{E}]) u^i = (u+1)^{\mathrm{rk}(\mathcal{E})} c(\mathcal{E}, \tfrac{u}{u+1}), \quad \textit{and}$$

$$\sum_{i\geq 0}\zeta_{X_E}([\textstyle\bigwedge^i \mathcal{E}^\vee])u^i = (u+1)^{\mathrm{rk}(\mathcal{E})}c(\mathcal{E})^{-1}c(\mathcal{E}, \tfrac{1}{u+1}).$$

Equivalently, we have

$$\sum_{i\geq 0}\zeta_{X_E}([\textstyle\bigwedge^{\mathrm{rk}(\mathcal{E})-i} \mathcal{E}])u^i = (u+1)^{\mathrm{rk}(\mathcal{E})}c(\mathcal{E}, \tfrac{1}{u+1}), \quad \textit{and}$$

$$\sum_{i\geq 0}\zeta_{X_E}([\textstyle\bigwedge^{\mathrm{rk}(\mathcal{E})-i} \mathcal{E}^\vee])u^i = (u+1)^{\mathrm{rk}(\mathcal{E})}c(\mathcal{E})^{-1}c(\mathcal{E}, \tfrac{u}{u+1}).$$

Proof We prove more strongly the T-equivariant versions of the statements in terms of the isomorphism $\zeta^T_{X_E}: K^0_T(X_E) \xrightarrow{\sim} A^\bullet_T(X_E)[\prod(1+t_i)^{-1}]$. It follows from the definition that, for each permutation $\sigma \in \mathfrak{S}_E$, there is a multi-subset I_σ of E with size $|I_\sigma| = \mathrm{rk}(\mathcal{E})$ such that $[\mathcal{E}]_\sigma = \sum_{i\in I_\sigma} T_i$ and $[\mathcal{E}^\vee]_\sigma = \sum_{i\in I_\sigma} T_i^{-1}$. We now compute the exterior powers and the Chern classes as in §2.5. We have $\left(\sum_{i\geq 0}[\bigwedge^i \mathcal{E}]u^i\right)_\sigma = \prod_{i\in I_\sigma}(1+uT_i)$ and likewise $\left(\sum_{i\geq 0}[\bigwedge^i \mathcal{E}^\vee]u^i\right)_\sigma = \prod_{i\in I_\sigma}(1+uT_i^{-1})$ for every permutation $\sigma \in \mathfrak{S}_E$, and thus

$$\begin{aligned}\left(\zeta^T_{X_E}\sum_{i\geq 0}([\textstyle\bigwedge^i \mathcal{E}])u^i\right)_\sigma &= \prod_{i\in I_\sigma}(1+u(1+t_i)) = (u+1)^{|I_\sigma|}\prod_{i\in I_\sigma}(1+t_i\tfrac{u}{u+1})\\ &= (u+1)^{\mathrm{rk}(E)}c^T(\mathcal{E}, \tfrac{u}{u+1})_\sigma,\end{aligned}$$

and

$$\begin{aligned}\left(\zeta^T_{X_E}\sum_{i\geq 0}([\textstyle\bigwedge^i \mathcal{E}^\vee])u^i\right)_\sigma &= \prod_{i\in I_\sigma}(1+u(1+t_i)^{-1}) = \frac{(u+1)^{|I_\sigma|}}{\prod_{i\in I_\sigma}(1+t_i)}\prod_{i\in I_\sigma}(1+t_i\tfrac{1}{u+1})\\ &= (u+1)^{\mathrm{rk}(\mathcal{E})}c^T(\mathcal{E})_\sigma^{-1}c^T(\mathcal{E}, \tfrac{1}{u+1})_\sigma,\end{aligned}$$

as desired. □

We note the following characterizing property of the isomorphism ζ_E of Theorem 10.1.

Corollary 10.6 *The ring isomorphism $\zeta_{X_E}: K(X_E) \to A^\bullet(X_E)$ is the unique ring homomorphism such that for any realization $L \subseteq \mathbb{C}^E$ of a matroid, the K-class $[\mathcal{O}_{W_L}]$ of the structure sheaf of the wonderful compactification is sent to the Chow class $[W_L]$.*

Proof That W_L is the vanishing loci of a section $\mathcal{O}_{X_E} \to \mathcal{Q}_L$ (Theorem 7.10) implies that we have the Koszul resolution

$$0 \to \det \mathcal{Q}_L^\vee \to \cdots \to \textstyle\bigwedge^2 \mathcal{Q}_L^\vee \to \mathcal{Q}_L^\vee \to \mathcal{O}_{X_E} \to \mathcal{O}_{W_L} \to 0,$$

and hence $[\mathcal{O}_{W_L}] = \sum_{i\geq 0}(-1)^i[\bigwedge^i \mathcal{Q}_L^\vee]$. Applying Proposition 10.5 then yields $\zeta_{X_E}([\mathcal{O}_{W_L}]) = [W_L]$. That this property characterizes ζ_{X_E} follows from [63, Propositions 2.32 and 5.13], which showed that Bergman classes of realizable loopless matroids, in fact those of loopless Schubert matroids, span $A^\bullet(X_E)$. □

Remark 10.7 Let $P \subset \mathbb{R}^E_{\geq 0}$ be a generalized permutohedron (defined in §2.7) that is contained in the nonnegative orthant. Such P defines a T-equivariant K-class $[\mathcal{E}_P]$ with simple Chern roots by

$$[\mathcal{E}_P]_\sigma = \mathbf{m}_\sigma \cdot \mathbf{T} \quad \text{for } \sigma \in \mathfrak{S}_E$$

where $\mathbf{m}_\sigma$ is the vertex of P maximizing the pairing $\langle -, v_0\mathbf{e}_{\sigma(0)} + \cdots + v_n\mathbf{e}_{\sigma(n)}\rangle$ for any $v_0 > \cdots > v_n$. Theorem 2.1.(a) implies that $[\mathcal{E}_P]$ is a well-defined element of $K^0_T(X_E)$ because each edge of such P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$, since the normal fan of P coarsens $\widetilde{\Sigma}_E$. For example, if M is a matroid we have $[\mathcal{S}^\vee_M] = [\mathcal{E}_{P(M)}]$. This suggests that many results in this section may generalize from matroids to generalized permutohedra contained in the nonnegative orthant, often called discrete polymatroids. A particular family of discrete polymatroids known as flag matroids are studied in §11.

10.2 Fink-Speyer's K-theoretic interpretation of Tutte polynomials

The authors of [50] expressed the Tutte polynomial of a matroid via the K-theory of the Grassmannian. To state their theorem, we need the following definition.

Definition 10.8 For a matroid M of rank r, let $y(M) \in K^0(\mathrm{Gr}(r; E))$ be the K-class determined by the following two properties:

(i) If $L \subseteq \mathbb{C}^E$ is a realization of M, then $y(M) = [\mathcal{O}_{\overline{T \cdot L}}]$, the K-class of the structure sheaf of the torus-orbit-closure in the Grassmannian, and
(ii) the assignment $M \mapsto y(M)$ is valuative.

The class $y(M)$ is well-defined because (i) for a realization $L \subseteq \mathbb{C}^E$ of a matroid M the K-class $[\mathcal{O}_{\overline{T \cdot L}}]$ only depends on the matroid M, and (ii) the assignment for realizable matroids $M \mapsto [\mathcal{O}_{\overline{T \cdot L}}]$ is valuative. For a proof see [103, Proposition A.5] and the remark following it. Our definition of $y(M)$ here agrees with the definition of $y(M)$ given via the T-equivariant K-theory of the Grassmannian in [50] because both satisfy the defining properties (i) and (ii) above.

Fink and Speyer established the following K-theoretic interpretation of Tutte polynomials in [50, Theorem 5.2]. Here, we show that applying the HRR-type formula (Theorem 10.1) to our unifying Tutte formula Theorem A recovers this result. Recall the notation that $\mathcal{S}$ and $\mathcal{Q}$ are the tautological subbundle and the quotient bundle, respectively, on the Grassmannian $\mathrm{Gr}(r; E)$.

Theorem 10.9 *Let M be a matroid with ground set E, and $T_M(u, v)$ its Tutte polynomial. Then,*

$$T_M(u, v) = \sum_{i=0}^{r} \sum_{j=0}^{|E|-r} \chi_{\mathrm{Gr}(r;E)}\big(y(M)[\det \mathcal{S}^\vee][\textstyle\bigwedge^i \mathcal{S}][\bigwedge^j \mathcal{Q}^\vee]\big)(u-1)^i(v-1)^j.$$

We prepare the proof with a lemma, which allows one to translate certain Euler characteristic computations on Grassmannians to those on permutohedral varieties.

Lemma 10.10 *Let M be a matroid of rank r with ground set E, and let $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$. Then, for the class $[\mathcal{E}_M] \in K_T^0(X_E)$ defined in Proposition 3.13, we have an equality of non-equivariant Euler characteristics*

$$\chi_{\mathrm{Gr}(r;E)}\big(y(M)[\mathcal{E}]\big) = \chi_{X_E}\big([\mathcal{E}_M]\big).$$

Proof For a fixed $[\mathcal{E}] \in K_T^0(\mathrm{Gr}(r; E))$, we note that the assignment $[\mathcal{E}] \mapsto [\mathcal{E}_M]$ is valuative by Proposition 5.8. Since $M \mapsto y(M)$ is also valuative, by Lemma 5.9 it suffices to verify the desired equality for M with a realization $L \subseteq \mathbb{C}^E$. The projection formula yields

$$\chi_{\mathrm{Gr}(r;E)}\big(y(M)[\mathcal{E}]\big) = \chi_{\mathrm{Gr}(r;E)}\big(\mathcal{O}_{\overline{T\cdot L}} \cdot [\mathcal{E}]\big) = \chi_{\overline{T\cdot L}}\big([\mathcal{E}]|_{\overline{T\cdot L}}\big).$$

As the normal fan of the base polytope $P(M)$ coarsens the fan Σ_E, the induced map of toric varieties $\psi\colon X_E \to X_{P(M)} \simeq \overline{T\cdot L}$ satisfies $\psi_*\psi^*[\mathcal{O}_{\overline{T\cdot L}}] = [\mathcal{O}_{\overline{T\cdot L}}]$ by [31, Theorem 9.2.5], so using the projection formula and Proposition 3.13 yields

$$\chi_{\overline{T\cdot L}}\big([\mathcal{E}]|_{\overline{T\cdot L}}\big) = \chi_{X_E}\big(\varphi_L^*[\mathcal{E}]\big) = \chi_{X_E}\big(\mathrm{crem}\,\varphi_L^*[\mathcal{E}]\big) = \chi_{X_E}\big([\mathcal{E}_M]\big)$$

as desired. $\square$

Proof of Theorem 10.9 By Lemma 10.10, we have

$$\sum_{i=0}^{r}\sum_{j=0}^{|E|-r} \chi_{\mathrm{Gr}(r;E)}\Big(y(M)[\det \mathcal{S}^\vee][\textstyle\bigwedge^i \mathcal{S}][\bigwedge^j \mathcal{Q}^\vee]\Big)(u-1)^i(v-1)^j =$$

$$\sum_{i=0}^{r}\sum_{j=0}^{|E|-r} \chi_{X_E}\Big([\det \mathcal{S}_M^\vee][\textstyle\bigwedge^i \mathcal{S}_M][\bigwedge^j \mathcal{Q}_M^\vee]\Big)(u-1)^i(v-1)^j,$$

which by Theorem 10.1 equals

$$\deg_\alpha\Big(\sum_{i,j} \zeta_{X_E}\big([\det \mathcal{S}_M^\vee][\textstyle\bigwedge^i \mathcal{S}_M][\bigwedge^j \mathcal{Q}_M^\vee]\big)(u-1)^i(v-1)^j\Big),$$

where we recall the notation that $\deg_\alpha(-) = \int_{X_E}(1+\alpha+\cdots+\alpha^n)\cdot(-)$. We now claim that

$$\sum_{i,j} \zeta_{X_E}([\det \mathcal{S}_M^\vee][\textstyle\bigwedge^i \mathcal{S}_M][\bigwedge^j \mathcal{Q}_M^\vee])(u-1)^i(v-1)^j$$
$$= u^r v^{|E|-r} c(\mathcal{S}_M^\vee, u^{-1})c(\mathcal{Q}_M^\vee, 1-v^{-1}).$$

Noting $[\det \mathcal{S}_M^\vee][\bigwedge^i \mathcal{S}_M] = [\bigwedge^{r-i} \mathcal{S}_M^\vee]$, we write

$$\sum_{i,j}[\det \mathcal{S}_M^\vee][\textstyle\bigwedge^i \mathcal{S}_M][\bigwedge^j \mathcal{Q}_M](u-1)^i(v-1)^j$$

$$= \Big(\sum_i [\textstyle\bigwedge^{r-i} \mathcal{S}_M^\vee](u-1)^i\Big)\Big(\sum_j [\textstyle\bigwedge^{i} \mathcal{Q}_M^\vee](v-1)^j\Big).$$

Because ζ_{X_E} is a ring homomorphism, and $\mathcal{S}_M^\vee$, $\mathcal{Q}_M^\vee$ have simple Chern roots, applying Proposition 10.5 thus verifies our claim. Now, specializing the equality for the polynomial $t_M(x, y, z, w)$ in Theorem A at $x = 1$ and $y = 0$, we obtain

$$\begin{aligned}\deg_\alpha(c_k(\mathcal{S}_M^\vee, z)c_\ell(\mathcal{Q}_M, w)) &= \sum_{i+k+\ell=n} \Big(\int_{X_E} \alpha^i c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M)\Big) z^k w^\ell \\ &= z^r(1+w)^{|E|-r} T_M(\tfrac{1}{z}, \tfrac{1}{1+w}).\end{aligned}$$

Setting $u = z^{-1}$ and $v = (1+w)^{-1}$ in this final formula then yields

$$u^r v^{|E|-r} \deg_\alpha(c(\mathcal{S}_M^\vee, u^{-1})c(\mathcal{Q}_M^\vee, 1 - v^{-1})) = T_M(u, v),$$

and we thus conclude the desired formula for $T_M(u, v)$. $\square$

10.3 Cameron-Fink's lattice-point-counting interpretation of Tutte polynomials

Let M be a matroid with ground set E. By using the relationship between toric geometry and Ehrhart-style lattice point counting [31, Ch. 9], and by recalling the fact that $[\mathcal{O}(D_{-P(M)})] = [\det \mathcal{S}_M^\vee]$ from Example 3.11, we have an equality of polynomials in $\mathbb{Q}[t, u]$

$$\begin{aligned}\chi_{X_E}(\mathcal{O}(t\alpha + u\beta)[\det \mathcal{S}_M^\vee]) &= \text{the number of lattice points inside } -P(M) + t\Delta + u\nabla \\ &= \text{the number of lattice points inside } P(M) + t\nabla + u\Delta,\end{aligned}$$

where $\Delta = \mathrm{Conv}(\mathbf{e}_i \mid i \in E)$ and $\nabla = -\Delta$ are the standard simplex and the negative standard simplex in $\mathbb{R}^E$, respectively. The authors of [25] denoted this polynomial $Q_M(t, u)$. With $\Psi\colon \mathbb{Q}[t, u] \to \mathbb{Q}[x, y]$ defined as the invertible linear map sending $\binom{t}{i}\binom{u}{j} \mapsto x^i y^j$ for all $i, j \geq 0$, [25, Theorem 3.2] expressed the Tutte polynomial of M in terms of the polynomial $Q'_M(x, y) \in \mathbb{Q}[x, y]$ defined by

$$Q'_M(x+1, y+1) = \Psi\big(Q_M(t, u)\big).$$

We show that applying our HRR-type formula (Theorem 10.1) to Theorem A recovers [25, Theorem 3.2]. Combined with Theorem 10.9, which was also obtained by applying Theorem 10.1 to Theorem A, this answers a conjecture of Cameron and Fink (see the discussion after [25, Theorem 3.4]) on the relationship between their expression for the Tutte polynomial of M and the K-theoretic computations in [50].

Theorem 10.11 *Let M be a matroid of rank r with ground set E, and $Q'_M(x, y) \in \mathbb{Q}[x, y]$ be the polynomial as defined above. Then we have*

$$Q'_M(x+1, y+1) = (x+y+1)^{-1}(y+1)^r(x+1)^{|E|-r} T_M\Big(\frac{x+y+1}{y+1}, \frac{x+y+1}{x+1}\Big).$$

Equivalently, letting t_M be the polynomial in Theorem A, we have $Q'_M(x+1, y+1) = t_M(x+1, y, 1, 0)$.

Proof Recall from Example 3.10 that $[\mathcal{O}(\alpha)] = [\mathcal{Q}_{U_{n,E}}]$ and $[\mathcal{O}(\beta)] = [\mathcal{S}^\vee_{U_{1,E}}]$. Setting $u = 0$ in the last two lines of Proposition 10.5, one has that if $[\mathcal{E}]$ has simple Chern roots, then $\zeta_{X_E}([\det \mathcal{E}]) = c(\mathcal{E})$ and $\zeta_{X_E}([\det \mathcal{E}^\vee]) = c(\mathcal{E})^{-1}$. In particular, since $\mathcal{S}^\vee_M$ has simple Chern roots, we have that $\zeta_{X_E}([\det \mathcal{S}^\vee_M]) = c(\mathcal{S}^\vee_M)$, and since $\mathcal{Q}^\vee_{U_{n,E}}$ and $\mathcal{S}^\vee_{U_{1,E}}$ have rank 1 (so taking det makes no change) and have simple Chern roots, we have that

$$\zeta_{X_E}([\mathcal{O}(\alpha)]) = c(\mathcal{Q}^\vee_{U_{n,E}})^{-1} = (1-\alpha)^{-1} \quad \text{and}$$
$$\zeta_{X_E}([\mathcal{O}(\beta)]) = c(\mathcal{S}^\vee_{U_{1,E}}) = (1+\beta).$$

Theorem 10.1 thus implies that

$$\begin{aligned} Q_M(t,u) &= \chi(\mathcal{O}(t\alpha + u\beta)[\det \mathcal{S}^\vee_M]) \\ &= \deg_\alpha((1-\alpha)^{-t}(\beta+1)^u c(\mathcal{S}^\vee_M)) = \int_{X_E} (1-\alpha)^{-t-1}(1+\beta)^u c(\mathcal{S}^\vee_M). \end{aligned}$$

Before applying the linear map $\Psi\colon \mathbb{Q}[t,u] \to \mathbb{Q}[x,y]$, we note the following observations. First, we have $(1-\alpha)^{-t-1} = (1+\alpha+\alpha^2+\cdots)^{t+1} = \sum_{i\geq 0} \alpha^i \binom{t+i}{i}$. Moreover, for each $i \geq 0$, the Vandermonde identity $\sum_{k=0}^{i} \binom{i}{i-k}\binom{t}{k} = \binom{t+i}{i}$ implies that $\Psi\left(\binom{t+i}{i}\right) = (x+1)^i$. Lastly, we have $(1+\beta)^u = \sum_{j\geq 0} \beta^j \binom{u}{j}$. Thus, we conclude that

$$\begin{aligned} Q'_M(x+1, y+1) = \Psi(Q_M(t,u)) &= \Psi\left(\int_{X_E} (1-\alpha)^{-t-1}(1+\beta)^u c(\mathcal{S}^\vee_M) \right) \\ &= \int_{X_E} \Big(\sum_{i\geq 0} \alpha^i (x+1)^i\Big)\Big(\sum_{j\geq 0} \beta^j y^j\Big) c(\mathcal{S}^\vee_M) \\ &= t_M(x+1, y, 1, 0) \end{aligned}$$

where the last equality follows from Theorem A. $\square$

10.4 Ehrhart and volume polynomials of generalized permutohedra

It was noted in [95, Theorem 11.3] that the formula for the number of lattice points of a generalized permutohedra P is obtained from the formula [95, Theorem 9.3] for the volume of the Minkowski sum $P + \Delta$ by "replacing powers with raised powers." Let us explain here how this phenomenon is another consequence of our Theorem D.

For a nonempty subset $S \subseteq E$, let $\Delta_S = \mathrm{Conv}(\mathbf{e}_i \mid i \in S) \subset \mathbb{R}^E$ be the S-standard simplex, which defines a divisor class $[D_{\Delta_S}] \in A^1(X_E)$ as denoted in §2.7. It is known that the set of divisor classes $\{[D_{\Delta_S}] \mid \emptyset \subsetneq S \subseteq E\}$ form a basis of $A^1(X_E)$; see for instance [14, §3.3]. In other words, for any generalized permutohedron $P \subset \mathbb{R}^E$, there exist integers y_S for $\emptyset \subsetneq S \subseteq E$ such that

$$[D_P] = (y_E - 1)[D_{\Delta_E}] + \sum_{\emptyset \subsetneq S \subsetneq E} y_S [D_{\Delta_S}].$$

We will soon see the convenience of using $(y_E - 1)$ instead of y_E for the coefficient of $[D_{\Delta_E}]$. Since $\chi(\mathcal{O}(D_P)) =$ (the number of lattice points inside P), we wish to compute

$$\chi\Big(\mathcal{O}(D_{\Delta_E})^{\otimes(y_E-1)} \otimes \bigotimes_{\emptyset \subsetneq S \subsetneq E} \mathcal{O}(D_{\Delta_S})^{\otimes y_S}\Big).$$

To do so, let us define a matroid H_S for each nonempty subset $S \subseteq E$ by $H_S = U_{|S|-1,S} \oplus U_{|E\setminus S|,E\setminus S}$. Since H_S has corank 1, similar computations as in Example 3.10 shows that $[\mathcal{Q}_{H_S}] = [\mathcal{O}(D_{\Delta_S})]$. Then, by Proposition 10.5 we have $\zeta_{X_E}([\mathcal{O}(D_{\Delta_S})]) = c(\mathcal{Q}_{H_S}^\vee)^{-1} = (1 - [D_{\Delta_S}])^{-1}$. Thus, applying Theorem 10.1 yields

$$\begin{aligned}
\chi\big(\mathcal{O}(D_P)\big) &= \deg_\alpha \Big((1-[D_{\Delta_E}])^{-y_E+1} \prod_{\emptyset \subsetneq S \subsetneq E} (1-[D_{\Delta_S}])^{-y_S}\Big) \\
&= \int_{X_E} (1-[D_{\Delta_E}])^{-y_E} \prod_{\emptyset \subsetneq S \subsetneq E} (1-[D_{\Delta_S}])^{-y_S} \\
&= \int_{X_E} \prod_{\emptyset \subsetneq S \subseteq E} (1-[D_{\Delta_S}])^{-y_S}
\end{aligned}$$

where for the second to last equality we noted $\alpha = [D_{\Delta_E}]$ so that $(1+\alpha+\cdots+\alpha^n) = (1-[D_{\Delta_E}])^{-1}$. Let us define a map $\widehat{\Psi}$, related to but different from the previous map Ψ, by

$$\begin{aligned}
&\widehat{\Psi}\colon \mathbb{Q}[y_S \mid \emptyset \subsetneq S \subseteq E] \to \mathbb{Q}[\hat{y}_S \mid \emptyset \subsetneq S \subseteq E] \\
&\text{where} \quad \prod_S \binom{y_S + i_S - 1}{i_S} \mapsto \prod_S \hat{y}_S^{i_S}.
\end{aligned}$$

Since $\binom{-y}{i} = (-1)^i \binom{y+i-1}{i}$, the map $\widehat{\Psi}$ is related to Ψ by $\widehat{\Psi} = \Psi \circ \operatorname{neg}$ where $\operatorname{neg}\colon \mathbb{Q}[y_S \mid \emptyset \subsetneq S \subseteq E] \to \mathbb{Q}[y_S \mid \emptyset \subsetneq S \subseteq E]$ is the involution defined by $\prod_S \binom{y_S}{i_S} \mapsto \prod_S (-1)^{i_S}\binom{-y_S}{i_S}$. Now, for a nilpotent ring element D, one has an identity $(1-D)^{-y} = \sum_{i\ge 0}(-1)^i\binom{-y}{i}D^i$ as polynomials in y. Thus, we have that

$$\widehat{\Psi}\left(\int_{X_E} \prod_{\emptyset \subsetneq S \subseteq E} (1-[D_{\Delta_S}])^{-y_S}\right) = \int_{X_E} \prod_{\emptyset \subsetneq S \subseteq E} \Big(\sum_{i\ge 0} [D_{\Delta_S}]^i \hat{y}_S^i\Big).$$

Let $V = \int_{X_E} \prod_{\emptyset \subsetneq S \subseteq E} \left(\sum_{i \geq 0} [D_{\Delta_S}]^i \hat{y}_S^i \right) \in \mathbb{Q}[\hat{y}_S \mid \emptyset \subsetneq S \subseteq E]$ be the polynomial above. The normalization (as defined in §9.4) of the polynomial V is

$$N(V) = \int_{X_E} \prod_{\emptyset \subsetneq S \subseteq E} \left(\sum_{i \geq 0} [D_{\Delta_S}]^i \frac{\hat{y}_S^i}{i!} \right) = \frac{1}{n!} \int_{X_E} \Big(\sum_{\emptyset \subsetneq S \subseteq E} \hat{y}_S [D_{\Delta_S}] \Big)^n,$$

which is the Euclidean volume of the polytope $P + \Delta_E$ when one sets $\hat{y}_S = y_S$ for all $S \subseteq E$. A formula for $N(V)$ is given in [95, Theorem 9.3]. See [14, Theorem 5.2.4] for a generalization and a proof via matroid theory. The operation of "replacing powers with raising powers" defined in [95] sends $\hat{y}^a/a!$ to $\binom{y+a-1}{a}$, which is the composition $\widehat{\Psi}^{-1} \circ N^{-1}$. Thus, we have recovered [95, Theorem 11.3], which stated that applying the operation of "replacing powers with raising powers" to the Euclidean volume polynomial $N(V)$ gives the polynomial that measures the number of lattice points of P.

10.5 Speyer's g-polynomial of a matroid

Speyer defined the g-polynomial of a (loopless and coloopless) $\mathbb{C}$-realizable matroid in [103, §3] via the K-theory of Grassmannians, and used the invariant to give bounds on the complexity of matroid polytope subdivisions. The g-polynomial of an arbitrary (loopless and coloopless) matroid was later defined in [50, Remark 6.4], and it remains open whether the positivity property of the g-polynomial for $\mathbb{C}$-realizable matroids persists for arbitrary matroids.

We give a Chow-theoretic formula for the g-polynomial. Our formula below proves [84, Conjecture 1] (and corrects for the missing global sign depending on the number of connected components).

Theorem 10.12 *Let M be a loopless and coloopless matroid of rank r on ground set E. Let* $\operatorname{comp}(M)$ *be the number of connected components of M. Then we have*

$$g_M(s) = (-1)^{\operatorname{comp}(M)} \sum_{i \geq 0} \deg_\alpha \Big(c(\mathcal{Q}_M^\vee) c_{r-i}(\mathcal{S}_M^\vee) c_{|E|-r}(\mathcal{Q}_M) \Big) (-s)^i.$$

Proof [50, Theorem 6.5] states that $(-1)^{\operatorname{comp}(M)} g_M(-s) = H_M(s)$ where

$$H_M(x + y - xy) = \sum_{i=0}^{r} \sum_{j=0}^{|E|-r} \chi_{Gr(r;E)} \Big(y(M) [\textstyle\bigwedge^i \mathcal{S}][\bigwedge^j \mathcal{Q}^\vee] \Big) (x-1)^i (y-1)^j.$$

Combining Lemma 10.10, Theorem 10.1, and Proposition 10.5 yields

$$\sum_{i=0}^{r} \sum_{j=0}^{|E|-r} \chi_{Gr(r;E)} \Big(y(M) [\textstyle\bigwedge^i \mathcal{S}][\bigwedge^j \mathcal{Q}^\vee] \Big) (x-1)^i (y-1)^j =$$

$$\sum_{i=0}^{r} \sum_{j=0}^{|E|-r} \deg_\alpha \Big(c(\mathcal{Q}_M^\vee) c_i(\mathcal{S}_M^\vee) c_j(\mathcal{Q}_M^\vee) \Big) x^{r-i} y^{|E|-r-j} (y-1)^j. \quad (2)$$

Here we have used that $c(\mathcal{Q}_M^\vee) = c(\mathcal{S}_M^\vee)^{-1}$, which follows from $[\mathcal{S}_M^\vee] + [\mathcal{Q}_M^\vee] = [\underline{\mathbb{C}}^E]$.

Now, note that the coefficient of s^i in the polynomial $H_M(s)$ is the same as the coefficient of x^i in $H_M(x+y-xy)$. To get the coefficient of x^i in the right-hand-side of (2), we set $y = 0$ and obtain the coefficient of x^i to be

$$(-1)^{|E|-r} \deg_\alpha \left(c(\mathcal{Q}_M^\vee) c_{r-i}(\mathcal{S}_M^\vee) c_{|E|-r}(\mathcal{Q}_M^\vee) \right)$$
$$= \deg_\alpha \left(c(\mathcal{Q}_M^\vee) c_{r-i}(\mathcal{S}_M^\vee) c_{|E|-r}(\mathcal{Q}_M) \right).$$

The desired formula for $g_M(s)$ follows. □

A similar formula holds for the generalization of the g-polynomial of a matroid to that of a matroid morphism (see [39, §7.1]), but we don't include the details here.

11 Flag matroids

Flag matroids generalize matroids, just as partial flag varieties generalize Grassmannians. Tutte polynomials of matroids were generalized to those of flag matroids via the K-theory of partial flag varieties in [26] and [39]. In this section, use the tools developed from our framework of tautological K-classes of matroids to give a Chow-theoretic formula for such generalizations of Tutte polynomials. As a result, we answer [39, Conjecture 7.7], and establish a log-concavity property for characteristic polynomials of flag matroids, positively answering [26, Conjecture 9.4].

11.1 Flag matroids

We review flag matroids, and extend few results in Sect. 10.2 about the K-theory of Grassmannians to that of partial flag varieties. We omit the proofs as they only require minor changes from proofs of analogous statements for matroids.

Definition 11.1 A **flag matroid** of rank $\boldsymbol{r} = (r_1, \dots, r_k)$ is a sequence $\boldsymbol{M} = (M_1, \dots, M_k)$ of matroids of ranks $\boldsymbol{r}$ on a common ground set E such that for all $i = 1, \dots, k-1$, any flat of M_i is a flat of M_{i+1}. A **realization** of a flag matroid $\boldsymbol{M}$ is a flag $\boldsymbol{L}\colon L_1 \subseteq \cdots \subseteq L_k \subseteq \mathbb{C}^E$ of linear subspaces such that L_i is a realization of M_i for all $i = 1, \dots, k$.

More generally, replacing partial flag varieties by generalized flag varieties of arbitrary finite Coxeter type gives rise to Coxeter matroids, introduced in [56, 57]. See [16] for a treatment. Flag matroids are the Coxeter matroids of type A in this framework. When the flag matroid has only two constituents (M_1, M_2), it is often called a **morphism of matroids**, denoted $M_1 \twoheadleftarrow M_2$. See [46] and references therein for a slightly more general definition of morphism of matroids.

The relation between matroids and the K-theory of Grassmannians generalize in the following way. See [26] for a survey with proofs. For a sequence of nonnegative integers $\boldsymbol{r}: 0 \le r_1 \le \cdots \le r_k \le |E|$, let $\mathrm{Fl}(\boldsymbol{r}; E)$ be the partial flag variety consisting of flags of linear subspaces of the respective dimensions in $\mathbb{C}^E$. The torus T acts on $\mathrm{Fl}(\boldsymbol{r}; E)$ via its standard action on $\mathbb{C}^E$. For a realization $\boldsymbol{L} \in \mathrm{Fl}(\boldsymbol{r}; E)$ of a flag matroid $\boldsymbol{M}$, we have that the torus-orbit-closure $\overline{T \cdot \boldsymbol{L}}$ is isomorphic to $X_{P(\boldsymbol{M})}$, where $P(\boldsymbol{M})$ is the **base polytope** of the flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$ defined as the Minkowski sum $\sum_{i=1}^k P(M_i)$ of the base polytopes of its constituent matroids. One has a commuting diagram

$$\begin{tikzcd} X_E \arrow[r] \arrow[dr, "\varphi_{\boldsymbol{L}}"'] & X_{P(\boldsymbol{M})} \simeq \overline{T \cdot \boldsymbol{L}} \arrow[r] \arrow[d, hook] & \prod_{i=1}^k X_{P(M_i)} \simeq \prod_{i=1}^k \overline{T \cdot L_i} \arrow[d, hook] \\ & \mathrm{Fl}(\boldsymbol{r}; E) \arrow[r, "\prod_i \pi_i"] & \prod_{i=1}^k \mathrm{Gr}(r_i; E). \end{tikzcd}$$

Thus, for a class $[\mathcal{E}^{(i)}] \in K_T^0(\mathrm{Gr}(r_i; E))$, the class $[\mathcal{E}^{(i)}_{M_i}] \in K_T^0(X_E)$ defined via Proposition 3.13 coincides with the class $\mathrm{crem}(\varphi_{\boldsymbol{L}} \circ \pi_i)^*[\mathcal{E}^{(i)}]$. The notion of valuativity generalizes to flag matroids. See [47] for an in-depth study of valuativity for Coxeter matroids in general, and see [19] for a study of subdivisions of base polytopes of flag matroids.

Similar to the class $y(M)$ of a matroid M in Definition 10.8, one can also define a K-class $y(\boldsymbol{M})$ in $K^0(\mathrm{Fl}(\boldsymbol{r}; E))$ of a flag matroid $\boldsymbol{M}$ by the following two determining properties: (i) If $\boldsymbol{L}$ is a realization of $\boldsymbol{M}$ then $y(\boldsymbol{M}) = [\mathcal{O}_{\overline{T \cdot \boldsymbol{L}}}]$, and (ii) the assignment $\boldsymbol{M} \mapsto y(\boldsymbol{M})$ is valuative. Its well-definedness follows from [26, Equation (8.7)] and [39, Remark 2.11]. See [26, Definition 8.19] for a definition via the T-equivariant K-theory of $\mathrm{Fl}(\boldsymbol{r}; E)$. The class $y(\boldsymbol{M})$ satisfies the following analogue of Lemma 10.10, whose proof is similar.[5]

Lemma 11.2 *Let $[\mathcal{E}^{(i)}]$ be a class in $K_T^0(\mathrm{Gr}(r_i; E))$ for each $i = 1, \dots, k$, and let $[\mathcal{E}^{(i)}_{M_i}] \in K_T^0(X_E)$ be as defined in Proposition 3.13. For a flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$ on E with ranks $\boldsymbol{r}$, we have*

$$\chi_{\mathrm{Fl}(\boldsymbol{r}; E)}\Big(y(\boldsymbol{M}) \cdot \prod_{i=1}^k \pi_i^*[\mathcal{E}^{(i)}]\Big) = \chi_{X_E}\Big(\prod_{i=1}^k [\mathcal{E}^{(i)}_{M_i}]\Big).$$

11.2 Flag-geometric Tutte polynomials of flag matroids

Generalizing the K-theoretic interpretation of Tutte polynomials of matroids in [50], the authors of [26] and [39] defined the **flag-geometric Tutte polynomials** of flag matroids. We give a Chow-theoretic formula for the flag-geometric Tutte polynomial of a matroid. Recall the shorthand that $\deg_\alpha(\xi) = \int_{X_E} (1 + \alpha + \cdots + \alpha^n) \cdot \xi$.

[5] One subtlety is that Lemma 5.9 does not generalize easily to flag matroids, but this is remedied by [47, Corollary 3.16]. Alternatively, one can prove both this lemma and the original one (Lemma 10.10) by using the Atiyah-Bott localization formula (Theorem 10.2.(a)) combined with a generalized form [73, Theorem 2.3] of Brion's formula [20].

Theorem 11.3 *Let $\boldsymbol{M} = (M_1, \dots, M_k)$ be a flag matroid on ground set E with rank sequence $(r_1, \dots, r_k)$. Then, the flag-geometric Tutte polynomial $KT_{\boldsymbol{M}}(x, y)$ of $\boldsymbol{M}$, as defined in* [26, *Definition* 8.23], *satisfies*

$$KT_{\boldsymbol{M}}(x, y)$$
$$= \sum_{i=0}^{r_k} \sum_{j=0}^{|E|-r_1} \deg_\alpha \Big(c(\mathcal{S}^\vee_{M_1}) \cdots c(\mathcal{S}^\vee_{M_{k-1}}) c_i(\mathcal{S}^\vee_{M_k}) c_j(\mathcal{Q}_{M_1}) \Big) x^{r_k - i} y^{|E| - r_1 - j} (1-y)^j$$

In particular, the ***flag-geometric characteristic polynomial*** *of $\boldsymbol{M}$, defined in* [26, §9] *as*

$$K\chi_{\boldsymbol{M}}(q) \coloneqq (-1)^{r_1 + \cdots + r_k} KT_{\boldsymbol{M}}(1-q, 0),$$

satisfies

$$(-1)^{r_1 + \cdots + r_k} K\chi_{\boldsymbol{M}}(q)$$
$$= \sum_{i=0}^{r_k} \deg_\alpha \Big(c(\mathcal{S}^\vee_{M_1}) \cdots c(\mathcal{S}^\vee_{M_{k-1}}) c_i(\mathcal{S}^\vee_{M_k}) c_{|E|-r_1}(\mathcal{Q}_{M_1}) \Big) (1-q)^{r_k - i},$$

and its coefficients have alternating signs.

Proof By [39, §6.1], the flag-geometric Tutte polynomial $KT_{\boldsymbol{M}}(x, y)$ is given by

$$KT_{\boldsymbol{M}}(u+1, v+1) = \sum_{i,j} \chi_{Fl(\boldsymbol{r}; E)} \Big(y(\boldsymbol{M}) [\det \mathcal{S}^\vee_1] \cdots [\det \mathcal{S}^\vee_k] [\textstyle\bigwedge^i \mathcal{S}_k] [\bigwedge^j \mathcal{Q}^\vee_1] \Big) u^i v^j,$$

where $\mathcal{S}_\ell$ and $\mathcal{Q}_\ell$ denotes the tautological bundles on $\mathrm{Gr}(r_\ell; E)$ pulled back to $Fl(\boldsymbol{r}; E)$ for $\ell = 1, \dots, k$. By Lemma 11.2, this equals

$$\sum_{i,j} \chi_{X_E} \Big([\det \mathcal{S}^\vee_{M_1}] \cdots [\det \mathcal{S}^\vee_{M_k}] [\textstyle\bigwedge^i \mathcal{S}_{M_k}] [\bigwedge^j \mathcal{Q}^\vee_{M_1}] \Big) u^i v^j.$$

Noting that $[\det \mathcal{S}^\vee_{M_k}][\bigwedge^i \mathcal{S}_{M_k}] = [\bigwedge^{\mathrm{rk}(\mathcal{S}^\vee_{M_k}) - i} \mathcal{S}^\vee_{M_k}]$, combining Theorem 10.1 and Proposition 10.5 yields the desired equalities for $KT_{\boldsymbol{M}}$ and $K\chi_{\boldsymbol{M}}$. Because $KT_{\boldsymbol{M}}(1+q, 0)$ has all nonnegative coefficients by Theorem 9.13, that the coefficients of $K\chi_{\boldsymbol{M}}(q)$ have alternating signs follows. $\square$

We now resolve [26, Conjecture 9.4], which stated that the flag-geometric characteristic polynomial of $\boldsymbol{M}$ form a log-concave sequence.

Corollary 11.4 *For a flag matroid $\boldsymbol{M} = (M_1, \dots, M_k)$, the (unsigned) coefficients of $K\chi_{\boldsymbol{M}}(q)$ form a log-concave sequence with no internal zeros.*

Proof As coefficients of $KT_{\boldsymbol{M}}(1-q, 0)$ have alternating signs, we show the stronger statement that the coefficients of $KT_{\boldsymbol{M}}(q, 0)$ are log-concave with no internal zeros (see [34] where this reduction is proved in the context of showing that h-vector log concavity implies f-vector log concavity). By Theorem 11.3, the homogenization of $KT_{\boldsymbol{M}}(q, 0)$ by an additional variable p is written as

$$\sum_{i=0}^{r_k} \Big(\int_{X_E} c(\mathcal{S}^\vee_{U_{n,E}}) c(\mathcal{S}^\vee_{M_1}) \cdots c(\mathcal{S}^\vee_{M_{k-1}}) c_i(\mathcal{S}^\vee_{M_k}) c_{|E|-r_1}(\mathcal{Q}_{M_1}) \Big) p^i q^{r_k - i}, \qquad (3)$$

since $c(\mathcal{S}^\vee_{U_{n,E}}) = 1 + \alpha + \cdots + \alpha^n$ from Example 3.10. This homogeneous polynomial is obtained by setting $q_0 = \cdots = q_{k-1} = q$ in $(\frac{\partial}{\partial u}^{|E|-r_1} f)|_{u=0}$ where f is the polynomial

$$\sum_{i=0}^{r_k} \Big(\int_{X_E} c(\mathcal{S}^\vee_{U_{n,E}}, q_0) c(\mathcal{S}^\vee_{M_1}, q_1) \cdots c(\mathcal{S}^\vee_{M_{k-1}}, q_{k-1}) c(\mathcal{S}^\vee_{M_k}, p) c(\mathcal{Q}_{M_1}, u) \Big).$$

Theorem 9.13 implies that f above is a denormalized Lorentzian polynomial. As taking partials and evaluating at zero preserves Lorentzian polynomials ([17, Theorem 2.25] and [17, Theorem 2.10]), and since setting variables equal to each other preserves denormalized Lorentzian polynomials [18, Lemma 4.8], we conclude that the polynomial in (3) is a denormalized Lorentzian polynomial. We thereby conclude that its coefficients form a log-concave nonnegative sequence with no internal zeros. □

We also resolve [39, Conjecture 7.7].

Corollary 11.5 *Let M be a loopless matroid of rank r on $E = \{0, 1, \ldots, n\}$, so that one has $U_{1,E} \twoheadleftarrow M$. Then we have*

$$KT_{U_{1,E},M}(x, 0) = x^r.$$

Proof By Theorem 11.3, we have

$$KT_{U_{1,E},M}(x, 0) = \sum_{i=0}^{r} \deg_\alpha \Big(c(\mathcal{S}^\vee_{U_{1,E}}) c_i(\mathcal{S}^\vee_M) c_n(\mathcal{Q}^\vee_{U_{1,E}}) \Big) x^{r-i} (-1)^n$$

since all other terms from $j \neq |E| - 1 = n$ vanish as we set $y = 0$ in $KT_{U_{1,E},M}(x, y)$. Moreover, since the dimension of X_E is n, the only term that survives is from $i = 0$. Noting that $c_n(\mathcal{Q}^\vee_{U_{1,E}}) = (-\alpha)^n$ from Example 3.10, we have $KT_{U_{1,E},M}(x, 0) = x^r$ as desired. □

11.3 Las Vergnas Tutte polynomials of morphisms of matroids

We now turn to Las Vergnas' Tutte polynomials of morphisms of matroids, which is different from the flag-geometric Tutte polynomials. See [39] for a geometric origin of the difference between the two generalizations. Las Vergnas introduced the following generalization of the Tutte polynomial to morphisms of matroids in [80]. See [81] for a survey of its properties.

Definition 11.6 Let M_1 and M_2 be matroids of rank r_1 and r_2 (respectively) on ground set E such that $M_1 \twoheadleftarrow M_2$. The **Las Vergnas Tutte polynomial** of (M_1, M_2) is a polynomial in three variables x, y, z defined by

$$\begin{aligned} &LVT_{M_1,M_2}(x, y, z) \\ &:= \sum_{A\subseteq E} (x-1)^{r_1-\mathrm{rk}_{M_1}(A)}(y-1)^{|A|-\mathrm{rk}_{M_2}(A)} z^{r_2-r_1-\mathrm{rk}_{M_2}(A)+\mathrm{rk}_{M_1}(A))}. \end{aligned}$$

To express LVT_{M_1,M_2} Chow-theoretically, it is convenient to have the following notation. Let $\mathcal{S}^\vee_{M_1,M_2}$ be a K-class on X_E whose equivariant K-class $[\mathcal{S}^\vee_{M_1,M_2}]^T$ is defined by

$$[\mathcal{S}^\vee_{M_1,M_2}]^T_\sigma = \sum_{i\in B_\sigma(M_2)\setminus B_\sigma(M_1)} T_i = [\mathcal{S}^\vee_{M_2}]_\sigma - [\mathcal{S}^\vee_{M_1}]_\sigma.$$

When (M_1, M_2) has a realization (L_1, L_2), it is equal to the pullback $\varphi^*_{(L_1,L_2)}(\mathcal{S}_2/\mathcal{S}_1)^\vee$, where $\mathcal{S}_2/\mathcal{S}_1$ is the quotient of the two tautological subbundles on $Fl(r_1, r_2; E)$.

Theorem 11.7 *Let M_1 and M_2 be matroids of rank r_1 and r_2 (respectively) on ground set E such that $M_1 \twoheadleftarrow M_2$. The Las Vergnas Tutte polynomial of the matroid morphism $M_1 \twoheadleftarrow M_2$ satisfies*

$$\begin{aligned} &LVT_{M_1,M_2}(x, y, z) = \\ &\sum_{i=0}^{r_1}\sum_{j=0}^{|E|-r_2}\sum_{k=0}^{r_2-r_1} \deg_\alpha\Big(c_i(\mathcal{S}^\vee_{M_1})c_j(\mathcal{Q}^\vee_{M_2})c_k(\mathcal{S}^\vee_{M_1,M_2})\Big)x^{r_1-i}y^{|E|-r_2-j}(y-1)^j \\ &\qquad\qquad \times (z+1)^{r_2-r_1-k}. \end{aligned}$$

Proof The partial flag variety $Fl(r_1, r_2; E)$ has tautological subbundles $\mathcal{S}_1$ and $\mathcal{S}_2$ with ranks r_1 and r_2 (respectively), and corresponding quotient bundles $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We also have the short exact sequence $0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_2/\mathcal{S}_1 \to 0$. It was shown in [39, Theorem 5.3] that

$$\begin{aligned} &LVT_{M_1,M_2}(u+1, v+1, w) \\ &= \sum_{i,j,k} \chi_{Fl(r_1,r_2;E)}\Big(y(\boldsymbol{M})[\det \mathcal{S}_2^\vee][\textstyle\bigwedge^i \mathcal{S}_1][\bigwedge^j \mathcal{Q}_2^\vee][\bigwedge^k(\mathcal{S}_2/\mathcal{S}_1)]\Big)u^i v^j w^k, \end{aligned}$$

where $\boldsymbol{M} = (M_1, M_2)$ denotes the two-step flag matroid. Now, applying Lemma 11.2 while noting that $\det \mathcal{S}_2^\vee \simeq \det \mathcal{S}_1^\vee \otimes \det(\mathcal{S}_2/\mathcal{S}_1)^\vee$, gives

$$\begin{aligned} &LVT_{M_1,M_2}(u+1, v+1, w) \\ &= \sum_{i,j,k} \chi_{X_{A_E}}\Big([\textstyle\bigwedge^{r_1-i}\mathcal{S}^\vee_{M_1}][\bigwedge^j \mathcal{Q}^\vee_{M_2}][\bigwedge^{r_2-r_1-k}\mathcal{S}^\vee_{M_1,M_2}]\Big)u^i v^j w^k. \end{aligned}$$

Applying Proposition 10.5 then yields

$$LVT_{M_1,M_2}(u+1, v+1, w) =$$
$$(u+1)^{r_1}(v+1)^{|E|-r_2}(w+1)^{r_2-r_1}\deg_\alpha\left(c(\mathcal{S}_{M_1}^\vee, \tfrac{1}{u+1})c(\mathcal{Q}_{M_2}^\vee, \tfrac{v}{v+1})c(\mathcal{S}_{M_1,M_2}^\vee, \tfrac{1}{w+1})\right).$$

Substituting $u = x-1$, $v = y-1$, and $w = z$ then yields the desired equality. $\square$

We remark that, despite this Chow-theoretic formula, [39, Conjecture 7.10] concerning the log-concavity of a specialization of the Las Vergnas Tutte polynomial remains open, since $\mathcal{S}_{M_1,M_2}^\vee$ is in general not the K-class of a nef vector bundle even when (M_1, M_2) has a realization.

Appendix I: Alternate proof of Theorem A via convolution formulas

We give another proof for Theorem A, different from the proof in §4, by using the base polytope properties of the tautological classes established in §5. Instead of establishing a deletion-contraction relation, we establish a recursive convolution formula for $\alpha^i\beta^j c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M)$, and show that it agrees with a new Tutte polynomial convolution formula whose proof was communicated to us by Alex Fink. As before, let $E = \{0, 1, \ldots, n\}$, and X_E the n-dimensional permutohedral variety. Important for us will be the following well-known formula, called the corank-nullity formula, for the Tutte polynomial of a matroid M of rank r

$$T_M(x, y) = \sum_{S\subseteq E}(x-1)^{r-\mathrm{rk}_M(S)}(y-1)^{|S|-\mathrm{rk}_M(S)}.$$

Theorem A *For a matroid M of rank r with ground set E, denote*

$$t_M(x, y, z, w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r}T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right),$$

where T_M is the Tutte polynomial of M. Then, we have

$$\sum_{i+j+k+\ell=n}\left(\int_{X_E}\alpha^i\beta^j c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M)\right)x^i y^j z^k w^\ell = t_M(x, y, z, w).$$

For a matroid M, note that $t_M(x, y, z, w)$ is a polynomial since the Tutte polynomial T_M always has no constant term. Let us denote

$$\widetilde{t}_M(x, y, z, w) = \sum_{i+j+k+\ell=n}\left(\int_{X_E}\alpha^i\beta^j c_k(\mathcal{S}_M^\vee)c_\ell(\mathcal{Q}_M)\right)x^i y^j z^k w^\ell.$$

We prove $\widetilde{t}_M(x, y, z, w) = t_M(x, y, z, w)$ in two steps. First, by using the matroid minor decomposition properties, we show that $\widetilde{t}_M(x, y, z, w)$ and $t_M(x, y, z, w)$ satisfy an identical recursive relation, which reduces the proof of Theorem A to the case where $x=y=0$. This case is precisely the content of Theorem 6.2, which we will give an alternate proof for using a computation in [103, Theorem 5.1], together with the valuativity and duality properties of tautological Chern classes of matroids.

We start with a recursive relation for $\widetilde{t}_M(x,y,z,w)$.

Lemma I.1 *Let M be a matroid with ground set E, and fix any element $e\in E$. Then, one has*

$$\widetilde{t}_M(x,y,z,w)=\widetilde{t}_M(0,y,z,w)+x\sum_{e\in S\subsetneq E}\widetilde{t}_{M|S}(0,y,z,w)\,\widetilde{t}_{M/S}(x,0,z,w),\quad \textit{and}$$

$$\widetilde{t}_M(x,y,z,w)=\widetilde{t}_M(x,0,z,w)+y\sum_{\substack{S\not\ni e\\ \emptyset\subsetneq S\subsetneq E}}\widetilde{t}_{M|S}(0,y,z,w)\,\widetilde{t}_{M/S}(x,0,z,w).$$

Proof Let us show the first statement (the second statement is proved similarly). Recall from Remark 2.4 that $\alpha=\sum_{e\in S\subsetneq E}[Z_S]$, where Z_S is the torus-invariant divisor of X_E corresponding to the ray $\operatorname{Cone}(\overline{\mathbf{e}}_S)$ of the fan Σ_E, and recall the notation that $c(\mathcal{E},u)=\sum_{i\geq 0}c_i(\mathcal{E})u^i$ denotes the Chern polynomial of a K-class $[\mathcal{E}]$ with formal variable u. For any integers $i\geq 1$ and $j\geq 0$, we first compute that

$$\begin{aligned}\int_{X_E}\alpha^i\beta^j c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w)&=\int_{X_E}\sum_{e\in S\subsetneq E}[Z_S]\alpha^{i-1}\beta^j c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w)\\&=\sum_{e\in S\subsetneq E}\int_{Z_S}\big(\alpha^{i-1}\beta^j c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w)\big)|_{Z_S}.\end{aligned}$$

Moreover, since $Z_S\simeq X_S\times X_{E\setminus S}$ and $A^\bullet(Z_S)\simeq A^\bullet(X_S)\otimes A^\bullet(X_{E\setminus S})$ by Proposition 5.2, applying the matroid minors decomposition formula (Proposition 5.3 and Corollary 5.4) yields that

$$\begin{aligned}&\sum_{e\in S\subsetneq E}\int_{Z_S}\big(\alpha^{i-1}\beta^j c(\mathcal{S}_M^\vee,z)c(\mathcal{Q}_M,w)\big)|_{Z_S}\\&=\sum_{e\in S\subsetneq E}\int_{X_S\times X_{E\setminus S}}\Big(\big(1\otimes\alpha_{E\setminus S}^{i-1}\big)\big(\beta_S^j\otimes 1\big)\big(c(\mathcal{S}_{M|S}^\vee,z)\otimes c(\mathcal{S}_{M/S}^\vee,z)\big)\\&\quad\times\big(c(\mathcal{Q}_{M|S},w)\otimes c(\mathcal{Q}_{M/S},w)\big)\Big)\\&=\sum_{e\in S\subsetneq E}\int_{X_S}\big(\beta_S^j c(\mathcal{S}_{M|S}^\vee,z)c(\mathcal{Q}_{M|S},w)\big)\cdot\int_{X_{E\setminus S}}\big(\alpha_{E\setminus S}^{i-1}c(\mathcal{S}_{M/S}^\vee,z)c(\mathcal{Q}_{M/S},w)\big).\end{aligned}$$

Thus, by rewriting $\widetilde{t}_M(x,y,z,w)$ as

$$\begin{aligned}&\widetilde{t}_M(x,y,z,w)\\&=\int_{X_E}\Big((1+\alpha x+\cdots+\alpha^n x^n)\cdot(1+\beta y+\cdots+\beta^n y^n)\cdot c(\mathcal{S}_M^\vee,z)\cdot c(\mathcal{Q}_M,w)\Big),\end{aligned}$$

we conclude that

$$\widetilde{t}_M(x,y,z,w) = \widetilde{t}_M(0,y,z,w) + x \sum_{e \in S \subsetneq E} \widetilde{t}_{M|S}(0,y,z,w)\, \widetilde{t}_{M/S}(x,0,z,w),$$

as desired. □

We now show that the polynomial $t_M(x, y, z, w)$ obeys the same recursive relation.

Lemma I.2 *Let M be a matroid with ground set E, and fix an element $e \in E$. Then one has*

$$t_M(x,y,z,w) = t_M(0,y,z,w) + x \sum_{e \in S \subsetneq E} t_{M|S}(0,y,z,w) t_{M/S}(x,0,z,w), \quad \textit{and}$$

$$t_M(x,y,z,w) = t_M(x,0,z,w) + y \sum_{\substack{e \notin S \\ \emptyset \subsetneq S \subsetneq E}} t_{M|S}(0,y,z,w) t_{M/S}(x,0,z,w).$$

From here to the end of this subsection, we include $\emptyset$ and E in summations unless otherwise stated, and allow a matroid M to have an empty ground set, in which case we write $M = \emptyset$ for the unique matroid on the ground set $\emptyset$ whose set of bases is $\{\emptyset\}$. By convention, we set $T_\emptyset(x, y) = 1$.

To prove the lemma, we first borrow some notation from [40]. For two functions f and g from the set of matroids with ground sets contained in E to a common ring, we define $f * g$ by

$$(f * g)(M) = \sum_{\emptyset \subseteq A \subseteq E} f(M|A) g(M/A).$$

Then, one can verify that $*$ is associative by computing that

$$(f_0 * \ldots * f_k)(M) = \sum_{\emptyset \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq E} f_1(M|A_1) f_2(M|A_2/A_1) \ldots f_k(M/A_k).$$

The function ν such that $\nu(\emptyset) = 1$ and $\nu(M) = 0$ for $M \neq \emptyset$ acts as the identity for $*$, as one easily checks

$$\nu * f = f * \nu = f \quad \text{for any } f.$$

We define $N_{(a,b)}(M) = a^{\mathrm{rk}_M} b^{\mathrm{crk}_M}$, where rk_M and crk_M denotes the rank and corank of M, respectively. This function satisfies

$$N_{(a,b)}(\emptyset) = 1 \text{ and } N_{(a,b)}(M) = N_{(a,b)}(M|A) N_{(a,b)}(M/A)$$

for all $\emptyset \subseteq A \subseteq E$. We note the following convolution formula. (The first part appears in [40, Sect. 5 Equation (3)] and the second part appears in [40, Proposition 3.6, proof of Theorem 5.10]).

Lemma I.3 *We have*

$$(N_{(a,b)} * N_{(c,d)})(M) = a^{\mathrm{rk}_M} d^{\mathrm{crk}_M} T_M(1 + \frac{c}{a}, 1 + \frac{b}{d}),$$

and in particular, denoting $\overline{N_{(a,b)}} = N_{(-a,-b)}$, *we have*

$$N_{(a,b)} * \overline{N_{(a,b)}} = \overline{N_{(a,b)}} * N_{(a,b)} = \nu.$$

Proof For the first part, both sides are simultaneously homogenous in a, c of degree rk_M and in b, d of degree crk_M, so it suffices to show the equality when $a = d = 1$. We have $N_{(1,b)}(M|A) = b^{|A|-\mathrm{rk}_M(A)}$ and $N_{(c,1)}(M/A) = c^{\mathrm{rk}_M - \mathrm{rk}_M(A)}$, so by the corank-nullity formula for the Tutte polynomial and then the definition of the convolution $*$, we have

$$T_M(1+c, 1+b) = \sum_{\emptyset \subseteq A \subseteq E} c^{\mathrm{rk}_M - \mathrm{rk}_M(A)} b^{|A|-\mathrm{rk}_M(A)} = (N_{(1,b)} * N_{(c,1)})(M)$$

as desired. The second part follows since $T_M(0, 0) = 0$ if $M \neq \emptyset$ and $T_\emptyset(0, 0) = 1$. $\square$

Proof of Lemma I.2 Write

$$g_M(x, y, z, w) = (x + y)t_M(x, y, z, w) = (y + z)^r (x + w)^{|E|-r} T_M(\frac{x + y}{y + z}, \frac{x + y}{x + w}),$$

so that we have to show

$$\frac{y}{x+y} g_M(x, y, z, w) = \sum_{e \in B} g_{M|B}(0, y, z, w) g_{M/B}(x, 0, z, w), \quad \text{and} \tag{4}$$

$$\frac{x}{x+y} g_M(x, y, z, w) = \sum_{e \notin B} g_{M|B}(0, y, z, w) g_{M/B}(x, 0, z, w). \tag{5}$$

Here, we used our convention for this subsection that summations include the $\emptyset$ and E cases unless stated otherwise. Now, define the functions

$$N_0 = N_{(-y-z,-y+w)}, \qquad N_1 = N_{(-z,w)}, \qquad N_2 = N_{(x-z,x+w)}.$$

Then we can directly check from the $N_{(a,b)} * N_{(c,d)}$ formula that

$$g_M(x, y, z, w) = (\overline{N_0} * N_2)(M), \quad g_M(0, y, z, w) = (\overline{N_0} * N_1)(M),$$
$$g_M(x, 0, z, w) = (\overline{N_1} * N_2)(M).$$

Therefore,

$$\begin{aligned} g_M(x, y, z, w) &= (\overline{N_0} * N_2)(M) = ((\overline{N_0} * N_1) * (\overline{N_1} * N_2))(M) \\ &= \sum_B g_{M|B}(0, y, z, w) g_{M/B}(x, 0, z, w), \end{aligned}$$

which is the sum of (4) and (5). Hence to conclude, we only need to verify (4). To simplify notation, for subsets $X \subseteq Y \subseteq E$ we will write X/Y for $M|X/Y$, which also equals $(M/Y)|X$. We compute

$$\begin{aligned}
&\sum_{e\in B} g_B(0,y,z,w)g_{M/B}(x,0,z,w)\\
&=\sum_{e\in B}\sum_{A\subseteq B\subseteq C} \overline{N_0}(A)N_1(B/A)\overline{N_1}(C/B)N_2(M/C)\\
&=\sum_{A\subseteq A\cup e\subseteq B\subseteq C} \overline{N_0}(A)N_1((A\cup e)/A)N_1(B/(A\cup e))\overline{N_1}(C/B)N_2(M/C)\\
&=\sum_{A\subseteq A\cup e\subseteq C} \overline{N_0}(A)N_1((A\cup e)/A)(N_1*\overline{N_1})(C/(A\cup e))N_2(M/C)\\
&=\sum_{A} \overline{N_0}(A)N_1((A\cup e)/A)N_2(M/(A\cup e)).
\end{aligned}$$

When $i \in A$ we have $N_1((A\cup i)/A) = 1$, and when $i \notin A$ then $(A\cup i)/A$ is a one element rank 1 matroid. For a 1 element matroid L we have $N_1(L) = -\frac{x}{x+y}\overline{N_0}(L) + \frac{y}{x+y}N_2(L)$ since we can check

$$\begin{aligned}
N_1(U_{0,1}) = w &= -\frac{x}{x+y}(y-w) + \frac{y}{x+y}(x+w)\\
&= -\frac{x}{x+y}\overline{N_0}(U_{0,1}) + \frac{y}{x+y}N_2(U_{0,1})\\
N_1(U_{1,1}) = -z &= -\frac{x}{x+y}(y+z) + \frac{y}{x+y}(x-z)\\
&= -\frac{x}{x+y}\overline{N_0}(U_{1,1}) + \frac{y}{x+y}N_2(U_{1,1}).
\end{aligned}$$

Therefore, we continue our computation as

$$\begin{aligned}
&\sum_{A} \overline{N_0}(A)N_1((A\cup e)/A)N_2(M/(A\cup e))\\
&=\sum_{e\in A} \overline{N_0}(A)N_2(M/A) - \frac{x}{x+y}\sum_{e\notin A}\overline{N_0}(A\cup e)N_2(M/(A\cup i))\\
&\quad+\frac{y}{x+y}\sum_{e\notin A}\overline{N_0}(A)N_2(M/A)\\
&=\frac{y}{x+y}\sum_{e\in A}\overline{N_0}(A)N_2(M/A) + \frac{y}{x+y}\sum_{e\notin A}\overline{N_0}(A)N_2(M/A)\\
&=\frac{y}{x+y}(\overline{N_0}*N_2)(M) = \frac{y}{x+y}g_M(x,y,z,w).
\end{aligned}$$

We have thus verified (4). $\square$

Proof of Theorem A When the ground set E has cardinality 1, the left-hand-side $\widetilde{t}_M(x,y,z,w)$ equals 1, and the right-hand-side $t_M(x,y,z,w)$ is also 1 because $T_{U_{1,\{0\}}}(u,v)=u$ and $T_{U_{0,\{0\}}}(u,v)=v$. Let us now induct on the cardinality of E. Let M be a matroid on E, and assume that the desired equality holds for all matroids on ground sets with cardinality less than $|E|$.

Since $\widetilde{t}_M(x,y,z,w)$ and $t_M(x,y,z,w)$ satisfy the same recursive relation given in Lemma I.1 and Lemma I.2, the induction hypothesis implies that it suffices to show $\widetilde{t}_M(0,y,z,w)=t_M(0,y,z,w)$ and $\widetilde{t}_M(x,0,z,w)=t_M(x,0,z,w)$. Applying the recursive relation and the induction hypothesis again, we find that it suffices to show $\widetilde{t}_M(0,0,z,w)=t_M(0,0,z,w)$. Noting that Tutte polynomials have no constant terms, we compute that $t_M(0,0,z,w)=z^r w^{|E|-r}(\beta(M)\frac{1}{z}+\beta(M^\perp)\frac{1}{w})$. We have thus reduced the proof to showing Theorem 6.2, reproduced below, for which we give an alternate proof. □

Theorem 6.2 *Let M be a matroid of rank r on ground set E. Then,*

$$\int_{X_E} c_{r-1}(\mathcal{S}_M^\vee)c_{|E|-r}(\mathcal{Q}_M)=\beta(M) \quad \textit{and} \quad \int_{X_E} c_r(\mathcal{S}_M^\vee)c_{|E|-r-1}(\mathcal{Q}_M)=\beta(M^\perp),$$

where we set by convention $c_{-1}(\mathcal{E})=0$ for a K-class $[\mathcal{E}]$.

In §6, we had derived Theorem 6.2 as an immediate consequence of Theorem A. Here, we give another proof that does not rely on Theorem A, but uses a geometric computation in [103, Theorem 5.1] and valuativity.

Alternate proof of Theorem 6.2 via geometry and valuativity Noting that Cremona involution is an isomorphism, one has from the matroid duality property (Proposition 5.11) that

$$\begin{aligned}\int_{X_E} c_r(\mathcal{S}_M^\vee)c_{|E|-r-1}(\mathcal{Q}_M) &= \int_{X_E} \operatorname{crem}\Big(c_r(\mathcal{S}_M^\vee)c_{|E|-r-1}(\mathcal{Q}_M)\Big)\\ &= \int_{X_E} c_r(\mathcal{Q}_{M^\perp})c_{|E|-r-1}(\mathcal{S}_{M^\perp}^\vee).\end{aligned}$$

Hence, the second equality in the theorem follows from the first, so we prove the first equality only.

When M has rank 0, the Tutte polynomial $T_M(x,y)$ has no x terms, so the claimed equality is satisfied. Suppose now $r\geq 1$. If $|E|=1$, so that $M=U_{1,\{0\}}$, then $\int_{X_E} c_0(\mathcal{S}_M^\vee)c_0(\mathcal{Q})=1$, whereas $\beta(M)=1$ since $T_{U_{1,\{0\}}}(x,y)=x$. Hence, we now suppose $|E|\geq 2$.

Because the assignment $M\mapsto c_{r-1}(\mathcal{S}_M^\vee)c_{|E|-r}(\mathcal{Q}_M)$ is valuative by Proposition 5.6, and the assignment $M\mapsto\beta(M)$ is also valuative [11, Corollary 5.7], Lemma 5.9 implies that it suffices to show the equality $\int_{X_E} c_{r-1}(\mathcal{S}_M^\vee)c_{|E|-r}(\mathcal{Q}_M)=\beta(M)$ for all matroids M that are realizable over $\mathbb{C}$. So, let $L\subseteq\mathbb{C}^E$ be a realization of a matroid M of rank $r\geq 1$ with $|E|\geq 2$. For $H\subset\mathbb{C}^E$ a generic hyperplane and $\ell\subset H$ a generic line in H, denote by $\Omega(\ell,H)$ the Schubert variety in $\mathrm{Gr}(r;E)$

consisting of $L \in \mathrm{Gr}(r; E)$ such that $\ell \subseteq L \subseteq H$. In [103, Theorem 5.1] it is shown that

$$\int_{\mathrm{Gr}(r;E)} [\overline{T \cdot L}] \cdot [\Omega(\ell, H)] = \beta(M).$$

Note that the Chow class $[\Omega(\ell, H)]$ is equal to $c_{r-1}(\mathcal{S}^\vee)c_{|E|-r}(\mathcal{Q})$, where $\mathcal{S}$ and $\mathcal{Q}$ are the tautological sub and quotient bundles of $\mathrm{Gr}(r; E)$, respectively (see for instance [44, §5.6.2]). Writing $\varphi_L \colon X_E \to \overline{T \cdot L} \subset \mathrm{Gr}(r; E)$ for the map as defined in §3.1, we have by the functoriality of Chern classes that

$$\int_{X_E} c_{r-1}(\mathcal{S}_L^\vee)c_{|E|-r}(\mathcal{Q}_L) = \int_{X_E} \mathrm{crem}\, \varphi_L^*[\Omega(\ell, H)].$$

We now break into two cases. First, suppose the matroid M is disconnected, say $M = M_1 \oplus M_2$ for matroids M_1 and M_2 on nonempty ground sets. Then, Proposition 5.12 states that $\dim P(M) < n$, so that $\dim \overline{T \cdot L} < n$. Thus, we have $\varphi_L^*[\Omega(\ell, H)] = 0$, as the pullback of the n-codimensional class $[\Omega(\ell, H)]$ to $\overline{T \cdot L}$ is already 0 by dimensional reason. We also have $\beta(M) = 0$ since $T_M(x, y) = T_{M_1}(x, y)T_{M_2}(x, y)$ and both T_{M_1} and T_{M_2} have no constant terms. Now, suppose M is connected, in which case Proposition 5.12 states that $\dim P(M) = \dim \overline{T \cdot L} = n$, so that φ_L is birational onto its image. Then, the push-pull formula implies that

$$\begin{aligned}\int_{X_E} \mathrm{crem}\, \varphi_L^*[\Omega(\ell, H)] &= \int_{\mathrm{Gr}(r;E)} (\varphi_{L*}[X_E]) \cdot [\Omega(\ell, H)] \\ &= \int_{\mathrm{Gr}(r;E)} [\overline{T \cdot L}] \cdot [\Omega(\ell, H)] = \beta(M).\end{aligned}$$

Thus, we have the desired equality in both cases. □

Appendix II: The tropical logarithmic Poincaré-Hopf theorem: representable case

A reformulation of the Poincaré-Hopf theorem states that the (topological) Euler characteristic $\chi(X)$ of a compact manifold is equal to the self-intersection number of its diagonal $\mathrm{diag}(X)$ in $X \times X$. In an attempt to create a tropical analogue, Rau computed the self-intersection number of the diagonal of the Bergman class of a matroid [96].

Theorem II.1 ([96, Theorem 1.1]) *Let M be a loopless matroid of rank r, and let* $\mathrm{diag}(\Delta_M)$ *be the Minkowski weight of constant weight* 1 *on the diagonal copy of Σ_M inside $\Sigma_M \times \Sigma_M$. Then, as a tropical subcycle of $\Delta_M \times \Delta_M$, its self-intersection number is given by*

$$\deg(\mathrm{diag}(\Delta_M)^2) = (-1)^{r-1}\beta(M).$$

In [96, Remark 1.7], the author expresses a desire for a classical counterpart to Theorem II.1. The goal in this section is to provide such a classical counterpart. We give a geometric proof of Theorem II.1 in the representable case, using the intuition gained from tautological bundles on matroids and reducing to a logarithmic version of the Poincaré Hopf theorem.

Proof of Theorem II.1 when M is representable Let $L \subseteq \mathbb{C}^E$ be a realization of the matroid M. The first step is to translate the tropical self-intersection $\deg(\mathrm{diag}(\Delta_M)^2)$ into a Chow-theoretic intersection. To do this, we recall that the tropical intersection $\mathrm{diag}(\Delta_M)^2$ is computed by expressing the diagonal Minkowski weight $[\mathrm{diag}(\Sigma_M)]$ as the intersection of the Minkowski weight $[\Sigma_M \times \Sigma_M]$ with $r-1$ piecewise linear functions. This is summarized in [96, Sect. 2] and uses [51, Proposition 3.10].

Next, the tropical intersection of a weighted fan with a piecewise linear function [3, Definition 3.4] mirrors the intersection of the corresponding Minkowski weight with a divisor on a toric variety ([76, Lemma 2.5] or [69, Theorem 27]). Thus, to compute the intersection $\mathrm{diag}(\Delta_M)^2$, we start with $\mathrm{diag}(W_L) \subset W_L \times W_L \subset X_E \times X_E$ and perform three steps:

(1) Refine the fan Σ_E^2 of $X_E \times X_E$ to $\widetilde{\Sigma}$ so that the piecewise linear functions used in [96, Proposition 2.6] are linear on each cone of the fan $\widetilde{\Sigma}$.
(2) Take the proper transform of $\mathrm{diag}(W_M)$ and $W_L \times W_L$ in $X_{\widetilde{\Sigma}}$ to get $\widetilde{\mathrm{diag}(W_L)}$ and $\widetilde{W_L \times W_L}$.
(3) Evaluate $\int_{\widetilde{W_L \times W_L}} [\widetilde{\mathrm{diag}(W_L)}]^2$ in Chow theory.

We know the final answer is independent of the choice of sufficiently refined $\widetilde{\Sigma}$ by the equivalence with the tropical intersection number, and this will also be implied by the proof below.

At this point, we will translate our question into the self-intersection of a section within the projectivization of a tautological bundle. Let ϕ be the map

$$\phi\colon X_{\widetilde{\Sigma}} \dashrightarrow X_E \times \mathbb{P}^n$$

given on the open torus $T \times T$ by $(x, y) \mapsto (x, x^{-1}y)$. Similarly, let

$$\begin{aligned}\phi_{\mathrm{trop}}\colon (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1}) \times (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1}) &\to (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1}) \times (\mathbb{Z}^{n+1}/\mathbb{Z}\mathbf{1})\\ (u, v) &\mapsto (u, -u+v).\end{aligned}$$

We can and will choose $\widetilde{\Sigma}$ so that it contains the fan obtained by $\phi_{\mathrm{trop}}^{-1}$ applied to the fan of $X_E \times \mathbb{P}^n$. This means ϕ is now a regular map $X_{\widetilde{\Sigma}} \xrightarrow{\phi} X_E \times \mathbb{P}^n$. We now claim to have the following diagram where the vertical arrows are all birational morphisms

$$\begin{tikzcd} \widetilde{\mathrm{diag}(W_L)} \arrow[r, hook] \arrow[d] & \widetilde{W_L \times W_L} \arrow[r, hook] \arrow[d] & X_{\widetilde{\Sigma}} \arrow[d, "\phi"] \\ W_L \times \{\mathbf{1}\} \arrow[r, hook] & \mathbb{P}(S_L)|_{W_L} \arrow[r, hook] & X_E \times \mathbb{P}^n \end{tikzcd}$$

The two things to check are that $\widetilde{\operatorname{diag}(W_L)}$ and $\widetilde{W_L \times W_L}$ map birationally onto $W_L \times \{\mathbf{1}\}$ and $\mathbb{P}(S_L)|_{W_L}$ respectively. This is possible because it suffices to check that this is true when restricted to the open torus $\phi|_{T\times T}$ as $\widetilde{\operatorname{diag}(W_L)}$ and $\widetilde{W_L \times W_L}$ are irreducible. To see that $(\widetilde{W_L \times W_L}) \cap (T \times T)$ maps into $\mathbb{P}(S_L)|_{W_L}$, we need our convention that the fiber of $\mathbb{P}(S_L) \to X_E$ over $t \in T$ is $t^{-1}\mathbb{P}(L) \subset \mathbb{P}^n$.

To proceed, we need to know that the pullback of the Chow class $[W_L \times \{\mathbf{1}\}]$ agrees with the Chow class of the proper transform, or equivalently that

$$[\widetilde{\operatorname{diag}(W_L)}] = \phi^*[W_L \times \{\mathbf{1}\}]. \tag{6}$$

To prove (6), one first notes that the wonderful compactification W_L intersects the torus orbits of the permutohedral toric variety X_E properly [71, Theorem 6.3]. This implies $W_L \times \{\mathbf{1}\}$ intersects the torus orbits of $X_E \times \mathbb{P}^n$ properly. Finally, applying the dimension count in Lemma II.2 below yields (6). Alternatively, it also is possible to deduce (6) from Lemma 9.8.

Applying (6) to the problem at hand, we obtain

$$\int_{\widetilde{W_L\times W_L}} [\widetilde{\operatorname{diag}(W_L)}]^2 = \int_{\widetilde{W_L\times W_L}} \phi^*[W_L \times \{\mathbf{1}\}]^2 =$$
$$\int_{\mathbb{P}(S_L)|_{W_L}} \phi_*\phi^*([W_L \times \{\mathbf{1}\}]^2) = \int_{\mathbb{P}(S_L)|_{W_L}} [W_L \times \{\mathbf{1}\}]^2.$$

At this point, one can use the formula for the class of $[W_L \times \{\mathbf{1}\}]$ as the projectivization of a subbundle [44, Proposition 9.13] and finish by a computation.

Instead of doing the computation, we will present a geometric proof, connecting the self intersection with the log-tangent sheaf and finally reducing to a logarithmic version of the Poincaré-Hopf theorem. To make this connection, we will need to show that

$$N_{(W_L\times\{\mathbf{1}\})/(\mathbb{P}(S_L)|_{W_L})} = T_{W_L}(-\log D_{W_L}). \tag{7}$$

To compute the normal bundle of $W_L \times \{\mathbf{1}\}$ in $\mathbb{P}(S_L)|_{W_L}$, we will express $W_L \times \{\mathbf{1}\}$ as the zero locus of a section of a vector bundle on $\mathbb{P}(S_L)|_{W_L}$. The locus $W_L \times \{\mathbf{1}\} \subset \mathbb{P}(S_L)|_{W_L}$ can be described as the locus in $\mathbb{P}(S_L)|_{W_L}$, where the universal line is parallel to $\mathbf{1}$. This is equivalently the zero locus of the map

$$\mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(-1) \to \pi^* S_L|_{W_L}/(\mathcal{O}_{\mathbb{P}(S_L)|_{W_L}} \cdot \mathbf{1}).$$

The target $\pi^* S_L|_{W_L}/(\mathcal{O}_{\mathbb{P}(S_L)|_{W_L}} \cdot \mathbf{1})$ is given taking the quotient of the inclusion of the constant section $\mathcal{O}|_{W_L} \cdot \mathbf{1} = \mathcal{O}|_{W_L} \cdot (1, \dots, 1)$ in $S_L|_{W_L} \subset \mathcal{O}_{W_L}^{n+1}$, and pulling back by the projection $\pi \colon \mathbb{P}(S_L)|_{W_L} \to W_L$. We have already taken the quotient $S_L|_{W_L}/(\mathcal{O}_{W_L} \cdot \mathbf{1})$ in Theorem 8.8 and identified it as $T_{W_L}(-\log D_{W_L})$.

Thus, $W_L \times \{\mathbf{1}\} \subset \mathbb{P}(S_L)|_{W_L}$ is the zero locus of the map

$$\mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(-1) \to \pi^* T_{W_L}(-\log D_{W_L}),$$

or equivalently the zero locus of a section of $\pi^* T_{W_L}(-\log D_{W_L}) \otimes \mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(1)$.

The restriction of a vector bundle to the zero locus of a section vanishing in proper codimension is the normal bundle of that section [44, Proposition-Definition 6.15]. Thus, the restriction of the vector bundle $\pi^* T_{W_L}(-\log D_{W_L}) \otimes \mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(1)$ to $W_L \times \{\mathbf{1}\}$ is the normal bundle $N_{(W_L \times \{\mathbf{1}\})/(\mathbb{P}(S_L)|_{W_L})}$. To perform the restriction, $\mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(1)$ restricts to the trivial bundle as the universal line is constant along $W_L \times \{\mathbf{1}\}$ and $\pi^* T_{W_L}(-\log D_{W_L})$ restricts to $T_{W_L}(-\log D_{W_L})$ as $W_L \times \{\mathbf{1}\}$ maps isomorphically to W_L under π. Therefore, $\pi^* T_{W_L}(-\log D_{W_L}) \otimes \mathcal{O}_{\mathbb{P}(S_L)|_{W_L}}(1)$ restricted to $W_L \times \{\mathbf{1}\}$ is $T_{W_L}(-\log D_{W_L})$, concluding our proof of (7).

Finally,

$$\int_{\mathbb{P}(S_L)|_{W_L}} [W_L \times \{\mathbf{1}\}]^2 = c_{\mathrm{top}}(N_{(W_L \times \{\mathbf{1}\})/\mathbb{P}(S_L)})$$

by [82], and by (7),

$$c_{\mathrm{top}}(N_{(W_L \times \{\mathbf{1}\})/\mathbb{P}(S_L)}) = c_{\mathrm{top}}(T_{W_L}(-\log D_{W_L})).$$

The top Chern class $c_{\mathrm{top}}(T_{W_L}(-\log D_{W_L}))$ is $c_{r-1}(\mathcal{S}_{\mathcal{L}}|_{W_L})$ by Theorem 8.8 and $\int_{X_E} c_{r-1}(\mathcal{S}_{\mathcal{L}}|_{W_L}) = \int_{X_E} c_{r-1}(\mathcal{S}_L) c_{|E|-r}(\mathcal{Q}_M)$ is equal to $(-1)^{r-1}\beta(M)$ by Theorem 6.2. $\square$

We chose to conclude $c_{\mathrm{top}}(T_{W_L}(-\log D_{W_L})) = (-1)^{r-1}\beta(M)$ using the framework given in this paper to be self-contained. However, there is a more classical approach to get the same result given in Remark 8.9, which uses a logarithmic version of the Poincaré Hopf theorem to relate the Chern class to the topological Euler characteristic of the hyperplane arrangement complement $W_L \backslash D_{W_L}$.

Lemma II.2 below was used in the proof of Theorem II.1 in the representable case as a substitute for Lemma 9.8, giving a geometric proof independent of tropical methods.

Lemma II.2 *Let $Y \subset T$ be an irreducible subvariety of a torus. Let X_Σ be a smooth toric variety compactifying T and $\overline{Y}$ be the closure of Y in X_Σ. Suppose $\overline{Y}$ intersects each torus orbit in X_Σ properly. Then, the following statement holds:*

Let $\widetilde{\Sigma}$ be a unimodular fan refining Σ, and $\pi : \widetilde{X} \to X$ be the corresponding map of toric varieties. Then, $\pi^{-1}(\overline{Y})$ is equal to the closure $\overline{\pi^{-1}(Y)}$ in $X_{\widetilde{\Sigma}}$. In particular, $\pi^*[\overline{Y}] = [\overline{\pi^{-1}(Y)}]$ in $A^\bullet(\pi^{-1}(\overline{Y})) = A^\bullet(\overline{\pi^{-1}(Y)})$, which implies equality in $A^\bullet(X_{\widetilde{\Sigma}})$.

Proof We clearly have $\pi^{-1}(\overline{Y}) \supset \overline{\pi^{-1}(Y)}$. To show the reverse inclusion, we first show $\dim(\pi^{-1}(\overline{Y} \backslash Y)) < \dim(Y)$. To do this, we will show for all positive-dimensional cones σ in Σ and the corresponding torus orbit O_σ, we must have $\dim(\pi^{-1}(O_\sigma \cap \overline{Y})) < \dim(Y)$.

By the assumption that $\overline{Y}$ intersects each torus orbit of X_Σ properly, $\dim(O_\sigma \cap \overline{Y}) \leq \dim(Y) - \dim(\sigma)$, where either equality holds or the intersection $O_\sigma \cap \overline{Y}$ is empty, in which case the dimension is understood to be $-\infty$. By [66, Proposition

2.14], the fibers over O_σ under $\pi\colon \widetilde{X} \to X$ have dimension at most $\dim(\sigma) - 1$. Therefore,

$$\dim(\pi^{-1}(O_\sigma \cap \overline{Y})) \le \dim(O_\sigma \cap \overline{Y}) + \dim(\sigma) - 1 < \dim(Y).$$

To finish, we first note that every irreducible component of $\pi^{-1}(\overline{Y})$ has dimension at least $\dim(Y)$ [44, Theorem 0.2], as $\pi^{-1}(\overline{Y})$ can be expressed as the intersection between the graph of π and $X_{\widetilde{\Sigma}} \times \overline{Y}$ inside the smooth variety $X_{\widetilde{\Sigma}} \times X_\Sigma$. Next, $\pi^{-1}(\overline{Y}) = \pi^{-1}(Y) \sqcup \bigcup_\sigma \pi^{-1}(O_\sigma \cap \overline{Y})$, where the union is over all positive-dimensional cones σ in Σ. Since we have just shown that $\pi^{-1}(O_\sigma \cap \overline{Y}) < \dim(Y)$, $\pi^{-1}(\overline{Y})$ must be irreducible. Since $\pi^{-1}(\overline{Y})$ is an irreducible variety containing $\overline{\pi^{-1}(Y)}$ and their dimensions agree, $\pi^{-1}(\overline{Y}) = \overline{\pi^{-1}(Y)}$.

To deduce $\pi^*[\overline{Y}] = [\overline{\pi^{-1}(Y)}]$, we note that $\pi^*[\overline{Y}]$ is a well-defined class in $A^\bullet(\pi^{-1}(\overline{Y})) = A^\bullet(\overline{\pi^{-1}(Y)})$ by construction of the cap product using [53, Definition 8.1.2], so it must be the fundamental class $[\overline{\pi^{-1}(Y)}]$. $\square$

Appendix III: Global Chern roots

In this section we show that one can decompose tautological K-classes of matroids as sums of classes of line bundles. The construction of these decompositions are analogous to considering successive quotients in filtrations of tautological bundles of Grassmannians, and likewise are not canonical. Moreover, they are not directly applicable for proving positivity statements because the line bundle summands are generally not nef. However, they relate the tautological K-classes of matroids to certain constructions in previous works [45, 51, 69]. Also, they are useful in computer computations, for instance in Macaulay2 [59], which has been helpful for the development of this paper.

Let M be a matroid of rank r on E. Fix a sequence $\boldsymbol{M} = (M_0, \dots, M_{n+1})$ consisting of matroids M_i of rank i on E such that $M_r = M$ and $B_\sigma(M_i) \subset B_\sigma(M_{i+1})$ for all permutations $\sigma \in \mathfrak{S}_E$ and $i = 0, \dots, n$. Such a sequence $\boldsymbol{M}$ is known as a "full flag matroid that lifts M" [16, Ch. 1]. One such $\boldsymbol{M}$ is the "full Higgs lift" of M which is obtained by defining

$$\text{the set of bases of } M_i = \left\{ S \in \binom{E}{i} \;\middle|\; S \text{ contains or is contained in a basis of } M \right\}$$

for all $0 \le i \le n+1$. For each $0 \le i \le n$, we express the differences $[\mathcal{S}_{M_{i+1}}] - [\mathcal{S}_{M_i}]$ and $[\mathcal{Q}_{M_i}] - [\mathcal{Q}_{M_{i+1}}]$ as K-classes of line bundles as follows. As denoted in §2.7, let $\mathcal{O}(D_P)$ be the T-equivariant line bundle of X_E corresponding to a lattice polytope $P \subset \mathbb{R}^E$ whose normal fan coarsens $\widetilde{\Sigma}_E$. For a matroid N with ground set E, we then have by the discussion in §2.7 that

$$[\mathcal{O}(D_{-P(N)})]_\sigma = \prod_{i \in B_\sigma(N)} T_i \quad \text{and} \quad [\mathcal{O}(D_{P(N^\perp)})]_\sigma = \prod_{i \notin B_\sigma(N)} T_i^{-1}.$$

Thus, since $B_\sigma(M_i) \subset B_\sigma(M_{i+1})$ for all $0 \leq i \leq n$ and permutations σ, we have that

$$\begin{aligned} [\mathcal{S}_{M_{i+1}}] - [\mathcal{S}_{M_i}] &= [\mathcal{O}(D_{-P(M_{i+1})})^\vee \otimes \mathcal{O}(D_{-P(M_i)})] \qquad \text{and} \\ [\mathcal{Q}_{M_i}] - [\mathcal{Q}_{M_{i+1}}] &= [\mathcal{O}(D_{P(M_i{}^\perp)}) \otimes \mathcal{O}(D_{P(M_{i+1}{}^\perp)})^\vee]. \end{aligned}$$

In particular, since $M_0 = U_{0,E}$ and $M_{n+1} = U_{n+1,E}$ so that $[\mathcal{S}_{M_0}] = [\mathcal{Q}_{M_{n+1}}] = 0$, we have that

$$\begin{aligned} [\mathcal{S}_M] &= \sum_{i=0}^{r-1} [\mathcal{O}(D_{-P(M_{i+1})})^\vee \otimes \mathcal{O}(D_{-P(M_i)})] \qquad \text{and} \\ [\mathcal{Q}_M] &= \sum_{j=r}^{|E|-1} [\mathcal{O}(D_{P(M_j{}^\perp)}) \otimes \mathcal{O}(D_{P(M_{j+1}{}^\perp)})^\vee] \qquad \text{as elements in } K_T^0(X_E). \end{aligned} \tag{8}$$

One might hope that this decomposition implied positivity properties of $[\mathcal{S}_M^\vee]$ and $[\mathcal{Q}_M]$. However, for example for $[\mathcal{S}_M^\vee]$, the line bundles $\mathcal{O}(D_{-P(M_{i+1})}) \otimes \mathcal{O}(D_{-P(M_i)})^\vee$ is nef if and only if $P(M_i)$ is a weak Minkowski summand of $P(M_{i+1})$ — see [12, §2.2 & §2.4] for a proof. This however seldom occurs: When a matroid M is connected after removing its loops and coloops, the only weak Minkowski summand of $P(M)$ is itself [90]. Hence, although the bundles $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ are globally generated and hence nef if L is a realization of M, it is unclear how to establish from the Chern roots listed here that the positivity property of $\mathcal{S}_L^\vee$ and $\mathcal{Q}_L$ as nef vector bundles persist for arbitrary (not necessarily realizable) matroids.

Remark III.1 Let $z_S \in A^1(X_E)$ denote the divisor class of the torus-invariant divisor $Z_S \subset X_E$ corresponding to a nonempty proper subset S of E, and denote $z_E = -\alpha \in A^1(X_E)$. Combining Remark 2.4 with a well-known description for the Chow ring of a smooth complete toric variety (see for instance [52, Ch. 5]), one has that the Chow ring of the permutohedral variety has a presentation

$$A^\bullet(X_E) = \frac{\mathbb{Z}[z_S \mid \emptyset \subsetneq S \subseteq E]}{\langle z_S z_{S'} \mid S \not\subseteq S' \text{ and } S \not\supseteq S'\rangle + \langle \sum_{i \in S \subseteq E} z_S \mid i \in E\rangle}.$$

Note that in this presentation, one has $\sum_{\emptyset \subsetneq S \subseteq E} z_S = \beta$ because it follows from the end of Remark 2.4 that $\alpha + \beta = \sum_{\emptyset \subsetneq S \subsetneq E} z_S$. For a matroid N of rank r on E, the translate $P' = -P(N) + r\mathbf{e}_0$ of its base polytope lies in the lattice $\mathbf{1}^\perp$. The support function $h_{P(N)}(x) = \max_{\mathbf{m} \in P(M)} \langle \mathbf{m}, x\rangle$ of the base polytope satisfies $h_{P(N)}(\mathbf{e}_S) = \mathrm{rk}_M(S)$ for any subset $S \subseteq E$, and hence the support function $h_{P'}$ of the translate P' satisfies $h_{P'}(\mathbf{e}_S) = \mathrm{rk}_N(S) - r$ if $0 \in S$ and $h_{P'}(\mathbf{e}_S) = \mathrm{rk}_N(S)$ otherwise. Thus, by the discussion in §2.7 and the fact that $\alpha = \sum_{0 \in S \subsetneq E} z_S$ (Remark 2.4), one has

$$\sum_{\emptyset \subsetneq S \subseteq E} \mathrm{rk}_N(S) z_S = [D_{-P(N)}] = [D_{P(N^\perp)}] \quad \text{as elements in } A^1(X_E),$$

where the last equality follows from the fact that $P(N^\perp)$ and $-P(N)$ are translates $P(N^\perp) = -P(N) + \mathbf{1}$ of each other. In particular, one can restate the decomposition of $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ into sums of line bundles in Equation (8) by stating that a possible collection of Chern roots for $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ is given by

$$\text{Chern roots of } [\mathcal{S}_M] = \Big\{ \sum_{\emptyset \subsetneq S \subseteq E} \big(-\operatorname{rk}_{M_{i+1}}(S) + \operatorname{rk}_{M_i}(S) \big) z_S \Big\}_{i=0,\dots,r-1} \quad \text{and}$$

$$\text{Chern roots of } [\mathcal{Q}_M] = \Big\{ \sum_{\emptyset \subsetneq S \subseteq E} \big(-\operatorname{rk}_{M_{i+1}}(S) + \operatorname{rk}_{M_i}(S) \big) z_S \Big\}_{i=r,\dots,n}.$$

The "modular filter" of two consecutive matroids M_i and M_{i+1} in the sequence $\boldsymbol{M}$ is defined as the collection $\mathscr{F}_i = \{S \subseteq E \mid \operatorname{rk}_{M_{i+1}}(S) - \operatorname{rk}_{M_i}(S) = 1\}$. Writing $\alpha_{\mathscr{F}_i} = \sum_{\substack{S \in \mathscr{F}_i \\ \emptyset \subsetneq S \subsetneq E}} z_S$, we have that the elements $\alpha - \alpha_{\mathscr{F}_i}$ for various i give a collection of Chern roots for $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$. These elements $\alpha - \alpha_{\mathscr{F}_i}$ appeared previously in [69, Remark 31] and [51], and the interpretation of them as Chern roots of a K-class first appeared in [45, Remark 7.2.6]. The elements $\alpha - \alpha_{\mathscr{F}_i}$ when $\mathscr{F}_i$ are principal filters were studied in [14] to give a generalization of a volume formula for generalized polyhedra [95, Corollary 9.4] and a simplified proof for the portion of the Hodge theory of matroids in [1] relevant to log-concavity.

Acknowledgements We would like to thank Alex Fink for helpful discussions on the convolution formula for Tutte polynomials, and we would like to thank Eric Katz for helpful discussions and for sharing unpublished notes of a deletion-contraction proof of [72, Proposition 5.2]. We would also like to thank the creators of Macaulay2 [59] for their helpful and free software, and Justin Chen for the Macaulay2 package on matroids [29], which was used extensively in the early stages of this project. We thank Graham Denham, Ahmed Ashref, and Avi Steiner for suggesting minor edits to an earlier draft of the paper. We thank the referee for a careful reading and helpful suggestions. The second and fourth authors were partially supported by the US National Science Foundation (DMS-2001854 and DMS-2001712).

References

1. Adiprasito, K., Huh, J., Katz, E.: Hodge theory for combinatorial geometries. Ann. Math. (2) **188**(2), 381–452 (2018). MR 3862944
2. Aguiar, M., Ardila, F.: Hopf monoids and generalized permutahedra. Mem. Am. Math. Soc. (in press)
3. Allermann, L., Rau, J.: First steps in tropical intersection theory. Math. Z. **264**(3), 633–670 (2010) (English)
4. Aluffi, P.: Differential forms with logarithmic poles and Chern-Schwartz-MacPherson classes of singular varieties. C. R. Acad. Sci. Paris, Sér. I Math. **329**(7), 619–624 (1999). MR 1717120
5. Aluffi, P.: Characteristic classes of singular varieties. In: Topics in Cohomological Studies of Algebraic Varieties. Trends Math., pp. 1–32. Birkhäuser, Basel (2005). MR 2143071
6. Aluffi, P.: Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. Pure Appl. Math. Q. **2**(4), 915–941 (2006) (English)
7. Anari, N., Liu, K., Gharan, S.O., Vinzant, C.: Log-Concave Polynomials iii: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids (2018)
8. Ardila, F.: The geometry of matroids. Not. Am. Math. Soc. **65**(8), 902–908 (2018). MR 3823027
9. Ardila, F., Klivans, C.J.: The Bergman complex of a matroid and phylogenetic trees. J. Comb. Theory, Ser. B **96**(1), 38–49 (2006). MR 2185977
10. Ardila, F., Sanchez, M.: Valuations and the Hopf Monoid of Generalized Permutahedra (2020)
11. Ardila, F., Fink, A., Rincón, F.: Valuations for matroid polytope subdivisions. Can. J. Math. **62**(6), 1228–1245 (2010)
12. Ardila, F., Castillo, F., Eur, C., Postnikov, A.: Coxeter submodular functions and deformations of Coxeter permutahedra. Adv. Math. **365**, 107039 (2020). MR 4064768

13. Ardila, F., Denham, G., Huh, J.: Lagrangian geometry of matroids. J. Am. Math. Soc. (in press). https://doi.org/10.1090/jams/1009
14. Backman, S., Eur, C., Simpson, C.: Simplicial generation of Chow rings of matroids. J. Eur. Math. Soc. (in press)
15. Berget, A., Spink, H., Tseng, D.: Log-concavity of matroid h-vectors and mixed Eulerian numbers (2020). arXiv:2005.01937
16. Borovik, A.V., Gelfand, I.M., White, N.: Coxeter Matroids. Progress in Mathematics, vol. 216. Birkhäuser Boston, Boston (2003). MR 1989953
17. Brändén, P., Huh, J.: Lorentzian polynomials. Ann. Math. (2) **192**(3), 821–891 (2020). MR 4172622
18. Brändén, P., Leake, J., Pak, I.: Lower bounds for contingency tables via Lorentzian polynomials (2020). https://arxiv.org/abs/2008.05907
19. Brandt, M., Eur, C., Zhang, L.: Tropical flag varieties. Adv. Math. (2021, in press). https://doi.org/10.1016/j.aim.2021.107695
20. Brion, M.: Points entiers dans les polyèdres convexes. Ann. Sci. Éc. Norm. Supér. (4) **21**(4), 653–663 (1988). MR 982338
21. Brion, M.: Equivariant Chow groups for torus actions. Transform. Groups **2**(3), 225–267 (1997). MR 1466694
22. Brylawski, T.: The broken-circuit complex. Trans. Am. Math. Soc. **234**(2), 417–433 (1977). MR 468931
23. Brylawski, T.: The Tutte polynomial. I. General theory. In: Matroid Theory and Its Applications, Liguori, Naples, pp. 125–275 (1982). MR 863010
24. Brylawski, T., Oxley, J.: The Tutte polynomial and its applications. In: Matroid Applications. Encyclopedia Math. Appl., vol. 40, pp. 123–225. Cambridge University Press, Cambridge (1992). MR 1165543
25. Cameron, A., Fink, A.: The Tutte polynomial via lattice point counting. J. Comb. Theory, Ser. A **188**, 105584 (2022). MR 4369644
26. Cameron, A., Dinu, R., Michalek, M., Seynnaeve, T.: Flag Matroids: Algebra and Geometry (2018). https://arxiv.org/abs/1811.00272
27. Castillo, F., Liu, F.: On the Todd class of the permutohedral variety. Algebraic Combin. (2020, in press). https://doi.org/10.5802/alco.157. arXiv:2005.01937
28. Catanese, F., Hoşten, S., Khetan, A., Sturmfels, B.: The maximum likelihood degree. Am. J. Math. **128**(3), 671–697 (2006). MR 2230921
29. Chen, J.: Matroids: a Macaulay2 package. J. Softw. Algebra Geom. **9**, 19–27 (2019)
30. Cohen, D., Denham, G., Falk, M., Varchenko, A.: Critical points and resonance of hyperplane arrangements. Can. J. Math. **63**(5), 1038–1057 (2011). MR 2866070
31. Cox, D.A., Little, J.B., Schenck, H.K.: Toric Varieties. Graduate Studies in Mathematics, vol. 124. Am. Math. Soc., Providence (2011). MR 2810322
32. Crapo, H.H.: A higher invariant for matroids. J. Comb. Theory **2**, 406–417 (1967). MR 215744
33. Crapo, H.H.: The Tutte polynomial. Aequ. Math. **3**, 211–229 (1969). MR 262095
34. Dawson, J.E.: A collection of sets related to the Tutte polynomial of a matroid. In: Graph Theory, Singapore, 1983. Lecture Notes in Math., vol. 1073, pp. 193–204. Springer, Berlin (1984). MR 761018
35. De Concini, C., Procesi, C.: Wonderful models of subspace arrangements. Sel. Math. New Ser. **1**(3), 459–494 (1995). MR 1366622
36. Demailly, J.-P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Algebraic Geom. **3**(2), 295–345 (1994). MR 1257325
37. Denham, G., Garrousian, M., Schulze, M.: A geometric deletion-restriction formula. Adv. Math. **230**(4–6), 1979–1994 (2012). MR 2927361
38. Derksen, H., Fink, A.: Valuative invariants for polymatroids. Adv. Math. **225**(4), 1840–1892 (2010)
39. Dinu, R., Eur, C., Seynnaeve, T.: K-theoretic Tutte polynomials of morphisms of matroids. J. Comb. Theory, Ser. A **181**, 105414 (2021)
40. Dupont, C., Fink, A., Moci, L.: Universal Tutte characters via combinatorial coalgebras. Algebraic Combin. **1**(5), 603–651 (2018) (en). MR 3887405
41. Edidin, D., Graham, W.: Equivariant intersection theory. Invent. Math. **131**(3), 595–634 (1998). MR 1614555
42. Edidin, D., Graham, W.: Localization in equivariant intersection theory and the Bott residue formula. Am. J. Math. **120**(3), 619–636 (1998). MR 1623412

43. Edmonds, J.: Submodular functions, matroids, and certain polyhedra. In: Combinatorial Structures and Their Applications, Proc. Calgary Internat. Conf., Calgary, Alta., 1969, pp. 69–87. Gordon and Breach, New York (1970). MR 0270945
44. Eisenbud, D., Harris, J.: 3264 and All That—a Second Course in Algebraic Geometry. Cambridge University Press, Cambridge (2016). MR 3617981
45. Eur, C.: The geometry of divisors on matroids. Ph.D. thesis, UC Berkeley (2020)
46. Eur, C., Huh, J.: Logarithmic concavity for morphisms of matroids. Adv. Math. **367**, 107094 (2020). MR 4078485
47. Eur, C., Sanchez, M., Supina, M.: The universal valuation of Coxeter matroids. Bull. Lond. Math. Soc. **53**(3), 798–819 (2021)
48. Feichtner, E.M., Sturmfels, B.: Matroid polytopes, nested sets and Bergman fans. Port. Math. **62**(4), 437–468 (2005) (English)
49. Feichtner, E.M., Yuzvinsky, S.: Chow rings of toric varieties defined by atomic lattices. Invent. Math. **155**(3), 515–536 (2004). MR 2038195
50. Fink, A., Speyer, D.E.: K-classes for matroids and equivariant localization. Duke Math. J. **161**(14), 2699–2723 (2012). MR 2993138
51. François, G., Rau, J.: The diagonal of tropical matroid varieties and cycle intersections. Collect. Math. **64**(2), 185–210 (2013) (English)
52. Fulton, W.: Introduction to Toric Varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton (1993). The William H. Roever Lectures in Geometry
53. Fulton, W.: Intersection Theory, 2nd edn. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 2. Springer, Berlin (1998). MR 1644323
54. Fulton, W., Lazarsfeld, R.: Positive polynomials for ample vector bundles. Ann. Math. **118**(1), 35–60 (1983)
55. Fulton, W., Sturmfels, B.: Intersection theory on toric varieties. Topology **36**(2), 335–353 (1997) (English)
56. Gelfand, I.M., Serganova, V.V.: Combinatorial geometries and the strata of a torus on homogeneous compact manifolds. Usp. Mat. Nauk **42**(2(254)), 107–134 (1987). 287. MR 898623
57. Gelfand, I.M., Serganova, V.V.: On the general definition of a matroid and a greedoid. Dokl. Akad. Nauk SSSR **292**(1), 15–20 (1987). MR 871945
58. Gelfand, I.M., Goresky, R.M., MacPherson, R.D., Serganova, V.V.: Combinatorial geometries, convex polyhedra, and Schubert cells. Adv. Math. **63**(3), 301–316 (1987)
59. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/
60. Groemer, H.: On the extension of additive functionals on classes of convex sets. Pac. J. Math. **75**(2), 397–410 (1978). MR 513905
61. Gubler, W.: A guide to tropicalizations. In: Algebraic and Combinatorial Aspects of Tropical Geometry. Proceedings Based on the CIEM Workshop on Tropical Geometry, International Centre for Mathematical Meetings (CIEM), Castro Urdiales, Spain, December 12–16, 2011, pp. 125–189. American Mathematical Society (AMS), Providence (2013) (English)
62. Hacking, P., Keel, S., Tevelev, J.: Compactification of the moduli space of hyperplane arrangements. J. Algebraic Geom. **15**(4), 657–680 (2006). MR 2237265
63. Hampe, S.: The intersection ring of matroids. J. Comb. Theory, Ser. B **122**, 578–614 (2017)
64. Heron, A.P.: Matroid polynomials. In: Combinatorics, Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972, pp. 164–202 (1972). MR 0340058
65. Hovanskiĭ, A.G.: Newton polyhedra, and the genus of complete intersections. Funkc. Anal. Prilozh. **12**(1), 51–61 (1978). MR 487230
66. Hu, Y., Liu, C.-H., Yau, S.-T.: Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces. Adv. Theor. Math. Phys. **6**(3), 457–506 (2002) (English)
67. Huh, J.: Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. J. Am. Math. Soc. **25**(3), 907–927 (2012) (English)
68. Huh, J.: The maximum likelihood degree of a very affine variety. Compos. Math. **149**(8), 1245–1266 (2013)
69. Huh, J.: Rota's conjecture and positivity of algebraic cycles in permutohedral varieties. Ph.D. thesis (2014)
70. Huh, J.: h-Vectors of matroids and logarithmic concavity. Adv. Math. **270**, 49–59 (2015) (English)

71. Huh, J.: Tropical geometry of matroids. In: Current Developments in Mathematics 2016. Papers Based on Selected Lectures Given at the Current Development Mathematics Conference, Harvard University, Cambridge, MA, USA, November 2016 pp. 1–46. International Press, Somerville (2016) (English)
72. Huh, J., Katz, E.: Log-concavity of characteristic polynomials and the Bergman fan of matroids. Math. Ann. **354**(3), 1103–1116 (2012). MR 2983081
73. Ishida, M.-N.: Polyhedral Laurent series and Brion's equalities. Int. J. Math. **1**(3), 251–265 (1990). MR 1078514
74. Joni, S.A., Rota, G.-C.: Coalgebras and bialgebras in combinatorics. Stud. Appl. Math. **61**(2), 93–139 (1979). MR 544721
75. Katz, E.: A tropical toolkit. Expo. Math. **27**(1), 1–36 (2009) (English)
76. Katz, E.: Tropical intersection theory from toric varieties. Collect. Math. **63**(1), 29–44 (2012) (English)
77. Katz, E.: Matroid theory for algebraic geometers. In: Nonarchimedean and Tropical Geometry, Simons Symp., pp. 435–517. Springer, Cham (2016). MR 3702317
78. Katz, E., Payne, S.: Realization spaces for tropical fans. In: Combinatorial Aspects of Commutative Algebra and Algebraic Geometry. The Abel Symposium 2009. Proceedings of the 6th Abel Symposium, Voss, Norway, June 1–4, 2009, pp. 73–88. Springer, Berlin (2011) (English)
79. Krajewski, T., Moffatt, I., Tanasa, A.: Hopf algebras and Tutte polynomials. Adv. Appl. Math. **95**, 271–330 (2018)
80. Las Vergnas, M.: Extensions normales d'un matroïde, polynôme de Tutte d'un morphisme. C. R. Acad. Sci. Paris, Sér. A-B **280**, A1479–A1482 (1975)
81. Las Vergnas, M.: On the Tutte polynomial of a morphism of matroids. Ann. Discrete Math. **8**, 7–20 (1980). Combinatorics 79, Proc. Colloq., Univ. Montréal, Montreal, Que., 1979, Part I
82. Lascu, A.T., Mumford, D., Scott, D.B.: The self-intersection formula and the 'formule-clef'. Math. Proc. Camb. Philos. Soc. **78**, 117–123 (1975) (English)
83. Lazarsfeld, R.: Positivity in Algebraic Geometry. I. Classical Setting: Line Bundles and Linear Series, vol. 48. Springer, Berlin (2004) (English)
84. López de Medrano, L., Rincón, F., Shaw, K.: Chern-Schwartz-MacPherson cycles of matroids. Proc. Lond. Math. Soc. (3) **120**(1), 1–27 (2020). MR 3999674
85. Macdonald, I.G.: Symmetric Functions and Hall Polynomials. 2nd edn. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York (2015). With contribution by A.V. Zelevinsky and a foreword by Richard Stanley. Reprint of the 2008 paperback edition. MR 3443860
86. Maclagan, D., Sturmfels, B.: Introduction to Tropical Geometry. Graduate Studies in Mathematics, vol. 161. Am. Math. Soc., Providence (2015)
87. MacPherson, R.D.: Chern classes for singular algebraic varieties. Ann. Math. (2) **100**, 423–432 (1974). MR 361141
88. Mason, J.H.: Matroids: unimodal conjectures and Motzkin's theorem. In: Combinatorics, Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972, pp. 207–220 (1972). MR 0349445
89. Murota, K.: Discrete Convex Analysis. SIAM Monographs on Discrete Mathematics and Applications. SIAM, Philadelphia (2003). MR 1997998
90. Nguyen, H.Q.: Semimodular functions and combinatorial geometries. Trans. Am. Math. Soc. **238**, 355–383 (1978). MR 491269
91. Nielsen, A.: Diagonalizably linearized coherent sheaves. Bull. Soc. Math. Fr. **102**, 85–97 (1974). MR 0366928
92. Orlik, P., Solomon, L.: Combinatorics and topology of complements of hyperplanes. Invent. Math. **56**(2), 167–189 (1980). MR 558866
93. Orlik, P., Terao, H.: The number of critical points of a product of powers of linear functions. Invent. Math. **120**(1), 1–14 (1995). MR 1323980
94. Oxley, J.: Matroid Theory, 2nd edn. Oxford Graduate Texts in Mathematics, vol. 21. Oxford University Press, Oxford (2011)
95. Postnikov, A.: Permutohedra, associahedra, and beyond. Int. Math. Res. Not. **2009**(6), 1026–1106 (2009) (English)
96. Rau, J.: The Tropical Poincaré-Hopf Theorem (2020). https://arxiv.org/abs/2007.11642
97. Rosu, I.: Equivariant K-theory and equivariant cohomology. Math. Z. **243**(3), 423–448 (2003). With an appendix by Allen Knutson and Rosu. MR 1970011
98. Rota, G.-C.: Combinatorial theory, old and new. In: Actes du Congrès International des Mathématiciens, Nice, 1970, Tome 3, pp. 229–233 (1971). MR 0505646

99. Schmitt, W.R.: Antipodes and incidence coalgebras. J. Comb. Theory, Ser. A **46**(2), 264–290 (1987). MR 914660
100. Schwartz, M.-H.: Classes caractéristiques définies par une stratification d'une variété analytique complexe. C. R. Acad. Sci. Paris **260**, 3262–3264 (1965). 3535–3537. MR 184254
101. Silvotti, R.: On a conjecture of Varchenko. Invent. Math. **126**(2), 235–248 (1996) (English)
102. Speyer, D.E.: Tropical linear spaces. SIAM J. Discrete Math. **22**(4), 1527–1558 (2008). MR 2448909
103. Speyer, D.E.: A matroid invariant via the K-theory of the Grassmannian. Adv. Math. **221**(3), 882–913 (2009). MR 2511042
104. Sturmfels, B.: Solving Systems of Polynomial Equations. CBMS Regional Conference Series in Mathematics, vol. 97. Am. Math. Soc., Providence (2002). Published for the Conference Board of the Mathematical Sciences, Washington, DC. MR 1925796
105. Teissier, B.: Du théorème de l'index de Hodge aux inégalités isopérimétriques. C. R. Acad. Sci. Paris, Sér. A-B **288**(4), A287–A289 (1979). MR 524795
106. Tutte, W.T.: A contribution to the theory of chromatic polynomials. Can. J. Math. **6**, 80–91 (1954). MR 61366
107. Varchenko, A.: Critical points of the product of powers of linear functions and families of bases of singular vectors. Compos. Math. **97**(3), 385–401 (1995). MR 1353281
108. Vezzosi, G., Vistoli, A.: Higher algebraic K-theory for actions of diagonalizable groups. Invent. Math. **153**(1), 1–44 (2003). MR 1990666
109. Welsh, D.J.A.: Matroid Theory. L. M. S. Monographs, vol. 8. Academic Press [Harcourt Brace Jovanovich, Publishers], London (1976). MR 0427112
110. White, N.L.: The basis monomial ring of a matroid. Adv. Math. **24**(3), 292–297 (1977). MR 437366