Colored Fermionic Vertex Models and Symmetric Functions

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Abstract. In this text we introduce and analyze families of symmetric functions arising as partition functions for colored fermionic vertex models associated with the quantized affine Lie superalgebra $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$. We establish various combinatorial results for these vertex models and symmetric functions, which include the following.

(1) We apply the fusion procedure to the fundamental R-matrix for $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ to obtain an explicit family of vertex weights satisfying the Yang–Baxter equation.
(2) We define families of symmetric functions as partition functions for colored, fermionic vertex models under these fused weights. We further establish several combinatorial properties for these symmetric functions, such as branching rules and Cauchy identities.
(3) We show that the Lascoux–Leclerc–Thibon (LLT) polynomials arise as special cases of these symmetric functions. This enables us to show both old and new properties about the LLT polynomials, including Cauchy identities, contour integral formulas, stability properties, and branching rules under a certain family of plethystic transformations.
(4) A different special case of our symmetric functions gives rise to a new family of polynomials called factorial LLT polynomials. We show they generalize the LLT polynomials, while also satisfying a vanishing condition reminiscent of that satisfied by the factorial Schur functions.
(5) By considering our vertex model on a cylinder, we obtain fermionic partition function formulas for both the symmetric and nonsymmetric Macdonald polynomials.
(6) We prove combinatorial formulas for the coefficients of the LLT polynomials when expanded in the modified Hall–Littlewood basis, as partition functions for a $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ vertex model.

Contents

CHAPTER 1

Introduction

The study of exactly solvable, or *integrable*, lattice models is a vast domain that originated at the interface between statistical and quantum mechanics well over 50 years ago; a classical introduction to the subject is Baxter's book [**5**]. Over the years it has sprouted multiple branches, some of which have since developed into full-fledged domains of research; a prominent example is the representation theory of quantum groups. A more recent, and so far less studied, branch connects integrable lattice models with the theory of symmetric functions. Earlier works in this direction include those of Kirillov–Reshetikhin [**45**], Fomin–Kirillov [**28, 29**], Lascoux–Leclerc–Thibon [**61**], Gleizer–Postnikov [**33**], Tsilevich [**84**], Lascoux [**59**], Zinn-Justin [**90**], Brubaker–Bump–Friedberg [**19**], and Korff [**52**].

The theory of symmetric functions studies remarkable families of symmetric and associated nonsymmetric polynomials with origins in diverse areas, such as the theory of finite groups, multivariate statistics, representation theory of Lie and p-adic groups, harmonic analysis on Riemannian homogeneous spaces, probability theory, mathematical physics, algebraic geometry, and enumerative combinatorics. Macdonald's treatise [**66**] provides a comprehensive introdution to this subject. Perhaps the most well-known families of symmetric and nonsymmetric functions also bear the name of Macdonald, and their special cases include other celebrated families of functions such as Schur, Jack, Hall–Littlewood, and q-Whittaker functions.

Although symmetric functions come from very different backgrounds, their structural properties show remarkable uniformity. They typically include elements such as:

- branching rules, which are combinatorial recipes for decomposing a polynomial of a family into similar polynomials with fewer variables;
- summation identities, including (skew) Cauchy and Pieri formulas;
- combinatorial procedures for computing structure constants, namely, the coefficients arising from decomposing products of functions in a family on a basis of similar functions;
- combinatorial understanding of transition matrices between different families of functions.

The theory of integrable vertex models has been shown to provide a convenient framework for simultaneously accessing each of the above features, for various families of functions.

- Branching rules for these functions typically follow directly from their representations as partition functions for a suitable lattice model under appropriate boundary data.[1]
- Summation identities of Cauchy and Pieri type were proved in many instances by applying the Yang–Baxter commutation relations underlying the solvable lattice model. For earlier examples, we refer to the works of Bump–McNamara–Nakasuji [**20**], Motegi–Sakai [**74, 73**], Borodin [**8**], Wheeler–Zinn-Justin [**86**], and Borodin–Petrov [**11**].

[1]An exception exists in the case of the general Macdonald polynomials. Partition function formulas for them were provided by Cantini–de Gier–Wheeler [**21**] and Borodin–Wheeler [**13**], which do not appear to directly relate to the corresponding branching rules.

- A lattice model approach to find structure constants was developed by Gleizer–Postnikov [**33**], Zinn-Justin [**89, 88, 87, 91**], Wheeler–Zinn-Justin [**87, 88**], and Knutson–Zinn-Justin [**50**]. It turns out to be closely related to the celebrated puzzle representation for the classical Littlewood–Richardson coefficients that goes back to Knutson–Tao [**47, 48**] and Knutson–Tao–Woodward [**49**].
- The Kostka–Foulkes transition matrix between the Schur and Hall–Littlewood symmetric functions was the subject of one of the earliest works [**45**] relating symmetric functions and integrable vertex models. Moreover, the Yang–Baxter elements in the Hecke algebra [**61**] were central in the approach developed by Lascoux for transitioning between bases of (non)symmetric functions; see [**60**] and references therein.

In addition to providing a new path to understanding previously known families of symmetric and nonsymmetric functions, integrable vertex models turned out to be useful for defining novel families that often have many of the desirable properties satisfied by the classical ones. Examples include the spin Hall–Littlewood functions and spin q-Whittaker polynomials. Their symmetric versions were introduced and analyzed in [**8, 11, 14**], and further studied by Garbali–de Gier–Wheeler [**31**] and Muccicioni–Petrov [**75**]; their nonsymmetric versions were introduced and analyzed by Borodin–Wheeler in [**12**].

The development of the theory of spin Hall–Littlewood functions and q-Whittaker polynomials was based on a thorough investigation of partition functions on semi-infinite strips built from vertex weights associated with the quantum affine algebra $U_q\big(\widehat{\mathfrak{sl}}(n+1)\big)$. The rank $n = 1$ case corresponded to symmetric functions, and the higher rank $n > 1$ case to nonsymmetric ones. It was essential to consider not only weights coming from the fundamental R-matrix for $U_q\big(\widehat{\mathfrak{sl}}(n+1)\big)$, but also those obtained by its fusion in both directions.[2]

The goal of the present work is to develop a general theory for symmetric functions associated with the quantum affine deformation of the general linear Lie superalgebra $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$, with $m = 1$. The choice $m = 1$ was dictated by the fact that it is the case most different from the one related to $U_q\big(\widehat{\mathfrak{sl}}(n+1)\big)$. In the latter case, the vertex models were *bosonic*, meaning that there is no constraint on the number of arrows that can exist along a given edge; in the former, they will be *fermionic*, meaning that there is an exclusion rule preventing arrows of the same color from occupying the same edge. Another reason for restricting to the case $m = 1$ is that, in this setting, basic Cauchy summation identities involve only *symmetric* functions for any choice of the rank n, while for $m > 1$ these identities must also incorporate nonsymmetric ones.[3] In this text, we will primarily focus on the $n > 1$ setting; the somewhat different $n = 1$ story will be treated separately in [**1**].

Understanding the relationship between the bosonic and fermionic theories has been a key motivation for us. At first glance, they appear quite different; for example, the basic objects in the rank 1 bosonic and fermionic situations are the Hall–Littlewood [**8, 86**] and big Schur symmetric functions [**87**], respectively. In fact, the nontrivial interplay between these two families of functions is observed throughout Chapter III of Macdonald's book [**66**]. We were able to observe a few further hidden points of contact as well. One is a surprising fact that cylindrical partition functions of the type considered in [**13**] yield symmetric and nonsymmetric Macdonald polynomials for *both* bosonic and fermionic vertex weights. Another is a color merging result, stating that partition functions

[2] In representation theoretic terms, this means specializing the universal R-matrix to the tensor product of symmetric powers of the fundamental representation and further analytically continuing in the exponents of those powers.

[3] It should be noted, however, that some of our results involve vertex models with $m > 1$; more details will be given below.

for fermionic vertex models can be obtained by partially anti-symmetrizing partition functions for certain bosonic vertex models of a higher rank.

The present work enjoys multiple connections with the recents works of Brubaker–Buciumas–Bump–Gustafsson [**18, 16, 17**], where vertex models related to $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ also played a central role and some of the same symmetric functions appeared, such as the Lascoux–Leclerc–Thibon (LLT) polynomials of [**62**]. While most of our results appear to be different from those proved there, we are hopeful that the relationship between these two directions will become stronger with time.

The development of the theory of spin Hall–Littlewood and q-Whittaker polynomials, as well as various other recent works in the subject, came hand-in-hand with a development of their probabilistic applications; for example, an interested reader might consult the lecture notes of Borodin–Petrov [**10**] in the rank 1 case and the paper of Borodin–Gorin–Wheeler [**9**] in the higher rank case. In the present work we shy away from the probabilistic direction, but the forthcoming paper [**1**] will include a substantial probabilistic component in the rank $n = 1$.

Let us now briefly describe our main results.

1. *Fused weights*: We start by applying the fusion procedure to the fundamental R-matrix of Bazhanov–Shadrikov [**6**] for the quantum affine superalgebra $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$. This yields an explicit family of vertex weights satisfying the Yang–Baxter equation, which are in general given by a (sort of elaborate) sum. Under various specializations of their parameters, we show that these weights simplify considerably, factoring completely.

2. *Symmetric functions*: We define families of symmetric rational functions $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and[4] $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as partition functions under certain boundary data for the lattice models associated with the $m = 1$ cases of the above (otherwise fully general) fused $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ vertex weights. These skew functions are indexed by n-tuples of signatures $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\big)$ and $\boldsymbol{\mu} = \big(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}\big)$, and they depend on four sets of parameters $(\mathbf{x}, \mathbf{r}; \mathbf{y}, \mathbf{s})$. We further establish several combinatorial properties for these functions. These include branching rules, which are direct consequences of the definition of the underlying lattice model; Cauchy identities, which are proven through Yang–Baxter commutation relations, together with a fully factorized expression for the partition function of our vertex model under domain-wall boundary data; and contour integral formulas, which are shown through a reduction to the bosonic $U_q\big(\widehat{\mathfrak{sl}}(n+1)\big)$ case of [**12**], using a color merging result.

3. *LLT polynomials*: We next analyze how these functions specialize under various degenerations of their parameter sets. In certain cases when only one (among four) of these sets remains generic, we show that our symmetric functions specialize to skew LLT polynomials $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$.

These polynomials can be viewed as q-deformations of products of skew Schur functions $s_{\lambda/\mu}(\mathbf{x})$, in that for $q = 1$ we have $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \prod_{j=1}^{n} s_{\lambda^{(j)}/\mu^{(j)}}(\mathbf{x})$. They were originally introduced in [**62**] as plethystic images of Schur functions under q-deformed power sum operators; by setting q to roots of unity, such images are useful for analyzing how compositions of $\mathrm{GL}_n(\mathbb{C})$-representations decompose into irreducible ones [**23, 41**]. However, since then, they have been found to be ubiquitous in algebra. Within algebraic combinatorics, they satisfy branching rules and Cauchy identities [**57**]; are generalizations of modified[5] Hall–Littlewood polynomials [**62**] and (after plethysm) certain classes of quasi-symmetric chromatic polynomials [**22, 39**]; and arise as coefficients of modified Macdonald polynomials when expanded in powers of q [**37**]. Within enumerative algebraic geometry, they arise

[4]We also introduce a third family $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, which will play a more subsidiary role in this text.

[5]Here, the term "modified" refers to the plethystic image under the map sending the sum $X = \sum_{i=1}^{\infty} x_i$ of formal variables (arguments for a symmetric function) to $(1-q)^{-1}X$.

in the analysis of certain flag varieties [**62, 81, 83, 35**] and of the Frobenius series for the space of diagonal harmonics [**38, 64, 36, 22**]. Within representation theory, they appear in the Fock space representation for $U_q\big(\widehat{\mathfrak{sl}}(n)\big)$ [**62, 63, 58**] and exhibit relations with the Kazhdan–Lusztig theory for affine Hecke algebras [**63, 35**].

Our interpretation of the LLT polynomials as partition functions for an integrable lattice model enables us to prove old and new combinatorial properties about them. These include branching rules under a family of plethystic substitutions; both standard and dual Cauchy identities (originally due to Lam [**57**]); their specializations to modified Hall–Littlewood polynomials (originally due to Lascoux–Leclerc–Thibon [**62**]); and contour integral formulas for them in terms of nonsymmetric Hall–Littlewood polynomials (originally implicitly due to Grojnowski–Haiman [**35**]).

Let us mention that the recent work of Corteel–Gitlin–Keating–Meza [**25**], which was pursued independently of ours, also provides a solvable lattice model for the LLT polynomials[6] that coincides with one of ours. They further use this vertex model to establish LLT Cauchy identities and degenerations to modified Hall–Littlewood polynomials.

4. *Factorial LLT polynomials*: By instead letting all but two of their parameter sets remain generic, our functions become inhomogeneous generalizations of the LLT polynomials that we call factorial LLT polynomials. Our reason for this terminology is that we show these polynomials possess a certain vanishing property, similar to those satisfied by factorial Schur and interpolation Macdonald polynomials. Vanishing properties of this type have proven to be central in the theory of (non)symmetric functions, as in many cases they fully characterize the underlying function; see the survey [**76**] of Okounkov. Although the factorial LLT polynomials are not fully characterized (but might "partially" be; see Question 1.4.6 below) by their vanishing property for general n, they are when $n = 1$, in which case we show that they in fact coincide with the factorial Schur functions.

5. *Macdonald polynomials*: By taking our $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ vertex model on a cylinder and suitably specializing parameters, we obtain partition function formulas for both the nonsymmetric and symmetric Macdonald polynomials. The nonsymmetric formula differs from the one provided in [**13**], as the latter was bosonic (and therefore involved an infinite sum), while ours here is fermionic (and therefore only involves a finite sum). Comparing this lattice model interpretation for the symmetric Macdonald polynomials with that for the LLT polynomials, we deduce a new proof of an expression (originally due to Haglund–Haiman–Loehr [**37**]) for the modified Macdonald polynomials as a linear combination of LLT ones.

6. *Transition coefficients*: We provide a combinatorial procedure for determining the transition coefficients of both modified Macdonald and LLT polynomials in the basis of modified Hall–Littlewood polynomials.[7] These coefficients are expressed through partition functions for a suitably specialized fused $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ vertex model under certain boundary data; it bears close similarities with the puzzle interpretation for other families of expansion coefficients studied in [**47, 49, 33, 89, 87, 88, 50, 91**]. Let us mention that not all weights for this lattice model are nonnegative as polynomials in the underlying parameters q and t, and they cannot be, since the expansion of modified Macdonald or LLT polynomials in the modified Hall–Littlewood basis can involve negative coefficients. By combining our results with known statements for the Kostka–Foulkes transition matrix between Schur and Hall–Littlewood functions, we deduce a (not manifestly nonnegative) lattice model representation for the expansion coefficients of the modified Macdonald and LLT polynomials in the Schur basis.

[6]The same vertex model was also proposed by Curran–Yost-Wolff–Zhang–Zhang [**27**], where the Yang–Baxter equation was confirmed for $n \leqslant 3$ but left open for $n > 3$.

[7]This also yields the transition coefficients between the (standard) Macdonald and Hall–Littlewood bases.

Over the past two decades, an extensive literature has developed surrounding the decomposition of LLT and modified Macdonald polynomials into the basis of Schur functions, as the resulting coefficients admit interpretations as fundamental invariants from algebraic geometry [**40, 83**] and representation theory [**62, 63, 35**]. Examples of such works include those of Leclerc–Thibon [**63**], Haiman [**40**], Grojnowski–Haiman [**35**], Assaf [**3, 4**], Blasiak [**7**], Alexandersson–Uhlin [**2**], and Foster [**30**]; however, the combinatorial procedures there seem to be of a different nature from ours, and it is not transparent to us how to match them. The transition coefficients between the modified Macdonald and modified Hall–Littlewood bases have also been considered recently in the work of Mellit [**68**], who interpreted them through conjugacy classes of nilpotent matrices over a finite field with specified Jordan form. However, enumerating over such classes is intricate, and so turning his interpretation into an explicit combinatorial algorithm for determining these coefficients seems to be difficult.

Having outlined our results, let us proceed to describe them in more detail. Throughout this text, we fix a complex number $q \in \mathbb{C}$.

1.1. Fused Weights

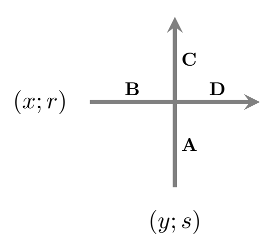
The vertex models we consider in this text will be ensembles of directed up-right paths on subdomains of the square lattice; each path in this ensemble will be labeled by a *color*, which is an index in $\{1, 2, \dots n\}$. Every vertex in the domain has some number of colored paths entering and exiting it and, depending on this local path configuration (or *arrow configuration*), the vertex is assigned a weight. The total ensemble weight is then given by the product of the weights of its vertices.

The arrow configuration at a vertex $v \in \mathbb{Z}^2$ will be indexed by an ordered sequence of four elements $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ in $\mathbb{Z}^n_{\geqslant 0}$. Here, $\mathbf{A} = (A_1, A_2, \dots, A_n)$ counts the number of paths of colors $1, 2, \dots, n$ vertically entering through v, respectively. In the same way, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ count the colored paths horizontally entering, vertically exiting, and horizontally exiting v, respectively. Thus, one can view $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ as the *states* of the south, west, north, and east edges adjacent to v, respectively.

In addition to depending on an arrow configuration $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$, the weight of a vertex $v \in \mathbb{Z}^2$ will also be governed by several complex parameters. The first among them consist in two pairs of *rapidity parameters* $(x; r)$ and $(y; s)$, which are associated with the row and column intersecting to form v, respectively; these rapidities $(x; r)$ and $(y; s)$ may vary across the domain but remain constant along rows or columns, respectively. The last is a *quantization parameter* q, which cannot vary and is fixed throughout the model. This produces five governing parameters, but the vertex weight will in fact only depend on x and y through their quotient $z = \frac{x}{y}$, which is sometimes referred to as a *spectral parameter*.

We will diagrammatically depict vertex weights by

$$W_{x/y;q}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) =$$



sometimes omitting the labels $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$, $(x; r)$, and $(y; s)$ from the diagram when convenient. We will also often omit the dependence of the quantization parameter q from W, by writing $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = W_{x/y;q}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ (where we recall the spectral parameter $z = \frac{x}{y}$).

Our first series of results concerns the derivation of an explicit family vertex weights that satisify the Yang–Baxter equation (Theorem 1.1.2 below); they are provided by the following definition. In what follows, for any vector $\mathbf{X} = (X_1, X_2, \ldots, X_k) \in \mathbb{R}^k$ we set $|\mathbf{X}| = \sum_{i=1}^{k} X_i$; we recall the *$q$-Pochhammer symbol* $(u;q)_k = \prod_{j=1}^{k} (1 - q^{j-1} u)$; and we set

(1.1.1)
$$\varphi(\mathbf{T}, \mathbf{U}) = \sum_{1 \leqslant i < j \leqslant k} T_i U_j, \qquad \text{for } \mathbf{T} = (T_1, T_2, \ldots, T_k) \in \mathbb{R}^k \text{ and } \mathbf{U} = (U_1, U_2, \ldots, U_k) \in \mathbb{R}^k.$$

Definition 1.1.1 (Definition 5.1.1 below)**.** If at least one of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ is not in $\{0,1\}^n$, then set $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0$. Otherwise define $\mathbf{V} = (V_1, V_2, \ldots, V_n) \in \mathbb{Z}_{\geqslant 0}^n$ by setting $V_j = \min\{A_j, B_j, C_j, D_j\}$ for each $j \in [1, n]$, and denote $|\mathbf{X}| = x$ for each $X \in \{A, B, C, D, V\}$. Then set

$$\begin{aligned} W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = (-1)^v z^{d-b} r^{2c-2a} s^{2d} q^{\varphi(\mathbf{D}-\mathbf{V}, \mathbf{C}) + \varphi(\mathbf{V}, \mathbf{A}) - av + cv} \frac{(q^{1-v} r^{-2} z; q)_v}{(q^{-v} s^2 r^{-2} z; q)_v} \frac{(r^2; q)_d}{(r^2; q)_b} \\ \times \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \sum_{p=0}^{\min\{b-v, c-v\}} \frac{(q^{-v} s^2 r^{-2} z; q)_{c-p} (q^v r^2 z^{-1}; q)_p (z; q)_{b-p-v}}{(s^2 z; q)_{c+d-p-v}} \\ \times (q^{-v} r^{-2} z)^p \sum_{\mathbf{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P}, \mathbf{P})}, \end{aligned}$$

where the last sum sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$ for each $i \in [1, n]$.

Let us mention three points concerning these vertex weights. The first is that they are nonzero only if $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$, a constraint we refer to by *arrow conservation*. This essentially forces any colored arrow entering a vertex to also exit it, thereby enabling the interpretation of our vertex models as ensembles of colored paths. The second is that they are nonzero only if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$. We refer this to as *fermionicity*, as it imposes an exclusion rule that prevents two distinct paths of the same color from passing along the same edge of the lattice.

The third concerns their origin. At $r = s = q^{-1/2}$ they arise as entries for the universal R-matrix of the quantum affine superalgebra $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ under its defining (or *fundamental*) representation, whose explicit form dates back to **[6]**. In the more general case when $r = q^{-L/2}$ and $s = q^{-M/2}$ are nonnegative integer powers of $q^{-1/2}$, we obtain them through the *fusion* procedure originating in the work **[53]** of Kulish–Reshetikhin–Sklyanin. In particular, in this case our W weights are entries of the above mentioned universal R-matrix, under the product of an L-fold and M-fold symmetric tensor power of the fundamental representation. These entries then happen to be rational functions in $r = q^{-L/2}$ and $s = q^{-M/2}$, enabling us to extend their definition to arbitrary $r, s \in \mathbb{C}$ by analytic continuation. For this reason, we refer to the weights $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ from Definition 1.1.1 as *fused weights*.

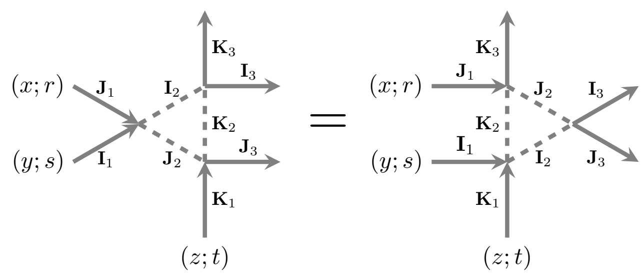
We implement this procedure in detail in Chapter 2, Chapter 3, Chapter 4, and Chapter 5 below. However, our exposition there will be less representation theoretic than what was described above. As has been done in several recent works **[26, 11, 12, 32, 9]** discussing fusion, we will instead proceed though an equivalent combinatorial (diagrammatic) framework to derive the fused weights. This fusion will in fact be applied on the quantum superalgebra $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ for arbitrary $m \geqslant 1$, with these more general fused weights given by Theorem 4.3.2 below. We only specialize to the case $m = 1$ in Chapter 5 when analytically continuing these weights in L and M (athough this can also be done for any $m \geqslant 1$).

As a consequence of this framework, we establish the following Yang–Baxter equation for our fused weights.

Theorem 1.1.2 (Proposition 5.1.4 below). *For any* $x, y, z, r, s, t \in \mathbb{C}$ *and* $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1, \mathbf{I}_3, \mathbf{J}_3, \mathbf{K}_3 \in \{0,1\}^n$, *we have*

$$\sum_{\mathbf{I}_2,\mathbf{J}_2,\mathbf{K}_2} W_{x/y}(\mathbf{I}_1,\mathbf{J}_1;\mathbf{I}_2,\mathbf{J}_2 \mid r,s) W_{x/z}(\mathbf{K}_1,\mathbf{J}_2;\mathbf{K}_2,\mathbf{J}_3 \mid r,t) W_{y/z}(\mathbf{K}_2,\mathbf{I}_2;\mathbf{K}_3,\mathbf{I}_3 \mid s,t)$$
$$= \sum_{\mathbf{I}_2,\mathbf{J}_2,\mathbf{K}_2} W_{y/z}(\mathbf{K}_1,\mathbf{I}_1;\mathbf{K}_2,\mathbf{I}_2 \mid s,t) W_{x/z}(\mathbf{K}_2,\mathbf{J}_1;\mathbf{K}_3,\mathbf{J}_2 \mid r,t) W_{x/y}(\mathbf{I}_2,\mathbf{J}_2;\mathbf{I}_3,\mathbf{J}_3 \mid r,s),$$

where both sums are over all $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2 \in \{0,1\}^n$. *Diagrammatically,*



where states along solid edges are fixed and those along dashed edges are summed over.

Although the fused weights given by Definition 1.1.1 might appear a bit unpleasant, the fact that they are governed by four parameters (q, z, r, s) makes them remarkably general. In particular, we show they factor completely under various specializations, including $n = 1$ (Example 8.1.1), $r = q^{-1/2}$ (Example 8.1.2), and a series of at least ten other degenerations (detailed in Section 8.2 and Section 8.3 below) that are depicted in Figure 1.1. In that chart, for any $\mathbf{X} = (X_1, X_2, \ldots, X_m) \in \mathbb{R}^m$ and any $1 \leqslant j \leqslant k \leqslant m$ we write $X_{[j,k]} = \sum_{i=j}^{k} X_i$. We also write $\mathbf{X} \geqslant \mathbf{Y}$ for any $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m) \in \mathbb{R}^m$ if $X_j \geqslant Y_j$ for each j.

Let us mention that directly establishing the Yang–Baxter equation for most of these degenerated weights (without realizing them as special cases of our general fused W ones) does not appear to be an immediate task. For example, this was done in Appendix A of **[25]** for the specializations $x^d q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})} \mathbf{1}_{v=0}$ depicted on the bottom-right of Figure 1.1, through an elaborate series of combinatorial reductions.

1.2. Symmetric Functions

The symmetric functions we consider in this text will be obtained from vertex models under the fused weights from Definition 1.1.1, on a semi-infinite strip with certain boundary data. To explain this further, we require some additional notation. In what follows, we fix $n \geqslant 1$ and denote the n-tuples $\mathbf{e}_0 = (0, 0, \ldots, 0) \in \{0,1\}^n$ and $\mathbf{e}_{[1,n]} = (1, 1, \ldots, 1) \in \{0,1\}^n$. We also define the

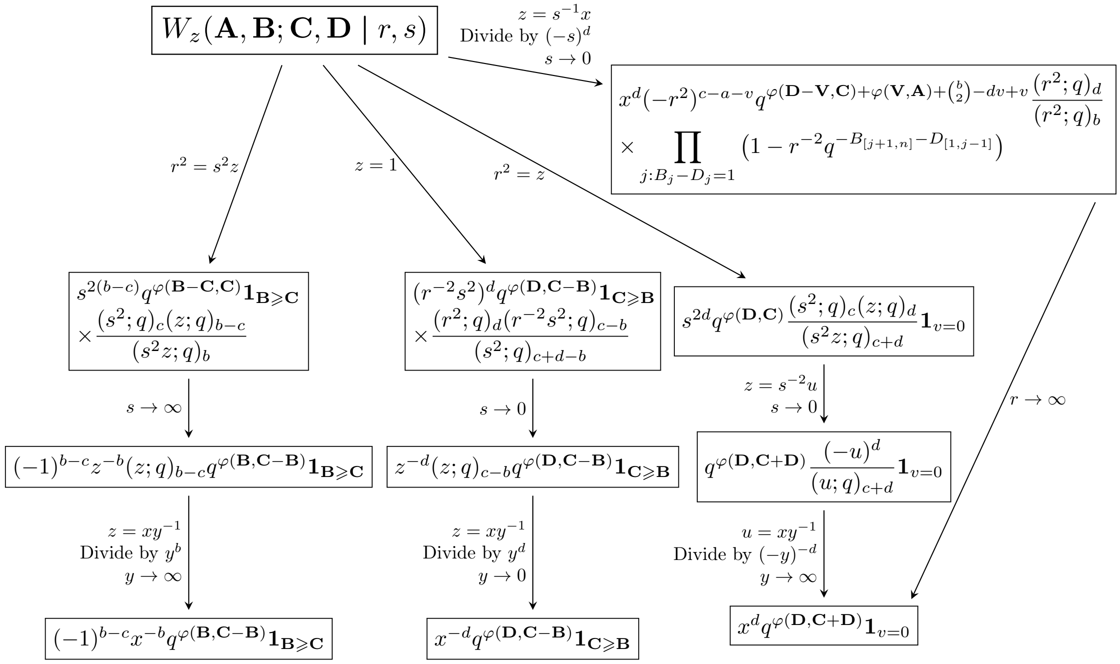


FIGURE 1.1. Depicted above are some fully factored cases of the fused weights $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$.

normalization $\widehat{W}_z$ of the fused weights W_z from Definition 1.1.1 by

$$\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) = \frac{(s^2z;q)_n}{s^{2n}(z;q)_n} W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s). \tag{1.2.1}$$

Then the W_z and $\widehat{W}_z$ weights always satisfy (see Example 5.1.2 and (5.1.5) below)

$$W_z(\mathbf{e}_0,\mathbf{e}_0;\mathbf{e}_0,\mathbf{e}_0 \mid r,s) = 1; \qquad \widehat{W}_z\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]} \mid r,s\big) = 1. \tag{1.2.2}$$

A *signature* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell) \in \mathbb{Z}_{\geqslant 0}^{\ell}$ is a finite (possibly empty) non-increasing sequence of nonnegative integers. Its *size* is $|\lambda| = \sum_{j=1}^{\ell}\lambda_j$ and its *length* is $\ell = \ell(\lambda)$. Let Sign_ℓ denote the set of all signatures of length ℓ; let $0^\ell \in \mathrm{Sign}_\ell$ denote the signature of length ℓ whose entries are all equal to 0; and let $\varnothing \in \mathrm{Sign}_0$ denote the empty signature, that is, the unique one of length 0. Further define $\mathrm{Sign} = \bigcup_{\ell=0}^{\infty}\mathrm{Sign}_\ell$, and for any $\lambda \in \mathrm{Sign}_\ell$ set

$$\mathfrak{T}(\lambda) = (\lambda_1+\ell,\lambda_2+\ell-1,\ldots,\lambda_\ell+1) \in \mathbb{Z}_{>0}^{\ell}, \tag{1.2.3}$$

whose entries are all distinct since λ is non-increasing. For example, $\mathfrak{T}(0^M) = (M,M-1,\ldots,1)$.

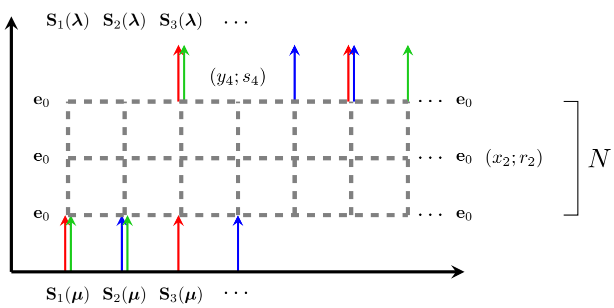
Let $\mathrm{SeqSign}_n$ denote the set of sequences of n signatures $\boldsymbol{\lambda} = \big(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(n)}\big)$ and, for any $M \geqslant 0$ let $\mathrm{SeqSign}_{n;M} \subset \mathrm{SeqSign}_n$ denote the set of those sequences such that $\ell\big(\lambda^{(j)}\big) = M$ for each $j \in [1,n]$. Further define the size of the sequence $\boldsymbol{\lambda}$ by $|\boldsymbol{\lambda}| = \sum_{j=1}^{n}\big|\lambda^{(j)}\big|$. For example, let $\mathbf{0}^M = \mathbf{0}^{(M;n)} \in \mathrm{SeqSign}_{n;M}$ denote the (size 0) sequence whose n signatures are all equal to $0^M \in \mathrm{Sign}_M$, and let $\boldsymbol{\varnothing} = \boldsymbol{\varnothing}_n \in \mathrm{SeqSign}_{n;0}$ denote the (size 0) sequence of n empty signatures.

For any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$, we introduce the following infinite sequence $\mathcal{S}(\boldsymbol{\lambda}) = \big(\mathbf{S}_1(\boldsymbol{\lambda}), \mathbf{S}_2(\boldsymbol{\lambda}), \ldots\big)$ of elements in $\{0,1\}^n$. For each $j \geqslant 1$, define $\mathbf{S}_j = \mathbf{S}_j(\boldsymbol{\lambda}) = (S_{1,j}, S_{2,j}, \ldots, S_{n,j}) \in \{0,1\}^n$ by setting $S_{i,j} = \mathbf{1}_{j \in \mathfrak{T}(\lambda^{(i)})}$ for every $i \in [1,n]$. For example, $\mathcal{S}(\boldsymbol{\varnothing}) = (\mathbf{e}_0, \mathbf{e}_0, \ldots)$; moreover, the first M entries of $\mathcal{S}(\mathbf{0}^M)$ are all $\mathbf{e}_{[1,n]}$, and the remaining ones are all $\mathbf{e}_0$. In general, we will use these sequences $\mathcal{S}(\boldsymbol{\lambda})$ to index boundary data for the vertex models associated with our symmetric functions.

Now fix an integer $N \geqslant 1$; finite sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$; and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. We will define functions $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ and $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ diagrammatically as *partition functions* for certain vertex models, that is, the sum of the total weights of all path ensembles with given boundary data; see Definition 7.1.1 below for an equivalent algebraic definition. In what follows, we fix an integer $M \geqslant 0$.

For any signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, define $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ as the partition function for the vertex model

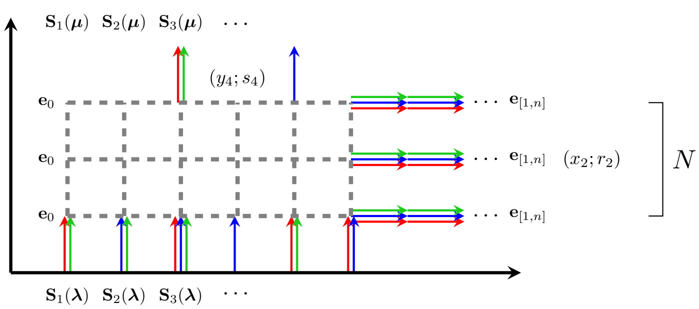
(1.2.4) $\qquad G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) =$



where the vertex weights are given by the W of Definition 1.1.1. The model here resides on a semi-infinite strip consisting of N rows. The rapidity pair in the i-th column is given by $(y_i; s_i)$, and that in the j-th row by $(x_j; r_j)$; the quantization parameter is q. No paths enter or exit through the left or right boundaries of the strip, but they enter along the bottom boundary as indexed by $\mathcal{S}(\boldsymbol{\mu})$ and exit along the top boundary as indexed by $\mathcal{S}(\boldsymbol{\lambda})$. All but finitely many vertices in this model have arrow configuration $(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0)$, which has weight 1 under W by (1.2.2); therefore, the partition function for this model is well-defined. We abbreviate $G_{\boldsymbol{\lambda}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\mathbf{0}^M}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$.

Next, for any signature sequences $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, define $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ as the partition function for the vertex model

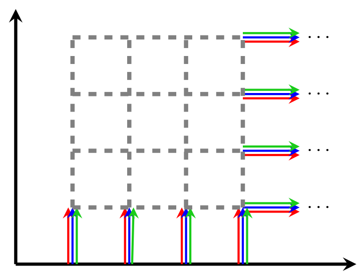
(1.2.5) $\qquad F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) =$



whose vertex weights are given by the $\widehat{W}$ of (1.2.1). The parameters and boundary data for the vertex model here are the same as that for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ above, except for two differences. First, the boundary data along the top and bottom boundaries are interchanged, that is, arrows enter along the bottom boundary as indexed by $\mathcal{S}(\boldsymbol{\lambda})$ and they exit along the top one as indexed by $\mathcal{S}(\boldsymbol{\mu})$. The second is that all horizontal edges in the strip sufficiently far to the right[8] are occupied by paths of all n colors. In this way, all but finitely many vertices in this model have arrow configuration $\big(\mathbf{e}_0, \mathbf{e}_{[1,n]}; \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$, which has weight 1 under $\widehat{W}$ by (1.2.2); therefore, the partition function for this model is also well-defined. We abbreviate $F_{\boldsymbol{\lambda}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = F_{\boldsymbol{\lambda}/\varnothing}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$.

Let us begin with an example for one of these functions. When $(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (\mathbf{0}^N, \varnothing)$, $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ becomes a partition function with *domain-wall boundary data*, given by

$$F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) =$$



.

Domain-wall partition functions of this general qualitative type have been studied extensively in the mathematical physics literature since the works of Korepin [**51**], Izergin [**42**], and Izergin–Coker–Korepin [**43**]. Such analyses have gained popularity over the past several decades, especially due to their connections with algebraic combinatorics, initially observed by Kuperberg [**56**] in the context of alternating sign matrices. In many cases, domain-wall partition functions are given explicitly by a determinant. We show the specific domain-wall partition function $F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ depicted above in fact admits the following fully factored form, which will be useful for establishing the Cauchy identity between F and G, given by Theorem 1.2.4 below.

Proposition 1.2.1 (Proposition 7.3.2 below)**.** *We have*

$$\begin{aligned} F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) &= \prod_{j=1}^{n} s_j^{2n(j-N)} r_j^{2n(j-N-1)} x_j^{n(N-j+1)} y_j^{-jn} (r_j^2;q)_n \\ &\quad \times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^{N} \prod_{j=1}^{N} (x_j y_i^{-1}; q)_n^{-1}. \end{aligned} \tag{1.2.6}$$

Let us mention that the above result was established as equation (42) in the work of Kulish–Ryasichenko [**54**] in the case $n = 1$, in which setting the above vertex model for $F_{\mathbf{0}^N}$ becomes a free-fermionic six-vertex model. For $n > 1$, Proposition 1.2.1 is, to the best of our knowledge, new.

We now proceed to more general properties of the F and G functions; the first concerns their symmetry. The following proposition shows that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ is symmetric under joint permutations of $\mathbf{x}$ and $\mathbf{r}$. It also shows that $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ is "almost symmetric," in that it multiplies

[8]In various previous works [**8, 11, 14, 12**], analogous functions were defined through vertex models in which all paths enter at the left boundary (instead of exit at the right one). We will also consider functions, denoted by $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ in Definition 7.1.1 below, defined by such boundary data in this text, but they will play less of a prominent role. Our reason for focusing on the $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ functions here is that they seem to pair better with the $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ ones for proving Cauchy identities. Still, under certain limits, we show that these F and H functions essentially coincide; see Proposition 10.4.1 below.

by an explicit factor under any such joint permutation. In the below, $\mathfrak{S}_N$ denotes the symmetric group on N elements, and we set $\sigma(\mathcal{I}) = \big(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(N)}\big)$ for any permutation $\sigma \in \mathfrak{S}_N$ and sequence $\mathcal{I} = (i_1, i_2, \ldots, i_N)$.

Proposition 1.2.2 (Proposition 7.2.2 below)**.** *For any $\sigma \in \mathfrak{S}_N$,*

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\sigma(\mathbf{x}); \sigma(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s});$$

$$F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\sigma(\mathbf{x}); \sigma(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) = F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \prod_{\substack{1 \leqslant i < j \leqslant N \\ \sigma(i) > \sigma(j)}} \frac{(r_j^2 x_i x_j^{-1}; q)_n}{(r_i^2 x_i^{-1} x_j; q)_n} \left(\frac{r_i^2 x_j}{r_j^2 x_i} \right)^n.$$

We next have the following branching rules for F and G.

Proposition 1.2.3 (Proposition 1.2.3 below)**.** *Suppose $N = K + L$, and denote $\mathbf{x}' = (x_1, \ldots, x_K)$, $\mathbf{x}'' = (x_{K+1}, \ldots, x_{K+L})$, $\mathbf{r}' = (r_1, \ldots, r_K)$, and $\mathbf{r}'' = (r_{K+1}, \ldots, r_{K+L})$. For any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Sign}_{n;M}$,*

$$\sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M}} G_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}),$$

Moreover, for any $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{Sign}_{n;M}$,

$$\sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M+L}} F_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}) F_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) = F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).$$

We further show the following Cauchy type identity between $F_{\boldsymbol{\lambda}}$ and $G_{\boldsymbol{\lambda}}$. It is in fact a special case of a more general skew Cauchy identity between the functions $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, but we will not state the latter here and instead refer to Theorem 7.3.1 below for its precise formulation.

Theorem 1.2.4 (Theorem 7.3.3 below). *Assume that there exists an integer $K > 1$ such that*

$$\sup_{k > K} \max_{\substack{1 \leqslant i \leqslant M \\ 1 \leqslant j \leqslant N}} \max_{\substack{a,b \in [0,n] \\ (a,b) \neq (n,0)}} \left| s_k^{2a+2b-2n} \frac{(s_k^2 u_j y_k^{-1}; q)_n (u_j y_k^{-1}; q)_a}{(u_j y_k^{-1}; q)_n (s_k^2 u_j y_k^{-1}; q)_a} \frac{(w_i y_k^{-1}; q)_b}{(s_k^2 w_i y_k^{-1}; q)_b} \right| < 1.$$

Then,

$$\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} F_{\boldsymbol{\lambda}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\lambda}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) = F_{\mathbf{0}^N}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{(t_i^2 u_j w_i^{-1}; q)_n}{t_i^{2n} (u_j w_i^{-1}; q)_n},$$

with $F_{\mathbf{0}^N}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ given by (1.2.6).

Let us conclude by mentioning that we also establish a contour integral representation for $G_{\boldsymbol{\lambda}}$ as Theorem 11.4.2 (and also for the more general skew function $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as Proposition 11.3.1 and Corollary 11.3.2) below. Although we will not precisely state the fully general version of this result in this introduction, we will provide its degeneration to the LLT case as Corollary 1.3.6 below.

1.3. LLT Polynomials

We next consider specializations for our symmetric functions $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ and $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ under some of the limit degenerations depicted in Figure 1.1. In particular, here we will be mainly focused on the bottom-right degeneration (with weight $x^d q^{\varphi(\mathbf{D}, \mathbf{C}+\mathbf{D})} \mathbf{1}_{v=0}$) shown there. In what follows, we fix a set of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

Define[9] $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$ and $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$ as partition functions for the vertex models shown in (1.2.4) and (1.2.5), respectively, but where for each $j \in [1,N]$ we replace the original W_{x_j/y_i} and $\widehat{W}_{x_j/y_i}$ weights in the j-th row of the model with the degenerated ones defined by

$$W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) \mapsto x_j^d q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})} \mathbf{1}_{v=0} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}};$$
$$\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) \mapsto x_j^{d-n} q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})-\binom{n}{2}} \mathbf{1}_{v=0} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}},$$

respectively. The below theorem then indicates that these specialized functions are given (up to global multiplicative factors) by the Lascoux–Leclerc–Thibon (LLT) polynomials $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};q)$ originally introduced in [**62**]; we refer to Section 9.1 and Section 9.2 below for the detailed definition of these polynomials. In what follows, for any sequence of signatures $\boldsymbol{\lambda} = \big(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(n)}\big) \in \mathrm{SeqSign}_n$, set

$$\psi(\boldsymbol{\lambda}) = \frac{1}{2} \sum_{1\leqslant i<j\leqslant n} \sum_{a\in\mathfrak{T}_i} \sum_{b\in\mathfrak{T}_j} \mathbf{1}_{a>b}, \tag{1.3.1}$$

where we have abbreviated $\mathfrak{T}_k = \mathfrak{T}\big(\lambda^{(k)}\big)$ for each index $k \in [1,n]$ (recall $\mathfrak{T}$ from (1.2.3)).

Theorem 1.3.1 (Parts 1 and 2 of Theorem 9.3.2 below). *The following statements hold.*

(1) For any sequences of signatures $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$,

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0) = q^{\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};q).$$

(2) For any sequences of signatures $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$,

$$\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0) = q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})+\binom{M}{2}\binom{n}{2}/2-\binom{M+N}{2}\binom{n}{2}/2} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q^{1-n}\mathbf{x}^{-1};q) \prod_{j=1}^{N} x_j^{n(j-M-N)}.$$

In addition to using the bottom-right limiting weights shown in Figure 1.1 to degenerate our symmetric functions to the LLT polynomials, it is also possible to use the bottom-middle and bottom-left ones depicted there. We refer to parts 3 and 4 of Theorem 9.3.2 below for the specific statements of these results.

The vertex model interpretation given by Theorem 1.3.1 is well-suited for establishing various properties about the LLT polynomials. We provide four here; the first, second, and fourth are known from [**57, 62, 35**], respectively, and the third appears to be new.

The first is a pair of Cauchy identities, originally shown as Theorem 35 and Proposition 36 of [**57**]. In what follows, for any signature sequence $\boldsymbol{\lambda} = \big(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(n)}\big) \in \mathrm{Sign}_n$, we define $\boldsymbol{\lambda}' = \big(\lambda'^{(1)},\lambda'^{(2)},\ldots,\lambda'^{(n)}\big) \in \mathrm{Sign}_n$ by setting each $\lambda'^{(j)}$ to be the dual of $\lambda^{(n-j)}$ (that is, obtained from the latter by transposing its Young diagram, as shown on the top of Figure 10.6 below).

Corollary 1.3.2 (Corollary 9.4.1 and Corollary 9.4.2 below)**.** *Fix sequences of complex numbers $\mathbf{x} = (x_1,x_2,\ldots,x_N)$ and $\mathbf{y} = (y_1,y_2,\ldots,y_M)$. If $|q| < 1$ and $|x_j|,|y_i| < 1$ for each $i \in [1,M]$ and $j \in [1,N]$, then*

$$\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q)\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{y};q) = \prod_{i=1}^{M}\prod_{j=1}^{N} (x_j y_i;q)_n^{-1}, \tag{1.3.2}$$

[9] Our reason for the notation $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$ (and $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$) is that this function is obtained from the original $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ (and $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$, respectively) by, after suitably normalizing, first sending each parameter in $\mathbf{s}$ to 0, and then sending each one in $\mathbf{r}$ and $\mathbf{y}$ to ∞ and 0, respectively. See (8.4.3) (and (8.4.4)) below.

and

$$\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q)\mathcal{L}_{\boldsymbol{\lambda}'}(\mathbf{y};q^{-1}) = \prod_{i=1}^{M}\prod_{j=1}^{N}(-q^{(n-1)/2}x_jy_i;q)_n. \tag{1.3.3}$$

The standard Cauchy identity (1.3.2) essentially follows directly from Theorem 1.2.4 and Theorem 1.3.1. The dual Cauchy identity (1.3.3) is shown[10] also using Theorem 1.2.4 and the second part of Theorem 1.3.1 expressing an LLT polynomial as a degeneration of a F function; however, we further require part 3 of Theorem 9.3.2 below that expresses a dual LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}'}(\mathbf{x};q^{-1})$ as a degeneration of a G function (under the bottom-middle limit depicted in Figure 1.1).

The second result is a *stability* property, originally shown as Theorem 6.6 of **[62]**, stating that if each signature in $\boldsymbol{\lambda}$ only contains one part, then the LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q)$ is a modified Hall–Littlewood polynomial $Q'_\lambda(\mathbf{x})$; we refer to the beginning of Section 9.6 for a definition of the latter.

Proposition 1.3.3 (Proposition 9.6.1 below)**.** *Fix a signature* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathrm{Sign}_n$, *and define* $\lambda^{(j)} = (\lambda_{n-j+1})\in\mathrm{Sign}_1$, *for each* $j\in[1,n]$. *Letting* $\boldsymbol{\lambda} = \big(\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(n)}\big)\in\mathrm{SeqSign}_{n;1}$, *we have* $\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q) = Q'_\lambda(\mathbf{x})$.

The use of the vertex model interpretation for the LLT polynomials (Theorem 1.3.1) in establishing this stability property is that there also exists a vertex model, due to Garbali–Wheeler **[32]**, for the modified Hall–Littlewood polynomials. The vertex weights for the latter model are quite similar to those of the former, enabling a comparison between the two that verifies Proposition 1.3.3.

Let us mention that Theorem 3.4 of the recent work **[25]** (pursued independently from ours) also establishes the first part of Theorem 1.3.1. Using this, Corollary 6.13 of **[25]** proves the standard Cauchy identity (1.3.2), and Proposition 5.1 of **[25]** proves the stability property Proposition 1.3.3.

Our third result provides a vertex model for the LLT polynomials under a certain family of plethysms. We will provide more detailed definitions in Section 9.5 below but, briefly, one views the skew LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};q)$ as an element in the ring of symmetric functions in $\mathbf{x}$. Defining the formal sum $X = \sum_{x\in\mathbf{x}} x$, for any variable u one lets $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big[(1-u)X\big]$ denote the image of $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};q)$ under the plethystic substitution $X\mapsto(1-u)X$.

Now let $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};r\mid 0;0)$ denote the partition function for the vertex model shown in (1.2.4), but where for each $j\in[1,N]$ we replace the original weight W_{x_j/y_i} in the j-th row of the model by the degenerated ones depicted in the top-right of Figure 1.1, namely,

$$\begin{aligned} W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r,s) \mapsto x_j^d(-r^2)^{c-a-v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\binom{b}{2}-dv+v}\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \\ \times\frac{(r^2;q)_d}{(r^2;q)_b}\prod_{j:B_j-D_j=1}\big(1-r^{-2}q^{-B_{[j+1,n]}-D_{[1,j-1]}}\big). \end{aligned}$$

Then, we have the following result.

Proposition 1.3.4 (Proposition 9.5.4 below)**.** *For any variable* r, *we have the plethystic identity*

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big[(1-r^{-2})X\big] = q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})}\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};r\mid 0;0).$$

[10] As discussed in **[57]**, it is alternatively possible to derive the dual Cauchy identity directly from the standard one by applying to it the involutory automorphism ω on the ring of symmetric functions that interchanges complete and elementary symmetric functions.

By combining Proposition 1.3.4 with Proposition 1.2.3, one can obtain explicit branching rules for the LLT polynomials under the plethystic substitutions $X \mapsto (1-u)X$. Let us mention that the skew LLT polynomials under such plethysms have been considered before in the algebraic combinatorics literature. For example, in certain cases when $u = q$ (that is, $r = q^{-1/2}$), it was shown by Carlsson–Mellit [**22**] that they coincide with chromatic quasisymmetric functions, as introduced by Shareshian–Wachs [**82, 81**], associated with the incomparability graph of a unit interval order.

The fourth result provides a contour integral formula for skew LLT polynomials in terms of nonsymmetric Hall–Littlewood polynomials. To state it, we must introduce some additional notation. For any sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}_{>0}^n$ and sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, let $\mathtt{f}_\mu^{(q)}(\mathbf{x} \mid 0)$ and $\mathtt{g}_\mu^{(q)}(\mathbf{x} \mid 0)$ denote certain normalizations of the nonsymmetric Hall–Littlewood polynomials, given more precisely in Definition 11.1.3 below. Moreover, for any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M}$, let $\Upsilon(\boldsymbol{\lambda})$ denote the set of sequences $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{nM}) \in \mathbb{Z}_{>0}^{nM}$ such that $\kappa^{(i)} = \big(\kappa_{(i-1)M+1}, \kappa_{(i-1)M+2}, \ldots, \kappa_{iM}\big)$ is a permutation of $\mathfrak{T}\big(\lambda^{(i)}\big)$ (where we recall $\mathfrak{T}$ from (1.2.3)), for each $i \in [1,n]$. For any $\kappa \in \Upsilon(\boldsymbol{\lambda})$, we also set

$$\mathrm{inv}_{\boldsymbol{\lambda}}(\kappa) = \sum_{i=1}^{n} \mathrm{inv}\left(\kappa^{(i)}\right), \qquad \text{where} \qquad \mathrm{inv}\left(\kappa^{(i)}\right) = \sum_{1 \leqslant j < k \leqslant M} \mathbf{1}_{\kappa_{(i-1)M+j} > \kappa_{(i-1)M+k}}.$$

Then we have the following theorem, which provides a relationship between the skew LLT polynomials $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and the nonsymmetric Hall–Littlewood polynomials $\mathtt{f}_\mu^{(q)}$ and $\mathtt{g}_\mu^{(q)}$. It was originally implicitly shown as equation (34) of [**35**].

THEOREM 1.3.5 (Corollary 11.5.3 below). *Fix $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_N) \subset \mathbb{C}$. Letting $\kappa = \Upsilon(\boldsymbol{\lambda})$ denote the unique element of $\Upsilon(\boldsymbol{\lambda})$ such that $\mathrm{inv}_{\boldsymbol{\lambda}}(\kappa) = 0$, we have*

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \frac{1}{(2\pi\mathbf{i})^{nM}} \frac{q^{\psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\lambda})}}{(1-q^{-1})^{nM}} \oint \cdots \oint \mathtt{f}_\kappa^{(q)}(\mathbf{u}^{-1} \mid 0) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_\nu^{(q)}(\mathbf{u} \mid 0) \\ \times \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{q^{-1}u_j - u_i} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1 - u_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},\end{aligned} \tag{1.3.4}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_{nM}) \subset \mathbb{C}$, and each u_i is integrated along a positively oriented, closed contour Γ_i satisfying the following two properties. First, each Γ_i contains 0 and does not contain $q^{-k}x_j^{-1}$ for all integers $k \in [1, nM-1]$ and $j \in [1, M]$. Second, the $\{\Gamma_i\}$ are mutually non-intersecting, and Γ_{i-1} is contained in both Γ_i and $q^{-1}\Gamma_i$ for each $i \in [2, nM]$.

Let us briefly indicate the relation between Theorem 1.3.5 and the results of [**35**]; we refer to Remark 11.5.4 below for a more detailed explanation. By (11.5.4) below, (1.3.4) can be reformulated as

$$\mathrm{Coeff}\big[\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}, s_\theta\big] = q^{\psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\lambda})} \mathrm{Coeff}\left[s_\theta \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_\nu ; \mathtt{g}_\kappa\right], \tag{1.3.5}$$

for any signature θ. Here, we recall the Schur functions s_λ and have abbreviated $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q)$ and $\mathtt{g}_\mu = \mathtt{g}_\mu^{(q)}(\mathbf{x} \mid 0)$. Moreover, for any (non)symmetric function F and basis $\{h_\mu\}$ for the space of (non)symmetric functions, we have also let $\mathrm{Coeff}[F; h_\lambda]$ denote the coefficient of h_λ in the expansion of F over $\{h_\mu\}$. The equality (1.3.5) was originally established as equation (34) of [**35**], which was central in their proof of nonnegativity for the expansion of the LLT polynomials in the Schur basis. Indeed, the coefficients in this expansion are given by the left side of (1.3.5), and those on the right side can be equated with matrix entries for the action of a certain operator on a submodule of the affine Hecke algebra. It is shown in [**35**] that this operator admits an algebro-geometric interpretation that can be used to establish nonnegativity for its matrix entries.

In the case when $\boldsymbol{\mu} = \mathbf{0}^M$, the anti-symmetrizing sum of $\mathtt{g}_\nu^{(q)}$ on the right side of (1.3.4) can be evaluated explicitly, giving rise to the following corollary.

Corollary 1.3.6 (Corollary 11.5.5 below)**.** *Adopting the notation of Theorem 1.3.5, we have*

$$\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q) = \frac{q^{\binom{n}{2}\binom{N}{2}/2-\psi(\boldsymbol{\lambda})}}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \prod_{1\leqslant i<j\leqslant nM} \frac{u_j-u_i}{u_j-qu_i} \prod_{k=0}^{n-1} \prod_{1\leqslant i<j\leqslant M} (qu_{Mk+i}-u_{Mk+j})$$
$$\times \mathtt{f}_\kappa^{(q)}(\mathbf{u}^{-1} \mid 0) \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1-u_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

where each u_i is integrated along Γ_i.

Under the condtion when all parts of any $\lambda^{(i)}$ are less by at least $M-1$ than all parts of any $\lambda^{(j)}$ for $i<j$, the nonsymmetric Hall–Littlewood function appearing in the integral from Corollary 1.3.6 factors completely. This gives rise to the following simplified formulas for certain LLT polynomials.

Corollary 1.3.7 (Corollary 11.5.6 below)**.** *Adopting the notation of Theorem 1.3.5; assume that $\lambda_k^{(i)} + M - 1 \leqslant \lambda_{k'}^{(j)}$ whenever $1 \leqslant i < j \leqslant n$ and $k, k' \in [1, M]$. Then,*

$$\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x};q) = \frac{q^{\binom{n}{2}\binom{N}{2}/2} - \psi(\boldsymbol{\lambda})}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \prod_{1\leqslant i<j\leqslant nM} \frac{u_j-u_i}{u_j-qu_i} \prod_{k=0}^{n-1} \prod_{1\leqslant i<j\leqslant M} (qu_{Mk+i}-u_{Mk+j})$$
$$\times \prod_{i=1}^{n} \prod_{j=1}^{M} u_{iM-j+1}^{j-\lambda_j^{(i)}-M-1} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1-u_i x_j} \prod_{i=1}^{nM} du_i,$$

where each u_i is integrated along Γ_i.

By applying a residue expansion of integral in Corollary 1.3.7, it might be possible to obtain a symmetrization formula for LLT polynomials satisfying the ordering constraint described there. However, we will not pursue this here.

1.4. Factorial LLT Polynomials

In Section 1.3 we explained how considering the degenerations of the F and G functions under the bottom-right limit depicted in Figure 1.1 gave rise to the LLT polynomials. Here we consider the degenerations of these functions under the limit depicted directly above it, namely, with weight $q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})}(-u)^d (u;q)_{c+d}^{-1} \mathbf{1}_{v=0}$. We will in fact be concerned with the F function, which requires a normalization of this weight as in (1.2.1). In what follows, we fix a finite set of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and an infinite set of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$.

Define $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0)$ as the partition functions for the vertex model shown in (1.2.5), but where for each $i \geqslant 1$ and $j \in [1, N]$ we replace the original $\widehat{W}_{x_j/y_i}$ weight at the vertex $(i,j) \in \mathbb{Z}_{>0}^2$ with the degenerated one given by

$$\widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) \mapsto (-x_j y_i^{-1})^{d-n} q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})-\binom{n}{2}} \frac{(x_j y_i^{-1};q)_n}{(x_j y_i^{-1};q)_{c+d}} \mathbf{1}_{v=0} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}.$$

Observe from the second part of Theorem 1.3.1 that, upon letting each of the y_i tend to ∞, this function $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x};\infty \mid \mathbf{y};0)$ degenerates to an LLT polynomial. Although $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid \mathbf{y};0)$ is not a polynomial in $\mathbf{x}$ and $\mathbf{y}$, a mild modification of it is. More specifically, denoting $\mathbf{z}^e = (z_1^e, z_2^e, \ldots)$ for any real number e and (possibly infinite) sequence $\mathbf{z} = (z_1, z_2, \ldots)$ (and recalling the sequence $\mathcal{S}(\boldsymbol{\lambda}) = \big(\mathbf{S}_1(\boldsymbol{\lambda}), \mathbf{S}_2(\boldsymbol{\lambda}), \ldots\big)$ from Section 1.2), the function

$$\check{\mathcal{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y}) = \mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x}^{-1};\infty \mid \mathbf{y}^{-1};0) \prod_{j=1}^{N} x_j^{n(j-N)} \prod_{i=1}^{\infty} y_i^{nN - \sum_{k=1}^{i} |\mathbf{S}_k(\boldsymbol{\lambda})|},$$

is quickly verified to be a polynomial in $(\mathbf{x},\mathbf{y})$ of total degree $|\boldsymbol{\lambda}|$. We refer to $\check{\mathcal{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y})$ as a *factorial LLT polynomial*, since we will show that it satisfies a vanishing property reminiscent of the one satisfied by factorial Schur (and interpolation Macdonald) polynomials.

These vanishing points for this LLT factorial polynomial will take the form $\{x_j = q^{n-\kappa_j-1} y_{\mathfrak{m}_j}\}$ for some integers $\mathfrak{m}_j \geqslant 1$ and $\kappa_j \in [0, n-1]$. It will be convenient to express such specializations through *marked sequences*, which are pairs $(\mathfrak{m}, \kappa)$ of integer sets of the same length, such the entries $\mathfrak{m}_1 > \mathfrak{m}_2 > \cdots > \mathfrak{m}_\ell$ of $\mathfrak{m}$ are decreasing and positive, and such that each entry of $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$ is in $[0, n-1]$. In what follows, we will often only refer to $\mathfrak{m}$ as the marked sequence and view κ as its *marking*, in that each entry $\mathfrak{m}_j \in \mathfrak{m}$ is *marked* by the corresponding entry $\kappa_j \in \kappa$.

To state the vanishing property, we require the notion of a splitting for a marked sequence.

Definition 1.4.1 (Definition 12.1.1 below)**.** Let $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_\ell)$ denote a marked sequence with marking $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$. A *splitting* of $(\mathfrak{m}, \kappa)$ is a family $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots, \mathfrak{m}^{(n)}\big)$ of decreasing subsequences of $\mathbb{Z}_{>0}$, such that the following two properties holds.

(1) For any $i \in [1, n]$, every entry $m \in \mathfrak{m}^{(i)}$ is equal to $\mathfrak{m}_j$, for some $j = j(m) \in [1, \ell]$.
(2) For any $j \in [1, \ell]$, there exist at least $n - \kappa_j$ distinct indices $i \in [1, n]$ such that $\mathfrak{m}_j \in \mathfrak{m}^{(i)}$.

Moreover, given two signatures $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_\ell) \in \mathrm{Sign}_\ell$ and $\mathfrak{n} = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k) \in \mathrm{Sign}_k$, we write $\mathfrak{m} \not\leq \mathfrak{n}$ if there exists an integer $j \geqslant 0$ such that $\mathfrak{m}_{\ell-j} > \mathfrak{n}_{k-j}$, where we set $\mathfrak{m}_i = \infty = \mathfrak{n}_i$ if $i \leqslant 0$. The vanishing property is then provided by the following theorem.

Theorem 1.4.2 (Theorem 12.1.2 and Remark 12.1.3 below). *Fix a marked sequence* $(\mathfrak{m}, \kappa)$ *of length* $\ell \leqslant N$ *and some* $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\big) \in \mathrm{SeqSign}_{n;N}$. *Denoting* $\mathfrak{l}^{(i)} = \mathfrak{T}\big(\lambda^{(i)}\big)$ *for each* i, *assume for any splitting* $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots, \mathfrak{m}^{(n)}\big)$ *of* $(\mathfrak{m}, \kappa)$ *that there exists an index* $h = h(\mathfrak{M}) \in [1, n]$ *such that* $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}^{(h)}$. *If* $x_j = q^{n-\kappa_j-1} y_{\mathfrak{m}_j}$ *holds for each* $j \in [1, \ell]$, *then* $\check{\mathcal{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y}) = 0$.

The proof of this theorem in Chapter 12 below makes use of the vertex model interpretation for $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x};\infty \mid \mathbf{y};0)$ by showing that, under the vanishing condition described in Theorem 1.4.2, there are no path ensembles with nonzero weight that contribute to this partition function. Based on our computer data, we were unable to find (for generic q and $\boldsymbol{\lambda}$) any additional vanishing points for $\check{\mathcal{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y})$ of the form $\{x_j = q^{k_j} y_{\mathfrak{m}_j}\}$ that are not listed in Theorem 1.4.2.

In general, it seems unlikely that the vanishing condition of Theorem 1.4.2 admits an efficient description in terms of the individual coordinates $\{\mathfrak{m}_j\}$ and $\{\mathfrak{l}_j\}$. Already in the case $N = n = \ell = 2$, such a description is quite intricate; see Lemma 12.1.7 below for the precise statement. However, it is still possible to explicitly list a subset of these coordinates, as indicated by the example below.

Example 1.4.3 (Example 12.1.6 below)**.** For $\kappa = 0^\ell$, Theorem 1.4.2 implies $\check{\mathcal{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y}) = 0$ when $x_j = q^{n-1} y_{\mathfrak{m}_j}$ for each $j \in [1, \ell]$, if there exists some $k \in [1, N]$ with $\max_{h \in [1,n]} \mathfrak{l}_{N-k}^{(h)} > \mathfrak{m}_{\ell-k}$.

In the case $n = 1$, we can in fact explicitly list all vanishing points prescribed by Theorem 1.4.2.

Example 1.4.4 (Example 12.1.4 below)**.** Assume $n = 1$, and abbreviate $\lambda = \lambda^{(1)}$ and $\mathfrak{l} = \mathfrak{l}^{(1)} = (\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_N)$. Then Theorem 1.4.2 implies $\check{\mathcal{F}}_\lambda(\mathbf{x} \mid \mathbf{y}) = 0$ when $x_j = y_{\mathfrak{m}_j}$ for each $j \in [1, \ell]$, if there exists some $k \in [1, n]$ with $\mathfrak{l}_{N-k} > \mathfrak{m}_{\ell-k}$.

One of the remarkable uses of vanishing properties is that in many cases they fully characterize the underlying (non)symmetric function, a fact originally observed by Sahi **[78, 79]** in the context of interpolation Macdonald polynomials. This phenomenon is useful for us as well. Indeed, the $n = 1$ case given by Example 1.4.4 of the vanishing property for the factorial LLT polynomials $\check{\mathcal{F}}_\lambda(\mathbf{x} \mid \mathbf{y})$ happens to coincide with that satisfied for the factorial Schur polynomials $s_\lambda(\mathbf{x} \mid \mathbf{y})$ (whose definition we recall as (12.1.2) below). The latter vanishing condition is known **[79]** to fully characterize the factorial Schur polynomials (among those of total degree $|\lambda|$ that are symmetric in $\mathbf{x}$), and so we deduce the following corollary equating them with the factorial LLT polynomials at $n = 1$.

Corollary 1.4.5 (Remark 12.1.5 and Proposition 10.4.1 below)**.** *If $n = 1$, $\check{\mathcal{F}}_\lambda(\mathbf{x} \mid \mathbf{y}) = s_\lambda(\mathbf{x} \mid \mathbf{y})$.*

Let us mention that Corollary 1.4.5 was also established as the $t = 0$ case of Theorem 1 in **[20]**.

The vanishing condition of Theorem 1.4.2 unfortunately does not fully characterize the factorial LLT polynomials $\check{\mathcal{F}}_{\boldsymbol{\lambda}}$ for $n \geqslant 2$. Indeed, denoting $\sigma(\boldsymbol{\lambda}) = \big(\lambda^{(\sigma(1))}, \lambda^{(\sigma(2))}, \ldots, \lambda^{(\sigma(n))}\big)$ for any $\sigma \in \mathfrak{S}_n$, a marked sequence $(\mathfrak{m}, \kappa)$ satisfies the vanishing condition described in Theorem 12.1.2 with respect to $\boldsymbol{\lambda}$ if and only if it does with respect to any $\sigma(\boldsymbol{\lambda})$. Thus, the $\big\{\check{\mathcal{F}}_{\sigma(\boldsymbol{\lambda})}(\mathbf{x} \mid \mathbf{y})\big\}$ over $\sigma \in \mathfrak{S}_n$ all satisfy the same vanishing condition but typically do not coincide up to a global factor.[11]

Still, one can ask whether this failure of characterization is exhaustive, in the following sense.

Question 1.4.6. Let $P(\mathbf{x}; \mathbf{y})$ denote a polynomial of total degree $|\boldsymbol{\lambda}|$, which is symmetric in $\mathbf{x}$. If $P(\mathbf{x}; \mathbf{y}) = 0$ for any $(\mathbf{x}; \mathbf{y})$ satisfying the conditions of Theorem 1.4.2, then do there exist constants $\{a_\sigma\}_{\sigma \in \mathfrak{S}_n} \subset \mathbb{C}$ such that $P(\mathbf{x}; \mathbf{y}) = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma \check{\mathcal{F}}_{\sigma(\boldsymbol{\lambda})}(\mathbf{x} \mid \mathbf{y})$?

We have not yet found a proof or counterexample to this statement.

1.5. Macdonald Polynomials

We next provide fermionic vertex model representations for the nonsymmetric and integral Macdonald polynomials, to which end we first require some notation. A *partition* is a finite, non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{Z}_{>0}^\ell$ of positive integers. A *(positive) composition* is a sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathbb{Z}_{>0}^\ell$ of positive integers (without any constraint on its ordering), and a *nonnegative composition* is a composition in which some of its entries may equal 0. For any (positive or nonnegative) composition μ, there exists a unique signature, denoted by μ^+, obtained from μ by permuting its entries into non-increasing order; μ^+ is a partition if μ is a positive composition. Moreover, for any $i \in [1, n]$ let $\mathbf{e}_i \in \{0, 1\}^n$ denote the n-tuple whose i-th coordinate is 1 and whose remaining coordinates are 0. Further let $\mathbf{e}_{[1,i]} \in \{0, 1\}^n$ denote the n-tuple whose first i elements are 1 and whose remaining ones are 0.

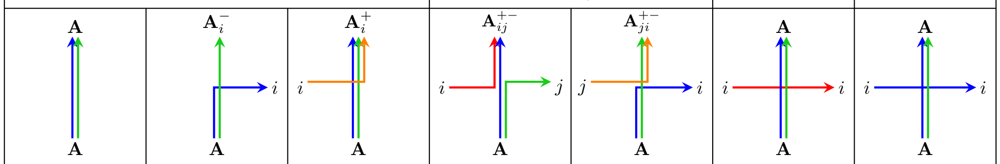
For any sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, nonnegative composition μ, and partition λ, let $f_\mu(\mathbf{x})$ denote the nonsymmetric Macdonald polynomial and $J_\lambda(\mathbf{x})$ denote the integral Macdonald polynomial, recalled in Section 13.1 and Section 15.1 below, respectively. Both of these functions depend on two parameters $q, t \in \mathbb{C}$, which are fixed throughout this section. Let us clarify that it will be

[11] A similar extensive, but non-characterizing, vanishing condition also holds for a different family of polynomials studied by Sahi–Salmasian–Serganova in **[80]**.

the Macdonald parameter t (and not q) that will correspond to the quantization parameter for our vertex models.

Next, we require the vertex weights for our model, which are obtained as the $r = q^{-1/2}$ specializations of the top-right limiting weights depicted in Figure 1.1. In this case, it is quickly verified (see Example 8.1.2 below) that arrow configurations admitting more than one arrow along a horizontal edge are assigned weight 0. Thus, any arrow configuration with nonzero weight must be of the form $(\mathbf{A}, \mathbf{e}_b; \mathbf{C}, \mathbf{e}_d)$ for some $b, d \in [0, n]$, and so we will abbreviate them by $(\mathbf{A}, b; \mathbf{C}, d)$.

These limiting vertex weights, which we denote by $L_x(\mathbf{A}, b; \mathbf{C}, d) = L_{x,t}(\mathbf{A}, b; \mathbf{C}, d)$ are then given diagrammatically by



$1 \leqslant i \leqslant n$			$1 \leqslant i < j \leqslant n$		$A_i = 0$	$A_i = 1$
$(\mathbf{A}, 0; \mathbf{A}, 0)$	$(\mathbf{A}, 0; \mathbf{A}_i^-, i)$	$(\mathbf{A}, i; \mathbf{A}_i^+, 0)$	$(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j)$	$(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$
1	$t^{A_{[i+1,n]}} x(1-t)$	1	$t^{A_{[j+1,n]}} x(1-t)$	0	$t^{A_{[i,n]}} x$	$-t^{A_{[i,n]}} x$

where arrow configurations not of the above form are assigned weight 0, and we have denoted

$$\mathbf{A}_i^+ = \mathbf{A} + \mathbf{e}_i; \qquad \mathbf{A}_j^- = \mathbf{A} - \mathbf{e}_j; \qquad \mathbf{A}_{ij}^{+-} = \mathbf{A} + \mathbf{e}_i - \mathbf{e}_j. \tag{1.5.1}$$

Let us mention that almost all of the L_x vertex weights shown above coincide with those given by equation (3.7) of **[13]**, which were used to provide bosonic partition function formulas for the nonsymmetric Macdonald polynomials. The one exceptional weight is the rightmost one (namely, $L_x(\mathbf{A}, i; \mathbf{A}, i)$ for $A_i = 1$), which differs from its counterpart in **[13]**. This contrasting vertex weight encapsulates the difference between the bosonic model used in **[13]** and the fermionic one we will use here to access the Macdonald polynomials; it will account for the finiteness of our partition function formulas here, and it will also enable us to compare these formulas with the LLT ones given by Theorem 1.3.1.

Next, for any nonnegative composition μ of length n, we introduce the following infinite sequence[12] $\mathcal{I}(\mu) = \big(\mathbf{I}_0(\mu), \mathbf{I}_1(\mu), \dots\big)$ of elements in $\{0,1\}^n$. For each $j \geqslant 0$, define $\mathbf{I}_j = \mathbf{I}_j(\mu) = (I_{1,j}, I_{2,j}, \dots, I_{n,j})$ so that $I_{k,j} = \mathbf{1}_{\mu_k = j}$ for every $k \in [1, n]$. For any $i \in [1, n]$ and $j \geqslant 0$, we further define the *twist parameters*

$$v_{i,j}(\mu) = q^{\mu_i - j} t^{\gamma_{i,j}(\mu)} \mathbf{1}_{\mu_i > j},$$

where the exponents $\gamma_{i,j}(\mu)$ are given by

$$\gamma_{i,j}(\mu) = -\#\{k < i : \mu_k > \mu_i\} + \#\{k > i : j \leqslant \mu_k < \mu_i\}.$$

[12]At certain points in this text (particularly when we consider only positive compositions), we might start the indexing for $\mathcal{I}(\mu)$ at 1 instead of 0, but we will make a point about such changes in convention whenever they arise.

We also define the normalization constant

$$\Omega_\mu(q,t) = \prod_{i=1}^{n} \prod_{j=0}^{\mu_i - 1} \frac{1}{1 - q^{\mu_i - j} t^{\alpha_{i,j}(\mu)+1}},$$

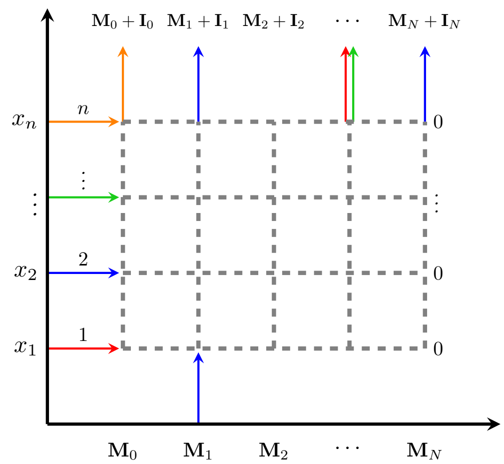
where the exponents $\alpha_{i,j}(\mu)$ are given by

$$\alpha_{i,j}(\mu) = \#\{k < i : \mu_k = \mu_i\} + \#\{k \neq i : j < \mu_k < \mu_i\} + \#\{k > i : j = \mu_k\}.$$

Under this notation, we have the following partition function formula for the nonsymmetric Macdonald polynomials, expressed diagrammatically.

THEOREM 1.5.1 (Theorem 13.3.2 below). *Denoting $N = \max_{i \in [1,n]} \mu_i$ and $v_{i,j} = v_{i,j}(\mu)$, we have*

(1.5.2) $$f_\mu(\mathbf{x}) = \Omega_\mu(q,t) \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{N} v_{i,j}^{M_{i,j}} \times$$



under the vertex weights $L_{x_j,t}(\mathbf{A}, b; \mathbf{C}, d)$ in row j. Here the sum is over all length N sequences $\mathcal{M} = (\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_N)$ of elements in $\{0,1\}^n$, with $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$, such that $\mathbf{I}_k(\mu) + \mathbf{M}_k \in \{0,1\}^n$ for each $k \geqslant 0$. Here, we have abbreviated $\mathbf{I}_k = \mathbf{I}_k(\mu)$ for each $k \geqslant 0$.

Up to the overall factor of $\Omega_\mu(q,t)$, one may interpret the right side of (1.5.2) as the partition function for our vertex model (under the L_x weights) on the cylinder obtained by identifying the top and bottom boundaries of the strip $\mathbb{Z}_{>0} \times [1,n]$. The sum over $\mathcal{M}$ allows our paths the option of "wrapping" around the cylinder at most once in each column; whenever a path of color i does this in column j, it contributes the twist parameter $v_{i,j}$ to the partition function.

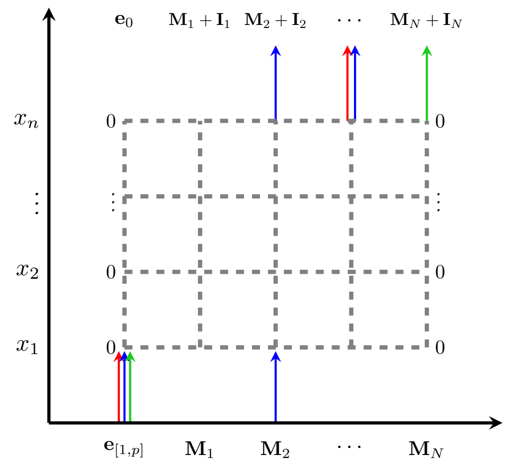
The next theorem states that the integral, symmetric Macdonald polynomials admit similar partition function formulas, but with slightly different boundary data. Here, a composition $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is *anti-dominant* if $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_p$.

THEOREM 1.5.2 (Theorem 15.1.2 below). *Fix an integer $p \in [1,n]$ and an anti-dominant, positive composition $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ of length p. Denoting $N = \max_{i \in [1,n]} \nu_i$ and*[13] $w_{i,j} =$

[13]Since ν is anti-dominant, the exponents $\gamma_{i,j}$ under this specialization are all equal to 0.

$v_{i,j}(\nu) = q^{\nu_i - j}\mathbf{1}_{\nu_i > j}$, *we have*

(1.5.3) $$J_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{N} w_{i,j}^{M_{i,j}} \times$$



under the vertex weights $L_{x_j,t}(\mathbf{A}, b; \mathbf{C}, d)$ *in row* j*. Here the sum is over all length* N *sequences* $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$ *of elements in* $\{0,1\}^n$*, with* $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$*, such that* $\mathbf{I}_k(\nu) + \mathbf{M}_k \in \{0,1\}^n$ *for each* $k \geqslant 1$*. Here, we have abbreviated* $\mathbf{I} = \mathbf{I}_k(\nu)$ *for each* $k \geqslant 1$*.*

As before, to the right of the first column, one may interpret the vertex model (1.5.3) as existing on a cylinder. However, the first column behaves differently; paths of colors $\{1, 2, \ldots, p\}$ vertically enter through it, and none vertically exit through it. In particular, no paths of colors in $\{p+1, p+2, \ldots, n\}$ exist in this model.

By comparing the vertex model representations Theorem 1.3.1 and Theorem 1.5.2 for the LLT and Macdonald polynomials, respectively, we deduce the following expression for the modified Macdonald polynomials $\widetilde{J}_\lambda(\mathbf{x})$ (recalled in Section 15.1 below) as linear combinations of skew LLT ones. It was originally established as Theorem 2.2, equation (23), and Proposition 3.4 of [**37**]. In the below, we recall $\mathcal{S}$ from Section 1.2.

Corollary 1.5.3 (Corollary 15.1.4 below)**.** *Let* ν *denote an anti-dominant composition of length* $p \leqslant n$*. For each* $i \in [1, n]$ *and* $j \geqslant 1$*, define*

$$u_{i,j} = u_{i,j}(\nu) = q^{\nu_i - j} t^{\beta_{i,j}(\nu)} \mathbf{1}_{\nu_i > j},$$

where $\nu_k = 0$ *for* $k > n$*, and the exponents* $\beta_{i,j}(\nu)$ *are given by*

$$\beta_{i,j}(\nu) = \frac{1}{2}\Big(\#\{k < i : \nu_k > j\} - p + i \Big).$$

For any infinite sequence $\mathcal{K} = (\mathbf{K}_1, \mathbf{K}_2, \ldots)$ *of elements in* $\{0,1\}^n$*, define* $\boldsymbol{\mu}(\mathcal{K}) \in \mathrm{SeqSign}_n$ *so that* $\mathcal{S}\big(\boldsymbol{\mu}(\mathcal{K})\big) = \mathcal{K}$*. Then,*

(1.5.4) $$\widetilde{J}_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} \mathcal{L}_{\boldsymbol{\mu}(\mathcal{M}_0 + \mathcal{I}_0(\nu)) / \boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{x}; q) \prod_{i=1}^{n} \prod_{j=1}^{\infty} u_{i,j}^{M_{i,j}},$$

where the sum is over all infinite sequences $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots)$ *of elements in* $\{0,1\}^n$*, with* $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$*, such that* $\mathbf{I}_k(\nu) + \mathbf{M}_k \in \{0,1\}^n$ *for each* $k \geqslant 1$*. Here, we have denoted* $\mathcal{M}_0 = (\mathbf{e}_0, \mathbf{M}_1, \mathbf{M}_2, \ldots)$*;* $\mathcal{I}_0(\nu) = \big(\mathbf{e}_0, \mathbf{I}_1(\nu), \mathbf{I}_2(\nu), \ldots\big)$*; and* $\mathcal{M}_p = \big(\mathbf{e}_{[1,p]}, \mathbf{M}_1, \mathbf{M}_2, \ldots\big)$*.*

Here, the reason for introducing the signature sequence $\boldsymbol{\mu}(\mathcal{K})$ is to account for the different indexing of the boundary data for the LLT and Macdonald vertex models; the former implements the shift $\mathfrak{T}$ from (1.2.3) (using $\mathcal{S}$), and the latter does not (using $\mathcal{I}$). Let us further mention that the sum on the right side of (1.5.4) is in fact finite, since $u_{i,j} = 0$ for $j \geqslant \nu_p$.

1.6. Transition Coefficients

Here we provide vertex model interpretations for the transition coefficients from both the LLT and modified Macdonald polynomials to the modified Hall–Littlewood basis of symmetric functions. In what follows, we fix Macdonald parameters $q, t \in \mathbb{C}$, and we recall that the LLT, modified Macdonald, and modified Hall–Littlewood polynomials are denoted by $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q)$, $\widetilde{J}_{\lambda/\mu}(\mathbf{x})$, and $Q'_\lambda(\mathbf{x})$, respectively. Then, for a signature $\lambda \in \mathrm{Sign}$ and signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, the above mentioned transition coefficients are implicitly defined by

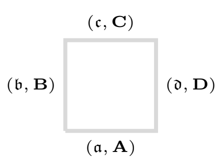
$$q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q) = \sum_\nu f^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) Q'_\nu(\mathbf{x}); \qquad \widetilde{J}_\lambda(\mathbf{x}) = \sum_\nu g^\nu_\lambda(q,t) Q'_\nu(\mathbf{x}), \tag{1.6.1}$$

where we recall ψ from (1.3.1), and both sums are over all partitions ν. Our reason for introducing the powers of q on the left side of (1.6.1) is to ensure that the transition coefficients $f^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ are polynomials in q (instead of in only $q^{1/2}$).

In this section we provide combinatorial formulas for these coefficients $f^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and g^ν_λ as partition functions for vertex models under certain weights. In order to state this result, we introduce two new types of vertex weights. Here, we depict these vertices as tiles instead of as the intersection between two lines; this will enable us to more visibly distinguish the two families of weights by shading the tiles under the second family and not shading them under the first.

Definition 1.6.1 (Definition 16.1.1 below)**.** Fix nonnegative integers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \geqslant 0$ and n-tuples $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$. Define $\mathbf{V} = (V_1, \ldots, V_n) \in \{0,1\}^n$ with $V_i = \min\{A_i, B_i, C_i, D_i\}$ for each $i \in [1, n]$. We introduce the lattice weights

(1.6.2)



$$= (-1)^{\mathfrak{c} + |\mathbf{V}|} q^\chi \frac{(q;q)_{\mathfrak{b} - |\mathbf{B}|}}{(q;q)_{\mathfrak{d} - |\mathbf{D}|}} \frac{(q^{\mathfrak{b} - |\mathbf{B}| - \mathfrak{a} + 1}; q)_{\mathfrak{c}}}{(q;q)_{\mathfrak{c}}} \mathbf{1}_{|\mathbf{B}| \leqslant \mathfrak{b}} \mathbf{1}_{|\mathbf{D}| \leqslant \mathfrak{d}} \mathbf{1}_{\mathfrak{a} + \mathfrak{d} = \mathfrak{b} + \mathfrak{c}}$$

$$\times \mathbf{1}_{\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}} \prod_{j : B_j - D_j = 1} \big(1 - q^{\mathfrak{d} - B_{[j+1,n]} - D_{[1,j-1]}}\big),$$

where the exponent $\chi \equiv \chi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is given by

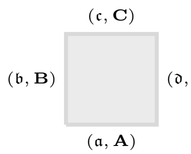
$$\chi = \binom{\mathfrak{d} - |\mathbf{D}|}{2} + \binom{\mathfrak{c} + 1}{2} - \big(\mathfrak{c} + |\mathbf{C}|\big)\mathfrak{d} + |\mathbf{V}|\big(\mathfrak{d} - |\mathbf{D}| + 1\big) + \varphi(\mathbf{D}, \mathbf{C}) + \varphi(\mathbf{V}, \mathbf{D} - \mathbf{B}).$$

Observe that there are two quadruples we consider here. The first is the quadruple $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ of elements in $\{0,1\}^n$, which satisfies $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$. We once again view $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ as indexing the up-right directed fermionic paths passing through the south, west, north, and east boundaries of a tile, respectively. The second is the quadruple $(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d})$ of integers, which instead satisfies $\mathfrak{a} + \mathfrak{d} = \mathfrak{b} + \mathfrak{c}$. In this way, we may view $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ as counting the numbers of directed down-right paths of the same bosonic color that pass through the bottom, left, top, and right boundaries of a tile, respectively. We provide depictions of these vertices in Example 1.6.6 below.

We will require a further set of weights, which are distinguished from those in (1.6.2) by means of shading.

Definition 1.6.2 (Definition 16.1.2 below)**.** Fix nonnegative integers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \geqslant 0$ and n-tuples $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$ with coordinates indexed by $[1, n]$. We define the weights

(1.6.3)



$$= \mathbf{1}_{|\mathbf{B}| = \mathfrak{b} \leqslant 1} \mathbf{1}_{|\mathbf{D}| \leqslant \mathfrak{d} \leqslant 1} \mathbf{1}_{\mathfrak{a} + \mathfrak{d} = \mathfrak{b} + \mathfrak{c}} \mathbf{1}_{\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}} \cdot W\big(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\big),$$

where (recalling $\mathbf{A}_i^+, \mathbf{A}_j^-, \mathbf{A}_{ij}^{+-}$ from (1.5.1)) the final function appearing in (1.6.3) is given by

(1.6.4)

$$W\big(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\big) = \begin{cases} q^{|\mathbf{A}|}, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (0, \mathbf{e}_0), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a}, \mathbf{A}), \quad (\mathfrak{d}, \mathbf{D}) = (0, \mathbf{e}_0), \\ 1, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (0, \mathbf{e}_0), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a} + 1, \mathbf{A}), \quad (\mathfrak{d}, \mathbf{D}) = (1, \mathbf{e}_0), \\ (1 - q^{A_i}) q^{A_{[i+1,n]}}, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (0, \mathbf{e}_0), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a} + 1, \mathbf{A}_i^-), \quad (\mathfrak{d}, \mathbf{D}) = (1, \mathbf{e}_i), \\ (1 - q^{A_j}) q^{A_{[j+1,n]}} \mathbf{1}_{A_i = 0}, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (1, \mathbf{e}_i), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a}, \mathbf{A}_{ij}^{+-}), \quad (\mathfrak{d}, \mathbf{D}) = (1, \mathbf{e}_j), \\ \mathbf{1}_{A_i = 0}, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (1, \mathbf{e}_i), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a}, \mathbf{A}_i^+), \quad (\mathfrak{d}, \mathbf{D}) = (1, \mathbf{e}_0), \\ (-1)^{A_i} q^{A_{[i,n]}}, & \text{if} \quad (\mathfrak{b}, \mathbf{B}) = (1, \mathbf{e}_i), \quad (\mathfrak{c}, \mathbf{C}) = (\mathfrak{a}, \mathbf{A}), \quad (\mathfrak{d}, \mathbf{D}) = (1, \mathbf{e}_i), \end{cases}$$

where we assume that $i < j$ whenever i and j both appear. In all cases of $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ not listed in (1.6.4), we set $W\big(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\big) = 0$.

Now we can state the following partition function formula for the coefficients $f_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\nu}$ from (1.6.1).

Theorem 1.6.3 (Theorem 16.1.3 below). *Fix integers $M \geqslant 0$ and $m \in [1, n]$; signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$; and a partition $\nu \in \mathrm{Sign}_m$, such that $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| = |\nu|$. Fix any integer*

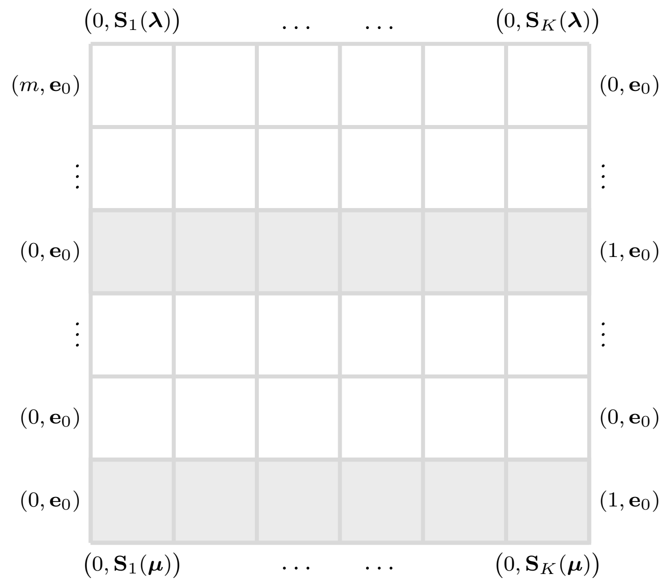
$$K \geqslant \max\Big\{ \max_{i \in [1,n]} \mathfrak{T}\big(\lambda^{(i)}\big), \nu_1 + 1 \Big\},$$

and write $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \cdots, \bar{\nu}_m)$ for the signature obtained by complementing ν in a $(K-1) \times m$ box; its parts are given by $\bar{\nu}_j = K - \nu_{m-j+1} - 1$, for each $j \in [1, m]$. Denote $\mathfrak{T}(\bar{\nu}) = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_m)$ so, for each $j \in [1, m]$,

$$\mathfrak{n}_j = \bar{\nu}_j + m - j + 1 = K - \nu_{m-j+1} + m - j. \tag{1.6.5}$$

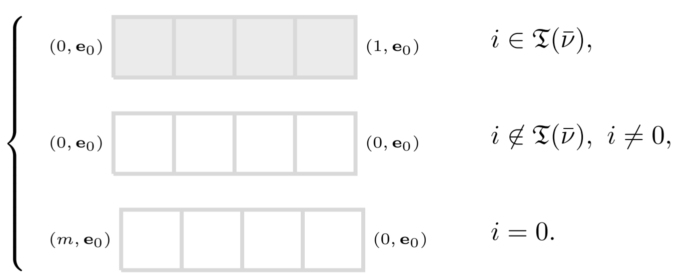
The coefficients (1.6.1) *are given by the partition function*

(1.6.6) $$f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \frac{(-1)^{|\bar{\nu}|}}{b_{\nu}(q)(q;q)_m} \times$$



consisting of $\mathfrak{n}_1 + 1$ *rows, where the* i*-th row of the lattice (counted from the top to bottom, starting at* $i = 0$*) takes the form*

(1.6.7)



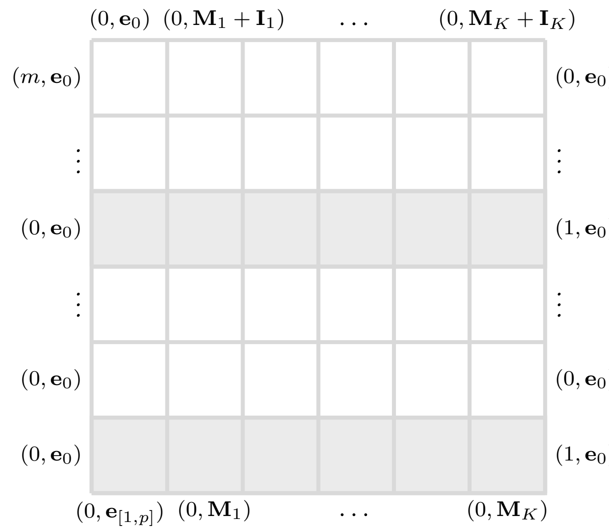
The constant $b_{\nu}(q)$ *appearing in* (1.6.6) *is given by* $b_{\nu}(q) = \prod_{j=1}^{\infty} (q;q)_{m_j(\nu)}$.

Since Corollary 1.5.3 provides an expansion for $\widetilde{J}_{\lambda}$ over the skew LLT polynomials, we deduce as a quick consequence of Theorem 1.6.3 the following combinatorial formula for the coefficients g^{ν}_{λ} from (1.6.1).

Corollary 1.6.4 (Corollary 16.2.1 below)**.** *Fix integers* $p, m \in [1, n]$*; an anti-dominant (positive) composition* λ *of length* p*; a partition* ν *of length* m*; and an integer* $K \geqslant \max\{\lambda_p, \nu_1 + 1\}$*. For any* *integers $i \in [1,p]$ and $j \geqslant 1$, set $w_{i,j} = q^{\lambda_i - j}\mathbf{1}_{\lambda_i > j}$. Then, the coefficient $g^{\nu}_{\lambda^+}(q,t)$ is given by*

(1.6.8)

$$g^{\nu}_{\lambda^+}(q,t) = \frac{(-1)^{|\bar{\nu}|}}{b_\nu(t)(t;t)_m} \sum_{\mathcal{M}} \prod_{i=1}^{p} \prod_{j=1}^{K} w_{i,j}^{M_{i,j}} \times$$



where the rows of the partition function (1.6.8) *are specified by* (1.6.5) *and* (1.6.7)*, using the same weights as in* (1.6.2) *and* (1.6.3) *but with the q there replaced by t here. The sum is over all sequences $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_K)$ of elements in $\{0,1\}^n$, with $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$, such that $M_{i,k} = 0$ for $i > p$ and $\mathbf{I}_k(\lambda) + \mathbf{M}_k \in \{0,1\}^n$ for each $k \in [1,K]$. Here, we abbreviated $\mathbf{I}_k = \mathbf{I}_j(\lambda)$ for each k, and we recall $b_\nu(t)$ from Theorem 1.6.3.*

Remark 1.6.5. Letting $Q_\lambda(\mathbf{x};t)$ and $J_\lambda(\mathbf{x};q,t)$ denote the Hall–Littlewood and integral Macdonald polynomials, respectively, we have

(1.6.9)
$$J_\lambda(\mathbf{x};q,t) = \sum_{\nu} g^{\nu}_{\lambda}(q,t) Q_\nu(\mathbf{x};t),$$

which follows from applying the plethystic substitution $X \mapsto (1-t)X$ in the second statement of (1.6.1), where $X = \sum_{x \in \mathbf{x}} x$. Thus, Corollary 1.6.4 equivalently yields a combinatorial formula for the expansion coefficients of the (standard) Macdonald polynomials in the Hall–Littlewood basis.

Let us briefly explain how Theorem 16.1.3 and Corollary 1.6.4 can be used to obtain (not manifestly nonnegative) combinatorial formulas for the expansion coefficients of the LLT and modified Macdonald polynomials in the Schur basis $s_\lambda(\mathbf{x})$, defined implicitly by

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};q) = \sum_{\nu} K^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) s_\nu(\mathbf{x}); \qquad \widetilde{J}_\lambda(\mathbf{x};q,t) = \sum_{\nu} K^{\nu}_{\lambda}(q,t) s_\lambda(\mathbf{x}).$$

We have by (6.8.4(i)) of **[66]** that

$$J_\lambda(\mathbf{x};q,q) = c_\lambda(q) s_\lambda(\mathbf{x}), \quad \text{where} \quad c_\lambda(q) = \prod_{(i,j)\in\lambda} (1 - q^{h_\lambda(i,j)}),$$

and $h_\lambda(i,j)$ denotes the hook length of cell (i,j) in λ. Thus, setting $q = t$ in (1.6.9) gives

$$s_\lambda(\mathbf{x}) = c_\lambda(q)^{-1} \sum_{\nu} g^{\nu}_{\lambda}(q,q) Q_\nu(\mathbf{x};q).$$

By Proposition 1.2 of [**44**], it follows that

$$Q'_\lambda(\mathbf{x}) = \sum_\nu b_\nu(q) c_\lambda(q)^{-1} g_\lambda^\nu(q,q) s_\nu(x).$$

Inserting this into (1.6.1) then yields

(1.6.10)
$$K^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \sum_\theta b_\nu(q) c_\theta(q)^{-1} g_\theta^\nu(q,q) f^\theta_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q); \qquad K^\nu_\lambda(q,t) = \sum_\theta b_\nu(q) c_\theta(q)^{-1} g_\theta^\nu(q,q) g^\theta_\lambda(q,t).$$

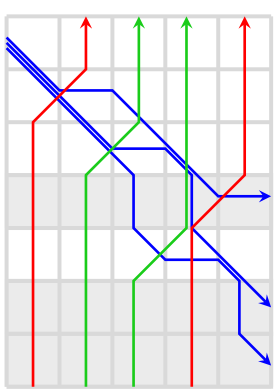
Since $f^\theta_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ and $g^\theta_\lambda(q,t)$ are given by Theorem 1.6.3 and Corollary 1.6.4, respectively, this provides explicit combinatorial formulas for the coefficients $K^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ and $K^\nu_\lambda(q,t)$. It was conjectured in [**62**] that $K^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) \in \mathbb{Z}_{\geqslant 0}[q]$; this was later proved in [**35**], although the proof there did not yield any combinatorial expression for the coefficients. As stated, we emphasize that the formulas (1.6.10) are not manifestly positive, since $f^\theta_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ and g^θ_λ can have negative coefficients. It is natural to hope that, with $f^\theta_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ and $g^\theta_\nu(q)$ in (1.6.10) both known explicitly, one might by suitable manipulation obtain a positive rule for $K^\nu_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ and $K^\nu_\lambda(q,t)$, but this is outside the scope of the present work.

We conclude with an example for Theorem 1.6.3. In what follows, we draw bosonic paths in the color blue. Unlike in previous diagrams, to simplify the figures, we depict paths as sometimes making diagonal steps, which are equivalent to making one step up or down and one step right. Let us mention that additional examples are provided in Section 16.3 below, which were generated by a `Mathematica` implementation of (1.6.6); the code is available from the authors upon request.

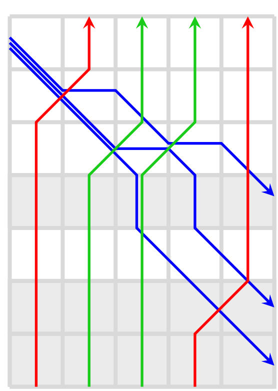
Example 1.6.6 (Example 16.3.3 below)**.** Let $n = 2$, $M = 2$, $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}\big)$, and $\boldsymbol{\mu} = \big(\mu^{(1)}, \mu^{(2)}\big)$ with $\lambda^{(1)} = (3,1)$, $\lambda^{(2)} = (2,2)$, $\mu^{(1)} = (2,0)$, and $\mu^{(2)} = (1,1)$. Let $\nu = (2,1,1)$ and choose $K = 5$, so that $\bar\nu = (3,3,2)$ and $\mathfrak{T}(\bar\nu) = (6,5,3)$. Given that $|\bar\nu| = 8$, $m = 3$, and $b_\nu(q) = (1-q)^2(1-q^2)$, we have

$$\frac{(-1)^{|\bar\nu|}}{b_\nu(q)(q;q)_m} = \frac{1}{(1-q)^3(1-q^2)^2(1-q^3)}. \tag{1.6.11}$$

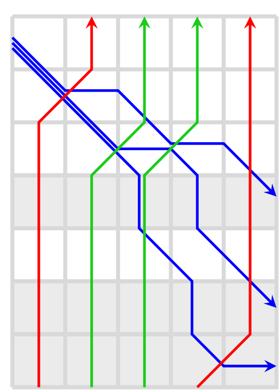
The formula (1.6.6) provides five non-vanishing lattice configurations, indicated below with their weights, where we have multiplied by the overall factor (1.6.11). Here, red is color 1 and green is color 2.



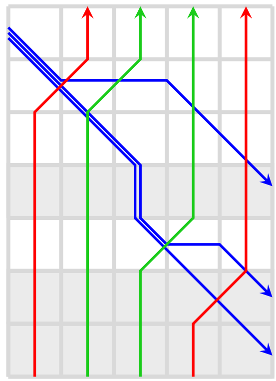
$q(1-q^2)$



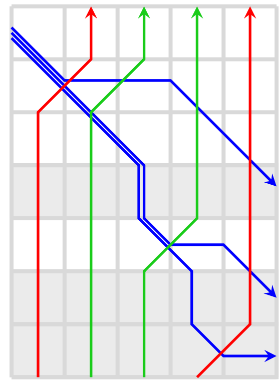
$\frac{-q^3}{1+q}$



$\frac{-q^2}{1+q}$



$\frac{q(1-q)}{1+q}$



$\frac{1-q}{1+q}$

We therefore find that $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = q(1-q^2) - \frac{q^3}{1+q} - \frac{q^2}{1+q} + \frac{q(1-q)}{1+q} + \frac{1-q}{1+q} = 1 - q^2 - q^3$.

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CHAPTER 2

$U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ Vertex Model

In this chapter we introduce and provide notation for the $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ vertex model that will serve as the basis for the fused model we will derive in this text. We further establish a "color merging" property for it, indicating how a $U_q\big(\widehat{\mathfrak{sl}}(m'|n')\big)$ model can be obtained from a $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ one, if $m \geqslant m'$ and $n \geqslant n'$.

2.1. Fundamental R-Matrix

The vertex weights for the model analyzed in this text will arise from applying the fusion procedure to the R-matrix associated with the fundamental representation of $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$. In this section we recall this R-matrix and the Yang–Baxter equation it satisfies. Throughout, we fix two integers $m \geqslant 1$ and $n \geqslant 0$ such that $m + n \geqslant 2$.

Let a, b denote two indices and let $V_a \simeq \mathbb{C}^{m+n} \simeq V_b$ denote two $(m+n)$-dimensional complex vector spaces, spanned by basis vectors $|0\rangle, |1\rangle, \ldots, |m+n-1\rangle$. For any complex number $z \in \mathbb{C}$, define the *fundamental R-matrix* $R_{ab}(z) = R_{a,b}(z) \in \operatorname{End}(V_a \otimes V_b)$ by setting

(2.1.1)
$$\begin{aligned}
R_{ab}(z) = &\sum_{i=0}^{m-1} R_z(i,i;i,i) E_a^{(ii)} \otimes E_b^{(ii)} + \sum_{j=m}^{m+n-1} R_z(j,j;j,j) E_a^{(jj)} \otimes E_b^{(jj)} \\
&+ \sum_{0 \leqslant i < j \leqslant m+n-1} R_z(i,j;i,j) E_a^{(jj)} \otimes E_b^{(ii)} + \sum_{0 \leqslant i < j \leqslant m+n-1} R_z(j,i;j,i) E_a^{(ii)} \otimes E_b^{(jj)} \\
&+ \sum_{0 \leqslant i < j \leqslant m+n-1} R_z(i,j;j,i) E_a^{(ji)} \otimes E_b^{(ij)} + \sum_{0 \leqslant i < j \leqslant m+n-1} R_z(j,i;i,j) E_a^{(ij)} \otimes E_b^{(ji)}.
\end{aligned}$$

Here, for any integers $i, j \in [0, m+n-1]$ and index $k \in \{a, b\}$, $E_k^{(ij)} \in \operatorname{End}(V_k)$ denotes the $(m+n) \times (m+n)$ matrix whose entries are all equal to 0, except for its (i,j)-entry, which is equal to 1; stated alternatively, it satisfies

$$E_k^{(ij)} |j\rangle = |i\rangle, \quad \text{and} \quad E_k^{(ij)} |i'\rangle = 0, \quad \text{for each } i' \neq i.$$

The R-matrix entries $R_z(i_1, j_1; i_2, j_2) = R_z^{(m;n)}(i_1, j_1; i_2, j_2)$ in (2.1.1) are defined by setting

(2.1.2)
$$\begin{aligned}
R_z(j,i;j,i) &= \frac{q(1-z)}{1-qz}; \qquad R_z(i,j;i,j) = \frac{1-z}{1-qz}; \\
R_z(j,i;i,j) &= \frac{1-q}{1-qz}; \qquad R_z(i,j;j,i) = \frac{z(1-q)}{1-qz},
\end{aligned}$$

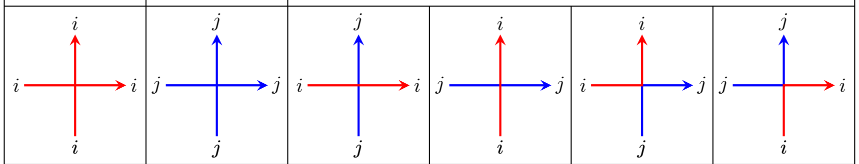
$i \in [0, m)$	$j \in [m, m+n)$	$0 \leqslant i < j < m+n$			
					
$(i,i;i,i)$	$(j,j;j,j)$	$(j,i;j,i)$	$(i,j;i,j)$	$(j,i;i,j)$	$(i,j;j,i)$
1	$\dfrac{z-q}{1-qz}$	$\dfrac{q(1-z)}{1-qz}$	$\dfrac{1-z}{1-qz}$	$\dfrac{1-q}{1-qz}$	$\dfrac{z(1-q)}{1-qz}$

FIGURE 2.1. The possible colored arrow configurations, and their vertex weights, are depicted above.

for $0 \leqslant i < j \leqslant m+n-1$;

$$R_z(i,i;i,i) = 1, \quad \text{for } i \in [0, m-1]; \qquad R_z(j,j;j,j) = \frac{z-q}{1-qz}, \quad \text{for } j \in [m, m+n-1]. \tag{2.1.3}$$

We also set $R_z(i_1, j_1; i_2, j_2) = 0$ for any $(i_1, j_1; i_2, j_2)$ not of the above form. These R-matrix entries were originally introduced under a different parameterization as equation (14) of **[6]**, but they appear essentially as above in Section 5.1 of **[87]**.

Remark 2.1.1. The $R_z(i_1, j_1; i_2, j_2)$ do not depend on n unless $i_1 = j_1 = i_2 = j_2 \geqslant m$, namely,

$$R_z^{(m;n)}(i_1, j_1; i_2, j_2) = R_z^{(m+n;0)}(i_1, j_1; i_2, j_2), \qquad \text{unless } i_1 = j_1 = i_2 = j_2 \in [m, m+n-1].$$

We interpret these entries $R_z(i_1, j_1; i_2, j_2)$ as *vertex weights* as follows. A *vertex* v is a transverse intersection between two oriented curves, typically (after a rotation, if necessary) a horizontal line ℓ_1 directed east and a vertical line ℓ_2 directed north. Associated with each of these two curves is a *rapidity parameter*, which is a complex number. Denoting that for ℓ_1 by $x \in \mathbb{C}$ and that for ℓ_2 by $y \in \mathbb{C}$, the *spectral parameter* of v is the ratio $\frac{y}{x}$.

Each of the four segments, also called *arrows*, of ℓ_1 and ℓ_2 adjacent to v is assigned a *color*, which is a label in $\{0, 1, \ldots, m+n-1\}$. Let i_1 and j_1 denote the colors of the vertical and horizontal segments entering v, respectively; similarly, let i_2 and j_2 denote the colors of the vertical and horizontal arrows exiting v, respectively. Then, $i_1, j_1, i_2, j_2 \in \{0, 1, \ldots, m+n-1\}$. We further assume that $\{i_1, j_1\} = \{i_2, j_2\}$ as multi-sets, so that the same number of arrows of any given color enters v as exits v; this is known as *arrow conservation*. We refer to the quadruple $(i_1, j_1; i_2, j_2)$ as the *arrow configuration* at v, and we view $R_z(i_1, j_1; i_2, j_2)$ as the weight of a vertex with arrow configuration $(i_1, j_1; i_2, j_2)$ and spectral parameter z. The possible arrow configurations are depicted in Figure 2.1. Under this convention, at most one arrow can occupy any edge; we will later remove this condition through fusion in Chapter 3 below.

Observe from (2.1.3) that vertices adjacent to four arrows of the same color i have different weights depending on whether $i \in [0, m-1]$ or $i \in [m, m+n-1]$. Throughout this text, we refer to colors in the interval $[0, m-1]$ as *bosonic* and to the remaining colors (in the interval $[m, m+n-1]$) as *fermionic*. This terminology will in a sense be later justified by Corollary 3.2.3 below, which implies that fermionic colors satisfy a certain exclusion property (which does not hold for bosonic colors).

The following proposition indicates that the R-matrix $R_{ab}(z)$ from (2.1.1) satisfies the Yang–Baxter equation. It was originally due to [**6**], but it can also be verified directly from the explicit forms (2.1.2) and (2.1.3) for its entries.

Proposition 2.1.2 ([**6**, Section 3])**.** *Let V_1, V_2, and V_3 denote three $(m+n)$-dimensional complex vector spaces, and let $x, y, z \in \mathbb{C}$ denote complex numbers. As operators on $V_1 \otimes V_2 \otimes V_3$, we have*

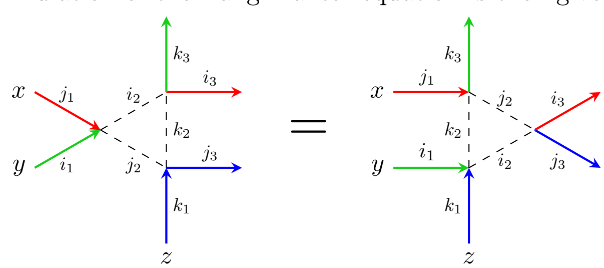
$$R_{12}\Big(\frac{y}{x}\Big) R_{13}\Big(\frac{z}{x}\Big) R_{23}\Big(\frac{z}{y}\Big) = R_{23}\Big(\frac{z}{y}\Big) R_{13}\Big(\frac{z}{x}\Big) R_{12}\Big(\frac{y}{x}\Big), \tag{2.1.4}$$

where R_{ij} acts as the identity on V_k, for $k \notin \{i, j\}$.

For fixed indices $i_1, j_1, k_1, i_3, j_3, k_3 \in [0, m+n-1]$, comparing the entries in (2.1.4) in the (i_1, j_1, k_1)-th row and (i_3, j_3, k_3)-column, we deduce that

$$\begin{aligned} &\sum_{i_2, j_2, k_2} R_{y/x}(i_1, j_1; i_2, j_2) R_{z/x}(k_1, j_2; k_2, j_3) R_{z/y}(k_2, i_2; k_3, i_3) \\ &\qquad = \sum_{i_2, j_2, k_2} R_{z/y}(k_1, i_1; k_2, i_2) R_{z/x}(k_2, j_1; k_3, j_2) R_{y/x}(i_2, j_2; i_3, j_3), \end{aligned} \tag{2.1.5}$$

where on both sides i_2, j_2, k_2 are each ranged over all indices in $[0, m+n-1]$. The diagrammatic interpretation of this formulation of the Yang–Baxter equation is then given by



where on either side of the equation is a family of vertices, and we view the weight of each family as the product of the weights of its constituent vertices. Along the solid edges the colors are fixed, and along the dashed ones they are summed over.

2.2. Domains, Boundary Data, and Partition Functions

In this section we introduce notation for vertex models on the general domains we consider. In what follows, we view any vertex $v \in \mathbb{Z}^2$ as a two-dimensional vector, so that vertices may be added or subtracted to form elements of $\mathbb{Z}^2$.

An *east-south path* is an ordered sequence of vertices $\mathbf{p} = (v_1, v_2, \ldots, v_k) \subset \mathbb{Z}^2$ such that $v_{i+1} - v_i \in \big\{(1,0), (0,-1)\big\}$ for each $i \in [1, k-1]$. We call k the *length* of $\mathbf{p}$ and refer to any pair of consecutive vertices (v_i, v_{i+1}) as an *edge* of $\mathbf{p}$; this edge is horizontal if $v_{i+1} - v_i = (1,0)$ and vertical if $v_{i+1} - v_i = (0,-1)$. Given two east-south paths $\mathbf{p} = (v_1, v_2, \ldots, v_k)$ and $\mathbf{p}' = (v_1', v_2', \ldots, v_k')$ of the same length, we write $\mathbf{p}' \geqslant \mathbf{p}$ (or equivalently $\mathbf{p} \leqslant \mathbf{p}'$) if $v_i' - v_i \in \mathbb{Z}_{\geqslant 0}^2$ for each $i \in [0, k]$, that is, if each v_i' is northeast of v_i. In this case, if we further have $v_0 = v_0'$ and $v_k = v_k'$ (namely, if $\mathbf{p}$ and $\mathbf{p}'$ share the same starting and ending points), then let $\mathcal{D}(\mathbf{p}, \mathbf{p}') \subset \mathbb{Z}^2$ denote the domain bounded by $\mathbf{p}$ and $\mathbf{p}'$. More specifically, it consists of those vertices $u \in \mathbb{Z}^2$ for which there exists some $i \in [1, k]$ such that $u - v_i, v_i' - u \in \mathbb{Z}_{\geqslant 0}^2$. A domain of the form $\mathcal{D}(\mathbf{p}, \mathbf{p}')$, for some east-south paths $\mathbf{p} \leqslant \mathbf{p}'$, is an *east-south domain*. We refer to the left side of Figure 2.2 for an example.

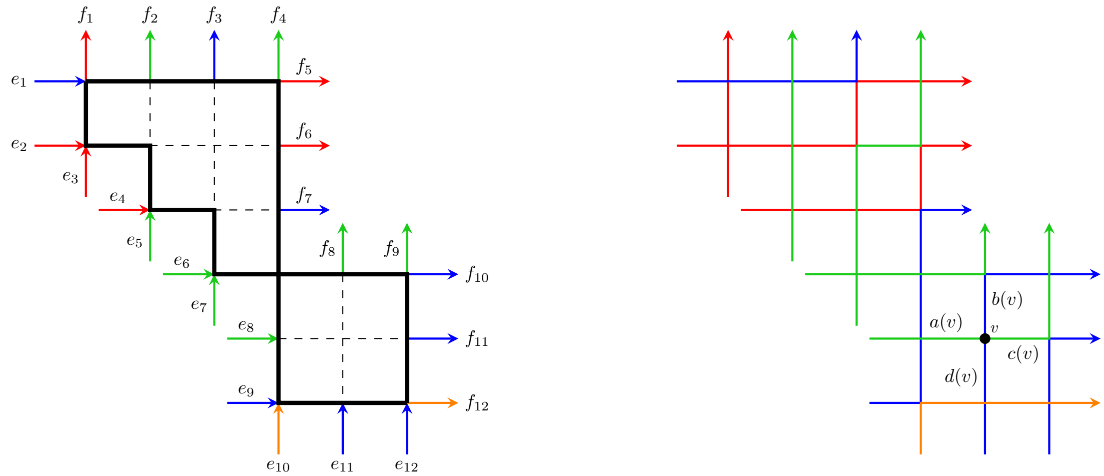


FIGURE 2.2. Shown to the left is an east-south domain with some boundary data, and shown to the right is a path ensemble on it.

Given an east-south domain $\mathcal{D} \subset \mathbb{Z}^2$, we call any $v \in \mathbb{Z}^2 \setminus \mathcal{D}$ a *boundary vertex* of $\mathcal{D}$ if there exists some $u \in \mathcal{D}$ that is adjacent to v through an edge of $\mathbb{Z}^2$; in this case, the edge connecting (v, u) is a *boundary edge*. This edge is *incoming* if $u - v \in \big\{(1,0),(0,1)\big\}$ and otherwise *outgoing* if $u - v \in \big\{(-1,0),(0,-1)\big\}$. Similarly, a boundary vertex $v \in \mathbb{Z}^2 \setminus \mathcal{D}$ is *incoming* or *outgoing* if it belongs to an incoming or outgoing boundary edge, respectively.

Any east-south domain $\mathcal{D}(\mathbf{p}, \mathbf{p}')$, whose paths $\mathbf{p}$ and $\mathbf{p}'$ are both of length k, has $k+1$ incoming and outgoing boundary vertices, and thus $k+1$ incoming and outgoing boundary edges. We index these incoming (and outgoing) boundary edges by $\{1, 2, \ldots, k+1\}$ from northwest to southeast, that is, so that the incoming boundary vertex in the i-th incoming edge is weakly northwest of that in the j-th one if and only if $i \leqslant j$ (and similarly for outgoing edges).

We next consider vertex models on an east-south domain $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}') \subset \mathbb{Z}^2$. A *path ensemble* on $\mathcal{D}$ is a *consistent* assignment of an arrow configution $\big(a(v), b(v); c(v), d(v)\big)$ to each vertex $v \in \mathcal{D}$; the consistency here means that for any $u, v \in \mathcal{D}$ we have $b(u) = d(v)$ if $u - v = (1,0)$ and $a(u) = c(v)$ if $u - v = (0,1)$. The arrow conservation condition $\big\{a(v), b(v)\big\} = \big\{c(v), d(v)\big\}$ implies that any path ensemble may be viewed as a collection of (possibly crossing) colored paths. Each such path emanates from and exits through a boundary vertex of $\mathcal{D}$, and no two paths share an edge; this last condition will later be removed in Section 5.1 below through fusion. We refer to the right side of Figure 2.2 for a depiction.

Boundary data for a path ensemble prescribes the colors of its incoming and outgoing paths. More specifically, for any sequences of indices $\mathfrak{E} = (e_1, e_2, \ldots, e_{k+1})$ and $\mathfrak{F} = (f_1, f_2, \ldots, f_{k+1})$ in $[0, m+n-1]$, a path ensemble has *boundary data* $(\mathfrak{E}; \mathfrak{F})$ if the following holds. For each $i \in [1, k+1]$, an arrow of color e_i enters through the i-th incoming edge of $\mathcal{D}$ and an arrow of color f_i exits through the i-th outgoing edge of $\mathcal{D}$; see the left side of Figure 2.2. We sometimes refer to $\mathfrak{E}$ as *entrance data* on $\mathcal{D}$ and to $\mathfrak{F}$ as *exit data*.

Given a set of spectral parameters[1] $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$ for each vertex of $\mathcal{D}$, the following definition provides notation for the partition function (sum of weights of all path ensembles) of a vertex model on $\mathcal{D}$ with boundary data $(\mathfrak{E}; \mathfrak{F})$, under the weights $R_z(a, b; c, d)$ from (2.1.2) and (2.1.3).

Definition 2.2.1. Let $\mathcal{D} \subset \mathbb{Z}^2$ denote an east-south domain. Fix boundary data $(\mathfrak{E}; \mathfrak{F})$ on $\mathcal{D}$, and a set of complex numbers $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$. Define the *partition function*

$$Z_{\mathcal{D}}^{(m;n)}(\mathfrak{E}; \mathfrak{F} \mid \mathbf{z}) = \sum \prod_{v \in \mathcal{D}} R_{z(v)}\big(a(v), b(v); c(v), d(v)\big),$$

where the sum is over all path ensembles on $\mathcal{D}$ with boundary data $(\mathfrak{E}; \mathfrak{F})$.

2.3. Color Merging

In this section we describe a way of merging colors of a $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ vertex model, analogous to the "colorblind projection" results from Section 2.3 of [**12**]. In particular, we show that certain signed sums of partition functions of such a vertex model is equal to the partition function for this model with several of its colors identified (as in Figure 2.3 below), thereby reducing its *rank* $m + n - 1$.

To explain this in more detail, we first require some notation. An *interval partition* of an integer set $I = \{a, a+1, \ldots, b\} \subset \mathbb{Z}_{\geqslant 0}$ is a collection of mutually disjoint integer intervals $\mathbb{J} = (J_0, J_1, \ldots, J_\ell)$ such that $\bigcup_{k=0}^{\ell} J_k = I$; here, the intervals in $\mathbb{J}$ are ordered so that any $j_i \in J_i$ is less than any $j_k \in J_k$ whenever $i < k$. Given an interval partition $\mathbb{J} = (J_0, J_1, \ldots, J_\ell)$ of I, we define the function $\theta_{\mathbb{J}} : I \to [0, \ell]$ by setting $\theta_{\mathbb{J}}(j) = k$ for each $j \in J_k$ and $k \in [0, \ell]$. In what follows, we will typically view $I = \{0, 1, \ldots, m+n-1\}$ as the set of colors in our model and $\theta_{\mathbb{J}}$ as a prescription for "merging" them, so that all colors in any J_k are identified and renamed to color k.

We further require a certain inversion count. To define it, for any sequence $\mathfrak{I} = (i_1, i_2, \ldots, i_k)$ of indices in $[0, m+n-1]$ and integer interval $J \subseteq [0, m+n-1]$, we set

$$\operatorname{inv}(\mathfrak{I}; J) = \sum_{1 \leqslant h < j \leqslant k} \mathbf{1}_{i_h \in J} \mathbf{1}_{i_j \in J} \mathbf{1}_{i_h > i_j}. \tag{2.3.1}$$

Observe for example that if $J = \{0, 1, \ldots, m+n-1\}$ then $\operatorname{inv}(\mathfrak{I}; J) = \operatorname{inv}(\mathfrak{I})$, which denotes the number of pairs of indices $(h, j) \in [1, k]^2$ such that $h < j$ and $i_h > i_j$.

Now we have the below proposition, which essentially states the following for any interval partition $\mathbb{J}$ of $[0, m+n-1]$ separating $[0, m-1]$ (bosonic colors) from $[m, m+n-1]$ (fermionic ones). Consider a $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ vertex model partition function, "symmetrize" it over all choices consistent with $\mathbb{J}$ of colors for its bosonic exiting arrows, and "anti-symmetrize" it over all choices consistent with $\mathbb{J}$ of colors for its fermionic entering arrows. This yields the partition function for the vertex model obtained from the original one by merging its colors as prescribed by $\theta_{\mathbb{J}}$.

In what follows, we recall the partition function $Z_{\mathcal{D}}^{(m;n)}(\mathfrak{E}; \mathfrak{F} \mid \mathbf{z})$ from Definition 2.2.1, and for any function $f : \mathbb{Z} \to \mathbb{Z}$ and sequence $\mathfrak{I} = (i_1, i_2, \ldots, i_k)$ we denote $f(\mathfrak{I}) = \big(f(i_1), f(i_2), \ldots, f(i_k)\big)$. We will establish this proposition in Section 2.4 below.

[1]Recall from Section 2.1 that the spectral parameter $z = z(i, j)$ at any vertex $(i, j) \in \mathcal{D}$ is set to be the ratio $x_j^{-1} y_i$, where x_j and y_i are rapidity parameters associated with row j and column i of the model, respectively. Although this constraint is central to the integrability of the underlying model, we will omit it in Definition 2.2.1 in order to ease notation (but later restore it in contexts where we directly require it, such as in Definition 3.1.1 below).

Proposition 2.3.1. *Fix integers $m \geqslant m' \geqslant 1$, $n \geqslant n' \geqslant 0$, and $k \geqslant 1$; an east-south domain $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}')$ with boundary paths $\mathbf{p}$ and $\mathbf{p}'$ of length k; and a set of complex numbers $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$. Let $\mathbb{J} = (J_0, J_1, \ldots, J_{m'+n'-1})$ denote an interval partition of $\{0, 1, \ldots, m+n-1\}$ such that*

$$\bigcup_{k=0}^{m'-1} J_k = \{0, 1, \ldots, m-1\}; \qquad \bigcup_{k=m'}^{m'+n'-1} J_k = \{m, m+1, \ldots, m+n-1\}. \tag{2.3.2}$$

For any fixed sequences of indices $\mathfrak{E} = (e_1, e_2, \ldots, e_{k+1})$ and $\mathfrak{F} = (f_1, f_2, \ldots, f_{k+1})$, with entries in $[0, m+n-1]$, constituting entrance and exit data on $\mathcal{D}$, respectively, we have

$$\sum_{\breve{\mathfrak{E}}, \breve{\mathfrak{F}}} Z_{\mathcal{D}}^{(m;n)}(\breve{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \mathbf{z}) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\breve{\mathfrak{E}}; J_i) - \operatorname{inv}(\breve{\mathfrak{F}}; J_i)} = Z_{\mathcal{D}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{E}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \mathbf{z}\big). \tag{2.3.3}$$

Here, the sum is over all sequences of indices $\breve{\mathfrak{E}} = (\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_{k+1})$ and $\breve{\mathfrak{F}} = (\breve{f}_1, \breve{f}_2, \ldots, \breve{f}_{k+1})$ with entries in $[0, m+n-1]$ such that $\theta_{\mathbb{J}}(\breve{\mathfrak{E}}) = \theta_{\mathbb{J}}(\mathfrak{E})$; $\theta_{\mathbb{J}}(\breve{\mathfrak{F}}) = \theta_{\mathbb{J}}(\mathfrak{F})$; $\breve{e}_i = e_i$ whenever $e_i \in [0, m-1]$; and $\breve{f}_i = f_i$ whenever $f_i \in [m, m+n-1]$.

For $n = n' = 0$, Proposition 2.3.1 essentially states that symmetrizing a $U_q\big(\widehat{\mathfrak{sl}}(m)\big)$ vertex model partition function over its exit data (in a way consistent with $\mathbb{J}$) in effect merges colors in the model, as prescribed by $\theta_{\mathbb{J}}$; this statement appeared as Proposition 4.11 of **[9]** (when all but one interval in $\mathbb{J}$ had length 1). The necessity of anti-symmetrizing the entering fermionic colors for this model appears to be a new effect present for $n > n' > 0$.

Now let us consider certain special cases of Proposition 2.3.1, first when $m' + n' = m + n$ and second (in a certain scenario of) when $m' + n' = m + n - 1$ and $|\mathcal{D}| = 1$. They will facilitate the proof of Proposition 2.3.1 when $|\mathcal{D}| = 1$, before proceeding to fully general domains.

Example 2.3.2. Suppose that $m' + n' = m + n$. Then, $m = m'$ and $n = n'$, which implies that each interval in $\mathbb{J}$ is a singleton. Thus $\theta_{\mathbb{J}}(i) = i$ for each $i \in [0, m+n-1]$, and so the left side of (2.3.3) is supported on the term $(\breve{\mathfrak{E}}, \breve{\mathfrak{F}}) = (\mathfrak{E}, \mathfrak{F})$. Since $\operatorname{inv}(\mathfrak{E}; J_i) = 0 = \operatorname{inv}(\mathfrak{F}; J_i)$ for each $i \in [0, m'+n'-1]$ (by (2.3.1), since each J_i is a singleton), this implies that both sides of (2.3.3) are equal $Z_{\mathcal{D}}^{(m;n)}(\mathfrak{E}; \mathfrak{F} \mid \mathbf{z})$, thereby verifying Proposition 2.3.1 in this case.

Lemma 2.3.3. *Suppose that $\mathcal{D} = \{v\}$ consists of a single vertex (so $k = 1$), and that $m' + n' = m + n - 1$. Further assume that there exists an index $h \in [0, m+n-1]$ with $e_1, e_2, f_1, f_2 \in \{h, h+1\}$ and*

$$J_i = \{i\}, \quad \textit{for } i \in [1, h-1]; \qquad J_h = \{h, h+1\}; \qquad J_i = \{i+1\}, \quad \textit{for } i \in [h+1, m+n-1]. \tag{2.3.4}$$

Then Proposition 2.3.1 holds.

Proof. Since $|\mathcal{D}| = 1$, the set $\mathbf{z} = \big(z(v)\big)$ consists of a single element, which we abbreviate $z = z(v)$. Then, for any entrance data $\breve{\mathfrak{E}} = (\breve{e}_1, \breve{e}_2)$ and exit data $\breve{\mathfrak{F}} = (\breve{f}_1, \breve{f}_2)$ on v, we have $Z_{\mathcal{D}}^{(m;n)}(\breve{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \mathbf{z}) = R_z(\breve{e}_2, \breve{e}_1; \breve{f}_1, \breve{f}_2)$. Moreover, since $\theta_{\mathbb{J}}(\breve{e}_i) = \theta_{\mathbb{J}}(e_i)$ and $\theta_{\mathbb{J}}(\breve{f}_i) = \theta_{\mathbb{J}}(f_i)$ for each $i \in \{1, 2\}$, we have $\theta_{\mathbb{J}}(e_i) = h = \theta_{\mathbb{J}}(f_i)$ for each i. In particular, the right side of (2.3.3) is equal to $R_z^{(m';n')}(h, h; h, h)$. Furthermore, by (2.3.2), $\mathbb{J}$ separates $[0, m-1]$ from $[m, m+n-1]$, so (2.3.4) implies that $h \neq m-1$. We will analyze the cases $h < m-1$ and $h \geqslant m$ separately.

Let us first suppose $h < m-1$, in which case $(m', n') = (m-1, n)$. Then, on the left side of (2.3.3) we have $\breve{e}_i = e_i$ for each i (since $e_i \leqslant h+1 \leqslant m-1$), and we sum over each

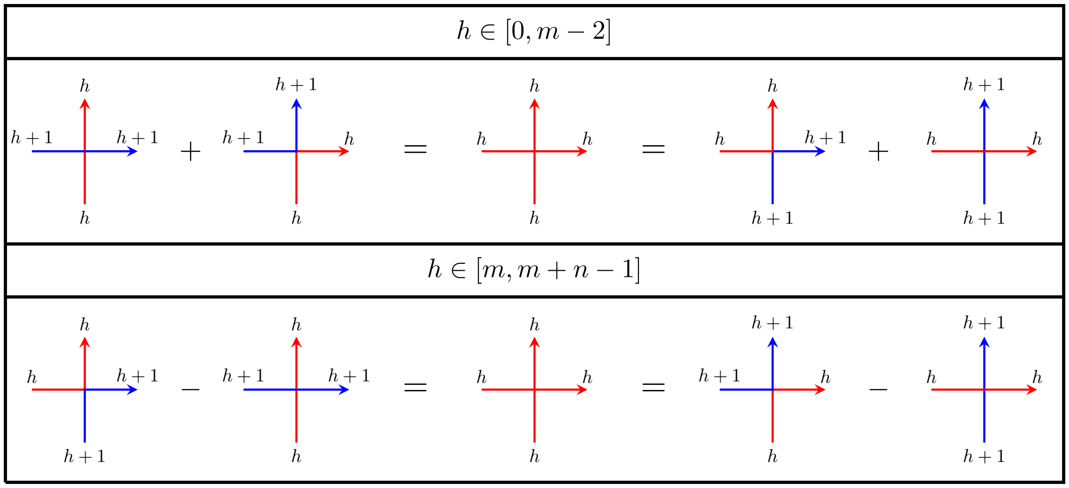


FIGURE 2.3. Shown above is the diagrammatic interpretation for (2.3.6) when $h < m-1$, and shown below is that for (2.3.8) when $h \geqslant m$.

$\breve{f}_i \in \{h, h+1\}$ (since $\theta_{\mathbb{J}}(\breve{f}_i) = \theta_{\mathbb{J}}(f_i) = h$). Also, as mentioned above, the right side of (2.3.3) is $R_z^{(m';n')}(h,h;h,h) = 1$, where the equality holds by the first statement of (2.1.3) (as $h < m-1 = m'$). Additionally $\operatorname{inv}(\breve{\mathfrak{E}}; J_i) = 0 = \operatorname{inv}(\mathfrak{F}; J_i)$ for each $i \in [m', m'+n'-1]$, since each entry of $\breve{\mathfrak{E}}$ and $\mathfrak{F}$ is h or $h+1$, and neither is contained in J_i for $i \geqslant m'$ (as $J_i \subseteq [m, m+n-1]$ if $i \geqslant m'$).

Hence, to show (2.3.3) for $h < m-1$ we must verify

$$\sum_{\breve{f}_1, \breve{f}_2 \in \{h, h+1\}} R_z^{(m;n)}(e_2, e_1; \breve{f}_1, \breve{f}_2) = R_z^{(m';n')}\big(h,h;h,h\big) = 1. \tag{2.3.5}$$

This follows by using (2.1.2) and (2.1.3) to check the four cases for (e_1, e_2) (equal to either (h,h), $(h+1,h+1)$, $(h,h+1)$, or $(h+1,h)$) individually. Indeed, for $h \leqslant m-2$ these definitions imply

$$\begin{aligned} &R_z^{(m;n)}(h,h;h,h) = 1 = R_z^{(m;n)}(h+1,h+1;h+1,h+1); \\ &R_z^{(m;n)}(h,h+1;h,h+1) + R_z^{(m;n)}(h,h+1;h+1,h) = 1; \\ &R_z^{(m;n)}(h+1,h;h,h+1) + R_z^{(m;n)}(h+1,h;h+1,h) = 1; \end{aligned} \tag{2.3.6}$$

see the top of Figure 2.3 for a depiction. This confirms (2.3.5) and thus the lemma for $h < m-1$.

So, let us suppose instead that $h \geqslant m$, in which case $(m', n') = (m, n-1)$. Then, on the left side of (2.3.3) we have $\breve{f}_i = f_i$ for each i (since $f_i \geqslant h \geqslant m$), and we sum over each $\breve{e}_i \in \{h, h+1\}$. Also, as mentioned above, the right side of (2.3.3) is $R_z^{(m;n)}(h,h;h,h) = \frac{z-q}{1-qz}$, where the equality follows from the second statement of (2.1.3). Additionally, $\operatorname{inv}(\breve{\mathfrak{E}}; J_i) = 0 = \operatorname{inv}(\mathfrak{F}; J_i)$ for $i \neq h$, but since now $h \in [m', m'+n'-1]$ we must take into account the facts that $\operatorname{inv}(\breve{\mathfrak{E}}; J_h) = \mathbf{1}_{\breve{e}_1 > \breve{e}_2}$ and $\operatorname{inv}(\mathfrak{F}; J_h) = \mathbf{1}_{f_1 > f_2}$.

Hence, to show (2.3.3) for $h \geqslant m$ we must verify

$$\sum_{\breve{e}_1, \breve{e}_2 \in \{h, h+1\}} (-1)^{\mathbf{1}_{\breve{e}_1 > \breve{e}_2} - \mathbf{1}_{f_1 > f_2}} R_z^{(m;n)}(\breve{e}_2, \breve{e}_1; f_1, f_2) = R_z^{(m';n')}\big(h,h;h,h\big) = \frac{z-q}{1-qz}. \tag{2.3.7}$$

As in the case $h < m-1$, this follows by using (2.1.2) and (2.1.3) to check the four cases for (f_1, f_2) individually. Indeed, for $h \geqslant m$ these definitions imply

$$
\begin{aligned}
&R_z^{(m';n')}(h,h;h,h) = \frac{z-q}{1-qz} = R_z^{(m';n')}(h+1,h+1;h+1,h+1); \\
&R_z^{(m';n')}(h+1,h;h,h+1) - R_z^{(m';n')}(h,h+1;h,h+1) = \frac{z-q}{1-qz}; \\
&R_z^{(m';n')}(h,h+1;h+1,h) - R_z^{(m';n')}(h+1,h;h+1,h) = \frac{z-q}{1-qz};
\end{aligned}
\tag{2.3.8}
$$

see the bottom of Figure 2.3. This confirms (2.3.7) and thus the lemma when $h \geqslant m$. □

We can now deduce the following corollary verifying Proposition 2.3.1 when $|\mathcal{D}| = 1$.

Corollary 2.3.4. *If $|\mathcal{D}| = 1$, then Proposition 2.3.1 holds.*

PROOF. Since $|\mathcal{D}| = 1$, we have $k = 1$. Thus, arrow conservation implies that any summand supported by the left side of (2.3.3) must satisfy $\{\breve{e}_1, \breve{e}_2\} = \{\breve{f}_1, \breve{f}_2\}$ as unordered multi-sets. Since $\breve{e}_i = e_i$ whenever $e_i \in [0, m-1]$ and $\breve{f}_i = f_i$ whenever $f_i \in [m, m+n-1]$, these unordered multi-sets are determined by $(\mathfrak{E}, \mathfrak{F})$. So, let $\{\breve{e}_1, \breve{e}_2\} = \{a, b\} = \{\breve{f}_1, \breve{f}_2\}$, where a and b are fixed by $(\mathfrak{E}, \mathfrak{F})$ (and, in particular, are not summed over on the left side of (2.3.3)).

Now recall from the explicit forms (2.1.2) and (2.1.3) of the weights $R_z(i, j; i', j')$ that they only depend on whether i, j, i', j' are bosonic (lie in $[0, m-1]$) or fermionic (lie in $[m, m+n-1]$), and on the relative ordering of $(i, j; i', j')$. Thus, we may assume in what follows that $a \in \{b-1, b\}$, that is, a and b either coincide or are consecutive. Since no color other than a or b gives rise to a nonzero contribution to either side of (2.3.3), we may further assume that each J_i is a singleton, except for possibly at most one, which would be given by $J_i = \{a, b\}$.

This reduces to the case when $m' + n' = m + n$ if either $a = b$ or a, b reside in different J_i, or when $m' + n' = m + n - 1$ if $a \neq b$ reside in the same J_i. The former scenario was verified as Example 2.3.2, and the latter was addressed by Lemma 2.3.3. □

2.4. Proof of Proposition 2.3.1

In this section we establish Proposition 2.3.1, which will follow from Corollary 2.3.4, together with induction on $|\mathcal{D}|$.

PROOF OF PROPOSITION 2.3.1. We induct on $|\mathcal{D}|$. If $|\mathcal{D}| = 1$, then the lemma is verified by Corollary 2.3.4. So, let us assume in what follows that it holds whenever $|\mathcal{D}| < r$ for some integer $r > 1$, and we will show it also holds whenever $|\mathcal{D}| = r$.

To that end, fix an east-south domain $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}') \subset \mathbb{Z}^2$ with $|\mathcal{D}| = r$. Then, there exists an east-south domain $\widetilde{\mathcal{D}} = \mathcal{D}(\widetilde{\mathbf{p}}, \widetilde{\mathbf{p}}') \subset \mathcal{D}$ obtained from $\mathcal{D}$ by removing one vertex $v \in \mathbf{p}$ on its southwest boundary; in particular, $|\widetilde{\mathcal{D}}| = |\mathcal{D}| - 1$.

First suppose that $\mathbf{p} \neq \mathbf{p}'$, in which case we may choose $v \notin \mathbf{p} \cap \mathbf{p}'$. Then, the northeast boundary $\widetilde{\mathbf{p}}'$ of $\widetilde{\mathcal{D}}$ is $\mathbf{p}'$; the southwest boundary $\widetilde{\mathbf{p}}$ of $\widetilde{\mathcal{D}}$ is obtained from $\mathbf{p}$ by replacing its corner at v, which is formed by a vertical edge followed by a horizontal one, with a corner formed by a horizontal edge followed by a vertical one. Let v be the K-th vertex in $\mathbf{p}$, for some index $K \in [1, k]$. So, under entrance data $\mathfrak{E} = (e_1, e_2, \ldots, e_{k+1})$, arrows of colors e_K and e_{K+1} horizontally and vertically enter through v, respectively. We refer to Figure 2.4 for a depiction, where there $K = 9$.

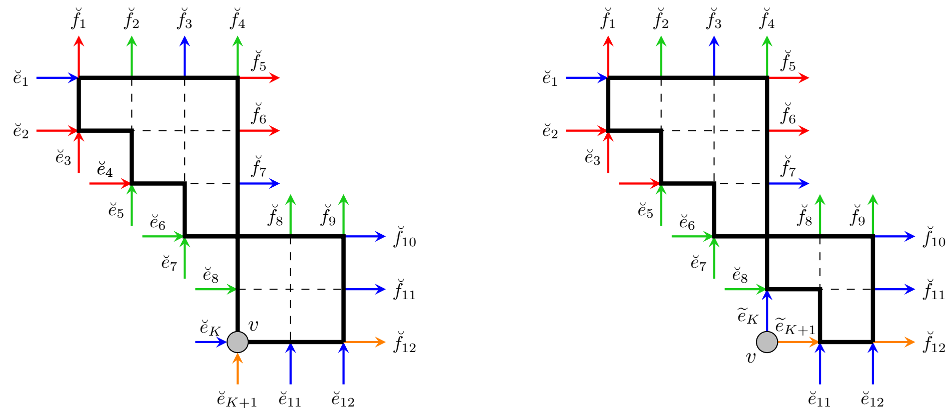


FIGURE 2.4. Shown to the left and right are the domains $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}')$ and $\widetilde{\mathcal{D}} = \mathcal{D}(\widetilde{\mathbf{p}}, \mathbf{p}')$, respectively, used in the proof of Proposition 2.3.1 if $\mathbf{p} \neq \mathbf{p}'$.

Letting $\widetilde{\mathbf{z}} = \big(z(v)\big)_{v \in \widetilde{\mathcal{D}}} = \mathbf{z} \setminus \big\{z(v)\big\}$, we have

$$Z_{\mathcal{D}}^{(m;n)}(\breve{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \mathbf{z}) = \sum_{\widetilde{e}_K = 0}^{m+n-1} \sum_{\widetilde{e}_{K+1}=0}^{m+n-1} Z_{\{v\}}^{(m;n)}\big((\breve{e}_K, \breve{e}_{K+1}); (\widetilde{e}_K, \widetilde{e}_{K+1}) \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}(\widetilde{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \widetilde{\mathbf{z}}), \tag{2.4.1}$$

for any sequences $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$ of indices in $[0, m+n-1]$; here, $\widetilde{\mathfrak{E}}$ is obtained from $\breve{\mathfrak{E}}$ by replacing $(\breve{e}_K, \breve{e}_{K+1})$ with $(\widetilde{e}_K, \widetilde{e}_{K+1})$, so that $\widetilde{\mathfrak{E}} = (\breve{e}_1, \ldots, \breve{e}_{K-1}, \widetilde{e}_K, \widetilde{e}_{K+1}, \breve{e}_{K+2}, \ldots, \breve{e}_{k+1})$.

Now we will essentially perform a signed sum of the right side of (2.4.1) over $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$. To that end, recall that $\mathfrak{E} = (e_1, e_2, \ldots, e_{k+1})$ and $\mathfrak{F} = (f_1, f_2, \ldots, f_{k+1})$, and set $\mathfrak{e} = (e_K, e_{K+1})$. Further fix pairs $\widehat{\mathfrak{e}} = (\widehat{e}_K, \widehat{e}_{K+1})$ of indices in $[0, m+n-1]$, and define $\widehat{\mathfrak{E}} = (\widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_{k+1})$ from $\mathfrak{E}$ by replacing $\mathfrak{e}$ with $\widehat{\mathfrak{e}}$, so that $\widehat{\mathfrak{E}} = (e_1, \ldots, e_{k-1}, \widehat{e}_K, \widehat{e}_{K+1}, e_{K+2}, \ldots, e_{k+1})$. Then, Corollary 2.3.4 yields

$$\sum_{\breve{\mathfrak{e}}} \sum_{\widetilde{\mathfrak{e}}} Z_{\{v\}}^{(m;n)}\big(\breve{\mathfrak{e}}; \widetilde{\mathfrak{e}} \mid z(v)\big) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\breve{\mathfrak{e}}, J_i) - \operatorname{inv}(\widehat{\mathfrak{e}}, J_i)} = Z_{\{v\}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{e}); \theta_{\mathbb{J}}(\widehat{\mathfrak{e}}) \mid z(v)\big), \tag{2.4.2}$$

where we sum over all $\breve{\mathfrak{e}} = (\breve{e}_K, \breve{e}_{K+1})$ and $\widetilde{\mathfrak{e}} = (\widetilde{e}_K, \widetilde{e}_{K+1})$ such that $\theta_{\mathbb{J}}(\breve{\mathfrak{e}}) = \theta_{\mathbb{J}}(\mathfrak{e})$; $\theta_{\mathbb{J}}(\widetilde{\mathfrak{e}}) = \theta_{\mathbb{J}}(\widehat{\mathfrak{e}})$; $\breve{e}_i = e_i$ whenever $e_i \in [0, m-1]$; and $\widetilde{e}_i = \widehat{e}_i$ whenver $\widehat{e}_i \in [m, m+n-1]$.

Moreover, since $|\widetilde{\mathcal{D}}| = |\mathcal{D}| - 1 = r - 1$, (2.3.3) applies and gives

$$\sum_{\widetilde{\mathfrak{E}}} \sum_{\breve{\mathfrak{F}}} Z_{\widetilde{\mathcal{D}}}^{(m;n)}\big(\widetilde{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \widetilde{\mathbf{z}}\big) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\widetilde{\mathfrak{E}}, J_i) - \operatorname{inv}(\mathfrak{F}, J_i)} = Z_{\widetilde{\mathcal{D}}}^{(m';n')}\big(\theta_{\mathbb{J}}(\widehat{\mathfrak{E}}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \widetilde{\mathbf{z}}\big), \tag{2.4.3}$$

where we sum over all $\widetilde{\mathfrak{E}} = (\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_{k+1})$ and $\breve{\mathfrak{F}} = (\breve{f}_1, \breve{f}_2, \ldots, \breve{f}_{k+1})$ such that $\theta_{\mathbb{J}}(\widetilde{\mathfrak{E}}) = \theta_{\mathbb{J}}(\widehat{\mathfrak{E}})$; $\theta_{\mathbb{J}}(\breve{\mathfrak{F}}) = \theta_{\mathbb{J}}(\mathfrak{F})$; $\widetilde{e}_i = \widehat{e}_i$ whenever $\widehat{e}_i \in [0, m-1]$; and $\breve{f}_i = f_i$ whenever $f_i \in [m, m+n-1]$.

We next combine (2.4.2) and (2.4.3). To that end, set $\breve{e}_i = \widetilde{e}_i$ for each $i \notin \{K, K+1\}$, so $\widetilde{\mathfrak{E}} = (\breve{e}_1, \ldots, \breve{e}_{K-1}, \widetilde{e}_K, \widetilde{e}_{K+1}, \breve{e}_{K+2}, \ldots, \breve{e}_{k+1})$ (as above), and for each $i \in [m', m'+n'-1]$ observe

$$\mathrm{inv}(\widetilde{\mathfrak{E}}; J_i) + \mathrm{inv}(\breve{\mathfrak{e}}; J_i) - \mathrm{inv}(\widehat{\mathfrak{e}}, J_i) = \mathrm{inv}(\widetilde{\mathfrak{E}}; J_i) + \mathrm{inv}(\breve{\mathfrak{e}}; J_i) - \mathrm{inv}(\widetilde{\mathfrak{e}}, J_i) = \mathrm{inv}(\breve{\mathfrak{E}}, J_i),$$

where the first equality follows from the fact that $\widetilde{e}_i = \widehat{e}_i$ whenever $\widetilde{e}_i \in [m, m+n-1]$ and the second follows from (2.3.1). This together with (2.4.2) and (2.4.3) yields, for a fixed choice of $\theta(\widehat{\mathfrak{e}}) = \big(\theta(\widehat{e}_K), \theta(\widehat{e}_{K+1})\big)$, that

$$\begin{aligned}\sum_{\breve{\mathfrak{E}}} \sum_{\breve{\mathfrak{F}}} \sum_{\widetilde{e}_K} \sum_{\widetilde{e}_{K+1}} Z_{\{v\}}^{(m;n)}\big(\breve{\mathfrak{e}}; \widetilde{\mathfrak{e}} \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}(\widetilde{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \widetilde{\mathbf{z}}) \prod_{i=m'}^{m'+n'-1} (-1)^{\mathrm{inv}(\breve{\mathfrak{E}}, J_i) - \mathrm{inv}(\mathfrak{F}, J_i)} \\ = Z_{\{v\}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{e}); \theta_{\mathbb{J}}(\widehat{\mathfrak{e}}) \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m';n')}\big(\theta_{\mathbb{J}}(\widehat{\mathfrak{E}}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \widetilde{\mathbf{z}}\big).\end{aligned} \tag{2.4.4}$$

Here, we sum over all $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$ such that $\theta_{\mathbb{J}}(\breve{\mathfrak{E}}) = \theta_{\mathbb{J}}(\mathfrak{E})$; $\theta_{\mathbb{J}}(\breve{\mathfrak{F}}) = \theta_{\mathbb{J}}(\mathfrak{F})$; $\breve{e}_i = e_i$ whenever $e_i \in [0, m-1]$; and $\breve{f}_i = f_i$ whenever $f_i \in [m, m+n-1]$. We further sum over all indices $\widetilde{e}_K$ and $\widetilde{e}_{K+1}$ in $[0, m+n-1]$ such that $\theta_{\mathbb{J}}(\widetilde{e}_K) = \theta_{\mathbb{J}}(\widehat{e}_K)$ and $\theta_{\mathbb{J}}(\widetilde{e}_{K+1}) = \theta_{\mathbb{J}}(\widehat{e}_{K+1})$. To combine (2.4.2) and (2.4.3) in this way, we have used the fact that the following hold for any index $i \in \{K, K+1\}$. The color $\widetilde{e}_i$ is summed over in (2.4.2) if and only if it is fixed in (2.4.3) (as both hold if $\widehat{e}_i \in [0, m-1]$), and that it is fixed in (2.4.2) if and only if it is summed over in (2.4.3) (both hold if $\widehat{e}_i \in [m, m+n-1]$).

Summing (2.4.4) over all choices of $\theta(\widehat{\mathfrak{e}}) = \big(\theta_{\mathbb{J}}(\widehat{e}_K), \theta_{\mathbb{J}}(\widehat{e}_{K+1})\big)$ then yields

$$\begin{aligned}\sum_{\breve{\mathfrak{E}}} \sum_{\breve{\mathfrak{F}}} \sum_{\widetilde{e}_K=0}^{m+n-1} \sum_{\widetilde{e}_{K+1}=0}^{m+n-1} Z_{\{v\}}^{(m;n)}\big(\breve{\mathfrak{e}}; \widetilde{\mathfrak{e}} \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}(\widetilde{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \widetilde{\mathbf{z}}) \prod_{i=m'}^{m'+n'-1} (-1)^{\mathrm{inv}(\breve{\mathfrak{E}}, J_i) - \mathrm{inv}(\mathfrak{F}, J_i)} \\ = \sum_{\theta_{\mathbb{J}}(\widehat{e}_K)} \sum_{\theta_{\mathbb{J}}(\widehat{e}_{K+1})} Z_{\{v\}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{e}); \theta_{\mathbb{J}}(\widehat{\mathfrak{e}}) \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m';n')}\big(\theta_{\mathbb{J}}(\widehat{\mathfrak{E}}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \widetilde{\mathbf{z}}\big),\end{aligned} \tag{2.4.5}$$

where on the left side $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$ are summed as previously, and on the right side $\theta_{\mathbb{J}}(\widehat{e}_K)$ and $\theta_{\mathbb{J}}(\widehat{e}_{K+1})$ are both summed over $[0, m'+n'-1]$. Then (2.3.3) follows from applying (2.4.1) on both sides of (2.4.5), thereby verifying the proposition if $\mathbf{p} \neq \mathbf{p}'$.

Hence, let us instead assume that $\mathbf{p} = \mathbf{p}'$, in which case $\mathcal{D} = \mathbf{p} = \mathbf{p}'$. Let v denote the common ending vertex of $\mathbf{p}$ and $\mathbf{p}'$; define the domain $\widetilde{\mathcal{D}} = \mathcal{D} \setminus \{v\} = \mathbf{p} \setminus \{v\}$; and set $\widetilde{\mathbf{z}} = \mathbf{z} \setminus \{z(v)\}$. Then, any entrance data $\breve{\mathfrak{E}} = (\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_{k+1})$ and exit data $\breve{\mathfrak{F}} = (\breve{f}_1, \breve{f}_2, \ldots, \breve{f}_{k+1})$ on $\mathcal{D}$ fix the colors of three among the four arrows adjacent to v; the fourth is then determined by arrow conservation. This gives rise to entrance data $\mathfrak{e}' = (e'_k, e'_{k+1})$ and exit data $\mathfrak{f}' = (f'_k, f'_{k+1})$ on v, which satisfies $(e'_{k+1}, f'_{k+1}) = (\breve{e}_{k+1}, \breve{f}_{k+1})$. It also satisfies either $e'_k = \breve{e}_k$ if the last edge of $\mathbf{p}$ is vertical, or $f'_k = \breve{f}_k$ if the last edge of $\mathbf{p}$ is horizontal. Let us assume in what follows that the former scenario $e'_k = \breve{e}_k$ holds, for the proof in the latter is entirely analogous. Then $(\breve{\mathfrak{E}}, \breve{\mathfrak{F}})$ induces entrance data $\widetilde{\mathfrak{E}} = (\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_k)$ and exit data $\widetilde{\mathfrak{F}} = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k)$ on $\widetilde{\mathcal{D}}$, which are in particular given by $\widetilde{\mathfrak{E}} = (\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_{k-1}, f'_k)$ and $\widetilde{\mathfrak{F}} = (\breve{f}_1, \breve{f}_2, \ldots, \breve{f}_k)$. Under this notation, we have

$$Z_{\mathcal{D}}^{(m;n)}(\breve{\mathfrak{E}}; \breve{\mathfrak{F}} \mid \mathbf{z}) = Z_{\{v\}}^{(m;n)}\big(\mathfrak{e}'; \mathfrak{f}' \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}\big(\widetilde{\mathfrak{E}}; \widetilde{\mathfrak{F}} \mid \widetilde{\mathbf{z}}\big). \tag{2.4.6}$$

We refer to Figure 2.5 for a depiction.

As previously for (2.4.1), we will essentially perform a signed sum of the right side of (2.4.6) over $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$. To that end, observe (using the same reasoning as that implemented in the derivation of (2.4.6)) that the boundary data $(\mathfrak{E}; \mathfrak{F})$ on $\mathcal{D}$ determines entrance data $\mathfrak{e}'' = (e''_k, e''_{k+1})$ and exit

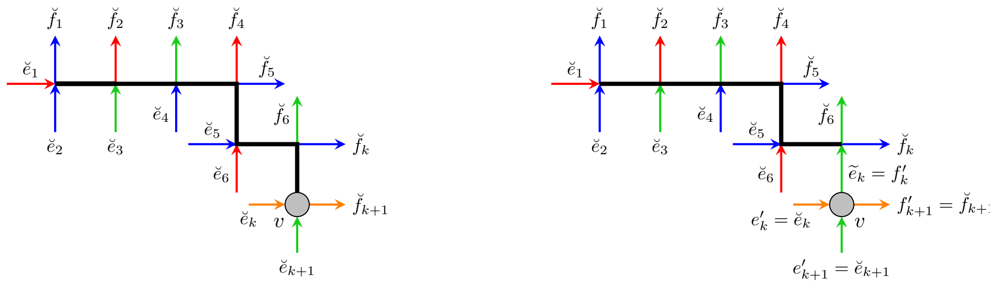


FIGURE 2.5. Shown to the left and right are the domains $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively, used in the proof of Proposition 2.3.1 if $\mathbf{p} = \mathbf{p}'$.

data $\mathfrak{f}'' = (f''_k, f''_{k+1})$ on v satisfying $e''_k = e_k$ and $(e''_{k+1}, f''_{k+1}) = (e_{k+1}, f_{k+1})$. Once again, this fixes entrace data $\widehat{\mathfrak{E}} = (\widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_k) = (e_1, e_2, \ldots, e_{k-1}, f''_k)$ and exit data $\widehat{\mathfrak{F}} = (\widehat{f}_1, \widehat{f}_2, \ldots, \widehat{f}_k) = (f_1, f_2, \ldots, f_k)$ on $\widetilde{\mathcal{D}}$. Then, Corollary 2.3.4 yields

$$\sum_{\mathfrak{e}'} \sum_{\mathfrak{f}'} Z_{\{v\}}^{(m;n)}\big(\mathfrak{e}'; \mathfrak{f}' \mid z(v)\big) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\mathfrak{e}', J_i) - \operatorname{inv}(\mathfrak{f}'', J_i)} = Z_{\{v\}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{e}''); \theta_{\mathbb{J}}(\mathfrak{f}'') \mid z(v)\big), \tag{2.4.7}$$

where we sum over all $\mathfrak{e}' = (e'_k, e'_{k+1})$ and $\mathfrak{f}' = (f'_k, f'_{k+1})$ such that $\theta_{\mathbb{J}}(\mathfrak{e}') = \theta_{\mathbb{J}}(\mathfrak{e}'')$; $\theta_{\mathbb{J}}(\mathfrak{f}') = \theta_{\mathbb{J}}(\mathfrak{f}'')$; $e'_i = e''_i$ whenever $e''_i \in [0, m-1]$; and $f'_i = f''_i$ whenever $f''_i \in [m, m+n-1]$.

Moreover, since $|\widetilde{\mathcal{D}}| = |\mathcal{D}| - 1 = r - 1$, (2.3.3) applies and gives

$$\sum_{\widetilde{\mathfrak{E}}} \sum_{\widetilde{\mathfrak{F}}} Z_{\widetilde{\mathcal{D}}}^{(m;n)}\big(\widetilde{\mathfrak{E}}; \widetilde{\mathfrak{F}} \mid \widetilde{\mathbf{z}}\big) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\widetilde{\mathfrak{E}}, J_i) - \operatorname{inv}(\widehat{\mathfrak{F}}, J_i)} = Z_{\widetilde{\mathcal{D}}}^{(m';n')}\big(\theta_{\mathbb{J}}(\widehat{\mathfrak{E}}); \theta_{\mathbb{J}}(\widehat{\mathfrak{F}}) \mid \widetilde{\mathbf{z}}\big), \tag{2.4.8}$$

where we sum over all $\widetilde{\mathfrak{E}} = (\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_k)$ and $\widetilde{\mathfrak{F}} = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k)$ such that $\theta_{\mathbb{J}}(\widetilde{\mathfrak{E}}) = \theta_{\mathbb{J}}(\widehat{\mathfrak{E}})$; $\theta_{\mathbb{J}}(\widetilde{\mathfrak{F}}) = \theta_{\mathbb{J}}(\widehat{\mathfrak{F}})$; $\widetilde{e}_i = \widehat{e}_i$ whenever $\widehat{e}_i \in [0, m-1]$; and $\widetilde{f}_i = \widehat{f}_i$ whenever $\widehat{f}_i \in [m, m+n-1]$.

We now combine (2.4.7) and (2.4.8). To that end, recall the correspondence between $\breve{\mathfrak{E}}$ and $(\widetilde{\mathfrak{E}}, \mathfrak{e}')$, and between $\breve{\mathfrak{F}}$ and $(\widetilde{\mathfrak{F}}, \mathfrak{f}')$, described above (2.4.6). We claim for each $i \in [0, m'+n'-1]$ that, whenever $(\mathfrak{e}', \mathfrak{f}')$ and $(\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{F}})$ satisfy arrow conservation on v and on $\widetilde{\mathcal{D}}$, respectively, we have

$$\operatorname{inv}(\breve{\mathfrak{E}}, J_i) - \operatorname{inv}(\widetilde{\mathfrak{E}}; J_i) - \operatorname{inv}(\mathfrak{e}'; J_i) = \operatorname{inv}(\breve{\mathfrak{F}}, J_i) - \operatorname{inv}(\widetilde{\mathfrak{F}}; J_i) - \operatorname{inv}(\mathfrak{f}'; J_i). \tag{2.4.9}$$

To verify (2.4.9), fix some $i \in [0, m'+n'-1]$. We may assume (for notational convenience) that each element of $\breve{\mathfrak{E}}, \widetilde{\mathfrak{E}}, \mathfrak{e}', \breve{\mathfrak{F}}, \widetilde{\mathfrak{F}}, \mathfrak{f}'$ is in J_i, for otherwise any elements not in J_i contained in at least one of these sets can be removed without effecting either side of (2.4.9). Then, since $\widetilde{e}_i = \breve{e}_i$ for each $i \in [1, k-1]$ and $(e'_k, e'_{k+1}) = (\breve{e}_k, \breve{e}_{k+1})$, (2.3.1) imples that the left side of (2.4.9) is equal to

$$\operatorname{inv}(\breve{\mathfrak{E}}, J_i) - \operatorname{inv}(\widetilde{\mathfrak{E}}; J_i) - \operatorname{inv}(\mathfrak{e}'; J_i) = \sum_{i=1}^{k-1} (\mathbf{1}_{\breve{e}_i > \breve{e}_k} + \mathbf{1}_{\breve{e}_i > \breve{e}_{k+1}} - \mathbf{1}_{\breve{e}_i > \widetilde{e}_k}). \tag{2.4.10}$$

Similarly, since $\widetilde{f}_i = \breve{f}_i$ for each $i \in [1,k]$ and $(f'_k, f'_{k+1}) = (\widetilde{e}_k, \breve{f}_{k+1})$, the right side of (2.4.9) equals

$$\operatorname{inv}(\breve{\mathfrak{F}}, J_i) - \operatorname{inv}(\widetilde{\mathfrak{F}}; J_i) - \operatorname{inv}(\mathfrak{f}'; J_i) = \sum_{i=1}^{k} \mathbf{1}_{\breve{f}_i > \breve{f}_{k+1}} - \mathbf{1}_{\widetilde{e}_k > \breve{f}_{k+1}}. \tag{2.4.11}$$

Next, arrow conservation for $(\mathfrak{e}', \mathfrak{f}')$ implies $(e'_k, e'_{k+1}) \in \big\{ (f'_k, f'_{k+1}), (f'_{k+1}, f'_k) \big\}$, so $(\breve{e}_k, \breve{e}_{k+1})$ is either $(\widetilde{e}_k, \breve{f}_{k+1})$ or $(\breve{f}_{k+1}, \widetilde{e}_k)$. Moreover, arrow conservation for $(\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{F}})$ yields $\{\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_{k-1}, \widetilde{e}_k\} = \{\breve{f}_1, \breve{f}_2, \ldots, \breve{f}_k\}$ as (unordered) multi-sets. From this, it is directly verified that the right sides of (2.4.10) and (2.4.11) are equal in both cases $(\breve{e}_k, \breve{e}_{k+1}) \in \big\{ (\widetilde{e}_k, \breve{f}_{k+1}), (\breve{f}_{k+1}, \widetilde{e}_k) \big\}$, establishing (2.4.9).

Now, observe that $\operatorname{inv}(\breve{\mathfrak{F}}, J_i) = \operatorname{inv}(\mathfrak{F}, J_i)$ holds for each $i \in [m', m'+n'-1]$, since $J_i \subseteq [m, m+n-1]$ and each $\breve{f}_j = f_j$ whenever $f_j \in [m, m+n-1]$. For the same reason, we also have for each $i \in [m', m'+n'-1]$ that $\operatorname{inv}(\widetilde{\mathfrak{F}}; J_i) = \operatorname{inv}(\widehat{\mathfrak{F}}; J_i)$ and $\operatorname{inv}(\mathfrak{f}'; J_i) = \operatorname{inv}(\mathfrak{f}''; J_i)$. Combining these three equalities with (2.4.9) yields for each $i \in [m', m'+n'-1]$ that

$$\operatorname{inv}(\widetilde{\mathfrak{E}}; J_i) - \operatorname{inv}(\widehat{\mathfrak{F}}; J_i) + \operatorname{inv}(\mathfrak{e}'; J_i) - \operatorname{inv}(\mathfrak{f}''; J_i) = \operatorname{inv}(\breve{\mathfrak{E}}, J_i) - \operatorname{inv}(\mathfrak{F}, J_i),$$

whenever $(\mathfrak{e}', \mathfrak{f}')$ and $(\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{F}})$ satisfy arrow conservation. This, (2.4.7), and (2.4.8) then together yield

$$\begin{aligned} &\sum_{\widetilde{\mathfrak{E}}} \sum_{\widetilde{\mathfrak{F}}} \sum_{\mathfrak{e}'} \sum_{\mathfrak{f}'} Z_{\{v\}}^{(m;n)}\big(\mathfrak{e}'; \mathfrak{f}' \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}(\widetilde{\mathfrak{E}}; \widetilde{\mathfrak{F}} \mid \widetilde{\mathbf{z}}) \prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\breve{\mathfrak{E}}, J_i) - \operatorname{inv}(\mathfrak{F}, J_i)} \\ &\qquad = Z_{\{v\}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{e}''); \theta_{\mathbb{J}}(\mathfrak{f}'') \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m';n')}\big(\theta_{\mathbb{J}}(\widehat{\mathfrak{E}}); \theta_{\mathbb{J}}(\widehat{\mathfrak{F}}) \mid \widetilde{\mathbf{z}}\big) = Z_{\mathcal{D}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{E}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \mathbf{z}\big), \end{aligned} \tag{2.4.12}$$

where to deduce the last equality we applied (2.4.6). Here, $(\mathfrak{e}', \mathfrak{f}')$ and $(\widetilde{\mathfrak{E}}, \widetilde{\mathfrak{F}})$ are summed as in (2.4.7) and (2.4.8), respectively, with the additional constraint that $\widetilde{e}_k = f'_k$. As in the derivation of (2.4.4), here we have used the fact that the index $\widetilde{e}_k = f'_k$ common to (2.4.7) and (2.4.8) is fixed in one if and only if it is summed over in the other.

Due to the correpsondences between $\breve{\mathfrak{E}}$ and $(\widetilde{\mathfrak{E}}, \mathfrak{e}')$ and between $\breve{\mathfrak{F}}$ and $(\widetilde{\mathfrak{F}}, \mathfrak{f}')$, we may instead sum the left side of (2.4.12) over $(\breve{\mathfrak{E}}, \breve{\mathfrak{F}})$. This gives

$$\begin{aligned} \sum_{\breve{\mathfrak{E}}} \sum_{\breve{\mathfrak{F}}} Z_{\{v\}}^{(m;n)}\big(\mathfrak{e}'; \mathfrak{f}' \mid z(v)\big) Z_{\widetilde{\mathcal{D}}}^{(m;n)}(\widetilde{\mathfrak{E}}; \widetilde{\mathfrak{F}} \mid \widetilde{\mathbf{z}}) &\prod_{i=m'}^{m'+n'-1} (-1)^{\operatorname{inv}(\breve{\mathfrak{E}}, J_i) - \operatorname{inv}(\mathfrak{F}, J_i)} \\ &= Z_{\mathcal{D}}^{(m';n')}\big(\theta_{\mathbb{J}}(\mathfrak{E}); \theta_{\mathbb{J}}(\mathfrak{F}) \mid \mathbf{z}\big), \end{aligned} \tag{2.4.13}$$

where we sum over all $\breve{\mathfrak{E}}$ and $\breve{\mathfrak{F}}$ such that $\theta_{\mathbb{J}}(\breve{\mathfrak{E}}) = \theta_{\mathbb{J}}(\mathfrak{E})$; $\theta_{\mathbb{J}}(\breve{\mathfrak{F}}) = \theta_{\mathbb{J}}(\mathfrak{F})$; $\breve{e}_i = e_i$ whenever $e_i \in [0, m-1]$; and $\breve{f}_i = f_i$ whenever $f_i \in [m, m+n-1]$. We now deduce (2.3.3) from applying (2.4.6) in (2.4.13); this establishes the proposition if $\mathbf{p} = \mathbf{p}'$. $\square$

CHAPTER 3

Fusion of Weights

The vertex weights provided in Chapter 2 are nonzero only if each edge incident to the vertex accommodates at most one colored arrow. We now proceed to remove this condition by applying the fusion procedure, which originated in **[53]**, to these weights by suitably concatenating rows of vertices whose spectral parameters are in a certain geometric progression.

3.1. Specialized Rectangular Partition Functions

In this section we provide notation for, and establish properties of, certain rectangular partition functions that will be relevant for the fusion procedure. To that end, we begin with the following partition function, which is the special case of Definition 2.2.1 when the domain $\mathcal{D}$ there is a $M \times L$ rectangle.

Definition 3.1.1. Fix integers $L, M \geqslant 1$ and sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_L)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_M)$. For any sequences $\mathfrak{A} = (a_1, a_2, \ldots, a_M)$, $\mathfrak{B} = (b_1, b_2, \ldots, b_L)$, $\mathfrak{C} = (c_1, c_2, \ldots, c_M)$, and $\mathfrak{D} = (d_1, d_2, \ldots, d_L)$ of indices in $[0, m+n-1]$, define $Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y}) = Z^{(m;n)}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y})$ by setting

$$Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y}) = \sum \prod_{i=1}^{M} \prod_{j=1}^{L} R_{y_i/x_j}\big(v_{i,j}, u_{i,j}; v_{i,j+1}, u_{i+1,j}\big), \tag{3.1.1}$$

where the sum is over all sequences $(u_{i,j})$ and $(v_{i,j})$ of indices in $[0, m+n-1]$ such that $v_{k,1} = a_k$, $u_{1,k} = b_k$, $v_{k,L+1} = c_k$, and $u_{M+1,k} = d_k$, for each k.

If there exist $x, y \in \mathbb{C}$ such that $\mathbf{x} = (x, qx, \ldots, q^{L-1}x)$ and $y = (q^{M-1}y, q^{M-2}y, \ldots, y)$, then we further denote $Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}) = Z_{x,y}^{(m;n)}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}) = Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y})$.

For any ordered set $\mathfrak{Z} = (z_1, z_2, \ldots, z_k)$, let $\overleftarrow{\mathfrak{Z}} = (z_k, z_{k-1}, \ldots, z_1)$ denote the reverse ordering of $\mathfrak{Z}$. Then, (3.1.1) denotes the partition function as in Definition 2.2.1 for the vertex model on the rectangular domain $[1, M] \times [1, L]$, whose entrance and exit data are given by $\overleftarrow{\mathfrak{B}} \cup \mathfrak{A}$ and $\mathfrak{C} \cup \overleftarrow{\mathfrak{D}}$, respectively. Here, we view x_k and y_k as the rapidities in the k-th row (from the bottom) and the k-th column (from the left) of this model, respectively, so that the spectral parameter $z(i,j)$ at $(i,j) \in [1, M] \times [1, L]$ is given by $x_j^{-1} y_i$. We refer to the left side of Figure 3.1 for a depiction.

Remark 3.1.2. For any ordered sequence $\mathcal{T} = (t_1, t_2, \ldots, t_K)$ and indices $1 \leqslant i \leqslant j \leqslant K$, define the restriction $\mathcal{T}_{[i,j]} = (t_i, t_{i+1}, \ldots, t_j)$. Then observe from the definition (3.1.1) for Z that, for any

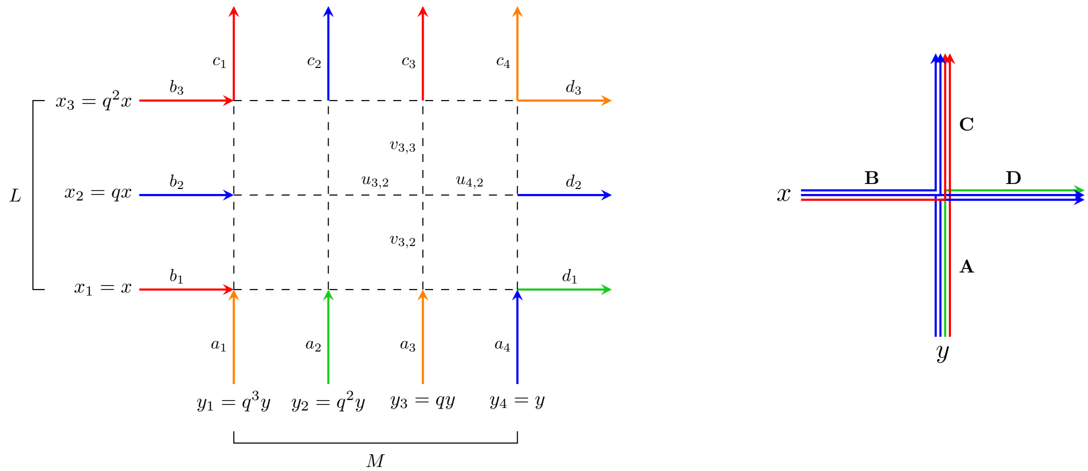


FIGURE 3.1. To the left is a diagrammatic interpretation for $Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D})$ and to the right is a depiction of a vertex for $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$.

integer $h \geqslant 0$, we have

(3.1.2)

$$Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y}) = \sum_{\mathfrak{I}} Z\big(\mathfrak{A}, \mathfrak{B}_{[1,h]}; \mathfrak{I}, \mathfrak{D}_{[1,h]} \mid \mathbf{x}_{[1,h]}, \mathbf{y}\big) Z\big(\mathfrak{I}, \mathfrak{B}_{[h+1,L]}; \mathfrak{C}, \mathfrak{D}_{[h+1,L]} \mid \mathbf{x}_{[h+1,L]}, \mathbf{y}\big);$$

$$Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{y}) = \sum_{\mathfrak{J}} Z\big(\mathfrak{A}_{[1,h]}, \mathfrak{B}; \mathfrak{C}_{[1,h]}, \mathfrak{J} \mid \mathbf{x}, \mathbf{y}_{[1,h]}\big) Z\big(\mathfrak{A}_{[h+1,M]}, \mathfrak{J}; \mathfrak{C}, \mathfrak{D}_{[h+1,M]} \mid \mathbf{x}, \mathbf{y}_{[h+1,M]}\big),$$

where $\mathfrak{I} = (i_1, i_2, \ldots, i_M)$ and $\mathfrak{J} = (j_1, j_2, \ldots, j_L)$ range over all sequences of indices in $[0, m+n-1]$. Indeed, in the former equality of (3.1.2), $\mathfrak{I}$ is interpreted as the ordered set of colors $(v_{1,h+1}, v_{2,h+1}, \ldots, v_{M,h+1})$ (from (3.1.1)) of vertical arrows intersecting the line $y = h + \frac{1}{2}$; a similar interpretation holds for $\mathfrak{J}$ in the latter statement of (3.1.2).

Next, for any index set $\mathcal{I} = (i_1, i_2, \ldots, i_\ell)$ and real number $k \in \mathbb{R}$, let $m_k(\mathcal{I})$ denote the number of indices $j \in [1, \ell]$ such that $i_j = k$. Furthermore, for any sequence $\mathbf{I} = (I_0, I_1, \ldots, I_k) \in \mathbb{Z}_{\geqslant 0}^{k+1}$, let $\mathcal{M}(\mathbf{I})$ denote the family of sequences $\mathcal{I} = (i_1, i_2, \ldots, i_{|\mathbf{I}|})$ with elements in $\{0, 1, \ldots, k\}$ such that $I_j = m_j(\mathcal{I})$ for each $j \in [0, k]$ (where we recall $|\mathbf{I}| = \sum_{j=0}^{k} I_j$).

The fused weights to be considered below will be closely related to the following linear combinations of rectangular partition functions. Here we adopt the convention that symbols of the form $\mathfrak{X} = (x_1, x_2, \ldots, x_K)$ denote ordered sequences of indices (colors) in $[0, m+n-1]$, as above, and those of the form $\mathbf{X} = (X_0, X_1, \ldots, X_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n}$ (in Chapter 5 and afterwards we will start the indexing at 1 instead of at 0) denote nonnegative compositions. The two will typically be related by stipulating $X_j = m_j(\mathfrak{X})$ for each index $j \in [0, m+n-1]$, that is, $\mathfrak{X} \in \mathcal{M}(\mathbf{X})$.

In the below, we recall for any sequence $\mathcal{X} = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell$ that $\operatorname{inv}(\mathcal{X})$ denotes the number of index pairs $(i, j) \in [1, \ell]^2$ such that $i < j$ and $x_i > x_j$.

Definition 3.1.3. Adopt the notation of Definition 3.1.1, and let $\mathbf{A} = (A_0, A_1, \ldots, A_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n}$ and $\mathbf{B} = (B_0, B_1, \ldots, B_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n}$ denote sequences of nonnegative integers such that

$|\mathbf{A}| = M$ and $|\mathbf{B}| = L$. Define

$$\mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) = \sum_{\substack{\mathfrak{A} \in \mathcal{M}(\mathbf{A}) \\ \mathfrak{B} \in \mathcal{M}(\mathbf{B})}} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}).$$

The following lemma provides a q-exchangeability property for these weights $\mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D})$, stating that they multiply by (explicit) powers of q upon permuting $\mathfrak{C}$ and $\mathfrak{D}$, assuming that $\max\{A_h, B_h\} \leqslant 1$ for each $h \in [m, m+n-1]$.

Lemma 3.1.4. *Adopt the notation of Definition 3.1.3, and assume that* $\max\{A_h, B_h\} \leqslant 1$ *for each index* $h \in [m, m+n-1]$. *For any permutations* $\mathfrak{C}' = (c'_1, c'_2, \ldots, c'_M)$ *and* $\mathfrak{D}' = (d'_1, d'_2, \ldots, d'_L)$ *of* $\mathfrak{C}$ *and* $\mathfrak{D}$, *respectively, we have*

$$q^{-\operatorname{inv}(\mathfrak{C}) - \operatorname{inv}(\overleftarrow{\mathfrak{D}})} \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) = q^{-\operatorname{inv}(\mathfrak{C}') - \operatorname{inv}(\overleftarrow{\mathfrak{D}}')} \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}', \mathfrak{D}'). \tag{3.1.3}$$

Proof. Since the symmetric group $\mathfrak{S}_n$ is generated by the transpositions $\{\mathfrak{s}_i\}_{i \in [1,n-1]}$ interchanging $(i, i+1)$, it suffices to establish (3.1.3) assuming that either $(\mathfrak{C}', \mathfrak{D}') = \big(\mathfrak{s}_k(\mathfrak{C}), \mathfrak{D}\big)$ or $(\mathfrak{C}', \mathfrak{D}') = \big(\mathfrak{C}, \mathfrak{s}_k(\mathfrak{D})\big)$ for some k. Let us assume that the former holds, as the proof in the latter case is entirely analogous (by replacing our use of the top diagram in Figure 3.2 below with the bottom one there). Thus, we must show that

$$\mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) = q^{\operatorname{sgn}(c_k - c_{k+1})} \mathcal{Z}_{x,y}\big(\mathbf{A}, \mathbf{B}; \mathfrak{C}', \mathfrak{D}\big). \tag{3.1.4}$$

We will in fact establish (3.1.4) under the more general hypothesis that only imposes $A_h \leqslant 1$ for each $h \in [m, m+n-1]$ (while allowing $B_h \geqslant 2$ for $h \in [m, m+n-1]$).

Let us reduce (3.1.4) to the case $M = 2$. Assuming (3.1.4) holds for $M = 2$ (and arbitrary $L \geqslant 1$), Definition 3.1.3 and repeated application of the second identity in (3.1.2) gives for arbitrary $M \geqslant 2$ that

$$\begin{aligned}
\mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) &= \sum_{\substack{\mathfrak{A} \in \mathcal{M}(\mathbf{A}) \\ \mathfrak{B} \in \mathcal{M}(\mathbf{B})}} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}) \\
&= \sum_{\substack{\mathfrak{A} \in \mathcal{M}(\mathbf{A}) \\ \mathfrak{B} \in \mathcal{M}(\mathbf{B})}} \sum_{\mathfrak{J}_1, \mathfrak{J}_2} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} Z_{x,q^{M-k+1}y}\big(\mathfrak{A}_{[1,k-1]}, \mathfrak{B}; \mathfrak{C}_{[1,k-1]}, \mathfrak{J}_1\big) \\
&\qquad \times Z_{x,q^{M-k-1}y}\big(\mathfrak{A}_{[k,k+1]}, \mathfrak{J}_1; \mathfrak{C}_{[k,k+1]}, \mathfrak{J}_2\big) Z_{x,y}\big(\mathfrak{A}_{[k+2,M]}, \mathfrak{J}_2; \mathfrak{C}_{[k+2,M]}, \mathfrak{D}\big) \\
&= \sum_{\substack{\mathfrak{A} \in \mathcal{M}(\mathbf{A}) \\ \mathfrak{B} \in \mathcal{M}(\mathbf{B})}} \sum_{\mathfrak{J}_1, \mathfrak{J}_2} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}}) + \operatorname{sgn}(c_k - c_{k+1})} Z_{x,q^{M-k+1}y}\big(\mathfrak{A}_{[1,k-1]}, \mathfrak{B}; \mathfrak{C}_{[1,k-1]}, \mathfrak{J}_1\big) \\
&\qquad \times Z_{x,q^{M-k-1}y}\Big(\mathfrak{A}_{[k,k+1]}, \mathfrak{J}_1; \mathfrak{s}_1\big(\mathfrak{C}_{[k,k+1]}\big), \mathfrak{J}_2\Big) Z_{x,y}\big(\mathfrak{A}_{[k+2,M]}, \mathfrak{J}_2; \mathfrak{C}_{[k+2,M]}, \mathfrak{D}\big) \\
&= q^{\operatorname{sgn}(c_k - c_{k+1})} \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{s}_k(\mathfrak{C}), \mathfrak{D}),
\end{aligned}$$

where $\mathfrak{J}_1, \mathfrak{J}_2$ are summed over $\{0, 1, \ldots, m+n-1\}^L$. This verifies the $M \geqslant 2$ case of (3.1.4) assuming that the $M = 2$ case holds. So, we may assume in what follows that $M = 2$ and thus that $k = 1$, in which case $\mathfrak{A} = (a_1, a_2)$ and $\mathfrak{C} = (c_1, c_2)$.

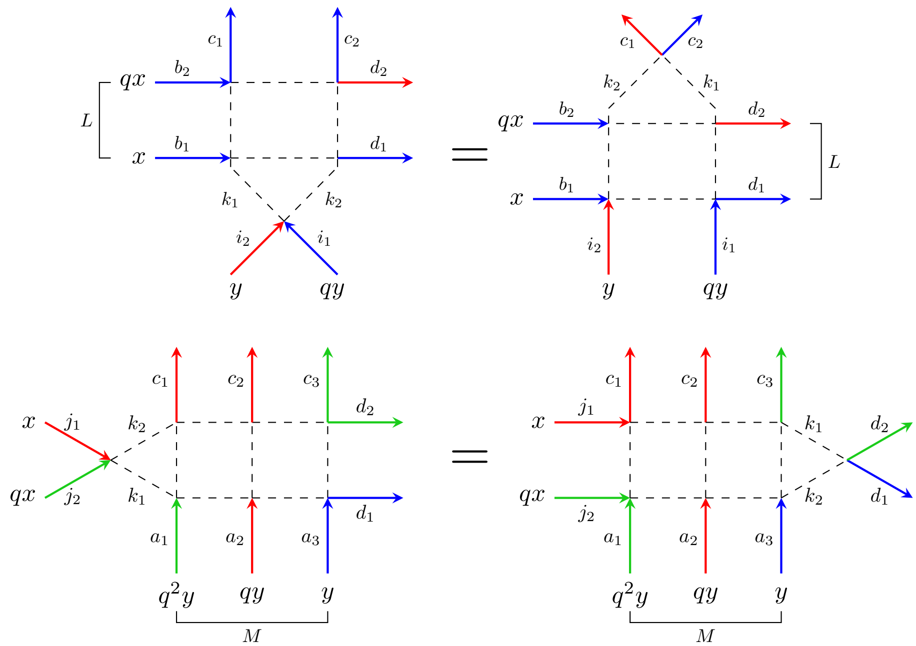


FIGURE 3.2. Shown on the top is a diagrammatic interpretation for (3.1.6); shown on the bottom is its rotated analog.

To that end, recalling the definition of $\mathcal{Z}$ from Definition 3.1.3, it suffices to show that

$$
\begin{aligned}
&q^{\mathbf{1}_{a_1>a_2}} Z_{x,y}\big((a_1,a_2),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) + q^{\mathbf{1}_{a_2>a_1}} Z_{x,y}\big((a_2,a_1),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) \\
&\qquad = q^{\operatorname{sgn}(c_1-c_2)}\Big(q^{\mathbf{1}_{a_1>a_2}} Z_{x,y}\big((a_1,a_2),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big) + q^{\mathbf{1}_{a_2>a_1}} Z_{x,y}\big((a_2,a_1),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big)\Big),
\end{aligned} \tag{3.1.5}
$$

which would impy (3.1.4) upon multiplying both sides by $q^{\mathrm{inv}(\overleftarrow{\mathfrak{B}})}$ and summing over $\mathfrak{B} \in \mathcal{M}(\mathbf{B})$.

Since (3.1.5) holds if $c_1 = c_2$, we will assume in what follows that $c_1 \neq c_2$, in which case (3.1.5) will follow from a suitable application of the Yang–Baxter equation. Indeed, applying (2.1.5) L times, we deduce for any fixed ordered pair of indices (i_1, i_2) constituting a permutation of $\mathfrak{A}$ that

$$
\begin{aligned}
&\sum_{k_1,k_2} R_q(i_1,i_2;k_1,k_2) Z_{x,y}\big((k_1,k_2),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) \\
&\qquad\qquad = \sum_{k_1,k_2} Z_{x,y}\big((i_2,i_1),\mathfrak{B};(k_2,k_1),\mathfrak{D}\big) R_q(k_1,k_2;c_1,c_2),
\end{aligned} \tag{3.1.6}
$$

where both sums range over all $k_1, k_2 \in [0, m+n-1]$; we refer to the top of Figure 3.2 for a depiction. By arrow conservation, any summand on either the left or right side is nonzero only if (k_1, k_2) is a permutation of $\mathfrak{A}$ or $\mathfrak{C}$, respectively. So, we may instead sum over $(k_1, k_2) \in \big\{(a_1, a_2), (a_2, a_1)\big\}$ on the left side of (3.1.6) and $(k_1, k_2) \in \big\{(c_1, c_2), (c_2, c_1)\big\}$ on the right side of (3.1.6).

Now, if $a_1 \neq a_2$, then by the explicit form (given by (2.1.1), (2.1.2), and (2.1.3)) for $R_{ab}(z)$, we have $R_q(i,j;i',j') = q^{\mathbf{1}_{i'>j'}}(q+1)^{-1}$ if $i' \neq j'$ and $\{i,j\} = \{i',j'\}$. So, (3.1.6) implies upon multiplying both sides by $q+1$ that

$$\begin{aligned} q^{\mathbf{1}_{a_1>a_2}} Z_{x,y}\big((a_1,a_2),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) &+ q^{\mathbf{1}_{a_2>a_1}} Z_{x,y}\big((a_2,a_1),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) \\ &= q^{\mathbf{1}_{c_1>c_2}}\Big(Z_{x,y}\big((i_2,i_1),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) + Z_{x,y}\big((i_2,i_1),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big)\Big), \end{aligned} \tag{3.1.7}$$

and, by similar reasoning, we find that

$$\begin{aligned} q^{\mathbf{1}_{a_1>a_2}} Z_{x,y}\big((a_1,a_2),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big) &+ q^{\mathbf{1}_{a_2>a_1}} Z_{x,y}\big((a_2,a_1),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big) \\ &= q^{\mathbf{1}_{c_2>c_1}}\Big(Z_{x,y}\big((i_2,i_1),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) + Z_{x,y}\big((i_2,i_1),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big)\Big). \end{aligned} \tag{3.1.8}$$

Then, (3.1.5) follows from comparing (3.1.7) and (3.1.8), since $\operatorname{sgn}(c_1-c_2) = \mathbf{1}_{c_1>c_2} - \mathbf{1}_{c_2>c_1}$.

If instead $a_1 = a_2$, then letting $a_1 = a = a_2$, we must have that $a < m$ (since $A_h \leqslant 1$ holds for each $h \in [m, m+n-1]$). Then, spin conservation implies that the left side of (3.1.6) is supported on the term $(k_1,k_2) = (a,a)$ and so, using the facts that $R_q(a,a;a,a) = 1$ for $a \in [0,m-1]$ and $R_q(k_1,k_2;c_1,c_2) = (q+1)^{-1}q^{\mathbf{1}_{c_1>c_2}}$, (3.1.6) yields

$$\begin{aligned} Z_{x,y}\big((a,a),&\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) \\ &= (q+1)^{-1}q^{\mathbf{1}_{c_1>c_2}}\Big(Z_{x,y}\big((a,a),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) + Z_{x,y}\big((a,a),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big)\Big). \end{aligned} \tag{3.1.9}$$

By similar reasoning,

$$\begin{aligned} Z_{x,y}\big((a,a),&\mathfrak{B};(c_2,c_1),\mathfrak{D}\big) \\ &= (q+1)^{-1}q^{\mathbf{1}_{c_2>c_1}}\Big(Z_{x,y}\big((a,a),\mathfrak{B};(c_1,c_2),\mathfrak{D}\big) + Z_{x,y}\big((a,a),\mathfrak{B};(c_2,c_1),\mathfrak{D}\big)\Big). \end{aligned} \tag{3.1.10}$$

Then, (3.1.9) and (3.1.10) together imply (3.1.5) again since $\operatorname{sgn}(c_1-c_2) = \mathbf{1}_{c_1>c_2} - \mathbf{1}_{c_2>c_1}$.

This addresses the case $(\mathfrak{C}',\mathfrak{D}') = \big(\mathfrak{s}_k(\mathfrak{C}),\mathfrak{D}\big)$; as mentioned above, we omit the proof in the alternative case $(\mathfrak{C}',\mathfrak{D}') = \big(\mathfrak{C},\mathfrak{s}_k(\mathfrak{D})\big)$. ☐

We next have the following lemma indicating a sort of exclusion principle for arrows of color $i \in [m, m+n-1]$, in that if the west and south boundaries of any vertex model corresponding to $\mathcal{Z}_{x,y}$ each admit at most one arrow of any color $i \in [m,m+n-1]$, then the same holds for its east and north boundaries.

Lemma 3.1.5. *Assume $n \geqslant 1$; fix $x,y \in \mathbb{C}$; and let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n}$ denote nonnegative integer sequences with coordinates indexed by $[0,m+n-1]$, such that $|\mathbf{A}| = M = |\mathbf{C}|$ and $|\mathbf{B}| = L = |\mathbf{D}|$. Assume that $\max\{A_i,B_i\} \leqslant 1$ for each $i \in [m,m+n-1]$ but that $\max\{C_i,D_i\} \geqslant 2$ for some $i \in [m,m+n-1]$. Then, for any sequences of indices $\mathfrak{C} = (c_1,c_2,\ldots,c_M) \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} = (d_1,d_2,\ldots,d_L) \in \mathcal{M}(\mathbf{D})$, we have that $\mathcal{Z}_{x,y}(\mathbf{A},\mathbf{B};\mathfrak{C},\mathfrak{D}) = 0$.*

Proof. It suffices to show that $\mathcal{Z}_{x,y}(\mathbf{A},\mathbf{B};\mathfrak{C},\mathfrak{D}) = 0$ if there exists some $h \in [m,m+n-1]$ such that either $C_h \geqslant 2$ or $D_h \geqslant 2$. Since the two cases are entirely analogous, let us assume in what follows that $C_h \geqslant 2$. By Lemma 3.1.4, we may further assume that $c_1 = h = c_2$ and, by following the first part of the proof of Lemma 3.1.4 (in particular, by using the second relation in (3.1.2)), we may assume that $M = 2$. To that end, letting $\mathbf{A} = \mathbf{e}_{a_1} + \mathbf{e}_{a_2} \in \mathbb{Z}_{\geqslant 0}^{m+n}$ for some indices $a_1,a_2 \in [0,m+n-1]$, it suffices to show that

$$q^{\mathbf{1}_{a_1>a_2}} Z_{x,y}\big((a_1,a_2),\mathfrak{B};(h,h),\mathfrak{D}\big) + q^{\mathbf{1}_{a_2>a_1}} Z_{x,y}\big((a_2,a_1),\mathfrak{B};(h,h),\mathfrak{D}\big) = 0. \tag{3.1.11}$$

This will follow from an application of the Yang–Baxter equation similar to the one used to show Lemma 3.1.4. To that end, we may first assume that $a_1 \neq a_2$ since otherwise we would either have $a_1 = h = a_2$, meaning $A_h = 2$ or $B_h \geqslant 2$ (by arrow conservation); both contradict the fact that $\max\{A_h, B_h\} \leqslant 1$, and so $a_1 \neq a_2$. Next, as in (3.1.6), L applications of (2.1.5) (again, see the top diagram in Figure 3.2) implies for any fixed permutation (i_1, i_2) of (a_1, a_2) that

$$(3.1.12) \quad \sum_{k_1, k_2} R_q(i_1, i_2; k_1, k_2) Z_{x,y}\big((k_1, k_2), \mathfrak{B}; (h, h), \mathfrak{D}\big) = Z_{x,y}\big((i_2, i_1), \mathfrak{B}; (h, h), \mathfrak{D}\big) R_q(h, h; h, h),$$

where the sum ranges over all pairs of indices $(k_1, k_2) \in \big\{(a_1, a_2), (a_2, a_1)\big\}$. Here, on the right side, we have used the fact that $R_q(k_1, k_2; h, h)$ is nonzero only if $k_1 = h = k_2$, by arrow conservation.

By the explicit form (given by (2.1.1), (2.1.2), and (2.1.3)) for $R_{ab}(z)$, we have $R_q(i, j; i', j') = q^{\mathbf{1}_{i'>j'}}(q+1)^{-1}$ if $i' \neq j'$ and $\{i, j\} = \{i', j'\}$, and also that $R_q(h, h; h, h) = 0$ for $h \in [m, m+n-1]$. So, (3.1.11) follows from multiplying both sides of (3.1.12) by $q+1$, thereby implying the lemma. □

3.2. Fused Weights and the Yang–Baxter Equation

In this section we define the fused weights for the $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ vertex model and show they satisfy the Yang–Baxter equation. To that end, we begin with the following definition for these weights.

Definition 3.2.1. Fix $L, M \in \mathbb{Z}_{\geqslant 1}$ and, for each index $X \in \{A, B, C, D\}$, fix an $(m+n)$-tuple $\mathbf{X} = (X_0, X_1, \ldots, X_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n}$ such that $|\mathbf{A}| = M = |\mathbf{C}|$ and $|\mathbf{B}| = L = |\mathbf{D}|$. Letting $\mathfrak{C} = (c_1, c_2, \ldots, c_M) \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_L) \in \mathcal{M}(\mathbf{D})$ denote the unique sequences such that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_M$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_L$, define the *fused weight*

$$\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathcal{R}_{x,y}^{(m;n)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{A_i}(q;q)_{B_i}}{(q;q)_{C_i}(q;q)_{D_i}},$$

where we recall that the right side is given by Definition 3.1.3.

Remark 3.2.2. Fix nonnegative integer sequences $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n-1}$ such that $|\mathbf{A}| = M = |\mathbf{C}|$; $|\mathbf{B}| = L = |\mathbf{D}|$; and $\max\{A_h, B_h\} \leqslant 1$ for each $h \in [m, m+n-1]$. Further fix index sequences $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$. By Lemma 3.1.4, we have

$$(3.2.1) \qquad \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) = q^{\operatorname{inv}(\mathfrak{C}) + \operatorname{inv}(\overleftarrow{\mathfrak{D}})} \mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i}(q;q)_{D_i}}{(q;q)_{A_i}(q;q)_{B_i}}.$$

Thus, summing over all $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$, and further using the fact that for any $\mathbf{X} = (X_0, X_1, \ldots, X_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n-1}$ we have

$$\sum_{\mathfrak{X} \in \mathcal{M}(\mathbf{X})} q^{\operatorname{inv}(\mathfrak{X})} = \frac{(q;q)_{|\mathbf{X}|}}{\prod_{i=0}^{m+n-1} (q;q)_{X_i}},$$

we deduce

$$\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \frac{\prod_{i=0}^{m+n-1} (q;q)_{A_i}}{(q;q)_M} \cdot \frac{\prod_{i=0}^{m+n-1} (q;q)_{B_i}}{(q;q)_L} \cdot \sum_{\substack{\mathfrak{C} \in \mathcal{M}(\mathbf{C}) \\ \mathfrak{D} \in \mathcal{M}(\mathbf{D})}} \mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}).$$

Thus, in the case $n = 0$, the fused weights described in Definition 3.2.1 coincide with those given by Definition 8.3 (see also equation (8.3)) of **[9]**.

Similar to in Section 2.1 for the case $L = 1 = M$, we view the quantity $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ as the weight of a vertex v whose row and column rapidities are given by x and y, respectively, with arrow configuration $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$. The latter point now means that, for each $i \in [0, m+n-1]$, A_i and B_i arrows of color i vertically and horizontally enter v, respectively; similarly, C_i and D_i arrows of color i vertically and horizontally exit v, respectively. Unlike in Section 2.1, the case $L, M > 1$ allows multiple arrows to exist along vertical and horizontal edges adjacent to v; we refer to the right side of Figure 3.1 for a depiction.

The following corollary, which follows directly from Lemma 3.1.5, shows that the weights $\mathcal{R}_{x,y}$ impose an exclusion restriction on colors $i \in [m, m+n-1]$ along any edge.

Corollary 3.2.3. *Assume $n \geqslant 1$; fix $x, y \in \mathbb{C}$; and let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n}$ denote nonnegative integer sequences with coordinates indexed by $[0, m+n-1]$. Assume that $\max\{A_i, B_i\} \leqslant 1$ for each $i \in [m, m+n-1]$. Then, $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 0$ unless $\max\{C_i, D_i\} \leqslant 1$ for each $i \in [m, m+n-1]$.*

Proof. This follows from Lemma 3.1.5 and Definition 3.2.1. □

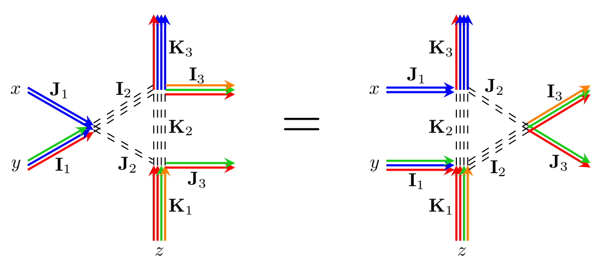
In what follows we will typically consider vertex models whose boundary data admits at most one arrow of any color $i \in [m, m+n-1]$ along any edge entering the domain. Then, Corollary 3.2.3 shows that this exclusion property is retained along interior or exiting boundary edges of the domain.

The following proposition now states that the fused $\mathcal{R}_{x,y}$ weights satisfy the Yang–Baxter equation.

Proposition 3.2.4. *Fix $L, M, N \in \mathbb{Z}_{\geqslant 1}$ and $x, y, z \in \mathbb{C}$. Fix $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1, \mathbf{I}_3, \mathbf{J}_3, \mathbf{K}_3 \in \mathbb{Z}_{\geqslant 0}^{m+n}$ such that $|\mathbf{I}_1| = L = |\mathbf{I}_3|$, $|\mathbf{J}_1| = M = |\mathbf{J}_3|$, and $|\mathbf{K}_1| = N = |\mathbf{K}_3|$. If $X_h \leqslant 1$ for any indices $h \in [m, m+n-1]$ and $\mathbf{X} = (X_0, X_1, \ldots, X_{m+n-1}) \in \{\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1\}$, then*

$$\begin{aligned}
&\sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} \mathcal{R}_{x,y}(\mathbf{I}_1, \mathbf{J}_1; \mathbf{I}_2, \mathbf{J}_2) \mathcal{R}_{x,z}(\mathbf{K}_1, \mathbf{J}_2; \mathbf{K}_2, \mathbf{J}_3) \mathcal{R}_{y,z}(\mathbf{K}_2, \mathbf{I}_2; \mathbf{K}_3, \mathbf{I}_3) \\
&\qquad = \sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} \mathcal{R}_{y,z}(\mathbf{K}_1, \mathbf{I}_1; \mathbf{K}_2, \mathbf{I}_2) \mathcal{R}_{x,z}(\mathbf{K}_2, \mathbf{J}_1; \mathbf{K}_3, \mathbf{J}_2) \mathcal{R}_{x,y}(\mathbf{I}_2, \mathbf{J}_2; \mathbf{I}_3, \mathbf{J}_3),
\end{aligned} \tag{3.2.2}$$

where both sums are over all $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2 \in \mathbb{Z}_{\geqslant 0}^{m+n}$ with $|\mathbf{I}_2| = L$, $|\mathbf{J}_2| = M$, and $|\mathbf{K}_2| = N$. Diagrammatically,



where states along the solid edges are fixed and those along dashed edges are summed over.

Proof. It will be useful here to introduce a gauge transformation $\widetilde{\mathcal{R}}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ of the $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ fused weights, given as follows. Recalling the notation of Definition 3.2.1, we set

$$\widetilde{\mathcal{R}}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i} (q;q)_{D_i}}{(q;q)_{A_i} (q;q)_{B_i}}. \tag{3.2.3}$$

Then, to establish (3.2.2), it suffices to show

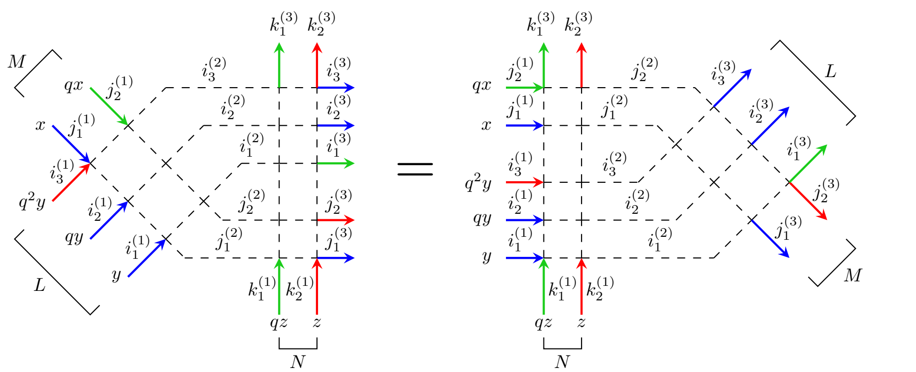
$$
\begin{aligned}
&\sum_{\mathbf{I}_2,\mathbf{J}_2,\mathbf{K}_2} \widetilde{\mathcal{R}}_{x,y}(\mathbf{I}_1,\mathbf{J}_1;\mathbf{I}_2,\mathbf{J}_2)\widetilde{\mathcal{R}}_{x,z}(\mathbf{K}_1,\mathbf{J}_2;\mathbf{K}_2,\mathbf{J}_3)\widetilde{\mathcal{R}}_{y,z}(\mathbf{K}_2,\mathbf{I}_2;\mathbf{K}_3,\mathbf{I}_3) \\
&\qquad = \sum_{\mathbf{I}_2,\mathbf{J}_2,\mathbf{K}_2} \widetilde{\mathcal{R}}_{y,z}(\mathbf{K}_1,\mathbf{I}_1;\mathbf{K}_2,\mathbf{I}_2)\widetilde{\mathcal{R}}_{x,z}(\mathbf{K}_2,\mathbf{J}_1;\mathbf{K}_3,\mathbf{J}_2)\widetilde{\mathcal{R}}_{x,y}(\mathbf{I}_2,\mathbf{J}_2;\mathbf{I}_3,\mathbf{J}_3),
\end{aligned} \tag{3.2.4}
$$

where the sum is as in (3.2.2).

To that end, by LMN applications of (2.1.5), we obtain for any fixed $\mathfrak{I}_h \in \mathcal{M}(\mathbf{I}_h)$, $\mathfrak{J}_h \in \mathcal{M}(\mathbf{J}_h)$, and $\mathfrak{K}_h \in \mathcal{M}(\mathbf{K}_h)$, for $h \in \{1,3\}$, that

$$
\begin{aligned}
&\sum_{\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2} Z_{x,y}\big(\overleftarrow{\mathfrak{I}}_1,\mathfrak{J}_1;\overleftarrow{\mathfrak{I}}_2,\mathfrak{J}_2\big) Z_{x,z}\big(\mathfrak{K}_1,\mathfrak{J}_2;\mathfrak{K}_2,\mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2,\mathfrak{I}_2;\mathfrak{K}_3,\mathfrak{I}_3) \\
&\qquad = \sum_{\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2} Z_{y,z}\big(\mathfrak{K}_1,\mathfrak{I}_1;\mathfrak{K}_2,\mathfrak{I}_2\big) Z_{x,z}(\mathfrak{K}_2,\mathfrak{J}_1;\mathfrak{K}_3,\mathfrak{J}_2) Z_{x,y}\big(\overleftarrow{\mathfrak{I}}_2,\mathfrak{J}_2;\overleftarrow{\mathfrak{I}}_3,\mathfrak{J}_3\big),
\end{aligned} \tag{3.2.5}
$$

where the sum is over all sequences $\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2$ of indices in $[0,m+n-1]$; diagrammatically,



To deduce (3.2.4) from (3.2.5), we will repeatedly first q-symmetrize the inputs of the Z-weights, thereby turning them into the $\mathcal{Z}$ weights of Definition 3.1.3. Then, we will use the q-exchangeability property given by Lemma 3.1.4 (together with Definition 3.2.1 and (3.2.3)) to turn the $\mathcal{Z}$-weights into the $\widetilde{\mathcal{R}}$ weights, which will give rise to (3.2.4).

To that end, multiplying both sides by $q^{\mathrm{inv}(\overleftarrow{\mathfrak{I}}_1)+\mathrm{inv}(\overleftarrow{\mathfrak{J}}_1)}$, summing over $\mathfrak{I}_1 \in \mathcal{M}(\mathbf{I}_1)$ and $\mathfrak{J}_1 \in \mathcal{M}(\mathbf{J}_1)$, and using Definition 3.1.3 then gives

$$
\begin{aligned}
&\sum_{\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2} \mathcal{Z}_{x,y}\big(\mathbf{I}_1,\mathbf{J}_1;\overleftarrow{\mathfrak{I}}_2,\mathfrak{J}_2\big) Z_{x,z}\big(\mathfrak{K}_1,\mathfrak{J}_2;\mathfrak{K}_2,\mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2,\mathfrak{I}_2;\mathfrak{K}_3,\mathfrak{I}_3) \\
&\quad = \sum_{\mathfrak{I}_1,\mathfrak{J}_1,\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2} q^{\mathrm{inv}(\overleftarrow{\mathfrak{I}}_1)+\mathrm{inv}(\overleftarrow{\mathfrak{J}}_1)} Z_{x,y}\big(\overleftarrow{\mathfrak{I}}_1,\mathfrak{J}_1;\overleftarrow{\mathfrak{I}}_2,\mathfrak{J}_2\big) Z_{x,z}\big(\mathfrak{K}_1,\mathfrak{J}_2;\mathfrak{K}_2,\mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2,\mathfrak{I}_2;\mathfrak{K}_3,\mathfrak{I}_3) \\
&\quad = \sum_{\mathfrak{I}_1,\mathfrak{J}_1,\mathfrak{I}_2,\mathfrak{J}_2,\mathfrak{K}_2} q^{\mathrm{inv}(\overleftarrow{\mathfrak{I}}_1)+\mathrm{inv}(\overleftarrow{\mathfrak{J}}_1)} Z_{y,z}\big(\mathfrak{K}_1,\mathfrak{I}_1;\mathfrak{K}_2,\mathfrak{I}_2\big) Z_{x,z}(\mathfrak{K}_2,\mathfrak{J}_1;\mathfrak{K}_3,\mathfrak{J}_2) Z_{x,y}\big(\overleftarrow{\mathfrak{I}}_2,\mathfrak{J}_2;\overleftarrow{\mathfrak{I}}_3,\mathfrak{J}_3\big),
\end{aligned} \tag{3.2.6}
$$

where we always sum over $\mathfrak{X}_h \in \mathcal{M}(\mathbf{X}_h)$, for any indices $X \in \{I, J, K\}$ and $h \in \{1, 3\}$. Multiplying both sides by $q^{\operatorname{inv}(\mathfrak{K}_1)}$; summing over $\mathfrak{K}_1 \in \mathcal{M}(\mathbf{K}_1)$; and using Definition 3.1.3 again then gives

$$\begin{aligned}
&\sum_{\mathfrak{K}_1, \mathfrak{I}_2, \mathfrak{J}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\mathfrak{K}_1)} \mathcal{Z}_{x,y}\big(\mathbf{I}_1, \mathbf{J}_1; \overleftarrow{\mathfrak{I}_2}, \mathfrak{J}_2\big) Z_{x,z}\big(\mathfrak{K}_1, \mathfrak{J}_2; \mathfrak{K}_2, \mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2, \mathfrak{I}_2; \mathfrak{K}_3, \mathfrak{I}_3) \\
&\qquad = \sum_{\mathfrak{J}_1, \mathfrak{I}_2; \mathfrak{J}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\overleftarrow{\mathfrak{J}}_1)} \mathcal{Z}_{y,z}\big(\mathbf{K}_1, \mathbf{I}_1; \mathfrak{K}_2, \mathfrak{I}_2\big) Z_{x,z}(\mathfrak{K}_2, \mathfrak{J}_1; \mathfrak{K}_3, \mathfrak{J}_2) Z_{x,y}\big(\overleftarrow{\mathfrak{I}_2}, \mathfrak{J}_2; \overleftarrow{\mathfrak{I}_3}, \mathfrak{J}_3\big).
\end{aligned} \tag{3.2.7}$$

By Lemma 3.1.5, we may restrict the sum on the left side of (3.2.7) over $(\mathfrak{I}_2, \mathfrak{J}_2)$ such that, for each index $h \in [m, m+n-1]$, we have that $\max\big\{m_h(\mathfrak{I}_2), m_h(\mathfrak{J}_2)\big\} \leqslant 1$. Similarly, we may restrict the sum on the right side of (3.2.7) over $(\mathfrak{J}_2, \mathfrak{K}_2)$ such that $\max\big\{m_h(\mathfrak{J}_2), m_h(\mathfrak{K}_2)\big\} \leqslant 1$ for each $h \in [m, m+n-1]$.

Now, fix nonnegative integer sequences $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n-1}$ such that $|\mathbf{A}| = M = |\mathbf{C}|$; $|\mathbf{B}| = L = |\mathbf{D}|$; and $\max\{A_h, B_h\} \leqslant 1$ for each $h \in [m, m+n-1]$. Further fix index sequences $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$. By Equation (3.2.1) and (3.2.3), we have

$$\mathcal{Z}_{x,y}(\mathbf{A}, \mathbf{B}; \mathfrak{C}, \mathfrak{D}) = q^{\operatorname{inv}(\mathfrak{C}) + \operatorname{inv}(\overleftarrow{\mathfrak{D}})} \widetilde{\mathcal{R}}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i} (q;q)_{D_i}}{(q;q)_{A_i} (q;q)_{B_i}}. \tag{3.2.8}$$

Repeated application of (3.2.8) in (3.2.7), and also decomposing each sum over an index set $\mathfrak{X}_2$ in (3.2.7) as a sum over the associated nonnegative composition $\mathbf{X}_2 \in \mathbb{Z}_{\geqslant 0}^{m+n}$ and one over $\mathfrak{X}_2 \in \mathcal{M}(\mathbf{X}_2)$, then gives

$$\begin{aligned}
&\sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} \widetilde{\mathcal{R}}_{x,y}\big(\mathbf{I}_1, \mathbf{J}_1; \mathbf{I}_2, \mathbf{J}_2\big) \\
&\qquad \times \sum_{\mathfrak{K}_1, \mathfrak{I}_2, \mathfrak{J}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\mathfrak{K}_1) + \operatorname{inv}(\overleftarrow{\mathfrak{I}_2}) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_2)} Z_{x,z}\big(\mathfrak{K}_1, \mathfrak{J}_2; \mathfrak{K}_2, \mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2, \mathfrak{I}_2; \mathfrak{K}_3, \mathfrak{I}_3) \\
&\quad = \sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} \widetilde{\mathcal{R}}_{y,z}\big(\mathbf{K}_1, \mathbf{I}_1; \mathbf{K}_2, \mathbf{I}_2\big) \\
&\qquad\qquad \times \sum_{\mathfrak{J}_1, \mathfrak{I}_2, \mathfrak{J}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\mathfrak{K}_2) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_2) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_1)} Z_{x,z}(\mathfrak{K}_2, \mathfrak{J}_1; \mathfrak{K}_3, \mathfrak{J}_2) Z_{x,y}\big(\overleftarrow{\mathfrak{I}_2}, \mathfrak{J}_2; \overleftarrow{\mathfrak{I}_3}, \mathfrak{J}_3\big),
\end{aligned} \tag{3.2.9}$$

where the sums over $\mathbf{X}_2$ range in $\mathbb{Z}_{\geqslant 0}^{m+n}$ and those over $\mathfrak{X}_2$ range in $\mathcal{M}(\mathbf{X}_2)$, for each $X \in \{I, J, K\}$; here, we have also used Corollary 3.2.3 to sum over all $\mathbf{X}_2$ (instead of only those satisfying whose h-th components are bounded above by 1 for each $h \in [m, m+n-1]$).

For fixed $\mathbf{X}_h$ for $X \in \{I, J, K\}$ and $h \in \{1, 2, 3\}$ (whose i-th component is bounded above by 1 for each $i \in [m, m+n-1]$), Lemma 3.1.5, Corollary 3.2.3, Definition 3.2.1, and (3.2.8) together yield for fixed $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2 \in \{0, 1\}^n$ that

$$\begin{aligned}
&\sum_{\mathfrak{K}_1, \mathfrak{I}_2, \mathfrak{J}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\mathfrak{K}_1) + \operatorname{inv}(\overleftarrow{\mathfrak{I}_2}) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_2)} Z_{x,z}\big(\mathfrak{K}_1, \mathfrak{J}_2; \mathfrak{K}_2, \mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2, \mathfrak{I}_2; \mathfrak{K}_3, \mathfrak{I}_3) \\
&\qquad = \sum_{\mathfrak{I}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\overleftarrow{\mathfrak{J}}_2)} \mathcal{Z}_{x,z}\big(\mathbf{K}_1, \mathbf{J}_2; \mathfrak{K}_2, \mathfrak{J}_3\big) Z_{y,z}(\mathfrak{K}_2, \mathfrak{I}_2; \mathfrak{K}_3, \mathfrak{I}_3) \\
&\qquad = \sum_{\mathfrak{I}_2, \mathfrak{K}_2} q^{\operatorname{inv}(\mathfrak{K}_2) + \operatorname{inv}(\overleftarrow{\mathfrak{I}_2}) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_3)} \widetilde{\mathcal{R}}_{x,z}\big(\mathbf{K}_1, \mathbf{J}_2; \mathbf{K}_2, \mathbf{J}_3\big) Z_{y,z}(\mathfrak{K}_2, \mathfrak{I}_2; \mathfrak{K}_3, \mathfrak{I}_3) \\
&\qquad = q^{\operatorname{inv}(\mathfrak{K}_3) + \operatorname{inv}(\overleftarrow{\mathfrak{I}}_3) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_3)} \widetilde{\mathcal{R}}_{x,z}\big(\mathbf{K}_1, \mathbf{J}_2; \mathbf{K}_2, \mathbf{J}_3\big) \widetilde{\mathcal{R}}_{y,z}(\mathbf{K}_2, \mathbf{I}_2; \mathbf{K}_3, \mathbf{I}_3).
\end{aligned} \tag{3.2.10}$$

By similar reasoning,

$$\begin{aligned}\sum_{\mathfrak{J}_1, \mathfrak{I}_2, \mathfrak{J}_2, \mathfrak{K}_2} & q^{\operatorname{inv}(\mathfrak{K}_2) + \operatorname{inv}(\overleftarrow{\mathfrak{I}}_2) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_1)} Z_{x,z}(\mathfrak{K}_2, \mathfrak{J}_1; \mathfrak{K}_3, \mathfrak{J}_2) Z_{x,y}\big(\overleftarrow{\mathfrak{I}_2}, \mathfrak{J}_2; \overleftarrow{\mathfrak{I}_3}, \mathfrak{J}_3\big) \\ &= q^{\operatorname{inv}(\mathfrak{K}_3) + \operatorname{inv}(\overleftarrow{\mathfrak{I}}_3) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}_3)} \widetilde{\mathcal{R}}_{x,z}\big(\mathbf{K}_2, \mathbf{J}_1; \mathbf{K}_3, \mathbf{J}_2\big) \widetilde{\mathcal{R}}_{x,y}(\mathbf{I}_2, \mathbf{J}_2; \mathbf{I}_3, \mathbf{J}_3).\end{aligned} \tag{3.2.11}$$

Now (3.2.4), and thus the proposition, follows from inserting (3.2.10) and (3.2.11) into (3.2.9). $\square$

CHAPTER 4

Evaluation of the Fused Weights

Observe that the fused weights $\mathcal{R}_{x,y}$ were defined in Definition 3.2.1 as (linear combinations of) certain rectangular partition functions with specialized rapidity parameters. In this chapter we provide a closed form for these quantities, given as Theorem 4.3.2 below.

4.1. The $U_q\big(\widehat{\mathfrak{sl}}(m)\big)$ Fused Weights

In this section we recall from **[15]** explicit forms for the fused weights of the $U_q\big(\widehat{\mathfrak{sl}}(m)\big)$-vertex model, thereby providing exact expressions for the weights $\mathcal{R}_{x,y}^{(m;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ from Definition 3.2.1 if $n=0$. To that end, following Theorem C.1.1 of **[12]**, for any $x,y\in\mathbb{C}$ and m-tuples $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathbb{Z}_{\geqslant 0}^m$ and $\mu=(\mu_1,\mu_2,\ldots,\mu_m)\in\mathbb{Z}_{\geqslant 0}^m$ of nonnegative integers, let

$$\Phi(\lambda,\mu;x,y)=\frac{(x;q)_{|\lambda|}(x^{-1}y;q)_{|\mu|-|\lambda|}}{(y;q)_{|\mu|}}\Big(\frac{y}{x}\Big)^{|\lambda|}q^{\varphi(\mu-\lambda,\lambda)}\prod_{i=1}^{m}\frac{(q;q)_{\mu_i}}{(q;q)_{\lambda_i}(q;q)_{\mu_i-\lambda_i}},\tag{4.1.1}$$

where we have recalled φ from (1.1.1).

The following proposition that was essentially originally established as equation (7.8) of **[15]**, but appears in its below form as Theorem 8.5 of **[9]**, provides an explicit form for the fused weights $\mathcal{R}_{x,y}^{(m;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ if $n=0$.

Proposition 4.1.1 ([**9**, Theorem 8.5])**.** *Assume $n=0$. Fix integers $L,M\geqslant 1$ and $x,y\in\mathbb{C}$; set $z=\frac{x}{y}$. Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\in\mathbb{Z}_{\geqslant 0}^m$ be such that $|\mathbf{A}|=M=|\mathbf{C}|$ and $|\mathbf{B}|=L=|\mathbf{D}|$; set $\mathbf{X}=(X_0,X_1,\ldots,X_{m-1})$ and $\check{\mathbf{X}}=(X_1,X_2,\ldots,X_{m-1})$, for each $X\in\{A,B,C,D\}$. Then, $\mathcal{R}_{x,y}^{(m;0)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})=\omega_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$, where $\omega_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})=\omega_{x,y}^{(L;M)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ is given by*

$$\begin{aligned}\omega_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})&=z^{|\check{\mathbf{D}}|-|\check{\mathbf{B}}|}q^{|\check{\mathbf{A}}|L-|\check{\mathbf{D}}|M}\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\\&\qquad\times\sum_{\check{\mathbf{P}}}\Phi(\check{\mathbf{C}}-\check{\mathbf{P}},\check{\mathbf{C}}+\check{\mathbf{D}}-\check{\mathbf{P}};q^{L-M}z,q^{-M}z)\Phi(\check{\mathbf{P}},\check{\mathbf{B}};q^{-L}z^{-1},q^{-L}),\end{aligned}\tag{4.1.2}$$

where $\check{\mathbf{P}}=(P_1,\ldots,P_{m-1})\in\mathbb{Z}_{\geqslant 0}^{m-1}$ is summed over $(m-1)$-tuples of nonnegative integers such that $P_i\leqslant\min\{B_i,C_i\}$, for each $i\in[1,m-1]$.

Let us provide an example for these weights in the case $L=1$, which will later be useful for us. In what follows, we recall for any set of real numbers $(X_0,X_1,\ldots,X_\ell)\in\mathbb{R}^{\ell+1}$ and indices $0\leqslant i\leqslant j\leqslant\ell$ that $X_{[i,j]}=\sum_{k=i}^{j}X_k$.

Example 4.1.2. Assume $L=1$, and abbreviate $\mathcal{R}_{x,y}(\mathbf{A},b;\mathbf{C},d)=\mathcal{R}_{x,y}\big(\mathbf{A},\mathbf{e}_b;\mathbf{C},\mathbf{e}_d\big)$, for any $\mathbf{A},\mathbf{C}\in\mathbb{Z}_{\geqslant 0}^m$ and $b,d\in\{0,1,\ldots,m-1\}$. Then, Proposition 4.1.1 (see also equation (B.4.1) of **[12]**) becomes

$$\mathcal{R}_{x,y}(\mathbf{A},j;\mathbf{A},j) = \frac{(x-q^{A_j}y)q^{A_{[j+1,m-1]}}}{x-q^M y}; \qquad \mathcal{R}_{x,y}(\mathbf{A},j;\mathbf{A}+\mathbf{e}_j-\mathbf{e}_k,k) = \frac{(1-q^{A_k})q^{A_{[k+1,m-1]}}x}{x-q^M y};$$

$$\mathcal{R}_{x,y}(\mathbf{A},k;\mathbf{A}+\mathbf{e}_k-\mathbf{e}_j,j) = \frac{(1-q^{A_j})q^{A_{[j+1,m-1]}}y}{x-q^M y},$$

for any $0 \leqslant j < k \leqslant M-1$, and $\mathcal{R}_{x,y}(\mathbf{A},b;\mathbf{C},d) = 0$ for any $(\mathbf{A},b;\mathbf{C},d)$ not of the above form.

4.2. Special Cases of the Fused Weights

In this section we evaluate the fused weights $\mathcal{R}_{x,y}^{(m;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ for general $n \geqslant 0$ under two special cases. The first assumes that $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ do not share any fermionic colors, in which case the weight is given by (4.1.2), as in the $n=0$ situation; the second assumes that $L=1$, in which case the weight is given by a modification of those in Example 4.1.2.

Lemma 4.2.1. *Fix $x,y \in \mathbb{C}$ and integers $L,M \geqslant 1$. Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n}$, whose coordinates are indexed by $[0,m+n-1]$; assume they satisfy $|\mathbf{A}| = M = |\mathbf{C}|$ and $|\mathbf{B}| = L = |\mathbf{D}|$. If there does not exist any index $h \in [m,m+n-1]$ such that $\max\{A_h,B_h,C_h,D_h\} \geqslant 1$, then $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \omega_{x,y}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$, where we recall the weight $\omega_{x,y}$ from* (4.1.2).

Proof. Recall from Definition 3.1.1 the partition function $Z_{x,y}^{(m;n)}$. By Definition 3.1.3, Definition 3.2.1, and Proposition 4.1.1, it suffices to show $Z_{x,y}^{(m;n)}(\mathfrak{A},\mathfrak{B};\mathfrak{C},\mathfrak{D}) = Z_{x,y}^{(m+n;0)}(\mathfrak{A},\mathfrak{B};\mathfrak{C},\mathfrak{D})$, for any $\mathfrak{A} \in \mathcal{M}(\mathbf{A})$, $\mathfrak{B} \in \mathcal{M}(\mathbf{B})$, $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$, and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$. To that end, observe since there does not exist any $h \in [m,m+n-1]$ such that $A_h,B_h,C_h,D_h \geqslant 1$, there does not exist any arrow configuration $(v_{i,j},u_{i,j};v_{i,j+1},u_{i+1,j})$ on the right side of (3.1.1) in the expansion of $Z_{x,y}^{(m;n)}$ that is of the form $(h,h;h,h)$ for some $h \in [m,m+n-1]$. Thus, by Remark 2.1.1, we have that $R_z^{(m;n)}(v_{i,j},u_{i,j};v_{i,j+1},u_{i+1,j}) = R_z^{(m+n;0)}(v_{i,j},u_{i,j};v_{i,j+1},u_{i+1,j})$ for any such vertex weight appearing there. Inserting this fact into (3.1.1) then yields $Z_{x,y}^{(m;n)}(\mathfrak{A},\mathfrak{B};\mathfrak{C},\mathfrak{D}) = Z_{x,y}^{(m+n;0)}(\mathfrak{A},\mathfrak{B};\mathfrak{C},\mathfrak{D})$, which as mentioned above implies the lemma. $\square$

Now let us evaluate the fused weights $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ in the case $L=1$. Letting $\mathbf{B} = \mathbf{e}_b$ and $\mathbf{D} = \mathbf{e}_d$ for some indices $b,d \in [0,m+n-1]$, this is done by Lemma 4.2.1 unless $b = d \in [m,m+n-1]$ and $A_b = C_b = 1$. This remaining situation is addressed through the following lemma.

Lemma 4.2.2. *Fix $x,y \in \mathbb{C}$ and an integer $M \geqslant 1$. Let $\mathbf{A} \in \mathbb{Z}_{\geqslant 0}^{m+n}$ satisfy $|\mathbf{A}| = M$, and denote $\mathbf{A} = (A_0,A_1,\ldots,A_{m+n-1})$; assume that $A_i \leqslant 1$ for each index $i \in [m,m+n-1]$. If $h \in [m,m+n-1]$ is such that $A_h = 1$, then*

$$\mathcal{R}_{x,y}(\mathbf{A},\mathbf{e}_h;\mathbf{A},\mathbf{e}_h) = q^{A_{[h+1,m+n-1]}}\frac{y-qx}{x-q^M y}.$$

Proof. Let $\mathfrak{A} = (a_1,a_2,\ldots,a_M) \in \mathcal{M}(\mathbf{A})$ denote the sequence of indices in $[0,m+n-1]$ such that $a_M = h$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{M-1}$. Then Definition 3.2.1, Definition 3.1.3, and Lemma 3.1.4 together imply that

$$\mathcal{R}_{x,y}(\mathbf{A},\mathbf{e}_h;\mathbf{A},\mathbf{e}_h) = q^{-\operatorname{inv}(\mathfrak{A})}\sum_{\mathfrak{A}' \in \mathcal{M}(\mathbf{A})} q^{\operatorname{inv}(\mathfrak{A}')}Z_{x,y}(\mathfrak{A}',h;\mathfrak{A},h). \tag{4.2.1}$$

Expanding $Z_{x,y}(\mathfrak{A}',h;\mathfrak{A},h)$ as on the right side of (3.1.1), and then using arrow conservation, the fact that $a_j \neq h$ for any $j < M$, and induction on $M-i$, we deduce that each nonzero vertex weight

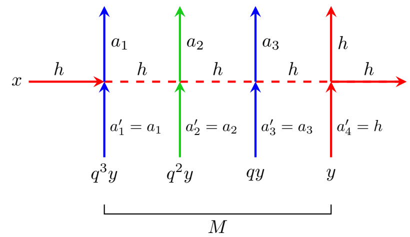


FIGURE 4.1. Shown above is a diagrammatic interpretation for (4.2.3).

$R_z(v_{i,1}, u_{i,1}; v_{i,2}, u_{i+1,1})$ there is associated with the arrow configuration $(a_i, h; a_i, h)$; we refer to Figure 4.1 for a depiction. Thus, only the $\mathfrak{A}' = \mathfrak{A}$ summand on the right side of (4.2.1) is nonzero, which yields

$$\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{e}_h; \mathbf{A}, \mathbf{e}_h) = Z_{x,y}(\mathfrak{A}, h; \mathfrak{A}, h). \tag{4.2.2}$$

Further let $\widetilde{\mathfrak{A}} = (a_1, a_2, \ldots, a_{M-1})$ denote the $(M-1)$-tuple obtained by removing the last entry of $\mathfrak{A}$. Using the fact that the rightmost arrow configuration $(v_{M,1}, u_{M,1}; v_{M,2}, u_{M+1,1})$ in the expansion of $Z_{x,y}(\mathfrak{A}, h; \mathfrak{A}, h)$ given by the right side of (3.1.1) is $(h, h; h, h)$ (since $a_M = h$), we find that

$$Z_{x,y}(\mathfrak{A}, h; \mathfrak{A}, h) = Z_{x,qy}\big(\widetilde{\mathfrak{A}}, h; \widetilde{\mathfrak{A}}, h\big) R_{y/x}(h, h; h, h). \tag{4.2.3}$$

Letting $\widetilde{\mathbf{A}} \in \mathbb{Z}_{\geqslant 0}^{m+n}$ be such that $\widetilde{\mathfrak{A}} \in \mathcal{M}\big(\widetilde{\mathbf{A}}\big)$, we have that $h \notin \widetilde{\mathfrak{A}}$ since $A_h = 1$ and $a_M = h$. Thus, arrow conservation again implies that, in the expansion of $Z_{x,y}\big(\widetilde{\mathfrak{A}}, h; \widetilde{\mathfrak{A}}, h\big)$ as on the right side of (3.1.1), each arrow configuration $(v_{i,1}, u_{i,1}; v_{i,2}, u_{i+1,1})$ is of the form $(a_i, h; a_i, h)$. Thus, similarly to in (4.2.2), we find by Definition 3.2.1, Definition 3.1.3, and Lemma 3.1.4 (and also the fact that $\operatorname{inv}\big(\widetilde{\mathfrak{A}}\big) = 0$) that

$$Z_{x,qy}\big(\widetilde{\mathfrak{A}}, h; \widetilde{\mathfrak{A}}, h\big) = \sum_{\widetilde{\mathfrak{A}}' \in \mathcal{M}(\widetilde{\mathbf{A}})} q^{\operatorname{inv}(\widetilde{\mathfrak{A}}')} Z_{x,qy}\big(\widetilde{\mathfrak{A}}', h; \widetilde{\mathfrak{A}}, h\big) = \mathcal{R}_{x,qy}\big(\widetilde{\mathbf{A}}, \mathbf{e}_h; \widetilde{\mathbf{A}}, \mathbf{e}_h\big).$$

Inserting this into (4.2.3) and using the identity (from (2.1.3)) that $R_{y/x}(h, h; h, h) = \frac{y-qx}{x-qy}$ yields

$$\mathcal{R}_{x,y}(\mathbf{A}, h; \mathbf{A}, h) = \mathcal{R}_{x,qy}\big(\widetilde{\mathbf{A}}, \mathbf{e}_h; \widetilde{\mathbf{A}}, \mathbf{e}_h\big) \frac{y - qx}{x - qy}. \tag{4.2.4}$$

Now observe, upon denoting $\widetilde{\mathbf{A}} = \big(\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_{m+n-1}\big)$, we have that $\widetilde{A}_h = 0$. Thus, Lemma 4.2.1, Proposition 4.1.1, and Example 4.1.2 together imply that

$$\mathcal{R}_{x,qy}\big(\widetilde{\mathbf{A}}, h; \widetilde{\mathbf{A}}, h\big) = q^{\widetilde{A}_{[h+1,m+n-1]}} \frac{x - qy}{x - q^M y} = q^{A_{[h+1,m+n-1]}} \frac{x - qy}{x - q^M y}.$$

Inserting this into (4.2.4) then yields the lemma. □

4.3. General Fused Weights

In this section we evaluate the general fused weights $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ through Theorem 4.3.2 below. This was done in Lemma 4.2.1 assuming that $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ do not share any fermionic colors. If instead these four sequences share a fermionic color $h \in [m, m+n-1]$, then the following lemma provides a recursion for the weight $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ that proceeds by removing a shared fermionic color h from $\mathbf{B}$ and $\mathbf{D}$.

Proposition 4.3.1. *Fix $x, y \in \mathbb{C}$ and $L, M \in \mathbb{Z}_{\geqslant 1}$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n}$, whose coordinates are indexed by $[0, m+n-1]$; suppose they satisfy $|\mathbf{A}| = M = |\mathbf{C}|$ and $|\mathbf{B}| = L = |\mathbf{D}|$. Assume that $\max\{A_i, B_i\} \leqslant 1$ for each index $i \in [m, m+n-1]$ and that there exists an index $h \in [m, m+n-1]$ such that $A_h = B_h = C_h = D_h = 1$. Letting $\widetilde{\mathbf{X}} = \mathbf{X} - \mathbf{e}_h$ for each $X \in \{B, D\}$, we have*

$$\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = q^{A_{[h+1,m+n-1]}} \frac{y - q^L x}{q^{L-1}x - q^M y} \mathcal{R}_{x,y}\big(\mathbf{A}, \widetilde{\mathbf{B}}; \mathbf{C}, \widetilde{\mathbf{D}}\big).$$

Proof. By Definition 3.1.3, Lemma 3.1.4, and Definition 3.2.1, we have for any fixed $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$ and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$ that

(4.3.1)

$$\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = q^{-\operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} \prod_{i=0}^{m+n-1} \frac{(q;q)_{A_i}(q;q)_{B_i}}{(q;q)_{C_i}(q;q)_{D_i}} \cdot \sum_{\substack{\mathfrak{A} \in \mathcal{M}(\mathbf{A}) \\ \mathfrak{B} \in \mathcal{M}(\mathbf{B})}} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}).$$

For each $X \in \{B, D\}$, let $\mathfrak{X} = (x_1, x_2, \ldots, x_L)$ and $\widetilde{\mathfrak{X}}' = (x_1, x_2, \ldots, x_{L-1})$. Then, we have by the definition of $Z_{x,y}$ (in particular, the first statement of (3.1.2)) that

$$Z_{x,y}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}) = \sum_{\mathfrak{I}} Z_{x,y}\big(\mathfrak{A}, \widetilde{\mathfrak{B}}'; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big) Z_{q^{L-1}x,y}\big(\mathfrak{I}, b_L; \mathfrak{C}, d_L\big),$$

where the sum is over all sequences $\mathfrak{I} = (i_1, i_2, \ldots, i_M)$ of indices in $[0, m+n-1]$. Inserting this into (4.3.1) yields

(4.3.2)

$$\begin{aligned}
\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i}(q;q)_{D_i}}{(q;q)_{A_i}(q;q)_{B_i}} & \\
= q^{-\operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} & \sum_{\mathfrak{A} \in \mathcal{M}(\mathbf{A})} \sum_{b_L=0}^{m+n-1} \sum_{\widetilde{\mathfrak{B}}' \in \mathcal{M}(\mathbf{B} - \mathbf{e}_{b_L})} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} \\
& \qquad \times \sum_{\mathfrak{I}} Z_{x,y}\big(\mathfrak{A}, \widetilde{\mathfrak{B}}'; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big) Z_{q^{L-1}x,y}(\mathfrak{I}, b_L; \mathfrak{C}, d_L) \\
= q^{-\operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} & \sum_{b_L=0}^{m+n-1} q^{B_{[0,b_L-1]}} \sum_{\mathfrak{I}} Z_{q^{L-1}x,y}(\mathfrak{I}, b_L; \mathfrak{C}, d_L) \\
& \times \sum_{\mathfrak{A} \in \mathcal{M}(\mathbf{A})} \sum_{\widetilde{\mathfrak{B}}' \in \mathcal{M}(\mathbf{B} - \mathbf{e}_{b_L})} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\widetilde{\mathfrak{B}}'})} Z_{x,y}\big(\mathfrak{A}, \widetilde{\mathfrak{B}}'; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big) \\
= q^{-\operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} & \sum_{b_L=0}^{m+n-1} q^{B_{[0,b_L-1]}} \\
& \times \sum_{\mathfrak{I}} Z_{q^{L-1}x,y}(\mathfrak{I}, b_L; \mathfrak{C}, d_L) \mathcal{Z}_{x,y}\big(\mathbf{A}, \mathbf{B} - \mathbf{e}_{b_L}; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big),
\end{aligned}$$

where in the last equalities we used the identity $\operatorname{inv}\big(\overleftarrow{\widetilde{\mathfrak{B}}'}\big) = \operatorname{inv}(\overleftarrow{\mathfrak{B}}) - B_{[0,b_L-1]}$ and applied Definition 3.1.3.

Now, let us suppose that $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$ is such that $d_L = h$. Since $\max\{A_j, B_j\} \leqslant 1$ for each $j \in [m, m+n-1]$, we have by Lemma 3.1.5 that $\mathcal{Z}_{x,y}\big(\mathbf{A}, \mathbf{B} - \mathbf{e}_{b_L}; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big) = 0$ unless $I_h \leqslant 1$, where $\mathbf{I} = (I_0, I_1, \ldots, I_{m+n-1}) \in \mathbb{Z}_{\geqslant 0}^{m+n}$ is such that $\mathfrak{I} \in \mathcal{M}(\mathbf{I})$. Furthermore, since $d_L = h$ and $h \in \mathfrak{C}$ (as $C_h = 1$), we have by arrow conservation that $Z_{q^{L-1}x,y}(\mathfrak{I}, b_L; \mathfrak{C}, d_L) = 0$ unless $I_h = 2 - \mathbf{1}_{b_L = h}$. Thus, for both of these quantities to be nonzero, we must have that $b_L = h$, and so $\mathbf{B} - \mathbf{e}_{b_L} = \widetilde{\mathbf{B}}$. Again by arrow conservation, this implies that $\mathbf{I} = \mathbf{C}$ for $Z_{x,y}(\mathfrak{I}, b_L; \mathfrak{C}, d_L) = Z_{x,y}(\mathfrak{I}, h; \mathfrak{C}, h)$ to be nonzero. Inserting these facts into (4.3.2), we deduce that

$$
\begin{aligned}
\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot & \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i} (q;q)_{D_i}}{(q;q)_{A_i} (q;q)_{B_i}} \\
&= q^{B_{[0,h-1]} - \operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} \sum_{\mathfrak{I} \in \mathcal{M}(\mathbf{C})} Z_{q^{L-1}x,y}(\mathfrak{I}, h; \mathfrak{C}, h) \mathcal{Z}_{x,y}\big(\mathbf{A}, \widetilde{\mathbf{B}}; \mathfrak{I}, \widetilde{\mathfrak{D}}'\big) \\
&= q^{B_{[0,h-1]} - \operatorname{inv}(\overleftarrow{\mathfrak{D}}) + \operatorname{inv}(\overleftarrow{\mathfrak{D}}')} \mathcal{R}_{x,y}\big(\mathbf{A}, \widetilde{\mathbf{B}}; \mathbf{C}, \widetilde{\mathbf{D}}\big) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i} (q;q)_{D_i}}{(q;q)_{A_i} (q;q)_{B_i}} \\
&\qquad \times \sum_{\mathfrak{I} \in \mathcal{M}(\mathbf{C})} q^{\operatorname{inv}(\mathfrak{I}) - \operatorname{inv}(\mathfrak{C})} Z_{q^{L-1}x,y}(\mathfrak{I}, h; \mathfrak{C}, h) \\
&= q^{B_{[0,h-1]} - D_{[0,h-1]}} \mathcal{R}_{x,y}\big(\mathbf{A}, \widetilde{\mathbf{B}}; \mathbf{C}, \widetilde{\mathbf{D}}\big) \mathcal{R}_{q^{L-1}x,y}(\mathbf{C}, \mathbf{e}_h; \mathbf{C}, \mathbf{e}_h) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i} (q;q)_{D_i}}{(q;q)_{A_i} (q;q)_{B_i}}.
\end{aligned} \tag{4.3.3}
$$

Here, in the second equality we applied Lemma 3.1.4 and Definition 3.2.1, together with the facts that $B_h = 1 = D_h$ and that the h-th coordinates of $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{D}}$ are equal to 0. In the last, we applied those two statements, Definition 3.1.3, and the identity $\operatorname{inv}(\overleftarrow{\mathfrak{D}}) = \operatorname{inv}\big(\overleftarrow{\mathfrak{D}}'\big) + D_{[0,h-1]}$ (as $d_L = h$).

By Lemma 4.2.2, we have that

$$
\mathcal{R}_{q^{L-1}x,y}(\mathbf{C}, \mathbf{e}_h; \mathbf{C}, \mathbf{e}_h) = q^{C_{[h+1,m+n-1]}} \frac{y - q^L x}{q^{L-1}x - q^M y}. \tag{4.3.4}
$$

Since arrow conservation and the fact that $B_h = 1 = D_h$ together imply that $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 0 = \mathcal{R}_{x,y}\big(\mathbf{A}, \widetilde{\mathbf{B}}; \mathbf{C}, \widetilde{\mathbf{D}}\big)$ unless

$$
\begin{aligned}
A_{[h+1,m+n-1]} - B_{[0,h-1]} &= A_{[h+1,m+n-1]} + B_{[h+1,m+n-1]} - L + 1 \\
&= C_{[h+1,m+n-1]} + D_{[h+1,m+n-1]} - L + 1 = C_{[h+1,m+n-1]} - D_{[0,h-1]},
\end{aligned}
$$

the proposition follows from inserting (4.3.4) into (4.3.3). $\square$

Repeated application of Proposition 4.3.1 reduces a general fused weight $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ to one satisfying the conditions of Lemma 4.2.1; this gives rise to the following theorem.

THEOREM 4.3.2. *Fix $x, y \in \mathbb{C}$ and integers $L, M \geqslant 1$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{m+n}$, with coordinates indexed by $[0, m+n-1]$, satisfy $|\mathbf{A}| = M = |\mathbf{C}|$, $|\mathbf{B}| = L = |\mathbf{D}|$; suppose that $\max\{A_j, B_j\} \leqslant 1$ for each index $j \in [m, m+n-1]$.*

1. *If $\max\{C_j, D_j\} \geqslant 2$ for some index $j \in [m, m+n-1]$, then $\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = 0$.*
2. *Otherwise, define $\mathbf{V} = (V_0, V_1, \ldots, V_{m+n-1}) \in \{0,1\}^{m+n}$ by setting $V_i = 0$ for $i \in [0, m-1]$ and $V_j = \min\{A_j, B_j, C_j, D_j\}$ for $j \in [m, m+n-1]$; let $v = |\mathbf{V}|$; and recall $\omega_{x,y}^{(L;M)}$ from*

(4.1.2). *Then, denoting* $z = \frac{x}{y}$, *we have*

$$\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = (-1)^v q^{\sum_{h:V_h=1} A_{[h+1,m+n-1]}-Mv} \frac{(q^{L-v+1}z;q)_v}{(q^{L-M-v}z;q)_v} \times \omega_{x,y}^{(L-v;M)}(\mathbf{A},\mathbf{B}-\mathbf{V};\mathbf{C},\mathbf{D}-\mathbf{V}).$$

Proof. The first statement of the theorem follows from Corollary 3.2.3, so it remains to establish the latter. To that end, observe by applying Proposition 4.3.1 v times (once for each $h \in \mathbf{V}$, in increasing order) that

(4.3.5)

$$\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \mathcal{R}_{x,y}(\mathbf{A},\mathbf{B}-\mathbf{V};\mathbf{C},\mathbf{D}-\mathbf{V}) q^{\sum_{h:V_h=1} A_{[h+1,m+n-1]}} \prod_{j=0}^{v-1} \frac{y - q^{L-j}x}{q^{L-j-1}x - q^M y}$$
$$= (-1)^v q^{\sum_{h:V_h=1} A_{[h+1,m+n-1]}-Mv} \mathcal{R}_{x,y}(\mathbf{A},\mathbf{B}-\mathbf{V};\mathbf{C},\mathbf{D}-\mathbf{V}) \prod_{j=0}^{v-1} \frac{1-q^{L-j}z}{1-q^{L-M-j-1}z}.$$

By Lemma 4.2.1, we have that $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B}-\mathbf{V};\mathbf{C},\mathbf{D}-\mathbf{V}) = \omega_{x,y}^{(L-v;M)}(\mathbf{A},\mathbf{B}-\mathbf{V};\mathbf{C},\mathbf{D}-\mathbf{V})$, since $\min\{A_j, B_j - V_j, C_j, D_j - V_j\} = 0$ for each $j \in [m, m+n-1]$. Inserting this into (4.3.5) yields the theorem. $\square$

CHAPTER 5

The $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ Specialization

In this chapter we give analytic continuations for the $m = 1$ specializations of the fused weights $\mathcal{R}_{x,y}^{(m;n)}$ from Definition 3.2.1 and provide a color merging result for them.

5.1. The $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ Weights and Analytic Continuation

In this section we consider the $m = 1$ specializations of the fused weights $\mathcal{R}_{x,y}$ from Definition 3.2.1; this corresponds to the case when one arrow color is bosonic and the remaining ones are all fermionic. The following definition provides a family of vertex weights W_z, which we will show as Proposition 5.1.3 below are analytic continuations (in $q^{-L/2}$ and $q^{-M/2}$) of these specialized fused $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ weights if each fermionic color appears at most once in $\mathbf{A}$ and $\mathbf{B}$.

In what follows, we recall the function φ from (1.1.1).

Definition 5.1.1. Fix an integer $n \geqslant 1$; complex numbers $r, s, z \in \mathbb{C}$; and n-tuples $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{Z}_{\geqslant 0}^n$. For each index $X \in \{A, B, C, D\}$, denote $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and set $|\mathbf{X}| = x$. Further define $\mathbf{V} = (V_1, V_2, \ldots, V_n) \in \mathbb{Z}_{\geqslant 0}^n$ by setting $V_j = \min\{A_j, B_j, C_j, D_j\}$ for each $j \in [1, n]$, and let $|\mathbf{V}| = v$. If $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ and $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \{0, 1\}^n$, then define $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = W_{z;q}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s)$

(5.1.1)

$$\begin{aligned} W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = (-1)^v z^{d-b} r^{2c-2a} s^{2d} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+cv} \frac{(q^{1-v} r^{-2} z; q)_v}{(q^{-v} s^2 r^{-2} z; q)_v} \frac{(r^2; q)_d}{(r^2; q)_b} \\ \times \sum_{p=0}^{\min\{b-v,c-v\}} \frac{(q^{-v} s^2 r^{-2} z; q)_{c-p} (q^v r^2 z^{-1}; q)_p (z; q)_{b-p-v}}{(s^2 z; q)_{c+d-p-v}} (q^{-v} r^{-2} z)^p \\ \times \sum_{\mathbf{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})}, \end{aligned}$$

where the last sum sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0, 1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$ for each $i \in [1, n]$. Otherwise, set $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = 0$.

Let us provide several examples of these W_z weights that will be useful for us later.

Example 5.1.2. Recall the notation $\mathbf{e}_0 = (0, 0, \ldots, 0) \in \mathbb{Z}^n$ and $\mathbf{e}_{[1,n]} = (1, 1, \ldots, 1) \in \mathbb{Z}^n$. For any $r, s, z \in \mathbb{C}$, we have that

(5.1.2)

$$W_z(\mathbf{e}_0,\mathbf{e}_0;\mathbf{e}_0,\mathbf{e}_0 \mid r, s) = 1; \qquad W_z(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]} \mid r, s) = \frac{s^{2n}(z; q)_n}{(s^2 z; q)_n};$$

$$W_z(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]};\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]} \mid r, s) = (s^2 r^{-2} z)^n \frac{(r^2 z^{-1}; q)_n}{(s^2 z; q)_n}.$$

Indeed, each of the three weights in (5.1.2) are of the form $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$, with $\mathbf{C} = \mathbf{V}$ (where, as in Definition 5.1.1, $\mathbf{V}$ denotes the n-tuple obtained by taking the entrywise minimum of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$). Thus, the sum over $\mathbf{P}$ on the right side of (5.1.1) is supported on the single term $\mathbf{P} = \mathbf{e}_0$, from which one quickly deduces (5.1.2). Through similar reasoning, generalizing the first two statements of (5.1.2), we have for any $\mathbf{B} \in \{0,1\}^n$ with $|\mathbf{B}| = b$ that

$$W_z(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0,\mathbf{B} \mid r,s) = \frac{s^{2b}(z;q)_b}{(s^2z;q)_b}. \tag{5.1.3}$$

For any $r,s,z \in \mathbb{C}$ and n-tuples $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{Z}_{\geqslant 0}^n$, we define a normalization of the weight $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ from Definition 5.1.1, given by

$$\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) = \frac{(s^2z;q)_n}{s^{2n}(z;q)_n} W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s). \tag{5.1.4}$$

Observe in particular by the second statement of (5.1.2) that

$$\widehat{W}_z\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]} \mid r,s\big) = 1. \tag{5.1.5}$$

Similarly to in Section 2.1, we interpret $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ (or $\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$, depending on the context) as a vertex weight in the following way. As there, a vertex v is the intersection between two directed transverse curves ℓ_1 (typically oriented east) and ℓ_2 (typically oriented north), but now associated with each curve is an ordered pair of rapidity parameters. Let the one associated with ℓ_1 be $(x;r)$ and that associated with ℓ_2 be $(y;s)$. The spectral parameter associated with the vertex v is then given by the ratio[1] $\frac{x}{y}$.

Each of the four edges (segments of the curves ℓ_1 and ℓ_2) adjacent to v may accommodate arrows of colors in $\{1,2,\ldots,n\}$. Let A_i, B_i, C_i, and D_i denote the numbers of arrows of any color $i \in [1,n]$ that vertically enter, horizontally enter, vertically exit, and horizontally exit v, respectively. As in Definition 5.1.1, define $\mathbf{X} = (X_1,X_2,\ldots,X_n)$ for each index $X \in \{A,B,C,D\}$. We will assume in what follows that $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \{0,1\}^n$, so that each edge accommodates at most one arrow of any given color (that is, all colors are fermionic). We will additionally impose $\mathbf{A}+\mathbf{B} = \mathbf{C}+\mathbf{D}$, which is a form of arrow conservation. The quadruple $(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ is the *(fused) arrow configuration* at v, and we interpret $W_{x/y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ (or $\widehat{W}_{x/y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$) as the weight of this vertex v. We diagrammatically depict such vertices as on the left side of Figure 5.1.

Observe that, unlike in Chapter 2, in Definition 5.1.1 we do not impose either $|\mathbf{A}| = |\mathbf{C}|$ or $|\mathbf{B}| = |\mathbf{D}|$, and we do not count arrows of the originally bosonic color 0 (which we might view as "empty"). Still, as the following proposition indicates, if r and s are negative half-integer powers of q, then the W_z weights can be matched with the $\mathcal{R}_z$ weights from Definition 3.2.1 under a change of variables.

Proposition 5.1.3. *Fix integers $n \geqslant 1$ and $L,M \geqslant 1$. Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{n+1}$ and, for each index $X \in \{A,B,C,D\}$, set $\mathbf{X} = (X_0,X_1,\ldots,X_n)$ and $\check{\mathbf{X}} = (X_1,X_2,\ldots,X_n)$. If $\check{\mathbf{A}},\check{\mathbf{B}} \in \{0,1\}^n$, $|\mathbf{A}| = M = |\mathbf{C}|$, and $|\mathbf{B}| = L = |\mathbf{D}|$, then*

$$\mathcal{R}_{x,y}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = W_{x/y}(\check{\mathbf{A}},\check{\mathbf{B}},\check{\mathbf{C}},\check{\mathbf{D}} \mid q^{-L/2},q^{-M/2}),$$

where the $W_{x/y}$ weights are given by Definition 5.1.1.

---

[1] Observe that this is slightly different from in Section 2.1, where the spectral parameter was instead given by $\frac{y}{x}$.

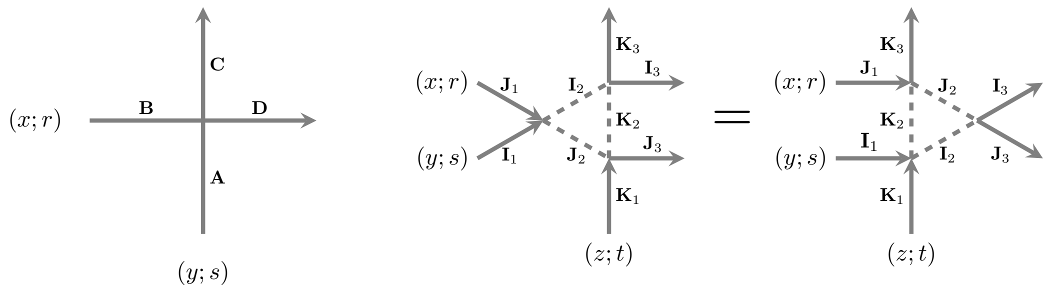


FIGURE 5.1. A fused vertex is shown to the left. The diagrammatic interpretation of the Yang–Baxter equation for the W_z weights is depicted to the right.

PROOF. If either $\mathbf{A}+\mathbf{B} \neq \mathbf{C}+\mathbf{D}$, $\check{\mathbf{C}} \notin \{0,1\}^n$, or $\check{\mathbf{D}} \notin \{0,1\}^n$, then Theorem 4.3.2 implies that $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = 0$, and so Definition 5.1.1 implies $\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = W_{x/y}(\check{\mathbf{A}},\check{\mathbf{B}};\check{\mathbf{C}},\check{\mathbf{D}})$. Hence, let us assume in what follows that $\mathbf{A}+\mathbf{B} = \mathbf{C}+\mathbf{D}$ and that $\check{\mathbf{C}},\check{\mathbf{D}} \in \{0,1\}^n$. Define $\mathbf{V} = (V_0, V_1, \ldots, V_n) \in \{0,1\}^{n+1}$ by setting $V_0 = 0$ and $V_j = \min\{A_j, B_j, C_j, D_j\}$ for each $j \in [1,n]$; we also let $\check{\mathbf{V}} = (V_1, V_2, \ldots, V_n) \in \{0,1\}^n$ and, for each index $X \in \{A,B,C,D,V\}$, we set $|\check{\mathbf{X}}| = x$.

Then, recalling the definition of $\omega_{x,y}$ from (4.1.2) and setting $z = \frac{x}{y}$, we find that

(5.1.6)
$$
\begin{aligned}
&\omega_{x,y}^{(L-v;M)}(\mathbf{A},\mathbf{B}-\mathbf{V},\mathbf{C},\mathbf{D}-\mathbf{V}) \\
&\quad = z^{d-b} q^{a(L-v)-(d-v)M} \sum_{p=0}^{\min\{b-v,c-v\}} \frac{(q^{L-M-v}z;q)_{c-p}(q^{v-L};q)_{d-v}}{(q^{-M}z;q)_{c+d-v-p}} \frac{(q^{v-L}z^{-1};q)_p (z;q)_{b-v-p}}{(q^{v-L};q)_{b-v}} \\
&\qquad\qquad \times q^{(L-v)(p-c)} z^p \sum_{\check{\mathbf{P}}} q^{\varphi(\check{\mathbf{D}}-\check{\mathbf{V}},\check{\mathbf{C}}-\check{\mathbf{P}})+\varphi(\check{\mathbf{B}}-\check{\mathbf{V}}-\check{\mathbf{P}},\check{\mathbf{P}})},
\end{aligned}
$$

where $\check{\mathbf{P}} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ is summed over all n-tuples such that $|\check{\mathbf{P}}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i\} = \min\{B_i - V_i, C_i - V_i\}$ for each $i \in [1,n]$ (and the last equality holds since $V_i \neq 0$ only if $A_i = B_i = C_i = D_i = 1$). Here, we have used the fact that all factors of the form $(q;q)_{B_i - V_i}(q;q)_{P_i}^{-1}(q;q)_{B_i-V_i-P_i}^{-1}$ and $(q;q)_{C_i+D_i-V_i-P_i}(q;q)_{D_i-V_i}^{-1}(q;q)_{C_i-P_i}^{-1}$ are equal to 1. Indeed, these follow from the facts that $B_i - V_i, C_i + D_i - V_i \in \{0,1\}$; the first holds since $A_i, B_i, C_i, D_i \in \{0,1\}$ and the latter holds since $C_i + D_i \geqslant 2$ implies $A_i = B_i = C_i = D_i = V_i = 1$ (as $A_i + B_i = C_i + D_i$ and $A_i, B_i, C_i, D_i \in \{0,1\}$).

Inserting (5.1.6) into Theorem 4.3.2 yields

$$
\begin{aligned}
\mathcal{R}_{x,y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) &= z^{d-b} q^{a(L-v)-(d-v)M} (-1)^v q^{\sum_{h:V_h=1} A_{[h+1,n]} - Mv} \frac{(q^{L-v+1}z;q)_v}{(q^{L-M-v}z;q)_v} \\
&\quad \times \sum_{p=0}^{\min\{b-v,c-v\}} \frac{(q^{L-M-v}z;q)_{c-p}(q^{v-L};q)_{d-v}}{(q^{-M}z;q)_{c+d-v-p}} \frac{(q^{v-L}z^{-1};q)_p (z;q)_{b-v-p}}{(q^{v-L};q)_{b-v}} \\
&\qquad \times q^{(L-v)(p-c)} z^p \sum_{\check{\mathbf{P}}} q^{\varphi(\check{\mathbf{D}}-\check{\mathbf{V}},\check{\mathbf{C}})+\varphi(\check{\mathbf{B}}-\check{\mathbf{D}}-\check{\mathbf{P}},\check{\mathbf{P}})},
\end{aligned}
$$

where we have also used the bilinearity of the function φ from (1.1.1). Further using the facts that $\sum_{h:V_h=1} A_{[h+1,n]} = \varphi(\check{\mathbf{V}}, \check{\mathbf{A}})$ and that $q^L = r^{-2}$ and $q^M = s^{-2}$, we deduce that

$$
\begin{aligned}
\mathcal{R}_{x,y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = z^{d-b} r^{-2a} s^{2d} (-1)^v q^{\varphi(\check{\mathbf{V}}, \check{\mathbf{A}}) - av} \frac{(q^{1-v} r^{-2} z; q)_v}{(q^{-v} s^2 r^{-2} z; q)_v} \\
\times \sum_{p=0}^{\min\{b-v,c\}} \frac{(q^{-v} s^2 r^{-2} z; q)_{c-p} (q^v r^2; q)_{d-v}}{(s^2 z; q)_{c+d-v-p}} \frac{(q^v r^2 z^{-1}; q)_p (z; q)_{b-p-v}}{(q^v r^2; q)_{b-v}} \\
\times r^{2c-2p} q^{v(c-p)} z^p \sum_{\check{\mathbf{P}}} q^{\varphi(\check{\mathbf{D}} - \check{\mathbf{V}}, \check{\mathbf{C}}) + \varphi(\check{\mathbf{B}} - \check{\mathbf{D}} - \check{\mathbf{P}}, \check{\mathbf{P}})},
\end{aligned}
$$

which yields the proposition in view of (5.1.1). $\square$

The following proposition states that these W_z weights satisfy the Yang–Baxter equation; we refer to the right side of Figure 5.1 for a depiction.

Proposition 5.1.4. *Fix an integer $n \geqslant 1$ and $x, y, z, r, s, t \in \mathbb{C}$. For any $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1, \mathbf{I}_3, \mathbf{J}_3, \mathbf{K}_3 \in \{0,1\}^n$, we have that*

(5.1.7)
$$
\begin{aligned}
\sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} W_{x/y}(\mathbf{I}_1, \mathbf{J}_1; \mathbf{I}_2, \mathbf{J}_2 \mid r, s) W_{x/z}(\mathbf{K}_1, \mathbf{J}_2; \mathbf{K}_2, \mathbf{J}_3 \mid r, t) W_{y/z}(\mathbf{K}_2, \mathbf{I}_2; \mathbf{K}_3, \mathbf{I}_3 \mid s, t) \\
= \sum_{\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2} W_{y/z}(\mathbf{K}_1, \mathbf{I}_1; \mathbf{K}_2, \mathbf{I}_2 \mid s, t) W_{x/z}(\mathbf{K}_2, \mathbf{J}_1; \mathbf{K}_3, \mathbf{J}_2 \mid r, t) W_{x/y}(\mathbf{I}_2, \mathbf{J}_2; \mathbf{I}_3, \mathbf{J}_3 \mid r, s),
\end{aligned}
$$

where both sums are over all $\mathbf{I}_2, \mathbf{J}_2, \mathbf{K}_2 \in \{0,1\}^n$.

Proof. Observe by the explicit form (5.1.1) for W that, for fixed $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1, \mathbf{I}_3, \mathbf{J}_3, \mathbf{K}_3 \in \{0,1\}^n$ and nonzero $x, y, z \in \mathbb{C}$, both sides of (5.1.7) are rational functions in r, s, and t. Thus, it suffices to verify (5.1.7) assuming that there exist integers $L, M, N \geqslant |\mathbf{I}|_3 + |\mathbf{J}_3| + |\mathbf{K}_3|$ such that $r = q^{-L/2}$, $s = q^{-M/2}$, and $t = q^{-N/2}$. We may further assume that $\mathbf{I}_1 + \mathbf{J}_1 + \mathbf{K}_1 = \mathbf{I}_3 + \mathbf{J}_3 + \mathbf{K}_3$, for otherwise both sides of (5.1.7) are equal to 0, by arrow conservation.

Define $\mathbf{I}'_1, \mathbf{J}'_1, \mathbf{K}'_1, \mathbf{I}'_3, \mathbf{J}'_3, \mathbf{K}'_3 \in \mathbb{Z}_{\geqslant 0}^{n+1}$ by setting $\mathbf{I}'_h = \big(L - |\mathbf{I}_h|, \mathbf{I}_h\big)$, $\mathbf{J}'_h = \big(M - |\mathbf{J}_h|, \mathbf{J}_h\big)$, and $\mathbf{K}'_h = \big(N - |\mathbf{K}_h|, \mathbf{K}_h\big)$, for each $h \in \{1,2,3\}$. Then, Proposition 3.2.4 implies that

$$
\begin{aligned}
\sum_{\mathbf{I}'_2, \mathbf{J}'_2, \mathbf{K}'_2} \mathcal{R}_{x,y}(\mathbf{I}'_1, \mathbf{J}'_1; \mathbf{I}'_2, \mathbf{J}'_2) \mathcal{R}_{x,z}(\mathbf{K}'_1, \mathbf{J}'_2; \mathbf{K}'_2, \mathbf{J}'_3) \mathcal{R}_{y,z}(\mathbf{K}'_2, \mathbf{I}'_2; \mathbf{K}'_3, \mathbf{I}'_3) \\
= \sum_{\mathbf{I}'_2, \mathbf{J}'_2, \mathbf{K}'_2} \mathcal{R}_{y,z}(\mathbf{K}'_1, \mathbf{I}'_1; \mathbf{K}'_2, \mathbf{I}'_2) \mathcal{R}_{x,z}(\mathbf{K}'_2, \mathbf{J}'_1; \mathbf{K}'_3, \mathbf{J}'_2) \mathcal{R}_{x,y}(\mathbf{I}_2, \mathbf{J}_2; \mathbf{I}_3, \mathbf{J}_3),
\end{aligned}
\tag{5.1.8}
$$

where both sums are over all $\mathbf{I}'_2, \mathbf{J}'_2, \mathbf{K}'_2 \in \mathbb{Z}_{\geqslant 0}^{n+1}$ such that $|\mathbf{I}'_2| = L$, $|\mathbf{J}'_2| = M$, and $|\mathbf{K}'_2| = N$. Since $\mathbf{I}_1, \mathbf{J}_1, \mathbf{K}_1 \in \{0,1\}^n$, the first statement of Theorem 4.3.2 implies that we may instead sum both sides of (5.1.8) over those $\mathbf{I}'_2, \mathbf{J}'_2, \mathbf{K}'_2$ satisfying the above conditions and also that $\check{\mathbf{I}}'_2, \check{\mathbf{J}}'_2, \check{\mathbf{K}}'_2 \in \{0,1\}^n$, where $\check{\mathbf{X}} = (X_1, X_2, \ldots, X_n)$ for any $\mathbf{X} = (X_0, X_1, \ldots, X_n) \in \mathbb{Z}_{\geqslant 0}^{n+1}$. Then, since $\check{\mathbf{I}}'_h = \mathbf{I}_h$, $\check{\mathbf{J}}'_h = \mathbf{J}_h$, and $\check{\mathbf{K}}'_h = \mathbf{K}_h$ for each index $h \in \{1,3\}$, Proposition 5.1.3 and (5.1.8) together imply (5.1.7). $\square$

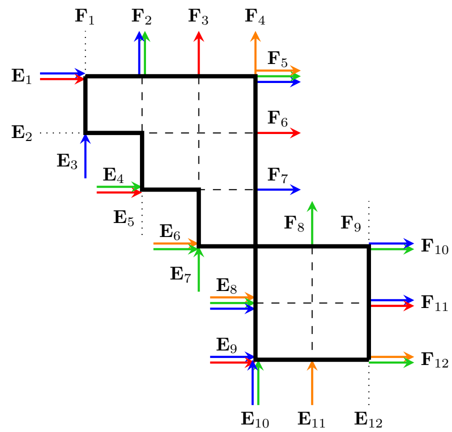


FIGURE 5.2. Shown above is an east-south domain with boundary data.

5.2. Fused Color Merging

In this section we establish a generalization to the fused setting of the color merging procedure, given by Proposition 2.3.1 in the case of fundamental weights. Throughout, we recall from Section 2.2 the notions of east-south paths and domains. We will once again consider vertex models on these domains, but with the difference that they will now be fused, in that paths may share edges.

To describe this in more detail, we first introduce the fused versions of the notation from Section 2.2. To that end, let $\mathcal{D} = \mathcal{D}(\mathbf{p}, \mathbf{p}') \subset \mathbb{Z}^2$ denote an east-south domain, whose boundary paths $\mathbf{p}$ and $\mathbf{p}'$ are of length k. A *(fused) path ensemble* on $\mathcal{D}$ is a consistent assignment of a (fused) arrow configuration[2] $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ to each vertex $v \in \mathcal{D}$; the consistency here means $\mathbf{B}(u) = \mathbf{D}(v)$ if $u - v = (1,0)$ and $\mathbf{A}(u) = \mathbf{C}(v)$ if $u - v = (0,1)$. Arrow conservation $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ implies that any path ensemble may be viewed as a collection of colored paths, which may cross or share edges; the condition $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$ indicates that two paths of the same color cannot share edges (but paths of different colors can).

For any $(k+1)$-tuples $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_{k+1})$ and $\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_{k+1})$ of elements in $\{0,1\}^n$, a path ensemble has boundary data $(\mathcal{E}; \mathcal{F})$ if the following holds for each $i \in [1, k+1]$ and $h \in [1, n]$. An arrow of color h enters through the i-th incoming edge in $\mathcal{D}$ if and only if the h-th coordinate of $\mathbf{E}_i$ is 1, and one exits through its i-th outgoing edge if and only if the h-th coordinate of $\mathbf{F}_i$ is 1. We refer to $\mathcal{E}$ as entrance data on $\mathcal{D}$ and $\mathcal{F}$ as exit data. For example, if red, blue, green, and orange are colors 1, 2, 3, and 4, respectively, then $\mathbf{E}_9 = (1,1,0,0)$ and $\mathbf{F}_5 = (0,1,1,1)$ in Figure 5.2.

The following definition provides notation for the partition functions of a vertex model with given boundary data, under the fused weights W_z from Definition 5.1.1.

Definition 5.2.1. Let $\mathcal{D} \subset \mathbb{Z}^2$ denote an east-south domain. For any boundary data $(\mathcal{E}; \mathcal{F})$ on $\mathcal{D}$ and any sets of complex numbers $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$, $\mathbf{r} = \big(r(v)\big)_{v \in \mathcal{D}}$, and $\mathbf{s} = \big(s(v)\big)_{v \in \mathcal{D}}$, define

$$W_{\mathcal{D}}^{(m;n)}(\mathcal{E}; \mathcal{F} \mid \mathbf{z} \mid \mathbf{r}, \mathbf{s}) = \sum \prod_{v \in \mathcal{D}} W_{z(v)}\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v) \mid r(v), s(v)\big),$$

[2] Throughout the remainder of this text, nearly all arrow configurations and path ensembles will be fused, so we will typically not mention this explicitly in what follows.

where the sum is over all (fused) path ensembles on $\mathcal{D}$ with boundary data $(\mathcal{E};\mathcal{F})$. We further set $W_{\mathcal{D}}^{(m;n)}(\mathcal{E};\mathcal{F} \mid \mathbf{z} \mid \mathbf{r},\mathbf{s}) = 0$ if there exists some $\mathbf{E}_i \in \mathcal{E}$ or $\mathbf{F}_i \in \mathcal{F}$ not in $\{0,1\}^n$.

To state the color merging result, we further require an inversion count, similar to (2.3.1) in the fundamental case. For any k-tuple $\mathcal{I} = (\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_k)$ of elements in $\{0,1\}^n$, with $\mathbf{I}_j = (I_{1,j}, I_{2,j}, \ldots, I_{n,j}) \in \{0,1\}^n$ for each $j \in [1,k]$, and any integer interval $J \subseteq [1,n]$, define

$$\operatorname{inv}(\mathcal{I};J) = \sum_{1\leqslant h<j\leqslant k} \sum_{1\leqslant a<b\leqslant n} \mathbf{1}_{a\in J}\mathbf{1}_{b\in J} I_{b,h} I_{a,j}. \tag{5.2.1}$$

Observe if $J = \{1,2,\ldots,n\}$, then $\operatorname{inv}(\mathcal{I};J) = \sum_{1\leqslant h<j\leqslant k} \varphi(\mathbf{I}_j, \mathbf{I}_h)$, where φ is given by (1.1.1).

Next, recall the notion of interval partitions from Section 2.3, and fix one $\mathbb{J} = (J_1, J_2, \ldots, J_\ell)$ of $\{1,2,\ldots,n\}$. Define the function $\vartheta_{\mathbb{J}} : \mathbb{Z}_{\geqslant 0}^n \to \mathbb{Z}_{\geqslant 0}^\ell$ by setting

$$\vartheta_{\mathbb{J}}(\mathbf{I}) = \left(\sum_{h\in J_1} I_h, \sum_{h\in J_2} I_h, \ldots, \sum_{h\in J_\ell} I_h \right), \qquad \text{for any } \mathbf{I} = (I_1, I_2, \ldots, I_n) \in \{0,1\}^n.$$

Similar to the function $\theta_{\mathbb{J}}$ defined in Section 2.3, $\vartheta_{\mathbb{J}}$ identifies all colors in any J_i and renames them to color i. For any sequence $\mathcal{I} = (\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_k)$ of elements in $\mathbb{Z}_{\geqslant 0}^n$, let $\vartheta(\mathcal{I}) = \big(\vartheta(\mathbf{I}_1), \vartheta(\mathbf{I}_2), \ldots, \vartheta(\mathbf{I}_k)\big)$.

We now have the following theorem providing color merging for the fused $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ model. Observe here that since the number of bosonic colors is $m = 1$, we do not symmetrize over the exit data for the model, as we did in Proposition 2.3.1 for general m.

THEOREM 5.2.2. *Fix integers $n \geqslant n' \geqslant 1$, and $k \geqslant 1$; an east-south domain $\mathcal{D} = \mathcal{D}(\mathbf{p},\mathbf{p}')$ with boundary paths $\mathbf{p}$ and $\mathbf{p}'$ of length k; and sets of complex numbers $\mathbf{z} = \big(z(v)\big)_{v\in\mathcal{D}}$, $\mathbf{r} = \big(r(v)\big)_{v\in\mathcal{D}}$, and $\mathbf{s} = \big(s(v)\big)_{v\in\mathcal{D}}$. Let $\mathbb{J} = (J_1, J_2, \ldots, J_{n'})$ denote an interval partition of $\{1,\ldots,n\}$, and let $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_{k+1})$ and $\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_{k+1})$ denote sequences of elements in $\{0,1\}^n$ constituting entrance and exit data on $\mathcal{D}$, respectively; assume that $\vartheta_{\mathbb{J}}(\mathbf{E}_i) \in \{0,1\}^{n'}$ for each i. Then,*

$$\sum_{\breve{\mathcal{E}}} W_{\mathcal{D}}^{(1;n)}(\breve{\mathcal{E}};\mathcal{F} \mid \mathbf{z} \mid \mathbf{r},\mathbf{s}) \prod_{i=1}^{n'} (-1)^{\operatorname{inv}(\breve{\mathcal{E}};J_i) - \operatorname{inv}(\mathcal{F};J_i)} = W_{\mathcal{D}}^{(1;n')}\big(\vartheta_{\mathbb{J}}(\mathcal{E}); \vartheta_{\mathbb{J}}(\mathcal{F}) \mid \mathbf{z} \mid \mathbf{r},\mathbf{s}\big), \tag{5.2.2}$$

where the sum is over all sequences $\breve{\mathcal{E}} = (\breve{\mathbf{E}}_1, \breve{\mathbf{E}}_2, \ldots, \breve{\mathbf{E}}_{k+1})$ of elements in $\{0,1\}^n$ such that $\vartheta_{\mathbb{J}}(\breve{\mathcal{E}}) = \vartheta_{\mathbb{J}}(\mathcal{E})$.

PROOF. Throughout this proof, we will assume that $|\mathcal{D}| = 1$ since, given this, the proof in the general case follows from an induction very similar to the one implemented in Section 2.4 (alternatively, one can use the same method as applied in the proof of the Yang-Baxter equation in Proposition 3.2.4).

Then letting $\mathcal{D} = \{v\}$, the sets $\mathbf{z}$, $\mathbf{r}$, and $\mathbf{s}$ each consist of one element, abbreviated by $z = z(v)$, $r = r(v)$, and $s = s(v)$, respectively. Hence, recalling $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2)$ and $\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2)$, we have for any entrance data $\breve{\mathcal{E}} = (\breve{\mathbf{E}}_1, \breve{\mathbf{E}}_2)$ on $\mathcal{D}$ that $W_{\mathcal{D}}^{(1;n)}(\breve{\mathcal{E}};\mathcal{F} \mid \mathbf{z} \mid \mathbf{r},\mathbf{s}) = W_z^{(1;n)}(\breve{\mathbf{E}}_2, \breve{\mathbf{E}}_1; \mathbf{F}_1; \mathbf{F}_2 \mid r,s)$ and $W_{\mathcal{D}}^{(1;n')}\big(\vartheta_{\mathbb{J}}(\mathcal{E}); \vartheta_{\mathbb{J}}(\mathcal{F}) \mid \mathbf{z} \mid \mathbf{r},\mathbf{s}\big) = W_z^{(1;n')}\big(\vartheta_{\mathbb{J}}(\mathbf{E}_2), \vartheta_{\mathbb{J}}(\mathbf{E}_1); \vartheta_{\mathbb{J}}(\mathbf{F}_1), \vartheta_{\mathbb{J}}(\mathbf{F}_2) \mid r,s\big)$. The explicit form (5.1.1) for W implies, for any fixed boundary data $(\mathcal{E};\mathcal{F})$ on v, that both sides of (5.2.2) are rational in r and s. Thus it suffices to verify (5.2.2) assuming $r = q^{-L/2}$ and $s = q^{-M/2}$, for some integers $L, M > n$.

Next, we use Proposition 5.1.3 to express both sides of (5.2.2) in terms of the weights $\mathcal{R}_{z,1}$ from Definition 3.2.1. To do this, let $\mathbb{J}' = (J_0, J_1, \ldots, J_{n'})$ denote the interval partition of $\{0,1,\ldots,n\}$

obtained by appending the singleton $J_0 = \{0\}$ to $\mathbb{J}$. As in the proof of Proposition 5.1.4, define $\mathbf{E}_1', \mathbf{E}_2', \breve{\mathbf{E}}_1', \breve{\mathbf{E}}_2', \mathbf{F}_1', \mathbf{F}_2' \in \mathbb{Z}_{\geqslant 0}^{n+1}$, with entries indexed by $[0, n+1]$, by setting $\mathbf{X}' = \big(L - |\mathbf{X}|, \mathbf{X}\big)$ for each index $X \in \{E_1, \breve{E}_1, F_2\}$ and $\mathbf{X}' = \big(M - |\mathbf{X}|, \mathbf{X}\big)$ for each $X \in \{E_2, \breve{E}_2, F_1\}$. Then, Proposition 5.1.3 implies

$$W_z^{(1;n)}(\breve{\mathbf{E}}_2, \breve{\mathbf{E}}_1; \mathbf{F}_1, \mathbf{F}_2 \mid r, s) = \mathcal{R}_{z,1}^{(1;n)}(\breve{\mathbf{E}}_2', \breve{\mathbf{E}}_1'; \mathbf{F}_1', \mathbf{F}_2');$$

$$W_z^{(m';n')}\big(\vartheta_{\mathbb{J}}(\mathbf{E}_2), \vartheta_{\mathbb{J}}(\mathbf{E}_1); \vartheta_{\mathbb{J}}(\mathbf{F}_1), \vartheta_{\mathbb{J}}(\mathbf{F}_2) \mid r, s\big) = \mathcal{R}_{z,1}^{(1;n')}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_2'), \vartheta_{\mathbb{J}'}(\mathbf{E}_1'); \vartheta_{\mathbb{J}'}(\mathbf{F}_1'), \vartheta_{\mathbb{J}}(\mathbf{F}_2')\big).$$

Moreover, recall the rectangular partition function $Z_{x,y}^{(m,n)}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D})$ from Definition 3.1.1, and observe by Definition 3.1.3, Lemma 3.1.4, and Definition 3.2.1 that

$$\begin{aligned}\mathcal{R}_{x,y}^{(1;n)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) \cdot \prod_{i=0}^{m+n-1} \frac{(q;q)_{C_i}(q;q)_{D_i}}{(q;q)_{A_i}(q;q)_{B_i}} \\ = q^{-\operatorname{inv}(\overleftarrow{\mathfrak{D}}) - \operatorname{inv}(\mathfrak{C})} \sum_{\mathfrak{A} \in \mathcal{M}(\mathbf{A})} \sum_{\mathfrak{B} \in \mathcal{M}(\mathbf{B})} q^{\operatorname{inv}(\mathfrak{A}) + \operatorname{inv}(\overleftarrow{\mathfrak{B}})} Z_{x,y}^{(1;n)}(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D}),\end{aligned} \tag{5.2.3}$$

for any fixed complex numbers $x, y \in \mathbb{C}$; integer sequences $\mathbf{A}, \mathbf{B} \in \{0,1\}^{n+1}$ and $\mathbf{C}, \mathbf{D} \in \mathbb{Z}_{\geqslant 0}^{n+1}$, with coordinates indexed by $[0, n]$; $\mathfrak{C} \in \mathcal{M}(\mathbf{C})$; and $\mathfrak{D} \in \mathcal{M}(\mathbf{D})$. Hence, to establish (5.2.2) we must show that[3]

$$\begin{aligned}\sum_{\breve{\mathcal{E}}} \sum_{\breve{\mathfrak{E}}_1'} \sum_{\breve{\mathfrak{E}}_2'} q^{\operatorname{inv}(\breve{\mathfrak{E}}_2') + \operatorname{inv}(\overleftarrow{\breve{\mathfrak{E}}_1'})} Z_{z,1}^{(1;n)}(\breve{\mathfrak{E}}_2', \breve{\mathfrak{E}}_1'; \mathfrak{F}_1', \mathfrak{F}_2') \prod_{i=1}^{n'} (-1)^{\operatorname{inv}(\breve{\mathcal{E}}; J_i) - \operatorname{inv}(\mathcal{F}; J_i)} \\ = \sum_{\mathfrak{E}_1''} \sum_{\mathfrak{E}_2''} q^{\operatorname{inv}(\mathfrak{E}_2'') + \operatorname{inv}(\overleftarrow{\mathfrak{E}_1''})} Z_{z,1}^{(1;n')}\big(\mathfrak{E}_2'', \mathfrak{E}_1''; \mathfrak{F}_1'', \mathfrak{F}_2''\big),\end{aligned} \tag{5.2.4}$$

where we sum $\breve{\mathcal{E}}$ as in (5.2.2); $\breve{\mathfrak{E}}_1'$ over $\mathcal{M}(\breve{\mathbf{E}}_1')$; $\breve{\mathfrak{E}}_2'$ over $\mathcal{M}(\breve{\mathbf{E}}_2')$; $\mathfrak{E}_1''$ over $\mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_1')\big)$; and $\mathfrak{E}_2''$ over $\mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_2')\big)$. Here, $\mathfrak{F}_1' \in \mathcal{M}(\mathbf{F}_1')$, $\mathfrak{F}_2' \in \mathcal{M}(\mathbf{F}_2')$, $\mathfrak{F}_1'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{F}_1')\big)$, and $\mathfrak{F}_2'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{F}_2')\big)$ are fixed by the condition $\operatorname{inv}(\mathfrak{X}) = 0$ for each $\mathfrak{X} \in \big\{\mathfrak{F}_1', \overleftarrow{\mathfrak{F}}_2', \mathfrak{F}_1'', \overleftarrow{\mathfrak{F}}_2''\big\}$. Recalling the function $\theta_{\mathbb{J}'}$ from Section 2.3, this ensures that $\theta_{\mathbb{J}'}(\mathfrak{F}_i') = \mathfrak{F}_i''$ for each $i \in \{1, 2\}$, since $\mathfrak{X}' \in \mathcal{M}(\mathbf{X}')$ implies $\theta_{\mathbb{J}'}(\mathfrak{X}') \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{X}')\big)$ for any sequence $\mathfrak{X}'$ of indices in $[0, n]$ and any $\mathbf{X} \in \mathbb{Z}_{\geqslant 0}^{n+1}$.

To establish (5.2.4), we will apply Proposition 2.3.1, using the fact that the $Z_{z,1}$ on both sides of (5.2.4) are partition functions (as in Definition 2.2.1) for a fundamental $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ six-vertex model on the rectangular domain $\mathcal{D}' = [1, M] \times [1, L] \subset \mathbb{Z}^2$. To implement this, let us fix $\mathfrak{E}_1'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_1')\big)$ and $\mathfrak{E}_2'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_2')\big)$. Then the $m = 1$ case of Proposition 2.3.1 yields

$$\sum_{\breve{\mathfrak{E}}'} Z_{z,1}^{(1;n)}(\breve{\mathfrak{E}}_2', \breve{\mathfrak{E}}_1; \mathfrak{F}_1', \mathfrak{F}_2') \prod_{i=1}^{n'} (-1)^{\operatorname{inv}(\breve{\mathfrak{E}}'; J_i) - \operatorname{inv}(\mathfrak{F}'; J_i)} = Z_{z,1}^{(1;n')}\big(\mathfrak{E}_2'', \mathfrak{E}_1''; \mathfrak{F}_1'', \mathfrak{F}_2''\big), \tag{5.2.5}$$

where $\breve{\mathfrak{E}}' = (\breve{\mathfrak{E}}_2', \breve{\mathfrak{E}}_1')$ is summed over all sequences of indices in $[0, n]$ such that $\theta_{\mathbb{J}'}(\breve{\mathfrak{E}}_i') = \mathfrak{E}_i''$ for each $i \in \{1, 2\}$, and we have set $\mathfrak{F}' = (\mathfrak{F}_1', \mathfrak{F}_2')$ (which satisfies $\theta_{\mathbb{J}'}(\mathfrak{F}') = \mathfrak{F}''$, as mentioned above).

[3]Fix an index $\mathbf{X} \in \{\mathbf{E}_1', \mathbf{E}_2', \mathbf{F}_1', \mathbf{F}_2'\}$; let $\breve{\mathbf{X}} = (\breve{X}_0, \breve{X}_1, \ldots, \breve{X}_n)$; and let $\mathbf{Y} = (Y_0, Y_1, \ldots, Y_{n'}) = \vartheta_{\mathbb{J}'}(\mathbf{X})$. Then, it is quickly verified since $\vartheta_{\mathbb{J}'}\big(\breve{\mathbf{X}}\big) = \vartheta_{\mathbb{J}'}(\mathbf{X}) = \mathbf{Y} \in \{0,1\}^{n'}$ that $\prod_{i=0}^{n} (q;q)_{\breve{X}_i} = \prod_{i=0}^{n'} (q;q)_{Y_i}$. In passing from (5.2.3) to (5.2.4), factors of this type were canceled on both sides.

We next match the powers of -1 and q in (5.2.4) and (5.2.5), to which end we claim for any $i \in [1, n']$ that

(5.2.6)

$$\operatorname{inv}(\breve{\mathfrak{E}}_2') = \operatorname{inv}(\mathfrak{E}_2''); \quad \operatorname{inv}\big(\overleftarrow{\breve{\mathfrak{E}}_1'}\big) = \operatorname{inv}\big(\overleftarrow{\mathfrak{E}_1''}\big); \quad \operatorname{inv}(\breve{\mathcal{E}}; J_i) = \operatorname{inv}(\breve{\mathfrak{E}}'; J_i); \quad \operatorname{inv}(\mathcal{F}; J_i) = \operatorname{inv}(\mathfrak{F}'; J_i).$$

Given (5.2.6), (5.2.4) follows by multiplying (5.2.5) by $q^{\operatorname{inv}(\breve{\mathfrak{E}}_2')+\operatorname{inv}(\overleftarrow{\breve{\mathfrak{E}}_1'})} = q^{\operatorname{inv}(\mathfrak{E}_2'')+\operatorname{inv}(\overleftarrow{\mathfrak{E}_1''})}$ and summing both sides over each $\mathfrak{E}_i'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_i')\big)$. Indeed, the left side of (5.2.4) is summed over $\breve{\mathcal{E}} = (\breve{\mathbf{E}}_2, \breve{\mathbf{E}}_1)$ and $\breve{\mathfrak{E}}_i'$ with each $\vartheta_{\mathbb{J}}(\breve{\mathbf{E}}_i) = \vartheta_{\mathbb{J}}(\mathbf{E}_i)$ (equivalently, $\vartheta_{\mathbb{J}'}(\breve{\mathbf{E}}_i') = \vartheta_{\mathbb{J}'}(\mathbf{E}_i')$) and $\breve{\mathfrak{E}}_i' \in \mathcal{M}(\breve{\mathbf{E}}_i')$. Since $\mathfrak{X}' \in \mathcal{M}(\mathbf{X}')$ implies $\theta_{\mathbb{J}'}(\mathfrak{X}') \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{X}')\big)$ for any $\mathfrak{X}'$ and $\mathbf{X}'$, this is equivalent to first summing over each $\breve{\mathfrak{E}}_i'$ with $\theta_{\mathbb{J}'}(\breve{\mathfrak{E}}_i') = \mathfrak{E}_i''$ for fixed $\mathfrak{E}_i'' \in \mathcal{M}\big(\vartheta_{\mathbb{J}'}(\mathbf{E}_i')\big)$ and then over each such $\mathfrak{E}_i''$.

Now let us establish (5.2.6). To that end, observe since each $\breve{\mathfrak{E}}_2' \in \mathcal{M}(\breve{\mathbf{E}}_2')$ and $\vartheta(\breve{\mathbf{E}}_2) = \vartheta(\mathbf{E}_2) \in \{0,1\}^{n'}$ that $|\breve{\mathfrak{E}}_2' \cap J_i| \leqslant 1$ for each $i \in [1, n']$. Thus, applying $\theta_{\mathbb{J}'}$ to $\breve{\mathfrak{E}}_2'$ does not change the relative ordering of its nonzero elements, which yields $\operatorname{inv}(\breve{\mathfrak{E}}_2') = \operatorname{inv}\big(\theta_{\mathbb{J}'}(\breve{\mathfrak{E}}_2')\big) = \operatorname{inv}(\mathfrak{E}_2'')$. This verifies the first statement of (5.2.6); the proof of the second is entirely analogous and therefore omitted.

To confirm the third, observe for any $i \in [1, n']$ that

$$\begin{aligned}\operatorname{inv}(\breve{\mathfrak{E}}'; J_i) &= \operatorname{inv}(\breve{\mathfrak{E}}_1'; J_i) + \operatorname{inv}(\breve{\mathfrak{E}}_2'; J_i) + \sum_{a \in \breve{\mathfrak{E}}_1'} \sum_{b \in \breve{\mathfrak{E}}_2'} \mathbf{1}_{a>b} \mathbf{1}_{a \in J_i} \mathbf{1}_{b \in J_i} \\ &= \sum_{a \in \breve{\mathfrak{E}}_1'} \sum_{b \in \breve{\mathfrak{E}}_2'} \mathbf{1}_{a>b} \mathbf{1}_{a \in J_i} \mathbf{1}_{b \in J_i} = \sum_{a \in \breve{\mathfrak{E}}_1} \sum_{b \in \breve{\mathfrak{E}}_2} \mathbf{1}_{a>b} \mathbf{1}_{a \in J_i} \mathbf{1}_{b \in J_i} = \operatorname{inv}(\breve{\mathcal{E}}; J_i).\end{aligned}$$

Here, to deduce first equality, we used (2.3.1); for the second, we used the fact that each $\operatorname{inv}(\breve{\mathfrak{E}}_h'; J_i) = 0$ (again since $|\breve{\mathfrak{E}}_h' \cap J_i| \leqslant 1$); for the third, we used the fact that each $\breve{\mathfrak{E}}_h' \cap J_i = \breve{\mathfrak{E}}_h \cap J_i$; and for the fourth we used (5.2.1) and the fact that each $\breve{\mathfrak{E}}_h \in \mathcal{M}(\breve{\mathbf{E}}_h)$. This confirms the third equality in (5.2.6); the proof of the fourth is omitted since it is very similar. This establishes (5.2.6), thus implying (5.2.4) and hence the theorem. □

CHAPTER 6

Transfer Operators

In this chapter we use the W_z (and $\widehat{W}_z$) weights from Definition 5.1.1 (and (5.1.4)) to define the actions of certain operators on a vector space; this will be used later in Chapter 7 to define and analyze properties of certain families of (symmetric) functions.

6.1. Single-Row and Double-Row Partition Functions

In this section we provide properties for partition functions of vertex models under the W_z or $\widehat{W}_z$ weights (from Definition 5.1.1 and (5.1.4), respectively) on domains consisting of one or two rows. This will later be useful in Section 6.2 to define certain operator actions on vector spaces.

To that end, we begin with the following definition for the single-row partition functions.

Definition 6.1.1. Fix an integer $M \geqslant 1$; complex numbers $x, r \in \mathbb{C}$; and finite sequences $\mathbf{y} = (y_1, y_2, \ldots, y_M)$ and $\mathbf{s} = (s_1, s_2, \ldots, s_M)$ of complex numbers. For any elements $\mathbf{B} \in \{0,1\}^n$ and $\mathbf{D} \in \{0,1\}^n$, and sequences $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M) \subset \{0,1\}^n$ and $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_M) \in \{0,1\}^n$, define the *single-row partition functions*

$$
\begin{aligned}
W_{x;\mathbf{y}}\big(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}\big) &= \sum_{\mathcal{J}} \prod_{i=1}^{M} W_{x/y_i}(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s); \\
\widehat{W}_{x;\mathbf{y}}\big(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}\big) &= \sum_{\mathcal{J}} \prod_{i=1}^{M} \widehat{W}_{x/y_i}(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s),
\end{aligned}
\tag{6.1.1}
$$

where both sums are over all sequences $\mathcal{J} = (\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_{M+1}) \subset \{0,1\}^n$ such that $\mathbf{J}_1 = \mathbf{B}$ and $\mathbf{J}_{M+1} = \mathbf{D}$; by arrow conservation, both of these sums are supported on at most one term.

These quantities can also be defined when $M = \infty$, in which case we set

$$
\begin{aligned}
W_{x;\mathbf{y}}\big(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}\big) &= \sum_{\mathcal{J}} \prod_{i=1}^{\infty} W_{x/y_i}(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s_i); \\
\widehat{W}_{x;\mathbf{y}}\big(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}\big) &= \sum_{\mathcal{J}} \prod_{i=1}^{\infty} \widehat{W}_{x/y_i}(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s_i),
\end{aligned}
\tag{6.1.2}
$$

where, in both equations $\mathcal{J} = (\mathbf{J}_1, \mathbf{J}_2, \ldots)$ is summed over all infinite sequences in $\{0,1\}^n$ such that $\mathbf{J}_1 = \mathbf{B}$ and $\mathbf{J}_i = \mathbf{D}$ for sufficiently large i; as above, arrow conservation implies that both sums on the right sides of (6.1.2) are supported on at most one term. In this $M = \infty$ case, we must further assume that parameters x, r, $\mathbf{y}$, and $\mathbf{s}$ are chosen so the infinite products on the right sides of (6.1.2) converge.

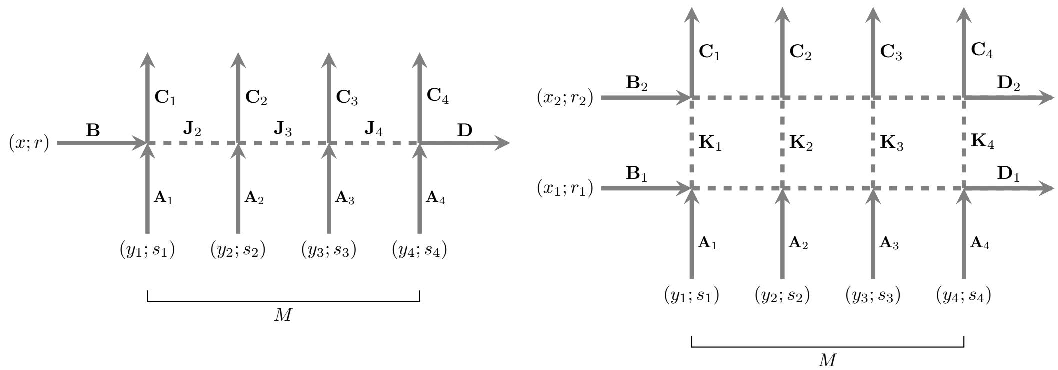


FIGURE 6.1. Shown to the left is a diagrammatic interpretation for (6.1.1); shown to the right is one for (6.1.3).

Observe that $W_{x;\mathbf{y}}$ and $\widehat{W}_{x;\mathbf{y}}$ are partition functions under the weights W_z and $\widehat{W}_z$, respectively, for the vertex model with M vertices in a row depicted on the left side of Figure 6.1. More explicitly, the boundary data for this vertex model is given as follows. The vertical entrance data (meaning the colors of the incoming vertical arrows) from west to east along this row is given by $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M)$; the horizontal entrance data is given by $\mathbf{B}$; the vertical exit data by $(\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_M)$; and the horizontal exit data by $\mathbf{D}$.

It will also be useful to define partition functions for two rows, one on top the other. This is given by the following definition.

Definition 6.1.2. Let M, $\mathbf{y}$, $\mathbf{s}$, $\mathcal{A}$, and $\mathcal{C}$ be as in Definition 6.1.1 (with M possibly infinite). Further let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{r} = (r_1, r_2)$ denote pairs of complex numbers. Then, for any sequences $\mathcal{B} = (\mathbf{B}_1, \mathbf{B}_2) \subset \{0,1\}^n$ and $\mathcal{D} = (\mathbf{D}_1, \mathbf{D}_2) \subset \{0,1\}^n$, define the *double-row partition functions*[1]

$$\begin{aligned} W_{\mathbf{x};\mathbf{y}}(\mathcal{A}, \mathcal{B}; \mathcal{C}, \mathcal{D} \mid \mathbf{r}, \mathbf{s}) &= \sum_{\mathcal{K}} W_{x_1;\mathbf{y}}(\mathcal{A}, \mathbf{B}_1; \mathcal{K}; \mathbf{D}_1 \mid r_1, \mathbf{s}) W_{x_2,\mathbf{y}}(\mathcal{K}, \mathbf{B}_2; \mathcal{C}, \mathbf{D}_2 \mid r_2, \mathbf{s}); \\ \widehat{W}_{\mathbf{x};\mathbf{y}}(\mathcal{A}, \mathcal{B}; \mathcal{C}, \mathcal{D} \mid \mathbf{r}, \mathbf{s}) &= \sum_{\mathcal{K}} \widehat{W}_{x_1;\mathbf{y}}(\mathcal{A}, \mathbf{B}_1; \mathcal{K}; \mathbf{D}_1 \mid r_1, \mathbf{s}) \widehat{W}_{x_2,\mathbf{y}}(\mathcal{K}, \mathbf{B}_2; \mathcal{C}, \mathbf{D}_2 \mid r_2, \mathbf{s}). \end{aligned} \tag{6.1.3}$$

Here $\mathcal{K} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_M)$ is summed over all sequences of elements in $\{0,1\}^n$, which we assume to satisfy $\mathbf{K}_i = \mathbf{e}_0$ for sufficiently large i if $M = \infty$, and we recall the single-row partition functions $W_{x_i;\mathbf{y}}$ from (6.1.1).

Observe that $W_{\mathbf{x};\mathbf{y}}$ and $\widehat{W}_{\mathbf{x};\mathbf{y}}$ are partition functions under the weights W_z and $\widehat{W}_z$, respectively, for the vertex model with $2M$ vertices, arranged in two rows with M vertices each, as depicted on the right side of Figure 6.1. More precisely, the vertical entrance data for this model is given by $(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M)$; the horizontal entrance data (from south to north) is given by $(\mathbf{B}_1, \mathbf{B}_2)$; the vertical exit data is given by $(\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_M)$; and the horizontal exit data is given by $(\mathbf{D}_1, \mathbf{D}_2)$.

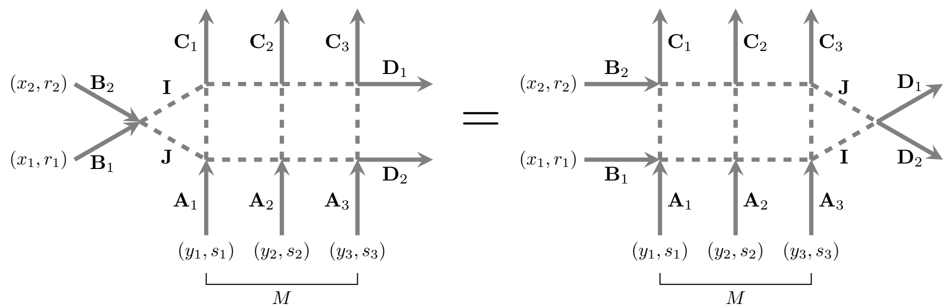
The following lemma is a quick consequence of the Yang–Baxter equation.

[1] Observe that the orderings $\mathcal{B} = (\mathbf{B}_1, \mathbf{B}_2)$ and $\mathcal{D} = (\mathbf{D}_1, \mathbf{D}_2)$ are the reverse of what they were in Section 5.2 (see Figure 5.2). The reason for this discrepancy is that our rapidity parameters (which were not relevant in Section 5.2) will be indexed from bottom to top, so we would like to make the ordering of $\mathcal{B}$ and $\mathcal{D}$ consistent with this.

Lemma 6.1.3. *Fix a finite integer $M \geqslant 1$ and sequences of complex numbers $\mathbf{x} = (x_1, x_2)$; $\mathbf{r} = (r_1, r_2)$; $\mathbf{y} = (y_1, y_2, \ldots, y_M)$; and $\mathbf{s} = (s_1, s_2, \ldots, s_M)$. For any $\{0,1\}^n$-sequences $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M)$; $\mathcal{B} = (\mathbf{B}_1, \mathbf{B}_2)$; $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_M)$; and $\mathcal{D} = (\mathbf{D}_1, \mathbf{D}_2)$, we have*

$$\begin{aligned} &\sum_{\mathbf{I},\mathbf{J}\in\{0,1\}^n} W_{x_2/x_1}(\mathbf{B}_1, \mathbf{B}_2; \mathbf{I}, \mathbf{J} \mid r_2, r_1) W_{(x_2,x_1);\mathbf{y}}\big(\mathcal{A}, (\mathbf{J}, \mathbf{I}); \mathcal{C}, (\mathbf{D}_2, \mathbf{D}_1) \mid (r_2, r_1), \mathbf{s}\big) \\ &\quad = \sum_{\mathbf{I},\mathbf{J}\in\{0,1\}^n} W_{(x_1,x_2);\mathbf{y}}\big(\mathcal{A}, (\mathbf{B}_1, \mathbf{B}_2); \mathcal{C}, (\mathbf{I}, \mathbf{J}) \mid (r_1, r_2), \mathbf{s}\big) W_{x_2/x_1}(\mathbf{I}, \mathbf{J}; \mathbf{D}_1, \mathbf{D}_2 \mid r_2, r_1). \end{aligned} \tag{6.1.4}$$

Diagrammatically,



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Proof. This follows from M applications of the Yang–Baxter equation Proposition 5.1.4. $\square$

6.2. Transfer Operators and Commutation Relations

The functions we will consider in this text are expressible in terms of actions of certain operators on vector spaces. In this section we define these operators and provide some relations they satisfy. Throughout this section, we fix an integer $n \geqslant 1$ and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$.

Let $\mathbb{V} = \mathbb{V}_{\mathbf{y};\mathbf{s}}$ denote the infinte-dimensional vector space spanned by basis vectors of the form $|\mathcal{A}\rangle$, where $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots)$ ranges over all *finitary* sequences of elements in $\{0,1\}^n$, namely, those that satisfy $\mathbf{A}_j = \mathbf{e}_0 = (0, 0, \ldots, 0)$ for all but finitely many j. Similarly, let $\mathbb{V}^* = \mathbb{V}^*_{\mathbf{y};\mathbf{s}}$ denote the space spanned by finitary dual vectors, namely, those of the form $\langle \mathcal{C}|$ over all finitary sequences of elements $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots)$ in $\{0,1\}^n$. We impose an inner product on $\mathbb{V}^* \times \mathbb{V}$ by first setting $\langle \mathcal{C}|\mathcal{A}\rangle = \mathbf{1}_{\mathcal{A}=\mathcal{C}}$, for any finitary $\mathcal{A}$ and $\mathcal{C}$, and then extending to all of $\mathbb{V}^* \times \mathbb{V}$ by bilinearity.

Now, for any complex numbers $x, r \in \mathbb{C}$ and sequences $\mathbf{B}, \mathbf{D} \in \{0,1\}^n$ we define *transfer operators* $\mathbb{T}_{\mathbf{B};\mathbf{D}} = \mathbb{T}_{\mathbf{B};\mathbf{D}}(x; r) : \mathbb{V} \to \mathbb{V}$ and $\widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}} = \widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}}(x; r) : \mathbb{V} \to \mathbb{V}$ by first setting, for any finitary sequence $\mathcal{A}$,

$$\mathbb{T}_{\mathbf{B};\mathbf{D}}|\mathcal{A}\rangle = \sum_{\mathcal{C}} W_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s})|\mathcal{C}\rangle; \qquad \widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}}|\mathcal{A}\rangle = \sum_{\mathcal{C}} \widehat{W}_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s})|\mathcal{C}\rangle, \tag{6.2.1}$$

where both sums are over all finitary sequences $\mathcal{C}$, and then extending their actions to all of $\mathbb{V}$ by linearity. Here, we recall the $M = \infty$ single-row partition functions $W_{x;\mathbf{y}}$ and $\widehat{W}_{x;\mathbf{y}}$ from (6.1.2) and assume the parameters $x, r, \mathbf{y}, \mathbf{s}$ are chosen such that the infinite products there converge.

By (6.2.1), we find for any finitary sequences $\mathcal{A}$ and $\mathcal{C}$ that

(6.2.2)
$$W_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}) = \big\langle \mathcal{C} \big| \mathbb{T}_{\mathbf{B};\mathbf{D}}(x; r) \big| \mathcal{A} \big\rangle; \qquad \widehat{W}_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}) = \big\langle \mathcal{C} \big| \widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}}(x; r) \big| \mathcal{A} \big\rangle,$$

and so $\mathbb{T}_{\mathbf{B};\mathbf{D}}$ and $\widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}}$ admit dual actions on $\mathbb{V}^*$ given by

$$\langle \mathcal{C} | \mathbb{T}_{\mathbf{B};\mathbf{D}} = \sum_{\mathcal{A}} W_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}) \langle \mathcal{A} |; \qquad \langle \mathcal{C} | \widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}} = \sum_{\mathcal{A}} \widehat{W}_{x;\mathbf{y}}(\mathcal{A}, \mathbf{B}; \mathcal{C}, \mathbf{D} \mid r, \mathbf{s}) \langle \mathcal{A} |. \tag{6.2.3}$$

Now, Lemma 6.1.3 will imply certain commutation relations between the operators $\mathbb{T}_{\mathbf{B};\mathbf{D}}$ and $\widehat{\mathbb{T}}_{\mathbf{B};\mathbf{D}}$ that will be useful for us. To explain them, for any $x, r \in \mathbb{C}$, we define the operators

$$\mathbb{B}(x;r) = \widehat{\mathbb{T}}_{\mathbf{e}_0;\mathbf{e}_{[1,n]}}(x;r); \qquad \mathbb{C}(x;r) = \mathbb{T}_{\mathbf{e}_{[1,n]};\mathbf{e}_0}; \qquad \mathbb{D}(x;r) = \mathbb{T}_{\mathbf{e}_0;\mathbf{e}_0}(x;r), \tag{6.2.4}$$

where we recall that $\mathbf{e}_0 = (0, 0, \ldots, 0)$ and $\mathbf{e}_{[1,n]} = (1, 1, \ldots, 1)$.

Observe that these operators are well-defined for any choices of parameters $(x; r; \mathbf{y}; \mathbf{s})$. Indeed, if $\mathbf{D} = \mathbf{e}_0$ then, since $\mathcal{A}$ and $\mathcal{C}$ are finitary, all but finitely many terms in the product appearing on the left side of the first equation of (6.1.2) are of the form $W_z(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r, s) = 1$; so the product defining the W weight there converges, implying that $\mathbb{C}$ and $\mathbb{D}$ are well-defined. A similar statement holds for the $\widehat{W}$ weights when $\mathbf{D} = \mathbf{e}_{[1,n]}$, implying that $\mathbb{B}$ is also well-defined.

The following two results, which are both consequences of (limits of) Lemma 6.1.3, provide commutation relations for these $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{D}$ operators.

Lemma 6.2.1. *Fix $x_1, x_2, r_1, r_2 \in \mathbb{C}$ and $\mathbf{y} = (y_1, y_2, \ldots) \subset \mathbb{C}$ and $\mathbf{s} = (s_1, s_2, \ldots) \subset \mathbb{C}$. Then, as operators on $\mathbb{V}$, we have that*

$$\begin{aligned}
&\mathbb{B}(x_1;r_1)\mathbb{B}(x_2;r_2) = \left(\frac{r_1^2 x_2}{r_2^2 x_1} \right)^n \frac{(r_2^2 x_1 x_2^{-1}; q)_n}{(r_1^2 x_1^{-1} x_2; q)_n} \cdot \mathbb{B}(x_2;r_2)\mathbb{B}(x_1;r_1); \\
&\mathbb{C}(x_1;r_1)\mathbb{C}(x_2;r_2) = \left(\frac{r_2^2 x_1}{r_1^2 x_2} \right)^n \frac{(r_1^2 x_1^{-1} x_2; q)_n}{(r_2^2 x_1 x_2^{-1}; q)_n} \cdot \mathbb{C}(x_2;r_2)\mathbb{C}(x_1;r_1); \\
&\mathbb{D}(x_1;r_1)\mathbb{D}(x_2;r_2) = \mathbb{D}(x_2;r_2)\mathbb{D}(x_1;r_1).
\end{aligned} \tag{6.2.5}$$

Proof. First observe for any $\mathbf{B}_1, \mathbf{B}_2, \mathbf{D}_1, \mathbf{D}_2 \in \{0,1\}^n$ and finitary $\mathcal{A}$ and $\mathcal{C}$ that

$$\begin{aligned}
\big\langle \mathcal{C} \big| \mathbb{T}_{\mathbf{B}_2;\mathbf{D}_2}(x_2;r_2) \mathbb{T}_{\mathbf{B}_1;\mathbf{D}_1}(x_1;r_1) \big| \mathcal{A} \big\rangle &= \sum_{\mathcal{K}} \big\langle \mathcal{C} \big| \mathbb{T}_{\mathbf{B}_2;\mathbf{D}_2}(x_2;r_2) \big| \mathcal{K} \big\rangle \big\langle \mathcal{K} \big| \mathbb{T}_{\mathbf{B}_1;\mathbf{D}_1}(x_1;r_1) \big| \mathcal{A} \big\rangle \\
&= \sum_{\mathcal{K}} W_{x_2;\mathbf{y}}(\mathcal{K}, \mathbf{B}_2; \mathcal{C}, \mathbf{D}_2 \mid r_2, \mathbf{s}) W_{x_1;\mathbf{y}}(\mathcal{A}, \mathbf{B}_1; \mathcal{K}, \mathbf{D}_1 \mid r_1, \mathbf{s}) \\
&= W_{(x_1,x_2);\mathbf{y}}\big(\mathcal{A}, (\mathbf{B}_1, \mathbf{B}_2); \mathcal{C}, (\mathbf{D}_1, \mathbf{D}_2) \mid (r_1, r_2), \mathbf{s}\big),
\end{aligned} \tag{6.2.6}$$

where in the second statement we applied (6.2.2) and in the last we applied (6.1.3). Similarly, we have that

$$\big\langle \mathcal{C} \big| \widehat{\mathbb{T}}_{\mathbf{B}_2;\mathbf{D}_2}(x_2;r_2) \widehat{\mathbb{T}}_{\mathbf{B}_1;\mathbf{D}_1}(x_1;r_1) \big| \mathcal{A} \big\rangle = \widehat{W}_{(x_1,x_2);\mathbf{y}}\big(\mathcal{A}, (\mathbf{B}_1, \mathbf{B}_2); \mathcal{C}, (\mathbf{D}_1, \mathbf{D}_2) \mid (r_1, r_2), \mathbf{s}\big). \tag{6.2.7}$$

Given these facts, (6.2.5) will follow from specializations of (6.1.4). In particular, let us apply Lemma 6.1.3 with the $(\mathbf{B}_1, \mathbf{B}_2)$ and $(\mathbf{D}_1, \mathbf{D}_2)$ there both equal to $(\mathbf{e}_0, \mathbf{e}_0)$. By arrow conservation, the sums on both sides of (6.1.4) are supported on the $(\mathbf{I}, \mathbf{J}) = (\mathbf{e}_0, \mathbf{e}_0)$ term. Since $W_z(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r, s) = 1$ by (5.1.2), this yields for any integer $M \geqslant 1$ that

$$\begin{aligned}
&W_{(x_2,x_1);\mathbf{y}_{[1,M]}}\big(\mathcal{A}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0) \mid (r_2, r_1), \mathbf{s}_{[1,M]}\big) \\
&\qquad = W_{(x_1,x_2);\mathbf{y}_{[1,M]}}\big(\mathcal{A}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0) \mid (r_1, r_2), \mathbf{s}_{[1,M]}\big),
\end{aligned} \tag{6.2.8}$$

for any finitary $\mathcal{A}$ and $\mathcal{C}$, where we have set $\mathcal{X}_{[1,M]} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M)$ for any sequence $\mathcal{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots)$. Taking the limit as M tends to ∞ on both sides of (6.2.8) yields

$$W_{(x_2,x_1);\mathbf{y}}\big(\mathcal{A}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}, (\mathbf{e}_0, \mathbf{e}_0) \mid (r_2, r_1), \mathbf{s}\big) = W_{(x_1,x_2);\mathbf{y}}\big(\mathcal{A}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}, (\mathbf{e}_0, \mathbf{e}_0) \mid (r_1, r_2), \mathbf{s}\big),$$

which by (6.2.6) yields the third statement of (6.2.5).

To establish the first statement of (6.2.5), we apply the $(\mathbf{B}_1, \mathbf{B}_2) = (\mathbf{e}_0, \mathbf{e}_0)$ and $(\mathbf{D}_1, \mathbf{D}_2) = \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big)$ case of Lemma 6.1.3; multiply both sides of (6.1.4) by

$$\prod_{j=1}^{M} \frac{(s_j^2 x_1 y_j^{-1}; q)_n}{s_j^{2n} (x_1 y_j^{-1}; q)_n} \frac{(s_j^2 x_2 y_j^{-1}; q)_n}{s_j^{2n} (x_2 y_j^{-1}; q)_n};$$

and use (5.1.4) to deduce that

$$\begin{aligned} &\sum_{\mathbf{I},\mathbf{J} \in \{0,1\}^n} \widehat{W}_{(x_2,x_1);\mathbf{y}_{[1,M]}}\Big(\mathcal{A}, (\mathbf{J}, \mathbf{I}); \mathcal{C}, \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big) \mid (r_2, r_1), \mathbf{s}_{[1,M]}\Big) W_{x_2/x_1}(\mathbf{e}_0, \mathbf{e}_0; \mathbf{I}, \mathbf{J} \mid r_2, r_1) \\ &\quad = \sum_{\mathbf{I},\mathbf{J} \in \{0,1\}^n} \widehat{W}_{(x_1,x_2);\mathbf{y}_{[1,M]}}\big(\mathcal{A}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}, (\mathbf{I}, \mathbf{J}) \mid (r_1, r_2), \mathbf{s}_{[1,M]}\big) W_{x_2/x_1}(\mathbf{I}, \mathbf{J}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid r_2, r_1). \end{aligned} \tag{6.2.9}$$

By arrow conservation, the left side of (6.2.9) is supported on the term $(\mathbf{I}, \mathbf{J}) = (\mathbf{e}_0, \mathbf{e}_0)$ and the right side is supported on the term $(\mathbf{I}, \mathbf{J}) = \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big)$. Using the expressions from (5.1.2) for the weights $W_z(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r, s)$ and $W_z\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid r, s\big)$, it follows from (6.2.9) that

$$\begin{aligned} &\widehat{W}_{(x_2,x_1);\mathbf{y}_{[1,M]}}\Big(\mathcal{A}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}, \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big) \mid (r_2, r_1), \mathbf{s}_{[1,M]}\Big) \\ &\quad = \widehat{W}_{(x_1,x_2);\mathbf{y}_{[1,M]}}\Big(\mathcal{A}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}, \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big) \mid (r_1, r_2), \mathbf{s}_{[1,M]}\Big) \bigg(\frac{r_1^2 x_2}{r_2^2 x_1}\bigg)^n \frac{(r_2^2 x_1 x_2^{-1}; q)_n}{(r_1^2 x_1^{-1} x_2; q)_n}. \end{aligned} \tag{6.2.10}$$

Now the first statement of (6.2.5) follows from taking the limit of (6.2.10) as M tends to ∞ and applying (6.2.7). The proof of the second statement in (6.2.5) is entirely analogous (obtained from the $(\mathbf{B}_1, \mathbf{B}_2) = \big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big)$ and $(\mathbf{D}_1, \mathbf{D}_2) = (\mathbf{e}_0, \mathbf{e}_0)$ case of Lemma 6.1.3) and is therefore omitted. ∎

Proposition 6.2.2. *Fix complex numbers $u, w, r, t \in \mathbb{C}$ and sequences $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$ of complex numbers. If there exists an integer $K > 0$ such that*

$$\sup_{j > K} \max_{\substack{a,b \in [0,n] \\ (a,b) \neq (n,0)}} \left| s_j^{2a+2b-2n} \frac{(s_j^2 u y_j^{-1}; q)_n (u y_j^{-1}; q)_a}{(u y_j^{-1}; q)_n (s_j^2 u y_j^{-1}; q)_a} \frac{(w y_j^{-1}; q)_b}{(s_j^2 w y_j^{-1}; q)_b} \right| < 1, \tag{6.2.11}$$

then, as operators on $\mathbb{V}$, we have

$$\mathsf{B}(u; r) \mathsf{D}(w; t) = \frac{(t^2 u w^{-1}; q)_n}{t^{2n} (u w^{-1}; q)_n} \cdot \mathsf{D}(w; t) \mathsf{B}(u; r). \tag{6.2.12}$$

Proof. For any integer $M \geqslant 1$ (including $M = \infty$); pairs $\mathbf{x} = (x_1, x_2)$ and $\mathbf{r} = (r_1, r_2)$ of complex numbers; and (finitary, if $M = \infty$) sequences $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_M)$ and $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_M)$

of elements in $\{0,1\}^n$, denote

(6.2.13)
$$\begin{aligned}\widetilde{W}_{\mathbf{x};\mathbf{y}}\big(\mathcal{A},(\mathbf{B}_1,\mathbf{B}_2);\mathcal{C},(\mathbf{D}_1,\mathbf{D}_2)\mid\mathbf{r},\mathbf{s}\big)&=\sum_{\mathcal{K}}\widehat{W}_{x_1;\mathbf{y}}(\mathcal{A},\mathbf{B}_1;\mathcal{K},\mathbf{D}_1\mid r_1,\mathbf{s})W_{x_2,\mathbf{y}}(\mathcal{K},\mathbf{B}_2;\mathcal{C},\mathbf{D}_2\mid r_2,\mathbf{s});\\ \widetilde{W}'_{\mathbf{x};\mathbf{y}}\big(\mathcal{A},(\mathbf{B}_1,\mathbf{B}_2);\mathcal{C},(\mathbf{D}_1,\mathbf{D}_2)\mid\mathbf{r},\mathbf{s}\big)&=\sum_{\mathcal{K}}W_{x_1;\mathbf{y}}(\mathcal{A},\mathbf{B}_1;\mathcal{K},\mathbf{D}_1\mid r_1,\mathbf{s})\widehat{W}_{x_2,\mathbf{y}}(\mathcal{K},\mathbf{B}_2;\mathcal{C},\mathbf{D}_2\mid r_2,\mathbf{s}),\end{aligned}$$

where $\mathcal{K}=(\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_M)$ is summed over all (finitary, if $M=\infty$) sequences of elements in $\{0,1\}^n$. Then, analogous reasoning as applied in (6.2.6) yields for any finitary $\mathcal{A}$ and $\mathcal{C}$ that

$$\begin{aligned}\big\langle\mathcal{C}\big|\mathbb{T}_{\mathbf{B}_2;\mathbf{D}_2}(x_2;r_2)\widehat{\mathbb{T}}_{\mathbf{B}_1;\mathbf{D}_1}(x_1;r_1)\big|\mathcal{A}\big\rangle&=\widetilde{W}_{\mathbf{x};\mathbf{y}}\big(\mathcal{A},(\mathbf{B}_1,\mathbf{B}_2);\mathcal{C},(\mathbf{D}_1,\mathbf{D}_2)\mid\mathbf{r},\mathbf{s}\big);\\ \big\langle\mathcal{C}\big|\widehat{\mathbb{T}}_{\mathbf{B}_2;\mathbf{D}_2}(x_2;r_2)\mathbb{T}_{\mathbf{B}_1;\mathbf{D}_1}(x_1;r_1)\big|\mathcal{A}\big\rangle&=\widetilde{W}'_{\mathbf{x};\mathbf{y}}\big(\mathcal{A},(\mathbf{B}_1,\mathbf{B}_2);\mathcal{C},(\mathbf{D}_1,\mathbf{D}_2)\mid\mathbf{r},\mathbf{s}\big).\end{aligned}$$

Therefore, to establish (6.2.12) it suffices to show under (6.2.11) that for any finitary $\mathcal{A}$ and $\mathcal{C}$ we have

(6.2.14)
$$\begin{aligned}&\widetilde{W}'_{(w,u);\mathbf{y}}\Big(\mathcal{A},(\mathbf{e}_0,\mathbf{e}_0);\mathcal{C},\big(\mathbf{e}_0,\mathbf{e}_{[1,n]}\big)\Big|(t,r),\mathbf{s}\Big)\\ &\qquad=\frac{(t^2uw^{-1};q)_n}{t^{2n}(uw^{-1};q)_n}\widetilde{W}_{(u,w);\mathbf{y}}\Big(\mathcal{A},(\mathbf{e}_0,\mathbf{e}_0);\mathcal{C},\big(\mathbf{e}_{[1,n]},\mathbf{e}_0\big)\Big|(r,t);\mathbf{s}\Big).\end{aligned}$$

To that end, we apply the $(\mathbf{B}_1,\mathbf{B}_2)=(\mathbf{e}_0,\mathbf{e}_0)$ and $(\mathbf{D}_1,\mathbf{D}_2)=\big(\mathbf{e}_0,\mathbf{e}_{[1,n]}\big)$ case of Lemma 6.1.3; multiply both sides of (6.1.4) by

$$\prod_{j=1}^{M}\frac{(s_j^2uy_j^{-1};q)_n}{s_j^{2n}(uy_j^{-1};q)_n};$$

use (5.1.4); and recall by arrow conservation that $W_{u/w}(\mathbf{e}_0,\mathbf{e}_0;\mathbf{I},\mathbf{J})\mid t,r)=0$ unless $\mathbf{I}=\mathbf{e}_0=\mathbf{J}$, in which case it is equal to 1 by (5.1.2), to deduce for any integer $M\geqslant 1$ that

(6.2.15)
$$\begin{aligned}&\widetilde{W}_{(u,w);\mathbf{y}_{[1,M]}}\Big(\mathcal{A}_{[1,M]},(\mathbf{e}_0,\mathbf{e}_0);\mathcal{C}_{[1,M]},\big(\mathbf{e}_{[1,n]},\mathbf{e}_0\big)\Big|(r,t),\mathbf{s}_{[1,M]}\Big)\\ &\quad=\sum_{\mathbf{I},\mathbf{J}\in\{0,1\}^n}\widetilde{W}'_{(w,u);\mathbf{y}_{[1,M]}}\big(\mathcal{A}_{[1,M]},(\mathbf{e}_0,\mathbf{e}_0);\mathcal{C}_{[1,M]},(\mathbf{I},\mathbf{J})\mid(t,r),\mathbf{s}_{[1,M]}\big)W_{u/w}(\mathbf{I},\mathbf{J};\mathbf{e}_0,\mathbf{e}_{[1,n]}\mid r,t),\end{aligned}$$

where we have set $\mathcal{X}_{[1,M]}=(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_M)$ for any sequence $\mathcal{X}=(\mathbf{X}_1,\mathbf{X}_2,\ldots)$.

We will show under (6.2.11) that the right side of (6.2.15) is asymptotically (in the limit as M tends to ∞) supported on the term $(\mathbf{I},\mathbf{J})=\big(\mathbf{e}_0,\mathbf{e}_{[1,n]}\big)$, in the sense that

(6.2.16)
$$\lim_{M\to\infty}\widetilde{W}'_{(w,u);\mathbf{y}_{[1,M]}}\big(\mathcal{A}_{[1,M]},(\mathbf{e}_0,\mathbf{e}_0);\mathcal{C}_{[1,M]},(\mathbf{I},\mathbf{J})\mid(t,r),\mathbf{s}_{[1,M]}\big)=0,\quad\text{unless }(\mathbf{I},\mathbf{J})=\big(\mathbf{e}_0,\mathbf{e}_{[1,n]}\big).$$

Given (6.2.16), (6.2.14) (and therefore the lemma) follows from first inserting (6.2.16) into limit of (6.2.15) as M tends to ∞, and then using the explicit form (given by the second statement of (5.1.2)) for $W_{u/w}\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]}\mid r,t\big)$. Thus, it remains to establish (6.2.16).

To that end, first observe from (5.1.3) and (5.1.4) that, for any integer $i\geqslant 1$ and sequence $\mathbf{B}\in\{0,1\}^n$ with $|\mathbf{B}|=b$,

$$\widehat{W}_{u/y_j}\big(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0,\mathbf{B}\mid r,s_j\big)=s_j^{2b-2n}\frac{(s_1^2uy_j^{-1};q)_n(uy_j^{-1};q)_b}{(uy_j^{-1};q)_n(s_j^2uy_j^{-1};q)_b};$$

$$W_{w/y_j}(\mathbf{e}_0, \mathbf{B}; \mathbf{e}_0, \mathbf{B} \mid t, s_j) = s_j^{2b} \frac{(wy_j^{-1}; q)_b}{(s_j^2 wy_j^{-1}; q)_b}.$$

So, (6.2.11) implies the existence of some real number $\varepsilon > 0$ such that, for sufficiently large K,

$$\sup_{j>K} \max_{\substack{\mathbf{I},\mathbf{J} \in \{0,1\}^n \\ (\mathbf{I},\mathbf{J}) \neq (\mathbf{e}_0, \mathbf{e}_{[1,n]})}} \left| \widehat{W}_{u/y_j}(\mathbf{e}_0, \mathbf{J}; \mathbf{e}_0, \mathbf{J} \mid r, s_j) W_{w/y_j}(\mathbf{e}_0, \mathbf{I}; \mathbf{e}_0, \mathbf{I} \mid t, s_j) \right| < 1 - \varepsilon. \tag{6.2.17}$$

We may also assume that K is sufficiently large so that $\mathbf{A}_j = \mathbf{e}_0 = \mathbf{C}_j$ for $j > K$ and so that $\sum_{j=1}^{\infty} |\mathbf{A}_j| < K$.

Inserting the definition (6.1.1) of the single-row partition functions W and $\widehat{W}$ into the second equation of (6.2.13) yields

$$\begin{aligned} &\left| \widetilde{W}'_{(w,u);\mathbf{y}_{[1,M]}} \big(\mathcal{A}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}_{[1,M]}, (\mathbf{I}, \mathbf{J}) \mid (t, r); \mathbf{s}_{[1,M]} \big) \right| \\ &\qquad \leqslant \sum_{\mathcal{K}} \sum_{\mathcal{J}} \sum_{\mathcal{I}} \prod_{m=1}^{M} \left| W_{w/y_m}(\mathbf{A}_m, \mathbf{I}_m; \mathbf{K}_m, \mathbf{I}_{m+1} \mid t, s_m) \widehat{W}_{u/y_m}(\mathbf{K}_m, \mathbf{J}_m; \mathbf{C}_m, \mathbf{J}_{m+1} \mid r, s_m) \right|, \end{aligned} \tag{6.2.18}$$

where $\mathcal{I} = (\mathbf{I}_1, {}_2, \ldots, \mathbf{I}_{M+1})$, $\mathcal{J} = (\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_{M+1})$, and $\mathcal{K} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_M)$ are each summed over all sequences of elements in $\{0,1\}^n$ such that $\mathbf{I}_{M+1} = \mathbf{I}$, $\mathbf{J}_{M+1} = \mathbf{J}$, and $\mathbf{I}_1 = \mathbf{e}_0 = \mathbf{J}_1$. Since $\mathbf{A}_m = \mathbf{e}_0 = \mathbf{C}_m$ for $m > K$, by arrow conservation any nonzero summand on the right side of (6.2.18) must satisfy $|\mathbf{I}_{m+1}| \leqslant |\mathbf{I}_m|$ and $|\mathbf{J}_{m+1}| \geqslant |\mathbf{J}_m|$. Thus, if $(\mathbf{I}, \mathbf{J}) \neq \big(\mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$, then (6.2.17) implies that

$$\prod_{m=K+1}^{M} \left| W_{w/y_m}(\mathbf{A}_m, \mathbf{I}_m; \mathbf{K}_m, \mathbf{I}_{m+1} \mid t, s_m) \widehat{W}_{u/y_m}(\mathbf{K}_m, \mathbf{J}_m; \mathbf{C}_m, \mathbf{J}_{m+1} \mid r, s_m) \right| < (1-\varepsilon)^{M-K}, \tag{6.2.19}$$

for any $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ supported by the sum on the right side of (6.2.18).

Furthermore, arrow conservation implies that $\sum_{j=1}^{M} |\mathbf{K}_j| \leqslant \sum_{j=1}^{M} |\mathbf{A}_j| < nK$ for any $\mathcal{K}$ supported on the right side of (6.2.18), from which it follows that there are at most $2^n \binom{M}{nK} \leqslant (2M)^{nK}$ choices of $\mathcal{K}$ for which the corresponding summand is nonzero. Given any fixed such $\mathcal{K}$, arrow conservation implies that there is at most one choice of $\mathcal{I}$ and $\mathcal{J}$ on which the right side of (6.2.18) is supported. Inserting this and (6.2.19) into (6.2.18), and further denoting

$$\Xi = \max_{m \in [1,K]} \max \left| W_{w/y_m}(\mathbf{I}_1, \mathbf{J}_1; \mathbf{I}_2, \mathbf{J}_2 \mid t, s_m) \widehat{W}_{u/y_m}(\mathbf{I}'_1, \mathbf{J}'_1; \mathbf{I}'_2, \mathbf{J}'_2 \mid r, s_m) \right|,$$

where the second maximum is taken over $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}'_1, \mathbf{I}'_2, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}'_1, \mathbf{J}'_2 \in \{0,1\}^n$ yields if $(\mathbf{I}, \mathbf{J}) \neq \big(\mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ that

$$\left| \widetilde{W}'_{(w,u);\mathbf{y}_{[1,M]}} \big(\mathcal{A}_{[1,M]}, (\mathbf{e}_0, \mathbf{e}_0); \mathcal{C}_{[1,M]}, (\mathbf{I}, \mathbf{J}) \mid (t, r); \mathbf{s}_{[1,M]} \big) \right| \leqslant \Xi^K (2M)^{nK} (1-\varepsilon)^{M-K}.$$

This gives (6.2.16) by letting M tend to ∞, which as mentioned above implies the lemma. □

Commutation relations between the operators $(\mathbb{B}, \mathbb{C})$ and $(\mathbb{C}, \mathbb{D})$ can also be derived as a result of Lemma 6.1.3, but we will not state them here since we will not require them in full generality.

6.3. Composition of $\mathbb{D}$ Operators

In this section we establish the following result for composing the operators $\mathbb{D}(x;r)$ from (6.2.4), which will be useful for analyzing principal specializations of symmetric functions in Section 7.2 below.

Proposition 6.3.1. *Fix complex numbers $x, r_1, r_2 \in \mathbb{C}$ and infinite sequences $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$ of complex numbers. As operators on $\mathbb{V}$, we have*

$$\mathbb{D}(r_1^{-2}x; r_2)\mathbb{D}(x; r_1) = \mathbb{D}(x; r_1 r_2). \tag{6.3.1}$$

Composition results similar to Proposition 6.3.1 also hold for the operators $\mathbb{B}$ and $\mathbb{C}$ from (6.2.4), but we will not pursue this here.

We will deduce Proposition 6.3.1 essentially by interpreting both sides of (6.3.1) as special cases of the rectangular partition functions given by Definition 3.1.1, and then applying the first statement of the branching-type result (3.1.2) for the latter quantities. To that end, we require the following partition function, which is for the vertex model consisting of horizontally adjacent rectangles whose rapidity parameters are in a geometric progression (as in Definition 3.1.1).

Definition 6.3.2. Fix integers $K, L \geqslant 1$; a set of positive integers $\mathbf{M} = (M_1, M_2, \ldots, M_K)$; a complex number $x \in \mathbb{C}$; and a set of complex numbers $\mathbf{y}_{[1,K]} = (y_1, y_2, \ldots, y_K)$. For each $i \in [1, K]$, define the sequence $\mathbf{w}^{(i)} = (q^{M_i - 1} y_i, q^{M_i - 2} y_i, \ldots, y_i)$, and further let

$$\mathbf{x} = (x, qx, \ldots, q^{L-1}x); \qquad \mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)} \cup \cdots \cup \mathbf{w}^{(K)},$$

where the union is ordered, so $\mathbf{y} = (q^{M_1-1}y_1, q^{M_1-2}y_1, \ldots, y_1, \ldots, q^{M_K-1}y_K, q^{M_K-2}y_K, \ldots, y_K)$.

Additionally, let $\mathfrak{B} = (b_1, b_2, \ldots, b_L)$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_L)$ denote sequences of indices in $[0, n]$. Also let $\mathcal{A} = \big(\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \ldots, \mathfrak{A}^{(K)}\big)$ and $\mathcal{C} = \big(\mathfrak{C}^{(1)}, \mathfrak{C}^{(2)}, \ldots, \mathfrak{C}^{(K)}\big)$, where for each $i \in [1, K]$ the $\mathfrak{A}^{(i)} = \big(a_1^{(i)}, a_2^{(i)}, \ldots, a_{M_i}^{(i)}\big)$ and $\mathfrak{C}^{(i)} = \big(c_1^{(i)}, c_2^{(i)}, \ldots, c_{M_i}^{(i)}\big)$ are sequences of indices in $[0, n]$. Moreover define (where the unions below are again ordered)

$$\mathfrak{A} = \mathfrak{A}^{(1)} \cup \mathfrak{A}^{(2)} \cup \cdots \mathfrak{A}^{(K)}; \qquad \mathfrak{C} = \mathfrak{C}^{(1)} \cup \mathfrak{C}^{(2)} \cup \cdots \mathfrak{C}^{(K)}.$$

Then, recalling the notation of Definition 3.1.1, denote the partition function

$$Z_{L;\mathbf{M}}\big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]}\big) = Z(\mathfrak{A}, \mathfrak{B}; \mathfrak{C}, \mathfrak{D} \mid \mathbf{x}, \mathbf{w}).$$

Observe under that $Z_{L;\mathbf{M}}\big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x, \mathbf{y}_{[1,K]}\big)$ is the partition function for the vertex model obtained by horizontally juxtaposing the K rectangles corresponding to $Z_{x,y_i}\big(\mathfrak{A}^{(i)}, \mathfrak{B}; \mathfrak{C}^{(i)}, \mathfrak{D}\big)$ from Definition 3.1.1 (see the left side of Figure 3.1). We refer to Figure 6.2 for a depiction.

We now have the following lemma that expresses $\big\langle \mathcal{C} \big| \mathbb{D}(x;r) \big| \mathcal{A} \big\rangle$ as linear combinations of the partition functions from Definition 6.3.2, if $r = q^{-L/2}$ and $s_i = q^{-M_i/2}$ for each $i \in [1, K]$ and some integers $L, M_1, M_2, \ldots, M_K \geqslant 0$.

Lemma 6.3.3. *Let $K, L, \mathbf{M}, x, \mathbf{y}_{[1,K]}, \mathbf{x}, \mathbf{w}$ be as in Definition 6.3.2, and define the sequences of indices $\mathfrak{B} = (0, 0, \ldots, 0) = \mathfrak{D}$ (where 0 appears with multiplicity L). Let $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots)$ and $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots)$ denote finitary sequences of elements in $\{0,1\}^n$; assume that $\mathbf{A}_i = \mathbf{e}_0 = \mathbf{C}_i$ for each $i > K$, and that $|\mathbf{A}_i|, |\mathbf{C}_i| \leqslant M_i$ for each $i \in [1, K]$. Set $\mathbf{A}_i' = \big(M_i - |\mathbf{A}_i|, \mathbf{A}_i\big)$ and $\mathbf{C}_i' = \big(M_i - |\mathbf{C}_i|, \mathbf{C}_i\big)$ for each $i \in [1, K]$; fix sequences of indices $\mathfrak{C}^{(i)} \in \mathcal{M}\big(\mathbf{C}_i'\big)$; and let $\mathcal{C} =$*

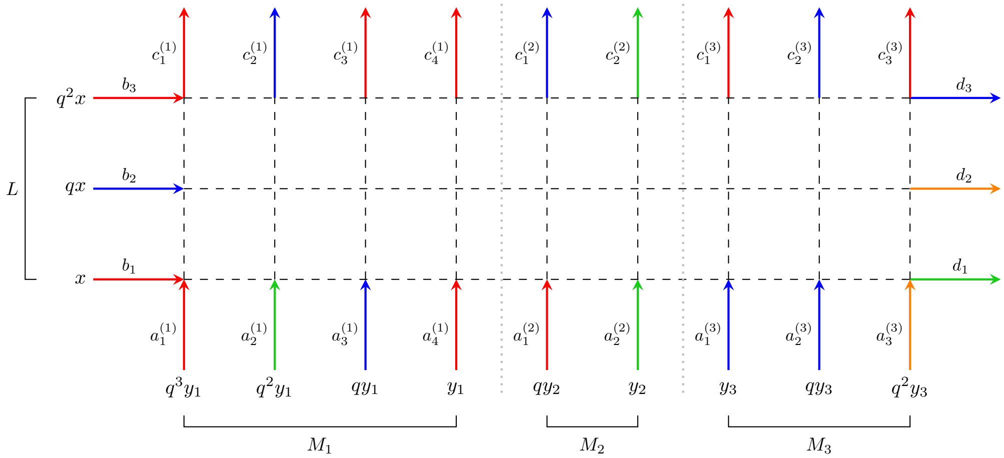


FIGURE 6.2. Shown above is a diagrammatic interpretation for $Z_{L;\mathbf{M}}(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]})$.

$\big(\mathfrak{C}^{(1)}, \mathfrak{C}^{(2)}, \ldots, \mathfrak{C}^{(K)}\big)$. *If $r = q^{-L/2}$ and $s_i = q^{-M_i/2}$ for each $i \in [1, K]$, then we have*

$$\big\langle \mathcal{A} \big| \mathbb{D}(x; r) \big| \mathcal{C} \big\rangle = \sum_{\mathfrak{A}^{(i)} \in \mathcal{M}(\mathbf{A}_i')} Z_{L;\mathbf{M}}\big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]}\big) \prod_{i=1}^{K} \Bigg(q^{\mathrm{inv}(\mathfrak{A}^{(i)}) - \mathrm{inv}(\mathfrak{C}^{(i)})} \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{A}^{(i)})}}{(q;q)_{m_j(\mathfrak{C}^{(i)})}} \Bigg),$$

where $\mathcal{A} = \big(\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \ldots, \mathfrak{A}^{(K)}\big)$, and each $\mathfrak{A}^{(i)}$ is summed over $\mathcal{M}\big(\mathbf{A}_i'\big)$.

PROOF. By (6.2.2) and (6.1.2), we have that

(6.3.2)
$$\big\langle \mathcal{A} \big| \mathbb{D}(x; r) \big| \mathcal{C} \big\rangle = \sum_{\mathcal{J}} \prod_{i=1}^{\infty} W_{x/y_i}\big(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s_i\big) = \sum_{\mathcal{J}} \prod_{i=1}^{K} W_{x/y_i}\big(\mathbf{A}_i, \mathbf{J}_i; \mathbf{C}_i, \mathbf{J}_{i+1} \mid r, s_i\big),$$

where in the second term $\mathcal{J} = (\mathbf{J}_0, \mathbf{J}_1, \ldots)$ is summed over all infinite sequences of elements in $\{0,1\}^n$ such that $\mathbf{J}_0 = \mathbf{e}_0 = \mathbf{J}_i$ for sufficiently large i, and in the third $\mathcal{J} = (\mathbf{J}_0, \mathbf{J}_1, \ldots \mathbf{J}_{K+1})$ is summed over all K-term sequences of elements in $\{0,1\}^n$ such that $\mathbf{J}_0 = \mathbf{e}_0 = \mathbf{J}_{K+1}$. Here, the second equality follows from the fact that $W_{x/y_i}(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r, s_i) = 1$ and $\mathbf{A}_i = \mathbf{C}_i = \mathbf{J}_i = \mathbf{e}_0$ for $i > K$ in the second term of (6.3.2) (where the latter holds by arrow conservation, since $\mathbf{A}_i = \mathbf{e}_0 = \mathbf{C}_i$ for $i > K$ and $\mathbf{J}_i = \mathbf{e}_0$ for sufficiently large i). By arrow conservation, the right side of (6.3.2) is supported on a single sequence, which we also call $\mathcal{J} = (\mathbf{J}_0, \mathbf{J}_1, \ldots, \mathbf{J}_{K+1})$.

Then, setting $\mathbf{J}_i' = \big(L - |\mathbf{J}_i|, \mathbf{J}_i\big) \in \mathbb{Z}_{\geqslant 0}^{n+1}$ and applying Proposition 5.1.3 gives

$$\big\langle \mathcal{A} \big| \mathbb{D}(x; r) \big| \mathcal{C} \big\rangle = \prod_{i=1}^{K} \mathcal{R}_{x,y_i}^{(1;n)}\big(\mathbf{A}_i', \mathbf{J}_i'; \mathbf{C}_i', \mathbf{J}_{i+1}' \mid r, s_i\big). \tag{6.3.3}$$

By Definition 3.1.3, Lemma 3.1.4, and Definition 3.2.1 (see also (3.2.1)), for any $\mathfrak{J}^{(i+1)} \in \mathcal{M}(\mathbf{J}'_{i+1})$ we have that

$$\mathcal{R}_{x,y_i}(\mathbf{A}'_i, \mathbf{J}'_i; \mathbf{C}'_i, \mathbf{J}'_{i+1} \mid r, s_i) = \sum_{\mathfrak{A}^{(i)} \in \mathcal{M}(\mathbf{A}'_i)} \sum_{\mathfrak{J}^{(i)} \in \mathcal{M}(\mathbf{J}'_i)} q^{\operatorname{inv}(\mathfrak{A}^{(i)}) - \operatorname{inv}(\mathfrak{C}^{(i)}) + \operatorname{inv}(\overleftarrow{\mathfrak{J}}^{(i)}) - \operatorname{inv}(\overleftarrow{\mathfrak{J}}^{(i+1)})}$$
$$\times Z_{x,y_i}\big(\mathfrak{A}^{(i)}, \mathfrak{J}^{(i)}; \mathfrak{C}^{(i)}, \mathfrak{J}^{(i+1)}\big) \cdot \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{A}^{(i)})}}{(q;q)_{m_j(\mathfrak{C}^{(i)})}}.$$

Inserting this into (6.3.3) and using the fact that $\operatorname{inv}\big(\mathfrak{J}^{(0)}\big) = 0 = \operatorname{inv}\big(\mathfrak{J}^{(K+1)}\big)$ (as $\mathbf{J}_0 = \mathbf{e}_0 = \mathbf{J}_{K+1}$), we deduce

$$\begin{aligned}\big\langle \mathcal{A} \big| \mathbb{D}(x;r) \big| \mathcal{C} \big\rangle = \sum_{\mathfrak{A}^{(i)} \in \mathcal{M}(\mathbf{A}'_i)} \sum_{\mathfrak{J}^{(i)} \in \mathcal{M}(\mathbf{J}'_i)} & Z_{x,y_i}\big(\mathfrak{A}^{(i)}, \mathfrak{J}^{(i)}; \mathfrak{C}^{(i)}, \mathfrak{J}^{(i+1)}\big) \\ & \times \prod_{i=1}^{K} \left(q^{\operatorname{inv}(\mathfrak{A}^{(i)}) - \operatorname{inv}(\mathfrak{C}^{(i)})} \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{A}^{(i)})}}{(q;q)_{m_j(\mathfrak{C}^{(i)})}} \right).\end{aligned} \tag{6.3.4}$$

Now, applying the second statement of (3.1.2) $K-1$ times, for the h there in $\big\{M_1, M_1 + M_2, \ldots, \sum_{i=1}^{K-1} M_i\big\}$ (see Figure 6.2) gives

$$\sum_{\mathfrak{J}^{(i)} \in \mathcal{M}(\mathbf{J}'_i)} \prod_{i=1}^{K} Z_{x,y_i}\big(\mathfrak{A}^{(i)}, \mathfrak{J}^{(i)}; \mathfrak{C}^{(i)}, \mathfrak{J}^{(i+1)} \mid r, s_i\big) = Z_{L;\mathbf{M}}\big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]}\big),$$

which upon insertion into (6.3.4) yields the lemma. □

Now we can establish Proposition 6.3.1.

Proof of Proposition 6.3.1. For any finitary $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots)$ and $\mathcal{C} = (\mathbf{C}_1, \mathbf{C}_2, \ldots)$, it suffices to show

$$\big\langle \mathcal{C} \big| \mathbb{D}(x; r_1 r_2) \big| \mathcal{A} \big\rangle = \big\langle \mathcal{C} \big| \mathbb{D}(r_1^{-2} x; r_2) \mathbb{D}(x; r_1) \big| \mathcal{A} \big\rangle.$$

Thus, since

$$\big\langle \mathcal{C} \big| \mathbb{D}(r_1^{-2} x; r_2) \mathbb{D}(x; r_1) \big| \mathcal{A} \big\rangle = \sum_{\mathcal{H}} \big\langle \mathcal{C} \big| \mathbb{D}(r_1^{-2} x; r_2) \big| \mathcal{H} \big\rangle \big\langle \mathcal{H} \big| \mathbb{D}(x; r_1) \big| \mathcal{A} \big\rangle,$$

where we sum over all finitary $\mathcal{H} = (\mathbf{H}_1, \mathbf{H}_2, \ldots)$, it suffices to show

$$\big\langle \mathcal{C} \big| \mathbb{D}(x; r_1 r_2) \big| \mathcal{A} \big\rangle = \sum_{\mathcal{H}} \big\langle \mathcal{C} \big| \mathbb{D}(r_1^{-2} x; r_2) \big| \mathcal{H} \big\rangle \big\langle \mathcal{H} \big| \mathbb{D}(x; r_1) \big| \mathcal{A} \big\rangle, \tag{6.3.5}$$

Now let $K > 0$ be such that $\mathbf{A}_j = \mathbf{e}_0 = \mathbf{C}_j$ for $j > K$. It then follows by arrow conservation that any finitary sequence $\mathcal{H}$ supported by the sum on the right side of (6.3.5) must also satisfy $\mathbf{H}_j = \mathbf{e}_0$ for $j > K$. Thus, from (6.2.4), (6.2.2), and Definition 6.1.1, together with the explicit form Definition 5.1.1 of the W weights, both sides of (6.3.5) are rational functions in the variables $r_1, r_2, s_1, s_2, \ldots, s_K$. So, to establish (6.3.5), we may assume in what follows there exist integers $L_1, L_2, M_1, M_2, \ldots, M_K \geqslant 1$ such that $r_j = q^{-L_j/2}$ for each $j \in \{1, 2\}$; such that $s_j = q^{-M_j/2}$ for each $j \in \{1, 2, \ldots, K\}$; and such that $M_j \geqslant |\mathbf{A}_j|, |\mathbf{C}_j|$ holds for each $j \in \{1, 2, \ldots, K\}$.

In this scenario, Lemma 6.3.3 applies. In particular, fix some finitary sequence $\mathcal{H} = (\mathbf{H}_1, \mathbf{H}_2, \ldots)$ of elements in $\{0,1\}^n$ as above; set $\mathbf{X}'_j = \big(M_j - |\mathbf{X}_j|, \mathbf{X}_j \big)$ for each index $X \in \{A, C, H\}$; and, letting $0^j = (0, 0, \ldots, 0) \in \mathbb{Z}^j$ for any integer $j \geqslant 1$, set

$$\mathfrak{B}^{(1)} = 0^{L_1} = \mathfrak{D}^{(1)}; \qquad \mathfrak{B}^{(2)} = 0^{L_2} = \mathfrak{D}^{(2)}; \qquad \mathfrak{B} = 0^{L_1+L_2} = \mathfrak{D}.$$

Then, since $r_1 = q^{-L_1/2}$ and $r_2 = q^{-L_2/2}$, Lemma 6.3.3 gives

$$\begin{aligned}
\big\langle \mathcal{C} \big| \mathbb{D}(x; r_1 r_2) \big| \mathcal{A} \big\rangle &= \sum_{\mathfrak{A}^{(i)} \in \mathcal{M}(\mathbf{A}'_i)} Z_{L_1+L_2;\mathbf{M}} \big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]} \big) \\
&\qquad \times \prod_{i=1}^{K} \left(q^{\operatorname{inv}(\mathfrak{A}^{(i)}) - \operatorname{inv}(\mathfrak{C}^{(i)})} \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{A}^{(i)})}}{(q;q)_{m_j(\mathfrak{C}^{(i)})}} \right); \\
\big\langle \mathcal{C} \big| \mathbb{D}(r_1^{-2} x; r_2) \big| \mathcal{H} \big\rangle &= \sum_{\mathfrak{H}^{(i)} \in \mathcal{M}(\mathbf{H}'_i)} Z_{L_2;\mathbf{M}} \big(\mathcal{H}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid q^{L_1} x; \mathbf{y}_{[1,K]} \big) \\
&\qquad \times \prod_{i=1}^{K} \left(q^{\operatorname{inv}(\mathfrak{H}^{(i)}) - \operatorname{inv}(\mathfrak{C}^{(i)})} \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{H}^{(i)})}}{(q;q)_{m_j(\mathfrak{C}^{(i)})}} \right); \\
\big\langle \mathcal{H} \big| \mathbb{D}(x; r_1) \big| \mathcal{A} \big\rangle &= \sum_{\mathfrak{A}^{(i)} \in \mathcal{M}(\mathbf{A}'_i)} Z_{L_1;\mathbf{M}} \big(\mathcal{A}, \mathfrak{B}; \mathcal{H}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]} \big) \\
&\qquad \times \prod_{i=1}^{K} \left(q^{\operatorname{inv}(\mathfrak{A}^{(i)}) - \operatorname{inv}(\mathfrak{H}^{(i)})} \prod_{j=0}^{n} \frac{(q;q)_{m_j(\mathfrak{A}^{(i)})}}{(q;q)_{m_j(\mathfrak{H}^{(i)})}} \right).
\end{aligned} \tag{6.3.6}$$

In first and third statements of (6.3.6), we sum over all sets $\mathcal{A} = \big(\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \ldots, \mathfrak{A}^{(K)} \big)$ with $\mathfrak{A}^{(j)} \in \mathcal{M}(\mathbf{A}'_j)$ for each j, and in the second we sum over all sets $\mathcal{H} = \big(\mathfrak{H}^{(1)}, \mathfrak{H}^{(2)}, \ldots, \mathfrak{H}^{(K)} \big)$ with $\mathfrak{H}^{(j)} \in \mathcal{M}(\mathbf{H}'_j)$ for each j. Moreover, in the first and second statements of (6.3.6), we have fixed a set $\mathcal{C} = \big(\mathfrak{C}^{(1)}, \mathfrak{C}^{(2)}, \ldots, \mathfrak{C}^{(K)} \big)$ with $\mathfrak{C}^{(j)} \in \mathcal{M}(\mathbf{C}'_j)$ for each j. Similarly, in the third statement of (6.3.6), we have fixed $\mathcal{H} = \big(\mathfrak{H}^{(1)}, \mathfrak{H}^{(2)}, \ldots, \mathfrak{H}^{(K)} \big)$ with $\mathfrak{H}^{(j)} \in \mathcal{M}(\mathbf{H}'_j)$ for each j.

Next, the first statement of (3.1.2) (applied with the h there equal to L_1 here) gives

$$\begin{aligned}
Z_{L_1+L_2;\mathbf{M}} \big(\mathcal{A}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]} \big) = \sum_{\mathcal{H}} & Z_{L_1;\mathbf{M}} \big(\mathcal{A}, \mathfrak{B}; \mathcal{H}, \mathfrak{D} \mid x; \mathbf{y}_{[1,K]} \big) \\
& \times Z_{L_2;\mathbf{M}} \big(\mathcal{H}, \mathfrak{B}; \mathcal{C}, \mathfrak{D} \mid q^{L_1} x; \mathbf{y}_{[1,K]} \big),
\end{aligned}$$

which together with (6.3.6) implies (6.3.5) and thus the proposition. $\square$

CHAPTER 7

Functions and Identities

In this chapter we define the functions we will study in this text and establish symmetry, branching, and skew Cauchy identities for them.

7.1. Symmetric Functions

In this section we define the symmetric functions we will study in this text as partition functions for vertex models with certain boundary conditions. Throughout this section, we fix infinite sequences $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$ of complex numbers.

Recall the vector spaces $\mathbb{V} = \mathbb{V}_{\mathbf{y};\mathbf{s}}$ and $\mathbb{V}^* = \mathbb{V}^*_{\mathbf{y};\mathbf{s}}$ from Section 6.2, which are spanned by basis vectors of the form $\{|\mathcal{A}\rangle\}$ and $\{\langle\mathcal{C}|\}$, respectively, where $\mathcal{A}$ and $\mathcal{C}$ range over all infinite, finitary sequences of elements in $\{0,1\}^n$. Let us introduce relabelings for these bases of $\mathbb{V}$ and $\mathbb{V}^*$ that will be more convenient for defining the symmetric functions below.

To that end, recall the notions of signatures and signature sequences from Section 1.2, as well as the shift operator $\mathfrak{T}$ from (1.2.3). In what follows, we will denote any signature $\lambda \in \mathrm{Sign}_\ell$ by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and any signature sequence $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$ by $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\big)$, even when not explicitly mentioned. For any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$, further recall from Section 1.2 the infinite, finitary sequence $\mathcal{S}(\boldsymbol{\lambda}) = \big(\mathbf{S}_1(\boldsymbol{\lambda}), \mathbf{S}_2(\boldsymbol{\lambda}), \ldots\big)$ of elements in $\{0,1\}^n$ defined as follows. For each $j \geqslant 1$, let $\mathbf{S}_j = \mathbf{S}_j(\boldsymbol{\lambda}) = (S_{1,j}, S_{2,j}, \ldots, S_{n,j}) \in \{0,1\}^n$, where $S_{i,j} = \mathbf{1}_{j \in \mathfrak{T}(\lambda^{(i)})}$ for every $i \in [1,n]$. Then $\mathcal{S}$ induces a bijection between $\mathrm{SeqSign}_n$ and finitary sequences of elements in $\{0,1\}^n$.

Denoting $|\boldsymbol{\lambda}\rangle = \big|\mathcal{S}(\boldsymbol{\lambda})\big\rangle$ and $\langle\boldsymbol{\lambda}| = \big\langle\mathcal{S}(\boldsymbol{\lambda})\big|$ for any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$, these vectors constitute bases of $\mathbb{V}$ and $\mathbb{V}^*$, respectively. Given this relabeling, we can define the following functions; in the below, we recall the operators $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{D}$ from (6.2.4).

Definition 7.1.1. Fix integers $N \geqslant 1$ and $M \geqslant 0$; sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$; and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. For any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, define

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \big\langle\boldsymbol{\lambda}\big|\mathbb{D}(x_N; r_N) \cdots \mathbb{D}(x_1; r_1)\big|\boldsymbol{\mu}\big\rangle.$$

We further abbreviate $G_{\boldsymbol{\lambda}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\mathbf{0}^M}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$.

Moreover, for any $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, define

$$F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \big\langle\boldsymbol{\mu}\big|\mathbb{B}(x_N; r_N) \cdots \mathbb{B}(x_1; r_1)\big|\boldsymbol{\lambda}\big\rangle;$$

$$H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \big\langle\boldsymbol{\lambda}\big|\mathbb{C}(x_N; r_N) \cdots \mathbb{C}(x_1; r_1)\big|\boldsymbol{\mu}\big\rangle,$$

We further abbreviate $F_{\boldsymbol{\lambda}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = F_{\boldsymbol{\lambda}/\varnothing}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ and $H_{\boldsymbol{\lambda}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}, \mathbf{s}) = H_{\boldsymbol{\lambda}/\varnothing}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$.

Remark 7.1.2. Although we will not pursue this here, one can define $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}, F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}, H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ for arbitrary sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Sign}_n$ (without stipulations on the lengths of their signatures) similarly to in Definition 7.1.1. All identities to be shown in Section 7.2 and Section 7.3 below will still hold in this more general setting. Under this convention, it is quickly verified that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = 0$ for any $(\mathbf{x},\mathbf{y},\mathbf{r},\mathbf{s})$ unless $\ell(\lambda^{(j)}) = \ell(\mu^{(j)})$ for each $j \in [1,n]$. Similarly, if $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, then $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = 0 = H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ unless $\ell(\lambda^{(j)}) = \ell(\mu^{(j)}) + N$ for each $j \in [1,n]$.

Let us describe diagrammatic interpretations of $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, and $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as partition functions for vertex models on the domain

$$\mathcal{D} = \mathcal{D}_N = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N\} \subset \mathbb{Z}_{>0}^2. \tag{7.1.1}$$

The following definition provides notation for sets of path ensembles on $\mathcal{D}_N$ with certain types of boundary data.

Definition 7.1.3. Fix $M, N \in \mathbb{Z}_{\geqslant 0}$ with $N \geqslant 1$, and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$.

If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, then let $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$ denote the set of path ensembles on $\mathcal{D}$ satisfying the following two properties; see the left side of Figure 7.1.

1. For every $c \in [1,n]$, one color c arrow vertically enters $\mathcal{D}$ through $(\mathfrak{m}, 1)$ for each $\mathfrak{m} \in \mathfrak{T}(\mu^{(c)})$.
2. For every $c \in [1,n]$, one color c arrow vertically exits $\mathcal{D}$ through $(\mathfrak{l}, N)$ for each $\mathfrak{l} \in \mathfrak{T}(\lambda^{(c)})$.

If $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, then let $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ denote[1] the set of path ensembles on $\mathcal{D}$ satisfying the following three properties for each $c \in [1,n]$; see the middle of Figure 7.1.

1. For every $c \in [1,n]$, one color c arrow vertically enters $\mathcal{D}$ through $(\mathfrak{l}, 1)$ for each $\mathfrak{l} \in \mathfrak{T}(\lambda^{(c)})$.
2. For every $c \in [1,n]$, one color c arrow vertically exits $\mathcal{D}$ through $(\mathfrak{m}, N)$ for each $\mathfrak{m} \in \mathfrak{T}(\mu^{(c)})$.
3. For every $c \in [1,n]$, one color c arrow horizontally exits $\mathcal{D}$ through (∞, j) for each $j \in [1,N]$.[2]

If $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, then let $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ denote the set of path ensembles on $\mathcal{D}$ satisfying the following three properties; see the right side of Figure 7.1.

1. For every $c \in [1,n]$, one color c arrow vertically enters $\mathcal{D}$ through $(\mathfrak{m}, 1)$ for each $\mathfrak{m} \in \mathfrak{T}(\mu^{(c)})$.
2. For every $c \in [1,n]$, one color c arrow vertically exits $\mathcal{D}$ through $(\mathfrak{l}, N)$ for each $\mathfrak{l} \in \mathfrak{T}(\lambda^{(c)})$.
3. For every $c \in [1,n]$, one color c arrow horizontally enters $\mathcal{D}$ through $(1, j)$ for each $j \in [1,N]$.

Now, fix finite sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$. We define the weight (with respect to either the vertex weights W_z from Definition 5.1.1 or $\widehat{W}_z$ from (5.1.4)) of any fused ensemble $\mathcal{E}$ on $\mathcal{D}$ to be

$$\begin{aligned} W(\mathcal{E} \mid \mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) &= \prod_{(i,j)\in\mathcal{D}} W_{x_j/y_i}\big(\mathbf{A}(i,j), \mathbf{B}(i,j); \mathbf{C}(i,j), \mathbf{D}(i,j) \mid r_j, s_i\big); \\ \widehat{W}(\mathcal{E} \mid \mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) &= \prod_{(i,j)\in\mathcal{D}} \widehat{W}_{x_j/y_i}\big(\mathbf{A}(i,j), \mathbf{B}(i,j); \mathbf{C}(i,j), \mathbf{D}(i,j) \mid r_j, s_i\big), \end{aligned} \tag{7.1.2}$$

where $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ denotes the arrow configuration under $\mathcal{E}$ at any vertex $v \in \mathcal{D}$, and we must assume that the above infinite products converge. Observe in particular that if $\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$ or $\mathcal{E} \in \mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ for some $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$, then the product defining $W(\mathcal{E})$ converges

[1] Observe here that N is fixed by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ and is therefore not incorporated in the notation for $\mathfrak{P}_F$.

[2] This means that every edge connecting (i,j) to $(i+1,j)$ for sufficiently large i contains an arrow of color c.

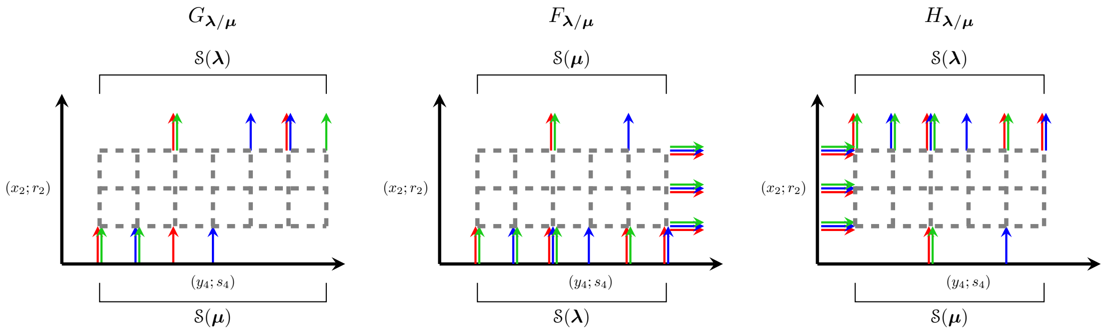


FIGURE 7.1. To the left, middle, and right are the vertex models $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; 3)$, $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$, and $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$, respectively. On the left, we have $\boldsymbol{\lambda} = \big((4,2),(4,4),(5,2)\big)$ and $\boldsymbol{\mu} = \big((1,0),(2,1),(0,0)\big)$, and on the middle and right we have $\boldsymbol{\lambda} = \big((2,2,1,0),(2,1,1,1),(1,0,0,0)\big)$ and $\boldsymbol{\mu} = \big((2),(4),(2)\big)$. Here, red is color 1, blue is color 2, and green is color 3.

since $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big) = (\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0)$ for all but finitely many $v \in \mathcal{D}$, and this arrow configuration has weight 1 under W by (5.1.2). Similarly, if $\mathcal{E} \in \mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ for some $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$, then the product defining $\widehat{W}(\mathcal{E})$ converges since $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big) = \big(\mathbf{e}_0, \mathbf{e}_{[1,n]}; \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ for all but finitely many $v \in \mathcal{D}$, and this arrow configuration has weight 1 under $\widehat{W}$ by (5.1.5).

Then, under the above notation, we have that

(7.1.3)

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \sum_{\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)} W(\mathcal{E} \mid \mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}); \quad H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \sum_{\mathcal{E} \in \mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})} W(\mathcal{E} \mid \mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s});$$

$$F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \sum_{\mathcal{E} \in \mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})} \widehat{W}(\mathcal{E} \mid \mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).$$

This indicates that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, and $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ can be interpreted as partition functions for the vertex models $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$ (under the weight W_z), $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (under $\widehat{W}_z$), and $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (under W_z), respectively. In each of these vertex models, the rapidity pair for the i-th column from the left is $(y_i; s_i)$ and that for the j-th row from the bottom is $(x_j; r_j)$; we refer to Figure 7.1 for depictions in all three cases.

7.2. Symmetry, Branching, and Principal Specializations

In this section we establish branching identities, symmetry relations, and results concerning principal specializations for the F, G, H functions from Definition 7.1.1. We begin with the former.

Proposition 7.2.1. *Fix integers $n, K, L, M \geqslant 1$; finite sequences of complex numbers $\mathbf{x}' = (x_1, \ldots, x_K)$, $\mathbf{x}'' = (x_{K+1}, \ldots, x_{K+L})$, $\mathbf{r}' = (r_1, \ldots, r_K)$, and $\mathbf{r}'' = (r_{K+1}, \ldots, r_{K+L})$; and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. Define $\mathbf{x} = \mathbf{x}' \cup \mathbf{x}'' = (x_1, x_2, \ldots, x_{K+L})$ and $\mathbf{r} = \mathbf{r}' \cup \mathbf{r}'' = (r_1, r_2, \ldots, r_{K+L})$. For any n-tuples of signatures $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Sign}_{n;M}$, we have*

$$\sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M}} G_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}), \tag{7.2.1}$$

Moreover, for any n-tuples of signatures $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;M+K+L}$ and $\boldsymbol{\mu} \in \mathrm{Sign}_{n;M}$, we have that

$$\begin{aligned}
&\sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M+L}} F_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}) F_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) = F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}); \\
&\sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M+K}} H_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) H_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}) = H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).
\end{aligned} \tag{7.2.2}$$

Proof. By Definition 7.1.1, we have that

$$\begin{aligned}
&G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \\
&\quad = \langle \boldsymbol{\lambda} \mid \mathbb{D}(x_{K+L}; r_{K+L}) \cdots \mathbb{D}(x_1; r_1) \big| \boldsymbol{\mu} \rangle \\
&\quad = \sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_n} \langle \boldsymbol{\lambda} \big| \mathbb{D}(x_{K+L}; r_{K+L}) \cdots \mathbb{D}(x_{K+1}; r_{K+1}) \big| \boldsymbol{\nu} \rangle \langle \boldsymbol{\nu} \big| \mathbb{D}(x_K; r_K) \cdots \mathbb{D}(x_1; r_1) \big| \boldsymbol{\mu} \rangle \\
&\quad = \sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_n} G_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}''; \mathbf{r}'' \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'; \mathbf{r}' \mid \mathbf{y}; \mathbf{s}),
\end{aligned}$$

which establishes (7.2.1) (by using the last statement of Remark 7.1.2 to restrict the sum over $\nu \in \mathrm{SeqSign}_{n;M}$). The proof of (7.2.2) is entirely analogous and is therefore omitted. $\square$

We next have the following proposition, which shows that G is symmetric in $\mathbf{x}$ and $\mathbf{r}$. It also shows that F and H are not exactly symmetric in those variables, but are up to explicit factors; this lack of exact symmetry is a consequence of the fact that $W_z\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid r, s\big) \neq 1$.

Proposition 7.2.2. *Fix integers $n, N \geqslant 1$; finite sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$; and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. For any n-tuples $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$ of signatures and permutation $\sigma \in \mathfrak{S}_N$, we have that*

$$\begin{aligned}
&G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\sigma(\mathbf{x}); \sigma(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}); \\
&F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\sigma(\mathbf{x}); \sigma(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) = F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \prod_{\substack{1 \leqslant i < j \leqslant N \\ \sigma(i) > \sigma(j)}} \frac{(r_j^2 x_i x_j^{-1}; q)_n}{(r_i^2 x_i^{-1} x_j; q)_n} \bigg(\frac{r_i^2 x_j}{r_j^2 x_i} \bigg)^n; \\
&H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\sigma(\mathbf{x}); \sigma(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) = H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \prod_{\substack{1 \leqslant i < j \leqslant N \\ \sigma(i) > \sigma(j)}} \frac{(r_i^2 x_i^{-1} x_j; q)_n}{(r_j^2 x_i x_j^{-1}; q)_n} \bigg(\frac{r_j^2 x_i}{r_i^2 x_j} \bigg)^n.
\end{aligned} \tag{7.2.3}$$

Proof. Since $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \langle \boldsymbol{\lambda} \big| \mathbb{D}(x_N; r_N) \cdots \mathbb{D}(x_1; r_1) \big| \boldsymbol{\mu} \rangle$, the first statement of (7.2.3) follows from the third statement of (6.2.5). So, it remains to establish the second and third ones there; since their proofs are very similar, we only detail the former.

To that end, since $\mathfrak{S}_N$ is generated by the transpositions $\{\mathfrak{s}_i\}$ for $i \in [1, N-1]$ interchanging $(i, i+1)$, we may assume that $\sigma = \mathfrak{s}_k$ for some $k \in [1, N-1]$. In this case, Definition 7.1.1 and the first statement of (6.2.5) together yield

$$\begin{aligned}
&F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathfrak{s}_k(\mathbf{x}); \mathfrak{s}_k(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) \\
&\quad = \langle \boldsymbol{\mu} \big| \mathbb{B}(x_N; r_N) \cdots \mathbb{B}(x_{k+2}; r_{k+2}) \mathbb{B}(x_k; r_k) \mathbb{B}(x_{k+1}; r_{k+1}) \mathbb{B}(x_{k-1}; r_{k-1}) \cdots \mathbb{B}(x_1; r_1) \big| \boldsymbol{\lambda} \rangle \\
&\quad = \bigg(\frac{r_k^2 x_{k+1}}{r_{k+1}^2 x_k} \bigg)^n \frac{(r_{k+1}^2 x_k x_{k+1}^{-1}; q)_n}{(r_k^2 x_k^{-1} x_{k+1}; q)_n} \langle \boldsymbol{\mu} \big| \mathbb{B}(x_N; r_N) \cdots \mathbb{B}(x_1; r_1) \big| \boldsymbol{\lambda} \rangle
\end{aligned}$$

$$= \left(\frac{r_k^2 x_{k+1}}{r_{k+1}^2 x_k} \right)^n \frac{(r_{k+1}^2 x_k x_{k+1}^{-1}; q)_n}{(r_k^2 x_k^{-1} x_{k+1}; q)_n} F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}),$$

which gives the second statement of (7.2.3). $\square$

The following proposition indicates that, if each entry of $\mathbf{r}$ is a negative half-integer power of q, then $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ may be recovered from a function of the form $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s})$, where $\mathbf{q}^{-1/2} = (q^{-1/2}, q^{-1/2}, \ldots, q^{-1/2})$. Here, $\mathbf{w}$ will be a union of *principal specializations*, that is, geometric progressions of the form $(x_i, q x_i, \ldots, q^{L_i - 1} x_i)$.

Proposition 7.2.3. *Fix integers $M \geqslant 0$ and $N \geqslant 1$; finite sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$; and infinite sequences of complex numbers $\mathbf{s} = (s_1, s_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. Assume for each $i \in [1, N]$ that there exists an integer $L_i \in \mathbb{Z}_{\geqslant 1}$ such that $r_i = q^{-L_i/2}$, and define $\mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)} \cup \cdots \cup \mathbf{w}^{(N)}$, where $\mathbf{w}^{(i)} = (x_i, q x_i, \ldots, q^{L_i - 1} x_i)$. Further set $\mathbf{q}^{-1/2} = (q^{-1/2}, q^{-1/2}, \ldots, q^{-1/2})$, where $q^{-1/2}$ appears with multiplicity N. Then, for any signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s})$.*

Proof. It suffices to establish the proposition for $N = 1$, as the result for general N follows from this together with the branching relation (7.2.1). Thus, we may assume in what follows that

$$\mathbf{x} = (x); \qquad \mathbf{r} = (r); \qquad r = q^{-L/2}; \qquad \mathbf{w} = (x, qx, \ldots, q^{L-1} x),$$

for some complex numbers $x, r \in \mathbb{C}$ and integer $L \in \mathbb{Z}_{\geqslant 1}$. Then, Definition 7.1.1 gives

$$\begin{aligned} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) &= \langle \boldsymbol{\lambda} | \mathbb{D}(x; r) | \boldsymbol{\mu} \rangle; \\ G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s}) &= \langle \boldsymbol{\lambda} | \mathbb{D}(q^{L-1} x; q^{-1/2}) \mathbb{D}(q^{L-2} x; q^{-1/2}) \cdots \mathbb{D}(x; q^{-1/2}) | \boldsymbol{\mu} \rangle. \end{aligned} \tag{7.2.4}$$

Moreover, Proposition 6.3.1 yields by induction on L that

$$\mathbb{D}(q^{L-1} x; q^{-1/2}) \mathbb{D}(q^{L-2} x; q^{-1/2}) \cdots \mathbb{D}(x; q^{-1/2}) = \mathbb{D}(x; q^{-L/2}) = \mathbb{D}(x; r),$$

which together with with (7.2.4) implies the proposition. $\square$

Results similar to Proposition 7.2.3 can also be derived for the functions $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, but we will not pursue this here.

7.3. Cauchy Identities

In this section we establish various Cauchy identities for the F and G functions. We begin with the following proposition, which provides the most general skew-Cauchy identity.

Theorem 7.3.1. *Fix integers $L, M \geqslant 0$ and $N \geqslant 1$; finite sequences $\mathbf{u} = (u_1, u_2, \ldots, u_N)$, $\mathbf{r} = (r_1, r_2, \ldots, r_N)$, $\mathbf{w} = (w_1, w_2, \ldots, w_M)$, and $\mathbf{t} = (t_1, t_2, \ldots, t_M)$ of complex numbers; and infinite sequences $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$ of complex numbers. Assume that there exists an integer $K > 1$ such that*

$$\sup_{k > K} \max_{\substack{1 \leqslant i \leqslant M \\ 1 \leqslant j \leqslant N}} \max_{\substack{a, b \in [0, n] \\ (a, b) \neq (n, 0)}} \left| s_k^{2a + 2b - 2n} \frac{(s_k^2 u_j y_k^{-1}; q)_n (u_j y_k^{-1}; q)_a}{(u_j y_k^{-1}; q)_n (s_k^2 u_j y_k^{-1}; q)_a} \frac{(w_i y_k^{-1}; q)_b}{(s_k^2 w_i y_k^{-1}; q)_b} \right| < 1. \tag{7.3.1}$$

Then for any $\boldsymbol{\mu} \in \operatorname{Sign}_{n;L+N}$ and $\boldsymbol{\nu} \in \operatorname{SeqSign}_{n;L}$, we have that

$$\begin{aligned}\sum_{\boldsymbol{\lambda} \in \operatorname{SeqSign}_{n;L+N}} & F_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) \\ &= \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{(t_i^2 u_j w_i^{-1}; q)_n}{t_i^{2n} (u_j w_i^{-1}; q)_n} \sum_{\boldsymbol{\kappa} \in \operatorname{SeqSign}_{n;L}} G_{\boldsymbol{\nu}/\boldsymbol{\kappa}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) F_{\boldsymbol{\mu}/\boldsymbol{\kappa}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).\end{aligned} \tag{7.3.2}$$

Proof. By Definition 7.1.1, we have that

$$\begin{aligned}\sum_{\boldsymbol{\lambda} \in \operatorname{SeqSign}_{n;L+N}} & F_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) \\ &= \sum_{\boldsymbol{\lambda} \in \operatorname{SeqSign}_{n;L+N}} \big\langle \boldsymbol{\nu} \big| \mathbb{B}(u_N; r_N) \cdots \mathbb{B}(u_1; r_1) \big| \boldsymbol{\lambda} \big\rangle \big\langle \boldsymbol{\lambda} \big| \mathbb{D}(w_M; t_M) \cdots \mathbb{D}(w_1; t_1) \big| \boldsymbol{\mu} \big\rangle \\ &= \big\langle \boldsymbol{\nu} \big| \mathbb{B}(u_N; r_N) \cdots \mathbb{B}(u_1; r_1) \mathbb{D}(w_M; t_M) \cdots \mathbb{D}(w_1; t_1) \big| \boldsymbol{\mu} \big\rangle,\end{aligned}$$

and by similar reasoning

$$\begin{aligned}\sum_{\boldsymbol{\kappa} \in \operatorname{SeqSign}_{n;L}} & G_{\boldsymbol{\nu}/\boldsymbol{\kappa}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; s) F_{\boldsymbol{\mu}/\boldsymbol{\kappa}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \\ &= \big\langle \boldsymbol{\nu} \big| \mathbb{D}(w_M; t_M) \cdots \mathbb{D}(w_1; t_1) \mathbb{B}(u_N; r_N) \cdots \mathbb{B}(u_1; r_1) \big| \boldsymbol{\mu} \big\rangle.\end{aligned}$$

These identities, together with MN applications of Proposition 6.2.2 (where the condition (6.2.11) there is verified by (7.3.1)) yield the theorem. □

Let us mention that $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ does not appear to satisfy a Cauchy identity when paired with either $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ or $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$; it instead would satisfy one when paired with a different function (which did not appear in Definition 7.1.1 above) of the form $\big\langle \boldsymbol{\lambda} \big| \mathbb{A}(x_N; r_N) \cdots \mathbb{A}(x_1; r_1) \big| \boldsymbol{\mu} \big\rangle$, where $\mathbb{A}(x; r) = \widehat{\mathbb{T}}_{\mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}}(x; r)$ (and we recall the latter from (6.2.1)). However, we will not describe this type of Cauchy identity in detail here, since we will not need it.

We will next derive from the general skew-Cauchy identity Theorem 7.3.1 a Cauchy identity, to which end we first require the following proposition that provides a factored form for the function $F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$. Its proof will appear in Section 7.4 below.

Proposition 7.3.2. *Fix an integer $N \geqslant 1$; finite sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_N)$; and infinite sequences of complex numbers $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. Recalling the signature sequence $\mathbf{0}^N = (0^N, 0^N, \ldots, 0^N) \in \operatorname{Sign}_{n;N}$, we have*

$$\begin{aligned}F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) &= \prod_{j=1}^{n} s_j^{2n(j-N)} r_j^{2n(j-N-1)} x_j^{n(N-j+1)} y_j^{-jn} (r_j^2; q)_n \\ &\quad \times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^{N} \prod_{j=1}^{N} (x_j y_i^{-1}; q)_n^{-1}.\end{aligned}$$

We next have the following consequence of Theorem 7.3.1.

THEOREM 7.3.3. *Adopt the notation and assumptions of Theorem 7.3.1 with $L = 0$. Then,*

$$\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} F_{\boldsymbol{\lambda}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\lambda}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s})$$

$$(7.3.3) \qquad = \prod_{j=1}^{n} s_j^{2n(j-N)} r_j^{2n(j-N-1)} u_j^{n(N-j+1)} y_j^{-jn} (r_j^2; q)_n \prod_{i=1}^{N} \prod_{j=1}^{N} (u_j y_i^{-1}; q)_n^{-1}$$

$$\times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 u_i^{-1} u_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{(t_i^2 u_j w_i^{-1}; q)_n}{t_i^{2n} (u_j w_i^{-1}; q)_n}.$$

PROOF. Applying Theorem 7.3.1 in the case when $\boldsymbol{\mu} = \mathbf{0}^N$ and $\boldsymbol{\nu} = \boldsymbol{\varnothing}$, it is quickly verified by the first identity in (7.1.3) (for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$) that the right side of (7.3.2) is supported on $\boldsymbol{\kappa} = \boldsymbol{\varnothing}$. This, together with the fact that $G_{\boldsymbol{\varnothing}/\boldsymbol{\varnothing}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) = 1$ (since all signatures in $\boldsymbol{\varnothing}$ are empty) yields

$$\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} F_{\boldsymbol{\lambda}}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) G_{\boldsymbol{\lambda}}(\mathbf{w}; \mathbf{t} \mid \mathbf{y}; \mathbf{s}) = \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{(t_i^2 u_j w_i^{-1}; q)_n}{t_i^{2n} (u_j w_i^{-1}; q)_n} F_{\mathbf{0}^N}(\mathbf{u}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).$$

This, together with Proposition 7.3.2, implies the theorem. $\square$

7.4. Proof of Proposition 7.3.2

In this section we establish Proposition 7.3.2. To that end, we begin with the following lemma indicating that, upon suitable normalization, $F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ is a polynomial of specified degree in $\mathbf{x} \cup \mathbf{y}$.

Lemma 7.4.1. *For any integers $n, N \geqslant 1$ and sequences of complex numbers $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ and $\mathbf{s} = (s_1, s_2, \ldots)$, there exists a constant $C = C_{n;N}(\mathbf{r}, \mathbf{s})$ such that the following holds. For any sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, there exists a polynomial $P(\mathbf{x}, \mathbf{y})$ of total degree $nN(N-1)$ in the variables $\mathbf{x}$ and $\mathbf{y}$, with coefficients in $\mathbb{C}(q, \mathbf{r}, \mathbf{s})$, such that*

$$(7.4.1) \qquad F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{k=0}^{n-1} (y_i - q^k x_j) = P(\mathbf{x}, \mathbf{y}) \prod_{j=1}^{N} x_j^n.$$

PROOF. Since by (7.1.3) $F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ is the partition function for the vertex model $\mathfrak{P}_F(\mathbf{0}^N / \boldsymbol{\varnothing})$ (from Definition 7.1.3; see also the middle of Figure 7.1) on the domain $\mathcal{D}_N = \mathbb{Z}_{>0} \times [1, N]$, under the weights $\widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ from Definition 5.1.1 and (5.1.4), it will be useful to analyze properties of these weights. In particular, let us show for any $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0, 1\}^n$ that $(z; q)_n \widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ is a polynomial of degree at most n in z, with coefficients in $\mathbb{C}(q, r, s)$. By (5.1.4), it suffices to show that $(s^2 z; q)_n W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ is a polynomial of degree at most n in z.

The polynomiality of this normalized weight follows from the definition (5.1.1) of W_z, since

$$\frac{(s^2 z; q)_n}{(s^2 z; q)_{c+d-p-v}} \in \mathbb{C}(q, s)[z]; \qquad \frac{(q^{-v} s^2 r^{-2} z; q)_{c-p}}{(q^{-v} s^2 r^{-2} z; q)_v} = (s^2 r^{-2} z; q)_{c-p-v} \in \mathbb{C}(q, r, s)[z].$$

Here, the first inclusion follows from the fact that $c + d - p - v \leqslant c + d - v \leqslant n$ (since if $\mathbf{C} = (C_1, C_2, \ldots, C_n) \in \{0, 1\}^n$ and $\mathbf{D} = (D_1, D_2, \ldots, D_n) \in \{0, 1\}^n$, then $c + d - v$ counts the number of indices $i \in [1, n]$ for which $\max\{C_i, D_i\} = 1$), and the second follows from the fact that $p \leqslant c - v$ (in the sum on the right side of (5.1.1)). The fact that $(s^2 z; q)_n W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ is of degree at most

n in z then follows from the fact that the degree (in z) of the numerator of any summand on the right side of (5.1.1) is at most that of the denominator. This verifies that $(z;q)_n \widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ is a polynomial of degree at most n in z.

Now, observe that $F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ is the partition function for the vertex model $\mathfrak{P}_F(\mathbf{0}^N/\varnothing)$ on the domain $[1,N]\times[1,N]$, whose weight at any vertex $(i,j) \in \mathcal{D}_N = \mathbb{Z}_{>0}\times[1,N]$ is given by $\widehat{W}_{x_j/y_i}\big(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r_j,s_i\big)$; here, we replaced the original domain $\mathcal{D}_N$ with $[1,N]\times[1,N]$, which is permitted as the arrow configuration at each vertex $(i,j)\in\mathcal{D}_N$ with $i>N$ under any path ensemble in $\mathfrak{P}_F(\mathbf{0}^N/\varnothing)$ is given by $\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]}\big)$, and $\widehat{W}_z\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]} \mid r_j,s_i\big)=1$ by (5.1.5). Thus, the polynomiality of $(z;q)_n\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ implies that

$$F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) \prod_{i=1}^{N}\prod_{j=1}^{N}(x_j y_i^{-1};q)_n \tag{7.4.2}$$

is a polynomial in the variables $\{x_j y_i^{-1}\}$, whose degree in any individual $x_j y_i^{-1}$ is at most n. Multiplying by $\prod_{j=1}^N y_j^{nN}$ yields that the left side of (7.4.1) is a polynomial in $\mathbf{x}$ and $\mathbf{y}$ of total degree at most nN^2.

It therefore remains to show that the left side of (7.4.1) is a multiple of x_j^n for each $j\in[1,N]$. To that end, observe from (5.1.1) and (5.1.4) that $z^{|\mathbf{D}|-|\mathbf{B}|}$ divides $(z;q)_n\widehat{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$. Now observe under each path ensemble $\mathcal{E}\in\mathfrak{P}_F(\mathbf{0}^N)$, with arrow configuration $\big(\mathbf{A}(v),\mathbf{B}(v);\mathbf{C}(v),\mathbf{D}(v)\big)$ at any vertex $v\in\mathcal{D}_N$, that for any $j\in[1,N]$ we have $\sum_{i=1}^N\big(|\mathbf{D}|(i,j)-|\mathbf{B}|(i,j)\big)=\big|\mathbf{D}(N,j)\big|-\big|\mathbf{B}(1,j)\big|=n$, since $\mathbf{B}(1,j)=\mathbf{e}_0$ and $\mathbf{D}(N,j)=\mathbf{e}_{[1,n]}$. Hence, x_j^n divides the numerator of (7.4.2), which upon multiplication by $\prod_{j=1}^N y_j^{nN}$ implies that it also divides the left side of (7.4.1).

Hence, this left side is a polynomial of degree nN^2 in $\mathbf{x}\cup\mathbf{y}$ and is a multiple of $\prod_{j=1}^N x_j^n$. This implies that there exists a polynomial $P\in\mathbb{C}(q,\mathbf{r},\mathbf{s})[\mathbf{x},\mathbf{y}]$ of degree $n(N^2-N)$ such that (7.4.1) holds. □

We must next determine the polynomial $P(\mathbf{x},\mathbf{y})$ in Lemma 7.4.1, which we do by specifying sufficiently many of its zeroes. To that end, the following symmetry relation for $F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ in $\mathbf{y}$ and $\mathbf{s}$ (similar to Proposition 7.2.2 for the $(\mathbf{x},\mathbf{r})$ variables) will be useful.

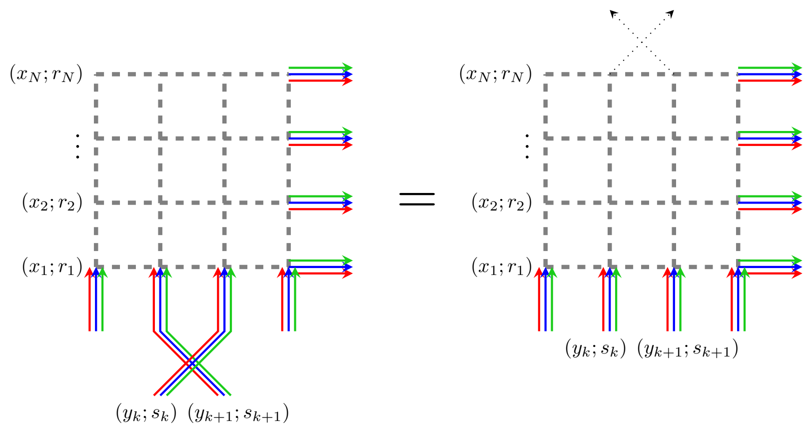
Lemma 7.4.2. *Fix integers $n,N\geqslant 1$; finite sequences of complex numbers $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ and $\mathbf{r}=(r_1,r_2,\ldots,r_N)$; and infinite sequences of complex numbers $\mathbf{y}=(y_1,y_2,\ldots)$ and $\mathbf{s}=(s_1,s_2,\ldots)$. For any permutation $\sigma\in\mathfrak{S}_N$, we have*

$$F_{\mathbf{0}^N}\big(\mathbf{x};\mathbf{r} \mid \sigma(\mathbf{y});\sigma(\mathbf{s})\big) = F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) \prod_{\substack{1\leqslant i<j\leqslant N\\ \sigma(i)>\sigma(j)}} \frac{(s_j^2 y_i y_j^{-1};q)_n}{(s_i^2 y_i^{-1} y_j;q)_n}\left(\frac{s_i^2 y_j}{s_j^2 y_i}\right)^n.$$

Proof. Since $\mathfrak{S}_N$ is generated by the transpositions $\{\mathfrak{s}_i\}$ for $i\in[1,N]$ interchanging $(i,i+1)$, we may assume that $\sigma=\mathfrak{s}_k$ for some $k\in[1,N-1]$. Then, by N applications of the Yang–Baxter equation Proposition 5.1.4, we obtain

$$\begin{aligned}\widehat{W}_{y_k/y_{k+1}}\big(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]};\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]} \mid s_k,s_{k+1}\big)F_{\mathbf{0}^N}\big(\mathbf{x};\mathbf{r} \mid \sigma(\mathbf{y});\sigma(\mathbf{s})\big)& \\ = F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})\widehat{W}_{y_k/y_{k+1}}\big(\mathbf{e}_0,\mathbf{e}_0;\mathbf{e}_0,\mathbf{e}_0 \mid s_k,s_{k+1}\big)&.\end{aligned} \tag{7.4.3}$$

Diagrammatically,



where the partition functions on both sides are with respect to the $\widehat{W}_z$ weights.

Next, by (5.1.2) and (5.1.4), we have

$$\widehat{W}_{y_k/y_{k+1}}\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid s_k, s_{k+1}\big) = \bigg(\frac{y_k}{s_k^2 y_{k+1}}\bigg)^n \frac{(s_k^2 y_k^{-1} y_{k+1}; q)_n}{(y_k y_{k+1}^{-1}; q)_n};$$
$$\widehat{W}_{y_k/y_{k+1}}\big(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid s_k, s_{k+1}\big) = \frac{(s_{k+1}^2 y_k y_{k+1}^{-1}; q)_n}{s_{k+1}^{2n} (y_k y_{k+1}^{-1}; q)_n},$$

which upon insertion into (7.4.3) yields

$$F_{\mathbf{0}^N}\big(\mathbf{x}; \mathbf{r} \mid \sigma(\mathbf{y}); \sigma(\mathbf{s})\big) = \bigg(\frac{s_k^2 y_{k+1}}{s_{k+1}^2 y_k}\bigg)^n \frac{(s_{k+1}^2 y_k y_{k+1}^{-1}; q)_n}{(s_k^2 y_k^{-1} y_{k+1}; q)_n} F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}).$$

This yields the lemma for $\sigma = \mathfrak{s}_k$, which as mentioned above implies it in general. $\square$

The following corollary then determines $F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$, up to an overall constant independent of $\mathbf{x}$ and $\mathbf{y}$.

Corollary 7.4.3. *For any integers $n, N \geqslant 1$ and sequences of complex numbers $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ and $\mathbf{s} = (s_1, s_2, \ldots)$, there exists a constant $C = C_{n;N}(\mathbf{r}, \mathbf{s})$ such that the following holds. For any sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, we have that*

$$F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = C \prod_{j=1}^{n} x_j^n \prod_{1 \leqslant i < j \leqslant N} \prod_{k=0}^{n-1} (x_i - q^k r_i^2 x_j)(y_i - q^k s_i^2 y_j) \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{k=0}^{n-1} (y_i - q^k x_j)^{-1}.$$

Proof. Let $\sigma_0 \in \mathfrak{S}_N$ denote the longest element, that is, it is defined so that $\sigma_0(j) = N - j + 1$ for each $j \in [1, N]$. Then, Proposition 7.2.2 and Lemma 7.4.2 give

$$\begin{aligned} F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) &= F_{\mathbf{0}^N}\big(\sigma_0(\mathbf{x}); \sigma_0(\mathbf{r}) \mid \mathbf{y}; \mathbf{s}\big) \prod_{1 \leqslant i < j \leqslant N} \frac{(r_i^2 x_i^{-1} x_j; q)_n}{(r_j^2 x_i x_j^{-1}; q)_n} \bigg(\frac{r_j^2 x_i}{r_i^2 x_j}\bigg)^n \\ F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) &= F_{\mathbf{0}^N}\big(\mathbf{x}; \mathbf{r} \mid \sigma_0(\mathbf{y}); \sigma_0(\mathbf{s})\big) \prod_{1 \leqslant i < j \leqslant N} \frac{(s_i^2 y_i^{-1} y_j; q)_n}{(s_j^2 y_i y_j^{-1}; q)_n} \bigg(\frac{s_j^2 y_i}{s_i^2 y_j}\bigg)^n. \end{aligned} \tag{7.4.4}$$

Hence, since Lemma 7.4.1 implies that

$$F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) \prod_{j=1}^{N} x_j^{-n} \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{k=0}^{n-1} (y_i - q^k x_j) \tag{7.4.5}$$

is a polynomial in $\mathbf{x}$ and $\mathbf{y}$ of total degree $nN(N-1)$, it follows from (7.4.4) that (7.4.5) is equal to 0 if there exist $1 \leqslant i < j \leqslant N$ and $k \in [0, n-1]$ such that either $x_i = q^k r_i^2 x_j$ or $y_i = q^k s_i^2 y_j$. This implies that

$$\prod_{1 \leqslant i < j \leqslant N} \prod_{k=0}^{n-1} (x_i - q^k r_i^2 x_j)(y_i - q^k s_i^2 y_j) \tag{7.4.6}$$

divides (7.4.5). The corollary then follows since both (7.4.5) and (7.4.6) are of total degree $nN(N-1)$ in $\mathbf{x}$ and $\mathbf{y}$. $\square$

We next state the following lemma, which will be established in Section 7.5 below, that provides an explicit form for the constant $C_{n;N}(\mathbf{r},\mathbf{s})$ from Corollary 7.4.3.

Lemma 7.4.4. *Adopting the notation of Corollary 7.4.3, we have*

$$C_{n;N}(\mathbf{r},\mathbf{s}) = \prod_{j=1}^{N} s_j^{2n(j-N)} r_j^{2n(j-N-1)} (r_j^2; q)_n.$$

Given the above, we can now quickly establish Proposition 7.3.2.

Proof of Proposition 7.3.2. By Corollary 7.4.3 and Lemma 7.4.4, we have

$$\begin{aligned} F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = \prod_{1 \leqslant i < j \leqslant N} \prod_{k=0}^{n-1} (x_i - q^k r_i^2 x_j)(y_i - q^k s_i^2 y_j) \prod_{j=1}^{N} s_j^{2n(j-N)} x_j^n r_j^{2n(j-N-1)} (r_j^2; q)_n \\ \times \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{k=0}^{n-1} (y_i - q^k x_j)^{-1}. \end{aligned}$$

This, together with the facts that

$$\begin{aligned} \prod_{1 \leqslant i < j \leqslant N} \prod_{k=0}^{n-1} (x_i - q^k r_i^2 x_j)(y_i - q^k s_i^2 y_j) = \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{j=1}^{N} (x_j y_j)^{n(N-j)}; \\ \prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{k=0}^{n-1} (y_i - q^k x_j) = \prod_{j=1}^{N} y_j^{nN} \prod_{i=1}^{N} \prod_{j=1}^{N} (x_j y_i^{-1}; q)_n, \end{aligned} \tag{7.4.7}$$

imply the proposition. $\square$

7.5. Proof of Lemma 7.4.4

In this section we establish Lemma 7.4.4, which explicitly determines the constant $C_{n;N}(\mathbf{r},\mathbf{s})$ from Corollary 7.4.3. To do this, it essentially suffices to evaluate $F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ for any choice of parameters $(\mathbf{x},\mathbf{y})$. Recalling the interpretation (7.1.3) of $F_{\mathbf{0}^N}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ as a partition function under the weights $\widehat{W}_{x_j/y_i}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r_j, s_i)$ (from (5.1.4)) for the vertex model $\mathfrak{P}_F(\mathbf{0}^N/\varnothing)$ (from Definition 7.1.3; see the middle of Figure 7.1), we will choose these parameters $(\mathbf{x},\mathbf{y})$ in such a way that this partition function *freezes*, that is, it admits at most one path ensemble with nonzero weight. The latter task will be facilitated through the following proposition, which provides a vanishing condition for the weights $\widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ when $r^2 = s^2 z$.

Lemma 7.5.1. *Fix complex numbers $s, r, z \in \mathbb{C}$ such that $r^2 = s^2 z$. For any $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \in \{0,1\}^n$, we have that $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0 = \widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ unless $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ and $\mathbf{B} \geqslant \mathbf{C}$.*

Proof. It suffices to show that the lemma holds for the weight $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$, for then (5.1.4) would imply that it also holds for $\widehat{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$. We may assume in what follows that $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$, for otherwise arrow conservation implies $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0$ for any $r, s, z \in \mathbb{C}$. Then, setting $r^2 = s^2 z$ in Definition 5.1.1 gives

(7.5.1)

$$\begin{aligned} W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = (-1)^v z^{c+d-a-b} s^{2c+2d-2a} q^{\varphi(\mathbf{D}-\mathbf{V}, \mathbf{C}) + \varphi(\mathbf{V}, \mathbf{A}) - av + cv} \\ \times \frac{(q^{1-v} s^{-2}; q)_v}{(q^{-v}; q)_v} \frac{(s^2 z; q)_d}{(s^2 z; q)_b} \sum_{p=0}^{\min\{b-v, c-v\}} \frac{(q^{-v}; q)_{c-p} (q^v s^2; q)_p (z; q)_{b-p-v}}{(s^2 z; q)_{c+d-p-v}} \\ \times q^{-pv} s^{-2p} \sum_{\boldsymbol{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P}, \mathbf{P})}, \end{aligned}$$

where the sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$, for each $i \in [1, n]$.

Since $c - p \geqslant v$, we have $(q^{-v}; q)_{c-p} = \mathbf{1}_{v=c-p} (q^{-v}; q)_v$. Thus, the sum over p on the right side of (7.5.1) is supported on the term $p = c - v$; since $P_i \leqslant C_i - V_i$ for each $i \in [1, n]$, this implies that the sum over $\mathbf{P}$ there is supported on $\mathbf{P} = \mathbf{C} - \mathbf{V}$. Since we must also have that $P_i \leqslant B_i - V_i$ for each $i \in [1, n]$, this implies that $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0$ unless $C_i \leqslant B_i$ for each $i \in [1, n]$ (that is, unless $\mathbf{B} \geqslant \mathbf{C}$). ☐

Remark 7.5.2. As Proposition 8.2.2 below, we will in fact derive a fully factored form for this weight $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ under the specialization $r^2 = s^2 z$, which will follow from the fact that the sum on the right side of (7.5.1) is supported on a single term.

The following corollary then evaluates $F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ when $s_j^2 x_j = r_j^2 y_j$ for each j.

Corollary 7.5.3. *Adopt the notation and assumptions of Corollary 7.4.3, and further assume for each $j \in [1, N]$ that $s_j^2 x_j = r_j^2 y_j$. Then,*

$$\begin{aligned} F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \prod_{j=1}^N s_j^{2n(j-N)} x_j^{n(N-j+1)} r_j^{2n(j-N-1)} y_j^{-jn} (r_j^2; q)_n \\ \times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^N \prod_{j=1}^N (x_j y_i^{-1}; q)_n^{-1}. \end{aligned}$$

Proof. Recall from (7.1.3) the interpretation of $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as a partition function under the $\widehat{W}_z$ weights (from (5.1.4)) for the vertex model $\mathfrak{P}_F(\mathbf{0}^N / \varnothing)$ (defined in Definition 7.1.3 and depicted in the middle of Figure 7.1). Thus, for any indices $c \in [1, n]$ and $j \in [1, N]$, one path of color c vertically enters the model through $(j, 1)$ and horizontally exits it through (N, j); we refer to Figure 7.2 for an example.

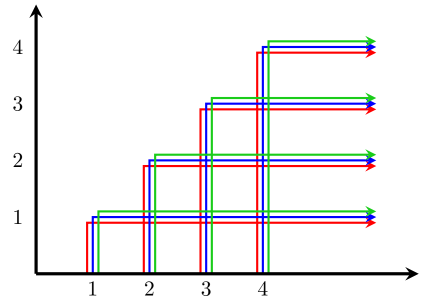


FIGURE 7.2. Shown above is the frozen vertex model correpsonding to the $F_{\mathbf{0}^N}$ when $s_j^2 x_j = r_j^2 y_j$ from the proof of Corollary 7.5.3.

By Lemma 7.5.1, $\widehat{W}_{x_j/y_j}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r_j, s_j) = 0$ unless $\mathbf{B} \geqslant \mathbf{C}$. From this fact, it is quickly verified by induction on N that this vertex model is frozen, that is, it admits only one path ensemble with a nonzero weight; it is given by the one depicted in Figure 7.2, in which to the left of the $(N+1)$-st column each path alternates between moving one space north and one space east (and after the $(N+1)$-st column, all paths proceed east). More specifically, in this ensemble, vertices (i, j) have arrow configuration $(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0)$ if $1 \leqslant i < j \leqslant N$; arrow configuration $\big(\mathbf{e}_{[1,n]}, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ if $1 \leqslant i = j \leqslant N$; arrow configuration $\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}\big)$ if $1 \leqslant j < i \leqslant N$; and arrow configuration $\big(\mathbf{e}_0, \mathbf{e}_{[1,n]}; \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ if $1 \leqslant j \leqslant N < i$. Hence,

(7.5.2)

$$
\begin{aligned}
F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \prod_{1 \leqslant i < j \leqslant N} & \widehat{W}_{x_j/y_i}\big(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r_j, s_i\big) \prod_{j=1}^{N} \widehat{W}_{x_j/y_j}\big(\mathbf{e}_{[1,n]}, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_{[1,n]} \mid r_j, s_j\big) \\
& \times \prod_{1 \leqslant j < i \leqslant N} \widehat{W}_{x_j/y_i}\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid r_j, s_i\big),
\end{aligned}
$$

where we have used the fact that $\widehat{W}_z\big(\mathbf{e}_0, \mathbf{e}_{[1,n]}; \mathbf{e}_0, \mathbf{e}_{[1,n]} \mid r, s\big) = 1$ (by (5.1.5)). By (5.1.2) and (5.1.4), we have

$$
\begin{aligned}
\widehat{W}_{x_j/y_i}(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0 \mid r_j, s_i) &= \frac{(s_i^2 x_j y_i^{-1}; q)_n}{s_i^{2n} (x_j y_i^{-1}; q)_n}; \\
\widehat{W}_{x_j/y_i}\big(\mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]}; \mathbf{e}_{[1,n]}, \mathbf{e}_{[1,n]} \mid r_j, s_i\big) &= \left(\frac{x_j}{r_j^2 y_i} \right)^n \frac{(r_j^2 x_j^{-1} y_i; q)_n}{(x_j y_i^{-1}; q)_n}.
\end{aligned}
\tag{7.5.3}
$$

We further have by (5.1.1) that

$$
W_z\big(\mathbf{e}_{[1,n]}, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_{[1,n]} \mid r, s\big) = \left(\frac{s^2 z}{r^2} \right)^n \frac{(r^2; q)_n}{(s^2 z; q)_n}. \tag{7.5.4}
$$

Indeed, under this weight, the n-tuple $\mathbf{C} \in \{0,1\}^n$ from Definition 5.1.1 is equal to $\mathbf{e}_0$. So, the sum over $\mathbf{P}$ on the right side of (5.1.1) is supported on $\mathbf{P} = \mathbf{e}_0$, from which (7.5.4) quickly follows.

Applying (7.5.4), together with (5.1.4), yields

$$\widehat{W}_{x_j/y_i}\big(\mathbf{e}_{[1,n]}, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_{[1,n]} \mid r_j, s_i\big) = \left(\frac{x_j}{r_j^2 y_i}\right)^n \frac{(r_j^2; q)_n}{(x_j y_i^{-1}; q)_n}.$$

Inserting this, together with (7.5.2), into (7.5.3) yields

$$\begin{aligned} F_{\mathbf{0}^N}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) &= \prod_{j=1}^N s_j^{2n(j-N)} x_j^{n(N-j+1)} r_j^{2n(j-N-1)} y_j^{-jn} (r_j^2; q)_n \\ &\qquad \times \prod_{1 \leqslant i < j \leqslant N} (s_i^2 x_j y_i^{-1}; q)_n (r_i^2 x_i^{-1} y_j; q)_n \prod_{i=1}^N \prod_{j=1}^N (x_j y_i^{-1}; q)_n^{-1} \\ &= \prod_{j=1}^N s_j^{2n(j-N)} x_j^{n(N-j+1)} r_j^{2n(j-N-1)} y_j^{-jn} (r_j^2; q)_n \\ &\qquad \times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^N \prod_{j=1}^N (x_j y_i^{-1}; q)_n^{-1}, \end{aligned}$$

where to deduce the second equality we used the fact that $s_i^2 x_j y_i^{-1} = r_i^2 x_i^{-1} x_j$ and $r_i^2 x_i^{-1} y_j = s_i^2 y_i^{-1} y_j$ (as $s_i^2 x_i = r_i^2 y_i$). This implies the corollary. □

Now we can quickly establish Lemma 7.4.4.

Proof of Lemma 7.4.4. This follows from Corollary 7.5.3 and the fact that

$$\begin{aligned} &\prod_{j=1}^N s_j^{2n(j-N)} x_j^{n(N-j+1)} r_j^{2n(j-N-1)} y_j^{-jn} (r_j^2; q)_n \\ &\qquad \times \prod_{1 \leqslant i < j \leqslant N} (r_i^2 x_i^{-1} x_j; q)_n (s_i^2 y_i^{-1} y_j; q)_n \prod_{i=1}^N \prod_{j=1}^N (x_j y_i^{-1}; q)_n^{-1} \\ &= \prod_{j=1}^N s_j^{2n(j-N)} r_j^{2n(j-N-1)} (r_j^2; q)_n \\ &\qquad \times \prod_{j=1}^n x_j^n \prod_{1 \leqslant i < j \leqslant N} \prod_{k=0}^{n-1} (x_i - q^k r_i^2 x_j)(y_i - q^k s_i^2 y_j) \prod_{i=1}^N \prod_{j=1}^N \prod_{k=0}^{n-1} (y_i - q^k x_j)^{-1}, \end{aligned}$$

where we applied (7.4.7). □

CHAPTER 8

Degenerations

Although the weights W_z from Definition 5.1.1 might at first glance appear a bit unpleasant, the fact that they are governed by four parameters (r, s, z, q) makes them quite general. We will see in this chapter how imposing certain relations among these parameters drastically simplifies the W_z weights, enabling them to factor completely. We will furthermore analyze the symmetric functions resulting from such simplifications and the Cauchy identities they satisfy.

8.1. Restricting Horizontal Arrows

In this section we explain two degenerations of the W_z weights from Definition 5.1.1 that are obtained by restricting the parameters so that at most one arrow is permitted along any horizontal edge. Throughout this section, we adopt the notation of Definition 5.1.1.

The first way of doing this is to set $n = 1$, as in the following example.

Example 8.1.1. Suppose $n = 1$, so that $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) \neq 0$ only if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{\mathbf{e}_0, \mathbf{e}_1\}$. Abbreviating $W(i, j; i', j') = W_z\big(\mathbf{e}_i, \mathbf{e}_j; \mathbf{e}_{i'}, \mathbf{e}_{j'} \mid r, s\big)$ for any $i, j, i', j' \in \{0, 1\}$ we have by (5.1.1) that

$$W(0,0;0,0) = 1; \qquad W(1,0;1,0) = \frac{1 - s^2 r^{-2} z}{1 - s^2 z}; \qquad W(1,0;0,1) = \frac{s^2 z (r^{-2} - 1)}{1 - s^2 z};$$
$$W(1,1;1,1) = \frac{s^2 (r^{-2} z - 1)}{1 - s^2 z}; \qquad W(0,1;0,1) = \frac{s^2 (1 - z)}{1 - s^2 z}; \qquad W(0,1;1,0) = \frac{1 - s^2}{1 - s^2 z},$$

and $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0$ otherwise. This recovers the vertex weights for a generalized free-fermionic six-vertex model studied in **[72]**. We refer to Figure 8.1 for a depiction.

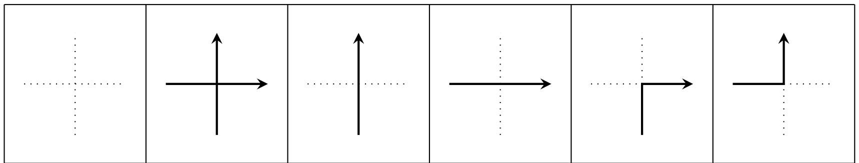
The second way of doing this is to set $r = q^{-1/2}$; this corresponds to the situation when $L = 1$ in Section 3.1, which again implies that any horizontal edge can be occupied by at most one arrow.

Example 8.1.2. Suppose $r = q^{-1/2}$. In this case, $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) \neq 0$ only if $\mathbf{A}, \mathbf{C} \in \{0, 1\}^n$ and $\mathbf{B}, \mathbf{D} \in \{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$, so let us assume this to be the case below. For any $i, j \in [0, n]$, abbreviate $W(\mathbf{A}, i; \mathbf{C}, j) = \mathbf{W}_z\big(\mathbf{A}, \mathbf{e}_i; \mathbf{C}, \mathbf{e}_j \mid r, s\big)$ and, as in (1.2.2) of **[12]**, set

$$\mathbf{A}_i^{+} = \mathbf{A} + \mathbf{e}_i; \qquad \mathbf{A}_j^{-} = \mathbf{A} - \mathbf{e}_j; \qquad \mathbf{A}_{ij}^{+-} = \mathbf{A} + \mathbf{e}_i - \mathbf{e}_j. \tag{8.1.1}$$

Then, it follows as a direct consequence of (5.1.1) that for each $\mathbf{A} \in \{0, 1\}^n$ and $i \in [1, n]$, we have (letting $|\mathbf{A}| = a$)

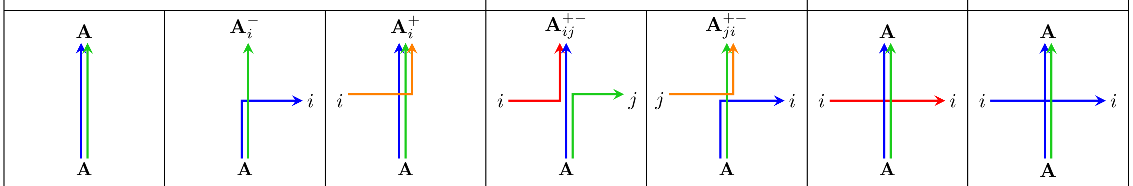
$$W(\mathbf{A}, 0; \mathbf{A}, 0) = \frac{1 - q^a s^2 z}{1 - s^2 z}; \quad W(\mathbf{A}, 0; \mathbf{A}_i^{-}, i) = q^{A_{[i+1,n]}} \frac{s^2 z (q - 1)}{1 - s^2 z}; \quad W(\mathbf{A}, i; \mathbf{A}_i^{+}, 0) = \frac{1 - q^a s^2}{1 - s^2 z}.$$



$(0,0;0,0)$	$(1,1;1,1)$	$(1,0;1,0)$	$(0,1;0,1)$	$(1,0;0,1)$	$(0,1;1,0)$
1	$\frac{s^2(r^{-2}z-1)}{1-s^2z}$	$\frac{1-s^2r^{-2}z}{1-s^2z}$	$\frac{s^2(1-z)}{1-s^2z}$	$\frac{s^2z(r^{-2}-1)}{1-s^2z}$	$\frac{1-s^2}{1-s^2z}$

FIGURE 8.1. The $n=1$ cases of the W_z weights, and their arrow configurations, are depicted above.

$1 \leqslant i \leqslant n$			$1 \leqslant i < j \leqslant n$		$A_i = 0$	$A_i = 1$



$(\mathbf{A},0;\mathbf{A},0)$	$\big(\mathbf{A},0;\mathbf{A}_i^-,i\big)$	$\big(\mathbf{A},i;\mathbf{A}_i^+,0\big)$	$\big(\mathbf{A},i;\mathbf{A}_{ij}^{+-},j\big)$	$\big(\mathbf{A},j;\mathbf{A}_{ji}^{+-},i\big)$	$(\mathbf{A},i;\mathbf{A},i)$	$(\mathbf{A},i;\mathbf{A},i)$
$\frac{1-q^a s^2 z}{1-s^2z}$	$q^{A_{[i+1,n]}}\frac{s^2z(q-1)}{1-s^2z}$	$\frac{1-q^a s^2}{1-s^2z}$	$q^{A_{[j+1,n]}}\frac{s^2z(q-1)}{1-s^2z}$	$q^{A_{[i+1,n]}}\frac{s^2(q-1)}{1-s^2z}$	$q^{A_{[i+1,n]}}\frac{s^2(1-z)}{1-s^2z}$	$q^{A_{[i+1,n]}}\frac{s^2(qz-1)}{1-s^2z}$

FIGURE 8.2. The $r=q^{-1/2}$ cases of the W_z weights, and their arrow configurations, are depicted above.

Moreover, for any $1 \leqslant i < j \leqslant n$, we have

$$W(\mathbf{A},i;\mathbf{A}_{ij}^{+-},j) = q^{A_{[j+1,n]}}\frac{s^2z(q-1)}{1-s^2z}; \qquad W(\mathbf{A},j;\mathbf{A}_{ji}^{+-},i) = q^{A_{[i+1,n]}}\frac{s^2(q-1)}{1-s^2z}.$$

Additionally,

$$W(\mathbf{A},i;\mathbf{A},i) = q^{A_{[i+1,n]}}\frac{s^2(1-z)}{1-s^2z}, \quad \text{if } A_i = 0; \qquad W(\mathbf{A},i;\mathbf{A};i) = q^{A_{[i+1,n]}}\frac{s^2(qz-1)}{1-s^2z}, \quad \text{if } A_i = 1.$$

We further have $W_z(\mathbf{A},b;\mathbf{C},d \mid r,s) = 0$ if $(\mathbf{A},b;\mathbf{C},d)$ is not of the above form, with $\mathbf{A},\mathbf{C} \in \{0,1\}^n$. We refer to Figure 8.2 for a depiction.

Remark 8.1.3. All vertex weights $W(\mathbf{A},i;\mathbf{B},j)$ in Figure 8.2, except for the rightmost one there (correpsonding to when $i=j$ and $A_i=1$), coincide with the $\widetilde{L}_{z/s}(\mathbf{A},i;\mathbf{B},j)$ weights from equation (2.5.1) of **[12]** (see also equation (B.4.7) there) for the $U_q\big(\widehat{\mathfrak{sl}}(n+1)\big)$-vertex model. This can be viewed as a consequence of Lemma 4.2.1, stating that the $U_q\big(\widehat{\mathfrak{sl}}(m|n)\big)$ weights $\mathcal{R}_{x,y}^{(m;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ coincide with the $U_q\big(\widehat{\mathfrak{sl}}(m+n)\big)$ ones if $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ share no fermionic colors.

8.2. Additional Simplifications

In this section we provide four additional special choices of the parameters (r, s, z) under which the sums from (5.1.1) describing the weights $W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ again simplify into factored products, but which now permit multiple arrows to exist along horizontal edges. Throughout this section, we adopt the notation of Definition 5.1.1.

Before detailing these degenerations, we begin with the following lemma that will be useful in their derivations. In what follows, we recall the function φ from (1.1.1).

Lemma 8.2.1. *Fix $\mathbf{X}, \mathbf{Y} \in \{0,1\}^n$, whose coordinates are both indexed by $[1,n]$, and define $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n) \in \{0,1\}^n$ by setting $Z_i = \min\{X_i, Y_i\}$ for each $i \in [1,n]$. Then,*

$$\varphi(\mathbf{X}, \mathbf{Y}) + \varphi(\mathbf{Y}, \mathbf{X}) = |\mathbf{X}||\mathbf{Y}| - |\mathbf{Z}|. \tag{8.2.1}$$

In particular, recalling that $\mathbf{e}_{[1,n]} = (1, 1, \ldots, 1) \in \{0,1\}^n$, we have

$$\varphi\big(\mathbf{X}, \mathbf{e}_{[1,n]}\big) + \varphi\big(\mathbf{e}_{[1,n]}, \mathbf{X}\big) = (n-1)|\mathbf{X}|. \tag{8.2.2}$$

Proof. Observe that (8.2.2) follows from the $\mathbf{Y} = \mathbf{e}_{[1,n]}$ case of (8.2.1) (since then $\mathbf{Z} = \mathbf{X}$ and $|\mathbf{Y}| = n$), so it suffices to establish the latter. To that end, observe from the definition (1.1.1) of φ that

$$\begin{aligned} \varphi(\mathbf{X}, \mathbf{Y}) + \varphi(\mathbf{Y}, \mathbf{X}) &= \sum_{1 \leqslant i < j \leqslant n} X_i Y_j + \sum_{1 \leqslant j < i \leqslant n} X_i Y_j \\ &= \sum_{j=1}^{n} Y_j \Bigg(\sum_{i \neq j} X_i \Bigg) = \sum_{j=1}^{n} Y_j \big(|\mathbf{X}| - X_j \big) = |\mathbf{X}||\mathbf{Y}| - \sum_{j=1}^{n} X_j Y_j = |\mathbf{X}||\mathbf{Y}| - |\mathbf{Z}|, \end{aligned}$$

where in the last equality we used the fact that $Z_j = \min\{X_j, Y_j\} = X_j Y_j$ for each $j \in [1,n]$ (as $X_j, Y_j \in \{0,1\}$). $\square$

Now, the following three propositions address simplifications of W_z under certain relations between the three parameters $(r, s, z) \in \mathbb{C}$. The source of these simplifications comes from the three factors $(q^{-v} s^2 r^{-2} z; q)_{c-p}$, $(z; q)_{b-p-v}$, and $(q^{1-v} r^{-2} z; q)_v (q^v r^2 z^{-1}; q)_p$ appearing on the right side of (5.1.1). If $s^2 r^{-2} z = 1$, $z = 1$, or $r^2 z^{-1} = 1$, then the first, second, or third of these quantities will be supported on a single value of p, respectively,[1] thereby giving rise to a factored form for W_z.

We begin with the $s^2 r^{-2} z = 1$ specialization, which is similar to the q-Hahn type specialization originally explained in Proposition 6.7 of **[8]** and Proposition 6.11 of **[11]** for the $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ vertex model, and later in Proposition 7 of **[55]** for the more general $U_q\big(\widehat{\mathfrak{sl}}(n)\big)$ vertex model.

Proposition 8.2.2. *For any $s, z \in \mathbb{C}$, we have*[2]

$$W_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid s z^{1/2}, s) = s^{2(b-c)} q^{\varphi(\mathbf{B}-\mathbf{C}, \mathbf{C})} \frac{(s^2; q)_c (z; q)_{b-c}}{(s^2 z; q)_b} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{B} \geqslant \mathbf{C}}. \tag{8.2.3}$$

Proof. We may assume in what follows that $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$, for otherwise both sides of (8.2.3) are equal to 0 by arrow conservation. We may further assume by Lemma 7.5.1 that $\mathbf{B} \geqslant \mathbf{C}$.

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[1] For the first quantity $(q^{-v} s^2 r^{-2}; q)_{c-p}$, this follows from the fact that we additionally have $c - v \geqslant p$.

[2] In the below, and throughout this text, quantities such as $z^{1/2}$ will not depend on the choice for the root of z.

Inserting $r = sz^{1/2}$ into (5.1.1), we have that

(8.2.4)
$$\begin{aligned}W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sz^{1/2}; s) = (-1)^v z^{c+d-a-b} s^{2c+2d-2a} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+cv} \\ \times \frac{(q^{1-v}s^{-2};q)_v}{(q^{-v};q)_v} \frac{(s^2 z;q)_d}{(s^2 z;q)_b} \sum_{p=0}^{\min\{b-v,c-v\}} \frac{(q^{-v};q)_{c-p}(q^v s^2;q)_p (z;q)_{b-p-v}}{(s^2 z;q)_{c+d-p-v}} \\ \times q^{-pv} s^{-2p} \sum_{\boldsymbol{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})},\end{aligned}$$

where the sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$, for each $i \in [1,n]$. As indicated at the end of the proof of Lemma 7.5.1, the sum on the right side of (8.2.4) is supported on the term $\mathbf{P} = \mathbf{C} - \mathbf{V}$. Inserting this into (8.2.4), and using the fact that $a + b = c + d$, yields

$$\begin{aligned}W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sz^{1/2}; s) = (-1)^v s^{2b-2c+2v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+v^2} \frac{(q^{1-v}s^{-2};q)_v}{(s^2 z;q)_b} \\ \times (q^v s^2;q)_{c-v} (z;q)_{b-c} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{C}+\mathbf{V},\mathbf{C}-\mathbf{V})}.\end{aligned}$$

Using the identities $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ and

$$(q^{1-v}s^{-2};q)_v = (-1)^v q^{-\binom{v}{2}} s^{-2v} (s^2;q)_v,$$

it follows that

$$\begin{aligned}W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sz^{1/2}; s) = s^{2b-2c} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+v^2} \frac{(z;q)_{b-c}}{(s^2 z;q)_b} \\ \times (s^2;q)_v (q^v s^2;q)_{c-v} q^{\varphi(\mathbf{V}-\mathbf{A},\mathbf{C}-\mathbf{V})-\binom{v}{2}}.\end{aligned} \tag{8.2.5}$$

Next, by the bilinearity of φ, we have that

(8.2.6)
$$\begin{aligned}\varphi(\mathbf{D}-\mathbf{V},\mathbf{C}) + \varphi(\mathbf{V},\mathbf{A}) + \varphi(\mathbf{V}-\mathbf{A},\mathbf{C}-\mathbf{V}) &= \varphi(\mathbf{D}-\mathbf{A},\mathbf{C}) + \varphi(\mathbf{V},\mathbf{A}) - \varphi(\mathbf{V},\mathbf{V}) + \varphi(\mathbf{A},\mathbf{V}) \\ &= \varphi(\mathbf{B}-\mathbf{C},\mathbf{C}) + av - v - \binom{v}{2},\end{aligned}$$

where in the second statement we have used the facts that $\mathbf{A}+\mathbf{B} = \mathbf{C}+\mathbf{D}$, that $\varphi(\mathbf{V},\mathbf{V}) = \binom{v}{2}$, and that $\varphi(\mathbf{A},\mathbf{V}) + \varphi(\mathbf{V},\mathbf{A}) = av - v$ (where the last statement holds by Lemma 8.2.1, since $\mathbf{V} \leqslant \mathbf{A}$).

Now the proposition follows from inserting (8.2.6) into (8.2.5) and using the identities $v^2 = 2\binom{v}{2} + v$ and $(s^2;q)_v (q^v s^2;q)_{c-v} = (s^2;q)_c$. $\square$

Next we consider the case $z = 1$.

Proposition 8.2.3. *For any $r, s \in \mathbb{C}$, we have*

$$W_1(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = (r^{-2}s^2)^d (r^2;q)_d \frac{(r^{-2}s^2;q)_{c-b}}{(s^2;q)_{c+d-b}} q^{\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{C} \geqslant \mathbf{B}}. \tag{8.2.7}$$

Proof. We may again assume in what follows that $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$, for otherwise arrow conservation implies $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = 0$ for any $r, s, z \in \mathbb{C}$. Then setting $z = 1$ in (5.1.1), we find that the factor $(z;q)_{b-p-v} = \mathbf{1}_{p=b-v}$ on the right side there implies that the sum is supported on the term $\mathbf{P} = \mathbf{B} - \mathbf{V}$. In particular, since we must have $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$, this implies that $W_1(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r,s)=0$ unless $\mathbf{C}\geqslant\mathbf{B}$. Thus, we will assume in in what follows that $\mathbf{C}\geqslant\mathbf{B}$, for otherwise both sides of (8.2.7) are equal to 0.

Then inserting $z=1$ into (5.1.1) (and using that it is supported on $\mathbf{P}=\mathbf{B}-\mathbf{V}$) yields

$$\begin{aligned}W_1(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r,s)&=(-1)^v r^{2c-2a}s^{2d}q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+cv}\frac{(q^{1-v}r^{-2};q)_v}{(q^{-v}s^2r^{-2};q)_v}\frac{(r^2;q)_d}{(r^2;q)_b}\\&\quad\times\frac{(q^{-v}s^2r^{-2};q)_{c+v-b}(q^vr^2;q)_{b-v}}{(s^2;q)_{c+d-b}}(q^{-v}r^{-2})^{b-v}q^{\varphi(\mathbf{V}-\mathbf{D},\mathbf{B}-\mathbf{V})}.\end{aligned}$$

Using the identities

$$\begin{aligned}&\frac{(q^{1-v}r^{-2};q)_v(q^vr^2;q)_{b-v}}{(r^2;q)_b}=\frac{(q^{1-v}r^{-2};q)_v}{(r^2;q)_v}=(-1)^vr^{-2v}q^{-\binom{v}{2}};\\&\frac{(q^{-v}s^2r^{-2};q)_{c+v-b}}{(q^{-v}s^2r^{-2};q)_v}=(r^{-2}s^2;q)_{c-b},\end{aligned}$$

it follows that

$$\begin{aligned}W_1(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r,s)&=r^{2c-2a-2b}s^{2d}q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\varphi(\mathbf{V}-\mathbf{D},\mathbf{B}-\mathbf{V})-av-bv+cv+v^2-\binom{v}{2}}\\&\quad\times(r^2;q)_d\frac{(r^{-2}s^2;q)_{c-b}}{(s^2;q)_{c+d-b}}.\end{aligned}\tag{8.2.8}$$

Next, by the bilinearity of φ, we find that

$$\begin{aligned}&\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\varphi(\mathbf{V}-\mathbf{D},\mathbf{B}-\mathbf{V})\\&\qquad=\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})+\varphi(\mathbf{V},\mathbf{A}+\mathbf{B}-\mathbf{C})+\varphi(\mathbf{D},\mathbf{V})-\varphi(\mathbf{V},\mathbf{V})\\&\qquad=\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})+\varphi(\mathbf{V},\mathbf{D})+\varphi(\mathbf{D},\mathbf{V})-\varphi(\mathbf{V},\mathbf{V}),\end{aligned}\tag{8.2.9}$$

where in the last equality we used the fact that $\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}$. Since $\mathbf{V}\leqslant\mathbf{D}$, Lemma 8.2.1 implies that $\varphi(\mathbf{V},\mathbf{D})+\varphi(\mathbf{D},\mathbf{V})=|\mathbf{D}||\mathbf{V}|-|\mathbf{V}|=dv-v$. Inserting this and the fact that $\varphi(\mathbf{V},\mathbf{V})=\binom{v}{2}$ into (8.2.9) yields

$$\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\varphi(\mathbf{V}-\mathbf{D},\mathbf{B}-\mathbf{V})=\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})+dv-v-\binom{v}{2}.$$

The proposition then follows from inserting this into (8.2.8) and applying the identities $a+b=c+d$ and $v^2=2\binom{v}{2}+v$. $\square$

Now we address the specialization $r^2z^{-1}=1$.

Proposition 8.2.4. *For any $s,z\in\mathbb{C}$, we have*

$$W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid z^{1/2},s)=s^{2d}q^{\varphi(\mathbf{D},\mathbf{C})}\frac{(s^2;q)_c(z;q)_d}{(s^2z;q)_{c+d}}\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{v=0}.$$

Proof. As in the proofs of Proposition 8.2.2 and Proposition 8.2.3, we may assume that $\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}$. Then, setting $r=z^{1/2}$ in (5.1.1) yields

$$\begin{aligned}W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid z^{1/2},s)&=(-1)^vz^{c+d-a-b}s^{2d}q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})-av+cv}\frac{(q^{1-v};q)_v}{(q^{-v}s^2;q)_v}\frac{(z;q)_d}{(z;q)_b}\\&\quad\times\sum_{p=0}^{\min\{b-v,c-v\}}\frac{(q^{-v}s^2;q)_{c-p}(q^v;q)_p(z;q)_{b-p-v}}{(s^2z;q)_{c+d-p-v}}q^{-vp}\sum_{\mathbf{P}}q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})},\end{aligned}$$

where the last sum sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$ for each $i \in [1,n]$. Since $(q^{1-v}; q)_v = \mathbf{1}_{v=0}$ and since $a+b = c+d$ (as $\mathbf{A}+\mathbf{B} = \mathbf{C}+\mathbf{D}$), it follows that

$$W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid z^{1/2}, s) = s^{2d} q^{\varphi(\mathbf{D},\mathbf{C})} \frac{(z;q)_d}{(z;q)_b} \mathbf{1}_{v=0} \sum_{p=0}^{\min\{b,c\}} \frac{(s^2;q)_{c-p} (q^v;q)_p (z;q)_{b-p}}{(s^2 z;q)_{c+d-p}} \times \sum_{\mathbf{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})}.$$

As $\mathbf{1}_{v=0}(q^v;q)_p = \mathbf{1}_{v=0}\mathbf{1}_{p=0}$, it follows that the above sum is supported on the term $\mathbf{P} = \mathbf{e}_0$, from which we deduce the proposition. $\square$

A fourth simplification arises in the limiting regime when (s,z) tends to $(0,\infty)$ in such a way that $x = sz$ is fixed. Ensuring the existence of this limit will require normalizing the weight by a factor of $(-s)^{-d}$.

Proposition 8.2.5. *For any fixed $x \in \mathbb{C}$, we have*

(8.2.10)
$$\lim_{s\to 0} (-s)^{-d} W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = x^d (-r^2)^{c-a-v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C}) + \varphi(\mathbf{V},\mathbf{A}) + \binom{b}{2} - dv + v} \frac{(r^2;q)_d}{(r^2;q)_b} \times \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \prod_{j: B_j - D_j = 1} \big(1 - r^{-2} q^{-B_{[j+1,n]} - D_{[1,j-1]}}\big).$$

Proof. As in the proof of Proposition 8.2.2, we may assume in what follows that $\mathbf{A}+\mathbf{B} = \mathbf{C}+\mathbf{D}$. Then, replacing the z in Definition 5.1.1 with $s^{-1}x$ yields

(8.2.11)
$$(-s)^d W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = (-1)^{v+d} s^b x^{d-b} r^{2c-2a} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C}) + \varphi(\mathbf{V},\mathbf{A}) - av + cv} \frac{(q^{1-v} r^{-2} s^{-1} x; q)_v}{(q^{-v} s r^{-2} x; q)_v} \times \frac{(r^2;q)_d}{(r^2;q)_b} \sum_{p=0}^{\min\{b-v,c-v\}} \frac{(q^{-v} s r^{-2} x; q)_{c-p} (q^v r^2 s x^{-1}; q)_p}{(sx;q)_{c+d-p-v}} \times (q^{-v} r^{-2} s^{-1} x)^p (s^{-1}x;q)_{b-p-v} \sum_{\mathbf{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})},$$

where the sum is over all n-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \{0,1\}^n$ such that $|\mathbf{P}| = p$ and $P_i \leqslant \min\{B_i - V_i, C_i - V_i\}$ for each $i \in [1,n]$. Next, observe that

$$\lim_{s\to 0} (-s)^v (q^{1-v} r^{-2} s^{-1} x; q)_v = q^{-\binom{v}{2}} r^{-2v} x^v;$$
$$\lim_{s\to 0} s^{b-p-v} (s^{-1}x;q)_{b-p-v} = q^{\binom{b-p-v}{2}} (-1)^{b-p-v} x^{b-p-v},$$

which upon insertion into (8.2.11) gives

(8.2.12)
$$\lim_{s\to 0} (-s)^d W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r, s) = (-1)^{v+b+d} x^d r^{2c-2a-2v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C}) + \varphi(\mathbf{V},\mathbf{A}) - av + cv - \binom{v}{2}} \times \frac{(r^2;q)_d}{(r^2;q)_b} \sum_{p=0}^{\min\{b-v,c-v\}} (-r^{-2})^p q^{\binom{b-p-v}{2} - pv} \sum_{\mathbf{P}} q^{\varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P})}.$$

By the bilinearity of φ and the fact that $\varphi(\mathbf{X},\mathbf{X}) = \binom{x}{2}$ whenever $|\mathbf{X}| = x$, we have

$$\begin{aligned}\binom{b-p-v}{2} - \binom{v}{2} + \varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P}) \\ &= \varphi(\mathbf{B}-\mathbf{P}-\mathbf{V},\mathbf{B}-\mathbf{P}-\mathbf{V}) - \varphi(\mathbf{V},\mathbf{V}) + \varphi(\mathbf{B}-\mathbf{D}-\mathbf{P},\mathbf{P}) \\ &= \varphi(\mathbf{B}-\mathbf{P},\mathbf{B}-\mathbf{P}) - \varphi(\mathbf{B}-\mathbf{P},\mathbf{V}) - \varphi(\mathbf{V},\mathbf{B}-\mathbf{P}) + \varphi(\mathbf{B}-\mathbf{P},\mathbf{P}) - \varphi(\mathbf{D},\mathbf{P}) \\ &= \binom{b}{2} - \varphi(\mathbf{P},\mathbf{B}) - (b-p-1)v - \varphi(\mathbf{D},\mathbf{P}),\end{aligned}$$

where in the last equality we also applied Lemma 8.2.1 (with the fact that $\mathbf{P} \leqslant \mathbf{B}-\mathbf{V}$). Upon insertion into (8.2.12) and using the fact that $a+b = c+d$, this yields

$$\begin{aligned}\lim_{s\to 0}(-s)^{-d} W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) = x^d(-r^2)^{c-a-v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\binom{b}{2}-dv+v} \frac{(r^2;q)_d}{(r^2;q)_b} \\ \times \sum_{p=0}^{\min\{b-v,c-v\}} (-r^{-2})^p \sum_{\mathbf{P}} q^{-\varphi(\mathbf{P},\mathbf{B})-\varphi(\mathbf{D},\mathbf{P})}.\end{aligned} \tag{8.2.13}$$

Next, observe again by the bilinearity of φ and Lemma 8.2.1, applied with the $(\mathbf{X},\mathbf{Y})$ there equal to $(\mathbf{P},\mathbf{V})$ here, gives

$$\begin{aligned}\varphi(\mathbf{P},\mathbf{B}) + \varphi(\mathbf{D},\mathbf{P}) &= \varphi(\mathbf{P},\mathbf{B}-\mathbf{V}) + \varphi(\mathbf{D}-\mathbf{V},\mathbf{P}) + \varphi(\mathbf{P},\mathbf{V}) + \varphi(\mathbf{V},\mathbf{P}) \\ &= \varphi(\mathbf{P},\mathbf{B}-\mathbf{V}) + \varphi(\mathbf{D}-\mathbf{V},\mathbf{P}) + pv,\end{aligned}$$

where we have further used the fact that the $\mathbf{Z}$ in Lemma 8.2.1 is equal to $\mathbf{e}_0$ here since $\mathbf{P} \leqslant \mathbf{B}-\mathbf{V}$. Inserting this into (8.2.13) then gives

$$\begin{aligned}\lim_{s\to 0}(-s)^{-d} W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) = x^d(-r^2)^{c-a-v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\binom{b}{2}-dv+v} \frac{(r^2;q)_d}{(r^2;q)_b} \\ \sum_{p=0}^{\min\{b-v,c-v\}} (-q^{-v}r^{-2})^p \sum_{\mathbf{P}} q^{\varphi(\mathbf{P},\mathbf{V}-\mathbf{B})+\varphi(\mathbf{V}-\mathbf{D},\mathbf{P})}.\end{aligned} \tag{8.2.14}$$

Now observe for any $w \in \mathbb{C}$ and $\mathbf{X},\mathbf{Y},\mathbf{Z} \in \{0,1\}^n$ (whose coordinates are indexed by $[1,n]$) that

$$\sum_{\substack{\mathbf{P}\in\{0,1\}^n \\ \mathbf{P}\leqslant\mathbf{Z}}} w^{|\mathbf{P}|} q^{-\varphi(\mathbf{P},\mathbf{X})-\varphi(\mathbf{Y},\mathbf{P})} = \prod_{j:Z_j=1} \big(1 + q^{-X_{[j+1,n]}-Y_{[1,j-1]}} w\big), \tag{8.2.15}$$

which holds since (1.1.1) implies $\varphi(\mathbf{P},\mathbf{X}) = \sum_{j:P_j=1}^n X_{[j+1,n]}$ and $\varphi(\mathbf{Y},\mathbf{P}) = \sum_{j:P_j=1} Y_{[1,j-1]}$. Applying (8.2.14), together with the

$$(w,\mathbf{X},\mathbf{Y},\mathbf{Z}) = \big(-q^{-v}r^{-2}, \mathbf{B}-\mathbf{V}, \mathbf{D}-\mathbf{V}, \min\{\mathbf{B}-\mathbf{V},\mathbf{C}-\mathbf{V}\}\big)$$

case of (8.2.15) (where $\min\{\mathbf{B}-\mathbf{V},\mathbf{C}-\mathbf{V}\} \in \{0,1\}^n$ denotes the entrywise minimum of $\mathbf{B}-\mathbf{V}$ and $\mathbf{C}-\mathbf{V}$) yields

$$\begin{aligned}\lim_{s\to 0}(-s)^{-d} W_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s) = x^d(-r^2)^{c-a-v} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\binom{b}{2}-dv+v} \frac{(r^2;q)_d}{(r^2;q)_b} \\ \times \prod_{j:\min\{B_j-V_j,C_j-V_j\}=1} \big(1 - r^{-2}q^{-B_{[j+1,n]}-D_{[1,j-1]}}\big),\end{aligned}$$

where we have used the fact that $V_{[1,j-1]} + V_{[j+1,n]} = |\mathbf{V}| - V_j = v$ holds for any $j \in [0, n]$ with $B_j - V_j = 1$ (as $B_j, V_j \in \{0,1\}$). Now the lemma follows from this, together with the fact that $\min\{B_j - V_j, C_j - V_j\} = 1$ holds if and only if $(B_j, C_j, V_j) = (1,1,0)$, which in turn holds if and only if $B_j - D_j = 1$ (as $A_j + B_j = C_j + D_j$ and $A_j, B_j, C_j, D_j \in \{0,1\}$). $\square$

Remark 8.2.6. It is quickly verified that the right side of (8.2.10) is a polynomial in r^{-2}, as the potential denominator in the term $(r^2;q)_b^{-1}(r^2;q)_d$ there can be shown to be canceled by (part of) the product $\prod_{j:B_j-D_j=1}\big(1 - r^{-2}q^{-B_{[j+1,n]}-D_{[1,j-1]}}\big)$.

8.3. Limit Degenerations

In this section we consider the special cases of the W_z weights given by Proposition 8.2.2, Proposition 8.2.3, and Proposition 8.2.4, and consider their limits as some of the parameters (r, s, z) tend to 0 or ∞. Although there are in principle many possible choices for these limit degenerations, we only consider two in each case; the first (given by Corollary 8.3.1, Corollary 8.3.3, and Corollary 8.3.5 below) applies the limit as one parameter tends to 0 or ∞, and the second (given by Corollary 8.3.2, Corollary 8.3.4, and Corollary 8.3.6 below) applies it as two do. We will later see as Theorem 9.3.2 below that, under the latter three double limits, the F, G, H functions from Definition 7.1.1 degenerate to the Lascoux–Leclerc–Thibon (LLT) polynomials. Throughout this section we again adopt the notation of Definition 5.1.1.

We begin by degenerating Proposition 8.2.2, first when s tends to ∞ and then when z tends to 0.

Corollary 8.3.1. *For any fixed $z \in \mathbb{C}$, we have*

$$\lim_{s\to\infty} W_z(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \mid sz^{1/2}; s) = (-1)^{b-c} q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})} z^{-b} (z;q)_{b-c} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}}.$$

Proof. Since for any fixed $z \in \mathbb{C}$ we have that

$$\lim_{s\to\infty} s^{2b-2c} \frac{(s^2;q)_c}{(s^2z;q)_b} = (-1)^{b-c} z^{-b} q^{\binom{c}{2}-\binom{b}{2}},$$

and, by the bilinearity of φ,

$$\varphi(\mathbf{B}-\mathbf{C},\mathbf{C}) + \binom{c}{2} - \binom{b}{2} = \varphi(\mathbf{B}-\mathbf{C},\mathbf{C}) + \varphi(\mathbf{C},\mathbf{C}) - \varphi(\mathbf{B},\mathbf{B}) = \varphi(\mathbf{B},\mathbf{C}-\mathbf{B}),$$

the corollary follows from Proposition 8.2.2. $\square$

Corollary 8.3.2. *For any fixed $x \in \mathbb{C}$, we have*

(8.3.1)

$$\lim_{y\to\infty} y^{-b}\bigg(\lim_{s\to\infty} W_{x/y}\big(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sx^{1/2}; sy^{1/2}\big)\bigg) = (-1)^{b-c} x^{-b} q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}}.$$

Proof. This follows by setting (s, z) to $\big(sy^{1/2}, \frac{x}{y}\big)$ in Corollary 8.3.1 and letting y tend to ∞. $\square$

Next we degenerate Proposition 8.2.3, first when $(r, s) = (sx^{1/2}, sy^{1/2})$ and s tends to 0, and then when y tends to 0.

Corollary 8.3.3. *For any fixed $r \in \mathbb{C}$, we have*

$$\lim_{s\to 0} W_1\big(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sx^{1/2}, sy^{1/2}\big) = x^{-d} y^{d} (x^{-1}y;q)_{c-b} q^{\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{C}\geqslant\mathbf{B}}.$$

Proof. This follows by setting (r,s) to $(sx^{1/2}, sy^{1/2})$ in Proposition 8.2.3 and letting s tend to 0. $\square$

Corollary 8.3.4. *For any fixed $z \in \mathbb{C}$, we have*

$$\lim_{y\to 0} y^{-d}\bigg(\lim_{s\to 0} W_1(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid sx^{1/2}, sy^{1/2})\bigg) = x^{-d} q^{\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathbf{C}\geqslant\mathbf{B}}. \tag{8.3.2}$$

Proof. This follows by letting y tend to 0 in Corollary 8.3.3. $\square$

Now we degenerate Proposition 8.2.4, first when s tends to 0 and then when y tends to 0.

Corollary 8.3.5. *For any fixed $z \in \mathbb{C}$, we have*

$$\lim_{s\to 0} s^{-2d} W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid z^{1/2}; s) = q^{\varphi(\mathbf{D},\mathbf{C})} (z;q)_d \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{v=0}.$$

Proof. This follows by letting s tend to 0 in Proposition 8.2.4. $\square$

Corollary 8.3.6. *For any fixed $x \in \mathbb{C}$, we have*

$$\lim_{y\to 0} (-y)^{d}\bigg(\lim_{s\to 0} s^{-2d} W_{x/y}\big(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid (xy^{-1})^{1/2}, s\big)\bigg) = x^{d} q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{v=0}. \tag{8.3.3}$$

Proof. This follows by setting $z = \frac{x}{y}$ in Corollary 8.3.5, letting y tend to 0, the fact that

$$\lim_{y\to 0} (-y)^d (xy^{-1};q)_d = x^d q^{\binom{d}{2}} = x^d q^{\varphi(\mathbf{D},\mathbf{D})},$$

and the bilinearity of φ. $\square$

Remark 8.3.7. It is quickly verified that one obtains the same results as in Corollary 8.3.2, Corollary 8.3.4, and Corollary 8.3.6 by taking the double limits of (5.1.1) in y and s simultaneously, assuming s^2y^{-1} tends to ∞ in Corollary 8.3.2 or that it tends to 0 in Corollary 8.3.4 and Corollary 8.3.6.

Before proceeding, it will be useful to provide one further degeneration of the fused weights $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ from Definition 5.1.1. It corresponds to the limit of the weight from from Proposition 8.2.4 as s^2 tends to 0 and z tends ∞ simultaneously, so that s^2z remains fixed; this weight will be useful in Section 10.4 and Chapter 12 below.

Corollary 8.3.8. *For any fixed $z \in \mathbb{C}$, we have*

$$\lim_{s\to 0} W_{z/s^2}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid s^{-1}z^{1/2}; s) = q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})} \frac{(-z)^d}{(z;q)_{c+d}} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{v=0}.$$

Proof. By Proposition 8.2.4, we have that

$$W_{z/s^2}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid s^{-1}z^{1/2}; s) = s^{2d} q^{\varphi(\mathbf{D},\mathbf{C})} \frac{(s^2;q)_c (s^{-2}z;q)_d}{(z;q)_{c+d}} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{v=0}.$$

Letting s tend to 0, using the fact that

$$\lim_{s\to 0} s^{2d} (s^{-2}z;q)_d = (-z)^d q^{\binom{d}{2}} = (-z)^d q^{\varphi(\mathbf{D},\mathbf{D})},$$

and applying the bilinearity of φ, we deduce the corollary. $\square$

8.4. Limiting Weights and Degenerated Functions

In this section we introduce notation for the weights and functions degenerated according to the limits considered in Section 8.3. In view of Corollary 8.3.1, Corollary 8.3.2, Corollary 8.3.4, Corollary 8.3.6, Corollary 8.3.8, and Proposition 8.2.5, we define the following limiting weights.

Definition 8.4.1. If $\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}$, then set

$$\begin{aligned}
\mathcal{W}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,\infty) &= (-1)^{b-c}x^{-b}y^{b}(xy^{-1};q)_{b-c}q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})}\mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}};\\
\mathcal{W}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,\infty) &= (-1)^{b-c}x^{-b}q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})}\mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}};\\
\mathcal{W}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid 0,0) &= x^{-d}q^{\varphi(\mathbf{D},\mathbf{C}-\mathbf{B})}\mathbf{1}_{\mathbf{C}\geqslant\mathbf{B}};\\
\mathcal{W}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0) &= x^{d}q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})}\mathbf{1}_{v=0};\\
\mathcal{W}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0) &= q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})}\frac{(-x)^{d}y^{-d}}{(xy^{-1};q)_{c+d}}\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{v=0};\\
\mathcal{W}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r) &= x^{d}(-r^{2})^{c-a-v}q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})+\varphi(\mathbf{V},\mathbf{A})+\binom{b}{2}-dv+v}\frac{(r^{2};q)_{d}}{(r^{2};q)_{b}}\\
&\qquad\times\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\prod_{j:B_j-D_j=1}\big(1-r^{-2}q^{-B_{[j+1,n]}-D_{[1,j-1]}}\big),
\end{aligned}\tag{8.4.1}$$

and

$$\begin{aligned}
\widehat{\mathcal{W}}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,\infty) &= (-1)^{b-c-n}x^{n-b}y^{b-n}\frac{(xy^{-1};q)_{b-c}}{(xy^{-1};q)_{n}}q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})+\binom{n}{2}}\mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}};\\
\widehat{\mathcal{W}}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,\infty) &= (-1)^{b-c-n}x^{n-b}q^{\varphi(\mathbf{B},\mathbf{C}-\mathbf{B})+\binom{n}{2}}\mathbf{1}_{\mathbf{B}\geqslant\mathbf{C}};\\
\widehat{\mathcal{W}}_{x}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0) &= x^{d-n}q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})-\binom{n}{2}}\mathbf{1}_{v=0},\\
\widehat{\mathcal{W}}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0) &= q^{\varphi(\mathbf{D},\mathbf{C}+\mathbf{D})-\binom{n}{2}}\frac{(-x)^{d-n}y^{n-d}(xy^{-1};q)_{n}}{(xy^{-1};q)_{c+d}}\mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{v=0},
\end{aligned}\tag{8.4.2}$$

Otherwise, set all of these quantities to 0.

Observe that the definitions (8.4.1) and (8.4.2) are consistent with (5.1.4). We did not introduce $\widehat{\mathcal{W}}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid 0,0)$, since it is ill-defined as $\mathcal{W}_x\big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]}\mid 0,0\big)=0$ (we also did not introduce $\widehat{\mathcal{W}}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r)$ since we will not use it).

The next definition provides the corresponding limit degenerations for the functions $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, $F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, and $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ from Definition 7.1.1. There are various additional degenerations (using all possible weights from Definition 8.4.1) that could be defined and considered, but here we only introduce the ones we will use in this text. In what follows, for any complex number $t\in\mathbb{C}$ and (possibly infinite) sequence $\mathbf{z}=(z_1,z_2,\ldots,z_k)\subset\mathbb{C}$, we define the sequence $t\mathbf{z}=(tz_1,tz_2,\ldots,tz_k)\subset\mathbb{C}$.

Definition 8.4.2. Fix integers $M\geqslant 0$ and $N\geqslant 1$; a complex number $r\in\mathbb{C}$; a finite sequence $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ of complex numbers; and an infinite sequence $\mathbf{y}=(y_1,y_2,\ldots)$ of complex numbers. For any sequences of n signatures $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, set

(8.4.3)

$$\begin{aligned}
\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) &= \lim_{s \to \infty} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; s\mathbf{x}^{1/2} \mid \mathbf{y}; s\mathbf{y}^{1/2}); \\
\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0; \mathbf{x} \mid 0; 0) &= \lim_{y \to 0} y^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|} \bigg(\lim_{s \to 0} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big((y, y, \ldots); s\mathbf{x}^{1/2} \mid (y, y, \ldots); s(y^{1/2}, y^{1/2}, \ldots)\big) \bigg); \\
\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) &= \lim_{y \to 0} (-y)^{|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|} \bigg(\lim_{s \to 0} s^{2|\boldsymbol{\mu}| - 2|\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}; y^{-1/2}\mathbf{x}^{1/2} \mid (y, y, \ldots); (s, s, \ldots)\big) \bigg); \\
\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0) &= \lim_{s \to 0} (-s)^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}; (r, r, \ldots) \mid (s, s, \ldots); (s, s, \ldots)\big).
\end{aligned}$$

Moreover, for any sequences of n signatures $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, set

(8.4.4)

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) &= \lim_{s \to \infty} H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; s\mathbf{x}^{1/2} \mid \mathbf{y}; s\mathbf{y}^{1/2}); \\
\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty) &= \lim_{y \to \infty} y^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}| + n\binom{M+1}{2} - n\binom{M+N+1}{2}} \\
&\qquad \times \bigg(\lim_{s \to \infty} H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}; s^{1/2}\mathbf{x} \mid (y, y, \ldots); s^{1/2}(y, y, \ldots)\big) \bigg); \\
\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) &= \lim_{y \to 0} (-y)^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}| + n\binom{M}{2} - n\binom{M+N}{2}} \\
&\qquad \times \bigg(\lim_{s \to 0} s^{2(|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| - n\binom{M}{2} + n\binom{M+N}{2})} F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}; y^{-1/2}\mathbf{x}^{1/2} \mid (y, y, \ldots); (s, s, \ldots)\big) \bigg); \\
\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0) &= \lim_{s \to 0} F_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\Big(\mathbf{x}; \big(s^{-1}(x_1 y_1^{-1})^{1/2}, s^{-1}(x_2 y_2^{-1})^{1/2}, \ldots\big) \mid s^2\mathbf{y}; (s, s, \ldots)\Big).
\end{aligned}$$

Remark 8.4.3. Let us briefly account for the powers of y (and s) appearing in the limits on the right sides of (8.4.3) and (8.4.4), taking $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty)$ as an example. As indicated in (7.1.3), $H_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ is the partition function under the weights $W_{x/y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s)$ for the vertex model $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ from Definition 7.1.3. By Corollary 8.3.2, the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty)$ are obtained as the limit of $y^{-b} W_{x/y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid sx^{1/2}, sy^{1/2})$ as s and y tend to ∞; in particular, a factor of y^{-b} is required as a normalization in these weights. Thus, in view of (7.1.3) (and recalling the notation around there), a normalization of $y^{-\xi(\boldsymbol{\lambda}/\boldsymbol{\mu})}$ is required in the definition of $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty)$, where $\xi(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is the common value over all $\mathcal{E} \in \mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ of

$$\begin{aligned}
\xi(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \sum_{(i,j) \in \mathcal{D}} \big|\mathbf{B}(i,j)\big| &= \sum_{c=1}^{n} \bigg(\sum_{\mathfrak{l} \in \mathfrak{T}(\lambda^{(c)})} \mathfrak{l} - \sum_{\mathfrak{m} \in \mathfrak{T}(\mu^{(c)})} \mathfrak{m} \bigg) \\
&= \sum_{c=1}^{n} \bigg(\sum_{i=1}^{M+N} \big(\lambda_i^{(c)} + M + N - i + 1\big) - \sum_{i=1}^{M} \big(\mu_i^{(c)} + M - i + 1\big) \bigg) \\
&= |\boldsymbol{\lambda}| + \binom{M+N+1}{2} n - |\boldsymbol{\mu}| - \binom{M+1}{2} n,
\end{aligned}$$

where $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ denotes the arrow configuration at any vertex $v \in \mathcal{D}$ under $\mathcal{E}$. This indeed recovers the normalizing factor of y on the right side of the second equation of (8.4.4); similar statements hold for the remaining limits in Definition 8.4.2.

Remark 8.4.4. Observe that the degenerated functions from Definition 8.4.2 are partition functions for vertex models under the limiting weights from Definition 8.4.1. Indeed, let us extend the definitions of the weights $W(\mathcal{E} \mid \mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ and $\widehat{W}(\mathcal{E} \mid \mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$ from (7.1.2) to when $\mathbf{x},\mathbf{y},\mathbf{r},\mathbf{s} \in \{0,\infty\}$, so that for example

$$W(\mathcal{E} \mid \mathbf{x};\infty \mid \infty;\infty) = \prod_{(i,j)\in\mathcal{D}} \mathcal{W}_{x_i}\big(\mathbf{A}(i,j),\mathbf{B}(i,j);\mathbf{C}(i,j),\mathbf{D}(i,j) \mid \infty,\infty\big),$$

for any path ensemble $\mathcal{E}$ on $\mathcal{D}$ (recall (7.1.1)), where $\big(\mathbf{A}(v),\mathbf{B}(v);\mathbf{C}(v),\mathbf{D}(v)\big)$ denotes the arrow configuration at any vertex $v \in \mathcal{D}$ under $\mathcal{E}$. Then, (7.1.3) continues to hold for $\mathbf{x},\mathbf{r},\mathbf{s},\mathbf{y} \in \{0,\infty\}$.

8.5. Degenerations of Cauchy Identities

In this section we analyze the special cases of the Cauchy identity Theorem 7.3.3 under the degenerations considered in Section 8.2 and Section 8.3. Throughout this section, we fix integers $N,M \geqslant 1$; finite sequences of complex numbers $\mathbf{u} = (u_1,u_2,\ldots,u_N)$, $\mathbf{r} = (r_1,r_2,\ldots,r_N)$, $\mathbf{w} = (w_1,w_2,\ldots,w_M)$, and $\mathbf{t} = (t_1,t_2,\ldots,t_M)$; and infinite sequences of complex numbers $\mathbf{y} = (y_1,y_2,\ldots)$ and $\mathbf{s} = (s_1,s_2,\ldots)$.

There are in principle numerous such degenerations that can be analyzed, but let us only restrict our attention to two, corresponding to the sums over $\boldsymbol{\lambda}$ of $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{u};\infty \mid 0;0)\mathcal{G}_{\boldsymbol{\lambda}}(\mathbf{w};\infty \mid 0;0)$ and $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{u};\infty \mid 0;0)\mathcal{G}_{\boldsymbol{\lambda}}(0;\mathbf{w} \mid 0;0)$. We will later see that these correspond to the Cauchy identity and the dual Cauchy identity for the LLT polynomials, respectively.

Corollary 8.5.1. *If*

$$\max_{\substack{1\leqslant i\leqslant M\\ 1\leqslant j\leqslant N}} \max_{\substack{a,b\in[0,n]\\ (a,b)\neq(n,0)}} \big|q^{\binom{a}{2}+\binom{b}{2}-\binom{n}{2}} u_j^{a-n} w_i^b\big| < 1, \tag{8.5.1}$$

then

$$\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{u};\infty \mid 0;0)\mathcal{G}_{\boldsymbol{\lambda}}(\mathbf{w};\infty \mid 0;0) = q^{-\binom{n}{2}\binom{N}{2}} \prod_{i=1}^{M}\prod_{j=1}^{N} (u_j^{-1}w_i;q^{-1})_n^{-1} \prod_{j=1}^{N} u_j^{n(j-N)}. \tag{8.5.2}$$

Proof. Fix $s,y \in \mathbb{C}$, and denote the infinite sequences $\mathbf{s} = (s,s,\ldots)$ and $\mathbf{y} = (y,y,\ldots)$. Applying Theorem 7.3.3, we obtain

$$\begin{aligned}
\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} & (-y^{-1}s^2)^{|\boldsymbol{\lambda}|+n\binom{N}{2}} F_{\boldsymbol{\lambda}}\big(\mathbf{u};y^{-1/2}\mathbf{u}^{1/2} \mid \mathbf{y};\mathbf{s}\big)(-y^{-1}s^2)^{-|\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}}\big(\mathbf{w};y^{-1/2}\mathbf{w}^{1/2} \mid \mathbf{y};\mathbf{s}\big) \\
&= (-y^{-1}s^2)^{n\binom{N}{2}} \prod_{k=1}^{N} s^{2n(k-N)}(s^2;q)_n^{N-k} \prod_{j=1}^{N} (u_jy^{-1};q)_n^{j-N} \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{y^n(u_jy^{-1};q)_n}{w_i^n(u_jw_i^{-1};q)_n} \\
&= \prod_{k=1}^{N} (s^2;q)_n^{N-k} \prod_{j=1}^{N} (-y)^{n(j-N)}(u_jy^{-1};q)_n^{j-N} \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{y^n(u_jy^{-1};q)_n}{w_i^n(u_jw_i^{-1};q)_n},
\end{aligned} \tag{8.5.3}$$

assuming there exists an integer $K > 1$ and a real number $\varepsilon > 0$ such that

$$\begin{aligned}
&\sup_{k>K} \max_{\substack{1\leqslant i\leqslant M\\1\leqslant j\leqslant N}} \max_{\substack{a,b\in[0,n]\\(a,b)\neq(n,0)}} |s|^{2a+2b-2n} \max_{|\mathbf{B}|=a} \big|(-y^{-1}s^2)^{n-a}\widehat{W}_{u_j/y}(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0,\mathbf{B}\mid y^{-1/2}u_j^{1/2};s)\big| \\
&\qquad\qquad\qquad\times \max_{|\mathbf{B}|=b} \big|(-y^{-1}s^2)^{-b}W_{w_i/y}(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0,\mathbf{B}\mid y^{-1/2}w_i^{1/2};s)\big| \\
&\qquad= \sup_{k>K} \max_{\substack{1\leqslant i\leqslant M\\1\leqslant j\leqslant N}} \max_{\substack{a,b\in[0,n]\\(a,b)\neq(n,0)}} \left| y^{a+b-n}\frac{(s^2u_jy^{-1};q)_n(u_jy^{-1};q)_a}{(u_jy^{-1};q)_n(s^2u_jy^{-1};q)_a}\frac{(w_iy^{-1};q)_b}{(s^2w_iy^{-1};q)_b}\right| < 1-\varepsilon,
\end{aligned} \tag{8.5.4}$$

where the additional factors of $(-y^{-1}s^2)^{n-a}$ and $(-y^{-1}s^2)^{-b}$ arise due to the correpsonding gauge factors on the left side of (8.5.3), and we have used (5.1.3) and (5.1.4) in the first equality in (8.5.4). In particular, since $\lim_{y\to 0}(-y)^k(xy^{-1};q)_k = x^k q^{\binom{k}{2}}$ for any $x \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geqslant 0}$, (8.5.1) implies the existence of a constant $\delta = \delta(\varepsilon) > 0$ such that (8.5.4) holds whenever $|s|, |y|, |s^2y^{-1}| < \delta$.

Now let us show that the left side of (8.5.3) converges uniformly over $s, y \in \mathbb{C}$ with $|s| \leqslant |y| \leqslant \delta$ (so that we may take the limit there as s and y to 0). To that end, observe under (8.5.4) that

$$\begin{aligned}
&\Big|(-y^{-1}s^2)^{|\boldsymbol{\lambda}|+n\binom{N}{2}}F_{\boldsymbol{\lambda}}\big(\mathbf{u};y^{-1/2}\mathbf{u}^{1/2}\mid\mathbf{y};\mathbf{s}\big)(-y^{-1}s^2)^{-|\boldsymbol{\lambda}|}G_{\boldsymbol{\lambda}}\big(\mathbf{w};y^{-1/2}\mathbf{w}^{1/2}\mid\mathbf{y};\mathbf{s}\big)\Big| \\
&\qquad\qquad\qquad < C\binom{|\boldsymbol{\lambda}|+N}{N}^{2nN}(1-\varepsilon)^{2|\boldsymbol{\lambda}|-2nN^2},
\end{aligned} \tag{8.5.5}$$

for any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}$ and some constant $C > 0$ independent of $\boldsymbol{\lambda}$. Indeed, any path ensemble $\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\varnothing;N)$ (from Definition 7.1.3; see the left side of Figure 7.1), has at least $|\boldsymbol{\lambda}| - nN^2$ vertices with arrow configuration of the form $(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0;\mathbf{B})$ for some $\mathbf{B} \in \{0,1\}^n$ with $|\mathbf{B}| > 0$, as there exist at most nN^2 vertices at which any of the nN paths in the ensemble can go up. Similarly, any path ensemble $\mathcal{E} \in \mathfrak{P}_F(\boldsymbol{\lambda}/\varnothing)$ (from Definition 7.1.3; see the middle of Figure 7.1) has at least $|\boldsymbol{\lambda}| - nN^2$ vertices with arrow configuration of the form $(\mathbf{e}_0,\mathbf{B};\mathbf{e}_0,\mathbf{B})$ with $|\mathbf{B}| < n$. Moreover, there are at most $\binom{|\boldsymbol{\lambda}|+N}{N}$ choices for any of the nN paths in $\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\varnothing;N) \cup \mathfrak{P}_F(\boldsymbol{\lambda}/\varnothing)$, which gives $\big|\mathfrak{P}_G(\boldsymbol{\lambda}/\varnothing;N)\mathfrak{P}_F(\boldsymbol{\lambda}/\varnothing)\big| \leqslant \binom{|\boldsymbol{\lambda}|+N}{N}^{2nN}$. Thus, (8.5.5) follows from first applying (8.5.4) (which holds for $|s| \leqslant |y| \leqslant \delta$) to bound the contribution of weights to the right of the K-th column in any such vertex model (where K is from (8.5.4)), and using the constant C to account for the products of weights in the leftmost K columns.

This implies that the sum on the left side of (8.5.3) converges uniformly for $|s| \leqslant |y| \leqslant \delta$ (and fixed $n, N \geqslant 1$). So, first letting s tend to 0 and then letting y tend to 0 in (8.5.3), it follows that

$$\begin{aligned}
&\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{F}_{\boldsymbol{\lambda}}\big(\mathbf{u};\infty\mid 0;0\big)\mathcal{G}_{\boldsymbol{\lambda}}\big(\mathbf{w};\infty\mid 0;0\big) \\
&\qquad\qquad = \lim_{y\to 0}\prod_{j=1}^{N}(-y)^{n(j-N)}(u_jy^{-1};q)_n^{j-N}\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{y^n(u_jy^{-1};q)_n}{w_i^n(u_jw_i^{-1};q)_n}.
\end{aligned} \tag{8.5.6}$$

Thus, since

$$\lim_{y\to 0}(-y)^n(u_jy^{-1};q)_n = u_j^n q^{\binom{n}{2}}; \qquad (u_jw_i^{-1};q)_n = (-u_jw_i^{-1})^n q^{\binom{n}{2}}(u_j^{-1}w_i;q^{-1})_n, \tag{8.5.7}$$

we deduce (8.5.2) from (8.5.6). ☐

Corollary 8.5.2. *We have that*

$$\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} (-1)^{|\boldsymbol{\lambda}|} \mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{u}; \infty \mid 0; 0) \mathcal{G}_{\boldsymbol{\lambda}}(0; \mathbf{t} \mid 0; 0) = q^{-\binom{n}{2}\binom{N}{2}} \prod_{j=1}^{N} u_j^{n(j-N)} \prod_{i=1}^{M} \prod_{j=1}^{N} (t_i^{-1} u_j^{-1}; q^{-1})_n.$$

Proof. Fix a complex number $y \in \mathbb{C}$, and denote the infinite sequence $\mathbf{s} = (s, s, \ldots)$. Applying Theorem 7.3.3, we obtain

$$\begin{aligned} \sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} & (-1)^{|\boldsymbol{\lambda}|} (-s^{-2})^{-|\boldsymbol{\lambda}| - n\binom{N}{2}} F_{\boldsymbol{\lambda}}(\mathbf{u}; s^{-1} \mathbf{u}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) (s^2)^{-|\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}}(\mathbf{s}^2; s\mathbf{t}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) \\ &= (-s^2)^{-n\binom{N}{2}} (s^4; q)_n^{\binom{N}{2}} \prod_{j=1}^{N} (s^{-2} u_j; q)_n^{j-N} \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{(t_i u_j; q)_n}{s^{2n} t_i^n (s^{-2} u_j; q)_n}, \end{aligned} \tag{8.5.8}$$

where here we do not require the convergence condition since the sum on the left side of (8.5.8) is finite, as there are only finitely many $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;N}$ so that $G_{\boldsymbol{\lambda}}(\mathbf{s}^2; s\mathbf{t}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) \neq 0$. Indeed, to verify the latter point, observe by induction on N that if there exists some index pair $(i, j) \in [1, n] \times [1, N]$ such that $\lambda_j^{(i)} > n(N+1)$, then any path ensemble in the vertex model $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$ (from Definition 7.1.3; see also the left side of Figure 7.1) describing $G_{\boldsymbol{\lambda}}$ must contain a vertex of arrow configuration $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ with $B_i > C_i$ (where $\mathbf{B} = (B_1, B_2, \ldots, B_n) \in \{0, 1\}^n$ and $\mathbf{C} = (C_1, C_2, \ldots, C_n) \in \{0, 1\}^n$). By Proposition 8.2.3, the weight under W_1 of such a vertex is 0, and so $G_{\boldsymbol{\lambda}}(\mathbf{s}^2; s\mathbf{t}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) = 0$ for all but finitely many $\boldsymbol{\lambda}$.

Now by Corollary 8.3.4, Corollary 8.3.6, Remark 8.3.7, Definition 8.4.1, and Definition 8.4.2, we deduce

$$\lim_{s \to 0} (-s^{-2})^{-|\boldsymbol{\lambda}| - n\binom{N}{2}} F_{\boldsymbol{\lambda}}(\mathbf{u}; s^{-1} \mathbf{u}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) = \mathcal{F}_{\lambda}(\mathbf{u}; \infty \mid 0; 0);$$
$$\lim_{s \to 0} (s^2)^{-|\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}}(\mathbf{s}^2; s\mathbf{t}^{1/2} \mid \mathbf{s}^2; \mathbf{s}^2) = \mathcal{G}_{\boldsymbol{\lambda}}(0; \mathbf{t} \mid 0; 0).$$

This, letting s tend to 0 in (8.5.8), and the facts that

$$\lim_{s \to 0} s^{2n} (s^{-2} u_j; q)_n = (-u_j)^n q^{\binom{n}{2}}; \qquad (t_i u_j; q)_n = q^{\binom{n}{2}} (-t_i u_j)^n (t_i^{-1} u_j^{-1}; q^{-1})_n,$$

then together imply the corollary. $\square$

CHAPTER 9

Degeneration to the LLT Polynomials

In this chapter we explain through Theorem 9.3.2 below (whose proof will appear in Chapter 10) how some of the degenerations of the G, F, and H functions from Definition 8.4.2 can be matched with the Lascoux–Leclerc–Thibon polynomials originally introduced in [**62**].

9.1. The LLT Polynomials

In this section we recall the definitions of the Lascoux–Leclerc–Thibon (LLT) polynomials introduced in [**62**]. Although these polynomials are typically indexed by partitions, here we will index them by signatures $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{nN}) \in \mathrm{Sign}_{nN}$, where $n, N \geqslant 1$ are integers that will be fixed throughout this section. Any such signature λ is associated with the partition obtained by removing the zero entries of λ, under which correpsondence the notions below will coincide with the ones from [**62**]. In particular, it is quickly verified that the definitions in this section will only depend on the nonzero entries of λ (and not on the number of its entries equal to 0).

Let us begin with some notation. Given two signatures $\lambda \in \mathrm{Sign}_\ell$ and $\mu \in \mathrm{Sign}_m$, we say $\mu \subseteq \lambda$ if $m \leqslant \ell$ and $\mu_i \leqslant \lambda_i$ for each $i \in [1, m]$. Moreover, for any sequences of n signatures $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$, we say $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$ if $\mu^{(i)} \subseteq \lambda^{(i)}$ for each $i \in [1, n]$. If $\mu \subseteq \lambda$, we call λ/μ a *skew-shape*; the *size* of this skew-shape is $|\lambda/\mu| = |\lambda| - |\mu| \geqslant 0$. Similarly, if $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$, we call $\boldsymbol{\lambda}/\boldsymbol{\mu}$ a *sequence of skew-shapes*; its size is $|\boldsymbol{\lambda}/\boldsymbol{\mu}| = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}| \geqslant 0$. Similarly to our convention for sequence of signatures, we will denote any sequence $\boldsymbol{\lambda}/\boldsymbol{\mu}$ of n skew-shapes by $\boldsymbol{\lambda}/\boldsymbol{\mu} = \big(\lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots, \lambda^{(n)}/\mu^{(n)}\big)$, unless mentioned otherwise. Any signature λ is associated with the skew-shape $\lambda/\varnothing$, and any sequence $\boldsymbol{\lambda}$ of signatures is associated with the sequence of skew-shapes $\boldsymbol{\lambda}/\boldsymbol{\varnothing}$.

We depict any signature $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ through its *Young diagram* $\mathcal{Y}(\lambda)$, which is a left-justified collection of $|\lambda|$ boxes arranged in $\ell(\lambda) = \ell$ rows, whose i-th row from the top contains λ_i boxes. The condition $\mu \subseteq \lambda$ is then equivalent to imposing that $\mathcal{Y}(\mu)$ is contained inside $\mathcal{Y}(\lambda)$, where these Young diagrams are translated so that their top-left corners coincide. The Young diagram $\mathcal{Y}(\lambda/\mu)$ for the skew-shape λ/μ is then given by $\mathcal{Y}(\lambda) \setminus \mathcal{Y}(\mu)$, obtained by removing the Young diagram of μ from that of λ; we refer to the left side of Figure 9.1 for a depiction, where there dotted part is $\mathcal{Y}(\mu)$ and the solid part is $\mathcal{Y}(\lambda/\mu)$.

A skew-shape λ/μ is *connected* if its Young diagram is edge-connected, that is, for any two boxes $b, b' \in \mathcal{Y}(\lambda/\mu)$ there exists a sequence of boxes $b_0 = b, b_1, \ldots, b_k = b' \in \mathcal{Y}(\lambda/\mu)$ such that b_i and b_{i+1} share an edge for each $i \in [0, k-1]$. A *ribbon* is then a connected skew-shape that does not contain any 2×2 block of boxes, and an *n-ribbon* is a ribbon of size n; we refer to the left side of Figure 9.1 for a tiling of a skew-shape by 3-ribbons. The *height* $\mathrm{ht}(\lambda/\mu)$ of an n-ribbon λ/μ is the number of rows it spans; following equation (15) of [**62**], the *spin* $\mathrm{sp}(\lambda/\mu)$ of λ/μ is defined by

$$\mathrm{sp}(\lambda/\mu) = \frac{\mathrm{ht}(\lambda/\mu) - 1}{2}. \tag{9.1.1}$$

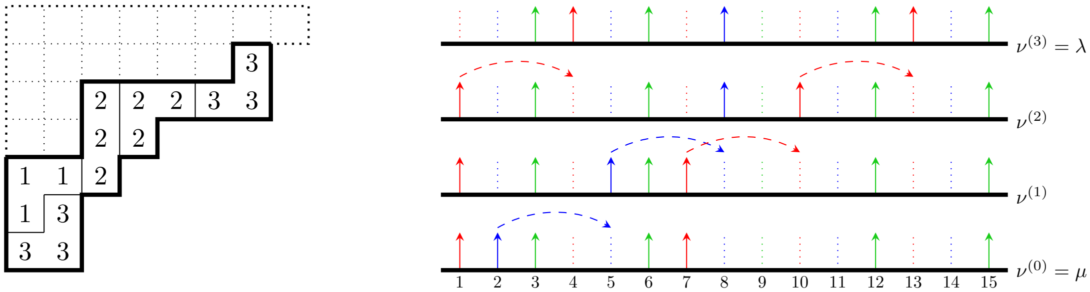

FIGURE 9.1. To the left is a 3-ribbon tableau for the skew shape λ/μ, where $\lambda = (8,7,7,4,3,2,2)$ and $\mu = (8,6,2,2)$; to the right is the associated sequence of colored Maya diagrams. Here, red is color 1, blue is color 2, and green is color 3.

A *horizontal n-ribbon strip* is a (not necessarily connected) skew-shape λ/μ that can be tiled by n-ribbons, each of whose top-right boxes is adjacent to the upper boundary of λ/μ. The spin $\mathrm{sp}(\lambda/\mu)$ of this horizontal n-ribbon strip is given by the sum of the spins of the n-ribbons composing it.

An *n-ribbon tableau* for a skew-shape λ/μ is a sequence of signatures $T = \big(\nu^{(0)}, \nu^{(1)}, \dots \nu^{(k)}\big)$ such that $\mu = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(k)} = \lambda$ and $\nu^{(i)}/\nu^{(i-1)}$ is a horizontal n-ribbon strip for each index $i \in [1,k]$. We associate such a tableau T with the function $T : \mathcal{Y}(\lambda/\mu) \to [1,k]$ defined by, for any box $b \in \mathcal{Y}(\lambda/\mu)$, setting $T(b) = i$ where i is such that $b \in \mathcal{Y}\big(\nu^{(i)}/\nu^{(i-1)}\big)$; see the left side of Figure 9.1 for a depiction. The spin $\mathrm{sp}(T)$ of T is then given by the sum of the spins of the horizontal n-ribbon strips composing it, namely,

$$\mathrm{sp}(T) = \sum_{i=1}^{k} \mathrm{sp}\big(\nu^{(i)}/\nu^{(i-1)}\big).$$

For example, the spin of the 3-ribbon tableaux shown on the left side of Figure 9.1 is 3.

For any (possibly infinite) sequence of complex numbers $\mathbf{x} = (x_1, x_2, \dots, x_N)$; skew-shape λ/μ; and n-ribbon tableau T for λ/μ, define the monomial

$$\mathbf{x}^T = \prod_{b \in \mathcal{Y}(\lambda/\mu)} x_{T(b)}^{1/n},$$

where we set $\mathbf{x}^T = 0$ if N is less than the number of horizontal n-ribbon strips composing T. Observe that $\mathbf{x}^T$ indeed defines a monomial since, for any index i, the number of boxes $b \in \mathcal{Y}(\lambda/\mu)$ such that $T(b) = i$ is $\big|\nu^{(i)}/\nu^{(i-1)}\big|$, which is a multiple of n (where this multiple is given by the number of n-ribbons composing $\nu^{(i)}/\nu^{(i-1)}$).

Now, for any complex number $q \in \mathbb{C}$ and finite sequence $\mathbf{x} = (x_1, x_2, \dots, x_N)$ of complex numbers, we can define the following polynomials, which originated (up to an overall factor of q) in equation (26) of **[62]**. For any signatures $\lambda, \mu \in \mathrm{Sign}$, the *Lascoux–Leclerc–Thibon (LLT) polynomial* $\mathcal{L}_{\lambda/\mu}^{(n)}(\mathbf{x}) = \mathcal{L}_{\lambda/\mu}^{(n)}(\mathbf{x}; q)$ is defined by setting

$$\mathcal{L}_{\lambda/\mu}^{(n)}(\mathbf{x}) = \sum_{T} q^{\mathrm{sp}(T)} \mathbf{x}^T, \tag{9.1.2}$$

where T is summed over all n-ribbon tableaux for λ/μ. Observe in particular that these functions are homogeneous of degree $n^{-1}\big(|\lambda| - |\mu|\big)$ in $\mathbf{x}$. The fact that they are symmetric in $\mathbf{x}$ is not immediate from the definition (9.1.2) but is known from Theorem 6.1 of **[62]**, and it will also follow as a consequence of (9.3.3) below (and Proposition 7.2.2).

Remark 9.1.1. The LLT polynomials $\mathcal{L}_{\lambda/\mu}^{(n)}$ satisfy a branching relation (similar to Proposition 7.2.1) in the following sense. If $\mathbf{x} = (x_1, x_2, \ldots, x_{K+M}) \subset \mathbb{C}$, and we denote $\mathbf{x}' = (x_1, x_2, \ldots, x_K) \subset \mathbb{C}$ and $\mathbf{x}'' = (x_{K+1}, x_{K+2}, \ldots, x_{K+M}) \subset \mathbb{C}$, then

$$\mathcal{L}_{\lambda/\mu}^{(n)}(\mathbf{x}) = \sum_{\nu \in \mathrm{Sign}_\ell} \mathcal{L}_{\lambda/\nu}^{(n)}(\mathbf{x}'')\mathcal{L}_{\nu/\mu}^{(n)}(\mathbf{x}'), \tag{9.1.3}$$

for any sufficiently large integer ℓ. Indeed, this follows from decomposing the ribbon tableau T for λ/μ on the right side of (9.1.2) into a union of ribbon tableaux (T', T''), where T' and T'' consist of those boxes in T with entries in the intervals $[1, K]$ and $[K+1, K+M]$, respectively. Then, the $\nu \in \mathbb{Y}$ on the right side of (9.1.3) is the partition such that T' and T'' are ribbon tableaux for ν/μ and λ/ν, respectively.

9.2. n-Quotients and Colored Maya Diagrams

The LLT polynomials defined in (9.1.2) are indexed by skew-shapes λ/μ, while our functions from Definition 7.1.1 are indexed by sequences of n skew-shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$. The correspondence between these two different indexings is given by setting $\boldsymbol{\lambda}/\boldsymbol{\mu}$ to be the n-quotient of λ/μ. In this section we recall this procedure in detail, as this will in any case be useful for degenerating our functions to the LLT polynomials. Throughout this section, we again fix an integer $n \geqslant 1$.

Recall from (1.2.3) that, for any $\lambda \in \mathrm{Sign}_\ell$, we set $\mathfrak{T}(\lambda) = (\lambda_1 + \ell, \lambda_2 + \ell - 1, \ldots, \lambda_\ell + 1) \in \mathbb{Z}_{>0}^\ell$, which is sometimes referred to as the *Maya diagram* for λ. We visually represent this Maya diagram $\mathfrak{T}(\lambda)$ through a particle configuration on the positive integer lattice $\mathbb{Z}_{>0}$ where a site $i \in \mathbb{Z}_{>0}$ is occupied by a particle, depicted by an arrow[1], if and only if $i \in \mathfrak{T}(\lambda)$. We further *color* this Maya diagram by associating with each arrow $i \in \mathfrak{T}(\lambda)$ the (unique) index $c(i) \in \{1, 2, \ldots, n\}$ such that n divides $i - c(i)$; this yields a *colored Maya diagram* (more specifically, an *n-colored Maya diagram*). The right side of Figure 9.1 depicts 3-colored Maya diagrams associated with all signatures in the 3-ribbon tableau $(8,6,2,2,0,0,0) \subseteq (8,6,2,2,2,1,0) \subseteq (8,6,5,4,3,1,0) \subseteq (8,7,7,4,3,2,2)$.

Any Maya diagram $\mathfrak{T}(\lambda)$ gives rise to a sequence of n Maya diagrams $\big(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \ldots, \mathcal{T}^{(n)}\big)$ by setting $j \in \mathcal{T}^{(i)}$ if and only if $n(j-1) + i \in \mathcal{T}$, for any indices $i \in [1, n]$ and $j \in \mathbb{Z}_{>0}$. In this way, one might view $\mathcal{T}^{(i)}$ as the restriction to color i of the Maya diagram $\mathfrak{T}(\lambda)$. Since $\mathfrak{T}$ induces a bijection between signatures and subsets of $\mathbb{Z}_{>0}$, there exists a sequence of n signatures $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\big) \in \mathrm{SeqSign}_n$ such that $\mathcal{T}^{(i)} = \mathfrak{T}\big(\lambda^{(i)}\big)$ for each $i \in [1, n]$; the sequence $\boldsymbol{\lambda}$ is called the *n-quotient* of λ. The n-quotient of a skew shape λ/μ is then given by the sequence of skew shapes $\boldsymbol{\lambda}/\boldsymbol{\mu} = \big(\lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots, \lambda^{(n)}/\mu^{(n)}\big)$ where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the n-quotients of λ and μ, respectively. The n-quotient for the skew shape λ/μ from Figure 9.1, which is given by $\big((3,1)/(1,0), (2)/(0), (1,1,0,0)/(1,1,0,0)\big)$, is depicted in Figure 9.2. Under this notation, we alternatively write the LLT polynomials from (9.1.2) by

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q) = \mathcal{L}_{\lambda/\mu}^{(n)}(\mathbf{x}). \tag{9.2.1}$$

[1] Typically, these are depicted by beads, but here we use arrows to emphasize the similarity with vertex models.

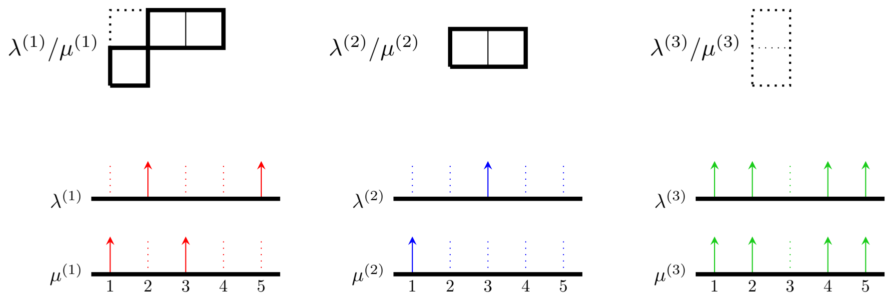

FIGURE 9.2. To the top is the 3-quotient for λ/μ, where $\lambda = (8, 7, 7, 4, 3, 2, 2)$ and $\mu = (8, 6, 2, 2)$. To the bottom are the Maya diagrams for the 3-quotients of λ and μ.

Now let us introduce some (nonstandard) terminology on how the notions from Section 9.1 (such as ribbons, horizontal ribbon strips, and ribbon tableaux) can be interpreted in terms of Maya diagrams. To that end, let $\mathcal{U} = (u_1, u_2, \ldots, u_k) \in \mathbb{Z}_{>0}^k$ denote a Maya diagram. An *n-jump* of $\mathcal{U}$ is a Maya diagram $\mathcal{T} = (t_1, t_2, \ldots, t_k) \in \mathbb{Z}_{>0}^k$ for which there exist indices $i, j \in [1, k]$ such that $u_i + n = t_j$ and $\mathcal{U} \setminus \{u_i\} = \mathcal{T} \setminus \{t_j\}$. Stated alternatively, $\mathcal{T}$ is obtained from $\mathcal{U}$ by having the particle at site u_i "jump" to the right by n spaces, so we call u_i the *jumper* of this n-jump. The quantity $\big|[u_i + 1, u_i + n - 1] \cap \mathcal{U}\big|$ then denotes the number of particles that the jumper jumps over.

For signatures $\lambda, \mu \in \mathrm{Sign}_N$, it is quickly verified that $\mathfrak{T}(\lambda)$ is an n-jump of $\mathfrak{T}(\mu)$ if and only if λ/μ is an n-ribbon. In this case, the number of particles jumped over is $\mathrm{ht}(\lambda/\mu) - 1 = 2\,\mathrm{sp}(\lambda/\mu)$, where we recall from Section 9.1 that $\mathrm{ht}(\lambda/\mu)$ denotes the height of the ribbon λ/μ and $\mathrm{sp}(\lambda/\mu)$ denotes its spin given by (9.1.1). We refer to the bottom of the right side of Figure 9.1 for an example, where $\mathfrak{T}\big(\nu^{(1)}\big) = (1, 3, 5, 6, 7, 12, 15)$ is a 3-jump of $\mathfrak{T}\big(\nu^{(0)}\big) = (1, 2, 3, 6, 7, 12, 15)$; the jumper there is 2, and it jumps over one particle at site 3.

Next, we say that $\mathcal{T}$ is obtained by a *series of n-jumps* from $\mathcal{U}$ if there exists a sequence $\mathcal{U} = \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m = \mathcal{T}$ of Maya diagrams such that $\mathcal{T}_i$ is an n-jump of $\mathcal{T}_{i-1}$ for each $i \in [1, m]$. Such a series is called *nonconflicting* if, denoting the jumper from $\mathcal{T}_{i-1}$ to $\mathcal{T}_i$ by $j_{i-1} \in \mathcal{T}_{i-1}$ for each $i \in [1, m]$, we have $j_0 < j_1 < \cdots < j_{m-1}$. It is quickly verified that λ/μ is a horizontal n-ribbon strip if and only if $\mathfrak{T}(\lambda)$ can be obtained by a nonconflicting series of n-jumps from $\mathfrak{T}(\mu)$. In this case, such a nonconflicting series must be unique, and $2\,\mathrm{sp}(\lambda/\mu)$ denotes the total number of particles jumped over in this series.

Thus, n-ribbon tableaux T of λ/μ are in bijection with sequences $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m)$ of Maya diagrams such that $\mathcal{T}_0 = \mathfrak{T}(\mu)$, $\mathcal{T}_m = \mathfrak{T}(\lambda)$, and $\mathcal{T}_i$ can be obtained by a nonconflicting series of n-jumps from $\mathcal{T}_{i-1}$ for each $i \in [1, m]$. In this case, $2\,\mathrm{sp}(T)$ gives the total number of particles jumped over in this sequence of nonconflicting series of n-jumps. For example, the left side of Figure 9.1 depicts a 3-ribbon tableau, with each $\mathfrak{T}\big(\nu^{(i)}\big)$ obtained by a non-conflicting series of n-jumps from $\mathfrak{T}\big(\nu^{(i-1)}\big)$; the total number of particles jumped over there is 6.

For a more detailed exposition on some of these notions (under slightly different notation), we refer to Section 2 and Section 3 of **[85]**.

9.3. Correspondences With LLT Polynomials

In this section we state Theorem 9.3.2 below, which indicates how some of the various specializations of the F, G, and H functions introduced in Definition 8.4.2 degenerate to the LLT polynomials given by (9.1.2) (or, equivalently, (9.2.1)). In what follows, for any sequence of signatures $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$, set

$$\psi(\boldsymbol{\lambda}) = \frac{1}{2} \sum_{1 \leqslant i < j \leqslant n} \sum_{a \in \mathfrak{T}_i} \sum_{b \in \mathfrak{T}_j} \mathbf{1}_{a > b}, \tag{9.3.1}$$

where we have abbreviated $\mathfrak{T}_k = \mathfrak{T}\big(\lambda^{(k)}\big)$ for each index $k \in [1, n]$.

Example 9.3.1. For any integer $N \geqslant 0$, we have

$$\psi(\mathbf{0}^N) = \frac{1}{2} \sum_{i=1}^{n} (n-i) \sum_{a=1}^{N} (a-1) = \frac{1}{2} \binom{n}{2} \binom{N}{2}. \tag{9.3.2}$$

Moreover, for any skew-shape λ/μ, let λ'/μ' denote its *dual*, whose Young diagram $\mathcal{Y}(\lambda'/\mu')$ is obtained by reflecting $\mathcal{Y}(\lambda/\mu)$ across the diagonal line $x + y = 0$ (where the top-left corner of the top-left box in $\mathcal{Y}(\lambda/\mu)$ is located at $(0,0)$); see the top of Figure 10.6 below for a depiction.[2] In addition, if $\boldsymbol{\lambda}/\boldsymbol{\mu}$ denotes the n-quotient of λ/μ, then we let $\boldsymbol{\lambda}'/\boldsymbol{\mu}'$ denote that of its dual[3] λ'/μ'.

Further recall for any signature sequence $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\big) \in \mathrm{SeqSign}_n$ that its reverse ordering is denoted by $\overleftarrow{\boldsymbol{\lambda}} = \big(\lambda^{(n)}, \lambda^{(n-1)}, \ldots, \lambda^{(1)}\big) \in \mathrm{SeqSign}_n$. We may view the latter as obtained from the former by reversing its colors (that is, interchanging colors i and $n-i+1$ for each $i \in [1, n]$).

We can now state the following theorem, which describes four ways of obtaining the LLT polynomials from the degenerated functions $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ from Definition 8.4.2.

Theorem 9.3.2. *For any fixed integers $N \geqslant 1$ and $M \geqslant 0$, and set $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ of complex numbers, the following statements hold.*

(1) For any sequences of signatures $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q). \tag{9.3.3}$$

(2) For any sequences of signatures $\boldsymbol{\lambda} \in \mathrm{Sign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have

$$\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\lambda}) + \binom{M}{2}\binom{n}{2}/2 - \binom{M+N}{2}\binom{n}{2}/2} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q^{1-n}\mathbf{x}^{-1}; q) \prod_{j=1}^{N} x_j^{n(j-M-N)}. \tag{9.3.4}$$

(3) For any sequences of signatures $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0; \mathbf{x} \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(q^{(n-1)/2}\mathbf{x}^{-1}; q^{-1}). \tag{9.3.5}$$

(4) For any sequences of signatures $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have

$$\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty) = q^{\psi(\overleftarrow{\boldsymbol{\mu}}) - \psi(\overleftarrow{\boldsymbol{\lambda}}) + \binom{M+N}{2}\binom{n}{2}/2 - \binom{M}{2}\binom{n}{2}/2} \mathcal{L}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(-\mathbf{x}^{-1}; q^{-1}) \prod_{j=1}^{N} x_j^{n(j-M-N-1)}. \tag{9.3.6}$$

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[2]In principle, there is an ambiguity in the numbers of entries equal to zero in λ' and μ'. However, we will ignore this point, since (as mentioned in the beginning of Section 9.1) $\mathcal{L}^{(n)}_{\lambda/\mu}(\mathbf{x}; q)$ does not depend on these numbers.

[3]Let us clarify that, under this notation, the i-th component $\lambda'^{(i)}$ of $\boldsymbol{\lambda}'$ happens to not be the dual of $\lambda^{(i)}$. Instead, it is that of $\lambda^{(n-i)}$ (and similarly for $\boldsymbol{\mu}$).

Remark 9.3.3. The skew-shapes λ/μ whose n-quotients are of the form $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, for some $M \geqslant 1$, are those for which λ and μ both have an empty n-core.[4] To allow for more general λ/μ, one must permit the signatures in $\boldsymbol{\lambda}$ (and $\boldsymbol{\mu}$) to have different lengths. The framework introduced above allows this with very minor modifications (see Remark 7.1.2), under which (9.3.3) and (9.3.5) continue to hold as written, and the exponents of q and the x_j on the right side of (9.3.4) become slightly different. However, to ease notation, we will not pursue this generalization here in detail.

In addition to the curiosity that the LLT polynomials can be recovered from the F, G, and H functions in numerous ways,[5] the four parts of Theorem 9.3.2 will have their own uses and interpretations. More specifically, we will see as Corollary 9.4.1 below that (9.3.3) and (9.3.4) together imply that pairing $\mathcal{G}_{\boldsymbol{\lambda}}(\mathbf{y}; \infty \mid 0; 0)$ and $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid 0; 0)$ recovers the Cauchy identity for the LLT polynomials. Similarly, by Corollary 9.4.2 below, (9.3.5) and (9.3.4) together imply pairing $\mathcal{G}_{\boldsymbol{\lambda}}(0; \mathbf{y} \mid 0; 0)$ and $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid 0; 0)$ recovers the dual LLT Cauchy identity. We will further show in Chapter 12 below that the degeneration of $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ given by the fourth part (9.3.6) of Theorem 9.3.2 admits a deformation that satisfies a vanishing property similar to those defining various families of interpolation polynomials.

Using the branching property Proposition 7.2.1, we can quickly reduce Theorem 9.3.2 to the following proposition, which verifies it in the case when $N = 1$; we will establish the latter result in Chapter 10 below.

Proposition 9.3.4. *If $N = 1$, then Theorem 9.3.2 holds.*

Proof of Theorem 9.3.2 Assuming Proposition 9.3.4. All four parts of Theorem 9.3.2 will follow from the facts that the $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, or $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ functions satisfy the same branching relation as do the $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ ones. We will only describe this in detail for the first two statements of Theorem 9.3.2 (assuming their counterparts in Proposition 9.3.4); the proofs of the remaining two statements there are entirely analogous and are therefore omitted.

We begin by verifying (9.3.3), assuming it holds when $N = 1$. To that end, observe by Remark 9.1.1 and (9.2.1) that if $\mathbf{x} = \mathbf{x}' \cup \mathbf{x}''$ then we have the branching relation

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M}} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}'') \mathcal{L}_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'). \tag{9.3.7}$$

Thus, since the same branching relation holds for $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ by Proposition 7.2.1 and (8.4.3), and since Proposition 9.3.4 implies that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_i; \infty \mid 0; 0) = \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_i)$ for each $i \in [1, N]$, we deduce Theorem 9.3.2 in general by induction on N.

The verification of (9.3.4) is largely similar, except that we must first introduce a normalized variant of $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid 0; 0)$ as suggested by (9.3.4) and then verify that its branching coincides with the one indicated by (9.3.7). More specifically, for any sequences of n signatures $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, define

$$\widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu}) + \binom{M+N}{2}\binom{n}{2}/2 - \binom{M}{2}\binom{n}{2}/2} \mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) \prod_{j=1}^{N} x_j^{n(M+N-j)}. \tag{9.3.8}$$

[4]The *n-core* of a signature $\lambda \in \mathrm{Sign}$ is a signature $\kappa \in \mathrm{Sign}$ of minimal size such that λ/κ admits an n-ribbon tableau. This signature κ is unique, up to its number of entries equal to 0.

[5]The list of degenerations in Theorem 9.3.2 from these functions to the LLT polynomials is likely not at all exhaustive, but we will not pursue further ones here.

We must show $\widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0) = \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$, assuming this holds when $N = 1$. As in the proof of (9.3.3), to do this it will suffice by induction to show that they satisfy the same branching property.

To that end, fix integers $K, L \geqslant 1$ such that $N = K + L$. Let $\mathbf{x}' = (x_1, x_2, \ldots, x_K)$ and $\mathbf{x}'' = (x_{K+1}, x_{K+2}, \ldots, x_{K+L})$, so that $\mathbf{x} = \mathbf{x}' \cup \mathbf{x}''$. Then, we find that

$$\begin{aligned}
\widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0) &= q^{\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\mu})+\binom{M+N}{2}\binom{n}{2}/2-\binom{M}{2}\binom{n}{2}/2} \prod_{j=1}^{N} x_j^{n(M+N-j)} \\
&\qquad \times \sum_{\boldsymbol{\nu}\in\mathrm{SeqSign}_{n;M+L}} \mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}';\infty \mid 0;0)\mathcal{F}_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'';\infty \mid 0;0) \\
&= \prod_{j=1}^{N} x_j^{n(M+N-j)} \prod_{j=1}^{K} x_j^{n(j-M-N)} \prod_{j=1}^{L} x_{j+K}^{n(j-M-L)} \\
&\qquad \times \sum_{\boldsymbol{\nu}\in\mathrm{SeqSign}_{n;M+L}} \widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}';\infty \mid 0;0)\widetilde{\mathcal{F}}_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'';\infty \mid 0;0) \\
&= \sum_{\boldsymbol{\nu}\in\mathrm{SeqSign}_{n}} \widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\nu}}(\mathbf{x}';\infty \mid 0;0)\widetilde{\mathcal{F}}_{\boldsymbol{\nu}/\boldsymbol{\mu}}(\mathbf{x}'';\infty \mid 0;0),
\end{aligned}$$

where in the first equality we used the definition (9.3.8) for $\widetilde{\mathcal{F}}$ and its branching (which follows from Proposition 7.2.1 and (8.4.4)); in the second we used (9.3.8) again; and in the third we used the fact that $N = K + L$.

By Remark 9.1.1, $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ satisfies the same branching identity, given by (9.3.7). Since by Proposition 9.3.4 we have $\widetilde{\mathcal{F}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0) = \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ when $N = 1$, it follows by induction on N that this equality holds in general. $\square$

9.4. Cauchy Identities for LLT Polynomials

In this section we establish, as consequences of Theorem 9.3.2 and the results from Section 8.5, two identities for the LLT polynomials that were first derived in [**57**]. The first is a Cauchy identity for the LLT polynomials and the second is a dual Cauchy identity.

Corollary 9.4.1 ([**57**, Theorem 35])**.** *Fix integers $N, M \geqslant 1$ and sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_M)$. If $|q| < 1$ and $|x_j|, |y_i| < 1$ for each $i \in [1, M]$ and $j \in [1, N]$, then*

$$\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x})\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{y}) = \prod_{i=1}^{M}\prod_{j=1}^{N} (x_j y_i; q)_n^{-1}.$$

Proof. Applying Corollary 8.5.1, with the $(\mathbf{u}, \mathbf{w})$ there equal to $(q^{1-n}\mathbf{x}^{-1}, \mathbf{y})$ here, we deduce that

(9.4.1)
$$\sum_{\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;N}} \mathcal{F}_{\boldsymbol{\lambda}}(q^{1-n}\mathbf{x}^{-1};\infty \mid 0;0)\mathcal{G}_{\boldsymbol{\lambda}}(\mathbf{y};\infty \mid 0;0) = q^{\binom{n}{2}\binom{N}{2}} \prod_{j=1}^{N} x_j^{n(N-j)} \prod_{i=1}^{M}\prod_{j=1}^{N} (q^{n-1}x_j y_i; q^{-1})_n^{-1},$$

where (8.5.1) holds since the bounds $|q| < 1$ and $\binom{a}{2} - \binom{n}{2} + (n-1)(n-a) \geqslant 0$ for $a \leqslant n$ give

$$\max_{\substack{1 \leqslant i \leqslant M \\ 1 \leqslant j \leqslant N}} \max_{\substack{a,b \in [0,n] \\ (a,b) \neq (n,0)}} \left| q^{\binom{a}{2} + \binom{b}{2} - \binom{n}{2}} q^{(n-1)(n-a)} x_j^{n-a} y_i^b \right| \leqslant \max_{\substack{1 \leqslant i \leqslant M \\ 1 \leqslant j \leqslant N}} \big\{ |x_j|, |y_i| \big\} < 1.$$

Since (9.3.3) and (9.3.4) imply that $\mathcal{G}_{\boldsymbol{\lambda}}(\mathbf{y}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\mathbf{0}^N)} \mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{y})$ and

$$\mathcal{F}_{\boldsymbol{\lambda}}(q^{1-n} \mathbf{x}^{-1}; \infty \mid 0; 0) = q^{-\psi(\boldsymbol{\lambda}) - \binom{n}{2}\binom{N}{2}/2} \mathcal{L}(\mathbf{x}) \prod_{j=1}^{N} q^{n(n-1)(N-j)} x_j^{n(N-j)}, \tag{9.4.2}$$

the corollary follows from (9.4.1), (9.3.2), and the fact that $(q^{n-1} z; q^{-1})_n = (z; q)_n$, for any $z \in \mathbb{C}$. ☐

Corollary 9.4.2 ([**57**, Proposition 36])**.** *Fix integers $N, M \geqslant 1$ and sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_M)$. Then,*

$$\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} \mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x}; q) \mathcal{L}_{\boldsymbol{\lambda}'}(\mathbf{y}; q^{-1}) = \prod_{i=1}^{M} \prod_{j=1}^{N} (-q^{(n-1)/2} x_j y_i; q)_n.$$

Proof. Applying Corollary 8.5.2, with the $(\mathbf{u}, \mathbf{t})$ there equal to $(q^{1-n} \mathbf{x}^{-1}, -q^{(n-1)/2} \mathbf{y}^{-1})$ here, we deduce that

$$\begin{aligned} &\sum_{\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}} (-1)^{|\boldsymbol{\lambda}|} \mathcal{F}_{\boldsymbol{\lambda}}(q^{1-n} \mathbf{x}^{-1}; \infty \mid 0; 0) \mathcal{G}_{\boldsymbol{\lambda}}(0; -q^{(n-1)/2} \mathbf{y}^{-1} \mid 0; 0) \\ &\qquad = q^{\binom{n}{2}\binom{N}{2}} \prod_{j=1}^{N} x_j^{n(N-j)} \prod_{i=1}^{M} \prod_{j=1}^{N} (-q^{(n-1)/2} x_j y_i; q^{-1})_n^{-1}. \end{aligned} \tag{9.4.3}$$

Since (9.3.5) and the homogeneity of the LLT polynomials $\mathcal{L}_{\boldsymbol{\lambda}}$ together imply that

$$(-1)^{|\boldsymbol{\lambda}|} \mathcal{G}_{\boldsymbol{\lambda}}(0; -q^{(n-1)/2} \mathbf{y}^{-1} \mid 0; 0) = (-1)^{|\boldsymbol{\lambda}|} q^{\psi(\boldsymbol{\lambda}) - \psi(\mathbf{0}^N)} \mathcal{L}_{\boldsymbol{\lambda}'}(-\mathbf{y}; q^{-1}) = q^{\psi(\boldsymbol{\lambda}) - \psi(\mathbf{0}^N)} \mathcal{L}_{\boldsymbol{\lambda}'}(\mathbf{y}; q^{-1}),$$

the corollary follows from (9.4.3), (9.4.2), and (9.3.2). ☐

9.5. Plethystic Substitution of LLT Polynomials

In this section we explain how the $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big((x_1, x_2, \ldots, x_N), r \mid 0; 0\big)$ functions from (8.4.3) can be interpreted as images of the LLT polynomials under a certain plethystic substitution. In what follows, we fix an infinite set of variables $\mathbf{x} = (x_1, x_2, \ldots)$; let $\Lambda(\mathbf{x})$ denote the ring of symmetric functions in $\mathbf{x}$; and set $\mathbf{x}_{[1,N]} = (x_1, x_2, \ldots, x_N)$ for any integer $N \geqslant 1$.

As defined in (8.4.3), the $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ are functions of finitely many variables and are therefore not directly elements of $\Lambda(\mathbf{x})$. However, the following lemma shows that the sequence of functions $\big\{\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}_{[1,N]}, r; \infty, 0)\big\}_{N \geqslant 1}$ satisfies the compatibility condition that makes it an inverse system in $\Lambda(\mathbf{x})$.

Lemma 9.5.1. *Fix integers $M \geqslant 0$ and $N \geqslant 1$; signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$; and a complex number $r \in \mathbb{C}$. For any integer $N \geqslant 1$, the function $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}_{[1,N]}; r \mid 0; 0)$ from (8.4.3) is a symmetric, homogeneous polynomial in $\mathbf{x}_{[1,N]}$ of degree $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$. Moreover, if $x_{N+1} = 0$, then $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N+1]}; r \mid 0; 0\big) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N]}; r \mid 0, 0\big)$.*

Proof. The symmetry of $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}_{[1,N]}; r \mid 0,0)$ follows from Proposition 7.2.2. To verify that it is homogenous and of degree $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$, recall from (7.1.3) and Remark 8.4.4 that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N]}; r \mid 0; 0\big)$ is the partition function under the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r)$ (from (8.4.1)) for the vertex model $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$ (from Definition 7.1.3; see also the left side of Figure 7.1) on the domain $\mathcal{D}_N = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N\}$. The explicit form (8.4.1) for these weights implies that they only depend on x through a factor of $x^{|\mathbf{D}|}$. Since for each path ensemble $\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)$, under which the arrow configuration at any vertex $v \in \mathcal{D}_N$ is denoted by $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$, we have $\sum_{v \in \mathcal{D}_N} \big|\mathbf{D}(v)\big| = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$, it follows that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}_{[1,N]}; r \mid 0; 0)$ is a homogeneous polynomial in $\mathbf{x}_{[1,N]}$ of degree $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$. This verifies the first statement of the lemma.

Next let us verify the second statement of the lemma. Once again, $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N+1]}; r \mid 0; 0\big)$ is the partition function under the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r)$ for $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N+1)$ on the domain $\mathcal{D}_{N+1} = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N+1\}$. Since spectral parameter x_{N+1} for this top row in this model is equal to 0, and since (8.4.1) implies that $\mathcal{W}_0(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r) = 0$ unless $|\mathbf{D}| = 0$, the arrow configurations $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ under any path ensemble supported by this vertex model must satisfy $\mathbf{D}(v) = \mathbf{e}_0$ whenever $v \in \mathbb{Z}_{>0} \times \{N+1\}$. In particular, paths in any such ensemble cannot move to the right in this top row, meaning that for any $v \in \mathbb{Z}_{>0} \times \{N+1\}$ we must have that $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ is of the form $(\mathbf{A}, \mathbf{e}_0; \mathbf{A}, \mathbf{e}_0)$, for some $\mathbf{A} \in \{0,1\}^n$. By the last statement of (8.4.1), $\mathcal{W}_x(\mathbf{A}, \mathbf{e}_0; \mathbf{A}, \mathbf{e}_0 \mid r) = 1$ for any $\mathbf{A}$, meaning that the weight of the topmost row of any ensemble $\mathcal{E}$ supported by $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N+1]}; r \mid 0; 0\big)$ is equal to 1.

Hence, we can omit this row from model, deducing from Remark 8.4.4 that

$$\begin{aligned}\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N+1]}; r \mid 0; 0\big) &= \sum_{\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N+1)} \mathcal{W}(\mathcal{E} \mid \mathbf{x}; r \mid \infty, 0) \\ &= \sum_{\mathcal{E} \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; N)} \mathcal{W}(\mathcal{E} \mid \mathbf{x}; r \mid \infty, 0) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}_{[1,N]}; r \mid 0; 0\big),\end{aligned}$$

and thereby establishing the lemma. ☐

Definition 9.5.2. For any $r \in \mathbb{C}$ and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$, let $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0) \in \Lambda(\mathbf{x})$ denote the element of $\Lambda(\mathbf{x})$ associated with the inverse system of symmetric, degree $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$ polynomials $\big\{\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0)\big\}_{N \geqslant 1}$.

Remark 9.5.3. Observe from the explicit forms (8.4.1) of the $\mathcal{W}$ weights (or, alternatively, as a consequence of Remark 8.3.7) that, for any $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$, we have

$$\lim_{r \to \infty} \mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r) = \mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0).$$

Thus, (7.1.3) and Remark 8.4.4 together imply for any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$ that

$$\lim_{r \to \infty} \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0). \tag{9.5.1}$$

This, together with (9.3.3), implies that we may view LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ as an element of $\Lambda(\mathbf{x})$.

Next, we briefly recall the notion of plethystic substituion. Letting $p_k(\mathbf{x}) \in \Lambda(\mathbf{x})$ denote the power sum $p_k(\mathbf{x}) = \sum_{i=1}^{\infty} x_i^k$ for each integer $k \geqslant 1$, the ring $\Lambda(\mathbf{x})$ is freely generated as an algebra by the $\big\{p_k(\mathbf{x})\big\}$. For any symmetric function $f \in \Lambda(\mathbf{x})$ and formal sum of monomials $A = \sum_{i=1}^{\infty} a_i$, define the *plethysm* $f[A]$ to be the image of $f(\mathbf{x})$ under the substitution $p_k(\mathbf{x}) = \sum_{i=1}^{\infty} a_i^k$. Let us mention that, under this notation, we will view additional parameters such as q and r below as variables. In particular, if for instance $B = (1 - r^{-2})A$, then

$$p_k[B] = \sum_{i=1}^{\infty} a_i^k - \sum_{i=1}^{\infty} r^{-2k} a_i^k = \sum_{i=1}^{\infty} (1 - r^{-2k}) a_i^k.$$

Now we can state the following consequence of Theorem 9.3.2 that expresses the function $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0) \in \Lambda(\mathbf{x})$ from Definition 9.5.2 as a certain plethystic substitution of the LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) \in \Lambda(\mathbf{x})$ from (9.1.2) (and (9.2.1)). In what follows, we recall the function ψ from (9.3.1).

Proposition 9.5.4. *Fix an integer $M \geqslant 0$ and signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$; further let r and q denote variables. Defining the formal sum $X = \sum_{i=1}^{\infty} x_i$, we have the plethystic identity*

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big[(1 - r^{-2})X\big] = q^{\psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\lambda})} \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0). \tag{9.5.2}$$

Proof. For the purposes of this proof, define the symmetric function $Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) \in \Lambda(\mathbf{x})$ by setting

$$Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0). \tag{9.5.3}$$

We will establish the plethystic identity

$$Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big[(1 - r^{-2})(1 - q)^{-1} X\big] = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0). \tag{9.5.4}$$

Let us first show the proposition assuming (9.5.4). To that end, observe by specializing $r = \infty$ in (9.5.4), and applying (9.5.1) and (9.3.3), yields

$$Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big[(1 - q)^{-1} X\big] = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(X).$$

Applying this with X replaced by $(1 - r^{-2})X$, together with (9.5.4), implies (9.5.2).

So, it remains to verify (9.5.4). To that end, observe that from the explicit form (8.4.1) of the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r)$ that they are rational functions of r. Hence, by (7.1.3) and Remark 8.4.4, the coefficients of $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0)$ when expanded in the monomial basis of $\Lambda(\mathbf{x})$ are also rational functions of r. Hence, it suffices to show that (9.5.4) holds for infinitely many values of r. In particular, we may assume that $r = q^{-L/2}$ for some integer $L \geqslant 1$.

In this case, $(1 - r^{-2})(1 - q)^{-1} = (1 - q^L)(1 - q)^{-1} = \sum_{j=0}^{L-1} q^j$. In particular, the plethysm mapping X to $(1 - r^{-2})(1 - q)^{-1} X$ is realized by replacing the variable sequence $\mathbf{x}$ with $\mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)} \cup \cdots$, where we have denoted the principal specialization $\mathbf{w}^{(j)} = (x_j, q x_j, \ldots, q^{L-1} x_j)$ for each $j \geqslant 1$. Thus, in view of the definition (9.5.3) of $Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, it suffices to show that

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; q^{-1/2} \mid 0; 0) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-L/2} \mid 0; 0).$$

If there exists some integer $N \geqslant 0$ such that $x_j = 0$ for $j > N$, then this follows from Proposition 7.2.3, from which we deduce the general case by letting N tend to ∞. $\square$

Remark 9.5.5. Plethysms of the LLT polynomials have been considered and studied previously in the algebraic combinatorics literature. In particular, in the case when each skew-shape $\lambda^{(i)}/\mu^{(i)}$ in the sequence $\boldsymbol{\lambda}/\boldsymbol{\mu}$ has size 1, the functions $Y_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ from (9.5.3) coincide with chromatic quasisymmetric functions **[81, 82]** associated with the incomparability graph of a unit interval order; see Remark 3.6 of **[22]** and Section 5.4 of **[39]**.

9.6. Stability

In this section we explain how Theorem 9.3.2 can be used to recover a *stability* property for the LLT polynomials, which was originally established in [**62**]. This property essentially states that the LLT polynomial $\mathcal{L}_{n\lambda}^{(n)}(\mathbf{x})$ from (9.1.3) becomes independent of n once $n \geqslant \ell(\lambda)$, at which point it coincides with a modified Hall–Littlewood polynomial. To define the latter, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$ and sequence of complex variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ (possibly with $N = \infty$), recall the Hall–Littlewood function $Q_\lambda(\mathbf{x}) = Q_\lambda(\mathbf{x}; q)$ from Section 3.2 of [**66**]. Then, (following, for instance, Definition 1.1 of [**44**]) set the *modified Hall–Littlewood polynomial* $Q'_\lambda(\mathbf{x})$ to be

$$Q'_\lambda(\mathbf{x}) = Q_\lambda\Bigg(\bigcup_{j=0}^{\infty} q^j \mathbf{x}\Bigg).$$

Equivalently, it is given by the plethystic substitution $Q'_\lambda[X] = Q_\lambda\big[(1-q)^{-1}X\big]$, where X denotes the formal sum $X = \sum_{i=1}^{\infty} x_i$.

Under this notation, we can state the following stability result, first shown in [**62**].

Proposition 9.6.1 ([**62**, Theorem 6.6])**.** *Fix integers $N \geqslant 1$ and $n \geqslant M \geqslant 1$; a set of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$; and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$ of length M. Define the signature $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n) \in \operatorname{Sign}_n$ such that $\widetilde{\lambda}_i = n\lambda_i$ for each $i \in [1, M]$ and $\widetilde{\lambda}_i = 0$ for $i \in [M+1, n]$. Then, $\mathcal{L}_{\widetilde{\lambda}}^{(n)}(\mathbf{x}) = Q'_\lambda(\mathbf{x})$.*

Remark 9.6.2. Through similar methods, one can also establish variants of Proposition 9.6.1 for the more general functions $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$ from Definition 7.1.1 (and not only their degenerations as LLT polnomials). Here, the stable limits will instead be fully fused analogs of the $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ vertex model partition functions $\mathsf{G}_{\lambda/\mu}$ introduced in Definition 4.3 of [**11**]. For the sake of brevity, we will not pursue this further here.

To establish Proposition 9.6.1, we require notation for partition functions similar to those given in Definition 5.2.1 (see Figure 5.2), under the weights $\mathcal{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ from (8.4.1); in what follows, we recall the notions of fused path ensembles, east-south domains,[6] and boundary data from Section 5.2. Let $\mathcal{D} \subset \mathbb{Z}^2$ denote some east-south domain, and fix some boundary data $(\mathcal{E}; \mathcal{F})$ on $\mathcal{D}$. Letting $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$ denote a set of complex numbers, define the partition function

$$\mathcal{W}_{\mathcal{D}}(\mathcal{E}; \mathcal{F} \mid \mathbf{z} \mid \infty, 0) = \sum \prod_{v \in \mathcal{D}} \mathcal{W}_{z(v)}\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v) \mid \infty, 0\big),$$

where the sum is over all ensembles on $\mathcal{D}$ with boundary data $(\mathcal{E}; \mathcal{F})$.

Certain entrance and exit data on the east-south domain $\mathcal{D}_N = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N\} \subset \mathbb{Z}^2_{\geqslant 0}$ will be of particular use to us. First, any entrance data $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ on $\mathcal{D}_N$ such that $\mathbf{E}_k = \mathbf{e}_0$ for $k > N$ corresponds to prohibiting paths from entering $\mathcal{D}_N$ along its bottom boundary $\{y = 1\}$. Second, for any (not necessarily non-increasing) sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell_{>0}$ of positive integers of length $\ell \leqslant n$, define the sequence[7] $\mathcal{I}(\lambda) = \big(\mathbf{I}_1(\lambda), \mathbf{I}_2(\lambda), \ldots\big)$ of elements in $\{0, 1\}^n$, such that the j-th entry of $\mathbf{I}_k(\lambda) \in \{0, 1\}^n$ equals $\mathbf{1}_{\lambda_j = k}$, for each $j \in [1, \ell]$ and $k > 0$ (and this entry equals 0 for each $j \in [\ell+1, n]$). In this way, $\mathcal{I}(\lambda)$ prescribes exit data on $\mathcal{D}_N$, under which one arrow of color j vertically exits through (λ_j, N), for each $j \in [1, \ell]$. In particular, any path ensemble with

[6]Here, we will allow our east-south domains to be infinite, in which scenario all results and notation from Section 5.2 continue to apply.

[7]The indexing here, which starts at 1, is slightly different from that in Section 1.5, which started at 0.

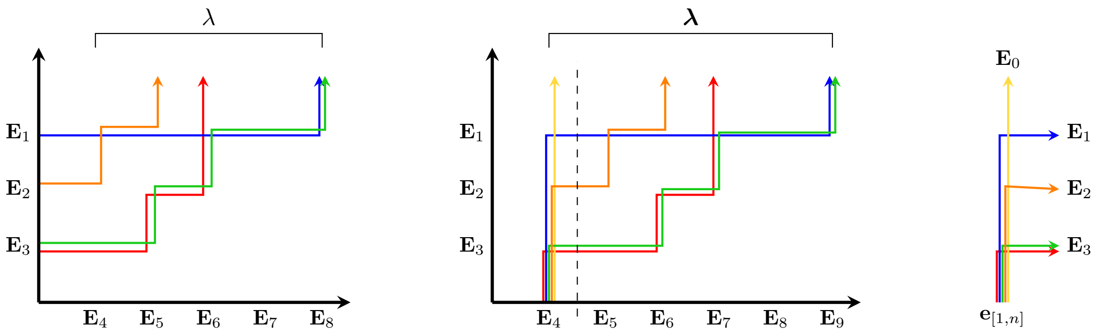


FIGURE 9.3. To the left is a path ensemble on $\mathcal{D}_3$ with boundary data $\big(\mathcal{E}; \mathcal{I}(\lambda)\big)$, where $\lambda = (3,5,5,2)$ and $\mathbf{E}_i = \mathbf{e}_0$ for $i > 3$. In the middle is a path ensemble counted by $\mathcal{G}_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} = \big((3),(5),(5),(2),(0)\big)$, whose leftmost column is depicted to the right. Here red, blue, green, orange, and yellow are colors 1, 2, 3, 4, and 5, respectively.

such exit data has at most one path of any given color; observe that this notation does not specify the order in which the colored paths enter through the left boundary of $\mathcal{D}_N$. We refer to the left side of Figure 9.3 for a depiction of an ensemble under such entrance and exit data.

Now we have the following lemma expressing the LLT polynomial $\mathcal{L}_{\widetilde{\lambda}}^{(n)}(\mathbf{x})$ as linear combinations of the partition functions $\mathcal{W}_{\mathcal{D}}(\mathcal{E}; \mathcal{F} \mid \mathbf{z} \mid \infty, 0)$, for $(\mathcal{E}; \mathcal{F})$ of the above form. In the below, we set $\operatorname{inv}(\mathcal{E}) = \operatorname{inv}\big(\mathcal{E}; [1,n]\big)$, where we recall the latter from (5.2.1).

Lemma 9.6.3. *Adopt the notation of Proposition 9.6.1, and define the variable set* $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}_N}$ *so that* $z(v) = z(i,j) = x_j$ *for each* $v = (i,j) \in \mathcal{D}_N$. *Then,*

$$\mathcal{L}_{\widetilde{\lambda}}^{(n)}(\mathbf{x}) = \sum_{\mathcal{E}} q^{\operatorname{inv}(\mathcal{E})} \mathcal{W}_{\mathcal{D}_N}\big(\mathcal{E}; \mathcal{I}(\lambda) \mid \mathbf{z} \mid \infty, 0\big) \prod_{j=1}^{N} q^{\binom{|\mathbf{E}_j|}{2}} x_{N-j+1}^{|\mathbf{E}_j|},$$

where we sum over all entrance data $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ *on* $\mathcal{D}_N$, *with* $\mathbf{E}_k = \mathbf{e}_0$ *for* $k > N$.

PROOF. Observe that the n-quotient $\widetilde{\boldsymbol{\lambda}} = \big(\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)}, \ldots, \widetilde{\lambda}^{(n)}\big)$ of $\widetilde{\lambda}$ is given by setting $\widetilde{\lambda}^{(i)} = (\lambda_{n-i+1}) \in \operatorname{Sign}_1$ for each $i \in [n-M+1, n]$ and $\widetilde{\lambda}^{(i)} = (0)$ for $i \in [1, n-M]$. Indeed, recalling $\mathfrak{T}$ from (1.2.3), we have

$$\mathfrak{T}(\widetilde{\lambda}) = (n\lambda_1 + n, n\lambda_2 + n - 1, \ldots, n\lambda_M + n - M + 1, n - M, n - M - 1, \ldots, 1).$$

Hence the content in Section 9.2 implies for each $i \in [n-M+1, n]$ that $\mathfrak{T}\big(\widetilde{\lambda}^{(i)}\big) = (\lambda_{n-i+1} + 1)$, meaning $\widetilde{\lambda}^{(i)} = (\lambda_{n-i+1})$, and for each $i \in [1, n-M]$ that $\mathfrak{T}\big(\widetilde{\lambda}^{(i)}\big) = (1)$, meaning $\widetilde{\lambda}^{(i)} = (0)$.

Since $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_M$, it follows from (9.3.1) that $\psi(\widetilde{\boldsymbol{\lambda}}) = \psi(\mathbf{0}^1) = 0$, meaning by (9.3.3) that $\mathcal{G}_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{x}; \infty \mid 0; 0) = \mathcal{L}_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{x}) = \mathcal{L}_{\widetilde{\lambda}}^{(n)}(\mathbf{x})$. Therefore, it suffices to show that

$$\mathcal{G}_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{x}; \infty \mid 0; 0) = \sum_{\mathcal{E}} q^{\operatorname{inv}(\mathcal{E})} \mathcal{W}_{\mathcal{D}_N}\big(\mathcal{E}; \mathcal{I}(\lambda) \mid \mathbf{z} \mid \infty, 0\big) \prod_{j=1}^{N} q^{\binom{|\mathbf{E}_j|}{2}} x_{N-j+1}^{|\mathbf{E}_j|}, \tag{9.6.1}$$

where we sum over all entrance data $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ on $\mathcal{D}_N$ such that $\mathbf{E}_k = \mathbf{e}_0$ for $k > N$.

To that end, observe from (7.1.3) and Remark 8.4.4 that $\mathcal{G}_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{x}; \infty \mid 0; 0)$ is the partition function under the $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ weights for the vertex model $\mathfrak{P}_G(\boldsymbol{\lambda}/\varnothing; N)$ from Definition 7.1.3 (see also the middle of Figure 9.3). Thus the second statement of (3.1.2) (or, diagrammatically, by separating out the leftmost column of the model, as on the middle and right of Figure 9.3) gives

$$\mathcal{G}_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{x}; \infty \mid 0; 0) = \sum_{\mathcal{E}} \mathcal{W}_{\mathcal{D}_N}\big(\mathcal{E}; \mathcal{I}(\lambda) \mid \mathbf{z} \mid \infty, 0\big) \zeta_N(\mathcal{E}; \mathbf{x}) \tag{9.6.2}$$

where we sum over all entrance data $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ on $\mathcal{D}_N$ such that $\mathbf{E}_k = \mathbf{e}_0$ for $k > N$. Here, $\zeta_N(\mathcal{E}; \mathbf{x})$ is defined to be the partition function on the column $\mathcal{D}_{1,N} = \{1\} \times \{1, 2, \ldots, N\} \subset \mathbb{Z}^2$ under the weight $\mathcal{W}_{x_j}\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ at any $v = (1, j) \in \mathcal{D}_{1,N}$, with the following boundary conditions. Its entrance data is $\big(\mathbf{e}_0, \mathbf{e}_0, \ldots, \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ (where $\mathbf{e}_0$ appears with multiplicity N), and its exit data is $(\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_N)$, where $\mathbf{E}_0 \in \{0,1\}^n$ is defined so that $\sum_{i=0}^N \mathbf{E}_i = \mathbf{e}_{[1,n]}$. We refer to the right side of Figure 9.3 for a depiction.

Thus, by (9.6.2), to establish (9.6.1) we must verify that

$$\zeta_N(\mathcal{E}; \mathbf{x}) = q^{\operatorname{inv}(\mathcal{E})} \prod_{j=1}^N q^{\binom{|\mathbf{E}_j|}{2}} x_{N-j+1}^{|\mathbf{E}_j|}. \tag{9.6.3}$$

To that end, observe by arrow conservation that there is a single path ensemble supported by $\zeta_N(\mathcal{E}; \mathbf{x})$, for any fixed sequence $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ of elements in $\{0,1\}^n$. Denote its arrow configuration at any vertex $v \in \mathcal{D}_{1,N}$ by $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$, which must satisfy $\mathbf{B}(1,j) = \mathbf{e}_0$ and $\mathbf{D}(1,j) = \mathbf{E}_{N-j+1}$ for each $j \in [1, N]$ (see the right side of Figure 9.3). Then, by the explicit form (8.4.1) for the $\mathcal{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ weights and the equality $\varphi(\mathbf{E}_{N-j+1}, \mathbf{E}_{N-j+1}) = \binom{|\mathbf{E}_{N-j+1}|}{2}$ (recall φ from (1.1.1)), we obtain

$$\begin{aligned}
\zeta_N(\mathcal{E}; \mathbf{x}) &= \prod_{j=1}^N \mathcal{W}_{x_j}\big(\mathbf{A}(1,j), \mathbf{e}_0; \mathbf{C}(1,j), \mathbf{E}_{N-j+1} \mid \infty, 0\big) \\
&= \prod_{j=1}^N x_j^{|\mathbf{E}_{N-j+1}|} q^{\binom{|\mathbf{E}_{N-j+1}|}{2}} q^{\varphi(\mathbf{E}_{N-j+1}, \mathbf{C}(1,j))} = q^{\operatorname{inv}(\mathcal{E})} \prod_{j=1}^N q^{\binom{|\mathbf{E}_{N-j+1}|}{2}} x_j^{|\mathbf{E}_{N-j+1}|}.
\end{aligned}$$

Here, we have used the fact that

$$\sum_{j=1}^N \varphi\big(\mathbf{E}_{N-j+1}, \mathbf{C}(1,j)\big) = \sum_{j=1}^N \varphi\left(\mathbf{E}_{N-j+1}, \sum_{i=0}^{N-j} \mathbf{E}_i\right) = \sum_{1 \leqslant i < j \leqslant N} \varphi(\mathbf{E}_j, \mathbf{E}_i) = \operatorname{inv}(\mathcal{E}),$$

where the first equality follows from arrow conservation; the second from the bilinearity of φ and the fact that $\varphi(\mathbf{E}_i, \mathbf{E}_0) = 0$ for $i > 0$, which holds since each color vertically exiting the leftmost column is less than each color horizontally exiting it (as $\widetilde{\lambda}^{(i)} = (0)$ if and only if $i \in [1, n-M]$); and the third from the definition of inv (see (5.2.1)). This yields (9.6.3) and thus the lemma. ☐

By Lemma 9.6.3, the LLT polynomial $\mathcal{L}_{\widetilde{\lambda}}^{(n)}(\mathbf{x})$ is given by a linear combination of partition functions for (fermionic) colored vertex models. To proceed, we first recall from **[32]** an expression for the modified Hall–Littlewood polynomial $Q'_\lambda(\mathbf{x})$ through (bosonic) uncolored vertex models (see Proposition 9.6.4 below). Then we will provide a color merging result, similar to (but slightly different from) Theorem 5.2.2, that equates linear combinations of colored vertex model partition functions with uncolored vertex model partition functions (see Lemma 9.6.6 below).

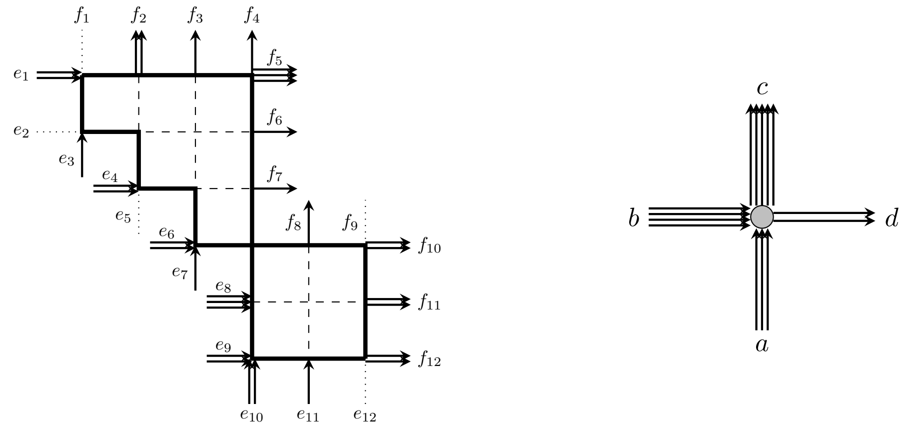


FIGURE 9.4. Shown to the left is an east-south domain with uncolored boundary data, and shown to the right is an uncolored vertex with $(a, b; c, d) = (3, 4; 5, 2)$.

To that end, we require some additional notation. For any nonnegative integers $a, b, c, d \in \mathbb{Z}_{\geqslant 0}$ and complex number $z \in \mathbb{C}$, define (similarly to in equation (59) of [**32**])

(9.6.4)

$$\mathcal{U}_z(a, b; c, d) = x^d q^{\binom{d}{2}} \frac{(q;q)_{a+b}}{(q;q)_a (q;q)_b} \mathbf{1}_{a+b=c+d}; \qquad \mathcal{U}'_z(a, b; c, d) = x^b q^{\binom{b}{2}} \frac{(q;q)_{a+b}}{(q;q)_a (q;q)_b} \mathbf{1}_{a+b=c+d}.$$

Once again, $\mathcal{U}_z$ and $\mathcal{U}'_z$ can be interpreted diagrammatically, now as uncolored vertex weights. As previously, a vertex v is the intersection between two transverse curves ℓ_1 and ℓ_2, and associated with each vertex is a spectral parameter z. Each of the four edges adjacent to v may accommodate any nonnegative number of *uncolored* arrows, that is, arrows are not assigned any color (equivalently, we may view all arrows to be of the same, unlabeled color). Let a, b, c, and d denote the numbers of arrows that vertically enter, horizontally enter, vertically exit, and horizontally exit v respectively.[8] We will assume that $a + b = c + d$, which is a form of arrow conservation. The quadruple $(a, b; c, d)$ is the *uncolored arrow configuration* at v, and we interpret $\mathcal{U}_z(a, b; c, d)$ (or $\mathcal{U}'_z(a, b; c, d)$) as the associated vertex weight. Such vertices are bosonic, since we allow multiple arrows (of the same, unlabeled color) to exist along edges; we refer to the right side of Figure 9.4 for a depiction.

A *(bosonic) uncolored path ensemble* on a domain $\mathcal{D} \subset \mathbb{Z}^2$ is a consistent assignment of a arrow configuration $\big(a(v), b(v); c(v), d(v)\big)$ to each vertex $v \in \mathcal{D}$, meaning that $b(u) = d(v)$ if $u - v = (1, 0)$ and $a(u) = c(v)$ if $u - v = (0, 1)$. Arrow conservation $a + b = c + d$ implies that any path ensemble may be viewed as a collection of colored paths, which may share edges but do not cross.

Now fix an east-south domain $\mathcal{D} \subset \mathbb{Z}^2$ whose boundary paths are both of length k (possibly infinite). For any $(k+1)$-tuples $\mathsf{E} = (e_1, e_2, \ldots, e_{k+1})$ and $\mathsf{F} = (f_1, f_2, \ldots, f_{k+1})$ of nonnnegative integers, an uncolored path ensemble has *uncolored boundary data* $(\mathsf{E}; \mathsf{F})$ if, for each $i \in [1, k+1]$, e_i arrows enter through the i-th incoming edge in $\mathcal{D}$ and f_i arrows exit through its i-th outgoing edge. We refer to E as uncolored entrance data on $\mathcal{D}$ and F as exit data. For example, $\mathsf{E} = (2, 0, 1, 2, 0, 2, 1, 3, 2, 2, 1, 0)$ and $\mathsf{F} = (0, 2, 1, 1, 3, 1, 1, 1, 0, 2, 2, 2)$ on the left side of Figure 9.4.

[8] Unlike as for the colored arrow configurations in Section 2.1, the entries of an uncolored arrow configuration $(a, b; c, d)$ denote the numbers of arrows entering or exiting the vertex (and do not index the colors of the arrows).

Next let us consider partition functions for uncolored path ensembles. Fix some east-south domain $\mathcal{D} \subset \mathbb{Z}^2$; some uncolored boundary data $(\mathsf{E}; \mathsf{F})$ on $\mathcal{D}$; and a set of complex numbers $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$. For each index $\mathcal{X} \in \{\mathcal{U}, \mathcal{U}'\}$, define the *uncolored partition function*

$$\mathcal{X}_{\mathcal{D}}(\mathsf{E}; \mathsf{F} \mid \mathbf{z}) = \sum \prod_{v \in \mathcal{D}} \mathcal{X}_{z(v)}\big(a(v), b(v); c(v), d(v)\big),$$

where the sum is over all uncolored path ensembles with boundary data $(\mathsf{E}; \mathsf{F})$.

Again, certain uncolored entrance and exit data on $\mathcal{D}_N \subset \mathbb{Z}^2_{\geqslant 0}$ will be of particular use to us. First, any uncolored entrance data $\mathsf{E} = (e_1, e_2, \ldots)$ on $\mathcal{D}_N$ with $e_k = 0$ for $k > N$ corresponds to prohibiting paths from entering $\mathcal{D}_N$ along its bottom boundary. Second, for any signature $\lambda \in \mathrm{Sign}$, define the sequence $\mathsf{I}(\lambda) = \big(m_1(\lambda), m_2(\lambda), \ldots\big)$ of nonnegative integers, which is obtained from the colored exit data $\mathcal{I}(\lambda)$ by ignoring the colors of all arrows (viewing them as uncolored). In particular, $m_k(\lambda)$ denotes the multiplicity of k in λ, for each integer $k \geqslant 1$.

Under this notation, we can recall from **[32]** the expression for modified Hall–Littlewood polynomials through such uncolored partition functions under the above boundary data.

Proposition 9.6.4 (**[32]**, Equation (66)])**.** *Fix integers $N, M \geqslant 1$; a partition $\lambda \in \mathrm{Sign}_M$; and a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Defining the variable set $\mathbf{z} = \big(z(v)\big)_{\mathcal{D}_N}$ so that $z(v) = z(i,j) = x_j$ for each $v = (i,j) \in \mathcal{D}_N$, we have*

$$Q'_\lambda(\mathbf{x}) = \sum_{\mathsf{E}} \mathcal{U}'_{\mathcal{D}_N}\big(\mathsf{E}; \mathsf{I}(\lambda) \mid \mathbf{x}\big) = \sum_{\mathsf{E}} \mathcal{U}_{\mathcal{D}_N}\big(\mathsf{E}; \mathsf{I}(\lambda) \mid \mathbf{x}\big) \prod_{i=1}^{N} q^{\binom{e_i}{2}} x_{N-i+1}^{e_i}, \tag{9.6.5}$$

where both sums are over all uncolored entrance data $\mathsf{E} = (e_1, e_2, \ldots)$ on $\mathcal{D}_N$, with $e_k = 0$ for $k > N$.

Remark 9.6.5. The first statement of (9.6.5) is what was established in and below equation (66) of **[32]** (after rotating the vertex model there by 180 degrees). The second follows from the first, together with the gauge transformation $\mathcal{U}'_z(a,b;c,d) = x^{b-d} q^{\binom{b}{2} - \binom{d}{2}} \mathcal{U}_z(a,b;c,d)$ relating the weights $\mathcal{U}_z$ and $\mathcal{U}'_z$ from (9.6.4).

We next state the below color merging result, which essentially states the following. Consider a colored vertex model partition function, and q-symmetrize its entrance data $\mathcal{E}$ over all "colorings" of some fixed uncolored entrance data E. This yields the partition function for the uncolored vertex model obtained from the original one by ignoring all of its arrow colors.

Lemma 9.6.6. *Fix an integer $k \geqslant 1$; an east-south domain $\mathcal{D} \subset \mathbb{Z}^2$ whose boundary paths are both length k; and a set of complex numbers $\mathbf{z} = \big(z(v)\big)_{v \in \mathcal{D}}$. Further fix a sequence of elements $\mathcal{F} = \big(\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k\big)$ in $\{0,1\}^n$ constituting (colored) exit data on $\mathcal{D}$, and sequences $\mathsf{E} = (e_1, e_2, \ldots, e_k) \in \mathbb{Z}^k_{\geqslant 0}$ and $\mathsf{F} = (f_1, f_2, \ldots, f_k) \in \mathbb{Z}^k_{\geqslant 0}$ constituting uncolored entrance and exit data on $\mathcal{D}$, respectively; assume that $|\mathbf{F}_i| = f_i$ for each $i \in [1,k]$ and that $\sum_{i=1}^k \mathbf{F}_i \leqslant \mathbf{e}_{[1,n]}$. Then,*

$$\sum_{\mathcal{E}} q^{\mathrm{inv}(\mathcal{E}) - \mathrm{inv}(\mathcal{F})} \mathcal{W}_{\mathcal{D}}(\mathcal{E}; \mathcal{F} \mid \mathbf{z} \mid \infty, 0) = \mathcal{U}_{\mathcal{D}}(\mathsf{E}; \mathsf{F} \mid \mathbf{z}), \tag{9.6.6}$$

where the sum is over all sequences $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k)$ of elements in $\{0,1\}^n$ such that $|\mathbf{E}_i| = e_i$ for each $i \in [1,k]$.

Proof (Outline). We only establish the lemma when $\mathcal{D} = \{v\}$ consists of a single vertex, as the general case follows from this one similarly to as in the proof of Proposition 2.3.1 in Section 2.4.

So, let us abbreviate $z = z(v)$; we may also assume that $e_1 + e_2 = f_1 + f_2$, as otherwise both sides of (9.6.6) are equal to 0 by arrow conservation.

In this case, $k = 2$, and so the sum in (9.6.6) is over all pairs $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2)$ of elements in $\{0,1\}^n$ such that $|\mathbf{E}_i| = e_i$, for each $i \in \{1, 2\}$. Moreover, by (5.2.1), we have $\operatorname{inv}(\mathcal{E}) = \varphi(\mathbf{E}_2, \mathbf{E}_1)$ and $\operatorname{inv}(\mathcal{F}) = \varphi(\mathbf{F}_2, \mathbf{F}_1)$. So, since $\mathcal{W}_{\mathcal{D}}(\mathcal{E}; \mathcal{F} \mid \mathbf{z} \mid \infty, 0) = \mathcal{W}_z(\mathbf{E}_2, \mathbf{E}_1; \mathbf{F}_1, \mathbf{F}_2 \mid \infty, 0)$ and $\mathcal{U}_{\mathcal{D}}(\mathsf{E}; \mathsf{F} \mid \mathbf{z}) = \mathcal{U}_z(e_2, e_1; f_1, f_2)$ it suffices by (9.6.4) to show that

$$\sum_{\mathcal{E}} q^{\varphi(\mathbf{E}_2, \mathbf{E}_1) - \varphi(\mathbf{F}_2, \mathbf{F}_1)} \mathcal{W}_z(\mathbf{E}_2, \mathbf{E}_1; \mathbf{F}_1, \mathbf{F}_2 \mid \infty, 0) = \mathcal{U}_z(e_2, e_1; f_1, f_2) = q^{\binom{f_2}{2}} z^{f_2} \frac{(q;q)_{e_1+e_2}}{(q;q)_{e_1}(q;q)_{e_2}},$$

where we sum over all pairs $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2)$ with $|\mathbf{E}_i| = e_i$, for each $i \in \{1, 2\}$. Recalling from (8.4.1) that $\mathcal{W}_z(\mathbf{E}_1, \mathbf{E}_1; \mathbf{F}_1, \mathbf{F}_2 \mid \infty, 0) = q^{\binom{f_2}{2} + \varphi(\mathbf{F}_2, \mathbf{F}_1)} z^{f_2}$, we must therefore verify

$$\sum_{\mathcal{E}} q^{\varphi(\mathbf{E}_2, \mathbf{E}_1)} = \frac{(q;q)_{e_1+e_2}}{(q;q)_{e_1}(q;q)_{e_2}}, \tag{9.6.7}$$

where we sum over all pairs $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2)$ with $|\mathbf{E}_i| = e_i$ for each $i \in \{1, 2\}$ and $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{F}_1 + \mathbf{F}_2$ (which may be imposed by arrow conservation).

To that end, let $\mathbf{E}_i = (E_{i,1}, E_{i,2}, \ldots, E_{I,n})$ for each $i \in \{1, 2\}$; since $\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2)$ is fixed, the condition $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{F}_1 + \mathbf{F}_2 \leqslant \mathbf{e}_{[1,n]}$ (which was imposed by the statement of the lemma) fixes the indices $j \in [1, n]$ for which we have $E_{1,j} + E_{2,j} = 0$ or $E_{1,j} + E_{2,j} = 1$. Indices j for which $E_{1,j} + E_{2,j} = 0$ impose $E_{1,j} = E_{2,j} = 0$ and do not contribute to the exponent $q^{\varphi(\mathbf{E}_2, \mathbf{E}_1)}$ on the left side of (9.6.7). Thus, we may remove such indices from consideration, thereby assuming that $n = e_1 + e_2$. Then, (9.6.7) is equivalent to

$$\sum_{\substack{\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{e}_{[1,n]} \\ |\mathbf{E}_1| = e_1 \\ |\mathbf{E}_2| = e_2}} \prod_{1 \leqslant j < k \leqslant n} q^{E_{1,k} E_{2,j}} = \frac{(q;q)_{e_1+e_2}}{(q;q)_{e_1}(q;q)_{e_2}}. \tag{9.6.8}$$

Observe in this sum that $\mathbf{E}_1$ is fixed by $\mathbf{E}_2$ and that $E_{1,k} E_{2,j} = 1$ holds if and only if $(E_{2,j}, E_{2,k}) = (1, 0)$ or, equivalently, $E_{2,j} > E_{2,k}$. Thus, (9.6.8) follows from the interpretation of the q-binomial coefficient $(q;q)_{e_1+e_2}(q;q)_{e_1}^{-1}(q;q)_{e_2}^{-1}$ as the generating series for inversions of $\{0,1\}^{e_1+e_2}$ strings with e_1 zeroes and e_2 ones. $\square$

Now we can establish Proposition 9.6.1.

Proof of Proposition 9.6.1. In what follows, we adopt the notation of Lemma 9.6.3. That lemma and the fact that $\operatorname{inv}\big(\mathcal{I}(\lambda)\big) = 0$ for any partition λ, together imply

$$\begin{aligned} \mathcal{L}_{\tilde{\lambda}}^{(n)}(\mathbf{x}) &= \sum_{\mathcal{E}} q^{\operatorname{inv}(\mathcal{E})} \mathcal{W}_{\mathcal{D}_N}\big(\mathcal{E}; \mathcal{I}(\lambda) \mid \mathbf{z} \mid \infty, 0\big) \prod_{j=1}^{N} q^{\binom{|\mathbf{E}_j|}{2}} x_{N-j+1}^{|\mathbf{E}_j|} \\ &= \sum_{\mathsf{E}} \prod_{j=1}^{N} q^{\binom{e_j}{2}} x_{N-j+1}^{e_j} \sum_{\mathcal{E}} q^{\operatorname{inv}(\mathcal{E}) - \operatorname{inv}(\mathcal{I}(\lambda))} \mathcal{W}_{\mathcal{D}_N}\big(\mathcal{E}; \mathcal{I}(\lambda) \mid \mathbf{z} \mid \infty, 0\big). \end{aligned} \tag{9.6.9}$$

Here, the sum in the second term is over all sequences $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ of elements in $\{0,1\}^n$ such that $\mathbf{E}_k = \mathbf{e}_0$ for $k > N$; the first sum in the third term is over all integer sequences $\mathsf{E} = (e_1, e_2, \ldots)$ such that $e_k = 0$ for $k > N$; and the second sum in the third term is over all sequences $\mathcal{E} = (\mathbf{E}_1, \mathbf{E}_2, \ldots)$ of elements in $\{0,1\}^n$ such that $|\mathbf{E}_i| = e_i$ for each $i \geqslant 1$.

Applying (9.6.6) in (9.6.9), we deduce

$$\mathcal{L}_{\widetilde{\lambda}}^{(n)}(\boldsymbol{x}) = \sum_{\mathsf{E}} \mathcal{U}_{\mathcal{D}_N}\big(\mathsf{E}; \mathsf{I}(\lambda) \mid \mathbf{z}\big) \prod_{j=1}^{N} q^{\binom{e_j}{2}} x_{N-j+1}^{e_j},$$

which together with Proposition 9.6.4 yields the proposition. $\square$

CHAPTER 10

Proof of Proposition 9.3.4

In this chapter we establish Proposition 9.3.4 (that is, the $N = 1$ case of Theorem 9.3.2). This consists in four parts, given by the corresponding parts of Theorem 9.3.2. We establish the first, second, third, and fourth in Section 10.1, Section 10.2, Section 10.3, and Section 10.4, respectively. Throughout this section, we adopt the notation of Theorem 9.3.2 and set $x_1 = x$.

10.1. LLT Polynomials From $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$

In this section we establish the first part of Proposition 9.3.4, given by the $N = 1$ case of (9.3.3). To that end, recall from (7.1.3) and Remark 8.4.4 that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ is the partition function under the $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ weights (from (8.4.1)) for the vertex model $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; 1)$ (from Definition 7.1.3; see also the left side of Figure 7.1) on the domain $\mathcal{D} = \mathcal{D}_1 = \mathbb{Z}_{>0} \times \{1\}$. Since this domain has one row, $\mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu}; 1)$ only consists in a single path ensemble, which we denote by $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. For each $j \geqslant 1$, let $\big(\mathbf{A}(j), \mathbf{B}(j); \mathbf{C}(j), \mathbf{D}(j)\big)$ denote the arrow configuration at the vertex $(j, 1) \in \mathcal{D}$ under $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$; we refer to the left side of Figure 10.1 for an example in the case when $\boldsymbol{\lambda} = \big((3,1),(2,2),(3,0)\big)$ and $\boldsymbol{\mu} = \big((1,0),(2,0),(2,0)\big)$.

Let us now "decompress" $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ to form a pair of colored Maya diagrams $\big(\mathfrak{T}(\lambda), \mathfrak{T}(\mu)\big)$, as follows. We first split each vertex $(j, 1)$ of $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ into the interval $\{nj - n + 1, nj - n + 2, \ldots, nj\}$ consisting of n integers. Then, if an arrow of color $c \in [1, n]$ enters through $(j, 1)$, we set $(j-1)n + c \in \mathfrak{T}(\mu)$; similarly, if an arrow of color c exits through $(j, 1)$, we set $(j-1)n + c \in \mathfrak{T}(\lambda)$. Observe from the content in Section 9.2 that $\lambda, \mu \in \mathrm{Sign}_{nM}$ are characterized by the property that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is the n-quotient of λ/μ. We refer to Figure 10.1 for an example, where $\lambda/\mu = (9,8,7,5,2,2)/(6,6,3,0,0,0)$.

Next we establish the following two lemmas expressing $\mathrm{sp}(\lambda/\mu)$ and $\psi(\boldsymbol{\lambda}/\boldsymbol{\mu})$ through $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$.

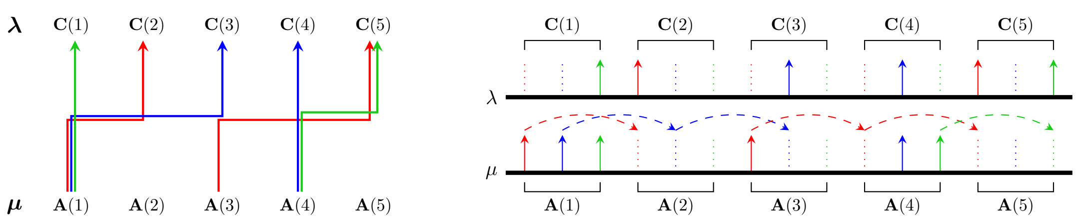


FIGURE 10.1. To the left is a vertex model for a 3-tuple of skew-shapes. To the right is its decompression into a pair of colored Maya diagrams.

Lemma 10.1.1. *If λ/μ is a horizontal n-ribbon strip, then (recalling φ from* (1.1.1)*) we have that*

$$\mathrm{sp}(\lambda/\mu) = \sum_{j=1}^{\infty} \varphi\big(\mathbf{D}(j), \mathbf{D}(j)\big) + \frac{1}{2} \sum_{j=1}^{\infty} \Big(\varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big) + \varphi\big(\mathbf{A}(j), \mathbf{B}(j)\big)\Big).$$

Proof. Recall from Section 9.2 that, if λ/μ is a horizontal n-ribbon strip, then $\mathrm{sp}(\lambda/\mu)$ equals half of the total number of particles jumped over in the nonconflicting series of n-jumps $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m$ that transforms the Maya diagram $\mathfrak{T}(\mu) = \mathcal{T}_0$ into $\mathfrak{T}(\lambda) = \mathcal{T}_m$. Observe that the paths in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ correpsond to the jump trajectories for particles in this nonconflicting series, in the sense that a horizontal arrow of color $c \in [1, n]$ vertically enters and horizontally exits through a vertex $(j, 1)$ in this model if and only if a particle at site $n(j-1) + c$ eventually jumps in this nonconflicting series.

Now, let us classify the ways in which a particle at some site $J \in \mathcal{T}_i$ can jump over one at another site $K \in \mathcal{T}_i$ into three types; in what follows we let $J = n(j-1) + a$ and $K = n(k-1) + b$, so that $k \in \{j, j+1\}$ (since all jumps are of size n, that is, $J < K < J + n$) with $a < b$ if $k = j$ and $a > b$ if $k = j + 1$. This jump is of type 1 if it satisfies $K \in \mathcal{T}_m = \mathfrak{T}(\lambda)$ and $k = j$; of type 2 if it satisfies $K \in \mathcal{T}_0 = \mathfrak{T}(\mu)$ and $k = j + 1$; and of type 3 if it satisfies neither of these two conditions. For instance, on the right side of Figure 10.1, a jump of the first type comes from the pair $(J, K) = (1, 3)$; a jump of the second type comes from $(J, K) = (5, 7)$; and a jump of the third type comes from $(J, K) = (2, 4)$.

For fixed $j, k \geqslant 1$, jumps of type 1 are in bijection with pairs of arrows at $(j, 1)$ in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, of distinct colors a and b, satisfying the following three conditions. First, $a < b$ (since $j = k$); second, the arrow of color a exits $(j, 1)$ horizontally (since it corresponds to the jumper at site $J = n(j-1)+a$); and, third, the arrow of color b exits $(j, 1)$ vertically (since $K = n(k-1)+b \in \mathfrak{T}(\lambda)$). Recalling the definition of φ from (1.1.1), the number of such pairs is given by $\varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big)$. By similar reasoning, jumps of type 2 are in bijection with pairs of arrows at $(k, 1)$, of distinct colors a and b, such that $a > b$ (since $k = j + 1$); the arrow of color a enters $(k, 1)$ horizontally (since the destination of the jumper at site $J = n(j-1) + a$ is $n(k-1) + a$); and the arrow of color b enters $(k, 1)$ vertically (since $K = n(k-1) + b \in \mathfrak{T}(\mu)$). Thus, there are $\varphi\big(\mathbf{A}(k), \mathbf{B}(k)\big)$ such pairs.

Next, we claim that the number of jumps of the third type is twice the number of unordered pairs (that is, subsets of size 2) of horizontal arrows in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ passing from $(j, 1)$ to $(j+1, 1)$ of distinct colors a and b. Indeed, recalling that $J = n(j-1) + a$ and $K = n(k-1) + b$, let us consider the cases $k \in \{j, j+1\}$ individually. If $k = j$, then we must have $K \notin \mathfrak{T}(\lambda)$ for the jump to not be of type 1. Thus, the particle at site K must eventually jump horizontally in the nonconflicting series $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m$, meaning that an arrow of color b must horizontally exit $(j, 1)$ in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then, we associate this jump with the unordered pair $\{a, b\}$ of arrows horizontally exiting $(j, 1)$ (the arrow of color a must horizontally exit this vertex during the jump). If instead $k = j + 1$, then we must have $K \notin \mathfrak{T}(\mu)$ for the jump to not be of type 2. Thus, the particle at site K must have jumped from site $K - n$ at some previous point in the nonconflicting series $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_m$, meaning that an arrow of color b must horizontally enter $(k, 1) = (j+1, 1)$ in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then, we again associate this jump with the unordered pair $\{a, b\}$ of arrows horizontally entering $(j+1, 1)$ (the arrow of color a must horizontally enter this vertex during the jump). This establishes the claim, so the number of jumps of the third type is given by $2\binom{|\mathbf{D}(j)|}{2} = 2\varphi\big(\mathbf{D}(j)\big)$.

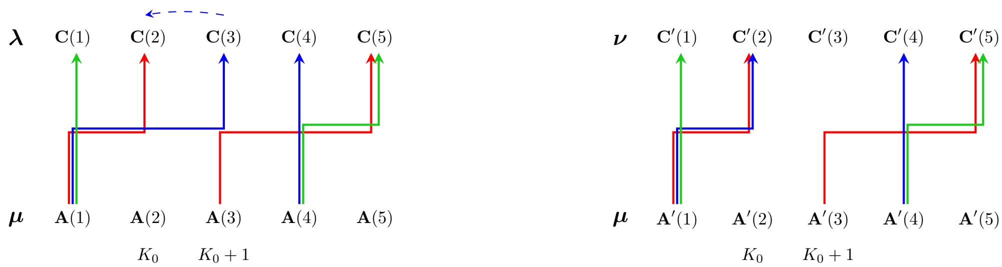


FIGURE 10.2. Shown above are the single-row path ensembles $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$ used in the proof of Lemma 10.1.2.

Summing over these three types, we deduce that the total number of jumps in the nonconflicting series $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ is given by

$$\sum_{j=1}^{\infty} \Big(2\varphi\big(\mathbf{D}(j), \mathbf{D}(j)\big) + \varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big) + \varphi\big(\mathbf{A}(j), \mathbf{B}(j)\big) \Big).$$

This implies the lemma, since $2\,\mathrm{sp}(\lambda/\mu)$ is the total number of these jumps. $\square$

Lemma 10.1.2. *Recalling ψ from* (9.3.1), *we have that*

$$\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu}) = \frac{1}{2} \sum_{j=1}^{\infty} \Big(\varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big) - \varphi\big(\mathbf{A}(j), \mathbf{B}(j)\big) \Big). \tag{10.1.1}$$

PROOF. We induct on $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$. If $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ then $\boldsymbol{\lambda} = \boldsymbol{\mu}$, so $\mathbf{B}(j) = \mathbf{e}_0 = \mathbf{D}(j)$ for each $j \geqslant 1$, which implies that both sides of (10.1.1) equal 0. Thus, let us assume in what follows that (10.1.1) holds whenever $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| < m$ for some integer $m > 0$, and we will show it also holds whenever $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| = m$.

To that end, fix $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$ such that $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| = m$ and $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$. Then, there exists a sequence $\boldsymbol{\nu} \in \mathrm{SeqSign}_n$ such that $|\boldsymbol{\nu}| - |\boldsymbol{\mu}| = m - 1$ and $\boldsymbol{\mu} \subseteq \boldsymbol{\nu} \subseteq \boldsymbol{\lambda}$. In particular, there exist indices $h \in [1, n]$ and $k \in [1, M]$ such that $\nu_j^{(i)} = \lambda_j^{(i)}$ for all $(i, j) \neq (h, k)$ and $\nu_k^{(h)} = \lambda_k^{(h)} - 1$. In what follows, we set $K_0 = \nu_k^{(h)} + M - k + 1 \in \mathfrak{T}\big(\nu^{(h)}\big)$, so that $K_0 + 1 = \lambda_k^{(h)} + M - k + 1 \in \mathfrak{T}\big(\lambda^{(h)}\big)$.

Now let us compare the single-row path ensembles $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$ on $\mathbb{Z}_{>0} \times \{1\}$. We may interpret $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$ as the ensemble obtained by moving the output of the color h arrow vertically exiting through $(K_0 + 1, 1)$ in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ to the left by one space to $(K_0, 1)$; see Figure 10.2 for a depiction, where there $K_0 = 2$ and the output $(K_0 + 1, 1)$ for the color 2 (blue) arrow is shifted to $(K_0, 1)$. In this way, $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$ coincide everywhere in all colors except for the h-th color, and and they also coincide in the h-th color at all vertices except for $(K_0, 1)$ and $(K_0 + 1, 1)$.

To make the latter point more precise, recall for each $j \geqslant 1$ that $\big(\mathbf{A}(j), \mathbf{B}(j); \mathbf{C}(j), \mathbf{D}(j)\big)$ denotes the arrow configuration at $(j, 1)$ under $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Similarly let $\big(\mathbf{A}'(j), \mathbf{B}'(j); \mathbf{C}'(j), \mathbf{D}'(j)\big)$ denote the arrow configuration at $(j, 1)$ under $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$. For each index $X \in \{A, B, C, D, A', B', C', D'\}$, set $\mathbf{X}(j) = \big(X_1(j), X_2(j), \ldots, X_n(j)\big) \in \{0, 1\}^n$. Then, it is quickly verified that $X_i(j) = X_i'(j)$ for any integers $i \in [1, n]$ and $j \geqslant 1$ and index $X \in \{A, B, C, D\}$, unless $i = h$ and $(X, j) \in$

$\big\{(C, K_0), (D, K_0), (B, K_0+1), (C, K_0+1)\big\}$. In those exceptional cases, we have that

$$\begin{aligned} C_h(K_0+1) &= 1 = C'_h(K_0); \qquad & C_h(K_0) &= 0 = C'_h(K_0+1); \\ D_h(K_0) &= 1 = B_h(K_0+1); \qquad & D'_h(K_0) &= 0 = B'_h(K_0+1). \end{aligned} \tag{10.1.2}$$

Now, observe that

$$\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\nu}) = \frac{1}{2}\big(C_{[h+1,n]}(K_0) - C_{[1,h-1]}(K_0+1)\big), \tag{10.1.3}$$

since any contribution to $\psi(\boldsymbol{\lambda})$ not in $\psi(\boldsymbol{\nu})$ in the right side of (9.3.1) comes from some pair $(K_0+1, K_0) \in \mathfrak{T}\big(\lambda^{(h)}\big) \times \mathfrak{T}\big(\lambda^{(j)}\big)$ for some $j \in [h+1, n]$, and any contribution to $\psi(\boldsymbol{\nu})$ not in $\psi(\boldsymbol{\lambda})$ comes from some pair $(K_0+1, K_0) \in \mathfrak{T}\big(\nu^{(i)}\big) \times \mathfrak{T}\big(\nu^{(h)}\big) = \mathfrak{T}\big(\lambda^{(i)}\big) \times \mathfrak{T}\big(\nu^{(h)}\big)$ for some $i \in [1, h-1]$.

Moreover, applying (10.1.1) for the $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ there equal to $(\boldsymbol{\nu}, \boldsymbol{\mu})$ here, we deduce that

$$\begin{aligned} \psi(\boldsymbol{\nu}) - \psi(\boldsymbol{\mu}) &= \frac{1}{2}\sum_{j=1}^{\infty}\Big(\varphi\big(\mathbf{D}'(j), \mathbf{C}'(j)\big) - \varphi\big(\mathbf{A}'(j), \mathbf{B}'(j)\big)\Big) \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\Big(\varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big) - \varphi\big(\mathbf{A}(j), \mathbf{B}(j)\big)\Big) \\ &\qquad + \frac{1}{2}\big(D_{[1,h-1]}(K_0) - C_{[h+1,n]}(K_0) - D_{[1,h-1]}(K_0+1) + A_{[1,h-1]}(K_0+1)\big), \end{aligned}$$

where in the last equality we used the matching between $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\mathcal{E}(\boldsymbol{\nu}/\boldsymbol{\mu})$ at all vertices and indices except those indicated in (10.1.2), together with the facts that

$$\begin{aligned} &\varphi\big(\mathbf{D}'(K_0), \mathbf{C}'(K_0)\big) = \varphi\big(\mathbf{D}(K_0), \mathbf{C}(K_0)\big) + D_{[1,h-1]}(K_0) - C_{[h+1,n]}(K_0); \\ &\varphi\big(\mathbf{D}'(K_0+1), \mathbf{C}'(K_0+1)\big) = \varphi\big(\mathbf{D}(K_0+1), \mathbf{C}(K_0+1)\big) - D_{[1,h-1]}(K_0+1); \\ &\varphi\big(\mathbf{A}'(K_0), \mathbf{B}'(K_0)\big) = \varphi\big(\mathbf{A}(K_0), \mathbf{B}(K_0)\big); \\ &\varphi\big(\mathbf{A}'(K_0+1), \mathbf{B}'(K_0+1)\big) = \varphi\big(\mathbf{A}(K_0+1), \mathbf{B}(K_0+1)\big) - A_{[1,h-1]}(K_0+1), \end{aligned}$$

which follow from (10.1.2) and the explicit form (1.1.1) for φ. Combining this with the identities $\mathbf{D}(K_0) = \mathbf{B}(K_0+1)$ and $\mathbf{A}(K_0+1) + \mathbf{B}(K_0+1) = \mathbf{C}(K_0+1) + \mathbf{D}(K_0+1)$ gives

$$\psi(\boldsymbol{\nu}) - \psi(\boldsymbol{\mu}) = \frac{1}{2}\sum_{j=1}^{\infty}\Big(\varphi\big(\mathbf{D}(j), \mathbf{C}(j)\big) - \varphi\big(\mathbf{A}(j), \mathbf{B}(j)\big)\Big) + \frac{1}{2}\big(C_{[1,h-1]}(K_0+1) - C_{[h+1,n]}(K_0)\big),$$

which, together with (10.1.3), yields (10.1.1) and thus the lemma. $\square$

We can now establish the first part of Proposition 9.3.4.

Proof of Part 1 of Proposition 9.3.4. Let us first assume that λ/μ is not a horizontal n-ribbon strip. Then, any series of n-jumps transforming $\mathfrak{T}(\mu)$ into $\mathfrak{T}(\lambda)$ must be conflicting, meaning that in any such series there exists a jumper at some site $J \geqslant 1$ such that its destination $J + rn \in \mathfrak{T}(\mu)$ (where $r \geqslant 1$ is some integer); stated alternatively, the destination of this jumper is occupied in $\mathfrak{T}(\mu)$. Letting $J + rn = n(k-1) + h$ for some integers $k \geqslant 2$ and $h \in [1, n]$, this means that horizontal and vertical arrows of color h both enter and exit through the vertex $(k, 1)$ in the path ensemble $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, that is, $A_h(k) = B_h(k) = C_h(k) = D_h(k) = 1$.

Due to the factor of $\mathbf{1}_{v=0}$ in the expression (8.4.1) for $\mathcal{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$, it follows that the weight of $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is 0, and so $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x, \infty \mid 0; 0) = 0$. Similarly, from the definition (9.1.2) (and

(9.2.1)) for $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, we also find that $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x) = 0$ if λ/μ is not a ribbon strip, thereby verifying the proposition in this case.

So, let us assume that λ/μ is a horizontal ribbon strip. Then, (9.1.2) implies that

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x) = \mathcal{L}^{(n)}_{\lambda/\mu}(x) = q^{\mathrm{sp}(\lambda/\mu)} x^{(|\lambda|-|\mu|)/n} = q^{\mathrm{sp}(\lambda/\mu)} x^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|}. \tag{10.1.4}$$

Additionally, since $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x,\infty;0;0)$ is the weight of $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ under $\mathcal{W}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid \infty,0)$, the explicit form from (the fourth statement of) (8.4.1) for this weight yields

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid 0;0) = \prod_{j=1}^{\infty} \mathcal{W}_x\big(\mathbf{A}(j),\mathbf{B}(j);\mathbf{C}(j),\mathbf{D}(j) \mid \infty,0\big) = \prod_{j=1}^{\infty} x^{|\mathbf{D}(j)|} q^{\varphi(\mathbf{D}(j),\mathbf{C}(j)+\mathbf{D}(j))}.$$

Using Lemma 10.1.1, Lemma 10.1.2, and the fact that $\sum_{j=1}^{\infty} \big|\mathbf{D}(j)\big| = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$, it follows that

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid 0;0) = q^{\mathrm{sp}(\lambda/\mu)+\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\mu})} x^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|},$$

which together with (10.1.4) implies the first part of the proposition. $\square$

10.2. LLT Polynomials From $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$

In this section, we establish the second part of Proposition 9.3.4, namely, the $N = 1$ case of (9.3.4). To that end, as in Section 10.1, recall from (7.1.3) and Remark 8.4.4 that $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid 0;0)$ is the partition function under the $\widehat{\mathcal{W}}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid \infty,0)$ weights from (8.4.2) for the vertex model $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (from Definition 7.1.3 and depicted in the middle of Figure 7.1) on the domain $\mathcal{D} = \mathcal{D}_1 = \mathbb{Z}_{>0} \times \{1\}$. Since this domain has one row, $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ only consists in a single path ensemble, which we denote by $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. For each $j \geqslant 1$, let $\big(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j);\widehat{\mathbf{C}}(j),\widehat{\mathbf{D}}(j)\big)$ denote the arrow configuration at the vertex $(j,1) \in \mathcal{D}$ under $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. In particular, there exists a minimal integer $L_0 > 1$ such that $\big(\widehat{\mathbf{A}}(K),\widehat{\mathbf{B}}(K);\widehat{\mathbf{C}}(K),\widehat{\mathbf{D}}(K)\big) = \big(\mathbf{e}_0,\mathbf{e}_{[1,n]};\mathbf{e}_0,\mathbf{e}_{[1,n]}\big)$ for all $K \geqslant L_0$; it is given by

$$L_0 = \max_{i\in[1,n]} \Big(\max \mathfrak{T}\big(\lambda^{(i)}\big)\Big) + 1. \tag{10.2.1}$$

See the left side of Figure 10.3 for an example when $\boldsymbol{\lambda} = \big((1,0),(2,0),(2,0)\big)$ and $\boldsymbol{\mu} = \big((1),(2),(0)\big)$.

Recall that M denotes the length of each signature in $\boldsymbol{\mu}$ and $M+1$ denotes that of each one in $\boldsymbol{\lambda}$. To show the $N = 1$ case of (9.3.4), we will compare $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ with $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$, for some sequences $\widetilde{\boldsymbol{\mu}},\mathring{\boldsymbol{\lambda}} \in \mathrm{SeqSign}_{n;M+1}$, where we recall the path ensemble $\mathcal{E}$ from Section 10.1. In particular, to define these sequences, set

$$\begin{aligned} \mathring{\lambda}^{(i)}_j &= \lambda^{(i)}_j + 1, \quad \text{for all } j \in [1,M+1]; \\ \widetilde{\mu}^{(i)}_1 = L_0 - M - 1, \quad &\text{and} \quad \widetilde{\mu}^{(i)}_j = \mu^{(i)}_{j-1} + 1, \quad \text{for all } j \in [2,M+1], \end{aligned} \tag{10.2.2}$$

for each $i \in [1,n]$. Stated alternatively, $\mathring{\boldsymbol{\lambda}} \in \mathrm{SeqSign}_{n;M+1}$ is defined by forming each $\mathfrak{T}\big(\mathring{\lambda}^{(i)}\big)$ through shifting every entry in $\mathfrak{T}\big(\lambda^{(i)}\big)$ to the right by one space, and $\widetilde{\boldsymbol{\mu}} \in \mathrm{SeqSign}_{n;M+1}$ is defined by forming each $\mathfrak{T}\big(\widetilde{\mu}^{(i)}\big)$ through first shifting every entry in $\mathfrak{T}\big(\mu^{(i)}\big)$ to the right by one space, and then appending L_0. Observe that defining $\mu^{(i)}$ in this way indeed gives rise to a valid signature (with non-increasing entries), since $\widetilde{\mu}^{(i)}_1 = L_0 - M - 1 \geqslant \lambda^{(i)}_1 + 1 \geqslant \mu^{(i)}_1 + 1 = \widetilde{\mu}^{(i)}_2$. For each $j \geqslant 1$, let $\big(\widetilde{\mathbf{A}}(j),\widetilde{\mathbf{B}}(j);\widetilde{\mathbf{C}}(j),\widetilde{\mathbf{D}}(j)\big)$ denote the arrow configuration at $(j,1) \in \mathcal{D}$ under $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$. We refer to the middle of Figure 10.3 for a depiction.

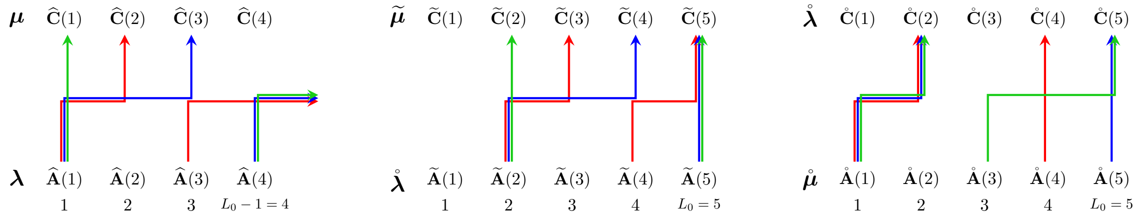


FIGURE 10.3. Shown above are the single-row path ensembles $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$, and $\mathcal{E}\big(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}}\big)$.

Observe in particular that the path ensembles $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ with $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$ nearly coincide, except for two differences. First, all vertices in the latter are shifted to the right by one with respect to the former. Second, arrows of all colors exit horizontally through $(L_0-1,1)$ (and through all vertices east of it) in $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, while arrows of all colors exit vertically through $(L_0,1)$ in $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$.

For this reason, the partition function of $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, under the weights $\widehat{\mathcal{W}}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0)$, will coincide with that of $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$, under the weights $\mathcal{W}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty,0)$, up to a factor coming from the relation (5.1.4) between $\mathcal{W}_x$ and $\widehat{\mathcal{W}}_x$. As shown in Section 10.1, partition functions of the latter type give rise to a factor of $q^{\mathrm{sp}(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}})}$. Since the LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ involves a power of $q^{\mathrm{sp}(\boldsymbol{\lambda}/\boldsymbol{\mu})}$, we must express these spins (in the exponents) through one another other.

To that end, we have the following lemma, where in the below $\lambda\in\mathrm{Sign}_{n(M+1)}$ and $\mu\in\mathrm{Sign}_{nM}$ are defined so that their n-quotients are $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively, and similarly $\mathring{\lambda},\widetilde{\mu}\in\mathrm{Sign}_{n(M+1)}$ are defined so that their n-quotients are $\mathring{\boldsymbol{\lambda}}$ and $\widetilde{\boldsymbol{\mu}}$, respectively.

Lemma 10.2.1. *If λ/μ is a horizontal n-ribbon strip, then*

$$\mathrm{sp}\left(\widetilde{\mu}/\mathring{\lambda}\right)=\mathrm{sp}(\lambda/\mu)+(n-1)\big(|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|-n\big)+(L_0-M)\binom{n}{2}.$$

PROOF. Recall from (10.2.2) that the sequence $\mathring{\boldsymbol{\lambda}}\in\mathrm{SeqSign}_{n;M+1}$ was defined from $\boldsymbol{\lambda}$ by increasing every entry of each of its signatures by one. Let us define $\mathring{\boldsymbol{\mu}}\in\mathrm{SeqSign}_{n;M+1}$ from $\boldsymbol{\mu}$ in a similar way, but where we also append a zero to each of its signatures so that they all have length $M+1$ (as in $\mathring{\boldsymbol{\lambda}}$). More specifically, for each $i\in[1,n]$, we set

$$\mathring{\mu}_j^{(i)}=\mu_j^{(i)}+1,\qquad\text{for each } j\in[1,M],\text{ and}\quad \mathring{\mu}_{M+1}^{(i)}=0. \tag{10.2.3}$$

Equivalently, it is defined by forming each $\mathfrak{T}\big(\mathring{\mu}^{(j)}\big)$ through first shifting every entry in $\mathfrak{T}\big(\mu^{(j)}\big)$ to the right two spaces, and then appending 1. We refer to the right side of Figure 10.3 for a depiction. Let $\mathring{\mu}\in\mathrm{Sign}_{n(M+1)}$ denote the signature whose n-quotient is $\mathring{\boldsymbol{\mu}}$.

Since $\boldsymbol{\mu}\subseteq\boldsymbol{\lambda}$, we have that $\mathring{\boldsymbol{\mu}}\subseteq\mathring{\boldsymbol{\lambda}}$. So, as $\mathcal{D}=\mathbb{Z}_{>0}\times\{1\}$ has one only row, there is a unique path ensemble $\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}})$ on $\mathcal{D}$ in the set $\mathfrak{P}_G(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}};1)$ from Definition 7.1.3. For each $j\geqslant 1$, let $\big(\mathring{\mathbf{A}}(j),\mathring{\mathbf{B}}(j);\mathring{\mathbf{C}}(j),\mathring{\mathbf{D}}(j)\big)$ denote the arrow configuration at the vertex $(j,1)\in\mathcal{D}$ under $\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}})$; see the right side of Figure 10.3 for a depiction. In what follows, the coordinates of any element $\mathbf{X}\in\{0,1\}^n$ will be indexed by $[1,n]$.

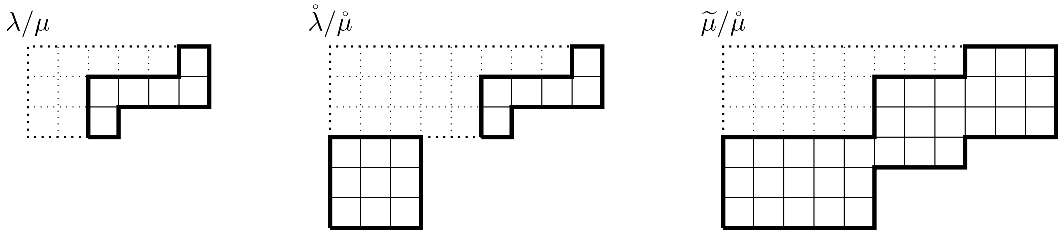


FIGURE 10.4. Shown to the left, middle, and right are the Young diagrams for skew-shapes λ/μ, $\mathring{\lambda}/\mathring{\mu}$, and $\widetilde{\mu}/\mathring{\mu}$ from Figure 10.3, respectively, where here $n = 3$.

Let us show that

$$\operatorname{sp}(\lambda/\mu) + \binom{n}{2} = \operatorname{sp}\left(\mathring{\lambda}/\mathring{\mu}\right) = \operatorname{sp}\left(\widetilde{\mu}/\mathring{\lambda}\right) + (n-1)\big(|\mathring{\boldsymbol{\lambda}}| - |\mathring{\boldsymbol{\mu}}|\big) + (M - L_0 + 1)\binom{n}{2}, \tag{10.2.4}$$

from which the lemma would follow since (10.2.2) and (10.2.3) imply

$$|\mathring{\boldsymbol{\lambda}}| = |\boldsymbol{\lambda}| + (M+1)n; \qquad |\mathring{\boldsymbol{\mu}}| = |\boldsymbol{\mu}| + Mn.$$

To establish the first relation in (10.2.4), observe by (10.2.2) and (10.2.3) that $\mathring{\lambda}$ is obtained from λ by increasing each entry by n, and $\mathring{\mu}$ is obtained from μ by first increasing each entry by n and then appending n entries equal to zero; see the left and middle parts of Figure 10.4 for depictions. Thus, the skew-shape $\mathring{\lambda}/\mathring{\mu}$ is obtained from λ/μ by adding an $n \times n$ block of boxes. Since this block is tiled by n ribbons of shape $1 \times n$, each of which has height $n - 1$, we deduce $\operatorname{sp}(\mathring{\lambda}/\mathring{\mu}) - \operatorname{sp}(\lambda/\mu) = \frac{n(n-1)}{2}$. This verifies the first statement of (10.2.4), so it suffices to establish the latter.

To that end, we induct[1] on $|\mathring{\boldsymbol{\lambda}}| - |\mathring{\boldsymbol{\mu}}|$, as in the proof of Lemma 10.1.2. If $|\mathring{\boldsymbol{\lambda}}| = |\mathring{\boldsymbol{\mu}}|$, then $\mathring{\lambda} = \mathring{\mu}$, and it suffices to show that

$$\operatorname{sp}\left(\widetilde{\mu}/\mathring{\mu}\right) = \big(L_0 - M - 1\big)\binom{n}{2}. \tag{10.2.5}$$

To do this, we claim that $\widetilde{\mu} \in \operatorname{Sign}_{(M+1)n}$ is obtained from $\mathring{\mu}$ by adding a $1 \times n$ ribbon in each of the leftmost $n\big(L_0 - M - 1\big)$ columns of its Young diagram; see the right side of Figure 10.4 for a depiction. Since each such ribbon has height $n - 1$, we would then deduce $\operatorname{sp}(\widetilde{\mu}/\mathring{\mu}) = \frac{(n-1)}{2} n\big(L_0 - M - 1\big)$, which implies (10.2.5).

To confirm the claim, we must show that

$$\widetilde{\mu}_j = n\big(L_0 - M - 1\big), \quad \text{for each } j \in [1, n]; \qquad \widetilde{\mu}_i = \mathring{\mu}_{i-n}, \quad \text{for each } i \in [n+1, Mn+n]. \tag{10.2.6}$$

The fact that $\widetilde{\boldsymbol{\mu}}$ is the n-quotient of $\widetilde{\mu}$ yields for any $j \in [1, n]$ that the j-th largest element in $\mathfrak{T}(\widetilde{\mu}) = \bigcup_{i=1}^{n} \big(n\mathfrak{T}(\widetilde{\mu}^{(i)}) - n + i\big)$ is

$$\widetilde{\mu}_j + (M+1)n - j + 1 = n \max \mathfrak{T}\big(\widetilde{\mu}^{(j)}\big) - j + 1 = nL_0 - j + 1,$$

[1]Here, we will ignore the relationship (10.2.2) between $\boldsymbol{\lambda}$ and $\mathring{\boldsymbol{\lambda}}$, as the former is not present in the second equality of (10.2.4). In doing so, we will establish a more general statement by allowing some entries of signatures in $\mathring{\boldsymbol{\lambda}}$ to be zero, which is in principle not permitted by (10.2.2).

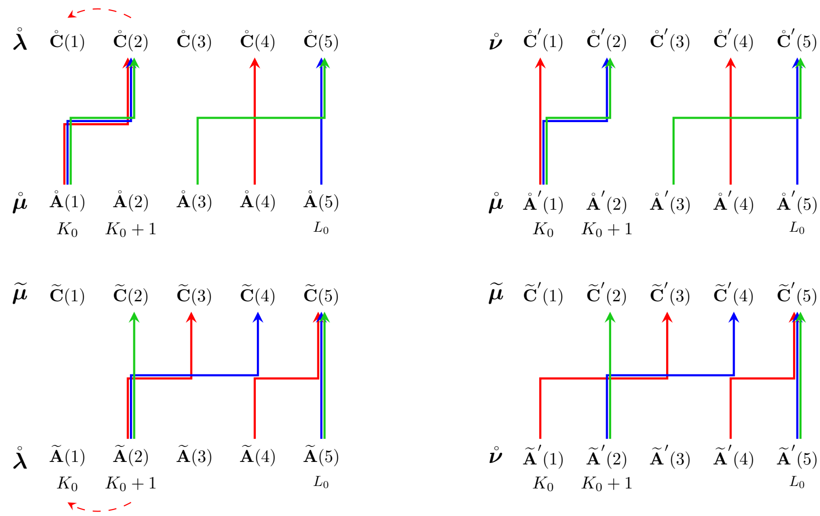


FIGURE 10.5. Shown above are the path ensembles $\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}})$, $\mathcal{E}(\mathring{\boldsymbol{\nu}}/\mathring{\boldsymbol{\mu}})$, $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$, and $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\nu}}\big)$.

where we have also used the fact that (10.2.2) implies $\max \mathfrak{T}\big(\widetilde{\mu}^{(i)}\big) = \widetilde{\mu}_1^{(i)} + M + 1 = L_0$ for each i. This establishes the first statement of (10.2.6). Further observe by (10.2.2) and (10.2.3) that $\mathring{\mu}_j^{(i)} = \mu_j^{(i)} + 1 = \widetilde{\mu}_{j-1}^{(i)}$ for each $j \in [2, M+1]$, which quickly implies that each entry in $\widetilde{\mu}$ (after removing the n largest ones equal to $L_0 - M - 1$) appears in $\mathring{\mu}$ with the same multiplicity. This implies the second statement of (10.2.6), thereby establishing (10.2.5) and hence the $|\mathring{\boldsymbol{\lambda}}| = |\mathring{\boldsymbol{\mu}}|$ case of (10.2.4).

Thus, let us assume in what follows that (10.2.4) holds whenever $|\mathring{\boldsymbol{\lambda}}| - |\mathring{\boldsymbol{\mu}}| < m$ for some integer $m > 0$, and we will show it also holds whenever $|\mathring{\boldsymbol{\lambda}}| - |\mathring{\boldsymbol{\mu}}| = m$. So, fix $\mathring{\boldsymbol{\lambda}} \in \mathrm{SeqSign}_{n;M+1}$ such that $|\mathring{\boldsymbol{\lambda}}| - |\mathring{\boldsymbol{\mu}}| = m$ and $\mathring{\boldsymbol{\mu}} \subseteq \mathring{\boldsymbol{\lambda}} \subseteq \widetilde{\boldsymbol{\mu}}$. Then, as in the proof of Lemma 10.1.2, there exists a sequence $\mathring{\boldsymbol{\nu}} \in \mathrm{SeqSign}_{n;M+1}$ such that $|\mathring{\boldsymbol{\nu}}| - |\mathring{\boldsymbol{\mu}}| = m - 1$ and $\mathring{\boldsymbol{\mu}} \subseteq \mathring{\boldsymbol{\nu}} \subseteq \mathring{\boldsymbol{\lambda}}$. In particular, there exist indices $h \in [1, n]$ and $k \in [1, M]$ such that $\mathring{\nu}_j^{(i)} = \mathring{\lambda}_j^{(i)}$ for all $(i, j) \neq (h, k)$ and $\mathring{\nu}_k^{(h)} = \mathring{\lambda}_k^{(h)} - 1$. In what follows, we set $K_0 = \mathring{\nu}_k^{(h)} + M - k + 2 \in \mathfrak{T}\big(\mathring{\nu}^{(h)}\big)$, so that $K_0 + 1 = \mathring{\lambda}_k^{(h)} + M - k + 2 \in \mathfrak{T}\big(\lambda^{(h)}\big)$.

Now let us compare the single-row path ensembles in $\big(\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}}), \mathcal{E}(\mathring{\boldsymbol{\nu}}/\mathring{\boldsymbol{\mu}})\big)$ and in $\big(\mathcal{E}(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}), \mathcal{E}(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\nu}})\big)$ on $\mathcal{D}_1$. We may interpret $\mathcal{E}(\mathring{\boldsymbol{\nu}}/\mathring{\boldsymbol{\mu}})$ as the ensemble obtained by moving the output of the color h arrow vertically exiting through $(K_0+1, 1)$ in $\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}})$ to the left by one space to $(K_0, 1)$. Similarly, we may interpret $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\nu}}\big)$ as the ensemble obtained by moving the input of the color h arrow vertically entering through $(K_0 + 1, 1)$ in $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$ to the left by one space to $(K_0, 1)$. We refer to Figure 10.5 for depictions in both cases, where there $K_0 = 1$. In this way, the ensembles in $\big(\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}}), \mathcal{E}(\mathring{\boldsymbol{\nu}}/\mathring{\boldsymbol{\mu}})\big)$ and in $\big(\mathcal{E}(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}), \mathcal{E}(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\nu}})\big)$ coincide everywhere in all colors except for the h-th color, and and they also coincide in the h-th color at all vertices except for $(K_0, 1)$ and $(K_0 + 1, 1)$.

To make this more precise, recall that $\big(\mathring{\mathbf{A}}(j), \mathring{\mathbf{B}}(j); \mathring{\mathbf{C}}(j), \mathring{\mathbf{D}}(j)\big)$ and $\big(\widetilde{\mathbf{A}}(j), \widetilde{\mathbf{B}}(j); \widetilde{\mathbf{C}}(j), \widetilde{\mathbf{D}}(j)\big)$ are the arrow configurations at any $(j, 1) \in \mathcal{D}_1$ under $\mathcal{E}(\mathring{\boldsymbol{\lambda}}/\mathring{\boldsymbol{\mu}})$ and $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$, respectively. For each

$j \geqslant 1$, further let $\big(\mathring{\mathbf{A}}'(j), \mathring{\mathbf{B}}'(j); \mathring{\mathbf{C}}'(j), \mathring{\mathbf{D}}'(j)\big)$ and $\big(\widetilde{\mathbf{A}}'(j), \widetilde{\mathbf{B}}'(j); \widetilde{\mathbf{C}}'(j), \widetilde{\mathbf{D}}'(j)\big)$ denote the arrow configurations at $(j,1) \in \mathcal{D}_1$ under $\mathcal{E}(\mathring{\boldsymbol{\nu}}/\mathring{\boldsymbol{\mu}})$ and $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\nu}}\big)$, respectively. See Figure 10.5 for depictions.

Then, it is quickly verified that $X_i(j) = X_i'(j)$ for any integers $i \in [1,n]$ and $j \geqslant 1$ and index $X \in \{\mathring{A}, \mathring{B}, \mathring{C}, \mathring{D}\}$, unless $i = h$ and $(X,j) \in \big\{(\mathring{C}, K_0), (\mathring{D}, K_0), (\mathring{B}, K_0+1), (\mathring{C}, K_0+1)\big\}$. It is further quickly verified that $X_i(j) = X_i'(j)$ for any $i \in [1,n]$, $j \geqslant 1$, and $X \in \big\{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\big\}$, unless $i = h$ and $(X,j) \in \big\{(\widetilde{A}, K_0), (\widetilde{D}, K_0), (\widetilde{A}, K_0+1), (\widetilde{B}, K_0+1)\big\}$. In those exceptional cases, we have

$$\begin{aligned} &\mathring{C}_h(K_0+1) = 1 = \mathring{C}_h'(K_0); \qquad \mathring{C}_h(K_0) = 0 = \mathring{C}_h'(K_0+1); \\ &\mathring{D}_h(K_0) = 1 = \mathring{B}_h(K_0+1); \qquad \mathring{D}_h'(K_0) = 0 = \mathring{B}_h'(K_0+1), \end{aligned} \tag{10.2.7}$$

and

$$\begin{aligned} &\widetilde{A}_h(K_0+1) = 1 = \widetilde{A}_h'(K_0); \qquad \widetilde{A}_h(K_0) = 0 = \widetilde{A}_h'(K_0+1); \\ &\widetilde{D}_h(K_0) = 0 = \widetilde{B}_h(K_0+1); \qquad \widetilde{D}_h'(K_0) = 1 = \widetilde{B}_h'(K_0+1). \end{aligned} \tag{10.2.8}$$

We again refer to Figure 10.5 for a depiction.

In particular, (10.2.7) and the bilinearity (and definition (1.1.1)) of φ together imply that

$$\begin{aligned} \varphi\big(\mathring{\mathbf{D}}(K_0), \mathring{\mathbf{D}}(K_0)\big) &= \varphi\big(\mathring{\mathbf{D}}'(K_0), \mathring{\mathbf{D}}'(K_0)\big) + \varphi\big(\mathring{\mathbf{D}}(K_0), \mathbf{e}_h\big) + \varphi\big(\mathbf{e}_h, \mathring{\mathbf{D}}(K_0)\big); \\ \varphi\big(\mathring{\mathbf{A}}(K_0+1), \mathring{\mathbf{B}}(K_0+1)\big) &= \varphi\big(\mathring{\mathbf{A}}'(K_0+1), \mathring{\mathbf{B}}'(K_0+1)\big) + \varphi\big(\mathring{\mathbf{A}}(K_0+1), \mathbf{e}_h\big); \\ \varphi\big(\mathring{\mathbf{D}}(K_0), \mathring{\mathbf{C}}(K_0)\big) &= \varphi\big(\mathring{\mathbf{D}}'(K_0), \mathring{\mathbf{C}}'(K_0)\big) - \varphi\big(\mathring{\mathbf{D}}(K_0), \mathbf{e}_h\big) + \varphi\big(\mathbf{e}_h, \mathring{\mathbf{C}}(K_0)\big); \\ \varphi\big(\mathring{\mathbf{D}}(K_0+1), \mathring{\mathbf{C}}(K_0+1)\big) &= \varphi\big(\mathring{\mathbf{D}}'(K_0+1), \mathring{\mathbf{C}}'(K_0+1)\big) + \varphi\big(\mathring{\mathbf{D}}(K_0+1), \mathbf{e}_h\big). \end{aligned}$$

So, by Lemma 10.1.1 we have

$$\begin{aligned} \operatorname{sp}\big(\mathring{\lambda}/\mathring{\mu}\big) &= \sum_{j=1}^{L_0-1} \varphi\big(\mathring{\mathbf{D}}(j), \mathring{\mathbf{D}}(j)\big) + \frac{1}{2}\sum_{j=2}^{L_0} \varphi\big(\mathring{\mathbf{A}}(j), \mathring{\mathbf{B}}(j)\big) + \frac{1}{2}\sum_{j=1}^{L_0-1} \varphi\big(\mathring{\mathbf{D}}(j), \mathring{\mathbf{C}}(j)\big) \\ &= \sum_{j=1}^{L_0-1} \varphi\big(\mathring{\mathbf{D}}'(j), \mathring{\mathbf{D}}'(j)\big) + \frac{1}{2}\sum_{j=2}^{L_0} \varphi\big(\mathring{\mathbf{A}}'(j), \mathring{\mathbf{B}}'(j)\big) + \frac{1}{2}\sum_{j=1}^{L_0-1} \varphi\big(\mathring{\mathbf{D}}'(j), \mathring{\mathbf{C}}(j)\big) \\ &\quad + \frac{1}{2}\Big(\varphi\big(\mathring{\mathbf{A}}(K_0+1) + \mathring{\mathbf{D}}(K_0) + \mathring{\mathbf{D}}(K_0+1), \mathbf{e}_h\big) + \varphi\big(\mathbf{e}_h, \mathring{\mathbf{C}}(K_0) + 2\mathring{\mathbf{D}}(K_0)\big)\Big) \\ &= \operatorname{sp}(\mathring{\nu}/\mathring{\mu}) + \frac{1}{2}\varphi\big(\mathring{\mathbf{A}}(K_0+1) + \mathring{\mathbf{D}}(K_0) + \mathring{\mathbf{D}}(K_0+1), \mathbf{e}_h\big) + \frac{1}{2}\varphi\big(\mathbf{e}_h, \mathring{\mathbf{C}}(K_0) + 2\mathring{\mathbf{D}}(K_0)\big). \end{aligned} \tag{10.2.9}$$

By similar reasoning, again using Lemma 10.1.1, (10.2.8), and the blinearity of φ, we deduce

$$\begin{aligned} \operatorname{sp}\big(\widetilde{\mu}/\mathring{\lambda}\big) = \operatorname{sp}\big(\widetilde{\mu}/\mathring{\nu}\big) &+ \frac{1}{2}\varphi\big(\mathbf{e}_h, \widetilde{\mathbf{B}}(K_0+1) - \widetilde{\mathbf{B}}(K_0) - 2\widetilde{\mathbf{D}}(K_0) - \widetilde{\mathbf{C}}(K_0)\big) \\ &- \frac{1}{2}\varphi\big(\widetilde{\mathbf{A}}(K_0+1) + 2\widetilde{\mathbf{D}}(K_0), \mathbf{e}_h\big). \end{aligned} \tag{10.2.10}$$

Subtracting (10.2.9) from (10.2.10) then yields

$$\operatorname{sp}\big(\widetilde{\mu}/\mathring{\lambda}\big) - \operatorname{sp}(\mathring{\lambda}/\mathring{\mu}) = \operatorname{sp}\big(\widetilde{\mu}/\mathring{\nu}\big) - \operatorname{sp}\big(\mathring{\nu}/\mathring{\mu}\big) - \frac{1}{2}\varphi(\mathbf{Y}, \mathbf{e}_h) - \frac{1}{2}\varphi(\mathbf{e}_h, \mathbf{Z}),$$

where

$$\begin{aligned}\mathbf{Y} &= \widetilde{\mathbf{A}}(K_0+1) + 2\widetilde{\mathbf{D}}(K_0) + \mathring{\mathbf{A}}(K_0+1) + \mathring{\mathbf{D}}(K_0) + \mathring{\mathbf{D}}(K_0+1); \\ \mathbf{Z} &= \widetilde{\mathbf{B}}(K_0) + 2\widetilde{\mathbf{D}}(K_0) + \widetilde{\mathbf{C}}(K_0) + \mathring{\mathbf{C}}(K_0) + 2\mathring{\mathbf{D}}(K_0) - \widetilde{\mathbf{B}}(K_0+1).\end{aligned} \tag{10.2.11}$$

Now, let us simplify $\mathbf{Y}$ and $\mathbf{Z}$. To do so, observe from (10.2.2) and (10.2.3) (see also Figure 10.3) that

$$\widetilde{\mathbf{A}}(j) = \mathring{\mathbf{C}}(j), \quad \text{for each } j \geqslant 1; \qquad \widetilde{\mathbf{C}}(j) = \mathring{\mathbf{A}}(j+1), \quad \text{for each } 1 \leqslant j \leqslant L_0 - 1. \tag{10.2.12}$$

By spin conservation we also have that

(10.2.13)

$$\mathring{\mathbf{B}}(j+1) = \mathring{\mathbf{D}}(j) = \sum_{i=1}^{j} \big(\mathring{\mathbf{A}}(i) - \mathring{\mathbf{C}}(i)\big), \quad \text{for each } j \geqslant 1;$$

$$\widetilde{\mathbf{B}}(j+1) = \widetilde{\mathbf{D}}(j) = \sum_{i=1}^{j} \big(\widetilde{\mathbf{A}}(i) - \widetilde{\mathbf{C}}(i)\big) = \sum_{i=1}^{j} \big(\mathring{\mathbf{C}}(i) - \mathring{\mathbf{A}}(i+1)\big), \quad \text{for each } 1 \leqslant j \leqslant L_0 - 1,$$

where in the last equality we used (10.2.12). In particular, (10.2.12) and (10.2.13) together imply

$$\mathring{\mathbf{D}}(K_0) + \widetilde{\mathbf{D}}(K_0) = \mathring{\mathbf{A}}(1) - \mathring{\mathbf{A}}(K_0+1); \qquad \mathring{\mathbf{D}}(K_0+1) + \widetilde{\mathbf{D}}(K_0) = \mathring{\mathbf{A}}(1) - \mathring{\mathbf{C}}(K_0+1);$$
$$\widetilde{\mathbf{A}}(K_0+1) = \mathring{\mathbf{C}}(K_0+1),$$

which upon insertion into (10.2.11) yields $\mathbf{Y} = 2\mathring{\mathbf{A}}(1) = 2\mathbf{e}_{[1,n]}$ (where the last equality follows from the fact that $\mathring{\mu}_{M+1}^{(i)} = 0$ for each $i \in [1,n]$, by (10.2.3)). Similarly, (10.2.12) and (10.2.13) together imply

$$\widetilde{\mathbf{B}}(K_0+1) = \widetilde{\mathbf{D}}(K_0); \quad \widetilde{\mathbf{B}}(K_0) + \mathring{\mathbf{D}}(K_0) = \mathring{\mathbf{A}}(1) - \mathring{\mathbf{C}}(K_0); \quad \widetilde{\mathbf{D}}(K_0) + \mathring{\mathbf{D}}(K_0) = \mathring{\mathbf{A}}(1) - \mathring{\mathbf{C}}(K_0),$$

which upon insertion into (10.2.11) yields $\mathbf{Z} = 2\mathring{\mathbf{A}}(1) = 2\mathbf{e}_{[1,n]}$.

Thus, $\mathbf{Y} = 2\mathbf{e}_{[1,n]} = \mathbf{Z}$, and so by the $\mathbf{X} = \mathbf{e}_h$ case of (8.2.2) we have that $\varphi(\mathbf{Y}, \mathbf{e}_h) + \varphi(\mathbf{e}_h, \mathbf{Z}) = 2n - 2$. Inserting this into (10.2.10) gives

$$\begin{aligned}\operatorname{sp}\big(\widetilde{\mu}/\mathring{\lambda}\big) - \operatorname{sp}(\mathring{\lambda}/\mathring{\mu}) &= \operatorname{sp}\big(\widetilde{\mu}/\mathring{\nu}\big) - \operatorname{sp}(\mathring{\nu}/\mathring{\mu}) - n + 1 \\ &= (n-1)\big(|\mathring{\boldsymbol{\mu}}| - |\mathring{\boldsymbol{\nu}}|\big) + \big(M - L_0 + 1\big)\binom{n}{2} - n + 1,\end{aligned}$$

where in the last equality we applied the ($\mathring{\boldsymbol{\lambda}} = \mathring{\boldsymbol{\nu}}$ case of) the second statement in (10.2.4). Recalling that $|\mathring{\boldsymbol{\lambda}}| = |\mathring{\boldsymbol{\nu}}| + 1$, it follows that

$$\operatorname{sp}\big(\widetilde{\mu}/\mathring{\lambda}\big) - \operatorname{sp}(\mathring{\lambda}/\mathring{\mu}) = (n-1)\big(|\mathring{\boldsymbol{\mu}}| - |\mathring{\boldsymbol{\lambda}}|\big) + \big(M - L_0 + 1\big)\binom{n}{2},$$

which establishes the second statement of (10.2.4) and therefore the lemma. □

Now we can establish the second part of Proposition 9.3.4.

Proof of Part 2 of Proposition 9.3.4. Let us first assume that λ/μ is not a horizontal n-ribbon strip. In this case, recall the signature sequence $\mathring{\boldsymbol{\mu}} \in \operatorname{SeqSign}_{n;M+1}$ defined in (10.2.3), which is the n-quotient of some $\mathring{\mu} \in \operatorname{Sign}_{(M+1)n}$. Since $\mathring{\boldsymbol{\lambda}}$ is obtained from $\boldsymbol{\lambda}$ by increasing every entry in each of its signatures by one, and since $\mathring{\boldsymbol{\mu}}$ is obtained from $\boldsymbol{\mu}$ in the same way (and by then appending a zero to each signature), it follows that $\mathring{\lambda}/\mathring{\mu}$ is also not a horizontal n-strip.

Now, recall from the proof of (10.2.5) that $\widetilde{\mu} \in \mathrm{Sign}_{(M+1)n}$ is obtained by appending a $1 \times n$ ribbon in each of the leftmost $n\big(L_0 - M - 1\big)$ columns in the Young diagram of $\mathring{\mu}$. In particular, $\widetilde{\mu}$ is the signature with maximal size such that $\widetilde{\mu}_1 \leqslant n\big(L_0 - M - 1\big)$ and $\widetilde{\mu}/\mathring{\mu}$ is a horizontal n-ribbon strip. Consequently, its Maya diagram $\mathfrak{T}(\widetilde{\mu})$ is obtained from $\mathfrak{T}(\mathring{\mu})$ by having each particle in $\mathfrak{T}(\mathring{\mu})$ perform n-jumps until colliding with another particle in $\mathfrak{T}(\mathring{\mu})$; stated alternatively, each particle $i \in \mathfrak{T}(\mathring{\mu})$ jumps to site $i + j_i n$, where $j_i \in \mathbb{Z}_{\geqslant 0}$ is such that $i + n, i + 2n, \ldots, i + j_i n \notin \mathfrak{T}(\mathring{\mu})$ but either $i + (j_i + 1)n \in \mathfrak{T}(\mathring{\mu})$ or $i + (j_i + 1)n > nL_0$. This characterization quickly implies that $\widetilde{\mu}/\nu$ is a horizontal n-ribbon strip for some signature $\nu \in \mathrm{Sign}_{n(M+1)}$ with $\mathring{\mu} \subseteq \nu \subseteq \widetilde{\mu}$ only if $\nu/\mathring{\mu}$ is. Hence, since $\mathring{\lambda}/\mathring{\mu}$ is not a horizontal n-ribbon strip, it follows that $\widetilde{\mu}/\mathring{\lambda}$ is not one as well.

Therefore, as explained in the proof of the first part of Proposition 9.3.4 in Section 10.1, there exists a vertex $(j, 1) \in \mathcal{D}_1$ and a color $h \in [1, n]$ such that horizontal and vertical arrows of color h both enter and exit through $(j, 1)$ in the path ensemble $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$. It follows that the same statement at $(j - 1, 1) \in \mathcal{D}$ holds in $\widehat{\mathcal{E}}\big(\boldsymbol{\lambda}/\boldsymbol{\mu}\big)$, since the latter is obtained from $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$ by first shifting each vertex to the left by one space and then modifying the horizontally and vertically exiting arrows at only the vertex $\big(L_0 - 1, 1\big) \in \mathcal{D}_1$. Hence, $\widehat{A}_h(j-1) = \widehat{B}_h(j-1) = \widehat{C}_h(j-1) = \widehat{D}_h(j-1) = 1$.

Due to the factor of $\mathbf{1}_{v=0}$ in the expression (8.4.2) for $\widehat{\mathcal{W}}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$, it follows that the weight of $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is 0, and so $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid 0; 0) = 0$. Similarly, from the definition (9.1.2) (and (9.2.1)) for $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, we also find that $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x) = 0$ if λ/μ is not a ribbon strip, thereby verifying the proposition in this case.

So, let us assume that λ/μ is a horizontal n-strip. Then the definition of $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid 0; 0)$ as the partition function, under the $\widehat{\mathcal{W}}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ weights, for the path ensemble $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (together with the explicit forms (8.4.2) for these weights) gives

(10.2.14)
$$\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid 0; 0) = \prod_{j=1}^{L_0 - 1} \widehat{\mathcal{W}}_x\big(\widehat{\mathbf{A}}(j), \widehat{\mathbf{B}}(j); \widehat{\mathbf{C}}(j), \widehat{\mathbf{D}}(j)\big) = \prod_{j=1}^{L_0 - 1} x^{|\widehat{\mathbf{D}}(j)| - n} q^{\varphi(\widehat{\mathbf{D}}(j), \widehat{\mathbf{C}}(j) + \widehat{\mathbf{D}}(j)) - \binom{n}{2}},$$

where here we have used the fact that $\big(\widehat{\mathbf{A}}(j), \widehat{\mathbf{B}}(j); \widehat{\mathbf{C}}(j), \widehat{\mathbf{D}}(j)\big) = \big(\mathbf{e}_0, \mathbf{e}_{[1,n]}; \mathbf{e}_0, \mathbf{e}_{[1,n]}\big)$ for $j \geqslant L_0 - 1$ and the weight under $\widehat{\mathcal{W}}_x$ of this arrow configuration is equal to 1 by (5.1.5) and (8.4.2).

Now, observe since $\widehat{\mathbf{B}}(1) = \mathbf{e}_0$ and $\widehat{\mathbf{B}}(j) = \mathbf{e}_{[1,n]}$ for $j > L_0$ that

$$\begin{aligned}
\sum_{j=2}^{L_0} \Big(n - \big|\widehat{\mathbf{B}}(j)\big|\Big) + n &= \sum_{j=1}^{\infty} \Big(n - \big|\widehat{\mathbf{B}}(j)\big|\Big) \\
&= \sum_{i=1}^{n} \left(\sum_{\mathfrak{l} \in \mathfrak{T}(\lambda^{(i)})} \mathfrak{l} - \sum_{\mathfrak{m} \in \mathfrak{T}(\mu^{(i)})} \mathfrak{m} \right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{M+1} \big(\lambda_j^{(i)} + M - j + 2\big) - \sum_{j=1}^{M} \big(\mu_j^{(i)} + M - j + 1\big) \right) \\
&= |\boldsymbol{\lambda}| - |\boldsymbol{\mu}| + n(M+1),
\end{aligned}$$

and so, since $\widehat{\mathbf{D}}(j) = \widehat{\mathbf{B}}(j+1)$ for each $j \geqslant 1$, we have

$$\sum_{j=1}^{L_0 - 1} \Big(n - \big|\widehat{\mathbf{D}}(j)\big|\Big) = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}| + nM.$$

Inserting this into (10.2.14) then yields

$$
\begin{aligned}
\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid 0;0) &= x^{|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|-nM} \prod_{j=2}^{L_0} q^{\varphi(\widehat{\mathbf{D}}(j-1),\widehat{\mathbf{C}}(j-1)+\widehat{\mathbf{D}}(j-1))-\binom{n}{2}} \\
&= x^{|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|-nM} \prod_{j=2}^{L_0-1} q^{\varphi(\widehat{\mathbf{D}}(j-1),\widehat{\mathbf{C}}(j-1)+\widehat{\mathbf{D}}(j-1))-\binom{n}{2}},
\end{aligned}
\tag{10.2.15}
$$

where in the second equality we used the identity $\varphi\big(\widehat{\mathbf{D}}(L_0-1),\widehat{\mathbf{C}}(L_0-1)+\widehat{\mathbf{D}}(L_0-1)\big) = \binom{n}{2}$, which holds since $\widehat{\mathbf{D}}(L_0-1) = \mathbf{e}_{[1,n]}$ and $\widehat{\mathbf{C}}(L_0-1) = \mathbf{e}_0$.

Now, since (10.2.2) implies for any indices $X \in \{A,B,C,D\}$ and $j \in [2, L_0-1]$ that $\widehat{\mathbf{X}}(j-1) = \widetilde{\mathbf{X}}(j)$, we deduce

$$
\sum_{j=2}^{L_0-1} \left(\varphi\big(\widehat{\mathbf{D}}(j-1),\widehat{\mathbf{C}}(j-1)+\widehat{\mathbf{D}}(j-1)\big) - \binom{n}{2} \right) = \sum_{j=1}^{L_0} \varphi\big(\widetilde{\mathbf{D}}(j),\widetilde{\mathbf{C}}(j)+\widetilde{\mathbf{D}}(j)\big) + \big(2-L_0\big)\binom{n}{2},
\tag{10.2.16}
$$

where we have additionally used the facts that $\widetilde{\mathbf{D}}(1) = \mathbf{e}_0 = \widetilde{\mathbf{D}}\big(L_0\big)$. Moreover, Lemma 10.1.1, Lemma 10.1.2, and Lemma 10.2.1 together yield

$$
\begin{aligned}
\sum_{j=1}^{L_0} \varphi\big(\widetilde{\mathbf{D}}(j),\widetilde{\mathbf{C}}(j)+\widetilde{\mathbf{D}}(j)\big) &= \mathrm{sp}(\widetilde{\mu}/\mathring{\lambda}) + \psi(\widetilde{\boldsymbol{\mu}}) - \psi(\mathring{\boldsymbol{\lambda}}) \\
&= \mathrm{sp}(\lambda/\mu) + (n-1)\big(|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|-n\big) + \big(L_0-M\big)\binom{n}{2} + \psi(\widetilde{\boldsymbol{\mu}}) - \psi(\mathring{\boldsymbol{\lambda}}).
\end{aligned}
\tag{10.2.17}
$$

Inserting (10.2.16) and (10.2.17) into (10.2.15), we obtain

$$
\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid 0;0) = x^{-nM}(q^{1-n}x^{-1})^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|} q^{\mathrm{sp}(\lambda/\mu)+\psi(\widetilde{\boldsymbol{\mu}})-\psi(\mathring{\boldsymbol{\lambda}})-M\binom{n}{2}}.
\tag{10.2.18}
$$

Next, the definition (9.3.1) of ψ implies by (10.2.2) that

$$
\psi(\mathring{\boldsymbol{\lambda}}) = \psi(\boldsymbol{\lambda}); \qquad \psi\big(\widetilde{\boldsymbol{\mu}}\big) - \psi(\boldsymbol{\mu}) = \frac{M}{2}\binom{n}{2},
\tag{10.2.19}
$$

where the latter holds since the summands (a,b) on the right side of (9.3.1) contributing to $\psi\big(\widetilde{\boldsymbol{\mu}}\big)$ but not $\psi(\boldsymbol{\mu})$ arise when $a = L_0$, and $1 \leqslant i < j \leqslant n$ and $b \in \mathfrak{T}\big(\mu^{(j)}\big)$ are arbitrary. Inserting these into (10.2.18) yields

$$
\begin{aligned}
\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid 0;0) &= q^{\mathrm{sp}(\lambda/\mu)+\psi(\boldsymbol{\mu})-M\binom{n}{2}/2-\psi(\boldsymbol{\lambda})} x^{-nM}\big(q^{1-n}x^{-1}\big)^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|} \\
&= q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})-M\binom{n}{2}/2} x^{-nM} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q^{1-n}x^{-1}),
\end{aligned}
$$

where the last equality follows from the definition (9.1.2) (and (9.2.1)) of $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. This establishes the second part of the proposition. $\square$

10.3. LLT Polynomials from $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0;\mathbf{x} \mid 0;0)$

In this section we establish the third part of Proposition 9.3.4, namely, that (9.3.5) holds for $N = 1$. This will follow from the first part of Proposition 9.3.4 (given by the $N = 1$ case of (9.3.3)), after applying a reversal and complementation procedure for the Maya diagrams of λ and μ.

To that end, we recall the unique path ensemble $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \in \mathfrak{P}_G(\boldsymbol{\lambda}/\boldsymbol{\mu};1)$ from Section 10.1, and the associated arrow configuration $\big(\mathbf{A}(j), \mathbf{B}(j); \mathbf{C}(j), \mathbf{D}(j)\big)$ at any vertex $(j,1) \in \mathcal{D}_1 = \mathbb{Z}_{>0} \times \{1\}$. Since $\big(\mathbf{A}(K), \mathbf{B}(K); \mathbf{C}(K), \mathbf{D}(K)\big) = (\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0)$ for sufficiently large K, there exists a maximal integer $L_0 > 0$ for which $\big(\mathbf{A}(L_0), \mathbf{B}(L_0); \mathbf{C}(L_0), \mathbf{D}(L_0)\big) \neq \big(\mathbf{e}_0, \mathbf{e}_0; \mathbf{e}_0, \mathbf{e}_0\big)$.

For any $j \in [1, L_0]$, define the elements $\mathbf{A}'(j), \mathbf{B}'(j), \mathbf{C}'(j), \mathbf{D}'(j) \in \{0,1\}^n$ by

$$\begin{aligned} \mathbf{A}'(j) &= \mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{A}}(L_0 - j + 1); \qquad & \mathbf{B}'(j) &= \overleftarrow{\mathbf{D}}(L_0 - j + 1); \\ \mathbf{C}'(j) &= \mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{C}}(L_0 - j + 1); \qquad & \mathbf{D}'(j) &= \overleftarrow{\mathbf{B}}(L_0 - j + 1), \end{aligned} \tag{10.3.1}$$

and define $\mathbf{A}'(j) = \mathbf{B}'(j) = \mathbf{C}'(j) = \mathbf{D}'(j) = \mathbf{e}_0$ for each $j > L_0$. Here, we have set $\overleftarrow{\mathbf{X}} = (X_n, X_{n-1}, \ldots, X_1)$ to be the order reversal of any $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \{0,1\}^n$. Additionally define $\mathbf{V}'(j) \in \{0,1\}^n$ by setting $V_i'(j) = \min\big\{A_i'(j), B_i'(j), C_i'(j), D_i'(j)\big\}$ for each $i \in [1,n]$. Let $x'(j) = \big|\mathbf{X}'(j)\big|$ for each index $X \in \{A,B,C,D,V\}$.

Then, since $\mathbf{A}(j) + \mathbf{B}(j) = \mathbf{C}(j) + \mathbf{D}(j)$, (10.3.1) implies $\mathbf{A}'(j) + \mathbf{B}'(j) = \mathbf{C}'(j) + \mathbf{D}'(j)$ for each $j > 0$. In particular, this arrow conservation implies the existence of a single-row path ensemble $\mathcal{E}'(\boldsymbol{\lambda}/\boldsymbol{\mu})$ on $\mathbb{Z}_{>0} \times \{1\}$ whose arrow configuration at $(j,1)$ is $\big(\mathbf{A}'(j), \mathbf{B}'(j); \mathbf{C}'(j), \mathbf{D}'(j)\big)$. In this way, $\mathcal{E}'(\boldsymbol{\lambda}/\boldsymbol{\mu})$ is obtained by first reversing the vertices and colors in $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and then *complementing* all vertical arrows, that is, interchanging a particle of any color with its absence. We refer to the middle of Figure 10.6 for a depiction.

As explained in Section 10.1, one may decompress the top and bottom boundaries of $\mathcal{E}'(\boldsymbol{\lambda}/\boldsymbol{\mu})$ to form two colored Maya diagrams. By (10.3.1), the decompression of the top boundary is obtained by first reversing $\mathfrak{T}(\lambda)$ and then complementing it (interchanging particles with empty sites); applying the same reversal and complementation procedure to $\mathfrak{T}(\mu)$ yields the bottom boundary of this decompression. We refer to the bottom part of Figure 10.6 for a depiction. As such, it is quickly verified (see, for example, (1.7) of **[66]**) that the top and bottom boundaries of this decompressed vertex model are given by $\mathfrak{T}(\lambda')$ and $\mathfrak{T}(\mu')$ (namely, the colored Maya diagrams for the duals of λ and μ), respectively. Hence $\mathcal{E}'(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \mathcal{E}(\boldsymbol{\lambda}'/\boldsymbol{\mu}')$, the unique path ensemble in $\mathfrak{P}_G(\boldsymbol{\lambda}'/\boldsymbol{\mu}';1)$.

Now, by (7.1.3) and Remark 8.4.4, $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0;x \mid 0;0)$ is the partition function for $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ under the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid 0,0)$. Hence, using the explicit form (8.4.1) for these weights, we obtain

$$\begin{aligned} \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0;x \mid 0;0) &= \prod_{j=1}^{L_0} \mathcal{W}_x\big(\mathbf{A}(j), \mathbf{B}(j); \mathbf{C}(j), \mathbf{D}(j) \mid 0,0\big) \\ &= \prod_{j=1}^{L_0} x^{-d(j)} q^{\varphi(\mathbf{D}(j), \mathbf{C}(j) - \mathbf{B}(j))} \mathbf{1}_{\mathbf{C}(j) \geqslant \mathbf{B}(j)}. \end{aligned}$$

Then (10.3.1) and the fact that $\sum_{j=1}^{L_0} d(j) = \sum_{j=1}^{\infty} d(j) = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$ together yield

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0;x \mid 0;0) = x^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|} \prod_{j=1}^{L_0} q^{\varphi(\overleftarrow{\mathbf{B}}'(j), \mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{C}}'(j) - \overleftarrow{\mathbf{D}}'(j))} \mathbf{1}_{\mathbf{e}_{[1,n]} \geqslant \overleftarrow{\mathbf{C}}'(j) + \overleftarrow{\mathbf{D}}'(j)}.$$

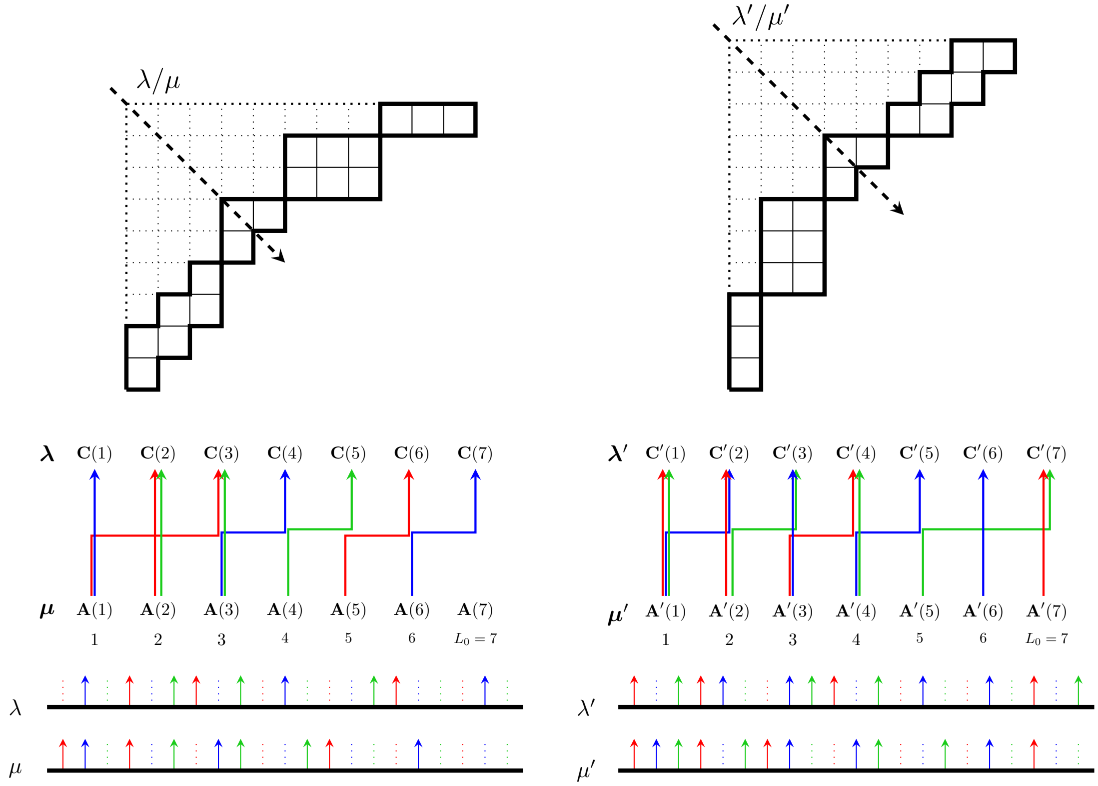


FIGURE 10.6. Shown above are the Young diagrams for skew-shapes λ/μ and λ'/μ'; shown to the bottom are their colored Maya diagrams; and shown to the middle are the single-row path ensembles $\mathcal{E}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\mathcal{E}'(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \mathcal{E}(\boldsymbol{\lambda}'/\boldsymbol{\mu}')$. Here, red is color 1, blue is color 2, and green is color 3.

Thus, since the definition (1.1.1) of φ implies

$$\varphi(\overleftarrow{\mathbf{X}}, \overleftarrow{\mathbf{Y}}) = \varphi(\mathbf{Y}, \mathbf{X}), \qquad \text{for any } \mathbf{X}, \mathbf{Y} \in \{0,1\}^n, \tag{10.3.2}$$

we find that

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0; \mathbf{x} \mid 0; 0) = x^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|} \prod_{j=1}^{L_0} q^{\varphi(\mathbf{e}_{[1,n]} - \mathbf{C}'(j) - \mathbf{D}'(j), \mathbf{B}'(j))} \mathbf{1}_{v'(j)=0}, \tag{10.3.3}$$

where we have also used the fact that $\mathbf{e}_{[1,n]} \geqslant \overleftarrow{\mathbf{C}}(j) + \overleftarrow{\mathbf{D}}(j)$ holds if and only if $v'(j) = 0$, for any $j \geqslant 1$. Indeed, since $A'_i(j), B'_i(j), C'_i(j), D'_i(j) \in \{0,1\}$, this follows from the fact that $C'_i(j) + D'_i(j) \leqslant 1$ holds for some $i \in [1,n]$ if and only if $V'_i(j) = 0$ does (using arrow conservation).

We now require the following lemma to analyze the power of q on the right side of (10.3.3).

Lemma 10.3.1. *If λ'/μ' is a horizontal n-ribbon strip, then*

$$\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\lambda}') - \psi(\boldsymbol{\mu}) + \psi(\boldsymbol{\mu}') - \sum_{j=1}^{L_0} \varphi\big(\mathbf{e}_{[1,n]}, \mathbf{B}'(j)\big) = \frac{n-1}{2}\big(|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|\big).$$

Proof. First observe using (10.3.1), (10.3.2), and the definition (1.1.1) of φ that

(10.3.4)

$$\sum_{j=1}^{L_0} \varphi\big(\mathbf{e}_{[1,n]}, \mathbf{B}'(j)\big) = \sum_{j=1}^{L_0} \varphi\big(\mathbf{D}(j), \mathbf{e}_{[1,n]}\big) = \sum_{i=1}^{n} (n-i) \sum_{j=1}^{L_0} D_i(j) = \sum_{i=1}^{n} (n-i)\Big(\big|\lambda^{(i)}\big| - \big|\mu^{(i)}\big|\Big),$$

where in the last equality we used the identity $\sum_{j=1}^{L_0} D_i(j) = \sum_{j=1}^{\infty} D_i(j) = \big|\lambda^{(i)}\big| - \big|\mu^{(i)}\big|$ for any $i \in [1,n]$. Now, we claim that

$$\begin{aligned} \psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\lambda}') + \sum_{i=1}^{n} (i-1)\big|\lambda^{(i)}\big| &= \frac{(n-1)|\boldsymbol{\lambda}|}{2} + \frac{1}{2}\binom{n}{2}\left(\binom{M}{2} - \binom{L_0 - M}{2}\right); \\ \psi(\boldsymbol{\mu}) - \psi(\boldsymbol{\mu}') + \sum_{i=1}^{n} (i-1)\big|\mu^{(i)}\big| &= \frac{(n-1)|\boldsymbol{\mu}|}{2} + \frac{1}{2}\binom{n}{2}\left(\binom{M}{2} - \binom{L_0 - M}{2}\right). \end{aligned} \tag{10.3.5}$$

Let us show the lemma assuming (10.3.5). To that end, subtracting the two equalities in (10.3.5) yields

$$\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\lambda}') - \psi(\boldsymbol{\mu}) + \psi(\boldsymbol{\mu}') + \sum_{i=1}^{n} (i-1)\Big(\big|\lambda^{(i)}\big| - \big|\mu^{(i)}\big|\Big) = \frac{(n-1)}{2}\big(|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|\big),$$

and so subtracting $\sum_{i=1}^{n} (n-1)\big(\big|\lambda^{(i)}\big| - \big|\mu^{(i)}\big|\big) = (n-1)\big(|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|\big)$ yields

$$\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\lambda}') - \psi(\boldsymbol{\mu}) + \psi(\boldsymbol{\mu}') + \sum_{i=1}^{n} (i-n)\Big(\big|\lambda^{(i)}\big| - \big|\mu^{(i)}\big|\Big) = \frac{(n-1)}{2}\big(|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|\big),$$

which implies the lemma by (10.3.4). Thus, it remains to establish (10.3.5); we will only verify the first statement there, as the proof of the latter is entirely analogous.

To that end, we induct on $|\boldsymbol{\lambda}|$, as in the proof of Lemma 10.1.2. Observe that if $\boldsymbol{\lambda} = \mathbf{0}^M$ then (9.3.2) imples that $\psi(\boldsymbol{\lambda}) = \frac{1}{2}\binom{n}{2}\binom{M}{2}$ and $\psi(\boldsymbol{\lambda}') = \frac{1}{2}\binom{n}{2}\binom{L_0-M}{2}$, which verifies (10.3.5). Thus, let us assume (10.3.5) holds whenever $|\boldsymbol{\lambda}| < m$ for some integer $m > 0$, and we will show it holds for $|\boldsymbol{\lambda}| = m$.

So, fix $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M}$ with $|\boldsymbol{\lambda}| = m$. Then, there exists a sequence of n signatures $\boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M}$ such that $|\boldsymbol{\nu}| = m-1$ and the following holds. There exists integers $h \in [1,n]$ and $k \in [1,M]$ such that $\nu_j^{(i)} = \lambda_j^{(i)}$ whenever $(i,j) \neq (h,k)$ and $\nu_k^{(h)} = \lambda_k^{(h)} - 1$. Stated alternatively, $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ coincide in every entry of every component, except for in one entry which is smaller by one in $\boldsymbol{\nu}$ than in $\boldsymbol{\lambda}$.

Letting $K_0 = \nu_k^{(h)} + M - k + 1 \in \mathfrak{T}\big(\nu^{(h)}\big)$, so that $K_0 + 1 = \lambda_k^{(h)} + M - k + 1 \in \mathfrak{T}\big(\lambda^{(h)}\big)$, we have

$$\psi(\boldsymbol{\lambda}) = \psi(\boldsymbol{\nu}) + \frac{1}{2}\sum_{i=h+1}^{n} \mathbf{1}_{K_0 \in \mathfrak{T}(\lambda^{(i)})} - \frac{1}{2}\sum_{i=1}^{h-1} \mathbf{1}_{K_0+1 \in \mathfrak{T}(\lambda^{(i)})}. \tag{10.3.6}$$

Here, the first sum on the right side arises since each appearance of K_0 in some $\mathfrak{T}\big(\lambda^{(i)}\big)$ with $i > h$ gives rise to the additional summand $(a,b) = (K_0+1, K_0) \in \mathfrak{T}\big(\lambda^{(h)}\big) \times \mathfrak{T}\big(\lambda^{(i)}\big)$ on the right side of the definition (9.3.1) of $\psi(\boldsymbol{\lambda})$ not present in the corresponding sum for $\psi(\boldsymbol{\nu})$. Similarly, the second sum arises since each appearance of K_0+1 in some $\mathfrak{T}\big(\lambda^{(i)}\big)$ for $i<h$ gives rise to the additional summand $(a,b)=(K_0+1,K_0)\in\mathfrak{T}\big(\nu^{(i)}\big)\times\mathfrak{T}\big(\nu^{(h)}\big)=\mathfrak{T}\big(\lambda^{(i)}\big)\times\mathfrak{T}\big(\nu^{(h)}\big)$ in the definition of $\psi(\boldsymbol{\nu})$ not present in the correpsonding sum for $\psi(\boldsymbol{\lambda})$. By similar reasoning and also using the facts that particles are complemented, colors are reversed, and positions are reversed in $\boldsymbol{\lambda}'$ and $\boldsymbol{\nu}'$ with respect to $\boldsymbol{\lambda}'$ and $\boldsymbol{\nu}'$, respectively, we deduce that

$$\begin{aligned}\psi(\boldsymbol{\lambda}')&=\psi(\boldsymbol{\nu}')+\frac{1}{2}\sum_{i=n-h+2}^{n}\mathbf{1}_{L_0-K_0\in\mathfrak{T}(\lambda'^{(i)})}-\frac{1}{2}\sum_{i=1}^{n-h}\mathbf{1}_{L_0-K_0+1\in\mathfrak{T}(\lambda'^{(i)})}\\&=\psi(\boldsymbol{\nu}')+\frac{1}{2}\sum_{i=1}^{h-1}\mathbf{1}_{K_0+1\notin\mathfrak{T}(\lambda^{(i)})}-\frac{1}{2}\sum_{i=h+1}^{n}\mathbf{1}_{K_0\notin\mathfrak{T}(\lambda^{(i)})}.\end{aligned}\tag{10.3.7}$$

Subtracting (10.3.7) from (10.3.6) yields

$$\begin{aligned}\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\lambda}')-\psi(\boldsymbol{\nu})+\psi(\boldsymbol{\nu}')&=\frac{1}{2}\sum_{i=h+1}^{n}\big(\mathbf{1}_{K_0\in\mathfrak{T}(\lambda^{(i)})}+\mathbf{1}_{K_0\notin\mathfrak{T}(\lambda^{(i)})}\big)\\&\quad-\frac{1}{2}\sum_{i=1}^{h-1}\big(\mathbf{1}_{K_0+1\in\mathfrak{T}(\lambda^{(i)})}+\mathbf{1}_{K_0+1\notin\mathfrak{T}(\lambda^{(i)})}\big)=\frac{n-1}{2}-(h-1),\end{aligned}$$

which upon adding to (10.3.5) (with the $\boldsymbol{\lambda}$ or $\boldsymbol{\mu}$ there equal to $\boldsymbol{\nu}$ here) gives

$$\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\lambda}')+\sum_{i=1}^{n}(i-1)\big|\nu^{(i)}\big|+h-1=\frac{(n-1)|\boldsymbol{\nu}|}{2}+\frac{n-1}{2}+\frac{1}{2}\binom{n}{2}\left(\binom{M}{2}-\binom{L_0-M}{2}\right).$$

This, together with the facts that $\big|\lambda^{(i)}\big|=\big|\nu^{(i)}\big|+\mathbf{1}_{i=h}$ and $|\boldsymbol{\lambda}|=|\boldsymbol{\nu}|+1$, imples the first statement of (10.3.5) and thus the lemma. $\square$

Now we can establish the third part of Proposition 9.3.4.

Proof of Part 3 of Proposition 9.3.4. As indicated in the proof of the first part of Proposition 9.3.4 from Section 10.1, if λ'/μ' is not a horizontal n-ribbon strip then there exists some integer $j\in[1,L_0]$ such that $v'(j)\neq 0$. By (10.3.3), this implies that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(0;x\mid 0;0)=0$; since the same holds for $\mathcal{L}_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(q^{(1-n)/2}x^{-1};q^{-1})$, this verifies (9.3.5). So, let us assume below that λ'/μ' is a horizontal n-ribbon strip.

Applying arrow conservation and the fact that $\mathbf{B}'(j+1)=\mathbf{D}'(j)$ for each $j\geqslant 1$ (and that $\mathbf{B}'(1)=\mathbf{D}(L_0)=\mathbf{B}(L_0+1)=\mathbf{e}_0$), we deduce

$$\begin{aligned}\sum_{j=1}^{\infty}\varphi\big(\mathbf{C}'(j)+\mathbf{D}'(j),\mathbf{B}'(j)\big)&=\sum_{j=1}^{\infty}\varphi\big(\mathbf{A}'(j)+\mathbf{B}'(j),\mathbf{B}'(j)\big)\\&=\sum_{j=1}^{\infty}\Big(\varphi\big(\mathbf{B}'(j),\mathbf{B}'(j)\big)+\varphi\big(\mathbf{A}'(j),\mathbf{B}'(j)\big)\Big)\\&=\sum_{j=1}^{\infty}\Big(\varphi\big(\mathbf{D}'(j),\mathbf{D}'(j)\big)+\varphi\big(\mathbf{A}'(j),\mathbf{B}'(j)\big)\Big)\\&=\operatorname{sp}\left(\lambda'/\mu'\right)-\psi\big(\boldsymbol{\lambda}'\big)+\psi\big(\boldsymbol{\mu}'\big),\end{aligned}$$

where in the last equality we applied Lemma 10.1.1, and Lemma 10.1.2. Inserting this into (10.3.3) (and using the bilinearity of φ) yields

$$\begin{aligned}\mathcal{G}_{\boldsymbol{\lambda}}(0; x \mid 0; 0) &= q^{\psi(\boldsymbol{\lambda}') - \psi(\boldsymbol{\mu}') - \mathrm{sp}(\lambda'/\mu')} x^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|} \prod_{j=1}^{\infty} q^{\varphi(\mathbf{e}_{[1,n]}, \mathbf{B}'(j))} \\ &= q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu}) - \mathrm{sp}(\lambda'/\mu')} (q^{(n-1)/2} x^{-1})^{|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|} = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(q^{(n-1)/2} x^{-1}; q^{-1}),\end{aligned}$$

where in the second equality we applied Lemma 10.3.1 and in the last we used the definition (9.1.2) (and (9.2.1)) of the LLT polynomial $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. This establishes the third part of Proposition 9.3.4. ☐

10.4. LLT Polynomials from $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty)$

In this section we establish the fourth part of Proposition 9.3.4, namely, that (9.3.6) holds for $N = 1$. Similarly to in Section 10.3, this will follow from the second part of Theorem 9.3.2, by expressing $\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid \infty; \infty)$ in terms of $\mathcal{F}^{(1/q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x; \infty \mid 0; 0)$; here, we write $\mathcal{H}^{(q)}$ to emphasize the dependence of $\mathcal{H}$ on q and $\mathcal{F}^{(1/q)}$ to indicate that the parameter q involved in the definition of $\mathcal{F}$ is replaced by q^{-1}. To implement this, we establish the following statement comparing the more general functions $\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty)$ and $\mathcal{F}^{(1/q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q^{n-1}\mathbf{x}; \infty \mid \mathbf{y}; 0)$ (from (8.4.4)). In the below, we recall ψ from (9.3.1).

Proposition 10.4.1. *Fix integers $N \geqslant 1$ and $M \geqslant 0$; sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$; and a nonzero complex number $q \in \mathbb{C}$. For any signature sequences $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+N}$ and $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, we have*

$$\begin{aligned}\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) &= (-1)^{n\binom{M+N}{2} - n\binom{M}{2}} q^{2\psi(\overleftarrow{\boldsymbol{\mu}}) - 2\psi(\overleftarrow{\boldsymbol{\lambda}}) + 2\binom{M+N}{2}\binom{n}{2} - 2\binom{M}{2}\binom{n}{2}} \mathcal{F}^{(1/q)}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(q^{n-1}\mathbf{x}; \infty \mid \mathbf{y}; 0) \\ &\quad \times \prod_{j=1}^{N} x_j^{-n} \prod_{i=1}^{n} \left(\prod_{j=1}^{M} y^{-1}_{\mu_j^{(i)} + M - j} \prod_{k=1}^{M+N} y_{\lambda_k^{(i)} + M + N - k} \right),\end{aligned}$$

Assuming Proposition 10.4.1, we can quickly establish the fourth part of Proposition 9.3.4.

Proof of 4 of Proposition 9.3.4 Assuming Proposition 10.4.1. For any $x \in \mathbb{C}$, Definition 8.4.1 implies

$$\begin{aligned}\widehat{\mathcal{W}}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0) &= \lim_{y \to \infty} (-y)^{d-n} \widehat{\mathcal{W}}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0); \\ \mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty) &= \lim_{y \to \infty} y^{-b} \mathcal{W}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty).\end{aligned} \tag{10.4.1}$$

Now, by Remark 8.4.4 and (7.1.3), $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0)$ and $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ are partition functions for the vertex model $\mathfrak{P}_F(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (recall Definition 7.1.3; see also the middle of Figure 7.1) under the vertex weights $\widehat{\mathcal{W}}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ and $\widehat{\mathcal{W}}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$, respectively. Thus, defining $\mathbf{y} = (y, y, \ldots)$, the first statement of (10.4.1) quickly implies (similarly to in Remark 8.4.3) that

$$\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = \lim_{y \to \infty} (-y)^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}| + n\binom{M}{2} - n\binom{M+N}{2}} \mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0). \tag{10.4.2}$$

Similarly, Remark 8.4.4 and (7.1.3) together indicate that $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty)$ and $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty)$ are partition functions for the vertex model $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (recall Definition 7.1.3; see also the right side of Figure 7.1) under the vertex weights $\mathcal{W}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty)$ and $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid$

$\infty, \infty)$, respectively. Thus, again letting $\mathbf{y} = (y, y, \ldots)$, the second statement of (10.4.1) quickly gives

$$\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty) = \lim_{y \to \infty} y^{|\boldsymbol{\mu}| - |\boldsymbol{\lambda}| + n\binom{M+1}{2} - n\binom{M+N+1}{2}} \mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty). \tag{10.4.3}$$

Applying the $\mathbf{y} = (y, y, \ldots)$ case of Proposition 10.4.1, and letting y tend to ∞, it follows from (10.4.2) and (10.4.3) that

(10.4.4)

$$\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \infty; \infty) = (-1)^{|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|} q^{2\psi(\overleftarrow{\boldsymbol{\mu}}) - 2\psi(\overleftarrow{\boldsymbol{\lambda}}) + 2\binom{M+N}{2}\binom{n}{2} - 2\binom{M}{2}\binom{n}{2}} \mathcal{F}^{(1/q)}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(q^{n-1}\mathbf{x}; \infty \mid 0; 0) \prod_{j=1}^{N} x_j^{-n}.$$

By (9.3.4), we have that

$$\mathcal{F}^{(1/q)}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(q^{n-1}\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\overleftarrow{\boldsymbol{\lambda}}) - \psi(\overleftarrow{\boldsymbol{\mu}}) + \binom{M+N}{2}\binom{n}{2}/2 - \binom{M}{2}\binom{n}{2}/2} \mathcal{L}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(\mathbf{x}^{-1}; q^{-1}) \prod_{j=1}^{N} (q^{n-1}x_j)^{n(j-M-N)}.$$

Together with (10.4.4) and the fact that $(-1)^{|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|} \mathcal{L}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(\mathbf{x}; q^{-1}) = \mathcal{L}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(-\mathbf{x}^{-1}; q^{-1})$ (which holds by the homogeneity of $\mathcal{L}$), this implies (9.3.6). $\square$

To establish Proposition 10.4.1, we require the following lemma stating a relation between the weights $\mathcal{W}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty)$ and $\widehat{\mathcal{W}}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$ from Definition 8.4.1, under complementation of horizontal arrows and reversal of colors; we refer to Figure 10.7 for a depiction. In what follows, we again write $\mathcal{W}^{(q)}_{x;y}$ to emphasize the dependence of $\mathcal{W}_{x;y}$ on q and $\mathcal{W}^{(1/q)}_{x;y}$ to indicate that its parameter q is replaced by q^{-1}.

Lemma 10.4.2. *For any $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$ and $x, y, q \in \mathbb{C}$, we have*

(10.4.5)

$$\begin{aligned} &\mathcal{W}^{(q)}_{x;y}(\overleftarrow{\mathbf{C}}, \mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{B}}; \overleftarrow{\mathbf{A}}, \mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{D}} \mid \infty; \infty) \\ &\quad = (-1)^c q^{\varphi(\mathbf{A}, \mathbf{e}_{[1,n]}) - \varphi(\mathbf{A}, \mathbf{B}) + \varphi(\mathbf{D}, \mathbf{C}) - \varphi(\mathbf{B}, \mathbf{B}) + \varphi(\mathbf{D}, \mathbf{D})} (q^{n-1}xy^{-1})^{b-d} \widehat{\mathcal{W}}^{(1/q)}_{q^{n-1}x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty; 0). \end{aligned}$$

Proof. Observe that if $\mathbf{A} + \mathbf{B} \neq \mathbf{C} + \mathbf{D}$, then both sides of (10.4.5) are equal to 0. Moreover, defining $\mathbf{V} = (V_1, V_2, \ldots, V_n) \in \{0,1\}^n$ by setting $V_i = \min\{A_i, B_i, C_i, D_i\}$ for each $i \in [1, n]$ and letting $v = |\mathbf{V}|$ (as in (5.1.1)), (10.4.5) also holds if $v \neq 0$. Indeed, in this case, the right side of (10.4.5) is equal to 0, by the $\mathbf{1}_{v=0}$ weight in the definition (8.4.1) of $\widehat{\mathcal{W}}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$. Furthermore, if $v \neq 0$, then there exists some $i \in [1, n]$ for which $V_i = 1$, in which case $A_i = 1 = B_i$, and so $A_i + B_i > 1$. Thus, $\mathbf{e}_{[1,n]} - \overleftarrow{\mathbf{B}} \geqslant \overleftarrow{\mathbf{A}}$ does not hold, and so the left side of (10.4.5) also equals 0, due to the $\mathbf{1}_{\mathbf{B} \geqslant \mathbf{C}}$ factor in the definition (8.4.1) of $\mathcal{W}_{x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty)$.

So, let us assume in what follows that $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ and $v = 0$. Then, (8.4.2) gives

$$\begin{aligned} \widehat{\mathcal{W}}^{(1/q)}_{q^{n-1}x;y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0) &= (-q^{n-1}xy^{-1})^{d-n} q^{\binom{n}{2} - \varphi(\mathbf{D}, \mathbf{C}+\mathbf{D})} \frac{(q^{n-1}xy^{-1}; q^{-1})_n}{(q^{n-1}xy^{-1}; q^{-1})_{a+b}} \\ &= (-q^{n-1}xy^{-1})^{d-n} q^{-\varphi(\mathbf{D}, \mathbf{C}) - \varphi(\mathbf{D}, \mathbf{D}) - \binom{n}{2}} (xy^{-1}; q)_{n-a-b}, \end{aligned} \tag{10.4.6}$$

where in the last equality we used the fact that $(q^{n-1}z;q^{-1})_n(q^{n-1}z;q^{-1})_k^{-1}=(z;q)_{n-k}$ for any $z\in\mathbb{C}$ and $k\in[0,n]$. Moreover, (8.4.1) gives

$$\begin{aligned}(10.4.7)\qquad &\mathcal{W}_{x;y}^{(q)}(\overleftarrow{\mathbf{C}},\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{B}};\overleftarrow{\mathbf{A}},\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{D}}\mid\infty;\infty)\\ &\quad=(-1)^{n-a-b}x^{b-n}y^{n-b}q^{\varphi(\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{B}},\overleftarrow{\mathbf{A}}+\overleftarrow{\mathbf{B}}-\mathbf{e}_{[1,n]})}(xy^{-1};q)_{n-a-b}.\end{aligned}$$

By (10.3.2), the bilinearity of φ, the fact that $\varphi\big(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]}\big)=\binom{n}{2}$, and the $\mathbf{X}=\mathbf{D}$ case of (8.2.2), we obtain

$$\begin{aligned}\varphi\big(\mathbf{e}_{[1,n]}-&\overleftarrow{\mathbf{B}},\overleftarrow{\mathbf{A}}+\overleftarrow{\mathbf{B}}-\mathbf{e}_{[1,n]}\big)\\ &=\varphi\big(\mathbf{A}+\mathbf{B}-\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]}-\mathbf{B}\big)\\ &=\varphi\big(\mathbf{B},\mathbf{e}_{[1,n]}\big)+\varphi\big(\mathbf{e}_{[1,n]},\mathbf{B}\big)-\varphi\big(\mathbf{A}+\mathbf{B},\mathbf{B}\big)+\varphi\big(\mathbf{A},\mathbf{e}_{[1,n]}\big)-\varphi\big(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]}\big)\\ &=(n-1)b-\binom{n}{2}+\varphi\big(\mathbf{A},\mathbf{e}_{[1,n]}\big)-\varphi\big(\mathbf{A},\mathbf{B}\big)-\varphi\big(\mathbf{B},\mathbf{B}\big).\end{aligned}$$

Together with (10.4.7), this gives

$$\begin{aligned}&\mathcal{W}_{x;y}^{(q)}(\overleftarrow{\mathbf{C}},\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{B}};\overleftarrow{\mathbf{A}},\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{D}}\mid\infty;\infty)\\ &\quad=(-1)^{n-a-b}x^{b-n}y^{n-b}q^{(n-1)b+\varphi(\mathbf{A},\mathbf{e}_{[1,n]})-\varphi(\mathbf{A},\mathbf{B})-\varphi(\mathbf{B},\mathbf{B})-\binom{n}{2}}(xy^{-1};q)_{n-a-b}.\end{aligned}$$

This, (10.4.6), and the fact that $a+b=c+d$ together imply the lemma. $\square$

Remark 10.4.3. The type of "horizontal complementation symmetry" described by Lemma 10.4.2 does not appear to admit a transparent generalization to the most general fused weights given by Definition 5.1.1. Indeed, if $\mathbf{B}\leqslant\mathbf{A},\mathbf{C},\mathbf{D}$ then the weight $W_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid r,s)$ factors completely, since the sum on the right side of (5.1.1) is supported on the $p=0$ term (as then $b-v=0$, since $\mathbf{V}=\mathbf{B}$ there). However, for generic choices of parameters, the weight $W_z(\mathbf{C},\mathbf{e}_{[1,n]}-\mathbf{B};\mathbf{A},\mathbf{e}_{[1,n]}-\mathbf{D}\mid r,s)$ does not factor under the condition $\mathbf{B}\leqslant\mathbf{A},\mathbf{C},\mathbf{D}$.

Now we can establish Proposition 10.4.1.

Proof of Proposition 10.4.1. Throughout this proof, let us assume for notational simplicity that $N=1$, as the proof for general N is entirely analogous (alternatively, given the $N=1$ case, the proof for general N can be established using the branching identities (7.2.2), as in the proof of Theorem 9.3.2 assuming Proposition 9.3.4 in Section 9.3).

Then, recall from (7.1.3) and Remark 8.4.4 that $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty\mid\mathbf{y};\infty)$ is the partition function under the $\mathcal{W}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid\infty;\infty)$ weights (from (8.4.1)) for the vertex model $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (from Definition 7.1.3) on the domain $\mathcal{D}=\mathcal{D}_1=\mathbb{Z}_{>0}\times\{1\}$. Since this domain has one row, $\mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ only consists in a single path ensemble, which we denote by $\mathcal{E}^*(\boldsymbol{\lambda}/\boldsymbol{\mu})$. For each $j\geqslant1$, let $\big(\mathbf{A}^*(j),\mathbf{B}^*(j);\mathbf{C}^*(j),\mathbf{D}^*(j)\big)$ denote the arrow configuration at the vertex $(j,1)\in\mathcal{D}$ under $\mathcal{E}^*(\boldsymbol{\lambda}/\boldsymbol{\mu})$. In particular, there exists a minimal integer $L_0>1$ (given explicitly by (10.2.1)) such that we have $\big(\mathbf{A}^*(K),\mathbf{B}^*(K);\mathbf{C}^*(K),\mathbf{D}^*(K)\big)=(\mathbf{e}_0,\mathbf{e}_0;\mathbf{e}_0,\mathbf{e}_0)$ for $K\geqslant L_0$; see the left side of Figure 10.7.

For any $j\geqslant1$, define $\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j),\widehat{\mathbf{C}}(j),\widehat{\mathbf{D}}(j)\in\{0,1\}^n$ by

(10.4.8)

$$\widehat{\mathbf{A}}(j)=\overleftarrow{\mathbf{C}}^*(j);\qquad\widehat{\mathbf{B}}(j)=\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{B}}^*(j);\qquad\widehat{\mathbf{C}}(j)=\overleftarrow{\mathbf{A}}^*(j);\qquad\widehat{\mathbf{D}}(j)=\mathbf{e}_{[1,n]}-\overleftarrow{\mathbf{D}}^*(j).$$

For each index $X\in\{A,B,C,D\}$, let $x^*(j)=\big|\mathbf{X}^*(j)\big|$ and $\widehat{x}=\big|\widehat{\mathbf{X}}\big|$.

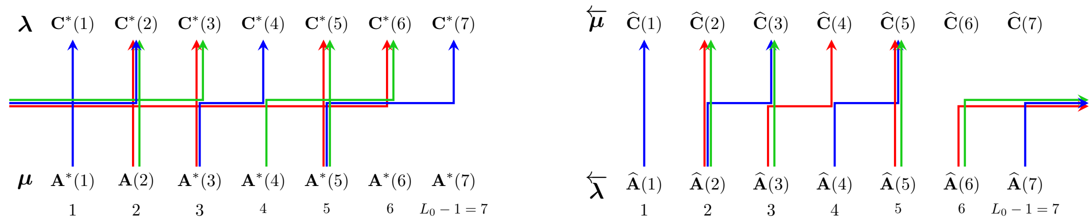


FIGURE 10.7. Shown to the left and right are examples of the single-row vertex models $\mathcal{E}^*(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\widehat{\mathcal{E}}\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$, respectively.

Then, since $\mathbf{A}^*(j)+\mathbf{B}^*(j) = \mathbf{C}^*(j)+\mathbf{D}^*(j)$, (10.4.8) implies each $\widehat{\mathbf{A}}(j)+\widehat{\mathbf{B}}(j) = \widehat{\mathbf{C}}(j)+\widehat{\mathbf{D}}(j)$. In particular, this yields the existence of a path ensemble $\widehat{\mathcal{E}}\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$ whose arrow configuration at $(j,1)$ is $\big(\widehat{\mathbf{A}}(j), \widehat{\mathbf{B}}(j); \widehat{\mathbf{C}}(j), \widehat{\mathbf{D}}(j)\big)$. It is quickly verified that this definition is consistent with the notation from Section 10.2, namely, that $\widehat{\mathcal{E}}\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$ is the unique element of $\mathfrak{P}_F\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$; see Figure 10.7.

Thus, $\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid \mathbf{y};\infty)$ and $\mathcal{F}^{(1/q)}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(q^{n-1}x;\infty \mid \mathbf{y};0)$ are partition functions, under the weights $\mathcal{W}^{(q)}_{x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid \infty,\infty)$ and $\widehat{\mathcal{W}}^{(1/q)}_{q^{n-1}x;y}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid \infty,0)$, for the path ensembles $\mathcal{E}^*(\boldsymbol{\lambda}/\boldsymbol{\mu})$ and $\widehat{\mathcal{E}}\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$, respectively. Therefore, Lemma 10.4.2 yields

(10.4.9)
$$\begin{aligned}
\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty \mid \mathbf{y};\infty) &= \prod_{j=1}^{L_0-1} \mathcal{W}^{(q)}_{x;y}\big(\mathbf{A}^*(j),\mathbf{B}^*(j);\mathbf{C}^*(j),\mathbf{D}^*(j) \mid \infty,\infty\big) \\
&= \prod_{j=1}^{L_0-1} (-1)^{\widehat{c}(j)} q^{\varphi(\widehat{\mathbf{A}}(j),\mathbf{e}_{[1,n]})+\varphi(\widehat{\mathbf{D}}(j),\widehat{\mathbf{C}}(j))-\varphi(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j))+\varphi(\widehat{\mathbf{D}}(j),\widehat{\mathbf{D}}(j))-\varphi(\widehat{\mathbf{B}}(j),\widehat{\mathbf{B}}(j))} \\
&\qquad\qquad \times (q^{n-1}xy_j^{-1})^{\widehat{b}(j)-\widehat{d}(j)} \widehat{\mathcal{W}}^{(1/q)}_{q^{n-1}x;y}\big(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j);\widehat{\mathbf{C}}(j),\widehat{\mathbf{D}}(j) \mid \infty,0\big) \\
&= \mathcal{F}^{(1/q)}_{\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}}(q^{n-1}x;\infty \mid \mathbf{y};0) \prod_{j=1}^{L_0-1} (-1)^{\widehat{c}(j)} q^{\varphi(\widehat{\mathbf{A}}(j),\mathbf{e}_{[1,n]})+\varphi(\widehat{\mathbf{D}}(j),\widehat{\mathbf{C}}(j))-\varphi(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j))} \\
&\qquad\qquad\qquad \times q^{\varphi(\widehat{\mathbf{D}}(j),\widehat{\mathbf{D}}(j))-\varphi(\widehat{\mathbf{B}}(j),\widehat{\mathbf{B}}(j))} (q^{n-1}xy_j^{-1})^{\widehat{b}(j)-\widehat{d}(j)}.
\end{aligned}$$

Now let us analyze the terms in the product appearing on the right side of (10.4.9). To that end, observe that

$$\sum_{j=1}^{L_0-1} \widehat{c}(j) = nM; \qquad \sum_{j=1}^{L_0-1} \varphi\big(\widehat{\mathbf{A}}(j),\mathbf{e}_{[1,n]}\big) = (M+1)\varphi\big(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]}\big) = (M+1)\binom{n}{2}, \tag{10.4.10}$$

where the first equality holds since $\boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$, and the second holds since $\sum_{j=1}^{L_0-1} \widehat{\mathbf{A}}(j) = (M+1)\mathbf{e}_{[1,n]}$ (as $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M+1}$). We also have that

$$\sum_{j=1}^{L_0-1} \Big(\varphi\big(\widehat{\mathbf{D}}(j),\widehat{\mathbf{D}}(j)\big) - \varphi\big(\widehat{\mathbf{B}}(j),\widehat{\mathbf{B}}(j)\big)\Big) = \varphi\big(\widehat{\mathbf{B}}(L_0),\widehat{\mathbf{B}}(L_0)\big) - \varphi\big(\widehat{\mathbf{B}}(1),\widehat{\mathbf{B}}(1)\big) = \binom{n}{2}, \tag{10.4.11}$$

where the first equality holds since each $\widehat{\mathbf{D}}(j)=\widehat{\mathbf{B}}(j+1)$, and the second holds as $\varphi\big(\widehat{\mathbf{B}}(1),\widehat{\mathbf{B}}(1)\big)=\varphi(\mathbf{e}_0,\mathbf{e}_0)=0$ and $\varphi\big(\widehat{\mathbf{B}}(L_0),\widehat{\mathbf{B}}(L_0)\big)=\varphi\big(\mathbf{e}_{[1,n]},\mathbf{e}_{[1,n]}\big)=\binom{n}{2}$. Moreover,

$$\begin{aligned}
&\prod_{j=1}^{L_0-1}(q^{n-1}x)^{\widehat{b}(j)-\widehat{d}(j)}=(q^{n-1}x)^{\widehat{b}(1)-\widehat{b}(L_0)}=q^{-2\binom{n}{2}}x^{-n};\\
&\prod_{j=1}^{L_0-1}y_j^{\widehat{d}(j)-\widehat{b}(j)}=\prod_{i=1}^{n}\prod_{j=1}^{L_0-1}y_j^{\widehat{D}_i(j)-\widehat{B}_i(j)}=\prod_{i=1}^{n}\left(\prod_{j=1}^{M}y^{-1}_{\mu_j^{(i)}+M-j}\prod_{k=1}^{M+1}y_{\lambda_k^{(i)}+M-k+1}\right),
\end{aligned}\tag{10.4.12}$$

where we have denoted $\widehat{\mathbf{X}}(j)=\big(\widehat{X}_1(j),\widehat{X}_2(j),\ldots,\widehat{X}_n(j)\big)$, for each index $X\in\{B,D\}$. The first equality of the first statement in (10.4.12) holds since each $\widehat{d}(j)=\widehat{b}(j+1)$; the second equality of the first statement holds since $\widehat{b}(1)=0$ and $\widehat{b}(L_0)=n$; the first equality of the second statement holds since $\widehat{d}(j)-\widehat{b}(j)=\sum_{j=1}^{L_0-1}\big(\widehat{D}_i(j)-\widehat{B}_i(j)\big)$; and the second equality of the second statement holds since $\widehat{D}_i(j)-\widehat{B}_i(j)=1$ if and only if $j\in\mathfrak{T}\big(\lambda^{(n-i+1)}\big)$ and $j\notin\mathfrak{T}\big(\mu^{(n-i+1)}\big)$, and $\widehat{D}_i(j)-\widehat{B}_i(j)=-1$ holds if and only if $j\in\mathfrak{T}\big(\mu^{(n-i+1)}\big)$ and $j\notin\mathfrak{T}\big(\lambda^{(n-i+1)}\big)$.

Together, (10.4.9), (10.4.10), (10.4.11), and (10.4.12) imply

$$\begin{aligned}
\mathcal{H}^{(q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x;\infty\mid\mathbf{y};\infty)&=(-1)^{nM}q^{M\binom{n}{2}}x^{-n}\mathcal{F}^{(1/q)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q^{n-1}x;\infty\mid\mathbf{y};\infty)\\
&\qquad\times\prod_{j=1}^{L_0-1}q^{\varphi(\widehat{\mathbf{D}}(j),\widehat{\mathbf{C}}(j))-\varphi(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j))}\prod_{i=1}^{n}\left(\prod_{j=1}^{M}y^{-1}_{\mu_j^{(i)}+M-j}\prod_{k=1}^{M+1}y_{\lambda_k^{(i)}+M-k+1}\right).
\end{aligned}$$

Thus, to establish the proposition it suffices to show that

$$\sum_{j=1}^{L_0-1}\Big(\varphi\big(\widehat{\mathbf{D}}(j),\widehat{\mathbf{C}}(j)\big)-\varphi\big(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j)\big)\Big)=2\psi(\overleftarrow{\boldsymbol{\mu}})-2\psi(\overleftarrow{\boldsymbol{\lambda}})+M\binom{n}{2}.\tag{10.4.13}$$

To that end, it will be useful to produce an analog of the path ensemble $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)$ from Section 10.2, following (10.2.2). So recalling the notation $\overleftarrow{\boldsymbol{\lambda}}=\big(\overleftarrow{\lambda}^{(1)},\overleftarrow{\lambda}^{(2)},\ldots,\overleftarrow{\lambda}^{(n)}\big)$ and $\overleftarrow{\lambda}^{(i)}=\big(\overleftarrow{\lambda}^{(i)}_1,\overleftarrow{\lambda}^{(i)}_2,\ldots,\overleftarrow{\lambda}^{(i)}_{M+1}\big)$ (and similarly for $\overleftarrow{\boldsymbol{\mu}}$ and the $\overleftarrow{\mu}^{(i)}$), following (10.2.2) we define the sequences $\mathring{\boldsymbol{\lambda}},\widetilde{\boldsymbol{\mu}}\in\mathrm{SeqSign}_{n;M+1}$ by setting

$$\begin{aligned}
&\mathring{\lambda}^{(i)}_j=\overleftarrow{\lambda}^{(i)}_j+1,\quad\text{for all } j\in[1,M+1];\\
&\widetilde{\mu}^{(i)}_1=L_0-M-1,\quad\text{and}\quad\widetilde{\mu}^{(i)}_j=\overleftarrow{\mu}^{(i)}_{j-1}+1,\quad\text{for all } j\in[2,M+1].
\end{aligned}$$

For any $j\geqslant 1$, let $\big(\widetilde{\mathbf{A}}(j),\widetilde{\mathbf{B}}(j);\widetilde{\mathbf{C}}(j),\widetilde{\mathbf{D}}(j)\big)$ denote the arrow configuration at the vertex $(j,1)$ under the path ensemble $\mathcal{E}\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}\big)\in\mathfrak{P}_G\big(\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}};1\big)$. We refer to Figure 10.3 for a depiction (where, in the present context, the left path ensemble there is $\widehat{\mathcal{E}}\big(\overleftarrow{\boldsymbol{\lambda}}/\overleftarrow{\boldsymbol{\mu}}\big)$, instead of $\widehat{\mathcal{E}}(\boldsymbol{\lambda}/\boldsymbol{\mu})$).

In particular, we have that $\widetilde{\mathbf{A}}(j)=\widehat{\mathbf{A}}(j-1)$ for each $j\geqslant 2$; that $\widetilde{\mathbf{B}}(j+1)=\widetilde{\mathbf{D}}(j)=\widehat{\mathbf{B}}(j)=\widehat{\mathbf{D}}(j-1)$ for each $j<L_0$; and that $\widetilde{\mathbf{A}}(K+1)=\widetilde{\mathbf{B}}(K+1)=\widetilde{\mathbf{D}}(K)=\mathbf{e}_0$ for each $K\geqslant L_0$. Thus,

$$\begin{aligned}
\sum_{j=1}^{L_0-1}\Big(\varphi\big(\widehat{\mathbf{D}}(j),\widehat{\mathbf{C}}(j)\big)-\varphi\big(\widehat{\mathbf{A}}(j),\widehat{\mathbf{B}}(j)\big)\Big)&=\sum_{j=1}^{\infty}\Big(\varphi\big(\widetilde{\mathbf{D}}(j),\widetilde{\mathbf{C}}(j)\big)-\varphi\big(\widetilde{\mathbf{A}}(j),\widetilde{\mathbf{B}}(j)\big)\Big)\\
&=2\psi\big(\widetilde{\boldsymbol{\mu}}\big)-2\psi\big(\mathring{\boldsymbol{\lambda}}\big)=2\psi\big(\overleftarrow{\boldsymbol{\mu}}\big)-2\psi\big(\overleftarrow{\boldsymbol{\lambda}}\big)+M\binom{n}{2},
\end{aligned}$$

where in the second equality we applied Lemma 10.1.2 (with the $\boldsymbol{\lambda}/\boldsymbol{\mu}$ there equal to $\widetilde{\boldsymbol{\mu}}/\mathring{\boldsymbol{\lambda}}$ here) and in the third we applied (10.2.19). This establishes (10.4.13) and thus the proposition. $\square$

CHAPTER 11

Contour Integral Formulas for $G_{\boldsymbol{\lambda}}$

In this chapter we establish contour integral formulas for the functions $G_{\boldsymbol{\lambda}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$. For simplicity, we only implement this in the homogeneous regime when all $y_i = 1$ and $s_i = s$, although more general results allowing for distinct $\{y_i\}$ and $\{s_i\}$ can be obtained similarly.

11.1. Nonsymmetric Functions and Integral Formulas

We will establish our contour integral formulas for $G_{\boldsymbol{\lambda}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s})$, with $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;M}$, by using the color merging result Theorem 5.2.2 to compare the $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ symmetric partition function $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ to certain $U_q\big(\widehat{\mathfrak{sl}}(1|nM)\big)$ nonsymmetric ones (which coincide with the $U_q\big(\widehat{\mathfrak{sl}}(nM+1)\big)$ nonsymmetric functions from **[12]**). Thus, we begin by recalling from **[12]** the definitions and properties of these nonsymmetric functions. Throughout this section, we fix an integer $n \geqslant 1$ (which in later sections will be set to nM) and a complex number $s \in \mathbb{C}$.

There will be two nonsymmetric functions of interest to us here, which will be denoted by $\mathtt{f}_\lambda$ and $\mathtt{g}_\lambda$. The first function $\mathtt{f}_\lambda$ will be defined as a certain transformation of the second one $\mathtt{g}_\lambda$, and the latter will be defined as the partition function for a particular $U_q\big(\widehat{\mathfrak{sl}}(1|n)\big)$ vertex model.

The weights for this vertex model, denoted by $M_{z,q,s}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$, will be given by a transformation of the $W_{z;q}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$ from Definition 5.1.1. More specifically, for any complex numbers $z,q \in \mathbb{C}$ and elements $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \{0,1\}^n$, set $M(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = M_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = M_{z,q,s}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ to be[1]

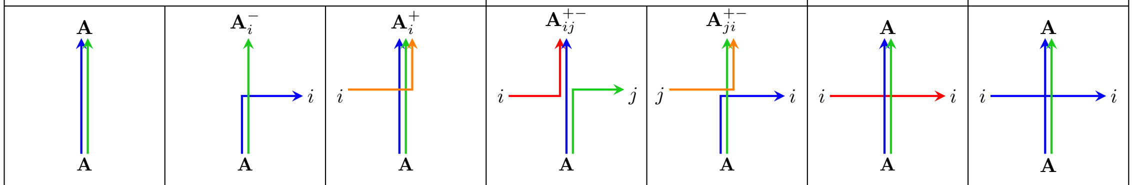
$$M(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = (-s)^{\mathbf{1}_{|\mathbf{B}|>0}} W_{s/z;1/q}^{(1;n)}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid q^{1/2},s^{-1}). \tag{11.1.1}$$

In particular, these M weights are (up to a change of variables and an overall multiplicative factor) given by the $r = q^{-1/2}$ cases of the $W(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D} \mid r,s)$, so Example 8.1.2 implies that $M_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = 0$ unless $\mathbf{B},\mathbf{D} \in \{\mathbf{e}_0,\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. Hence, in what follows, let us abbreviate $M_z = (\mathbf{A},b;\mathbf{C},d) = M_z(\mathbf{A},\mathbf{e}_b;\mathbf{C},\mathbf{e}_d)$ for any $b,d \in \{0,1,\ldots,n\}$. These weights are given more explicitly through the following definition.

Definition 11.1.1. For any $\mathbf{A},\mathbf{C} \in \mathbb{Z}_{\geqslant 0}^n$ and $b,d \in \{0,1,\ldots,n\}$, set $M(\mathbf{A},b;\mathbf{C},d) = 0$ unless $\mathbf{A},\mathbf{C} \in \{0,1\}^n$. Letting $|\mathbf{A}| = a$ and $\mathbf{A}_i^+,\mathbf{A}_j^-,\mathbf{A}_{ij}^{+-}$ be as in (8.1.1) for each $\mathbf{A} \in \{0,1\}^n$ and $i \in [1,n]$, set

(11.1.2)
$$M(\mathbf{A},0;\mathbf{A},0) = \frac{q^{-a}-sz}{1-sz}; \quad M(\mathbf{A},0;\mathbf{A}_i^-,i) = q^{-A_{[i+1,n]}}\frac{1-q^{-1}}{1-sz}; \quad M(\mathbf{A},i;\mathbf{A}_i^+,0) = \frac{z(s^2-q^{-a})}{1-sz}.$$

[1]These M weights are closely related to those given by equation (2.2.6) of **[12]**; see Remark 11.1.5 below.



$1 \leqslant i \leqslant n$			$1 \leqslant i < j \leqslant n$		$A_i = 0$	$A_i = 1$
$(\mathbf{A}, 0; \mathbf{A}, 0)$	$(\mathbf{A}, 0; \mathbf{A}_i^-, i)$	$(\mathbf{A}, i; \mathbf{A}_i^+, 0)$	$(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j)$	$(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$
$\dfrac{q^{-a} - sz}{1 - sz}$	$q^{-A_{[i+1,n]}} \dfrac{1 - q^{-1}}{1 - sz}$	$\dfrac{z(s^2 - q^{-a})}{1 - sz}$	$q^{-A_{[j+1,n]}} \dfrac{s(q^{-1} - 1)}{1 - sz}$	$q^{-A_{[i+1,n]}} \dfrac{z(q^{-1} - 1)}{1 - sz}$	$q^{-A_{[i+1,n]}} \dfrac{z - s}{1 - sz}$	$q^{-A_{[i+1,n]}} \dfrac{q^{-1}s - z}{1 - sz}$

FIGURE 11.1. The vertex weights M_z, and their arrow configurations, are depicted above. Here red, blue, green, and orange denote the colors 1, 2, 3, and 4, respectively.

Moreover, for any $1 \leqslant i < j \leqslant n$, set

$$M(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j) = q^{-A_{[j+1,n]}} \frac{s(q^{-1} - 1)}{1 - sz}; \qquad M(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i) = q^{-A_{[i+1,n]}} \frac{z(q^{-1} - 1)}{1 - sz}.$$

Additionally set

(11.1.3)

$$M(\mathbf{A}, i; \mathbf{A}, i) = q^{-A_{[i+1,n]}} \frac{z - s}{1 - sz}, \quad \text{if } A_i = 0; \qquad M(\mathbf{A}, i; \mathbf{A}; i) = q^{-A_{[i+1,n]}} \frac{q^{-1}s - z}{1 - sz}, \quad \text{if } A_i = 1.$$

We further set $M(\mathbf{A}, b; \mathbf{C}, d) = 0$ if $(\mathbf{A}, b; \mathbf{C}, d)$ is not of the above form.

As before, we interpret $M_z(\mathbf{A}, b; \mathbf{C}, d)$ as the weight associated with a vertex v whose arrow configuration is $(\mathbf{A}, \mathbf{e}_b; \mathbf{C}, \mathbf{e}_d)$ and whose spectral parameter is z. We refer to Figure 11.1 for a depiction of these weights; it is quickly verified that (11.1.1) holds for them.

It will additionally be useful to set notation for ensemble weights (as in (7.1.2)). So, let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ denote (possibly infinite) sequences of complex numbers. For any path ensemble $\mathcal{E}$ on some finite domain $\mathcal{D} \subset \mathbb{Z}_{>0}^2$, define the ensemble weight of $\mathcal{E}$ with respect to M by

$$M(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) = \prod_{(i,j) \in \mathcal{D}} M_{x_j / y_i}\big(\mathbf{A}(i,j), \mathbf{B}(i,j); \mathbf{C}(i,j), \mathbf{D}(i,j)\big), \tag{11.1.4}$$

where $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ denotes the arrow configuration under $\mathcal{E}$ at any vertex $v \in \mathcal{D}$.

Next, for $K, N \in \mathbb{Z}_{\geqslant 1}$, we introduce a family of path ensembles on the rectangular domain $\mathcal{D}_{K,N} = \{1, 2, \ldots, K\} \times \{1, 2, \ldots, N\}$.

Definition 11.1.2. Fix an integer $n \geqslant 1$; let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{>0}^n$ denote a composition of length n; and set $L = L(\lambda) = \max_{j \in [1,n]} \lambda_j$. Let $\mathfrak{Q}_{\mathfrak{g}}(\lambda)$ denote the set of path ensembles on $\mathcal{D}_{L,n}$ with the following boundary data; we refer to the left side of Figure 11.2 for a depiction.

(1) For each $c \in [1, n]$, one color c arrow vertically enters $\mathcal{D}_{L,n}$ at $(L - \lambda_c + 1, 1)$.
(2) For each $c \in [1, n]$, one color c arrow horizontally exits $\mathcal{D}_{L,n}$ at (L, c).

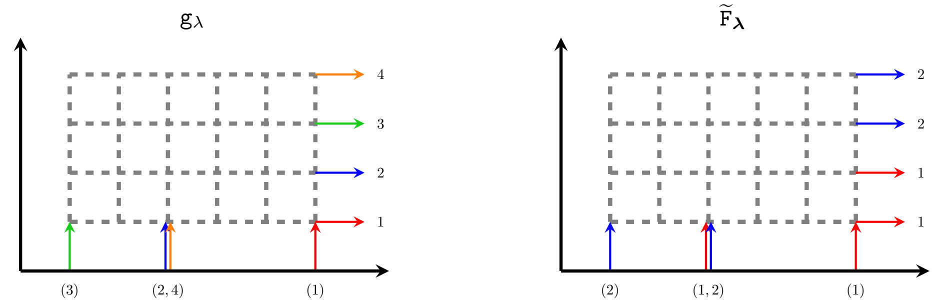


FIGURE 11.2. Depicted to the left and right are the vertex models $\mathfrak{Q}_{\mathtt{g}}(\lambda)$ and $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$, respectively, for $\lambda = (1, 4, 6, 4)$ we have $\boldsymbol{\lambda} = \big((2, 0), (4, 3)\big)$. Here red, blue, green, and orange are colors 1, 2, 3, and 4, respectively.

Now we can define the functions $\mathtt{f}_\lambda$ and $\mathtt{g}_\lambda$. In what follows, for any sequence $\mathcal{X} = (x_1, x_2, \ldots, x_\ell)$ we recall $\overleftarrow{\mathcal{X}} = (x_\ell, x_{\ell-1}, \ldots, x_1)$ denotes the reverse ordering of $\mathcal{X}$. We further recall that if $\mathcal{X} \in \mathbb{R}^\ell$ that $\operatorname{inv}(\mathcal{X})$ denotes the number of index pairs $(i, j) \in [1, \ell]^2$ such that $i < j$ and $x_i > x_j$.

Definition 11.1.3. Fix an integer $n \geqslant 1$; a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$; and a (positive) composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{>0}$ of length n. Further let $\mathbf{y} = (1, 1, \ldots)$. Define $\mathtt{g}_\lambda(\mathbf{x}) = g^{(q)}_\lambda(\mathbf{x} \mid q)$ by setting

$$\mathtt{g}_\lambda(\mathbf{x}) = \sum_{\mathcal{E} \in \mathfrak{Q}_{\mathtt{g}}(\lambda)} M(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}). \tag{11.1.5}$$

Stated alternatively, it is the partition function for $\mathfrak{Q}_{\mathtt{g}}(\lambda)$ under the weights M_{x_j} in the j-th row.

Further define $\mathtt{f}_\lambda(\mathbf{x}) = \mathtt{f}^{(q)}_\lambda(\mathbf{x} \mid s)$ from $\mathtt{g}$ by

$$\mathtt{f}^{(q)}_\lambda(\mathbf{x} \mid s) = s^n q^{-\operatorname{inv}(\overleftarrow{\lambda})} \mathtt{g}^{(1/q)}_{\overleftarrow{\lambda}}\big(\overleftarrow{\mathbf{x}}^{-1} \mid s^{-1}\big) \prod_{i=1}^{n} \big(x_i(q-1)\big)^{-1} \prod_{j=0}^{\infty} (s^{-2}; q^{-1})_{m_j(\lambda)}. \tag{11.1.6}$$

In addition to the fuctions $\mathtt{f}_\lambda$ and $\mathtt{g}_\lambda$, an additional pair of (symmetric) functions given by the below definition will be useful for us.

Definition 11.1.4. Fix an integer $n \geqslant 1$; a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$; and two (positive) compositions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of length n. Define the signature sequences

$$\boldsymbol{\lambda} = \big((\lambda_1 - 1), (\lambda_2 - 1), \ldots, (\lambda_n - 1)\big) \in \mathrm{SeqSign}_{n;1};$$
$$\boldsymbol{\mu} = \big((\mu_1 - 1), (\mu_2 - 1), \ldots, (\mu_n - 1)\big) \in \mathrm{SeqSign}_{n;1},$$

as well as the sequences of complex numbers

$$s^{-1}\mathbf{x} = (s^{-1}x_1, s^{-1}x_2, \ldots, s^{-1}x_n); \qquad \mathbf{y} = (1, 1, \ldots);$$
$$\mathbf{q}^{-1/2} = (q^{-1/2}, q^{-1/2}, \ldots, q^{-1/2}); \qquad \mathbf{s} = (s, s, \ldots).$$

Then, define $\widetilde{\mathtt{G}}_{\lambda/\mu}(\mathbf{x}) = \widetilde{\mathtt{G}}^{(q)}_{\lambda/\mu}(\mathbf{x} \mid s)$ by

$$\widetilde{\mathtt{G}}_{\lambda/\mu}(\mathbf{x}) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(s^{-1}\mathbf{x}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s}), \tag{11.1.7}$$

and $\mathtt{G}_{\lambda/\mu}(\mathbf{x}) = \mathtt{G}^{(q)}_{\lambda/\mu}(\mathbf{x} \mid s)$ by

$$\mathtt{G}_{\lambda/\mu}(\mathbf{x}) = (-s)^{|\lambda|-|\mu|} q^{\mathrm{inv}(\mu)-\mathrm{inv}(\lambda)} \widetilde{\mathtt{G}}^{(1/q)}_{\overleftarrow{\lambda}/\overleftarrow{\mu}}(\overleftarrow{\mathbf{x}}^{-1} \mid s^{-1}) \prod_{j=0}^{\infty} \frac{(s^2;q)_{m_j(\mu)}}{(s^2;q)_{m_j(\lambda)}}, \tag{11.1.8}$$

where we recall that $m_j(\nu)$ denote the multiplicity of j in ν.

Remark 11.1.5. Let us briefly indicate the relation between the functions from Definition 11.1.3 and Definition 11.1.4 and some of those introduced in [**12**]. Here, we let λ denote a composition of length n and $\widetilde{\lambda}$ denote the composition obtained from subtracting 1 from each entry of λ.

First, the $\mathtt{g}_\lambda(\mathbf{x})$ from Definition 11.1.4 coincides with the $g_{\widetilde{\lambda}}(\mathbf{x})$ given by Definition 3.4.6 of [**12**]. To see this, first observe that since the vertex model $\mathfrak{Q}_{\mathtt{g}}(\lambda)$ from Definition 11.1.2 contains at most one arrow of any color, the rightmost weight depicted in Figure 11.1 (with arrow configuration $(\mathbf{A}, i; \mathbf{A}, i)$ for $A_i = 1$) is irrelevant for evaluating the partition function (11.1.5) for $\mathtt{g}_\lambda$; see also Remark 8.1.3. Omitting this weight from consideration, the remaining ones from Definition 11.1.1 match with those given by equation (2.2.6) of [**12**] (after reflecting them into the y-axis). Thus, the partition function expression for $g_{\widetilde{\lambda}}(\mathbf{x})$ given by equation (3.4.10) of [**12**] matches (after reflection into the vertical line $x = \frac{L+1}{2}$) the expression (11.1.5) for $g_\lambda(\mathbf{x})$.

Then, equation (1.4.1) of [**12**] implies that the $\mathtt{f}_\lambda(\mathbf{x})$ from Definition 11.1.4 coincides with the $f_{\widetilde{\lambda}}(\mathbf{x})$ from Definition 3.4.3 of [**12**] (after taking into account the fact that the function inv there is defined slightly differently from how it is here). One can further show that the $\mathtt{G}_{\lambda/\mu}(\mathbf{x})$ from Definition 11.1.4 coincides with the $G_{\widetilde{\lambda}/\widetilde{\mu}}(\mathbf{x})$ given by Definition 4.4.1 of [**12**]. Given the partition function representation for $G_{\widetilde{\lambda}/\widetilde{\mu}}(\mathbf{x})$ from equation (4.4.2) of [**12**], the proof of this equality very closely follows that of Proposition 5.6.1 of [**12**], so we will not provide it here.

Now, let us state the following integral formula for $\mathtt{G}_{\lambda/\mu}$ from [**12**], which will be useful for us in what follows.

Proposition 11.1.6 ([**12**, Equation (9.5.2)])**.** *Fix an integer $N \geqslant 1$; compositions λ and μ of length n; and a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. We have that*

$$\mathtt{G}_{\lambda/\mu}(\mathbf{x}) = \frac{1}{(2\pi\mathbf{i})^n} \frac{q^{\binom{n+1}{2}-nN}}{(q-1)^n} \oint \cdots \oint \mathtt{f}_\mu(\mathbf{u}^{-1}) \mathtt{g}_\lambda(\mathbf{u}) \prod_{1 \leqslant i < j \leqslant n} \frac{u_j - u_i}{u_j - qu_i} \prod_{i=1}^{n} \prod_{j=1}^{N} \frac{u_i - qx_j}{u_i - x_j} \prod_{i=1}^{n} \frac{du_i}{u_i}, \tag{11.1.9}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n) \subset \mathbb{C}$, and each u_i is integrated along a positively oriented, closed contour γ_i satisfying the following two properties. First, each γ_i contains s and the x_j, and does not contain s^{-1}. Second, the $\{\gamma_i\}$ are mutually non-intersecting, and γ_{i+1} contains both γ_i and $q\gamma_i$ for each $i \in [1, n-1]$.

Let us mention two additional points. First, the value of the right side of (11.1.9) does not depend on whether the contours γ_i contains 0 or ∞, as it can be verified that those points do not constitute poles of the integrand. Second, whenever interpreting contour integrals such as those on the right side of (11.1.9), we either assume that the underlying parameters enable the existence of such contours or view the integrals as sums of residues (that can be analytically continued to regimes of parameters for which the contours no longer exist).

Although Proposition 11.1.6 provides an integral formula for $\mathtt{G}_{\lambda/\mu}$, it will be a bit more useful for us to have one for $\widetilde{\mathtt{G}}_{\lambda/\mu}$. This is done through the following corollary, which will be stated with the parameter n above replaced by m here (since we will eventually apply it for $m = nN$, where N is the number of $\mathbf{x}$ variables in the function $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s})$).

Corollary 11.1.7. *Fix integers $m, N \geqslant 1$; compositions κ and ν of length m; a complex number $s \in \mathbb{C}$; and a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. We have that*

$$\begin{aligned}\widetilde{\mathtt{G}}_{\kappa/\nu}(\mathbf{x}) = \frac{1}{(2\pi\mathbf{i})^m} \frac{(-s)^{|\kappa|-|\nu|} q^{\binom{m+1}{2}}}{(q-1)^m} \oint \cdots \oint \mathtt{g}_\nu(\mathbf{u}) \mathtt{f}_\kappa(\mathbf{u}^{-1}) \prod_{1 \leqslant i < j \leqslant m} \frac{u_j - u_i}{u_j - q u_i} \\ \times \prod_{i=1}^{m} \prod_{j=1}^{N} \frac{1 - q u_i x_j}{1 - u_i x_j} \prod_{i=1}^{m} \frac{du_i}{u_i},\end{aligned} \tag{11.1.10}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m) \subset \mathbb{C}$, and each u_i is integrated along a negatively oriented, closed contour γ_i satisfying the following properties. First, each γ_i contains s^{-1} and the x_j^{-1}, and does not contain s. Second, the $\{\gamma_i\}$ are mutually non-intersecting, and γ_{i-1} contains both γ_i and $q^{-1}\gamma_i$ for each $i \in [2, m]$.

Proof. Applying Proposition 11.1.6 with the integration variables $\mathbf{u}$ there replaced by $\overleftarrow{\mathbf{u}}$ here, and the parameters $(x_i, q, s, \lambda, \mu, n)$ there equal to $\big(x_{N-i+1}^{-1}, q^{-1}, s^{-1}, \overleftarrow{\kappa}, \overleftarrow{\nu}, m\big)$ here, and also using the facts that

$$\prod_{1 \leqslant i < j \leqslant m} \frac{u_i - u_j}{u_i - q^{-1} u_j} = q^{\binom{m}{2}} \prod_{1 \leqslant i < j \leqslant m} \frac{u_j - u_i}{u_j - q u_i}; \qquad \prod_{i=1}^{m} \prod_{j=1}^{N} \frac{u_i - q^{-1} x_j^{-1}}{u_i - x_j^{-1}} = q^{-mN} \prod_{i=1}^{m} \prod_{j=1}^{N} \frac{1 - q u_i x_j}{1 - u_i x_j},$$

we deduce that

$$\begin{aligned}\mathtt{G}^{(1/q)}_{\overleftarrow{\kappa}/\overleftarrow{\nu}}(\overleftarrow{\mathbf{x}}^{-1} \mid s^{-1}) = \frac{1}{(2\pi\mathbf{i})^m} \frac{1}{(q-1)^m} \oint \cdots \oint \mathtt{f}^{(1/q)}_{\overleftarrow{\nu}}(\overleftarrow{\mathbf{u}}^{-1} \mid s^{-1}) \mathtt{g}^{(1/q)}_{\overleftarrow{\kappa}}(\overleftarrow{\mathbf{u}} \mid s^{-1}) \\ \times \prod_{1 \leqslant i < j \leqslant m} \frac{u_j - u_i}{u_j - q u_i} \prod_{i=1}^{m} \prod_{j=1}^{N} \frac{1 - q u_i x_j}{1 - u_i x_j} \prod_{i=1}^{m} \frac{du_i}{u_i},\end{aligned}$$

where each u_i is integrated along γ_i. By (11.1.6), and using the facts that

$$\begin{aligned}(s^{-2}; q^{-1})_{m_j(\kappa)} = (-s^{-2})^{m_j(\kappa)} q^{-\binom{m_j(\kappa)}{2}} (s^2; q)_{m_j(\kappa)}; \\ \sum_{j=0}^{\infty} m_j(\kappa) = m; \qquad \operatorname{inv}(\kappa) + \operatorname{inv}\big(\overleftarrow{\kappa}\big) + \sum_{j=0}^{\infty} \binom{m_j(\kappa)}{2} = \binom{m}{2},\end{aligned} \tag{11.1.11}$$

we therefore obtain

$$\begin{aligned}\mathtt{G}^{(1/q)}_{\overleftarrow{\kappa}/\overleftarrow{\nu}}(\overleftarrow{\mathbf{x}}^{-1} \mid s^{-1}) = \frac{1}{(2\pi\mathbf{i})^m} (-s)^m q^{\binom{m}{2} - \operatorname{inv}(\kappa)} \oint \cdots \oint \mathtt{f}^{(1/q)}_{\overleftarrow{\nu}}(\overleftarrow{\mathbf{u}}^{-1} \mid s^{-1}) \mathtt{f}^{(q)}_{\kappa}(\mathbf{u}^{-1} \mid s) \\ \times \prod_{j=0}^{\infty} \frac{1}{(s^2; q)_{m_j(\kappa)}} \prod_{1 \leqslant i < j \leqslant m} \frac{u_j - u_i}{u_j - q u_i} \prod_{i=1}^{m} \prod_{j=1}^{N} \frac{1 - q u_i x_j}{1 - u_i x_j} \prod_{i=1}^{m} \frac{du_i}{u_i^2}.\end{aligned}$$

Again applying (11.1.6), it follows that

$$\mathtt{G}^{(1/q)}_{\overleftarrow{\kappa}/\overleftarrow{\nu}}(\overleftarrow{\mathbf{x}}^{-1} \mid s^{-1}) = \frac{1}{(2\pi\mathbf{i})^m}(1-q^{-1})^{-m} q^{\binom{m}{2}+\operatorname{inv}(\nu)-\operatorname{inv}(\kappa)} \oint \cdots \oint \mathtt{g}^{(q)}_{\nu}(\mathbf{u} \mid s) \mathtt{f}^{(q)}_{\kappa}(\mathbf{u}^{-1} \mid s)$$
$$\times \prod_{j=0}^{\infty} \frac{(s^2;q)_{m_j(\nu)}}{(s^2;q)_{m_j(\kappa)}} \prod_{1 \leqslant i<j \leqslant m} \frac{u_j - u_i}{u_j - qu_i} \prod_{i=1}^{m}\prod_{j=1}^{N} \frac{1-qu_i x_j}{1-u_i x_j} \prod_{i=1}^{m} \frac{du_i}{u_i}.$$

Then we deduce the corollary by first multiplying both sides by

$$(-s)^{|\kappa|-|\nu|} q^{\operatorname{inv}(\overleftarrow{\nu})-\operatorname{inv}(\overleftarrow{\kappa})} \prod_{j=0}^{\infty} \frac{(s^{-2};q^{-1})_{m_j(\kappa)}}{(s^{-2};q^{-1})_{m_j(\nu)}} = (-s)^{|\kappa|-|\nu|} q^{\operatorname{inv}(\kappa)-\operatorname{inv}(\nu)} \prod_{j=0}^{\infty} \frac{(s^2;q)_{m_j(\kappa)}}{(s^2;q)_{m_j(\nu)}},$$

where to deduce the equality we used (11.1.11), and then applying (11.1.8). $\square$

11.2. Applications of Color Merging

In this section we provide two applications of the color merging result Theorem 5.2.2, which will be useful for deriving contour integral representations for $G_{\boldsymbol{\lambda}}$. The first expresses $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as an anti-symmetrization over ν of the functions $\widetilde{\mathtt{G}}_{\kappa/\nu}$; together with Corollary 11.1.7, this will imply a contour integral formula for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ involving an anti-symmetrization over ν of the functions $\mathtt{g}_{\nu}$. The second expresses this anti-symmetrization of $\mathtt{g}_{\nu}$ as another partition function $\widetilde{\mathtt{F}}_{\boldsymbol{\mu}}$.

To define the latter, we begin with the following vertex model $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$. It is similar to $\mathfrak{Q}_{\mathtt{g}}(\lambda)$ from Definition 11.1.2, but with three differences. First, instead of consisting of n paths all of different colors, the model $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$ consists of nM paths, with M paths of each color $1, 2, \ldots, n$. Second, along the east edge of the domain, the colors of the horizontally exiting paths are (from bottom to top) given by $1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots n$, where each color appears with multiplicity M. Third, along the bottom edge of the domain, the boundary data is indexed slightly differently, using the shifts $\mathfrak{T}(\lambda^{(i)})$ from (1.2.3). In this way, the model $\mathfrak{Q}_{\mathtt{F}}$ will be essentially obtained from $\mathfrak{Q}_{\mathtt{g}}$ by merging all colors in intervals of the form $\{(k-1)M+1, (k-1)M+2, \ldots, kM\}$ to a single color $k \in [1,n]$. We refer to Figure 11.2 for a depiction.

Definition 11.2.1. Fix an integer $n \geqslant 1$; let $\boldsymbol{\lambda} \in \operatorname{SeqSign}_{n;M}$ denote a signature sequence; and set $K = K(\boldsymbol{\lambda}) = \max_{i \in [1,n]} \max \mathfrak{T}(\lambda^{(i)})$, where we recall $\mathfrak{T}$ from (1.2.3). Let $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$ denote the set of path ensembles on $\mathcal{D}_{K,nM} = [1,K] \times [1,nM]$ with the following boundary data; we refer to the right side of Figure 11.2 for a depiction.

1. For each $c \in [1,n]$, one color c arrow vertically enters $\mathcal{D}_{K,nM}$ at $(K - \mathfrak{l} + 1, 1)$, for all $\mathfrak{l} \in \mathfrak{T}(\lambda^{(c)})$.
2. For each $c \in [1,n]$ and $j \in [1,M]$, one color c arrow vertically exits $\mathcal{D}_{K,nM}$ at $\big(K, (c-1)M + j\big)$.

Now let us define the partition function $\widetilde{\mathtt{F}}_{\boldsymbol{\lambda}}$. In what follows, we recall the M weights from Definition 11.1.1 and (11.1.4).

Definition 11.2.2. Fix integers $n, M \geqslant 1$; a complex number $s \in \mathbb{C}$; two sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_{nM})$ and $\mathbf{y} = (y_1, y_2, \ldots)$; and a signature sequence $\boldsymbol{\lambda} \in \operatorname{SeqSign}_{n;M}$. Then, define $\widetilde{\mathtt{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y})$ by setting

$$\widetilde{\mathtt{F}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y}) = \sum_{\mathcal{E} \in \mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})} M(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}).$$

Stated alternatively, it is the partition function for the vertex model $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$ under the weights M_{x_j/y_i} at any vertex (i,j). If $\mathbf{y}=(1,1,\ldots)$, we abbreviate $\widetilde{\mathtt{F}}(\mathbf{x})=\widetilde{\mathtt{F}}(\mathbf{x}\mid\mathbf{y})$.

To proceed, we require a certain set of compositions $\Upsilon(\boldsymbol{\lambda})$, which can be interpreted as those compositions κ satisfying the following property. The boundary data along the bottom edge for the vertex model $\mathfrak{Q}_{\mathtt{g}}(\kappa)$ reduces to that of $\mathfrak{Q}_{\mathtt{F}}(\boldsymbol{\lambda})$ upon merging all colors in intervals of the form $\big\{(k-1)M+1,(k-1)M+2,\ldots,kM\big\}$ and identifying them as color k.

Definition 11.2.3. For any integers $n,M\geqslant 1$ and signature sequence $\boldsymbol{\lambda}\in\mathrm{SeqSign}_{n;M}$, let $\Upsilon(\boldsymbol{\lambda})$ denote the set of length nM compositions $\kappa=(\kappa_1,\kappa_2,\ldots,\kappa_{nM})$ such that, for each $i\in[1,n]$, the length M composition $\kappa^{(i)}=\big(\kappa_{(i-1)M+1},\kappa_{(i-1)M+2},\ldots,\kappa_{iM}\big)$ is a permutation of $\mathfrak{T}\big(\lambda^{(i)}\big)$. Moreover, for any $\kappa\in\Upsilon(\boldsymbol{\lambda})$, set

$$\operatorname{inv}_{\boldsymbol{\lambda}}(\kappa)=\sum_{i=1}^{n}\operatorname{inv}\left(\kappa^{(i)}\right)=\sum_{i=1}^{n}\sum_{1\leqslant h<j\leqslant M}\mathbf{1}_{\kappa_h^{(i)}>\kappa_j^{(i)}}.$$

Now we can state the following anti-symmetrization identities.

Lemma 11.2.4. *Fix integers $n,M,N\geqslant 1$; a complex number s; a sequence of complex numbers $\mathbf{x}=(x_1,x_2,\ldots,x_{nN})$; two sequences of signatures $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathrm{SeqSign}_{n;M}$; and any composition $\kappa\in\Upsilon(\boldsymbol{\lambda})$. Setting $\mathbf{q}^{-1/2}=(q^{-1/2},q^{-1/2},\ldots,q^{-1/2})$ (of length nM); $\mathbf{y}=(1,1,\ldots)$; and $\mathbf{s}=(s,s,\ldots)$, we have*

$$\begin{aligned}&\sum_{\nu\in\Upsilon(\boldsymbol{\mu})}(-1)^{\operatorname{inv}_{\boldsymbol{\mu}}(\nu)}\widetilde{\mathtt{G}}_{\kappa/\nu}(\mathbf{x})=(-1)^{\operatorname{inv}_{\boldsymbol{\lambda}}(\kappa)}G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(s^{-1}\mathbf{x};\mathbf{q}^{-1/2}\mid\mathbf{y};\mathbf{s});\\&\sum_{\nu\in\Upsilon(\boldsymbol{\mu})}(-1)^{\operatorname{inv}_{\boldsymbol{\lambda}}(\nu)}\mathtt{g}_{\nu}(\mathbf{x})=(-1)^{n\binom{M}{2}}\widetilde{\mathtt{F}}_{\boldsymbol{\mu}}(\mathbf{x}),\end{aligned}\tag{11.2.1}$$

where for the latter statement we assume $N=M$.

Proof. The proofs of both statements of (11.2.1) will follow in an entirely analogous way from Theorem 5.2.2; so, let us only establish the first one there.

We will deduce it as an application of Theorem 5.2.2, so let us first match the notation here with that there; in what follows, we let $K=\max_{i\in[1,n]}\max\mathfrak{T}\big(\lambda^{(i)}\big)$. Then, observe that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(s^{-1}\mathbf{x};\mathbf{q}^{-1/2}\mid\mathbf{y};\mathbf{s})$ and $\widetilde{\mathtt{G}}_{\kappa/\nu}(\mathbf{x})$ are both partition functions for vertex models on the rectangle $\mathcal{D}_{K,nN}=\{1,2,\ldots,K\}\times\{1,2,\ldots,nN\}$, under the weights $W_{x_j/s}\big(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}\mid q^{-1/2},s\big)$ in the j-th row. However, the boundary data for these models are slightly different.

In particular, Definition 7.1.3 (see also the left side of Figure 7.1) implies that the model describing $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ has, for each $c\in[1,n]$, one color c arrow vertically entering through $(\mathfrak{m},1)$ for every $\mathfrak{m}\in\mathfrak{T}\big(\mu^{(c)}\big)$, and one color c arrow vertically exiting through $(\mathfrak{l},nN)$ for every $\mathfrak{l}\in\mathfrak{T}\big(\lambda^{(c)}\big)$. Recalling the notation introduced in Section 5.2, we denote this boundary data by $(\mathcal{E};\mathcal{F})=\big(\mathcal{E}(\boldsymbol{\mu});\mathcal{F}(\boldsymbol{\lambda})\big)$. Similarly, (11.1.7) and Definition 7.1.3 together imply that the vertex model describing $\widetilde{\mathtt{G}}_{\kappa/\nu}$ has, for each $c\in[1,nN]$, one color c arrow vertically entering through $(\nu_c,1)$ and one color c arrow vertically exiting through (κ_c,nN). Denote this boundary data by $\big(\breve{\mathcal{E}}(\kappa);\breve{\mathcal{F}}(\nu)\big)$. Thus, defining the sequence of complex numbers $\mathbf{z}=\big(z(v)\big)_{v\in\mathcal{D}}$ such that $z(v)=s^{-1}x_j$ for any $v=(i,j)\in\mathcal{D}$, and also recalling the partition function $W_{\mathcal{D}}^{(m;n)}(\mathcal{E};\mathcal{F} \mid \mathbf{z} \mid \mathbf{r},\mathbf{s})$ from Definition 5.2.1, we have

$$
\begin{aligned}
G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(s^{-1}\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) &= W_{\mathcal{D}}^{(1;n)}\big(\mathcal{E}(\boldsymbol{\mu});\mathcal{F}(\boldsymbol{\lambda}) \mid \mathbf{z} \mid \mathbf{q}^{-1/2},\mathbf{s}\big);\\
\widetilde{\mathtt{G}}_{\kappa/\nu}(\mathbf{x}) &= W_{\mathcal{D}}^{(1;nN)}\big(\breve{\mathcal{E}}(\nu);\breve{\mathcal{F}}(\kappa) \mid \mathbf{z} \mid \mathbf{q}^{-1/2},\mathbf{s}\big).
\end{aligned}
\tag{11.2.2}
$$

Now, recall the notions of interval partitions from Section 2.3 and the associated merging prescription $\vartheta_{\mathbb{J}}$ from Section 5.2. Then, Definition 11.2.3 quickly implies that $\Upsilon(\boldsymbol{\mu})$ is equivalently given by the set of compositions ν such that $\vartheta_{\mathbb{J}}\big(\breve{\mathcal{E}}(\nu)\big) = \mathcal{E}(\boldsymbol{\mu})$, where $\mathbb{J} = (J_1, J_2, \ldots, J_n)$ is the interval partition of $\{1,2,\ldots,nM\}$ defined by setting $J_i = \big\{(i-1)M+1,(i-1)M+2,\ldots,iM\big\}$ for each $i \in [1,n]$. Under this notation, we further have that $\mathrm{inv}_{\boldsymbol{\mu}}(\nu) = \sum_{i=1}^{n} \mathrm{inv}\,\big(\breve{\mathcal{E}}(\nu);J_i\big)$ (and $\mathrm{inv}_{\boldsymbol{\lambda}}(\kappa) = \sum_{i=1}^{n} \mathrm{inv}\,\big(\breve{\mathcal{F}}(\kappa);J_i\big)$), where we recall the latter from (5.2.1).

Thus, Theorem 5.2.2 gives

$$
\sum_{\nu\in\Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)-\mathrm{inv}_{\boldsymbol{\lambda}}(\kappa)} W_{\mathcal{D}}^{(1;nN)}\big(\breve{\mathcal{E}}(\nu);\breve{\mathcal{F}}(\kappa) \mid \mathbf{z} \mid \mathbf{q}^{-1/2},\mathbf{s}\big) = W_{\mathcal{D}}^{(1;n)}\big(\mathcal{E}(\boldsymbol{\mu});\mathcal{F}(\boldsymbol{\lambda}) \mid \mathbf{z} \mid \mathbf{q}^{-1/2},\mathbf{s}\big),
$$

which together with (11.2.2), implies the first statement of (11.2.1).

As mentioned above, the proof of the second is entirely analogous and is therefore omitted. However, let us briefly indicate the source of the factor $(-1)^{n\binom{M}{2}}$ there. Observe from Definition 11.1.2 that the boundary data along the right edge of the vertex model $\mathfrak{Q}_{\mathtt{g}}(\nu)$ is (from top to bottom, as in Section 5.2; see Figure 5.2) in the reverse order $nM, nM-1, \ldots, 1$. Hence, its inversion count under Definition 11.2.3 is equal to $n\binom{M}{2}$, which accounts for this exponent of -1. ☐

11.3. Integral Formulas for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$

In this section we establish two contour integral formulas for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$. The first is given as follows.

Proposition 11.3.1. *Fix integers $n, M, N \geqslant 1$; signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$; a complex number $s \in \mathbb{C}$; and sequences of complex numbers $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Denote $\mathbf{s} = (s, s, \ldots)$ and $\mathbf{y} = (1, 1, \ldots)$, and let $\kappa \in \Upsilon(\boldsymbol{\lambda})$ denote the unique element of $\Upsilon(\boldsymbol{\lambda})$ such that $\mathrm{inv}_{\boldsymbol{\lambda}}(\kappa) = 0$. Then,*

$$
\begin{aligned}
G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = \frac{(-s)^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|}}{(2\pi\mathbf{i})^{nM}} \frac{q^{\binom{nM+1}{2}}}{(q-1)^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}(\mathbf{u}^{-1}) \sum_{\nu\in\Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_{\nu}(\mathbf{u})\\
\times \prod_{1\leqslant i<j\leqslant nM} \frac{u_j-u_i}{u_j-qu_i} \prod_{i=1}^{nM}\prod_{j=1}^{N} \frac{1-sr_j^{-2}u_ix_j}{1-su_ix_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},
\end{aligned}
\tag{11.3.1}
$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_{nM}) \subset \mathbb{C}$, and each u_i is integrated along a negatively oriented, closed contour γ_i satisfying the following three properties. First, each γ_i contains s^{-1} and $q^{-k}s^{-1}x_j^{-1}$ for all integers $k \in [1, nM-1]$ and $j \in [1, M]$. Second, each γ_i does not contain s. Third, the $\{\gamma_i\}$ are mutually non-intersecting, and γ_{i-1} contains both γ_i and $q^{-1}\gamma_i$ for each $i \in [2, nM]$.

Proof. Since both sides of (11.3.1) are rational functions in $\mathbf{r}$, it suffices to establish the theorem assuming there exist integers $L_1, L_2, \ldots, L_M \geqslant 1$ such that $r_i = q^{-L_i/2}$ for each i.

Let us first assume that each $L_i = 1$, that is, we have $\mathbf{r} = \mathbf{q}^{-1/2}$ where have denoted $\mathbf{q}^{-1/2} = (q^{-1/2}, q^{-1/2}, \ldots, q^{-1/2})$ (where $q^{-1/2}$ appears with multiplicity N). Applying Corollary 11.1.7 with the $(\mathbf{x}, \kappa, \nu, m)$ there equal to $(s\mathbf{x}, \kappa, \nu, nM)$ here for some fixed $\nu \in \Upsilon(\boldsymbol{\mu})$; multiplying both sides of the resulting (11.1.6) by $(-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)}$; summing over $\nu \in \Upsilon(\boldsymbol{\mu})$; and applying Lemma 11.2.4 gives

(11.3.2)

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(\mathbf{x}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s}\big) = \frac{(-s)^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|}}{(2\pi\mathbf{i})^{nM}} \frac{q^{\binom{nM+1}{2}}}{(q-1)^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}(\mathbf{u}^{-1}) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_{\nu}(\mathbf{u}) \\ \times \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1 - qsu_i x_j}{1 - su_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

where each u_i is integrated along γ_i. This establishes (11.3.1) if $\mathbf{r} = \mathbf{q}^{-1/2}$.

Now assume that each $r_i = q^{-L_i/2}$ for some integers $L_1, L_2, \ldots, L_N \geqslant 1$, and set $\mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)} \cup \cdots \cup \mathbf{w}^{(N)}$, where $\mathbf{w}^{(i)} = (x_i, qx_i, \ldots, q^{L-1}x_i)$ for each $i \in [1, N]$. By Proposition 7.2.3, we have that $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{q}^{-1/2} \mid \mathbf{y}; \mathbf{s})$, and so applying (11.3.2), with the $\mathbf{x}$ there equal to $\mathbf{w}$ here, gives

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{w}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \frac{(-s)^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|}}{(2\pi\mathbf{i})^{nM}} \frac{q^{\binom{nM+1}{2}}}{(q-1)^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}(\mathbf{u}^{-1}) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_{\nu}(\mathbf{u}) \\ \times \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1 - q^{L_i} su_i x_j}{1 - su_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

where again each u_i is integrated along γ_i. Since $q^{L_i} = r_i^{-2}$, this confirms the theorem if each $r_i = q^{-L_i/2}$ for some $L_i \in \mathbb{Z}_{\geqslant 1}$, which (as mentioned previously) establishes it in general. □

The nonsymmetric functions $\mathtt{f}_{\kappa}$ and $\mathtt{g}_{\nu}$ appearing in the integrand on the right side of (11.3.1) admit explicit (but elaborate) summation formulas, which are given by Theorem 1.5.5 of **[12]**. Thus Proposition 11.3.1 provides an in principle fully explicit, although quite intricate, contour integral representation for $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

Part of this intricacy is contained in the sum over $\nu \in \Upsilon(\boldsymbol{\mu})$ on the right side of (11.3.1). Using the second statement of (11.2.1), we can express this sum in terms of the partition function $\widetilde{\mathtt{F}}_{\boldsymbol{\mu}}$.

Corollary 11.3.2. *Adopting the notation of Proposition 11.3.1, we have that*

(11.3.3)

$$G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \frac{(-1)^{n\binom{M}{2}}}{(2\pi\mathbf{i})^{nM}} \frac{(-s)^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|} q^{\binom{nM+1}{2}}}{(q-1)^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}(\mathbf{u}^{-1}) \widetilde{\mathtt{F}}_{\boldsymbol{\mu}}(\mathbf{u}) \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \\ \times \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1 - sr_j^{-2} u_i x_j}{1 - su_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i}.$$

where each u_i is integrated along γ_i.

Proof. This follows from inserting the second statement of (11.2.1) into Proposition 11.3.1. □

11.4. Integral Formulas for $G_{\boldsymbol{\lambda}}$

One benefit of Corollary 11.3.2 over Proposition 11.3.1 is that, when $\boldsymbol{\mu} = \mathbf{0}^N$, the following proposition indicates that the function $\widetilde{\mathtt{F}}_{\boldsymbol{\mu}}(\mathbf{x})$ factors completely. Its proof will appear in Section 11.6 below.

Proposition 11.4.1. *For any integer $N \geqslant 1$ and set of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$,*

$$\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x}) = q^{-\binom{n}{2}N^2}(1-q^{-1})^{nN}(s^2;q)_n^{\binom{N}{2}} \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (q^{-1}x_{Nk+j} - x_{Nk+i}) \prod_{j=1}^{nN}(1-sx_j)^{-N}.$$

Applying Proposition 11.4.1 in Corollary 11.3.2, we deduce the following, more concise, contour integral representation for $G_{\boldsymbol{\lambda}}$.

Theorem 11.4.2. *Adopting the notation of Proposition 11.3.1, we have*

(11.4.1)

$$G_{\boldsymbol{\lambda}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = \frac{(-s)^{|\boldsymbol{\lambda}|}}{(2\pi\mathbf{i})^{nM}}(s^2;q)_n^{\binom{M}{2}} \oint \cdots \oint \mathtt{f}_\kappa(\mathbf{u}^{-1}) \prod_{i=1}^{nM}\prod_{j=1}^{N} \frac{1 - sr_j^{-2}u_i x_j}{1 - su_i x_j} \prod_{j=1}^{nM}(1-su_j)^{-M}$$
$$\times \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant M} (qu_{Mk+i} - u_{Mk+j}) \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

where each u_i is integrated along γ_i.

Proof. Since

$$q^{\binom{nM+1}{2} - \binom{n}{2}M^2}(1-q^{-1})^{nM} = q^{n\binom{M}{2}}(q-1)^{nM};$$
$$(-q)^{n\binom{M}{2}} \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant M} (q^{-1}u_{Mk+j} - u_{Mk+i}) = \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant M} (qu_{Mk+i} - u_{Mk+j}),$$

this follows from inserting Proposition 11.4.1 into the $\boldsymbol{\mu} = \mathbf{0}^M$ case of Corollary 11.3.2. ☐

The function $\mathtt{f}_\kappa(\mathbf{u}^{-1})$ appearing on the right side of (11.4.1) can be simplified in the special case when κ is *anti-dominant*, meaning that $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_{nM}$. In this case, it is shown in **[12]** that $\mathtt{f}_\kappa(\mathbf{w})$ admits an factored form.

Lemma 11.4.3 ([**12**, Proposition 5.1.1])**.** *Fix an integer $M \geqslant 1$; a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_M)$; a complex number $s \in \mathbb{C}$; and an anti-dominant composition κ of length M. We have*

$$\mathtt{f}_\kappa(\mathbf{x}) = \prod_{j=1}^{\infty}(s^2;q)_{m_j(\kappa)} \prod_{i=1}^{M} \frac{1}{1-sx_i}\left(\frac{x_i - s}{1 - sx_i}\right)^{\kappa_i - 1}. \qquad (11.4.2)$$

This, together with Theorem 11.4.2, implies the following simpler expression for $G_{\boldsymbol{\lambda}}$ under a certain "ordering constraint" on $\boldsymbol{\lambda}$.

Theorem 11.4.4. *Adopting the notation of Proposition 11.3.1, assume that $\lambda_1^{(i)} + M - 1 \leqslant \lambda_M^{(j)}$ whenever $1 \leqslant i < j \leqslant n$. For each integer $k \geqslant 1$, let $m_k\big(\mathfrak{T}(\boldsymbol{\lambda})\big)$ denote the number of indices $i \in [1,n]$ such that $k \in \mathfrak{T}\big(\lambda^{(i)}\big)$. Then, we have*

$$G_{\boldsymbol{\lambda}}(\mathbf{x};\mathbf{r} \mid \mathbf{y};\mathbf{s}) = \frac{(-s)^{|\boldsymbol{\lambda}|}}{(2\pi\mathbf{i})^{nM}}(s^2;q)_n^{\binom{M}{2}} \prod_{j=1}^{\infty}(s^2;q)_{m_j(\mathfrak{T}(\boldsymbol{\lambda}))}$$

$$\times \oint \cdots \oint \prod_{i=1}^{n} \prod_{j=1}^{M} \left(1 - s u_{iM-j+1}\right)^{\lambda_j^{(i)} - j} \left(u_{iM-j+1} - s\right)^{j - \lambda_j^{(i)} - M - 1}$$

$$\times \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1 - s r_j^{-2} u_i x_j}{1 - s u_i x_j} \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{u_j - q u_i} \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant M} (q u_{Mk+i} - u_{Mk+j}) \prod_{i=1}^{nM} du_i,$$

where each u_i is integrated along γ_i.

Proof. Let us first show that κ is anti-dominant. To that end, observe since $\operatorname{inv}_{\boldsymbol{\lambda}}(\kappa) = 0$ that the sequence $\kappa^{(i)} = \left(\kappa_{(i-1)M+1}, \kappa_{(i-1)M+2}, \ldots, \kappa_{iM}\right)$ is non-decreasing for each $i \in [1, n]$. Moreover, since $\kappa \in \Upsilon(\boldsymbol{\lambda})$, each $\kappa^{(i)}$ is a permutation of $\mathfrak{T}\big(\lambda^{(i)}\big)$, which implies (as $\kappa^{(i)}$ is non-decreasing) that

$$\kappa_{(i-1)M+k} = \lambda_{M-k+1}^{(i)} + k, \qquad \text{for each } i \in [1, n] \text{ and } k \in [1, M]. \tag{11.4.3}$$

Then, since $\lambda_1^{(i)} + M - 1 \leqslant \lambda_M^{(j)}$ holds if $i < j$, we have for any $1 \leqslant i < j \leqslant n$ and $k, k' \in [1, M]$ that

$$\kappa_{(i-1)M+k} = \lambda_{M-k+1}^{(i)} + k \leqslant \lambda_1^{(i)} + M \leqslant \lambda_M^{(j)} + 1 \leqslant \lambda_{M-k'+1}^{(j)} + k' \leqslant \kappa_{(j-1)M+k'}.$$

This, together with the fact that each $\kappa^{(i)}$ is non-decreasing, implies that κ is anti-dominant.

Thus, Lemma 11.4.3 applies and, together with (11.4.3), yields

$$\begin{aligned} \mathtt{f}_\kappa(\mathbf{u}^{-1}) &= \prod_{j=1}^{\infty} (s^2; q)_{m_j(\kappa)} \prod_{j=1}^{nM} \frac{u_j}{u_j - s} \left(\frac{1 - s u_j}{u_j - s}\right)^{\kappa_j - 1} \\ &= \prod_{j=1}^{\infty} (s^2; q)_{m_j(\mathfrak{T}(\boldsymbol{\lambda}))} \prod_{i=1}^{n} \prod_{j=1}^{M} \frac{u_{iM-j+1}}{u_{iM-j+1} - s} \left(\frac{1 - s u_{iM-j+1}}{u_{iM-j+1} - s}\right)^{\lambda_j^{(i)} + M - j}. \end{aligned}$$

Upon insertion into Theorem 11.4.4, this gives the theorem. □

11.5. Degenerations of the Integral Formulas

In this section we provide various degenerations of the integral formulas from Section 11.4. We begin with the case when $n = 1$.

Corollary 11.5.1. *Fix integers $N, M \geqslant 1$; a signature $\lambda \in \operatorname{Sign}_M$; a complex number $s \in \mathbb{C}$; and sequences of complex numbers $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Denote $\boldsymbol{\lambda} = (\lambda) \in \operatorname{SeqSign}_{1;M}$; $\mathbf{s} = (s, s, \ldots)$; and $\mathbf{y} = (1, 1, \ldots)$. Then, we have that*

(11.5.1)

$$G_{\boldsymbol{\lambda}}(\mathbf{x}; \mathbf{r} \mid \mathbf{y}; \mathbf{s}) = \frac{(-s)^{|\boldsymbol{\lambda}|}}{(2\pi \mathbf{i})^M} (1 - s^2)^{\binom{M+1}{2}} \oint \cdots \oint \prod_{j=1}^{M} (1 - s u_{M-j+1})^{\lambda_j - j} (u_{M-j+1} - s)^{j - \lambda_j - M - 1}$$

$$\times \prod_{1 \leqslant i < j \leqslant M} (u_i - u_j) \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{1 - u_i r_j^{-2} x_j}{1 - u_i x_j} \prod_{i=1}^{M} du_i,$$

where each u_i is integrated along the contour γ_i from Proposition 11.3.1.

Proof. This follows from the $n = 1$ case of Theorem 11.4.4 since $\prod_{j=1}^{\infty} (s^2; q)_{m_j(\mathfrak{T}(\lambda))} = (1 - s^2)^M$, which holds since λ has length M and since all entries in $\mathfrak{T}(\lambda)$ are mutually distinct. □

Remark 11.5.2. One can use (11.5.1) to identify the $n = 1$ case of $G_{\boldsymbol{\lambda}}$ with a generic supersymmetric Schur function, but we will not pursue this here since this $n = 1$ scenario will be addressed in substantially more detail in the forthcoming work [**1**].

Next, we consider the degenerations to the LLT case of Proposition 11.3.1, Theorem 11.4.2, and Theorem 11.4.4. We begin with the former, in which case the result reduces to one originally implicitly shown in [**35**]. In what follows, we recall the function ψ from (9.3.1).

Corollary 11.5.3 ([**35**, Equation (34)])**.** *Adopting the notation of Proposition 11.3.1, we have*

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \frac{1}{(2\pi\mathbf{i})^{nM}} \frac{q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})}}{(1-q^{-1})^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}^{(q)}(\mathbf{u}^{-1} \mid 0) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_{\nu}^{(q)}(\mathbf{u} \mid 0) \times \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{q^{-1}u_j - u_i} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1-u_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i}, \tag{11.5.2}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_{nM}) \subset \mathbb{C}$*, and each* u_i *is integrated along a positively oriented, closed contour* Γ_i *satisfying the following two properties. First, each* Γ_i *contains* 0 *and does not contain* $q^{-k}x_j^{-1}$ *for all integers* $k \in [1, nM-1]$ *and* $j \in [1, M]$*. Second, the* $\{\Gamma_i\}$ *are mutually non-intersecting, and* Γ_{i-1} *is contained in both* Γ_i *and* $q^{-1}\Gamma_i$ *for each* $i \in [2, nM]$*.*

Proof. Define the infinite sequence $\mathbf{y} = (1, 1, \ldots)$. By (9.3.3), (9.5.1), and the last statement of (8.4.3), we have

$$\begin{aligned} q^{\psi(\boldsymbol{\lambda})-\psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) &= \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) \\ &= \lim_{r \to \infty} \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0) \\ &= \lim_{r \to \infty} \Big(\lim_{s \to 0} (-s)^{|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|} G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\big(s^{-1}\mathbf{x}; (r, r, \ldots) \mid \mathbf{y}; (s, s, \ldots)\big) \Big). \end{aligned} \tag{11.5.3}$$

We then deduce the corollary by inserting this into (11.3.1) (with the $\mathbf{x}$ there replaced by $s^{-1}\mathbf{x}$ here and the s there equal to 0 here), with the contours there reversed. $\square$

Remark 11.5.4. Let us briefly explain how Corollary 11.5.3 and equation (34) of [**35**] are equivalent, for the latter is not directly stated as a contour integral. In what follows, we recall the Schur polynomial $s_\lambda(\mathbf{z})$ associated with any signature λ and (possibly infinite) set of variables $\mathbf{z} = (z_1, z_2, \ldots)$. Moreover, for any (non)symmetric function F in $\mathbf{z}$ and basis $\{h_\mu\}$ for the space of (non)symmetric functions in $\mathbf{z}$, we let $\mathrm{Coeff}[F; h_\lambda]$ denote the coefficient of h_λ in the expansion of F over $\{h_\mu\}$.

First observe by the Cauchy identity for Schur polynomials that

$$\sum_{\theta \in \mathrm{Sign}_N} s_\theta(\mathbf{x}) s_\theta(\mathbf{u}) = \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1-u_i x_j}.$$

Upon insertion into (11.5.2), this gives

$$\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \frac{q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})}}{(1-q^{-1})^{nM}} \sum_{\theta \in \mathrm{Sign}_N} s_\theta(\mathbf{x}) \frac{1}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \mathtt{f}_{\kappa}^{(q)}(\mathbf{u}^{-1} \mid 0) \prod_{1 \leqslant i < j \leqslant nM} \frac{u_j - u_i}{q^{-1}u_j - u_i} \times s_\theta(\mathbf{u}) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_{\nu}^{(q)}(\mathbf{u} \mid 0) \prod_{i=1}^{nM} \frac{du_i}{u_i}.$$

Consequently, for any $\theta \in \mathrm{Sign}_N$ we have

$$\mathrm{Coeff}\big[\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}); s_\theta(\mathbf{x})\big] = \frac{q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})}}{(1-q^{-1})^{nM}} \frac{1}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \mathtt{f}_\kappa^{(q)}(\mathbf{u}^{-1} \mid 0) \prod_{1 \leqslant i<j \leqslant nM} \frac{u_j - u_i}{q^{-1}u_j - u_i}$$
$$\times s_\theta(\mathbf{u}) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_\nu^{(q)}(\mathbf{u} \mid 0) \prod_{i=1}^{nM} \frac{du_i}{u_i}.$$

Next, it follows from Theorem 9.4.1 of **[12]** that for any polynomial $H \in \mathbb{C}(q)[\mathbf{u}]$ we have

$$\mathrm{Coeff}\Big[H(\mathbf{u}), \mathtt{g}_\kappa^{(q)}(\mathbf{u} \mid 0)\Big] = \frac{q^{\binom{nM}{2}}}{(1-q^{-1})^{nM}} \frac{1}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint H(\mathbf{u}) \mathtt{f}_\kappa^{(q)}(\mathbf{u}^{-1} \mid 0)$$
$$\times \prod_{1 \leqslant i<j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

from which we deduce that

$$\mathrm{Coeff}\big[\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}), s_\theta(\mathbf{x})\big] = q^{\psi(\boldsymbol{\mu})-\psi(\boldsymbol{\lambda})} \mathrm{Coeff}\Bigg[s_\theta(\mathbf{u}) \sum_{\nu \in \Upsilon(\boldsymbol{\mu})} (-1)^{\mathrm{inv}_{\boldsymbol{\mu}}(\nu)} \mathtt{g}_\nu^{(q)}(\mathbf{u} \mid 0); \mathtt{g}_\kappa^{(q)}(\mathbf{u} \mid 0)\Bigg]. \tag{11.5.4}$$

Now, Theorem 5.9.4 of **[12]** indicates that $\mathtt{g}_\lambda^{(q)}(\mathbf{x} \mid 0)$ is given, after a suitable normalization and change of variables, by a nonsymmetric Hall–Littlewood polynomial (namely, the $q = 0$ degenerations of the nonsymmetric Macdonald polynomials introduced in **[67]**). Under this identification, (11.5.4) coincides with equation (34) of **[35]**.

We next have the following corollary of Theorem 11.4.2, which simplifies Corollary 11.5.3 in the case when $\boldsymbol{\mu} = \mathbf{0}^M$.

Corollary 11.5.5. *Adopting the notation of Proposition 11.3.1, we have*

$$\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x}) = \frac{q^{\binom{n}{2}\binom{N}{2}/2-\psi(\boldsymbol{\lambda})}}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \prod_{1 \leqslant i<j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{k=0}^{n-1} \prod_{1 \leqslant i<j \leqslant M} (qu_{Mk+i} - u_{Mk+j})$$
$$\times \mathtt{f}_\kappa^{(q)}(\mathbf{u}^{-1} \mid 0) \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1 - u_i x_j} \prod_{i=1}^{nM} \frac{du_i}{u_i},$$

where each u_i is integrated along the Γ_i from Corollary 11.5.3.

Proof. This follows from inserting the $\boldsymbol{\mu} = \mathbf{0}^N$ case of (11.5.3) into Theorem 11.4.2 (with the $\mathbf{x}$ there replaced by $s^{-1}\mathbf{x}$ here, and the s there set to 0 here), with the contours there reversed, and Example 9.3.1. ∎

Under the ordering constraint described by Theorem 11.4.4, this nonsymmetric Hall–Littlewood function factors completely, giving rise to the following simplified integral formulas for certain LLT polynomials.

Corollary 11.5.6. *Adopting the notation of Proposition 11.3.1, assume that $\lambda_1^{(i)} + M - 1 \leqslant \lambda_M^{(j)}$ whenever $1 \leqslant i < j \leqslant n$. Then, we have*

$$\mathcal{L}_{\boldsymbol{\lambda}}(\mathbf{x}) = \frac{q^{\binom{n}{2}\binom{N}{2}/2-\psi(\boldsymbol{\lambda})}}{(2\pi\mathbf{i})^{nM}} \oint \cdots \oint \prod_{1 \leqslant i<j \leqslant nM} \frac{u_j - u_i}{u_j - qu_i} \prod_{k=0}^{n-1} \prod_{1 \leqslant i<j \leqslant M} (qu_{Mk+i} - u_{Mk+j})$$

$$\times \prod_{i=1}^{n} \prod_{j=1}^{M} u_{iM-j+1}^{j-\lambda_j^{(i)}-M-1} \prod_{i=1}^{nM} \prod_{j=1}^{N} \frac{1}{1-u_i x_j} \prod_{i=1}^{nM} du_i,$$

where each u_i is integrated along the Γ_i from Corollary 11.5.3.

Proof. This follows from inserting the $\boldsymbol{\mu} = \mathbf{0}^M$ case of (11.5.3) into Theorem 11.4.4 (with the $\mathbf{x}$ there replaced by $s^{-1}\mathbf{x}$ here, and the s there set to 0 here), with the contours there reversed, and Example 9.3.1. $\square$

11.6. Proof of Proposition 11.4.1

In this section we establish Proposition 11.4.1, which provides an explicit form for $\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x})$. To that end, it will in fact be useful to analyze the more general quantity given by $\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y})$ from Definition 11.2.2. So, throughout, we fix integers $n, N \geqslant 1$ and sequences of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$.

We will establish the following proposition evaluating $\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y})$, from which Proposition 11.4.1 directly follows.

Proposition 11.6.1. *We have that*

$$\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = q^{-\binom{n}{2}N^2}(1-q^{-1})^{nN} \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (q^{-1}x_{Nk+j} - x_{Nk+i})(y_i - q^k s^2 y_j)$$
$$\times \prod_{i=1}^{N} y_i^n \prod_{i=1}^{N} \prod_{j=1}^{nN} (y_i - s x_j)^{-1}.$$

Proof of Proposition 11.4.1 Assuming Proposition 11.6.1. This follows from setting each y_i equal to 1 in Proposition 11.6.1. $\square$

The proof of Proposition 11.6.1 will be similar to that of Proposition 7.3.2 in Section 7.4, by (after suitable normalization) realizing $\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y})$ as a polynomial that is characterized by a specific set of zeroes guaranteed by the Yang–Baxter equation. To that end, we have the following lemma.

Lemma 11.6.2. *There exists a constant $C = C_{n;N}(s;q)$ such that*

$$\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = C \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (x_{Nk+j} - q x_{Nk+i})(y_i - q^k s^2 y_j) \prod_{i=1}^{N} y_i^n \prod_{i=1}^{N} \prod_{j=1}^{nN} (y_i - s x_j)^{-1}.$$

Proof (Outline). Since this proof is similar to that of Corollary 7.4.3, we only outline it.

To that end, following Lemma 7.4.1, we first claim that

$$\mathtt{Z}(\mathbf{x} \mid \mathbf{y}) = \widetilde{\mathtt{F}}_{\mathbf{0}^N} \prod_{i=1}^{N} \prod_{j=1}^{nN} (y_i - s x_j), \tag{11.6.1}$$

is a polynomial in $(\mathbf{x}, \mathbf{y})$ of total degree at most nN^2. To see this, observe from Definition 11.2.2 that $\mathtt{Z}(\mathbf{x} \mid \mathbf{y})$ is the partition function for the model $\mathfrak{Q}_{\mathtt{F}}(\mathbf{0}^N)$ from Definition 11.2.1, whose weight at any vertex (i,j) in the domain $\mathcal{D} = \mathcal{D}_{N,nN} = [1, N] \times [1, nN]$ is given by

$$\widetilde{M}_{x_j;y_i}(\mathbf{A}, b; \mathbf{C}, d) = (y_i - s x_j) M_{x_j/y_i}(\mathbf{A}, b; \mathbf{C}, d).$$

By Definition 11.1.1, these weights are always linear in $(\mathbf{x}, \mathbf{y})$, from which it follows that $\mathtt{Z}(\mathbf{x} \mid \mathbf{y})$ is a polynomial in $(\mathbf{x}, \mathbf{y})$ of total degree at most $|\mathcal{D}| = nN^2$.

Next, we identify a divisibility property for the polynomial $\mathtt{Z}$, which will follow from certain exchange relations. To that end, recalling the transposition $\mathfrak{s}_i$ of $(i, i+1)$, we have

$$
\begin{aligned}
&\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = \widetilde{\mathtt{F}}_{\mathbf{0}^N}\big(\mathbf{x} \mid \mathfrak{s}_i(\mathbf{y})\big) \prod_{j=0}^{n-1} \frac{y_i - q^j s^2 y_{i+1}}{y_{i+1} - q^j s^2 y_i}, \qquad \text{for } i \in [1, N]; \\
&\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = \widetilde{\mathtt{F}}_{\mathbf{0}^N}\big(\mathfrak{s}_i(\mathbf{x}) \mid \mathbf{y}\big) \frac{x_{i+1} - q x_i}{x_i - q x_{i+1}}, \qquad \text{for } i \in [1, nN] \text{ with } i \notin \{N, 2N, \ldots, nN\}.
\end{aligned}
\tag{11.6.2}
$$

We omit the proof of (11.6.2), since its first and second statements follow from applications of the Yang–Baxter equation very similar to the ones used to show (the $s_j = s$ case of) Lemma 7.4.2 and (the $r_j = q^{-1/2}$ case of) Lemma 6.2.1, respectively.

Thus, by (11.6.2), (11.6.1), and the polynomiality of $\mathtt{Z}$, we deduce that

$$
\prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (x_{Nk+j} - q x_{Nk+i}) \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (y_i - q^k s^2 y_j) \quad \text{divides} \quad \mathtt{Z}(\mathbf{x} \mid \mathbf{y}).
$$

We further claim for each $i \in [1, N]$ that y_i^n divides $\mathtt{Z}(\mathbf{x} \mid \mathbf{y})$. To see this, observe that any path ensemble $\mathcal{E} \in \mathfrak{Q}_{\mathtt{F}}(\mathbf{0}^N)$ satisfies the following property. For each $i \in [1, N]$, there exist $n(i-1)$ arrows horizontally entering the i-th column of $\mathcal{E}$ and in arrows horizontally exiting it (see the left side of Figure 11.3). Thus, this i-th column contains at least n distinct vertices whose arrow configurations are of the form $(\mathbf{A}, 0; \mathbf{A}_i^-, i)$. Since (11.1.2) implies that y_i divides the weight of this configuration under $\widetilde{M}_{x;y_i}$, it follows that y_i^n divides $\mathtt{Z}(\mathbf{x} \mid \mathbf{y})$.

Hence, there exists a constant $C = C_{n;N}(s;q)$ such that

$$
\mathtt{Z}(\mathbf{x} \mid \mathbf{y}) = C \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (x_{Nk+j} - q x_{Nk+i}) \prod_{k=0}^{n-1} \prod_{1 \leqslant i < j \leqslant N} (y_i - q^k s^2 y_j) \prod_{i=1}^{N} y_i^n,
$$

since both sides are of degree nN^2, and the right side divides the left. This, together with (11.6.1), implies the lemma. $\square$

Now we can establish Proposition 11.6.1.

Proof of Proposition 11.6.1. In view of Lemma 11.6.2, it suffices to determine $C_{n;N}(s;q)$. To that end, as in the proof of Corollary 7.5.3, we will use a special choice for the parameters $(\mathbf{x}, \mathbf{y})$ that freezes the partition function $\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y})$, enabling us to evaluate it.

Let us set $x_j = q^{-1}s$ for each $j \in [1, nN]$ and $y_i = 1$ for each $i \in [1, n]$. Under this specialization, Lemma 11.6.2 yields

$$
\widetilde{\mathtt{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = C (q^{-1}s)^{n\binom{N}{2}} (1-q)^{n\binom{N}{2}} (s^2; q)_n^{\binom{N}{2}} (1 - q^{-1}s^2)^{-nN^2}.
\tag{11.6.3}
$$

To evaluate $\widetilde{\mathtt{F}}_{\mathbf{0}^N}$ directly under this specialization observe, since $(x_j, y_i) = (q^{-1}s, 1)$, that (11.1.3) gives $M_{x_j/y_i,s}(\mathbf{A}, h; \mathbf{A}; h) = 0$ whenever $A_h = 1$. From this, it is quickly verified that there is a unique path ensemble with nonzero weight in the vertex model $\mathfrak{Q}_{\mathtt{F}}(\mathbf{0}^N)$ from Definition 11.2.1. It is the one depicted on the right side of Figure 11.3, where the path of color k entering the domain $\mathcal{D} = \mathcal{D}_{N,nN} = [1, N] \times [1, nN]$ at $(j, 1)$ proceeds as north until it reaches $(j, kN - j + 1)$, and then proceeds east until it exits $\mathcal{D}$ at $(N, kN - j + 1)$. In particular, under this ensemble, for any $k \in [1, n]$ the arrow configuration at any vertex $(i, kN - j + 1) \in \mathcal{D}$ is $(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k,n]}, 0)$ if $1 \leqslant i < j \leqslant N$; $(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k+1,n]}, k)$ if $1 \leqslant i = j \leqslant N$; and is $(\mathbf{e}_{[k+1,n]}, k; \mathbf{e}_{[k+1,n]}, k)$ if $1 \leqslant j < i \leqslant N$.

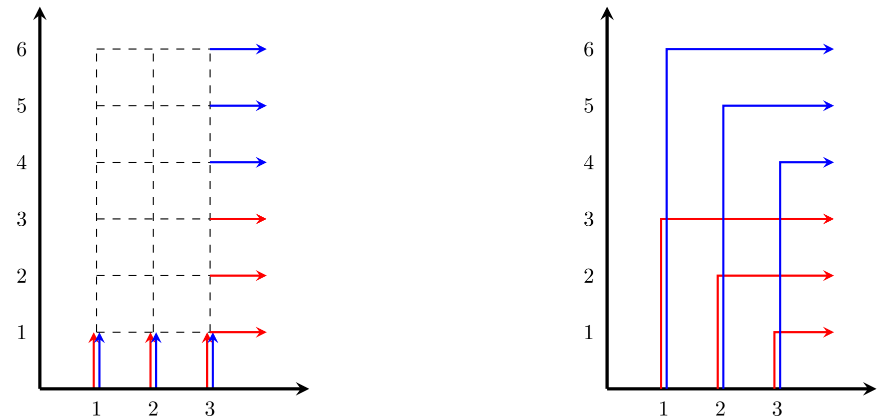


FIGURE 11.3. Shown to the left is the vertex model for $\widetilde{\mathrm{F}}_{\mathbf{0}^M}(\mathbf{x} \mid \mathbf{y})$. Shown to the right is the frozen path ensemble corresponding to this function when each $x_j = q^{-1}s$ and $y_i = 1$.

This gives

$$\begin{aligned}\widetilde{\mathrm{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = \prod_{k=1}^{n} \prod_{i=1}^{N} M_{s/q}\big(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k,n]}, 0\big)^{N-i} \prod_{k=1}^{n} \prod_{j=1}^{N} M_{s/q}\big(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k+1,n]}, k\big) \\ \times \prod_{k=1}^{n} \prod_{i=1}^{N} M_{s/q}\big(\mathbf{e}_{[k+1,n]}, k; \mathbf{e}_{[k+1,n]}, k\big)^{i-1}.\end{aligned} \tag{11.6.4}$$

By (11.1.2) and (11.1.3), we have for any $k \in [1, n]$ that

$$M_{s/q}\big(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k,n]}, 0\big) = q^{k-n-1} \frac{1 - q^{n-k}s^2}{1 - q^{-1}s^2}; \qquad M_{s/q}\big(\mathbf{e}_{[k,n]}, 0; \mathbf{e}_{[k+1,n]}, k\big) = q^{k-n-1} \frac{q-1}{1 - q^{-1}s^2};$$

$$M_{s/q}\big(\mathbf{e}_{[k+1,n]}, k; \mathbf{e}_{[k+1,n]}, k\big) = q^{k-n-1} \frac{s(1-q)}{1 - q^{-1}s^2},$$

which by (11.6.4) gives

$$\widetilde{\mathrm{F}}_{\mathbf{0}^N}(\mathbf{x} \mid \mathbf{y}) = q^{-\binom{n+1}{2}N^2} (-s)^{n\binom{N}{2}} (q-1)^{n\binom{N+1}{2}} (s^2; q)_n^{\binom{N}{2}} (1 - q^{-1}s^2)^{-nN^2}.$$

Comparing this with (11.6.3) yields

$$C = q^{n\binom{N}{2} - \binom{n+1}{2}N^2} (q-1)^{nN},$$

which implies the proposition upon insertion into Lemma 11.6.2. $\square$

CHAPTER 12

Vanishing Properties

Since (9.3.6) implies that the polynomials $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x};\infty \mid \mathbf{y};\infty)$ from Definition 8.4.2 degenerate to the LLT ones as each y_i tends to ∞, we may view the $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x};\infty \mid \mathbf{y};\infty)$ as inhomogeneous deformations of the LLT polynomials. In this chapter we will show that these inhomogeneous polynomials further satisfy a vanishing property, which appears similar to ones satisfied by the families of factorial Schur or interpolation Macdonald polynomials [**76, 79**]. Thus, we may interpret the $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x};\infty \mid \mathbf{y};\infty)$ as[1] "factorial LLT polynomials." In the case $n = 1$, the vanishing condition for these polynomials coincides with that for the factorial Schur functions, which characterizes them completely.

12.1. Zeroes of $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$

Let $N \geqslant 1$ denote an integer, and $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ denote sequences of complex numbers. In this section we will state a result, given by Theorem 12.1.2 below, that provides a family of vanishing points for the functions $\mathcal{H}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x};\infty \mid \mathbf{y};\infty)$ from (8.4.4).

These vanishing points will take the form $\{x_j = q^{-\kappa_j} y_{\mathfrak{m}_j}\}$ for some integers $\mathfrak{m}_j \geqslant 1$ and $\kappa_j \in [0, n-1]$. It will be convenient to express such specializations through *marked sequences*, which are pairs $(\mathfrak{m}, \kappa)$ of integer sets of the same length, such the entries $\mathfrak{m}_1 > \mathfrak{m}_2 > \cdots > \mathfrak{m}_\ell$ of $\mathfrak{m}$ are decreasing and positive, and such that each entry of $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$ is in $[0, n-1]$. In what follows, we will often only refer to $\mathfrak{m}$ as the marked sequence and view κ as its *marking*, in that each entry $\mathfrak{m}_j \in \mathfrak{m}$ is *marked* by the corresponding entry $\kappa_j \in \kappa$.

To proceed, we require the notion of a splitting for a marked sequence.

Definition 12.1.1. Let $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_\ell)$ denote a marked sequence with marking given by $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$. A *splitting* of $\mathfrak{m}$ (more precisely, of $(\mathfrak{m}, \kappa)$) is a sequence $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots, \mathfrak{m}^{(n)}\big)$ of Maya diagrams (that is, decreasing subsets of $\mathbb{Z}_{>0}$) such that the following two properties holds.

1. Every entry $m \in \mathfrak{m}^{(i)}$ of any Maya diagram in $\mathfrak{M}$ is equal to $\mathfrak{m}_j$, for some $j = j(m) \in [1, \ell]$.
2. For any $j \in [1, \ell]$, there exist at least $n - \kappa_j$ distinct indices $i \in [1, n]$ such that $\mathfrak{m}_j \in \mathfrak{m}^{(i)}$.

Moreover, given two signatures $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_\ell) \in \mathrm{Sign}_\ell$ and $\mathfrak{n} = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k) \in \mathrm{Sign}_k$, we (nonstandardly) write $\mathfrak{m} \not\leq \mathfrak{n}$ if there exists an integer $j \geqslant 0$ such that $\mathfrak{m}_{\ell-j} > \mathfrak{n}_{k-j}$, where we set $\mathfrak{m}_i = \infty = \mathfrak{n}_i$ if $i \leqslant 0$. This is equivalent to stipulating that the Young diagram for $\mathfrak{m}$ not be contained in that of $\mathfrak{n}$, if the two are superimposed to share the same bottom left corner.[2]

Now we can state the following vanishing result, which will be established in Section 12.3 below. Here, we recall the function $\mathfrak{T}$ from (1.2.3).

[1]In this chapter, we largely consider the $\mathcal{H}_{\boldsymbol{\lambda}}$ functions, instead of the $\mathcal{F}_{\boldsymbol{\lambda}}$ ones from Section 1.4. However, Remark 12.1.3 below indicates their equivalence.

[2]Observe that this superimposition is different from the one used in Chapter 9, where there two Young diagrams were always superimposed to share their top left corners.

THEOREM 12.1.2. *Fix integers $N \geqslant \ell \geqslant 1$; a marked sequence $(\mathfrak{m}, \kappa)$, with coordinates indexed by $[1, \ell]$; and a signature sequence $\boldsymbol{\lambda} \in \mathrm{SeqSign}_{n;N}$. Denoting $\mathfrak{l}^{(i)} = \mathfrak{T}(\lambda^{(i)})$ for each $i \in [1, n]$, assume for any splitting $\mathfrak{M} = (\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots, \mathfrak{m}^{(n)})$ of $(\mathfrak{m}, \kappa)$ that there exists an index $h = h(\mathfrak{M}) \in [1, n]$ such that $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}^{(h)}$. If $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ are sequences of complex numbers such that $x_j = q^{-\kappa_j} y_{\mathfrak{m}_j}$ for each $j \in [1, \ell]$, then $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) = 0$.*

Remark 12.1.3. By Theorem 12.1.2 and Proposition 10.4.1, the function $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0)$ from (8.4.4) also satisfies a vanishing condition. In particular, adopting the notation and assumptions of Theorem 12.1.2, but assuming instead that $x_j = q^{\kappa_j - n+1} y_{\mathfrak{m}_j}$ for each $k \in [1, \ell]$, we have $\mathcal{F}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; 0) = 0$. Thus, these $\mathcal{F}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid \mathbf{y}; 0)$ may also be viewed as factorial LLT functions. Moreover, Theorem 7.3.3 can be used to show that they also satisfy a Cauchy identity, similarly to other families of interpolation polynomials [**70, 77**].

The vanishing condition prescribed by Theorem 12.1.2 appears similar to ones satisfied by various families of symmetric functions, such as factorial Schur polynomials [**34, 65**] and interpolation Macdonald polynomials [**78, 79, 76**]. Although $\mathcal{H}_{\boldsymbol{\lambda}}$ is not exactly symmetric in $\mathbf{x}$, it is quickly verified from Proposition 7.2.2 that (recalling $\mathcal{S}(\boldsymbol{\lambda})$ from Section 7.1) its normalization

$$\check{\mathcal{H}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y}) = \mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}^{-1}; \infty \mid \mathbf{y}^{-1}; \infty) \prod_{j=1}^{N} x_j^{-j} \prod_{i=1}^{\infty} y_i^{nN - \sum_{k=1}^{i-1} |\mathbf{S}_k(\boldsymbol{\lambda})|}, \tag{12.1.1}$$

is a polynomial in $\mathbf{x}$ and $\mathbf{y}$ of total degree $|\boldsymbol{\lambda}|$ that is symmetric in $\mathbf{x}$. Due to the resemblance between the vanishing properties for $\check{\mathcal{H}}$ and the factorial Schur polynomials, and the facts that the former degenerate to the LLT polynomials (by (9.3.6)) and the latter to the Schur polynomials, one might view the functions $\check{\mathcal{H}}_{\boldsymbol{\lambda}}(\mathbf{x} \mid \mathbf{y})$ as "factorial variants" of LLT polynomials.

Before proceeding to the proof of Theorem 12.1.2, let us analyze several consequences of it; throughout these examples, we adopt the notation of Theorem 12.1.2. We begin with the case $n = 1$.

Example 12.1.4. Suppose that $n = 1$, and abbreviate $\lambda = \lambda^{(1)}$ and $\mathfrak{l} = \mathfrak{l}^{(1)} = (\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_N)$. Then, we must have $\kappa = 0^\ell$, so Definition 12.1.1 indicates that the unique splitting of $\mathfrak{m}$ is $\mathfrak{M} = (\mathfrak{m})$. Thus, recalling $\check{\mathcal{H}}$ from (12.1.1), Theorem 12.1.2 indicates that $\check{\mathcal{H}}_\lambda(\mathbf{x} \mid \mathbf{y}) = 0$ when $x_j = y_{\mathfrak{m}_j}$ for each $j \in [1, \ell]$, if there exists some $k \in [1, n]$ such that $\mathfrak{l}_{N-k} > \mathfrak{m}_{\ell-k}$.

Remark 12.1.5. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, Example 12.1.4 in fact enables us to identify $\check{\mathcal{H}}_\lambda(\mathbf{x} \mid \mathbf{y})$ in the case $n = 1$ as a *factorial Schur function* $s_\lambda(\mathbf{x} \mid \mathbf{y})$. Introduced as equation (6.4) of [**65**] and equation (4) of [**34**], the latter is defined by

$$s_\lambda(\mathbf{x} \mid \mathbf{y}) = \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^{-1} \det \left[\prod_{k=1}^{\lambda_i + N - i} (x_j - y_k) \right]_{1 \leqslant i,j \leqslant N}. \tag{12.1.2}$$

Indeed, by Theorem 2.1 of [**71**], these functions vanish under the same specializations described by Example 12.1.4. Moreover, by Theorem 3.1 of [**79**], $s_\lambda(\mathbf{x} \mid \mathbf{y})$ is (up to a constant factor) is the unique symmetric polynomial in $\mathbf{x}$ of degree $|\lambda|$ satisfying this vanishing property. Thus, $\check{\mathcal{H}}_\lambda(\mathbf{x} \mid \mathbf{y}) = (-1)^{|\lambda|} s_\lambda(\mathbf{x} | \mathbf{y})$, as it is quickly verified that the coefficient of $\prod_{i=1}^n x_i^{\lambda_i}$ in both equals $(-1)^{|\lambda|}$.

For general n and N, it appears that a complete coordinate-wise description (as in Example 12.1.4) for the vanishing points described by Theorem 12.1.2 would be intricate to state. However, we can still identify some of these points, which are similar to those considered in Example 12.1.4.

Example 12.1.6. Let us analyze when a marked sequence $(\mathfrak{m}, \kappa)$ satisfies the condition of Theorem 12.1.2 if $\kappa = 0^{\ell}$. In this case, $\mathfrak{m}$ admits the unique splitting $\mathfrak{M} = (\mathfrak{m}, \mathfrak{m}, \ldots, \mathfrak{m}) \in \mathrm{SeqSign}_n$, since each entry $\mathfrak{m}_j \in \mathfrak{m}$ must appear in $n - \kappa_j = n$ signatures of $\mathfrak{M}$. In particular, $(\mathfrak{m}, 0^{\ell})$ satisfies the condition from Theorem 12.1.2 if and only if there exists an index $h \in [1, n]$ for which $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}$. Thus, $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) = 0$ when $x_j = y_{\mathfrak{m}_j}$ for each $j \in [1, \ell]$, if there exists an index $k \in [1, N]$ such that $\max_{h \in [1,n]} \mathfrak{l}_{N-k}^{(h)} > \mathfrak{m}_{\ell - k}$.

The following lemma provides an explicit coordinate-wise description of the vanishing points described in Theorem 12.1.2 in the simplest ($N > 1$) case[3] not provided by the examples above, namely, $N = 2 = n$.

Lemma 12.1.7. *Adopt the notation of Theorem 12.1.2, and suppose $N = n = \ell = 2$. In each of the following four cases, we have $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) = 0$.*

(1) We have that $(x_1, x_2) = (y_{\mathfrak{m}_1}, y_{\mathfrak{m}_2})$ and either

$$\max \big\{ \mathfrak{l}_1^{(1)}, \mathfrak{l}_1^{(2)} \big\} > \mathfrak{m}_1, \quad \textit{or} \quad \max \big\{ \mathfrak{l}_2^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_2. \tag{12.1.3}$$

(2) We have that $(x_1, x_2) = (q^{-1} y_{\mathfrak{m}_1}, y_{\mathfrak{m}_2})$ and either

$$\min \big\{ \mathfrak{l}_1^{(1)}, \mathfrak{l}_1^{(2)} \big\} > \mathfrak{m}_1, \quad \textit{or} \quad \max \big\{ \mathfrak{l}_2^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_2. \tag{12.1.4}$$

(3) We have that $(x_1, x_2) = (y_{\mathfrak{m}_1}, q^{-1} y_{\mathfrak{m}_2})$ and both

$$\mathfrak{l}_2^{(1)} > \mathfrak{m}_2, \quad \textit{or} \quad \max \big\{ \mathfrak{l}_1^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_1. \tag{12.1.5}$$

and

$$\mathfrak{l}_2^{(2)} > \mathfrak{m}_2, \quad \textit{or} \quad \max \big\{ \mathfrak{l}_1^{(2)}, \mathfrak{l}_2^{(1)} \big\} > \mathfrak{m}_1. \tag{12.1.6}$$

(4) We have that $(x_1, x_2) = (q^{-1} y_{\mathfrak{m}_1}, q^{-1} y_{\mathfrak{m}_2})$ and either

$$\min \big\{ \mathfrak{l}_1^{(2)}, \mathfrak{l}_2^{(1)} \big\} > \mathfrak{m}_1, \quad \textit{or} \quad \min \big\{ \mathfrak{l}_1^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_1, \quad \textit{or} \quad \min \big\{ \mathfrak{l}_2^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_2. \tag{12.1.7}$$

Proof. The four parts of the lemma correpsond to the cases when κ from Theorem 12.1.2 is equal to $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, respectively. Let us analyze when $\mathfrak{m}$ satisfies the vanishing condition indicated there in each of these four cases separately.

The first scenario $\kappa = (0,0)$ was addressed by Example 12.1.6, in which case $\mathfrak{m}$ satisfies the vanishing condition if and only if (12.1.3) holds. By Theorem 12.1.2, this verifies the first part of the lemma.

In the second scenario $\kappa = (1,0)$, a signature sequence $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)} \big)$ is a splitting of $\mathfrak{m}$ if and only if $\mathfrak{m}_1 \in \mathfrak{m}^{(1)} \cup \mathfrak{m}^{(2)}$ and $\mathfrak{m}_2 \in \mathfrak{m}^{(1)} \cap \mathfrak{m}^{(2)}$. Thus, we may assume that either $\mathfrak{M} = \big((\mathfrak{m}_1, \mathfrak{m}_2), (\mathfrak{m}_2) \big)$ or $\mathfrak{M} = \big((\mathfrak{m}_2), (\mathfrak{m}_1, \mathfrak{m}_2) \big)$. In the former case, there exists some $h \in \{1, 2\}$ with $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}^{(h)}$ if and only if either $\max \big\{ \mathfrak{l}_2^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_2$ or $\mathfrak{l}_1^{(1)} > \mathfrak{m}_1$. In the latter case, such an i exists if and only if either $\max \big\{ \mathfrak{l}_2^{(1)}, \mathfrak{l}_2^{(2)} \big\} > \mathfrak{m}_2$ or $\mathfrak{l}_1^{(2)} > \mathfrak{m}_1$. Thus, if $\kappa = (1,0)$, all splittings $\mathfrak{M}$ of $(\mathfrak{m}, \kappa)$ satisfy the condition from Theorem 12.1.2 if and only if (12.1.4) holds. This verifies the second of the lemma.

[3]We will also assume that $\ell = 2$, from which the $\ell = 1$ case can be recovered by setting $\mathfrak{m}_1 = \infty$.

In the third scenario $\kappa = (0,1)$, a signature sequence $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}\big)$ is a splitting of $\mathfrak{m}$ if and only if $\mathfrak{m}_1 \in \mathfrak{m}^{(1)} \cap \mathfrak{m}^{(2)}$ and $\mathfrak{m}_2 \in \mathfrak{m}^{(1)} \cup \mathfrak{m}^{(2)}$. Thus, we may assume that either $\mathfrak{M} = \big((\mathfrak{m}_1, \mathfrak{m}_2), (\mathfrak{m}_1)\big)$ or $\mathfrak{M} = \big((\mathfrak{m}_1), (\mathfrak{m}_1, \mathfrak{m}_2)\big)$. In the former case, there exists $h \in \{1,2\}$ with $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}^{(h)}$ if and only if (12.1.5) holds; in the latter case, such an i exists if and only if (12.1.6) holds. Thus, if $\kappa = (0,1)$, all splittings $\mathfrak{M}$ of $(\mathfrak{m}, \kappa)$ satisfy the condition from Theorem 12.1.2 if and only if both (12.1.5) and (12.1.6) hold. This verifies the third part of the lemma.

In the fourth scenario $\kappa = (1,1)$, a signature sequence $\mathfrak{M} = \big(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}\big)$ is a splitting of $\mathfrak{m}$ if and only if $\mathfrak{m}^{(1)} \cup \mathfrak{m}^{(2)} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. We may assume each $\mathfrak{m}_j$ appears in at most one signature of $\mathfrak{M}$, so $\mathfrak{M} = \big\{\big((\mathfrak{m}_1, \mathfrak{m}_2), \varnothing\big), \big((\mathfrak{m}_1), (\mathfrak{m}_2)\big), \big((\mathfrak{m}_2), (\mathfrak{m}_1)\big), \big(\varnothing, (\mathfrak{m}_1, \mathfrak{m}_2)\big)\big\}$. If $\mathfrak{M} = \big((\mathfrak{m}_1, \mathfrak{m}_2), \varnothing\big)$, then there exists an $h \in \{1,2\}$ with $\mathfrak{l}^{(h)} \not\leq \mathfrak{m}^{(h)}$ if and only if either $\mathfrak{l}_2^{(1)} > \mathfrak{m}_2$ or $\mathfrak{l}_1^{(1)} > \mathfrak{m}_1$. Similarly, if $\mathfrak{M} = \big(\varnothing, (\mathfrak{m}_1, \mathfrak{m}_2)\big)$, then such an i exists if and only if either $\mathfrak{l}_2^{(2)} > \mathfrak{m}_2$ or $\mathfrak{l}_1^{(2)} > \mathfrak{m}_1$. If instead $\mathfrak{M} = \big((\mathfrak{m}_1), (\mathfrak{m}_2)\big)$, then we must have $\mathfrak{l}_2^{(1)} > \mathfrak{m}_1$ or $\mathfrak{l}_2^{(2)} > \mathfrak{m}_2$, and if $\mathfrak{M} = \big((\mathfrak{m}_2), (\mathfrak{m}_1)\big)$ then must have $\mathfrak{l}_2^{(1)} > \mathfrak{m}_2$ or $\mathfrak{l}_2^{(2)} > \mathfrak{m}_1$. Hence, if $\kappa = (1,1)$, all splittings $\mathfrak{M}$ of $(\mathfrak{m}, \kappa)$ satisfy the condition from Theorem 12.1.2 if and only if (12.1.7) holds. This verifies the fourth part of the lemma. $\square$

12.2. Blocking Vertices

In this section we introduce the notion of a blocking vertex with respect to a path ensemble, which will be useful for the proof of Theorem 12.1.2. We will only require them here in the case $n = 1$ (that is, when there is only one fermionic color), so throughout this section we will assume this holds.

Then, any sequence of $n = 1$ skew-shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ consists of a single skew-shape λ/μ. So, in what follows, we identify $\boldsymbol{\lambda}/\boldsymbol{\mu} = \lambda/\mu$; for instance, recalling Definition 7.1.3 we write $\mathfrak{P}_H(\lambda/\mu) = \mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\mu})$ (and $\mathfrak{P}_H(\lambda) = \mathfrak{P}_H(\boldsymbol{\lambda}/\boldsymbol{\varnothing})$). Moreover, since $n = 1$, any arrow configuration $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ is of the form $(\mathbf{e}_a, \mathbf{e}_b; \mathbf{e}_c, \mathbf{e}_d)$ for some $a, b, c, d \in \{0,1\}$. Hence, we will also abbreviate $(a, b; c, d) = (\mathbf{e}_a, \mathbf{e}_b; \mathbf{e}_c, \mathbf{e}_d)$, as well as the weight $\mathcal{W}_z(a, b; c, d) = \mathcal{W}_z(\mathbf{e}_a, \mathbf{e}_b; \mathbf{e}_c, \mathbf{e}_d \mid \infty, \infty)$ from (8.4.1), for any $a, b, c, d \in \{0,1\}$. Recalling Remark 8.4.4, we also abbreviate the weight of any path ensemble $\mathcal{E}$ under $\mathcal{W}_z(a, b; c, d)$ by $W(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) = W(\mathcal{E} \mid \mathbf{x}; \infty \mid \mathbf{y}; \infty)$.

Now recall from (7.1.3) and Remark 8.4.4 that, for any $\lambda, \mu \in \mathrm{Sign}_n$, the quantity $\mathcal{H}_{\lambda/\mu}(\mathbf{x}; \infty \mid \mathbf{y}; \infty)$ is the partition function under the $\mathcal{W}_{x/y}(a, b; c, d)$ weights for the vertex model $\mathfrak{P}_H(\lambda/\mu)$. Due to the factor of $(xy^{-1}; q)_{b-c}$ present in these weights, if $x = y$ then $\mathcal{W}_{x/y}(a, b; c, d) = 0$ whenever $(b, c) = (1, 0)$. In particular, if a path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda/\mu)$ has nonzero weight $W(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) \neq 0$, and there exists some vertex (i, j) with $x_j = y_i$, then the arrow configuration $(a, b; c, d)$ at (i, j) must satisfy $(b, c) \neq (1, 0)$. Thus, we introduce the following definition.

Definition 12.2.1. Let $\mathcal{E}$ denote a path ensemble on some domain $\mathcal{D} \subseteq \mathbb{Z}^2_{>0}$; for each vertex $u \in \mathcal{D}$, let $\big(a(u), b(u); c(u), d(u)\big)$ denote the arrow configuration at u under $\mathcal{E}$. We call any $v \in \mathcal{D}$ a *blocking vertex* with respect to $\mathcal{E}$ if $\big(b(v), c(v)\big) \neq (1, 0)$.

Observe in particular that any path in $\mathcal{E}$ horizontally entering some blocking vertex $v \in \mathcal{D}$ must exit v vertically. In this sense, v "blocks" the horizontal trajectory of this path. We refer to Figure 12.1 for a depiction, where the orange crosses at $(2, 1)$ and $(4, 3)$ are blocking with respect to $\mathcal{E}$ but the green one at $(5, 4)$ is not.

In this section we establish the following proposition, which indicates how blocking vertices may arrange themselves in a path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda/\mu)$.

Proposition 12.2.2. *Fix a signature* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathrm{Sign}_N$*, and denote* $\mathfrak{l} = \mathfrak{T}(\lambda) \in \mathrm{Sign}_N$*. For any fixed path ensemble* $\mathcal{E} \in \mathfrak{P}_H(\lambda)$*, there does not exist a set of* $K \leqslant N$ *blocking vertices* $(v_1, v_2, \ldots, v_K) \subset \mathcal{D}_N = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N\}$ *with respect to* $\mathcal{E}$ *satisfying the following two properties. In the below we set* $v_k = (i_k, j_k)$ *for each* $k \in [1, K]$*.*

(1) We have $1 \leqslant j_1 < j_2 < \cdots < j_K \leqslant N$ *and* $1 \leqslant i_1 < i_2 < \cdots < i_K$*.*
(2) We have $i_K < \mathfrak{l}_{N-K+1}$*.*

Stated alternatively, Proposition 12.2.2 indicates that, for any vertex sequence $(v_1, v_2, \ldots, v_K)$ satisfying its conditions and path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda)$, there must exist some $j \in [1, K]$ for which $\big(b(v_j), c(v_j)\big) = (1, 0)$.

Proof of Proposition 12.2.2. In what follows, for any path ensemble $\mathcal{E}$ on some domain $\mathcal{D}$, we denote the arrow configuration at any vertex $v \in \mathcal{D}$ under $\mathcal{E}$ by $\big(a_{\mathcal{E}}(v), b_{\mathcal{E}}(v); c_{\mathcal{E}}(v), d_{\mathcal{E}}(v)\big)$. Throughout this proof, we call any vertex set $(v_1, v_2, \ldots, v_K) \in \mathcal{D}_N$ satisfying the two properties listed in the propositon an *increasing blocking vertex set* with respect to $\mathcal{E}$. We must show that no path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda)$ admits an increasing blocking vertex set.

To that end, we induct on $N+K \geqslant 2$. If $N+K = 2$, then $N = 1 = K$, so there is a unique path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda)$ consisting of one path that horizontally enters $\mathcal{D}_1$ through the vertex $(1,1)$ and proceeds east until it reaches $(\mathfrak{l}_1, 1)$, where it vertically exits $\mathcal{D}_1$. Thus, $\big(b_{\mathcal{E}}(i,1), c_{\mathcal{E}}(i,1)\big) = (1,0)$ holds for each $i \in [0, \mathfrak{l}_1 - 1]$, and hence there does not exist a blocking vertex satisfying the second condition in the proposition. In particular, no increasing blocking vertex set exists.

So, let us suppose that the proposition holds whenever $N + K < m$ for some integer $m \geqslant 3$, and we will show it also holds if $N + K = m$. Suppose to the contrary that there exists a path ensemble $\mathcal{E} \in \mathfrak{P}_H(\lambda)$ that admits an increasing blocking vertex set $\mathcal{V} = (v_1, v_2, \ldots, v_K) \subset \mathcal{D}_N$; for each $k \in [1, K]$, set $v_k = (i_k, j_k) \in \mathcal{D}_N$.

Let $\mathcal{E}'$ denote the restriction of $\mathcal{E}$ to the subdomain $\mathcal{D}_{N-1} = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N-1\} \subset \mathcal{D}_N$ (that is, to the bottommost $N-1$ rows of $\mathcal{D}_N$). Then, there exists a signature $\nu \in \mathrm{Sign}_{N-1}$ such that $\mathcal{E}' \in \mathfrak{P}_H(\nu)$. Let $\mathfrak{T}(\nu) = \mathfrak{n} = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_{N-1}) \in \mathrm{Sign}_{N-1}$, which denote the x-coordinates of the locations where paths in $\mathcal{E}'$ vertically exit the row $\mathbb{Z}_{>0} \times \{N-1\}$; see Figure 12.1 for a depiction. Since these paths (in $\mathcal{E}$) exit the row $\mathbb{Z}_{>0} \times \{N\}$ through the x-coordinates $(\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_{N-1})$, we have the interlacing property

$$\mathfrak{l}_{j+1} \leqslant \mathfrak{n}_j \leqslant \mathfrak{l}_j, \qquad \text{for any index } j \in [1, N-1], \tag{12.2.1}$$

which follows from the fact that no two paths may share an edge.

Now, let us show that $\mathcal{E}'$ admits an increasing blocking vertex set in each of the cases $j_K \neq N$ and $j_K = N$ separately. If $j_K \neq N$, then we claim that the vertex set $\mathcal{V} = (v_1, v_2, \ldots, v_K) \subset \mathcal{D}_{N-1}$ is increasing blocking with respect to $\mathcal{E}'$. Indeed, the fact that each of its vertices is blocking with respect to $\mathcal{E}'$ follows from the fact that they are with respect to $\mathcal{E}$. Moreover, the facts that $1 \leqslant j_1 < j_2 < \cdots < j_K \leqslant N-1$ and $1 \leqslant i_1 < i_2 < \cdots < i_K$ follow from the facts that $\mathcal{V}$ is increasing blocking with respect to $\mathcal{E}$ and that $j_K \neq N$. Additionally, the bound $i_K < \mathfrak{n}_{N-K}$ follows from the fact that $i_K < \mathfrak{l}_{N-K+1} \leqslant \mathfrak{n}_{N-K}$, where the first inequality holds again since $\mathcal{V}$ is increasing blocking with respect to $\mathcal{E}$ and the second holds by (12.2.1). This confirms that the vertex set $\mathcal{V}$ is increasing blocking with respect to $\mathcal{E}'$ if $j_K \neq N$, which contradicts the inductive hypothesis that no path ensemble in $\mathfrak{P}_H(\nu)$ can admit such a set (since $\ell(\nu) + K = N + K - 1 = m - 1 < m$).

So, let us assume instead that $j_K = N$, in which case we claim that the vertex set $\mathcal{V}' = (v_1, v_2, \ldots, v_{K-1}) \subset \mathcal{D}_{N-1}$ is increasing blocking with respect to $\mathcal{E}'$. Again, the facts that each vertex $v_i \in \mathcal{V}'$ is blocking and that $1 \leqslant j_1 < j_2 < \cdots < j_{K-1} \leqslant N-1$ and $1 \leqslant i_1 < i_2 < \cdots < i_{K-1}$

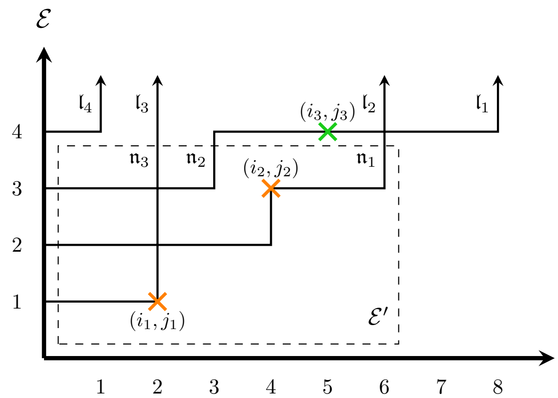


FIGURE 12.1. Depicted above is an attempt for an increasing blocking vertex set, given by the orange and green crosses. However, only the orange crosses are blocking with respect to $\mathcal{E}$; the green one is not.

follow from the fact that $\mathcal{V}$ is increasing blocking with respect to $\mathcal{E}$. Thus, it remains to verify the bound $i_{K-1} < \mathfrak{n}_{N-K+1}$. To that end, since $i_{K-1} < i_K$, it suffices to show that $i_K \leqslant \mathfrak{n}_{N-K+1}$.

Assume to the contrary that this is false, so $\mathfrak{n}_{N-K+1} < i_K < \mathfrak{l}_{N-K+1}$, where the last inequality holds since $\mathcal{V}$ is increasing blocking with respect to $\mathcal{E}$. Since there exists a path in $\mathcal{E}$ that enters the row $\mathbb{Z}_{>0} \times \{N\}$ vertically through the vertex $(\mathfrak{n}_{N-K+1}, N)$ and proceeds east until it reaches $(\mathfrak{l}_{N-K+1}, N)$, we must have $\big(b_{\mathcal{E}}(i), c_{\mathcal{E}}(i)\big) = (1, 0)$ for any $i \in (\mathfrak{n}_{N-K+1}, \mathfrak{l}_{N-K+1})$; see Figure 12.1, where $(N, K) = (4, 3)$ and $\big\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\big\} = \big\{(2, 1), (4, 3), (5, 4)\big\}$. In particular, this holds for $i = i_K$, contradicting the fact that $v_K = (i_K, N)$ is a blocking vertex with respect to $\mathcal{E}$.

Hence, $i_K \leqslant \mathfrak{n}_{N-K+1}$, so $\mathcal{V}'$ is increasing blocking with respect to $\mathcal{E}'$. This again contradicts the inductive hypothesis that no path ensemble in $\mathfrak{P}_H(\nu)$ can admit such a set, thereby establishing the proposition. $\square$

12.3. Proof of Theorem 12.1.2

In this section we establish Theorem 12.1.2, which will largely follow from Proposition 12.2.2.

PROOF OF THEOREM 12.1.2. Throughout this proof, we abbreviate the vertex model $\mathfrak{P}_H(\boldsymbol{\lambda}) = \mathfrak{P}_H(\boldsymbol{\lambda}/\varnothing)$ from Definition 7.1.3; the vertex weights $\mathcal{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathcal{W}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, \infty)$ from (8.4.1); and the weight $W(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) = W(\mathcal{E} \mid \mathbf{x}; \infty \mid \mathbf{y}; \infty)$ of any path ensemble $\mathcal{E}$ from Remark 8.4.4. We moreover recall from Section 12.2 the $n = 1$ notation $\mathfrak{P}_H(\lambda) = \mathfrak{P}_H(\boldsymbol{\lambda})$ if $\boldsymbol{\lambda} = (\lambda) \in \mathrm{SeqSign}_1$. Let us also assume throughout this proof that $x_{N-j+1} = y_{\mathfrak{m}_j}$ (instead of $x_j = y_{\mathfrak{m}_j}$) for each $j \in [1, \ell]$, which we may do since Proposition 7.2.2 implies that $\mathcal{H}_{\boldsymbol{\lambda}}$ is symmetric in $\mathbf{x}$ up to a factor.

Now, assume to the contrary that $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) \neq 0$. Since (7.1.3) and (8.4.4) together imply that $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty)$ is the partition function, under the weights $\mathcal{W}_z$, for the vertex model $\mathfrak{P}_H(\boldsymbol{\lambda})$, there must exist some path ensemble $\mathcal{E} \in \mathfrak{P}_H(\boldsymbol{\lambda})$ on $\mathcal{D}_N = \mathbb{Z}_{>0} \times \{1, 2, \ldots, N\}$ with nonzero weight $W(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) \neq 0$. Let $\big(\mathbf{A}(v), \mathbf{B}(v); \mathbf{C}(v), \mathbf{D}(v)\big)$ denote the arrow configuration at any vertex $v \in \mathcal{D}_N$ under $\mathcal{E}$, and set $\mathbf{X}(v) = \big(X_1(v), X_2(v), \ldots, X_n(v)\big) \in \{0, 1\}^n$ for each index $X \in \{A, B, C, D\}$.

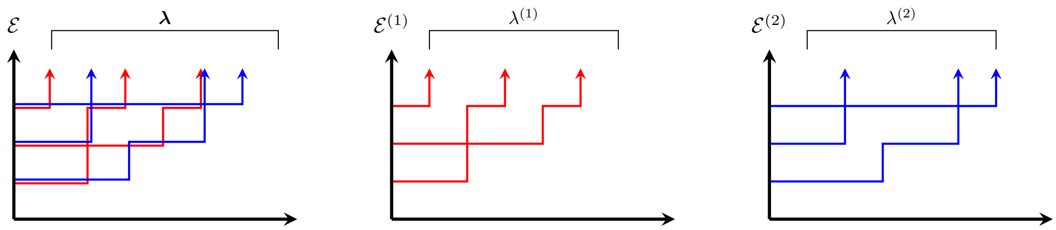


FIGURE 12.2. Shown to the left, middle, and right are examples of the path ensembles $\mathcal{E}$, $\mathcal{E}^{(1)}$, and $\mathcal{E}^{(2)}$, respectively, from the proof of Theorem 12.1.2.

For each $i \in [1, n]$, let $\mathcal{E}^{(i)} \in \mathfrak{P}_H\big(\lambda^{(i)}\big)$ denote the restriction to color i of $\mathcal{E}$. Stated alternatively, the arrow configuration at any vertex $v \in \mathcal{D}_N$ under $\mathcal{E}^{(i)}$ is given by $\big(A_i(v), B_i(v); C_i(v), D_i(v)\big)$. In this way, each $\mathcal{E}^{(i)}$ is a path ensemble with one fermionic color, and so the framework of Section 12.2 applies to it. We refer to Figure 12.2 for a depiction, where there red is color 1 and blue is color 2.

Under this notation, for each $j \in [1, \ell]$, let $\mathcal{I}_j \subseteq [1, n]$ denote the set of colors $i \in [1, n]$ for which $(\mathfrak{m}_j, N - j + 1) \in \mathcal{D}_N$ is a blocking vertex with respect to $\mathcal{E}^{(i)}$. We claim that $|\mathcal{I}_j| \geqslant n - \kappa_j$. Indeed, assume to the contrary that there exists a subset $\mathcal{U} \subseteq [1, n]$ of $\kappa_j + 1$ distinct colors such that $(\mathfrak{m}_j, N - j + 1) \in \mathcal{D}_N$ is not a blocking vertex with respect to $\mathcal{E}^{(u)}$ for each $u \in \mathcal{U}$; so, $B_u(\mathfrak{m}_j, N - j + 1) - C_u(\mathfrak{m}_j, N - j + 1) = 1$ for each such u. Thus, abbreviating the arrow configuration under $\mathcal{E}$ at $(\mathfrak{m}_j, N - j + 1) \in \mathcal{D}_N$ by

$$(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \big(\mathbf{A}(\mathfrak{m}_j, N - j + 1), \mathbf{B}(\mathfrak{m}_j, N - j + 1); \mathbf{C}(\mathfrak{m}_j, N - j + 1), \mathbf{D}(\mathfrak{m}_j, N - j + 1)\big),$$

we deduce either that $|\mathbf{B}| - |\mathbf{C}| \geqslant \kappa_j + 1$ or that $\mathbf{B} \geqslant \mathbf{C}$ does not hold. Recalling that $x_{N-j+1} = q^{-\kappa_j} y_{\mathfrak{m}_j}$, it follows from the factor of

$$(x_{N-j+1} y_{\mathfrak{m}_j}^{-1}; q)_{|\mathbf{B}|-|\mathbf{C}|} \mathbf{1}_{\mathbf{B} \geqslant \mathbf{C}} = (q^{-\kappa_j}; q)_{|\mathbf{B}|-|\mathbf{C}|} \mathbf{1}_{\mathbf{B} \geqslant \mathbf{C}},$$

appearing in the vertex weight $\mathcal{W}_{x_{N-j+1}; y_{\mathfrak{m}_j}}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty; \infty)$ (from the first statement of (8.4.1)) that this weight equals 0. This contradicts the fact that $\mathcal{E}$ has nonzero weight $W(\mathcal{E} \mid \mathbf{x} \mid \mathbf{y}) \neq 0$ under $\mathcal{W}_z$, thus verifying $|\mathcal{I}_j| \geqslant n - \kappa_j$.

Next, for each $i \in [1, n]$, let $\mathcal{V}^{(i)}$ denote the set of vertices of the form $(\mathfrak{m}_j, N - j + 1) \in \mathcal{D}_N$, for some $j \in [1, \ell]$, that are blocking with respect to $\mathcal{E}^{(i)}$. We order the vertices in $\mathcal{V}^{(i)}$ by

$$\mathcal{V}^{(i)} = \big(v_1^{(i)}, v_2^{(i)}, \ldots, v_{\ell_i}^{(i)}\big) = \Big\{\big(\mathfrak{m}_{j_1^{(i)}}, N - j_1^{(i)} + 1\big), \big(\mathfrak{m}_{j_2^{(i)}}, N - j_2^{(i)} + 1\big), \ldots, \big(\mathfrak{m}_{j_{\ell_i}^{(i)}}, N - j_{\ell_i}^{(i)} + 1\big)\Big\},$$

so that $j_1^{(i)} > j_2^{(i)} > \cdots > j_{\ell_i}^{(i)}$, which since $\mathfrak{m}$ is decreasing implies

$$\mathfrak{m}_{j_1^{(i)}} < \mathfrak{m}_{j_2^{(i)}} < \cdots < \mathfrak{m}_{j_{\ell_i}^{(i)}}.$$

Hence, the blocking vertex set $\mathcal{V}^{(i)} = \big(v_1^{(i)}, v_2^{(i)}, \ldots, v_{\ell_i}^{(i)}\big)$ (with respect to $\mathcal{E}^{(i)}$) satisfies the first property listed in Proposition 12.2.2, as does its truncation $\mathcal{V}_K^{(i)} = \big(v_1^{(i)}, v_2^{(i)}, \ldots, v_K^{(i)}\big)$, for any $K \in [1, \ell_i]$. Thus, that proposition implies none of the $\mathcal{V}_K^{(i)}$ can satisfy the second property listed there, meaning that $\mathfrak{m}_{j_K^{(i)}} \leqslant \mathfrak{l}_{N-K+1}^{(i)}$ holds for each $K \in [1, \ell_i]$. So, letting

$$\mathfrak{m}^{(i)} = \Big(\mathfrak{m}_{j_1^{(i)}}, \mathfrak{m}_{j_2^{(i)}}, \ldots, \mathfrak{m}_{j_{\ell^{(i)}}}\Big),$$

we deduce that

$$\mathfrak{m}^{(i)} \not\prec \mathfrak{l}^{(i)} \qquad \text{does not hold for any } i \in [1, n]. \tag{12.3.1}$$

Now, observe that $\mathfrak{M} = \left(\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots, \mathfrak{m}^{(n)}\right)$ is a splitting of $(\mathfrak{m}, \kappa)$. Indeed, it satisfies the first condition listed in Definition 12.1.1, since each entry of every $\mathfrak{m}^{(i)}$ is also an entry of $\mathfrak{m}$. It also satisfies the second, since $|\mathcal{I}_j| \geqslant n - \kappa_j$ for each $j \in [1, \ell]$ implies that any $\mathfrak{m}_j$ is an element in at least $n - \kappa_j$ of the $\mathfrak{m}^{(i)}$.

Thus, since the marked sequence $(\mathfrak{m}, \kappa)$ satisfies the condition of the theorem, there must exist some $h \in [1, n]$ such that $\mathfrak{l}^{(h)} \not\prec \mathfrak{m}^{(h)}$. This contradicts (12.3.1), and so $\mathcal{H}_{\boldsymbol{\lambda}}(\mathbf{x}; \infty \mid \mathbf{y}; \infty) = 0$. $\square$

CHAPTER 13

Vertex Models for Nonsymmetric Macdonald Polynomials

In this chapter we provide an expression, given by Theorem 13.3.2 below, for nonsymmetric Macdonald polynomais in terms of fermionic, colored vertex model partition functions. This result is similar to Theorem 4.2 of **[13]** (which established an expression for these polynomials through a bosonic partition function) and will be established in Chapter 14 below, largely following the framework of **[13]**. Throughout this section, we fix complex numbers $q, t \in \mathbb{C}$.

13.1. Cherednik–Dunkl Operators and Nonsymmetric Polynomials

In what follows, we use the same conventions as in **[13]** (which are in turn based on reversing all indices in the notation of **[69]**) in defining the nonsymmetric Macdonald polynomials $f_\mu(\mathbf{x}) = f_\mu(x_1, \ldots, x_n)$, for any composition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of complex variables. We begin by recalling the definition of the *Hecke algebra* of type A_{n-1}. It is the algebra generated by the family of generators $T_1, T_2, \ldots T_{n-1}$, which satisfy the relations

$$(T_i - t)(T_i + 1) = 0, \quad \text{for } i \in [1, n-1]; \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } i \in [1, n-2], \tag{13.1.1}$$

as well as the commutativity property

$$[T_i, T_j] = 0, \quad \text{for all } i, j \text{ such that } |i - j| > 1. \tag{13.1.2}$$

A well-known realization of this algebra is its *polynomial representation.* In this representation, one identifies the abstract generator T_i and its inverse with explicit *Demazure–Lusztig* operators on $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \ldots, x_n]$, the ring of polynomials in n variables, by setting for each $1 \leqslant i \leqslant n-1$

$$T_i \mapsto t - \frac{x_i - t x_{i+1}}{x_i - x_{i+1}} (1 - \mathfrak{s}_i), \qquad T_i^{-1} \mapsto t^{-1} \left(1 - \frac{x_i - t x_{i+1}}{x_i - x_{i+1}} (1 - \mathfrak{s}_i) \right), \tag{13.1.3}$$

where, as previously, $\mathfrak{s}_i$ denotes the transposition operator on neighboring variables, namely

$$\mathfrak{s}_i \cdot h(x_1, x_2, \ldots, x_n) = h(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n),$$

for any polynomial $h \in \mathbb{C}[\mathbf{x}]$. Let us introduce a further operator ω, defined by setting

$$\omega \cdot h(x_1, \ldots, x_n) = h(x_2, x_3, \ldots, x_n, q x_1), \tag{13.1.4}$$

for any $h \in \mathbb{C}[\mathbf{x}]$. Collectively, the operators $T_1, \ldots, T_{n-1}, \omega$ give a polynomial representation of the affine Hecke algebra of type A_{n-1}. The *Cherednik–Dunkl operators* Y_i generate an Abelian subalgebra of the affine Hecke algebra. They are given for each $1 \leqslant i \leqslant n$ by

$$Y_i = T_{i-1} \cdots T_1 \cdot \omega \cdot T_{n-1}^{-1} \cdots T_i^{-1}, \tag{13.1.5}$$

and for each $i, j \in [1, n]$ satisfy the commutation relation

$$[Y_i, Y_j] = 0. \tag{13.1.6}$$

In view of their commutativity, one can seek to jointly diagonalize the operators (13.1.5). This brings us to the definition of the nonsymmetric Macdonald polynomials; see Proposition 2.3 of [**13**]. In the below, for any composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and sequence of n variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we denote $\mathbf{x}^\lambda = \prod_{i=1}^N x_i^{\lambda_i}$.

Definition 13.1.1. Introduce the following two orders on length n compositions $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$. The first is the dominance order, denoted $<$, and given by

$$\nu < \mu \iff \nu \neq \mu, \qquad \text{and} \qquad \sum_{i=1}^{j} \nu_i \leqslant \sum_{i=1}^{j} \mu_i, \qquad \text{for each } j \in [1, n].$$

The second order, denoted $\prec$, is given by

$$\nu \prec \mu \iff \Big(\nu^+ < \mu^+ \quad \text{or} \quad \nu^+ = \mu^+, \ \nu < \mu \Big),$$

where λ^+ is the unique partition that can be obtained by permuting the parts of a composition λ. The nonsymmetric Macdonald polynomials $f_\mu(\mathbf{x}) = f_\mu(x_1, x_2, \dots, x_n) = f_\mu(x_1, x_2, \dots, x_n; q, t)$ are the unique family of polynomials in $\mathbb{C}[\mathbf{x}]$ satisfying the triangularity property

$$f_\mu(\mathbf{x}) = \mathbf{x}^\mu + \sum_{\overleftarrow{\nu} \prec \overleftarrow{\mu}} c_{\mu,\nu}(q,t) \mathbf{x}^\nu, \qquad \text{for some } c_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t), \tag{13.1.7}$$

with the sum taken over all compositions ν such that $\overleftarrow{\nu} \prec \overleftarrow{\mu}$, as well as the eigenvalue equation

$$Y_i \cdot f_\mu(\mathbf{x}) = y_i(\mu; q, t) f_\mu(\mathbf{x}), \quad \text{for each } i \in [1, n], \tag{13.1.8}$$

with the eigenvalues on the right-hand side of (13.1.8) given by

$$y_i(\mu; q, t) = q^{\mu_i} t^{\eta_i(\mu) + i - 1}; \quad \eta_i(\mu) = -\#\{j < i : \mu_j > \mu_i\} - \#\{j > i : \mu_j \geqslant \mu_i\}. \tag{13.1.9}$$

Remark 13.1.2. In the current work, it will be convenient to make use of an alternative characterization of the nonsymmetric Macdonald polynomials. Let $\mathbb{C}^D[\mathbf{x}] = \mathbb{C}^D[x_1, \dots, x_n]$ be the set of all polynomials in $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of total degree $\leqslant D$. Viewing the latter as a vector space, we can write $\mathbb{C}^D[\mathbf{x}] = \mathrm{Span}\{\mathbf{x}^\mu\}_{|\mu| \leqslant D}$.

Introduce a generating series $Y(w) = \sum_{i=1}^n Y_i w^{i-1}$ of the operators (13.1.5), which can be viewed as a linear operator $Y(w) : \mathbb{C}^D[\mathbf{x}] \to \mathbb{C}^D[\mathbf{x}]$. The corresponding eigenvalues $y(\mu; q, t; w) = \sum_{i=1}^n y_i(\mu; q, t) w^{i-1}$ are pairwise distinct for all $|\mu| \leqslant D$, with q, t, w living inside some open set in $\mathbb{C}$. The simplicity of the spectrum of $Y(w)$ (this simplicity plays an important role in Cherednik's theory of double affine Hecke algebras, *cf.* [**24**, Theorem 8.2(i)]) uniquely determines each eigenvector $f_\mu(\mathbf{x})$, with $|\mu| \leqslant D$, up to a multiplicative constant. One can therefore define f_μ as the unique polynomial solution of the equations (13.1.8) such that $f_\mu \in \mathbb{C}^{|\mu|}[\mathbf{x}]$ and $\mathrm{Coeff}[f_\mu; \mathbf{x}^\mu] = 1$, where $\mathrm{Coeff}[h; \mathbf{x}^\mu]$ denotes the coefficient of $\mathbf{x}^\mu$ in any $h \in \mathbb{C}[\mathbf{x}]$. We will tacitly assume this definition in what follows.

13.2. Fermionic L-Matrix and Row Operators

Our subsequent partition functions will be constructed in terms of another object, which we term the *L-matrix*. For any $x \in \mathbb{C}$, its entries $L_x(\mathbf{A}, b; \mathbf{C}, d)$ will be obtained from the specializations of the weights $W_{x;q}(\mathbf{A}, b; \mathbf{C}, d \mid q^{-1/2}, s)$ given in Example 8.1.2 (where here we sometimes emphasize their dependence on q in the subscript), by setting

$$L_x(\mathbf{A}, b; \mathbf{C}, d) = \lim_{s \to 0} (-s)^{-\mathbf{1}_{d>0}} W_{x/s;t}(\mathbf{A}, b; \mathbf{C}, d \mid t^{-1/2}, s). \tag{13.2.1}$$

$1 \leqslant i \leqslant n$			$1 \leqslant i < j \leqslant n$		$A_i = 0$	$A_i = 1$
$(\mathbf{A}, 0; \mathbf{A}, 0)$	$(\mathbf{A}, 0; \mathbf{A}_i^-, i)$	$(\mathbf{A}, i; \mathbf{A}_i^+, 0)$	$(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j)$	$(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$	$(\mathbf{A}, i; \mathbf{A}, i)$
1	$t^{A_{[i+1,n]}} x(1-t)$	1	$t^{A_{[j+1,n]}} x(1-t)$	0	$t^{A_{[i,n]}} x$	$-t^{A_{[i,n]}} x$

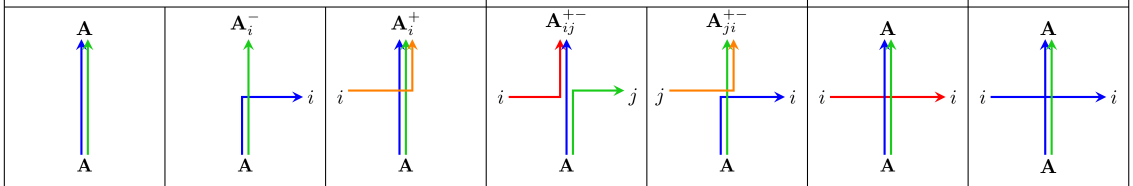


FIGURE 13.1. The L_x weights, and their arrow configurations, are depicted above. Here red, blue, green, and orange denote the colors 1, 2, 3, and 4, respectively.

Observe here that we have replaced the quantization parameter q used for the weights in previous chapters with the Macdonald parameter t here; this is because the symbol q historically plays a different role in the context of Macdonald polynomials. Further observe that these L_x weights are given by the $r = q^{-1/2}$ cases of the $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r)$ weights from (8.4.1) and are more explicitly through the following definition.

Definition 13.2.1. For any $\mathbf{A}, \mathbf{C} \in \mathbb{Z}_{\geqslant 0}^n$ and $b, d \in \{0, 1, \ldots, n\}$, set $L_x(\mathbf{A}, b; \mathbf{C}, d) = 0$ unless $\mathbf{A}, \mathbf{C} \in \{0,1\}^n$. Letting $\mathbf{A}_i^+, \mathbf{A}_j^-, \mathbf{A}_{ij}^{+-}$ be as in (8.1.1), for each $\mathbf{A} \in \{0,1\}^n$ and $i \in [1, n]$ set

$$L_x(\mathbf{A}, 0; \mathbf{A}, 0) = 1; \qquad L_x(\mathbf{A}, 0; \mathbf{A}_i^-, i) = t^{A_{[i+1,n]}} x(1-t);$$
$$L_x(\mathbf{A}, i; \mathbf{A}_i^+, 0) = 1; \qquad L_x(\mathbf{A}, i; \mathbf{A}; i) = (-1)^{A_i} t^{A_{[i,n]}} x. \tag{13.2.2}$$

Moreover, for any $1 \leqslant i < j \leqslant n$, set

$$L_x(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j) = t^{A_{[j+1,n]}} x(1-t); \qquad L_x(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i) = 0. \tag{13.2.3}$$

We further set $L_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid r, s) = 0$ if $(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ is not of the above form.

As before, we interpret $L_x(\mathbf{A}, b; \mathbf{C}, d)$ as the weight associated with a vertex v whose arrow configuration is $(\mathbf{A}, \mathbf{e}_b; \mathbf{C}, \mathbf{e}_d)$. We refer to Figure 13.1 for a depiction of these weights; it is quickly verified that (13.2.1) holds for them.

Remark 13.2.2. The weights from Definition 13.2.1 admit a simple dependence on the parameter x. If $d \geqslant 1$, then $L_x(\mathbf{A}, b; \mathbf{C}, d)$ is a linear, homogeneous polynomial in x. If instead $d = 0$, then $L_x(\mathbf{A}, b; \mathbf{C}, d)$ does not depend on x. Put another way, we obtain a factor of x every time a path horizontally exits through a vertex.

Given these weights, we next define row transfer operators on a vector space, as in (the homogeneous specializations of) Section 6.1 and Section 6.2. To that end, we first require the associated single row partition functions, which are the analogs of those appearing in Definition 6.1.1. In what follows, we will typically index finitary sequences $\mathcal{X} = (\mathbf{X}_0, \mathbf{X}_1, \ldots)$ starting from 0 instead of from 1.

Definition 13.2.3. Fix a complex number $x \in \mathbb{C}$. For any finitary sequences $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots)$ and $\mathcal{C} = (\mathbf{C}_0, \mathbf{C}_1, \ldots)$ of elements in $\{0,1\}^n$, and indices $b, d \in [0, n]$, define

$$L_x(\mathcal{A}, b; \mathcal{C}, d) = \sum_{\mathfrak{J}} \prod_{i=0}^{\infty} L_x(\mathbf{A}_i, j_i; \mathbf{C}_i, j_{i+1}), \tag{13.2.4}$$

where the sum is over all sequences $\mathfrak{J} = (j_0, j_1, \ldots)$ of indices in $[0, n]$ such that $j_0 = b$ and $j_i = d$ for sufficiently large i (we assume that the infinite product on the right side of (13.2.4) converges). By arrow conservation, the sum in (13.2.4) is supported on at most one term.

As in Definition 6.1.1, $L_x(\mathcal{A}, b; \mathcal{C}, d)$ is the partition function under the weights L_x for the vertex model on a row, whose vertical entrance data is given by $(\mathbf{A}_0, \mathbf{A}_1, \ldots)$; horizontal entrance data by $\mathbf{e}_b$; vertical exit data by $(\mathbf{C}_0, \mathbf{C}_1, \ldots)$; and horizontal exit data by $\mathbf{e}_d$. See the left side of Figure 6.1 for the depiction of a similar row.

Next, we define row operators. As in Section 6.2, let $\mathbb{V}$ denote the infinite-dimensional vector space spanned by basis vectors of the form $|\mathcal{A}\rangle$, where $\mathcal{A}$ ranges over all finitary sequences of elements in $\{0,1\}^n$. Similarly, let $\mathbb{V}^*$ denote the space spanned all dual basis vectors of the form $\langle \mathcal{C}|$, over finitary $\mathcal{C}$. We impose an inner product on $\mathbb{V}^* \times \mathbb{V}$ determined by setting $\langle \mathcal{C} | \mathcal{A} \rangle = \mathbf{1}_{\mathcal{A}=\mathcal{C}}$ for any finitary $\mathcal{A}$ and $\mathcal{C}$.

Definition 13.2.4. For any complex number $x \in \mathbb{C}$ and index $i \in [1, n]$, define the operators $\mathsf{C}_i(x) : \mathbb{V} \to \mathbb{V}$ and $\mathsf{D}(x) : \mathbb{V} \to \mathbb{V}$ by, for any finitary sequence $\mathcal{A}$, setting

$$\mathsf{C}_i(x)|\mathcal{A}\rangle = \sum_{\mathcal{C}} L_x(\mathcal{A}, i; \mathcal{C}, 0)|\mathcal{C}\rangle; \qquad \mathsf{D}(x)|\mathcal{A}\rangle = \sum_{\mathcal{C}} L_x(\mathcal{A}, 0; \mathcal{C}, 0)|\mathcal{C}\rangle,$$

where both sums are over all finitary sequences $\mathcal{C}$ of elements in $\{0,1\}^n$. In view of the inner product between $\mathbb{V}^*$ and $\mathbb{V}$, the operators $\mathsf{C}_i(x)$ and $\mathsf{D}(x)$ admit dual actions on $\mathbb{V}^*$ given by

$$\langle \mathcal{C}|\mathsf{C}_i(x) = \sum_{\mathcal{A}} L_x(\mathcal{A}, i; \mathcal{C}, 0)\langle \mathcal{A}|; \qquad \langle \mathcal{C}|\mathsf{D}(x) = \sum_{\mathcal{A}} L_x(\mathcal{A}, 0; \mathcal{C}, 0)\langle \mathcal{A}|.$$

In particular, under C_i one color i arrow horizontally enters the vertex model and none horizontally exit, and under D no arrows horizontally enter or exit the vertex model.

By virtue of the ($r = q^{-1/2} = s$ case of) Yang–Baxter equation, Proposition 5.1.4, one can now derive the following commutation relations of the row operators C_i through the following proposition. We omit its proof, which is very similar to that of Theorem 3.2.1 of **[12]** (see also the proof of Proposition 6.2.2).

Proposition 13.2.5. *For any complex numbers $x, y \in \mathbb{C}$, we have*

$$\mathsf{D}(x)\mathsf{D}(y) = \mathsf{D}(y)\mathsf{D}(x). \tag{13.2.5}$$

Moreover, for any integers $0 \leqslant i < j \leqslant n$ (where here we set $\mathsf{C}_0(z) = \mathsf{D}(z)$ for any $z \in \mathbb{C}$), we have

$$\mathsf{C}_i(x)\mathsf{C}_j(y) = \frac{x - ty}{x - y}\, \mathsf{C}_j(y)\mathsf{C}_i(x) - \frac{(1-t)y}{x - y}\, \mathsf{C}_j(x)\mathsf{C}_i(y). \tag{13.2.6}$$

13.3. Partition Function Formula for $f_\mu(\mathbf{x})$

In this section we provide an expression for $f_\mu(\mathbf{x})$ in terms of the C_i operators from Definition 13.2.4. Before doing so, we require some definitions regarding the vector space $\mathbb{V}$. For any nonnegative composition $\lambda \in \mathbb{Z}_{\geqslant 0}^p$ of length $p \in [1, n]$, define the sequence $\mathcal{I}(\lambda) = \big(\mathbf{I}_0(\lambda), \mathbf{I}_1(\lambda), \ldots\big)$

of elements in $\{0,1\}^n$ so that the k-th coordinate of $\mathbf{I}_j(\lambda)$ is given by $\mathbf{1}_{\lambda_k=j}$, for each $k \in [1,p]$ and $j \geqslant 0$.

Definition 13.3.1. Let μ be a nonnegative composition of length n, and let its largest part be $N = \max_{i\in[1,n]} \mu_i$. Fix a linear operator $\mathsf{X} \in \mathrm{End}(\mathbb{V})$ and a set of indeterminates $\mathbf{v} = (v_{i,j})$, which we will refer to as *twist parameters* below, where (i,j) ranges over all integer pairs $(i,j) \in [1,n]\times[0,N]$. Define the linear form

$$\langle \mathsf{X} \rangle_\mu \equiv \langle \mathsf{X} \rangle_\mu(\mathbf{v}) := \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=0}^{N} v_{i,j}^{M_{i,j}} \big\langle \mathcal{M} + \mathcal{I}(\mu) \big| \mathsf{X} \big| \mathcal{M} \big\rangle, \tag{13.3.1}$$

where we sum over sequences $\mathcal{M} = (\mathbf{M}_0, \mathbf{M}_1, \ldots)$ of elements in $\{0,1\}^n$ such that $\mathbf{M}_k = \mathbf{e}_0$ for $k > N$, and we have denoted $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j}) \in \{0,1\}^n$ for each $j \geqslant 0$. Observe that this sum is supported on only finitely many nonzero terms.

Now we can state the following theorem providing an expression for the nonsymmetric Macdonald polynomial $f_\lambda(\mathbf{x})$ through the fermionic row operators C_i from Definition 13.2.4; its proof will appear in Section 13.4 and Chapter 14 below. Let us mention that a similar bosonic expression for f_λ appeared as Theorem 4.2 of **[13]**, and our results and proofs here are largely based on those there. However, one subtantial difference between that result and the following one is that the latter provides a finite summation identity for f_λ, while the one from **[13]** involved an infinite sum.

THEOREM 13.3.2. *Fix a nonnegative composition μ of length n, set $N = \max_{i\in[1,n]} \mu_i$. Then, $f_\mu(\mathbf{x})$ is given in terms of the linear form* (13.3.1) *by*

$$f_\mu(\mathbf{x}) = \Omega_\mu(q,t) \big\langle \mathsf{C}_n(x_n) \cdots \mathsf{C}_1(x_1) \big\rangle_\mu, \tag{13.3.2}$$

where for $i \in [1,n]$ and $j \in [1,N]$ the underlying parameters $v_{i,j}$ are chosen to be

$$v_{i,j} = q^{\mu_i - j} t^{\gamma_{i,j}(\mu)} \mathbf{1}_{\mu_i > j}, \tag{13.3.3}$$

with exponents $\gamma_{i,j}(\mu)$ given by

$$\gamma_{i,j}(\mu) = -\#\{k < i : \mu_k > \mu_i\} + \#\{k > i : j \leqslant \mu_k < \mu_i\}. \tag{13.3.4}$$

The normalization factor $\Omega_\mu(q,t)$ appearing on the right-hand side of (13.3.2) *is given by*

$$\Omega_\mu(q,t) = \prod_{i=1}^{n} \prod_{j=0}^{\mu_i - 1} \frac{1}{1 - q^{\mu_i - j} t^{\alpha_{i,j}(\mu)+1}}, \tag{13.3.5}$$

where we have defined

$$\alpha_{i,j}(\mu) = \#\{k < i : \mu_k = \mu_i\} + \#\{k \neq i : j < \mu_k < \mu_i\} + \#\{k > i : j = \mu_k\}. \tag{13.3.6}$$

13.4. Proof of Theorem 13.3.2

In this section we establish Theorem 13.3.2, assuming the following two propositions. The first, which we will establish in Section 14.4 below, is a *cyclic relation* that will essentially amount to the eigenvalue equation (13.1.8) for f_μ. The second, which we will establish in Section 14.5 below, fixes the normalization (13.1.7) for f_μ. Here, we recall the exponents $\gamma_{i,j}$ and the quantity $\Omega_\mu(q,t)$ from (13.3.4) and (13.3.5), respectively.

Proposition 13.4.1. *For any integer $i \in [1, n]$, we have*

$$\begin{aligned}(13.4.1)\qquad &\big\langle \mathsf{C}_i(qx_i)\mathsf{C}_n(x_n)\cdots\mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_{i-1})\cdots\mathsf{C}_1(x_1)\big\rangle_\mu \\ &\qquad = q^{\mu_i}t^{\gamma_{i,0}(\mu)}\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_{i-1})\cdots\mathsf{C}_1(x_1)\mathsf{C}_i(x_i)\big\rangle_\mu,\end{aligned}$$

where the twist parameters associated with the linear form (13.3.1) *are given by* (13.3.3)–(13.3.4).

Proposition 13.4.2. *We have*

$$(13.4.2)\qquad \mathrm{Coeff}\Big[\Omega_\mu(q,t)\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu;\mathbf{x}^\mu\Big] = 1,$$

where the twist parameters associated with the linear form (13.3.1) *are given by* (13.3.3)–(13.3.4).

Before proceeding to the proof of Theorem 13.3.2 (assuming the two results above), we first modify (13.1.8) slightly, by transferring the string of Hecke generators $T_{i-1}\cdots T_1$ contained within the Cherednik–Dunkl operator Y_i to the right-hand side of the relation. One obtains

$$(13.4.3)\qquad \omega\cdot T_{n-1}^{-1}\cdots T_i^{-1}\cdot f_\mu(\mathbf{x}) = y_i(\mu;q,t)T_1^{-1}\cdots T_{i-1}^{-1}\cdot f_\mu(\mathbf{x}).$$

We wish to substitute the Ansatz (13.3.2) into (13.4.3), and check its validity. In order to do so, we require the following auxiliary result.

Proposition 13.4.3. *For any integers $i \in [1, n-1]$ and $1 \leqslant j < k \leqslant n$, we have*

$$(13.4.4)\qquad T_i^{-1}\cdot\mathsf{C}_k(x_{i+1})\mathsf{C}_j(x_i) = t^{-1}\mathsf{C}_j(x_{i+1})\mathsf{C}_k(x_i).$$

Proof. This follows by direct calculation, using the explicit form (13.1.3) of T_i^{-1}. We compute

$$\begin{aligned}T_i^{-1}\cdot\mathsf{C}_k(x_{i+1})\mathsf{C}_j(x_i) &= t^{-1}\left(\frac{(t-1)x_i}{x_{i+1}-x_i}\mathsf{C}_k(x_{i+1})\mathsf{C}_j(x_i) + \frac{x_{i+1}-tx_i}{x_{i+1}-x_i}\mathsf{C}_k(x_i)\mathsf{C}_j(x_{i+1})\right)\\ &= t^{-1}\mathsf{C}_j(x_{i+1})\mathsf{C}_k(x_i),\end{aligned}$$

where the final equality follows from the exchange relation (13.2.6). ∎

Now we can establish Theorem 13.3.2, assuming Proposition 13.4.1 and Proposition 13.4.2.

Proof of Theorem 13.3.2 Assuming Proposition 13.4.1 and Proposition 13.4.2. By Remark 13.1.2, it suffices to verify that our Ansatz (13.3.2) satisfies both the eigenvalue equation (13.1.8) and the normalization constraint (13.4.2), with $\Omega_\mu(q,t)$. The latter follows by Proposition 13.4.2, so it remains to establish (13.1.8) or, equivalently, (13.4.3).

To that end, let us examine the left-hand side of (13.4.3), assuming the substitution (13.3.2). It becomes

$$\begin{aligned}&\Omega_\mu(q,t)\cdot\omega\cdot T_{n-1}^{-1}\cdots T_i^{-1}\cdot\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu\\ &\qquad = \Omega_\mu(q,t)t^{i-n}\cdot\omega\cdot\big\langle \mathsf{C}_i(x_n)\mathsf{C}_n(x_{n-1})\cdots\mathsf{C}_{i+1}(x_i)\mathsf{C}_{i-1}(x_{i-1})\cdots\mathsf{C}_1(x_1)\big\rangle_\mu,\end{aligned}$$

where to deduce the equality we used the linearity of the form (13.3.1), together with $n-i$ applications of the identity (13.4.4) to transfer the operator C_i towards the left of the operator product. Finally we act explicitly with ω, which by (13.1.4) yields

$$\begin{aligned}(13.4.5)\qquad &\Omega_\mu(q,t)\cdot\omega\cdot T_{n-1}^{-1}\cdots T_i^{-1}\cdot\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu\\ &\qquad = \Omega_\mu(q,t)t^{i-n}\cdot\big\langle \mathsf{C}_i(qx_1)\mathsf{C}_n(x_n)\cdots\mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_i)\cdots\mathsf{C}_1(x_2)\big\rangle_\mu.\end{aligned}$$

Similarly, under the substitution (13.3.2), the right-hand side of (13.4.3) is given by

$$
\begin{aligned}
(13.4.6)\quad & y_i(\mu;q,t)\Omega_\mu(q,t)\cdot T_1^{-1}\cdots T_{i-1}^{-1}\cdot \big\langle \mathsf{C}_n(x_n)\cdots \mathsf{C}_1(x_1)\big\rangle_\mu, \\
& = t^{1-i} y_i(\mu;q,t)\Omega_\mu(q,t)\cdot \big\langle \mathsf{C}_n(x_n)\cdots \mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_i)\cdots \mathsf{C}_1(x_2)\mathsf{C}_i(x_1)\big\rangle_\mu.
\end{aligned}
$$

where to deduce the equality we again used the linearity of the form (13.3.1), together with $i-1$ applications of (13.4.4) to send the operator C_i towards the right of the operator product.

Matching the right-hand sides of (13.4.5) and (13.4.6), and applying $\mathfrak{s}_{i-1}\cdots\mathfrak{s}_1$ to both sides of the resulting equation, we are left with the task of proving that

$$
\begin{aligned}
(13.4.7)\quad & \big\langle \mathsf{C}_i(qx_i)\mathsf{C}_n(x_n)\cdots \mathsf{C}_{i+1}(x_{i+1})C_{i-1}(x_{i-1})\cdots \mathsf{C}_1(x_1)\big\rangle_\mu \\
& = t^{n-2i+1} y_i(\mu;q,t)\big\langle \mathsf{C}_n(x_n)\cdots \mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_{i-1})\cdots \mathsf{C}_1(x_1)\mathsf{C}_i(x_i)\big\rangle_\mu.
\end{aligned}
$$

Using the explicit form of the eigenvalue $y_i(\mu;q,t)$ from (13.1.9), together with the fact that

$$
n-i+\eta_i(\mu) = -\#\{j<i:\mu_j>\mu_i\} + \#\{j>i:\mu_j<\mu_i\} \equiv \gamma_{i,0}(\mu),
$$

(13.4.7) follows from Proposition 13.4.1. This establishes (13.4.3) and therefore the theorem. □

CHAPTER 14

Proof of Proposition 13.4.1 and Proposition 13.4.2

By the content in Section 13.4, to establish Theorem 13.3.2 it suffices to show the cyclic relation Proposition 13.4.1 and the normalization condition Proposition 13.4.2. In this chapter we perform these tasks, closely following the proofs of equation (4.14) and Section 4.10 of [**13**], respectively.

14.1. Column Operators and Diagrammatic Notation

Our main tool for proving Proposition 13.4.1 will be another class of objects called column operators, constructed in terms of the following single-column partition functions.

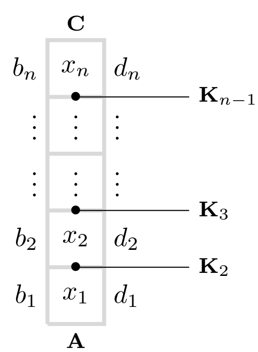
Definition 14.1.1. Fix two sequences $\mathfrak{B} = (b_1, b_2, \ldots, b_n)$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_n)$ of indices in $[0, n]$, as well as two elements $\mathbf{A}, \mathbf{C} \in \{0, 1\}^n$. For any set of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, let

$$L_{\mathbf{x}}(\mathbf{A}, \mathfrak{B}; \mathbf{C}, \mathfrak{D}) = \sum_{\mathcal{K}} \prod_{i=1}^{n} L_{x_i}(\mathbf{K}_i, b_i; \mathbf{K}_{i+1}, d_i), \tag{14.1.1}$$

where we sum over all sequences $\mathcal{K} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_{n+1})$ of elements in $\{0, 1\}^n$ such that $\mathbf{K}_1 = \mathbf{A}$ and $\mathbf{K}_{n+1} = \mathbf{C}$. By arrow conservation, the sum in (14.1.1) is supported on at most one term. We also set $L_{\mathbf{x}}(\mathbf{A}, \mathfrak{B}; \mathbf{C}, \mathfrak{D}) = 0$ if either $\mathbf{A}$ or $\mathbf{C}$ is not in $\{0, 1\}^n$.

In particular, $L_{\mathbf{x}}(\mathbf{A}, \mathfrak{B}; \mathbf{C}, \mathfrak{D})$ denotes the partition function for a column with vertical entrance data $\mathbf{A}$; horizontal entrance data $(b_1, b_2, \ldots, b_n)$; vertical exit data $\mathbf{C}$; and horizontal exit data $(d_1, d_2, \ldots, d_n)$. In what follows, instead of using the algebraic expressions (14.1.1) for partition functions, it will often be convenient to express them diagrammatically, for example by

(14.1.2) $\qquad L_{\mathbf{x}}(\mathbf{A}, \mathfrak{B}; \mathbf{C}, \mathfrak{D}) :=$



where the indices in the diagram on the right side of (14.1.2) are as follows. The x_i denote the spectral parameters assigned to each vertex (represented by a face in (14.1.2)); the indices $(b_1, b_2, \ldots, b_n)$ along the left boundary denote the colors of the arrows horizontally entering the column; the indices $(d_1, d_2, \ldots, d_n)$ along the right boundary denote the colors of the arrows horizontally exiting the colum; the n-tuples $\mathbf{A}$ and $\mathbf{C}$ denote the boundary data for vertical arrows along the bottom and top boundaries, respectively; and the n-tuples $\mathbf{K}_i$ along the middle of the column denote the collection of vertical arrows proceeding from vertex $i-1$ to vertex i.

Next, to define column operators, let $W \cong \mathbb{C}^{n+1}$ be an $(n+1)$-dimensional vector space with basis $\{|0\rangle, |1\rangle, \dots, |n\rangle\}$. The basis of the dual vector space W^* will be written as $\{\langle 0|, \langle 1|, \dots, \langle n|\}$ with $\langle i|j\rangle = \mathbf{1}_{i=j}$. We wish to consider an n-fold tensor product of W, denoted by

$$\mathbb{W} = W_1 \otimes \cdots \otimes W_n = \operatorname{Span}\left\{ \bigotimes_{k=1}^{n} |i_k\rangle_k \right\}_{0 \leqslant i_1, \dots, i_n \leqslant n} \equiv \operatorname{Span}\{\, |\mathfrak{I}\rangle \,\}_{\mathfrak{I} \in [0,n]^n},$$

where we have set $\mathfrak{I} = (i_1, i_2, \dots, i_n)$. We now define operators acting on $\mathbb{W}$.

Definition 14.1.2. Fix sequences $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ of complex numbers, and fix an element $\mathbf{C} \in \{0,1\}^n$. We introduce the *column operator* $\Psi_{\mathbf{v}}(\mathbf{C}) = \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{C}) \in \operatorname{End}(\mathbb{W})$, defined by setting for any sequence $\mathfrak{D} = (d_1, d_2, \dots, d_n)$ of indices in $[0,n]$

$$\Psi_{\mathbf{v}}(\mathbf{C})\, |\mathfrak{D}\rangle = \sum_{\mathfrak{B}} \left(\sum_{\mathbf{M} \in \{0,1\}^n} \prod_{k=1}^{n} v_k^{M_k} L_{\mathbf{x}}(\mathbf{M}, \mathfrak{B}; \mathbf{M}+\mathbf{C}, \mathfrak{D}) \right) |\mathfrak{B}\rangle, \tag{14.1.3}$$

and then extending its action to all of $\mathbb{W}$ by linearity. The first sum in (14.1.3) is over all sequences $\mathfrak{B} = (b_1, b_2, \dots, b_n)$ of indices in $[0,n]$, and the second over all elements $\mathbf{M} = (M_1, M_2, \dots, M_n) \in \{0,1\}^n$.

The reason for introducing the column operators (14.1.3) is that they offer an alternative algebraic setup for operator products of the form (13.3.2), which will be convenient for studying the cyclic relation (13.4.1). In particular, we note the following, where in the below we recall from Section 13.3 the sequence $\big(\mathbf{I}_0(\lambda), \mathbf{I}_1(\lambda), \dots\big)$ of elements in $\{0,1\}^n$.

Proposition 14.1.3. *Fix a composition μ of length n, and let $N = \max_{i \in [1,n]} \mu_i$. Then, denoting $\mathbf{v}^{(j)} = (v_{1,j}, v_{2,j}, \dots, v_{n,j})$ for $j \in [0,N]$, where the parameters $v_{i,j}$ are as in* (13.3.3)*, we have*

$$\big\langle \mathsf{C}_n(x_n) \cdots \mathsf{C}_1(x_1) \big\rangle_\mu = \big\langle 1, \dots, n \big| \Psi_{\mathbf{v}^{(0)}}\big(\mathbf{I}_0(\mu)\big) \cdots \Psi_{\mathbf{v}^{(N)}}\big(\mathbf{I}_j(\mu)\big) \big| 0, \dots, 0 \big\rangle. \tag{14.1.4}$$

Proof. By the expression of the row operators from Definition 13.2.4 and the linear form (13.3.1), we obtain the following partition function representation of $\big\langle \mathsf{C}_1(x_1) \cdots \mathsf{C}_n(x_n) \big\rangle_\mu$:

$$\big\langle \mathsf{C}_n(x_n) \cdots \mathsf{C}_1(x_1) \big\rangle_\mu = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=0}^{N} v_{i,j}^{M_{i,j}} \times \begin{array}{c|c|c|c|c|c|c|c} & \mathbf{M}_0 + \mathbf{I}_0(\mu) & & \cdots & \cdots & & \mathbf{M}_N + \mathbf{I}_N(\mu) & \\ \hline n & x_n & & \cdots & \cdots & & x_n & 0 \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline 2 & x_2 & & \cdots & \cdots & & x_2 & 0 \\ \hline 1 & x_1 & & \cdots & \cdots & & x_1 & 0 \\ \hline & \mathbf{M}_0 & & \cdots & \cdots & & \mathbf{M}_N & \end{array} \tag{14.1.5}$$

where we sum over all sequences $\mathcal{M} = (\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_N)$ of elements in $\{0,1\}^n$, with $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$ for each $j \in [0, N]$. Under this diagrammatic notation, summation over all possible states is implied at each internal lattice edge.

Now we read the partition function (14.1.5) column-by-column, starting from the rightmost and working to the left. We distribute the product $\prod_{i=1}^{n} \prod_{j=0}^{N} v_{i,j}^{M_{i,j}}$ over the columns by assigning the factor $\prod_{i=1}^{n} v_{i,j}^{M_{i,j}}$ to column j, and for each $0 \leqslant j \leqslant N$ we compute separately the sums over $\mathbf{M}_j = (M_{1,j}, \ldots, M_{n,j})$ (which only affects the boundary states of column j). Referring to the definition (14.1.3) of $\Psi_{\mathbf{v}^{(j)}}\big(\mathbf{I}_j(\mu)\big)$, equation (14.1.4) is nothing more than the algebraic realization of this column-by-column decomposition. $\square$

The partition function (14.1.5) is useful for visualizing the structure of $\big\langle \mathsf{C}_1(x_1) \ldots \mathsf{C}_n(x_n) \big\rangle_\mu$, but it is still not in the most convenient form. In particular, we would like to blend the summation over $\mathcal{M}$ into our graphical conventions. With that in mind, we set up the following notation.

Definition 14.1.4. Let $\Psi_{\mathbf{v}}(\mathbf{C}) = \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{C})$ be a column operator as defined in (14.1.3). We shall simplify the graphical representation of its components by for any $\mathfrak{B} = (b_1, b_2, \ldots, b_n)$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_n)$ writing

$$
\langle \mathfrak{B} |\, \Psi_{\mathbf{v}}(\mathbf{C})\, | \mathfrak{D} \rangle = \sum_{\mathbf{M} \in \{0,1\}^n} \prod_{k=1}^{n} v_k^{M_k} \times
\begin{array}{ccc}
 & \mathbf{M}+\mathbf{C} & \\
b_n & \boxed{x_n} & d_n \\
\vdots & \boxed{\vdots} & \vdots \\
\vdots & \boxed{\vdots} & \vdots \\
b_2 & \boxed{x_2} & d_2 \\
b_1 & \boxed{x_1} & d_1 \\
 & \mathbf{M} &
\end{array}
=
\begin{array}{ccc}
 & \mathbf{C} & \\
b_n & \boxed{x_n} & d_n \\
\vdots & \boxed{\vdots} & \vdots \\
\vdots & \boxed{\vdots} & \vdots \\
b_2 & \boxed{x_2} & d_2 \\
b_1 & \boxed{x_1} & d_1
\end{array}
$$

where $\mathbf{M} = (M_1, M_2, \ldots, M_n) \in \{0,1\}^n$, and the dependence on $\mathbf{v}$ is kept implicit in the object appearing on the right-hand side. Observe here the double line along the top and bottom of the column on the right side.

14.2. Explicit Computation of Column Operator Components

In this section we turn to the question of explicitly computing the matrix elements of the linear operators (14.1.3). We begin with some auxiliary definitions for admissible vectors, color data, and coordinates.

Definition 14.2.1. For any sequences $\mathfrak{B} = (b_1, b_2, \ldots, b_n)$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_n)$ of indices in $[0, n]$, we say that the pair $(\mathfrak{B}, \mathfrak{D})$ is *admissible* if for each $k \in [1, n]$ we have

$$0 \leqslant m_k(\mathfrak{D}) \leqslant m_k(\mathfrak{B}) \leqslant 1, \tag{14.2.1}$$

where $m_k(\mathfrak{B})$ and $m_k(\mathfrak{D})$ denote the multiplicities of k in $\mathfrak{B}$ and $\mathfrak{D}$, respectively.

Definition 14.2.2. Given an admissible pair of sequences $(\mathfrak{B}, \mathfrak{D})$, as in Definition 14.2.1, we introduce two disjoint sets $\mathcal{P} \subset \{1, \ldots, n\}$ and $\mathcal{Q} \subset \{1, \ldots, n\}$, where $p \in \mathcal{P}$ if and only if $m_p(\mathfrak{B}) = 1$

and $m_p(\mathfrak{D}) = 0$, while $p \in \mathcal{Q}$ if and only if $m_p(\mathfrak{B}) = m_p(\mathfrak{D}) = 1$. We refer to $(\mathcal{P}, \mathcal{Q})$ as the *color data* associated to $(\mathfrak{B}, \mathfrak{D})$.

Definition 14.2.3. Let $\mathfrak{B} = (b_1, b_2, \ldots, b_n)$ and $\mathfrak{D} = (d_1, d_2, \ldots, d_n)$ be sequences such that the pair $(\mathfrak{B}, \mathfrak{D})$ is admissible, as in Definition 14.2.1; also let $(\mathcal{P}, \mathcal{Q})$ be their associated color data, as in Definition 14.2.2. Introduce another two vectors

$$\mathcal{I} = (i_p)_{p \in \mathcal{P} \cup \mathcal{Q}}, \qquad \mathcal{J} = (j_p)_{p \in \mathcal{Q}}, \tag{14.2.2}$$

such that

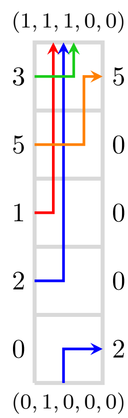
$$b_{i_p} = p, \quad \text{for each } p \in \mathcal{P} \cup \mathcal{Q}; \qquad d_{j_p} = p, \quad \text{for each } p \in \mathcal{Q}. \tag{14.2.3}$$

We shall call $(\mathcal{I}, \mathcal{J})$ the *coordinates* of $(\mathfrak{B}, \mathfrak{D})$.

Example 14.2.4. In order to illustrate the above definitions, let us choose $n = 5$, and $\mathfrak{B} = (0, 2, 1, 5, 3)$, $\mathfrak{D} = (2, 0, 0, 0, 5)$. These vectors clearly satisfy the requirements (14.2.1), and are thus admissible. We find that $m_p(\mathfrak{B}) = 1$ and $m_p(\mathfrak{D}) = 0$ for $p \in \{1, 3\}$, while $m_p(\mathfrak{B}) = m_p(\mathfrak{D}) = 1$ for $p \in \{2, 5\}$. Hence, $\mathcal{P} = \{1, 3\}$ and $\mathcal{Q} = \{2, 5\}$ is the color data associated to $(\mathfrak{B}, \mathfrak{D})$. Finally, one finds $\mathcal{I} = (i_1, i_2, i_3, i_5) = (3, 2, 5, 4)$ by reading the positions of $\{1, 2, 3, 5\}$ in $\mathfrak{B}$, and $\mathcal{J} = (j_2, j_5) = (1, 5)$ by reading the positions of $\{2, 5\}$ in $\mathfrak{D}$.

To illustrate how we will make use of such quantities, consider the following example of the column (14.1.2) for $n = 5$, $\mathbf{A} = (0, 1, 0, 0, 0)$, $\mathbf{C} = (1, 1, 1, 0, 0)$, and $(\mathfrak{B}, \mathfrak{D})$ as above, given by

$$L(\mathbf{A}, \mathfrak{B}; \mathbf{C}, \mathfrak{D}) =$$



where we omit the spectral parameters $\mathbf{x} = (x_1, x_2, \ldots, x_5)$ in both the weight $L = L_{\mathbf{x}}$ and the column diagram. In this picture, $\mathfrak{B}$ and $\mathfrak{D}$ label the colors (read from bottom to top) of the arrows entering and exiting through the left and right boundaries of the column, respectively. The admissibility of $(\mathfrak{B}, \mathfrak{D})$ translates into the fact that each color $\{1, 2, \ldots, 5\}$ appears at most once on the left or right of the tower, with the requirement that all colors that appear on the right must have also appeared on the left. Then $\mathcal{P} \cup \mathcal{Q}$ gives the set of all colors entering via the left edges of the tower, $\mathcal{Q}$ gives the set of all colors leaving via the right edges. Here red, blue, green, and orange are colors 1, 2, 3, and 5 respectively.

Now the following proposition provides explicit forms for the matrix entries of column operators.

Proposition 14.2.5. *Let* $\mathfrak{B} = (b_1, b_2, \ldots, b_n)$, $\mathfrak{D} = (d_1 d_2, \ldots, j_n)$ *be sequences of indices in* $[0, n]$ *such that the pair* $(\mathfrak{B}, \mathfrak{D})$ *is admissible, as in Definition 14.2.1. Associate to it the color data* $(\mathcal{P}, \mathcal{Q})$ *in the same way as in Definition 14.2.2. Let* $(\mathcal{I}, \mathcal{J})$ *be the coordinates of* $(\mathfrak{B}, \mathfrak{D})$*, as in equations*

(14.2.2)–(14.2.3). *Fix sequences* $\mathbf{x} = (x_1, x_2, \dots, x_n)$ *and* $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *of* n *complex numbers such that* $v_r = 0$ *for all* $r \notin \mathcal{P} \cup \mathcal{Q}$, *and define the* $\{0,1\}^n$ *element*

$$\mathbf{e}_{\mathcal{P}} = \sum_{p \in \mathcal{P}} \mathbf{e}_p.$$

Under the above set of assumptions, we have

(14.2.4)

$$\big\langle \mathfrak{B} \big| \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}}) \big| \mathfrak{D} \big\rangle = \prod_{\substack{p > \ell \\ p \in \mathcal{P} \cup \mathcal{Q} \\ \ell \in \mathcal{Q}}} \mathbf{1}_{i_p \neq j_\ell} \prod_{p \in \mathcal{P}} t^{g(p)} \prod_{p \in \mathcal{Q}} x_{j_p} \prod_{\substack{p \in \mathcal{Q} \\ i_p = j_p}} (1 - v_p t^{f(p)+1}) \prod_{\substack{p \in \mathcal{Q} \\ i_p \neq j_p}} v_p^{\mathbf{1}_{i_p > j_p}} t^{h(p)} (1-t),$$

where we have defined the combinatorial exponents

$$f(p) = \#\{\ell \in \mathcal{Q} : \ell < p\}; \qquad g(p) = \#\{\ell \in \mathcal{Q} : \ell < p, i_p < j_\ell\}; \\ h(p) = \#\big\{\ell \in \mathcal{Q} : \ell < p, j_\ell \in (i_p, j_p)\big\}, \tag{14.2.5}$$

with the interval (i_p, j_p) *appearing in* $h(p)$ *to be interpreted in a cyclic sense; namely, for all integers* $1 \leqslant a, b \leqslant n$ *we define*

$$(a,b) := \begin{cases} \{a+1, \dots, b-1\}, & a < b, \\ \{a+1, \dots, n\} \cup \{1, \dots, b-1\}, & a > b, \\ \varnothing, & a = b. \end{cases}$$

Proof. We compute the components by calculating

$$\big\langle \mathfrak{B} \big| \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}}) \big| \mathfrak{D} \big\rangle = \sum_{\mathbf{M} \in \{0,1\}^n} \prod_{k=1}^n v_k^{M_k} L_{\mathbf{x}}(\mathbf{M}, \mathfrak{B}; \mathbf{M} + \mathbf{e}_{\mathcal{P}}, \mathfrak{D}), \tag{14.2.6}$$

where we set $\mathbf{M} = (M_1, M_2, \dots, M_n) \in \{0,1\}^n$, and the $L_{\mathbf{x}}$ here is the weight of the column (14.1.2) (with $\mathbf{A} = \mathbf{M}$ and $\mathbf{C} = \mathbf{M} + \mathbf{e}_{\mathcal{P}}$). We begin by remarking that this column has weight zero if $b_k > d_k \geqslant 1$ for any $k \in [1, n]$, due to the vanishing of the fifth weight in Figure 13.1 (namely, $L_x(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i) = 0$ for any $1 \leqslant i < j \leqslant n$). This gives rise to the product of indicator functions present in (14.2.4). In the rest of the proof, we assume that the constraints imposed by these indicators are always obeyed.

Noting that the vertex weights from Definition 13.2.1 are factorized across different colors, *i.e.*, over the n components of $\mathbf{A}$, we find that the n sums in (14.2.6), over $M_1, M_2, \dots, M_n \geqslant 0$, can be computed independently of each other (once $\mathfrak{B}$ and $\mathfrak{D}$ are fixed). This leads to the factorization

$$\big\langle \mathfrak{B} \big| \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}}) \big| \mathfrak{D} \big\rangle = \prod_{p \in \mathcal{P}} \phi_p(v_p) \prod_{\substack{p \in \mathcal{Q} \\ i_p = j_p}} \chi_p(v_p) \prod_{\substack{p \in \mathcal{Q} \\ i_p \neq j_p}} \psi_p(v_p), \tag{14.2.7}$$

where we have introduced the functions

$$\phi_p(v) = t^{g(p)}; \qquad \chi_p(v) = x_{j_p}(1 - v t^{f(p)+1}); \qquad \psi_p(v) = x_{j_p} v^{\mathbf{1}_{i_p > j_p}} t^{h(p)} (1-t), \tag{14.2.8}$$

and for notational convenience we have omitted the dependence of these functions on $\mathbf{x}$.

The quantities in (14.2.8) can be obtained by tracing paths of a fixed color; see Figure 14.1. Pictorially, the first quantity $\phi_p(v)$ in (14.2.8) corresponds to a color p path that entered through the left boundary of the column and then exited through its top. The second quantity $\chi_p(v)$ in (14.2.8) corresponds to a color p path that entered through the left boundary, circled 0 or 1 times over the column, and then exited on its right through the *same* row as where it entered. The third

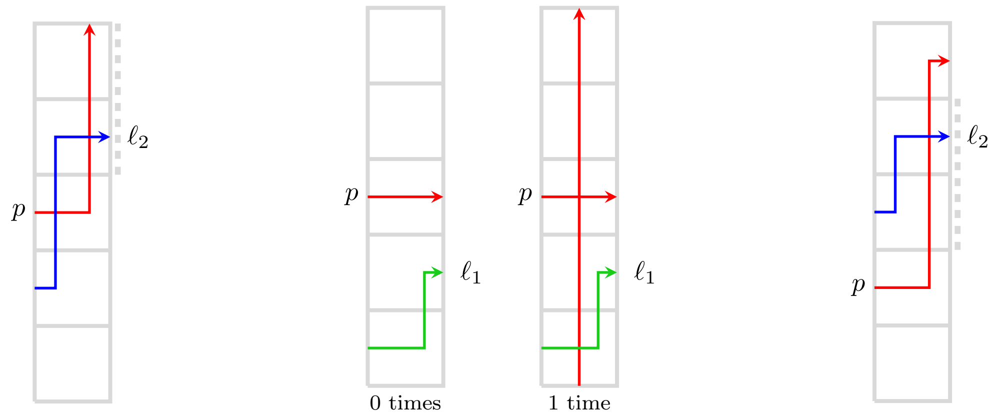


FIGURE 14.1. Left panel: a color p path enters via the left boundary, and exits via the top. $g(p)$ counts the number of colors $\ell_2 < p$ that exit through one of the shaded right edges, which are situated strictly above the position where p entered.

Middle panel: a color p path enters via the left boundary, and circles 0 or 1 times around it before exiting via the right edge directly opposite its entry point. $f(p)$ counts the number of colors $\ell_1 < p$ which enter from the left of the column and leave anywhere on its right.

Right panel: a color p path of enters via the left boundary, and then exits via a right edge in a different row to its starting point. $h(p)$ counts the number of colors $\ell_2 < p$ that exit through one of the shaded right edges, which are situated strictly between the initial and final locations of path p (the dashed interval on the right boundary).

quantity $\psi_p(v)$ in (14.2.8) corresponds to a color p path that entered through the left boundary of the column, and then exited on its right but via a *different* row from the one where it entered (after wrapping around the column, if required).

The result (14.2.4) now follows by inserting (14.2.8) into (14.2.7). $\square$

14.3. Column Rotation

In this section we define a specific "rotation" operation which acts on the column operator components. The key result will be the fact that, up to a simple overall factor depending on t and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the components (14.2.4) remain invariant under this operation.

Proposition 14.3.1. *Assuming the setup in Proposition 14.2.5, consider the effect of rotating the edge states of the column, which is equivalent to sending* $(b_1, \ldots, b_{n-1}, b_n) \mapsto (b_n, b_1, \ldots, b_{n-1})$ *and* $(d_1, \ldots, d_{n-1}, d_n) \mapsto (d_n, d_1, \ldots, d_{n-1})$ *in* (14.2.4)*, together with the corresponding cyclic shift of the row parameters* $(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1})$*. We find that*

$$\frac{\langle b_1, \ldots, b_n |\, \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\, | d_1, \ldots, d_n \rangle}{\langle b_n, b_1, \ldots, b_{n-1} |\, \mathfrak{s}_{n-1} \cdots \mathfrak{s}_1 \cdot \Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\, | d_n, d_1, \ldots, d_{n-1} \rangle} = \kappa_{\mathbf{v};t}(\mathcal{I}, \mathcal{J}), \tag{14.3.1}$$

where the right-hand side is given by

$$\kappa_{\mathbf{v};t}(\mathcal{I},\mathcal{J}) = \prod_{\substack{(p,\ell)\in\mathcal{P}\times\mathcal{Q}\\ p>\ell}} t^{\mathbf{1}_{j_\ell=n}-\mathbf{1}_{i_p=n}} \prod_{p\in\mathcal{Q}} v_p^{\mathbf{1}_{i_p=n}-\mathbf{1}_{j_p=n}}. \tag{14.3.2}$$

Proof. We can explicitly compute the ratio on the left-hand side of (14.3.1) by using the formula (14.2.4) for the column components. The main observation is that, under the proposed rotation, almost all of the factors present in (14.2.4) remain invariant.

Let us recall that $\mathcal{P}\cup\mathcal{Q}$ is the set of colors entering on the n left edges of the column, while $\mathcal{Q}$ is the set of colors exiting via the n right edges of the column. The vector $(i_p)_{p\in\mathcal{P}\cup\mathcal{Q}}$ records the positions where colors traverse left edges, while $(j_p)_{p\in\mathcal{Q}}$ records the positions where colors traverse right edges.

First, we examine the term $\prod_{p\in\mathcal{Q}} x_{j_p}$ in (14.2.4), where the product ranges over all colors which exit via the right edges of the column. The required cyclic rotation of the exiting positions is achieved by the replacement $j_p \mapsto j_p+1 \pmod n$, which is clearly negated by the cyclic permutation $\mathfrak{s}_{n-1}\cdots\mathfrak{s}_1$, which sends $x_i \mapsto x_{i-1 \pmod n}$. Therefore this term cancels out in the ratio on the left-hand side of (14.3.1).

Similarly, it is clear from their definition that both of the exponents $f(p)$ and $h(p)$ in (14.2.5) remain unchanged under the rotation $i_p \mapsto i_p+1 \pmod n$ for $p\in\mathcal{P}\cup\mathcal{Q}$ and $j_p\mapsto j_p+1 \pmod n$ for $p\in\mathcal{Q}$. This allows us to cancel all factors involving those exponents in the ratio (14.3.1), and we then read

$$\frac{\langle b_1,\dots,b_n|\,\Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\,|d_1,\dots,d_n\rangle}{\langle b_n,b_1,\dots,b_{n-1}|\,\mathfrak{s}_{n-1}\cdots\mathfrak{s}_1\cdot\Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\,|d_n,d_1,\dots,d_{n-1}\rangle} = \frac{\prod_{p\in\mathcal{P}} t^{g(p)}\prod_{p\in\mathcal{Q}} v_p^{\mathbf{1}_{i_p>j_p}}}{\prod_{p\in\mathcal{P}} t^{\tilde{g}(p)}\prod_{p\in\mathcal{Q}} v_p^{\mathbf{1}_{\tilde{i}_p>\tilde{j}_p}}}, \tag{14.3.3}$$

where we have defined $\tilde{i}_p = i_p+1 \pmod n$, $\tilde{j}_p = j_p+1 \pmod n$ and

$$\tilde{g}(p) = \#\{\ell\in\mathcal{Q} : \ell<p, \tilde{i}_p<\tilde{j}_\ell\}. \tag{14.3.4}$$

We are able to simplify the right-hand side of (14.3.3) yet further, by noticing that it is only in the case $i_p=n$ for some $p\in\mathcal{P}\cup\mathcal{Q}$, or $j_p=n$ for some $p\in\mathcal{Q}$, where we see a discrepancy between the numerator and denominator. In those cases, one has $\tilde{i}_p=1$ or $\tilde{j}_p=1$, which can cause inequalities of the form $i_p>j_\ell$ or $i_\ell<j_p$ that previously held to now be violated. Analyzing these cases yields the final result, given by

$$\frac{\langle b_1,\dots,b_n|\,\Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\,|d_1,\dots,d_n\rangle}{\langle b_n,b_1,\dots,b_{n-1}|\,\mathfrak{s}_{n-1}\cdots\mathfrak{s}_1\cdot\Psi_{\mathbf{v};\mathbf{x}}(\mathbf{e}_{\mathcal{P}})\,|d_n,d_1,\dots,d_{n-1}\rangle} = \prod_{\substack{(p,\ell)\in\mathcal{P}\times\mathcal{Q}\\ p>\ell}} t^{\mathbf{1}_{j_\ell=n}-\mathbf{1}_{i_p=n}} \prod_{p\in\mathcal{Q}} v_p^{\mathbf{1}_{n=i_p>j_p}-\mathbf{1}_{i_p<j_p=n}},$$

where we can simplify the exponent in the second product by noting that $\mathbf{1}_{n=i_p>j_p}-\mathbf{1}_{i_p<j_p=n} = \mathbf{1}_{i_p=n}-\mathbf{1}_{j_p=n}$. □

Remark 14.3.2. In the calculations that follow, it is more useful to recast the right-hand side of (14.3.1) in terms of the color data $\mathcal{P}$, $\mathcal{Q}$ and the entries b_n, d_n, without making reference to the coordinates of Definition 14.2.3. We find that

$$\kappa_{\mathbf{v};t} = \frac{t^{\#\{p\in\mathcal{P}:p>d_n\}\mathbf{1}_{d_n\geqslant 1}}}{t^{\#\{p\in\mathcal{Q}:b_n>p\}\mathbf{1}_{b_n\in\mathcal{P}}}}\cdot\frac{(v_{b_n})^{\mathbf{1}_{b_n\in\mathcal{Q}}}}{(v_{d_n})^{\mathbf{1}_{d_n\geqslant 1}}}, \tag{14.3.5}$$

with the same $\mathcal{P}$ and $\mathcal{Q}$ as in Definition 14.2.2.

14.4. Proof of Proposition 13.4.1

We are now ready to return to the proof of Proposition 13.4.1. Fix a composition μ of length n, and set $N = \max_{i \in [1,n]} \mu_i$. For each integer $a \in [1, N]$, further fix a sequence $\mathfrak{K}^{(a)} = \big(k_1^{(a)}, \ldots, k_n^{(a)}\big)$ of indices in $[0, n]$. We introduce a function Z_{l} obtained by concatenating N column operators, as

$$
Z_{\mathrm{l}}\Big[\mathfrak{K}^{(1)}, \mathfrak{K}^{(2)}, \ldots, \mathfrak{K}^{(N)}\Big] = \begin{array}{c|c|c|c|c|c|c|c}
 & \mathbf{I}_0(\mu) & & \cdots & \cdots & & \mathbf{I}_N(\mu) & \\
\hline
i & qx_i \bullet & k_i^{(1)} & \cdots & \cdots & k_i^{(N)} \bullet & qx_i & 0 \\
\hline
n & x_n \bullet & k_n^{(1)} & \cdots & \cdots & k_n^{(N)} \bullet & x_n & 0 \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
1 & x_1 \bullet & k_1^{(1)} & \cdots & \cdots & k_1^{(N)} \bullet & x_1 & 0 \\
\hline
\end{array} \tag{14.4.1}
$$

where the indices in this picture are as follows. The colors of the arrows horizontally entering the rows from bottom to top are $1, 2, \ldots, i-1, i+1, i+2, \ldots, n, i$. In what follows, we will label each row according to the color of the arrow entering through its left edge; accordingly, the top row will continue to be called the i-th row. For each $b \in [1, n]$, the internal arrow colors in the b-th row are indexed from left to right by $\big(k_b^{(1)}, \ldots, k_b^{(N)}\big)$. These colors are fixed, not summed over.

We introduce a similar function Z_{r}, given by

$$
Z_{\mathrm{r}}\Big[\mathfrak{K}^{(1)}, \mathfrak{K}^{(2)}, \ldots, \mathfrak{K}^{(N)}\Big] = \begin{array}{c|c|c|c|c|c|c|c}
 & \mathbf{I}_0(\mu) & & \cdots & \cdots & & \mathbf{I}_N(\mu) & \\
\hline
n & x_n \bullet & k_n^{(1)} & \cdots & \cdots & k_n^{(N)} \bullet & x_n & 0 \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
1 & x_1 \bullet & k_1^{(1)} & \cdots & \cdots & k_1^{(N)} \bullet & x_1 & 0 \\
\hline
i & x_i \bullet & k_i^{(1)} & \cdots & \cdots & k_i^{(N)} \bullet & x_i & 0 \\
\hline
\end{array} \tag{14.4.2}
$$

where again the left arrow colors are ordered sequentially from bottom to top, with the exception of state i; it is omitted from $(1, \ldots, n)$ and transferred to the bottom row. This time, the bottom row is called the i-th row. As before, we assign the colors $\big(k_b^{(1)}, \ldots, k_b^{(N)}\big)$ to the internal arrows in the b-th row, for each $b \in [1, n]$.

We note that Z_{l} and Z_{r} are equivalent under application of the rotation operation of Section 14.3 to each of the $N+1$ columns that are present (up to the variable shift $qx_i \mapsto x_i$ in the i-th row). Z_{l} and Z_{r} also serve as refinements of the left and right-hand sides of (13.4.1), respectively. One sees that

$$\left\langle \mathsf{C}_i(qx_i)\mathsf{C}_n(x_n)\cdots\mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_{i-1})\cdots\mathsf{C}_1(x_1)\right\rangle_\mu = \sum Z_{\mathrm{l}}\Big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\Big], \tag{14.4.3}$$

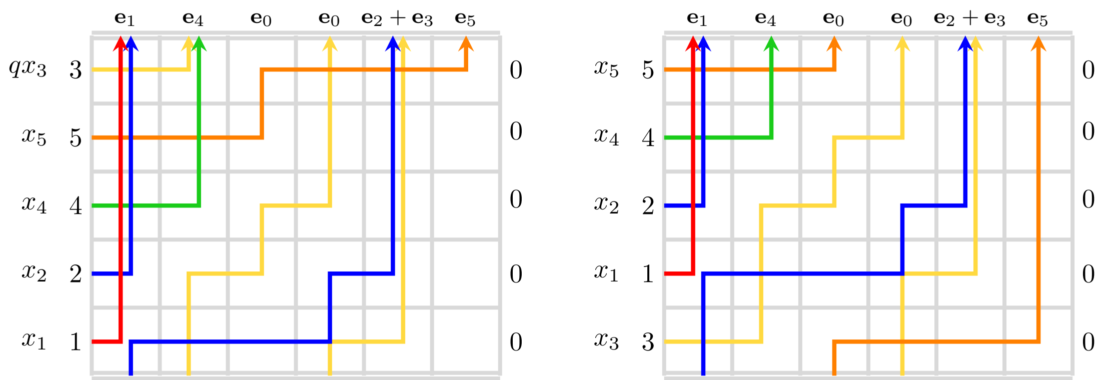
$$\left\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_{i+1}(x_{i+1})\mathsf{C}_{i-1}(x_{i-1})\cdots\mathsf{C}_1(x_1)\mathsf{C}_i(x_i)\right\rangle_\mu = \sum Z_{\mathrm{r}}\Big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\Big], \tag{14.4.4}$$

where both sums are over all length n sequences $\{\mathfrak{K}^{(a)}\}_{a\in[1,N]}$ of indices in $[0,n]$.

Example 14.4.1. Let us give an explicit example of the correspondence between (14.4.3) and (14.4.4), in the case $n=5$, $i=3$ and $\mu=(0,4,4,1,5)$. In this case we have

$$\mathbf{I}_0(\mu)=\mathbf{e}_1,\qquad \mathbf{I}_1(\mu)=\mathbf{e}_4,\qquad \mathbf{I}_2(\mu)=\mathbf{e}_0=\mathbf{I}_3(\mu),\qquad \mathbf{I}_4(\mu)=\mathbf{e}_2+\mathbf{e}_3,\qquad \mathbf{I}_5(\mu)=\mathbf{e}_5.$$

We extract a sample term from the sums (14.4.3) and (14.4.4) given by



where the configuration on the left is one term in the sum (14.4.3), while the configuration on the right is one term in the sum (14.4.4). The configuration on the right is obtained from the configuration on the left by rotating each of the $N+1$ columns; namely, by bumping each row upwards by one step, and transferring the top row to the bottom of the lattice (and then dividing the spectral parameter in the new bottommost row by q).

Proposition 14.4.2. *For each integer $a\in[1,N]$, let $\mathfrak{K}^{(a)}=\big(k_1^{(a)},k_2^{(a)},\dots,k_n^{(a)}\big)$ denote a sequence of indices in $[0,n]$, such that $Z_{\mathrm{r}}\big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\big]\neq 0$. Then $Z_{\mathrm{l}}\big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\big]$ is also non-vanishing, and we have*

$$\frac{Z_{\mathrm{l}}\Big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\Big]}{Z_{\mathrm{r}}\Big[\mathfrak{K}^{(1)},\mathfrak{K}^{(2)},\dots,\mathfrak{K}^{(N)}\Big]} = q^{\mu_i}t^{\gamma_{i,0}(\mu)}, \tag{14.4.5}$$

where the underlying parameters $v_{i,j}$ are given by (13.3.3).

Proof. The proof is based on $N+1$ applications of (14.3.1) and (14.3.5), noting that any of the $N+1$ columns present in Z_{l} differs from the corresponding column in Z_{r} by the rotation operation of Proposition 14.3.1. The ratio $Z_{\mathrm{l}}/Z_{\mathrm{r}}$ will thus be of the form $q^a t^b$ for some $a,b\in\mathbb{Z}$; let us proceed to calculate these exponents.

For the purpose of calculating the right-hand side of (14.4.5), Proposition 14.3.1 tells us that it is sufficient to focus on the internal edge states of the i-th row; in particular we will be interested in those $k_i^{(j)}$ which take non-zero values, since these contribute non-trivially to the right-hand side of

(14.3.1). As we will not need to specify $k_b^{(j)}$ for $b \neq i$, we hereafter lighten the notation by writing $k_i^{(j)} = K_j$ for all $1 \leqslant j \leqslant N$. We also set $K_0 = i$ and $K_{N+1} = 0$.

By iterating (14.3.5) over the $N+1$ columns and assigning a factor of q to each integer $1 \leqslant j \leqslant N$ for which $K_j \geqslant 1$,[1] we find that

$$\frac{Z_{\mathrm{l}}}{Z_{\mathrm{r}}} = \prod_{j=0}^{N} \frac{(v_{K_j,j})^{\mathbf{1}_{K_j \in \mathcal{Q}_j}}}{(v_{K_{j+1},j})^{\mathbf{1}_{K_{j+1} \geqslant 1}}} \cdot \frac{t^{\#\{a \in \mathcal{P}_j : a > K_{j+1}\} \mathbf{1}_{K_{j+1} \geqslant 1}}}{t^{\#\{a \in \mathcal{Q}_j : a < K_j\} \mathbf{1}_{K_j \in \mathcal{P}_j}}} \cdot \prod_{j=1}^{N} q^{\mathbf{1}_{K_j \geqslant 1}}, \tag{14.4.6}$$

where $\mathcal{P}_j$, $\mathcal{Q}_j$ are the color data associated to the j-th column of Z_{l}. Since $\mathcal{P}_j$ is the set of colors which exit column j via its top edge and $\mathcal{Q}_j$ is the set of colors exiting via columns $j+1, \dots, N$, we can write

$$\mathcal{P}_j = \{a : \mu_a = j\}, \qquad \mathcal{Q}_j = \{a : \mu_a > j\}. \tag{14.4.7}$$

The right-hand side of (14.4.6) becomes

$$\frac{Z_{\mathrm{l}}}{Z_{\mathrm{r}}} = \frac{(v_{i,0})^{\mathbf{1}_{i \in \mathcal{Q}_0}}}{t^{\#\{a < i : \mu_a > 0\} \mathbf{1}_{i \in \mathcal{P}_0}}} \prod_{j=1}^{N} \frac{(v_{K_j,j})^{\mathbf{1}_{K_j \in \mathcal{Q}_j}}}{(v_{K_j,j-1})^{\mathbf{1}_{K_j \geqslant 1}}} \cdot \frac{t^{\#\{a > K_j : \mu_a = j-1\} \mathbf{1}_{K_j \geqslant 1}}}{t^{\#\{a < K_j : \mu_a > j\} \mathbf{1}_{K_j \in \mathcal{P}_j}}} \cdot q^{\mathbf{1}_{K_j \geqslant 1}}, \tag{14.4.8}$$

where we have used the fact that $K_0 = i$ and $K_{N+1} = 0$ to redistribute the factors in the product.

Now let us compute the j-th term in the product on the right-hand side of (14.4.8). It is clearly sufficient to restrict our attention to j such that $K_j \geqslant 1$, since for each j such that $K_j = 0$ the factors inside the product are all equal to 1; we tacitly assume $K_j \geqslant 1$ in what follows. Invoking (for the first time) the explicit form of the parameters (13.3.3), we find that

$$\frac{(v_{K_j,j})^{\mathbf{1}_{K_j \in \mathcal{Q}_j}}}{(v_{K_j,j-1})^{\mathbf{1}_{K_j \geqslant 1}}} = \begin{cases} q^{-1} t^{\gamma_{K_j,j}(\mu) - \gamma_{K_j,j-1}(\mu)}, & \mu_{K_j} > j, \\ \\ q^{-1} t^{-\gamma_{K_j,j-1}(\mu)}, & \mu_{K_j} = j, \end{cases} \tag{14.4.9}$$

where the case $\mu_{K_j} < j$ never appears (it would mean that the color K_j has traversed beyond column μ_{K_j}, which is forbidden). From (13.3.4), we can write down the t exponents appearing in (14.4.9). In the case $\mu_{K_j} > j$ (equivalent to $K_j \in \mathcal{Q}_j$), one has

$$\begin{aligned} \gamma_{K_j,j}(\mu) - \gamma_{K_j,j-1}(\mu) &= \#\{a > K_j : j \leqslant \mu_a < \mu_{K_j}\} - \#\{a > K_j : j-1 \leqslant \mu_a < \mu_{K_j}\} \\ &= -\#\{a > K_j : j-1 = \mu_a\}. \end{aligned}$$

Similarly, for $\mu_{K_j} = j$ (equivalent to $K_j \in \mathcal{P}_j$), we find that

$$\begin{aligned} -\gamma_{K_j,j-1}(\mu) &= \#\{a < K_j : \mu_a > \mu_{K_j}\} - \#\{a > K_j : j-1 \leqslant \mu_a < \mu_{K_j}\} \\ &= \#\{a < K_j : \mu_a > j\} - \#\{a > K_j : j-1 = \mu_a\}. \end{aligned}$$

Using these facts in (14.4.9), we read

$$\frac{(v_{K_j,j})^{\mathbf{1}_{K_j \in \mathcal{Q}_j}}}{(v_{K_j,j-1})^{\mathbf{1}_{K_j \geqslant 1}}} = q^{-1} \cdot \begin{cases} t^{-\#\{a > K_j : \mu_a = j-1\}}, & K_j \in \mathcal{Q}_j, \\ \\ t^{\#\{a < K_j : \mu_a > j\} - \#\{a > K_j : \mu_a = j-1\}}, & K_j \in \mathcal{P}_j. \end{cases} \tag{14.4.10}$$

---

[1] We obtain a power of q for every horizontal step by a path in the i-th row of Z_{l}, due to the q-shifted argument of $\mathcal{C}_i(qx_i)$ and Remark 13.2.2.

We can now see that in either case, the t exponents cancel perfectly with the remaining factors in the j-th term of the product (14.4.8), and hence

$$\frac{(v_{K_j,j})^{\mathbf{1}_{K_j \in \mathcal{Q}_j}}}{(v_{K_j,j-1})^{\mathbf{1}_{K_j \geqslant 1}}} \cdot \frac{t^{\#\{a>K_j:\mu_a=j-1\}\mathbf{1}_{K_j\geqslant 1}}}{t^{\#\{a<K_j:\mu_a>j\}\mathbf{1}_{K_j\in\mathcal{P}_j}}} = q^{-\mathbf{1}_{K_j\geqslant 1}}.$$

Returning to the expression (14.4.8), we have shown that

$$\frac{Z_{\mathrm{l}}}{Z_{\mathrm{r}}} = \frac{(v_{i,0})^{\mathbf{1}_{i\in\mathcal{Q}_0}}}{t^{\#\{a<i:\mu_a>0\}\mathbf{1}_{i\in\mathcal{P}_0}}}, \tag{14.4.11}$$

which expresses the remarkable fact that the ratio $Z_{\mathrm{l}}/Z_{\mathrm{r}}$ does not depend on any of the values of the colors $k_b^{(j)}$, not even those for which $b = i$. Finally, we check that

$$\frac{(v_{i,0})^{\mathbf{1}_{i\in\mathcal{Q}_0}}}{t^{\#\{a<i:\mu_a>0\}\mathbf{1}_{i\in\mathcal{P}_0}}} = \begin{cases} v_{i,0} = q^{\mu_i} t^{\gamma_{i,0}(\mu)}, & i \in \mathcal{Q}_0, \\[2ex] t^{-\#\{a<i:\mu_a>0\}} = q^{\mu_i} t^{\gamma_{i,0}(\mu)}, & i \in \mathcal{P}_0, \end{cases} \tag{14.4.12}$$

where in the latter case, we have noted that $i \in \mathcal{P}_0$ implies $\mu_i = 0$. $\square$

Now we can quickly establish Proposition 13.4.1.

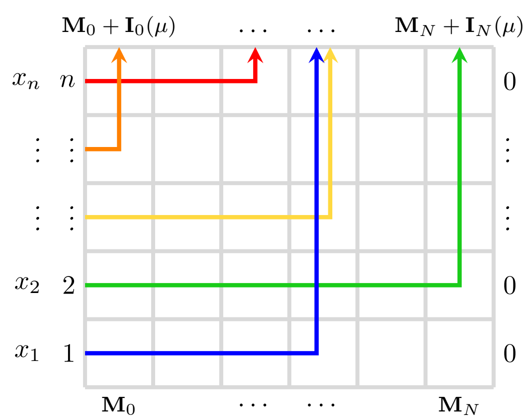
Proof of Proposition 13.4.1. Since we have a one-to-one pairing of each non-vanishing term in the sum (14.4.3) with a corresponding term in (14.4.4), and we have also shown that the two terms have the correct proportionality constant (14.4.5) irrespective of the values of the internal states $k_b^{(j)}$, (13.4.1) follows immediately. $\square$

14.5. Proof of Proposition 13.4.2

In this section we establish Proposition 13.4.2. To that end, we make the following claim.

Proposition 14.5.1. *The quantity* $\mathrm{Coeff}[\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\rangle_\mu; \mathbf{x}^\mu]$ *is generically non-vanishing and is given by the weight of the unique lattice configuration of the form*

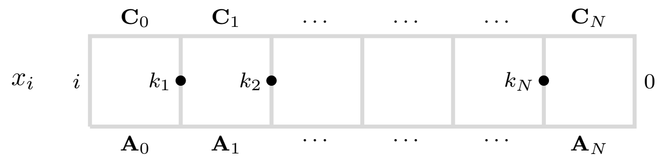
(14.5.1) $$\sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=0}^{N} v_{i,j}^{M_{i,j}} \times$$



in which path i travels straight for μ_i consecutive horizontal steps, before turning and exiting at the top of the μ_i-th column, for all $1 \leqslant i \leqslant n$.

Proof. In what follows let $N = \max_{i \in [1,n]} \mu_i$. Let i be the largest integer such that $\mu_i = N$ (there may be other integers j such that $\mu_j = N$, and we assume that $i > j$ for all such j). Then there is no other path which travels further, horizontally, than the path of color i does in order to reach its final destination in column N.

Given that we wish to calculate $\mathrm{Coeff}[\langle \mathsf{C}_n(x_n) \cdots \mathsf{C}_1(x_1) \rangle_\mu; \mathbf{x}^\mu]$, we restrict our attention to lattice configurations which give rise to a factor of $x_i^{\mu_i} = x_i^N$. We claim that the only possible way in which this factor is obtained is when the path of color i travels horizontally for N consecutive steps before turning into the N-th column. In other words, isolating the i-th row of the lattice,



and labelling its internal edges as $k_1, \dots, k_N$ as shown above, we claim that the only possible contribution to x_i^N comes in the case $k_1 = \cdots = k_N = i$. To prove this, we will assume that some other lattice configuration exists that gives rise to the factor x_i^N, in which not all $k_1, \dots, k_N$ are equal to i, and show that we obtain a contradiction.

To begin, we note that all $k_1, \dots, k_N$ must be nonzero, or else we cannot recover the required degree in x_i. Assume that for some integer p_1 one has a lattice configuration in which $k_{p_1} = i$ and $k_{p_1+1} = j_1$, where $i < j_1$ (note that one cannot have $i > j_1 \geqslant 1$, since the corresponding vertex weight in Definition 13.2.1 vanishes). Now the path of color j_1 must turn out of the i-th row somewhere before the N-th column, since by assumption i is the largest color such that $\mu_i = N$. Let the column where this turning happens be labelled p_2. Then we must have $k_{p_2} = j_1$ and $k_{p_2+1} = j_2$, where $j_1 < j_2$. We again find that the path of color j_2 must turn out of the i-th row somewhere before the N-th column. One may now iterate this reasoning, ultimately arriving at the contradiction that k_N will be forced to assume a value greater than i, which is impossible.

We have thus proved the required statement about the path of color i, completely constraining its motion to the i-th row, and it plays no role in configurations of the remaining rows. Then let us define $\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_{n-1}) = (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n)$, *i.e.*, we omit the i-th part from μ. One may now apply a similar reasoning to the largest integer j such that $\mu_j = \max_{i \in [1,n-1]} \hat{\mu}_i$, arriving at the same conclusion for the path of color j; namely, it is forced to take $\max_{i \in [1,n-1]} \hat{\mu}_i$ consecutive horizontal steps before turning into the μ_j-th column, where it terminates. By iterating this argument over each of the colors, we arrive at the statement of the proposition. $\square$

Now we can establish Proposition 13.4.2.

Proof of Proposition 13.4.2. Let us compute the weight of the configuration shown in (14.5.1). To do this, we use the formula (14.2.4) to calculate the weight of each of the columns in (14.5.1), and multiply them together. Each of the columns that appear in the frozen configuration (14.5.1) take a much simpler form than the generic column components (14.2.4); we denote the weight of the j-th column by X_j, and remark that it is a function only of the colors $\{a : \mu_a = j\}$ which exit via its top and those $\{b : \mu_b > j\}$ which horizontally traverse it. We are able to write

$$\mathrm{Coeff}\Big[\big\langle \mathsf{C}_n(x_n) \cdots \mathsf{C}_1(x_1) \big\rangle_\mu; \mathbf{x}^\mu\Big] = \prod_{j=0}^{N} X_j(\mathcal{P}_j; \mathcal{Q}_j), \tag{14.5.2}$$

with $\mathcal{P}_j$ and $\mathcal{Q}_j$ as given by (14.4.7), and where we have defined the function

$$X_j(\mathcal{P};\mathcal{Q}) = \prod_{p\in\mathcal{Q}}(1 - v_{p,j}t^{f(p)+1}) = \prod_{p\in\mathcal{Q}}(1 - v_{p,j}t^{\#\{\ell\in\mathcal{Q}:\ell<p\}+1}). \tag{14.5.3}$$

Rewriting the product (14.5.3) more explicitly, we have

$$\begin{aligned} \mathrm{Coeff}\Big[\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu;\mathbf{x}^\mu\Big] &= \prod_{j=0}^{N}\prod_{i:\mu_i>j}(1 - v_{i,j}t^{\#\{\ell<i:j<\mu_\ell\}+1}) \\ &= \prod_{i=1}^{n}\prod_{j=0}^{\mu_i-1}(1 - v_{i,j}t^{\#\{\ell<i:j<\mu_\ell\}+1}). \end{aligned} \tag{14.5.4}$$

To conclude our calculation, we recall the explicit form (13.3.3) of the parameters $v_{i,j}$. They contribute a factor of $t^{\gamma_{i,j}(\mu)}$ with $\gamma_{i,j}(\mu)$ given by (13.3.4), which can be combined with the $t^{\#\{\ell<i:j<\mu_\ell\}}$ term appearing in (14.5.4). Collecting the exponents proceeds as follows:

$$\begin{aligned} \#\{\ell<i:j<\mu_\ell\}+\gamma_{i,j}(\mu) &= \#\{\ell<i:j<\mu_\ell\} - \#\{k<i:\mu_k>\mu_i\} + \#\{k>i:j\leqslant\mu_k<\mu_i\} \\ &= \#\{k<i:j<\mu_k\leqslant\mu_i\} + \#\{k>i:j\leqslant\mu_k<\mu_i\} \\ &= \#\{k<i:\mu_k=\mu_i\} + \#\{k\neq i:j<\mu_k<\mu_i\} + \#\{k>i:j=\mu_k\} \\ &= \alpha_{i,j}(\mu), \end{aligned}$$

with $\alpha_{i,j}(\mu)$ given by (13.3.6). Putting everything together, we have shown that

$$\mathrm{Coeff}\Big[\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu;\mathbf{x}^\mu\Big] = \prod_{i=1}^{n}\prod_{j=0}^{\mu_i-1}(1 - q^{\mu_i-j}t^{\alpha_{i,j}(\mu)+1}) = \frac{1}{\Omega_\mu(q,t)}, \tag{14.5.5}$$

with $\Omega_\mu(q,t)$ given by (13.3.5). □

CHAPTER 15

Vertex Models for Symmetric Macdonald Polynomials

In this chapter we describe a symmetrization procedure which yields the symmetric Macdonald polynomials, in their integral form. Throughout this section we fix complex numbers $q, t \in \mathbb{C}$ and adopt the diagrammatic conventions introduced in Section 14.1.

15.1. Partition Function for Integral Macdonald Polynomials

In this section we first provide an expression for the integral, symmetric Macdonald polynomials in terms of the $\mathsf{D}(x)$ operators from Definition 13.2.4, which implies a decomposition of these integral Macdonald polynomials into sums of the $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ functions from Definition 7.1.1. To define the former, for any signature $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathrm{Sign}$ and sequence $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ of complex numbers, recall the Macdonald polynomial $P_\lambda(\mathbf{x}) = P_\lambda(\mathbf{x}; q, t)$ from (6.4.7) of **[66]**. We further recall from equation (6.8.3) of **[66]** the *integral, symmetric Macdonald polynomial* $J_\lambda(\mathbf{x}) = J_\lambda(\mathbf{x}; q, t)$, defined by setting

$$J_\lambda(\mathbf{x}) = c_\lambda(q,t) P_\lambda(x), \qquad \text{where} \quad c_\lambda(q,t) = \prod_{b \in \mathcal{Y}(\lambda)} (1 - q^{a(b)} t^{l(b)+1}), \tag{15.1.1}$$

and the product is over all boxes b in the Young diagram $\mathcal{Y}(\lambda)$ for λ. Here, $a(b)$ and $l(b)$ denote the *arm* and *leg* lengths of $b \in \mathcal{Y}(\lambda)$, given by the number of boxes in $\mathcal{Y}(\lambda)$ east and south of b, respectively; see equation (6.6.14) of **[66]**.

Now let us define the following polynomial, which we will eventually show to coincide with the integral Macdonald polynomial, in terms of the $\mathsf{D}(x)$ operators. In what follows, we recall the sequence $\mathcal{I}(\nu) = \big(\mathbf{I}_0(\nu), \mathbf{I}_1(\nu), \ldots\big)$ of elements in $\{0,1\}^n$ from Section 13.3, for any composition ν of length at most n.

Definition 15.1.1. Fix an integer $p \in [1, n]$; a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of complex numbers; and a (positive) composition $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ of length p. Denoting $\nu_k = 0$ for $k > p$ and letting $N = \max_{i \in [1,p]} \nu_i$, define the set of twist parameters $\mathbf{w} = (w_{i,j})$, where (i, j) ranges over all pairs of integers $(i, j) \in [1, n] \times [1, N]$, by setting

$$w_{i,j} = q^{\nu_i - j} \mathbf{1}_{\nu_i > j}. \tag{15.1.2}$$

Then, define

$$\mathcal{P}_\nu(\mathbf{x}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{N} w_{i,j}^{M_{i,j}} \Big\langle \mathcal{M} + \mathcal{I}(\nu) \big| \mathsf{D}(x_n) \cdots \mathsf{D}(x_1) \big| \mathcal{M} + \big(\mathbf{e}_{[1,p]}, \mathbf{e}_0, \mathbf{e}_0, \ldots\big) \Big\rangle, \tag{15.1.3}$$

where we sum over all sequences $\mathcal{M} = (\mathbf{M}_0, \mathbf{M}_1, \ldots)$ of elements in $\{0,1\}^n$ such that $\mathbf{M}_k = \mathbf{e}_0$ for $k = 0$ and each $k > N$; we also set $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j}) \in \{0,1\}^n$ for each $j \in [1, N]$, and $\mathbf{e}_{[1,p]} = (1^p, 0^{n-p}) \in \{0,1\}^n$ to be the vector whose first p coordinates are equal to 1 and whose last $n - p$ coordinates are equal to 0.

In particular, $\mathcal{P}_\nu(\mathbf{x})$ is diagrammatically given by the partition function

$$\mathcal{P}_\nu(\mathbf{x}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{N} w_{i,j}^{M_{i,j}} \times \begin{array}{c|c|c|c|c|c|c|c}
 & \mathbf{e}_0 & \mathbf{M}_1 + \mathbf{I}_1 & \cdots & \cdots & & \mathbf{M}_N + \mathbf{I}_N & \\
\hline
0 & x_n & & \cdots & \cdots & & x_n & 0 \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
0 & x_2 & & \cdots & \cdots & & x_2 & 0 \\
\hline
0 & x_1 & & \cdots & \cdots & & x_1 & 0 \\
\hline
 & \mathbf{e}_{[1,p]} & \mathbf{M}_1 & \cdots & \cdots & & \mathbf{M}_N &
\end{array} \tag{15.1.4}$$

where we have abbreviated $\mathbf{I}_j = \mathbf{I}_j(\nu)$ for each $j \geqslant 1$. Here, colors $1, 2, \ldots, p$ all enter at the base of the leftmost column, and this column does not have cylindrical boundary conditions (while the others do).

Now we can state the following theorem, equating the integral, symmetric Macdonald polynomial $J_\lambda(\mathbf{x})$ with the partition function $\mathcal{P}_\lambda(\mathbf{x})$ from Definition 15.1.1. We will establish this result in Section 15.3 below.

THEOREM 15.1.2. *Fix an integer $p \in [1, n]$; a composition $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ of length p that is anti-dominant, namely, $1 \leqslant \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_p$; and a set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of complex numbers. Then, $J_{\nu^+}(\mathbf{x}) = \mathcal{P}_\nu(\mathbf{x})$, where we recall that ν^+ denotes the dominant ordering of ν.*

Remark 15.1.3. Observe by (13.2.1) that the $\big\langle \mathcal{M} + \mathcal{I}(\nu) \big| \mathsf{D}(x_n) \cdots \mathsf{D}(x_1) \big| \mathcal{M} + \big(\mathbf{e}_{[1,p]}, \mathbf{e}_0, \mathbf{e}_0, \ldots\big) \big\rangle$ appearing in (15.1.3) are specializations of the functions $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ from[1] Definition 7.1.1, whose r and s parameters are all equal to $t^{-1/2}$ and 0, respectively (and whose quantization parameter is equal to t). In particular, Theorem 15.1.2 decomposes the integral Macdonald polynomial $J_{\nu^+}(\mathbf{x})$ as a linear combination of such functions.

Before proceeding to the proof of Theorem 15.1.2, let us explain how it can be used to recover an expression from **[37]** for the modified Macdonald polynomials as a linear combination of skew LLT polynomials. To define the former, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$ and sequence of complex variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, let the *modified Macdonald polynomial* $\widetilde{J}_\lambda(\mathbf{x})$ be

$$\widetilde{J}_\lambda(\mathbf{x}) = J_\lambda\left(\bigcup_{j=0}^{\infty} t^j \mathbf{x} \right). \tag{15.1.5}$$

Equivalently, (see, for instance, equation (4) of **[37]**) it is given by the plethystic substitution $\widetilde{J}_\lambda[X] = J_\lambda\big[(1-t)^{-1} X\big]$, where X denotes the formal sum $X = \sum_{i=1}^{N} x_i$. In the below, we recall the sequence $\mathcal{S}(\boldsymbol{\lambda})$ of elements in $\{0,1\}^n$ for any $\boldsymbol{\lambda} \in \{0,1\}^n$ from Section 7.1.

[1]Here, the constituent signatures in $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$ and in $\boldsymbol{\mu} \in \mathrm{SeqSign}_n$ have different lengths (see Remark 7.1.2)

Corollary 15.1.4 ([**37**, Theorem 2.2, Equation (23), and Proposition 3.4])**.** *Fix integers $p, n, N \geqslant 1$ with $p \leqslant n$; an anti-dominant composition ν of length p; and a sequence of complex numbers $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. For each $1 \leqslant i \leqslant n$ and $j \geqslant 1$, define*

$$u_{i,j} = u_{i,j}(\nu) = q^{\nu_i - j} t^{\beta_{i,j}(\nu)} \mathbf{1}_{\nu_i > j},$$

where $\nu_k = 0$ for $k > p$, and the exponents $\beta_{i,j}(\nu)$ are given by

$$\beta_{i,j}(\nu) = \frac{1}{2}\Big(\#\{k < i : \nu_k > j\} - p + i \Big).$$

For any infinite sequence $\mathcal{K} = (\mathbf{K}_1, \mathbf{K}_2, \ldots)$ of elements in $\{0,1\}^n$, define $\boldsymbol{\mu}(\mathcal{K}) \in \mathrm{SeqSign}_n$ so that $\mathcal{S}\big(\boldsymbol{\mu}(\mathcal{K})\big) = \mathcal{K}$. Then,

$$\widetilde{J}_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} \mathcal{L}_{\boldsymbol{\mu}(\mathcal{I}_0(\nu)+\mathcal{M}_0)/\boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{x}; q) \prod_{i=1}^{n} \prod_{j=1}^{\infty} u_{i,j}^{M_{i,j}}, \tag{15.1.6}$$

where the sum is over all infinite sequences $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots)$ of elements in $\{0,1\}^n$, with $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$, such that $\mathbf{I}_k(\nu) + \mathbf{M}_k \in \{0,1\}^n$ for each $k \geqslant 1$. Here, we have denoted $\mathcal{M}_0 = (\mathbf{e}_0, \mathbf{M}_1, \mathbf{M}_2, \ldots)$; $\mathcal{I}_0(\nu) = \big(\mathbf{e}_0, \mathbf{I}_1, \mathbf{I}_2, \ldots\big)$; and $\mathcal{M}_p = \big(\mathbf{e}_{[1,p]}, \mathbf{M}_1, \mathbf{M}_2, \ldots\big)$.

Proof. By (15.1.3) and Theorem 15.1.2, we have for any sequence of complex variables $\mathbf{y} = (y_1, y_2, \ldots, y_K)$ that

$$J_{\nu^+}(\mathbf{y}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{\infty} w_{i,j}^{M_{i,j}} \big\langle \mathcal{M}_0 + \mathcal{I}_0(\nu) \big| \mathsf{D}(y_K) \cdots \mathsf{D}(y_1) \big| \mathcal{M}_p \big\rangle,$$

where the sum is as in (15.1.6), and we have used the fact that $w_{i,j} = 0$ for $j > \nu_p$. Moreover, by Definition 7.1.1, (13.2.1), Definition 13.2.3, Definition 13.2.4, and the last statement of (8.4.3) we have

$$\mathcal{G}_{\boldsymbol{\mu}(\mathcal{M}_0+\mathcal{I}_0(\nu))/\boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{y}; q^{-1/2} \mid 0; 0) = \big\langle \mathcal{M}_0 + \mathcal{I}_0(\nu) \big| \mathsf{D}(y_K) \cdots \mathsf{D}(y_1) \big| \mathcal{M}_p \big\rangle,$$

and so

$$J_{\nu^+}(\mathbf{y}) = \sum_{\mathcal{M}} \mathcal{G}_{\boldsymbol{\mu}(\mathcal{M}_0+\mathcal{I}_0(\nu))/\boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{y}; q^{-1/2} \mid 0; 0) \prod_{i=1}^{n} \prod_{j=1}^{\infty} w_{i,j}^{M_{i,j}}. \tag{15.1.7}$$

Taking $\mathbf{y} = \bigcup_{j=1}^{N} \{x_j, q x_j, \ldots\}$ in (15.1.7), and then appyling (15.1.5) and (the $L_i = \infty$ limiting case of) Proposition 7.2.3, yields

$$\widetilde{J}_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} \mathcal{G}_{\boldsymbol{\mu}(\mathcal{M}_0+\mathcal{I}_0(\nu))/\boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{x}; \infty \mid 0; 0) \prod_{i=1}^{n} \prod_{j=1}^{\infty} w_{i,j}^{M_{i,j}}. \tag{15.1.8}$$

By the first statement Theorem 9.3.2 (and recalling the function ψ from (9.3.1)), it follows that

$$\widetilde{J}_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} q^{\psi(\boldsymbol{\mu}(\mathcal{M}_0+\mathcal{I}_0(\nu))) - \psi(\boldsymbol{\mu}(\mathcal{M}_p))} \mathcal{L}_{\boldsymbol{\mu}(\mathcal{M}_0+\mathcal{I}_0(\nu))/\boldsymbol{\mu}(\mathcal{M}_p)}(\mathbf{x}; q) \prod_{i=1}^{n} \prod_{j=1}^{\infty} w_{i,j}^{M_{i,j}}. \tag{15.1.9}$$

Let us analyze the exponent of q appearing on the right side of (15.1.9), assuming that $M_{i,j} = 0$ whenever $i > p$ (which we may do, since $w_{i,j} = 0$ whenever $i > p$). Since ν is anti-dominant, we have from (9.3.1) that

$$\psi\Big(\boldsymbol{\mu}\big(\mathcal{I}_0(\nu)+\mathcal{M}_0\big)\Big) = \frac{1}{2}\sum_{1\leqslant i<h\leqslant n}\sum_{j=\nu_h+1}^{\infty} M_{i,j} + \frac{1}{2}\sum_{1\leqslant h<i\leqslant n}\sum_{j=1}^{\nu_h-1} M_{i,j} + \frac{1}{2}\sum_{1\leqslant h<i\leqslant n}\sum_{1\leqslant j<k} M_{h,k}M_{i,j};$$

$$\psi\big(\boldsymbol{\mu}(\mathcal{M}_p)\big) = \frac{1}{2}\sum_{1\leqslant h<i\leqslant n}\sum_{1\leqslant j<k} M_{h,k}M_{i,j} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{\infty}(p-i)M_{i,j},$$

meaning that

(15.1.10)

$$\psi\Big(\boldsymbol{\mu}\big(\mathcal{I}_0(\nu)+\mathcal{M}_0\big)\Big) - \psi\big(\boldsymbol{\mu}(\mathcal{M}_p)\big) = \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{\infty} M_{i,j}\Big(\#\{h>i:\nu_h<j\} + \#\{h<i:\nu_h>j\} - p + i\Big).$$

Since ν is anti-dominant, there exists some $k>i$ with $\nu_k<j$ only if $\nu_i<j$. This would imply $w_{i,j}=0$, in which case the (i,j) summand on the right side of (15.1.9) only contributes if $M_{i,j}=0$. Hence, we may assume in what follows that $M_{i,j}=0$ whenever there exists some $k>i$ with $\nu_k<j$; inserting this into (15.1.10) yields

$$\psi\Big(\boldsymbol{\mu}\big(\mathcal{I}_0(\nu)+\mathcal{M}_0\big)\Big) - \psi\big(\boldsymbol{\mu}(\mathcal{M}_p)\big) = \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{\infty} M_{i,j}\Big(\#\{k<i:\nu_k>j\} - p + i\Big) = \sum_{i=1}^{p}\sum_{j=1}^{\infty}\beta_{i,j},$$

which yields the corollary upon insertion into (15.1.9), since $u_{i,j}=q^{\beta_{i,j}}w_{i,j}$. □

15.2. Identifying the Macdonald Polynomial

In this section we show that $\mathcal{P}_\nu(\mathbf{x})$ coincides with a symmetric Macdonald polynomial, up to a constant factor that we determine in Section 15.3 below. We first quickly show that it is symmetric in $\mathbf{x}$, through the following lemma.

Lemma 15.2.1. *The polynomial $\mathcal{P}_\nu(\mathbf{x})$ is symmetric in $\mathbf{x}=(x_1,x_2,\ldots,x_n)$.*

Proof. By (15.1.3), $\mathcal{P}_\nu(\mathbf{x})$ is given by an appropriate linear form acting on $\mathsf{D}(x_1)\ldots\mathsf{D}(x_n)$. Since the $\mathsf{D}(x_i)$ operators commute by (13.2.5), the symmetry is immediate. □

We next show that any $\mathcal{P}_\nu(\mathbf{x})$ is in the span of all nonsymmetric Macdonald polynomials $f_\mu(\mathbf{x})$ with $\mu^+=\nu^+$. To that end, we begin by providing an alternative expression for the latter nonsymmetric polynomials involving both C_i and D operators. This will be given by Corollary 15.2.3 below, to establish which we first show the following proposition that essentially "separates out" the leftmost column from the remaining part of the $\mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)$ type partition function for this nonsymmetric polynomial f_μ from Theorem 13.3.2. In what follows, we let μ be a composition formed by concatenating p non-zero parts and $n-p$ zeros, namely,

(15.2.1) $$\mu=(\nu_1,\ldots,\nu_p,0^{n-p}),\qquad \text{with } \nu_i\geqslant 1, \text{ for all } i\in[1,p].$$

Proposition 15.2.2. *Let $\mu=(\mu_1,\ldots,\mu_n)$ and $\nu=(\nu_1,\ldots,\nu_p)$ be two compositions related as in* (15.2.1)*, and denote the largest part of μ by $N=\max_{i\in[1,n]}\mu_i=\max_{i\in[1,p]}\nu_i$. Fix a set $\mathbf{v}=(v_{i,j})$ of complex numbers, where (i,j) ranges over all integer pairs $(i,j)\in[1,n]\times[0,N]$, such that $v_{i,j}=0$ for any $(i,j)\in[p+1,n]\times[0,N]$. Then, we have*

(15.2.2) $$\big\langle \mathsf{C}_n(x_n)\cdots\mathsf{C}_1(x_1)\big\rangle_\mu(\mathbf{v}) = \prod_{i=1}^{p}(1-v_{i,0}t^i)x_i\cdot\big\langle \mathsf{D}(x_n)\cdots\mathsf{D}(x_{p+1})\mathsf{C}_p(x_p)\cdots\mathsf{C}_1(x_1)\big\rangle'_\nu(\mathbf{v}),$$

with the two linear forms are given by

$$\big\langle \mathsf{X} \big\rangle_{\mu}(\mathbf{v}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=0}^{N} v_{i,j}^{M_{i,j}} \big\langle \mathcal{M} + \mathcal{I}(\mu) \big| \mathsf{X} \big| \mathcal{M} \big\rangle; \qquad \big\langle \mathsf{X} \big\rangle'_{\nu}(\mathbf{v}) = \sum_{\mathcal{M}} \prod_{i=1}^{n} \prod_{j=1}^{N} v_{i,j}^{M_{i,j}} \big\langle \mathcal{M} + \mathcal{I}(\nu) \big|' \mathsf{X} \big| \mathcal{M} \big\rangle',$$

where the first and second sums are over all sequences of $\{0,1\}^n$ *elements* $\mathcal{M} = (\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_N)$ *and* $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_N)$*, respectively, setting* $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j}) \in \{0,1\}^n$ *for each* j*. Here, the primed states have the same definition as in Section 13.3, but with all tensor products starting at* $j = 1$.

Proof. This is a consequence of expanding the partition function on the left hand side of (15.2.2) with respect to its zeroth column. This column is pictured as

$$(15.2.3) \qquad \sum_{\mathbf{M}_0} \prod_{k=1}^{n} v_{k,0}^{M_{k,0}} \quad \begin{array}{r|c|l} & \mathbf{M}_0 + (0^p, 1^{n-p}) & \\ \hline n & x_n & d_n \\ \vdots & \vdots & \vdots \\ \hline p+1 & x_{p+1} & d_{p+1} \\ \hline p & x_p & d_p \\ \vdots & \vdots & \vdots \\ \hline 2 & x_2 & d_2 \\ \hline 1 & x_1 & d_1 \\ \hline & \mathbf{M}_0 & \end{array}$$

where $(d_1, d_2, \ldots, d_n) \in \{0, 1, \ldots, p\}^n$. One can see that there is only one possible choice of $(d_1, d_2, \ldots, d_n)$ which yields a non-vanishing result; namely, $(d_1, d_2, \ldots, d_n) = (1, \ldots, p, 0^{n-p})$. Indeed, in view of the vanishing of the fifth weight in Figure 13.1 (that is, $L_x(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i) = 0$ for $1 \leqslant i < j \leqslant n$) we see that necessarily $d_{p+1} = \cdots = d_n = 0$ in (15.2.3). This constrains $(d_1, d_2, \ldots, d_p)$ to be a permutation of $(1, 2, \ldots, p)$, and the only permutation which does not lead to vertices of the form $(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i)$ for $1 \leqslant i < j \leqslant n$ is precisely $(d_1, d_2, \ldots, d_p) = (1, 2, \ldots, p)$.

Using the third and last two weights in Figure 13.1, the weight of the column is then readily computed as

$$\sum \prod_{i=1}^{p} (-v_{i,0} t^i)^{M_{i,0}} x_i = \prod_{i=1}^{p} (1 - v_{i,0} t^i) x_i,$$

where the left side is summed over all $(M_{1,0}, M_{2,0}, \ldots, M_{p,0}) \in \{0,1\}^p$. We now recognize this as the formula on the right hand side of (15.2.2). $\square$

Corollary 15.2.3. *Let* $\mu = (\mu_1, \ldots, \mu_n)$ *and* $\nu = (\nu_1, \ldots, \nu_p)$ *be compositions which satisfy* (15.2.1)*, with* ν *anti-dominant. The nonsymmetric Macdonald polynomial* $f_\mu(\mathbf{x})$ *is given by*

$$(15.2.4) \qquad f_\mu(\mathbf{x}) = \prod_{i=1}^{p} (1 - t^i w_{i,0}) x_i \cdot \Omega_\nu(q,t) \big\langle \mathsf{D}(x_n) \cdots \mathsf{D}(x_{p+1}) \mathsf{C}_p(x_p) \cdots \mathsf{C}_1(x_1) \big\rangle'_{\nu}(\mathbf{w})$$

where the parameters $\mathbf{w} = (w_{i,j})$ *for* $(i,j) \in [1,p] \times [1,N]$ *are chosen to be as in* (15.1.2)*. The normalization factor appearing on the right side of* (15.2.4) *is given by* (13.3.5) *and* (13.3.6)*.*

Proof. This follows from our previous formula (13.3.2), combined with (15.2.2), together with the fact that the exponent $\gamma_{i,j}$ from (13.3.4) is equal to 0 since ν anti-dominant. $\square$

Now we can establish the following proposition, indicating that $\mathcal{P}_\nu(\mathbf{x})$ is in the linear span of certain nonsymmetric Macdonald polynomials.

Proposition 15.2.4. *For each partition $\lambda = \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, let $\mathcal{V}_\lambda = \mathrm{Span}_{\mathbb{C}}\{f_\mu(\mathbf{x})\}_{\mu^+=\lambda}$. For any anti-dominant composition $\nu = (\nu_1, \ldots, \nu_p)$ such that $\nu_i \geqslant 1$ for each $i \in [1,p]$, we have $\mathcal{P}_\nu(\mathbf{x}) \in \mathcal{V}_\kappa$, where $\kappa = (\nu^+, 0^{n-p})$ is the partition obtained by sorting parts into dominant order, and then appending $n-p$ zeros.*

Proof. The proof proceeds by expanding over the configurations of the leftmost column in (15.1.4). Performing that expansion, we obtain

$$\mathcal{P}_\nu(x_1, \ldots, x_n) = (1-t)^p \sum_{\mathfrak{D}} t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})} \prod_{i=1}^{n} x_i^{\mathbf{1}_{d_i \geqslant 1}} \cdot \big\langle \mathsf{C}_{d_n}(x_n) \cdots \mathsf{C}_{d_1}(x_1) \big\rangle'_\nu(\mathbf{w}), \tag{15.2.5}$$

where $\mathfrak{D} = (d_1, \ldots, d_n)$ is summed over all permutations of the vector $(1, \ldots, p, 0^{n-p})$; we have let $\mathrm{inv}'(\overleftarrow{\mathfrak{D}})$ denote[2] the number of pairs $1 \leqslant i < j \leqslant n$ such that $1 \leqslant d_i < d_j$; and we have set $\mathsf{C}_0(x) = \mathsf{D}(x)$. Indeed, the factor $(1-t)^p \prod_{i=1}^n x_i^{\mathbf{1}_{d_i \geqslant 1}}$ arise from to the factors of $(1-t)x$ in the second and fourth weights in Figure 13.1, and the $t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})}$ factor arises as the product of the $t^{A_{[d_i+1,n]}}$ exponents in these weights.

Examining the right side of (15.2.5), we see that the $\mathfrak{D} = (1, \ldots, p, 0^{n-p})$ term is known already; up to overall multiplicative constants, that term is a nonsymmetric Macdonald polynomial, by (15.2.4). Our aim is to show that all terms in the sum (15.2.5) are related to the leading one, $(d_1, \ldots, d_n) = (1, \ldots, p, 0^{n-p})$, under the action of Hecke generators (13.1.3). To that end, we will use that for any polynomial $g \in \mathbb{C}[\mathbf{x}]$ we have the identity

$$T_i^{-1} \cdot \Big(x_i g(\mathbf{x})\Big) = t^{-1} x_{i+1} T_i \cdot g(\mathbf{x}), \tag{15.2.6}$$

which follows from (13.1.3). Using the commutativity of the Hecke generator T_i^{-1} with the multiplication by $x_i x_{i+1}$ (which follows from (13.1.3), since $[\mathfrak{s}_i, x_i x_{i+1}] = 0$) and the exchange relation (13.4.4), we see that for $1 \leqslant d_i < d_{i+1}$ we have

$$\begin{aligned}
T_i^{-1} \cdot t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d_k \geqslant 1}} \cdot \big\langle \mathsf{C}_{d_n}(x_n) \cdots \mathsf{C}_{d_1}(x_1) \big\rangle'_\nu(\mathbf{w}) \\
= t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d_k \geqslant 1}} T_i^{-1} \cdot \big\langle \mathsf{C}_{d_n}(x_n) \cdots \mathsf{C}_{d_1}(x_1) \big\rangle'_\nu(\mathbf{w}) \\
= t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}'})} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d'_k \geqslant 1}} \big\langle \mathsf{C}_{d'_n}(x_n) \cdots \mathsf{C}_{d'_1}(x_1) \big\rangle'_\nu(\mathbf{w}),
\end{aligned}$$

where $\mathfrak{D}' = (d'_1, \ldots, d'_n) = \mathfrak{s}_i(\mathfrak{D}) = (d_1, \ldots, d_{i-1}, d_{i+1}, d_i, d_{i+2}, \ldots, d_n)$. Similarly, using (15.2.6) and (13.4.4) (in which we act on both sides by T_i), we find that for $d_i \geqslant 1$ and $d_{i+1} = 0$ there holds

[2] Observe that this is slightly different from the definition of $\mathrm{inv}(\overleftarrow{\mathfrak{D}})$, which would count distinct indices i, j with $d_i = 0$ and $d_j > 0$.

$$T_i^{-1} \cdot t^{\operatorname{inv}'(\overleftarrow{\mathfrak{D}})} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d_k \geqslant 1}} \cdot \big\langle \mathsf{C}_{d_n}(x_n) \cdots \mathsf{C}_{d_1}(x_1) \big\rangle_{\nu}'(\mathbf{w})$$
$$= t^{\operatorname{inv}'(\overleftarrow{\mathfrak{D}})-1} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d_k' \geqslant 1}} T_i \cdot \big\langle \mathsf{C}_{d_n}(x_n) \cdots \mathsf{C}_{d_1}(x_1) \big\rangle_{\nu}'(\mathbf{w})$$
$$= t^{\operatorname{inv}'(\overleftarrow{\mathfrak{D}'})} \prod_{k=1}^{n} x_k^{\mathbf{1}_{d_k' \geqslant 1}} \big\langle \mathsf{C}_{d_n'}(x_n) \cdots \mathsf{C}_{d_1'}(x_1) \big\rangle_{\nu}'(\mathbf{w}),$$

where again $\mathfrak{D}' = (d_1', \ldots, d_n') = \mathfrak{s}_i(\mathfrak{D})$. Letting $\mu = (\nu, 0^{n-p})$, these two exchange relations and Corollary 15.2.3 together yield

$$\begin{aligned}
\mathcal{P}_\nu(\mathbf{x}) &= (1-t)^p \left(\sum_{\sigma \in \mathfrak{S}_n} T_\sigma^{-1} \right) \cdot t^{\binom{p}{2}} \prod_{i=1}^{p} x_i \cdot \big\langle \mathsf{D}(x_n) \cdots \mathsf{D}(x_{p+1}) \mathsf{C}_p(x_p) \cdots \mathsf{C}_1(x_1) \big\rangle_{\nu}'(\mathbf{w}) \\
&= \frac{(1-t)^p t^{\binom{p}{2}}}{\Omega_\nu(q,t)} \prod_{i=1}^{p} (1 - t^i w_{i,0})^{-1} \left(\sum_{\sigma \in \mathfrak{S}_n} T_\sigma^{-1} \right) \cdot f_\mu(\mathbf{x}),
\end{aligned} \tag{15.2.7}$$

where we recall the $w_{i,j}$ from (15.1.2), and for any permutation $\sigma \in \mathfrak{S}_n$ with reduced word form $\sigma = \mathfrak{s}_{i_1} \cdots \mathfrak{s}_{i_\ell}$ we have set $T_\sigma = T_{i_1} \cdots T_{i_\ell}$ (which is well-defined by the last relation in (13.1.1)). Here, the factor of $t^{\binom{p}{2}}$ follows from the fact that $\operatorname{inv}'(\overleftarrow{\mathfrak{D}'}) = \binom{p}{2}$ if $\mathfrak{D}' = (1, 2, \ldots, p, 0^{n-p})$.

Now $f_\mu(\mathbf{x}) \in \mathcal{V}_\kappa$, with $\kappa = (\nu^+, 0^{n-p})$. Moreover, it quickly follows from (13.1.1) and the recursive properties of nonsymmetric Macdonald polynomials with respect to the Hecke generators T_i^{-1} (see Theorem 4.2 of [**46**]) that $T_\sigma^{-1} h \in \mathcal{V}_\kappa$ for any $h \in \mathcal{V}_\kappa$ and $\sigma \in \mathfrak{S}_n$. This, together with (15.2.7), implies the theorem. □

Corollary 15.2.5. *For any anti-dominant composition $\nu = (\nu_1, \ldots, \nu_p)$ such that $\nu_i \geqslant 1$ for each $i \in [1, p]$, let $\kappa = (\nu^+, 0^{n-p})$. Up to an (at this stage) unspecified multiplicative constant, we have*

$$\mathcal{P}_\nu(\mathbf{x}) = d_\nu(q,t) P_\kappa(\mathbf{x}; q, t),$$

where $P_\kappa(\mathbf{x}; q, t)$ denotes a symmetric Macdonald polynomial.

Proof. This follows from the fact that, up to an overall multiplicative constant, the only polynomial which satisfies the properties of Lemma 15.2.1 and Proposition 15.2.4 is $P_\kappa(\mathbf{x}; q, t)$. □

15.3. Identifying the Leading Coefficient

In this section we establish Theorem 15.1.2, which will follow from Corollary 15.2.5, together with an analysis of the leading coefficient of $\mathcal{P}_\nu(\mathbf{x})$.

Proof of Theorem 15.1.2. We must show that

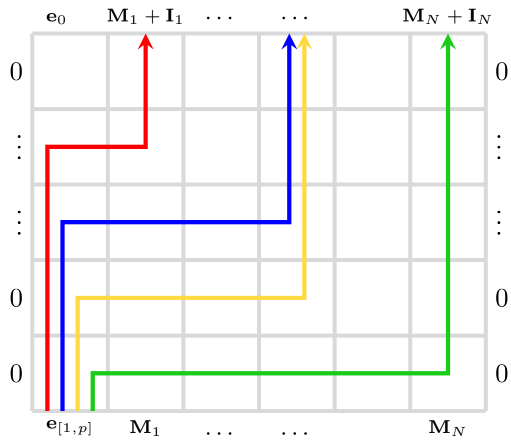
$$
J_{\nu^+}(\mathbf{x}) = \sum_{\mathcal{M}} \prod_{i=1}^{p} \prod_{j=1}^{N} w_{i,j}^{M_{i,j}} \times
\begin{array}{c|c|c|c|c|c|c|c}
 & \mathbf{e}_0 & \mathbf{M}_1+\mathbf{I}_1 & \cdots & \cdots & & \mathbf{M}_N+\mathbf{I}_N & \\
\hline
0 & x_n & & \cdots & \cdots & & x_n & 0 \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
0 & x_2 & & \cdots & \cdots & & x_2 & 0 \\
\hline
0 & x_1 & & \cdots & \cdots & & x_1 & 0 \\
\hline
 & \mathbf{e}_{[1,p]} & \mathbf{M}_1 & \cdots & \cdots & & \mathbf{M}_N &
\end{array}
\tag{15.3.1}
$$

where we abbreviate $\mathbf{I}_j = \mathbf{I}_j(\nu)$ for each $j \geqslant 1$, and for each $(i,j) \in [1,p] \times [1,N]$ the twist parameter $w_{i,j}$ is given by $w_{i,j} = \mathbf{1}_{\nu_i > j} q^{\nu_i - j}$ (recall (15.1.2)).

As Proposition 15.2.4, we proved that (15.3.1) gives a symmetric Macdonald polynomial, modulo normalization. So, it remains to verify that the coefficient of the $\prod_{i=1}^{p} x_i^{\nu_i^+} = \prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$ monomial is $c_{\nu^+}(q,t)$ from (15.1.1), in order to verify the normalization.

Observe that any path ensemble in the above partition function contributing to the coefficient of this monomial $\prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$ cannot have any path take a horizontal step in some row $k > p$. Otherwise, Remark 13.2.2 implies that such a step would contribute a factor of x_k to the weight of this ensemble, and so it cannot give rise to the monomial $\prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$. In particular, in any such path ensemble, each path exits the leftmost column of the model in one of the rows $1, 2, \ldots, p$.

Given this, the procedure for analyzing the coefficient of $\prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$ closely parallels the argument in Section 14.5. In particular, one can show (we omit the proof, since it is entirely analogous to that of Proposition 14.5.1) that there is an *almost* unique configuration in the partition function (15.3.1) which gives rise to the monomial $\prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$; these occur when the path of color i makes a right turn out of column zero into row $p - i' + 1$ for some i' with $\nu_{i'} = \nu_i$, and then takes ν_i consecutive horizontal steps in that row, before making a vertical step into its destination column. The "almost" part of the claim comes parts of equal size in ν; if a part is repeated k times in ν, then there are $k!$ possibilities for the corresponding colors to permute themselves in any way, before making the right turn out of the zeroth column. We thus need to compute the weight of configurations of the form



and perform the aforementioned summation over permutations of colors in the zeroth column, for parts of repeated size.

Denoting the colors of the arrows exiting the leftmost column, from bottom to top, by $\mathfrak{D} = (d_1, d_2, \ldots, d_n)$, we find that the weight of this column is given by $(1-t)^p t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})} \prod_{i=1}^{p} x_i$, where (as in the proof of Proposition 15.2.4) $\mathrm{inv}'(\overleftarrow{\mathfrak{D}})$ denotes the number of pairs $1 \leqslant i < j \leqslant n$ such that $1 \leqslant d_i < d_j$. Indeed, the factor of $(1-t)^p \prod_{i=1}^{p} x_i$ arises from to the factors of $(1-t)x$ in the second weight in Figure 13.1, and the $t^{\mathrm{inv}'(\overleftarrow{\mathfrak{D}})}$ factor arises as the product of the $t^{A_{[d_i+1,n]}}$ exponents in these weights.

Moreover, consulting equation (14.2.4) for the column weights, the contribution to this configuration weight coming from the remaining N columns is equal to

$$\prod_{i=1}^{p} x_i^{\nu_{p-i+1}-1} \cdot \prod_{j=1}^{N} \prod_{i:\nu_i>j} \left(1 - w_{i,j} t^{\#\{k<i:\nu_k>j\}+1}\right).$$

In particular, this quantity is independent of $\mathfrak{D}$ satisfying $\nu_{d_1} \geqslant \nu_{d_2} \geqslant \cdots \geqslant \nu_{d_p}$ and $d_{p+1} = d_{p+2} = \cdots = d_n = 0$, such that $\nu_{d_{K_i+1}} = \nu_{d_{K_i+2}} = \cdots = \nu_{d_{K_{i+1}}}$, where $K_i = \sum_{j=1}^{i-1} m_j(\nu)$, for each $i \in [1,p]$.

Taking the product of the contributions from the leftmost column and the remaining columns; summing over $\mathfrak{D}$ satisfying the three properties listed above; and using the fact that $\sum_{\sigma \in \mathfrak{S}_k} t^{\mathrm{inv}(\sigma)} = (1-t)^{-k}(t;t)_k$ for any $k \geqslant 0$ yields that the coefficient of $\prod_{i=1}^{p} x_i^{\nu_{p-i+1}}$ in $\mathcal{P}_\nu(\mathbf{x})$ is equal to

$$\prod_{i=1}^{p} (t;t)_{m_i(\nu)} \prod_{j=1}^{N} \prod_{i:\nu_i>j} \left(1 - w_{i,j} t^{\#\{k<i:\nu_k>j\}+1}\right). \tag{15.3.2}$$

Further observe that, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$, we may write the quantity $c_\lambda(q,t)$ from (15.1.1) as

$$c_\lambda(q,t) = \prod_{i=1}^{p} \prod_{j=1}^{\lambda_i-1} \left(1 - q^{\lambda_i - j} t^{\#\{k>i:\lambda_k \geqslant j\}+1}\right) \cdot \prod_{i=1}^{p} \left(1 - t^{\#\{k<i:\lambda_k=\lambda_i\}+1}\right), \tag{15.3.3}$$

where the first product comes from analyzing all boxes in (15.1.1) which have positive arm length, while the second product deals with all boxes that have arm length equal to zero. Now the theorem follows from the fact that (15.3.2) and (15.3.3) coincide for $\lambda = \nu^+$. $\square$

CHAPTER 16

Expansion of LLT Polynomials in the Modified Hall–Littlewood Basis

In this chapter we provide a combinatorial formula, as partition functions for a fused $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ vertex model, for the expansion coefficients of the LLT polynomials in the modified Hall–Littlewood basis. This result is given by Theorem 16.1.3, and its proof will appear in Chapter 17 below. Throughout this chapter, we fix an integer $n \geqslant 1$. Moreover, for any (possibly infinite) set of variables $\mathbf{x} = (x_1, x_2, \ldots, x_k)$; complex number $r \in \mathbb{C}$; signature $\lambda \in \mathrm{Sign}_\ell$; and signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_n$, we recall the LLT polynomials $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ from (9.2.1) and (9.1.2); the (standard) Hall–Littlewood polynomials $Q_\lambda(\mathbf{x})$ from Section 3.2 of [**66**]; the modified Hall–Littlewood polynomials $Q'_\lambda(\mathbf{x})$ from Section 9.6; and the functions $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0)$ from (8.4.3). We additionally recall the sequences $\mathcal{S}(\boldsymbol{\lambda}) = \big(\mathbf{S}_1(\boldsymbol{\lambda}), \mathbf{S}_2(\boldsymbol{\lambda}), \ldots\big)$ and $\mathcal{I}(\lambda) = \big(\mathbf{I}_0(\lambda), \mathbf{I}_1(\lambda), \ldots\big)$ of elements in $\{0,1\}^n$ from Section 7.1 and Section 13.3, respectively (assuming in the latter that $\ell(\lambda) \leqslant n$).

16.1. Combinatorial Formula for Expansion Coefficients

As explained in Definition 9.5.2, the polynomials $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ and $\mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ (and, by Proposition 9.6.1, also $Q'_\lambda(\mathbf{x})$) satisfy the compatibility condition of Lemma 9.5.1 that enables us to view them as elements in the ring $\Lambda(\mathbf{x})$ of symmetric functions in an infinite set of variables $\mathbf{x} = (x_1, x_2, \ldots)$. Since (by, for example, Chapter 3.2 of [**66**]) the (modified) Hall–Littlewood functions form a basis of $\Lambda(\mathbf{x})$, there exist coefficients $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) \in \mathbb{C}(q)$ such that

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = \sum_{\nu} f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) Q'_\nu(\mathbf{x}), \tag{16.1.1}$$

with the sum taken over all partitions ν. Recalling the function ψ from (9.3.1), (9.3.3) implies that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = q^{\psi(\boldsymbol{\lambda}) - \psi(\boldsymbol{\mu})} \mathcal{L}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$, and so (up to an explicit power of q) the $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ provide the expansion coefficients of the LLT polynomials in the modified Hall–Littlewood basis.

The main result to be stated in this section is an explicit combinatorial formula for these coefficients $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$ and partitions ν; see Theorem 16.1.3 below. In order to state our main formula, we introduce two types of vertex weights; their origin will become clear in Chapter 17 below.

Definition 16.1.1. Fix nonnegative integers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \geqslant 0$ and n-tuples $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \{0,1\}^n$. Define $\mathbf{V} = (V_1, \ldots, V_n) \in \{0,1\}^n$ with $V_i = \min\{A_i, B_i, C_i, D_i\}$ for each $i \in [1, n]$. We introduce the lattice weights

$$
\begin{array}{ccc} & (\mathfrak{c}, \mathbf{C}) & \\ (\mathfrak{b}, \mathbf{B}) & \square & (\mathfrak{d}, \mathbf{D}) \\ & (\mathfrak{a}, \mathbf{A}) & \end{array}
= (-1)^{\mathfrak{c}+|\mathbf{V}|} q^{\chi} \frac{(q;q)_{\mathfrak{b}-|\mathbf{B}|}}{(q;q)_{\mathfrak{d}-|\mathbf{D}|}} \frac{(q^{\mathfrak{b}-|\mathbf{B}|-\mathfrak{a}+1};q)_{\mathfrak{c}}}{(q;q)_{\mathfrak{c}}} \mathbf{1}_{|\mathbf{B}|\leqslant \mathfrak{b}} \mathbf{1}_{|\mathbf{D}|\leqslant \mathfrak{d}} \mathbf{1}_{\mathfrak{a}+\mathfrak{d}=\mathfrak{b}+\mathfrak{c}}
$$
$$
\times \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \prod_{j: B_j - D_j = 1} \big(1 - q^{\mathfrak{d} - B_{[j+1,n]} - D_{[1,j-1]}}\big), \tag{16.1.2}
$$

where the exponent $\chi \equiv \chi(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d};\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ is given by

$$
\chi = \binom{\mathfrak{d} - |\mathbf{D}|}{2} + \binom{\mathfrak{c}+1}{2} - \big(\mathfrak{c} + |\mathbf{C}|\big)\mathfrak{d} + |\mathbf{V}|\big(\mathfrak{d} - |\mathbf{D}| + 1\big) + \varphi(\mathbf{D},\mathbf{C}) + \varphi(\mathbf{V}, \mathbf{D} - \mathbf{B}).
$$

Aside from the vanishing constraints imposed by the indicator functions in (16.1.2), these vertex weights have additional vanishing properties that result from the q-Pochhammer function $(q^{\mathfrak{b}-|\mathbf{B}|-\mathfrak{a}+1};q)_{\mathfrak{c}}$ and the factor $\big(1 - q^{\mathfrak{d}-B_{[j+1,n]}-D_{[1,j-1]}}\big)$ present in the final product.

We will require a further set of weights, whose notation are distinguished from those in (16.1.2) by means of shading.

Definition 16.1.2. As in Definition 16.1.1, let us fix nonnegative integers $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d} \geqslant 0$ and n-tuples $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \{0,1\}^n$ with coordinates indexed by $[1,n]$. We define the weights

$$
\begin{array}{ccc} & (\mathfrak{c}, \mathbf{C}) & \\ (\mathfrak{b}, \mathbf{B}) & \blacksquare & (\mathfrak{d}, \mathbf{D}) \\ & (\mathfrak{a}, \mathbf{A}) & \end{array}
= \mathbf{1}_{|\mathbf{B}|=\mathfrak{b}\leqslant 1} \mathbf{1}_{|\mathbf{D}|\leqslant \mathfrak{d}\leqslant 1} \mathbf{1}_{\mathfrak{a}+\mathfrak{d}=\mathfrak{b}+\mathfrak{c}} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \cdot W\big(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d};\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\big), \tag{16.1.3}
$$

where (recalling $\mathbf{A}_i^{+}, \mathbf{A}_j^{-}, \mathbf{A}_{ij}^{+-}$ from (8.1.1)) the final function appearing in (16.1.3) is given by

(16.1.4)

$$
W\big(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d};\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\big) =
\begin{cases}
q^{|\mathbf{A}|}, & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (0,\mathbf{e}_0), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a},\mathbf{A}), \quad (\mathfrak{d},\mathbf{D}) = (0,\mathbf{e}_0), \\
1, & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (0,\mathbf{e}_0), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a}+1,\mathbf{A}), \quad (\mathfrak{d},\mathbf{D}) = (1,\mathbf{e}_0), \\
(1-q^{A_i}) q^{A_{[i+1,n]}}, & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (0,\mathbf{e}_0), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a}+1,\mathbf{A}_i^{-}), \quad (\mathfrak{d},\mathbf{D}) = (1,\mathbf{e}_i), \\
(1-q^{A_j}) q^{A_{[j+1,n]}} \mathbf{1}_{A_i=0} & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (1,\mathbf{e}_i), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a},\mathbf{A}_{ij}^{+-}), \quad (\mathfrak{d},\mathbf{D}) = (1,\mathbf{e}_j), \\
\mathbf{1}_{A_i=0}, & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (1,\mathbf{e}_i), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a},\mathbf{A}_i^{+}), \quad (\mathfrak{d},\mathbf{D}) = (1,\mathbf{e}_0), \\
(-1)^{A_i} q^{A_{[i,n]}}, & \text{if} \quad (\mathfrak{b},\mathbf{B}) = (1,\mathbf{e}_i), \quad (\mathfrak{c},\mathbf{C}) = (\mathfrak{a},\mathbf{A}), \quad (\mathfrak{d},\mathbf{D}) = (1,\mathbf{e}_i),
\end{cases}
$$

where we assume that $i < j$ whenever i and j both appear. In all cases of $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d};\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ not listed in (16.1.4), we set $W\big(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d};\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\big) = 0$.

Observe that there are two quadruples we consider in Definition 16.1.1 and Definition 16.1.2. The first is the quadruple $(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})$ of elements in $\{0,1\}^n$, which is stipulated to satisfy $\mathbf{A}+\mathbf{B}=$

$\mathbf{C}+\mathbf{D}$. We once again view $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ as indexing the up-right directed fermionic paths passing through the south, west, north, and east boundaries of a tile, respectively. The second is the quadruple $(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d})$ of integers, which is stipulated to satisfy $\mathfrak{a}+\mathfrak{d}=\mathfrak{b}+\mathfrak{c}$. In this way, we may view $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ as counting the numbers of directed down-right paths (that is, they proceed in a different direction from the fermionic paths) of the same bosonic color that pass through the bottom, left, top, and right boundaries of a tile, respectively. We provide depictions of these tiles in Section 16.3 below.

Now we present the main result of this section, Theorem 16.1.3; its proof will appear in Chapter 17 below.

THEOREM 16.1.3. *Fix integers $M \geqslant 0$ and $m \in [1, n]$; signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \operatorname{SeqSign}_{n;M}$; and a partition $\nu \in \operatorname{Sign}_m$, such that $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}| = |\nu|$. Fix any integer*

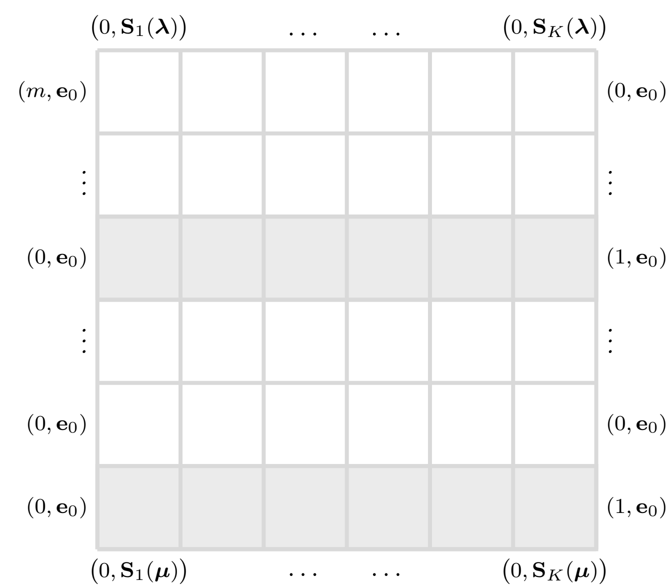
$$K \geqslant \max\Big\{ \max_{i \in [1,n]} \mathfrak{T}\big(\lambda^{(i)}\big), \nu_1 + 1 \Big\},$$

and write $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \cdots, \bar{\nu}_m)$ for the signature obtained by complementing ν in a $(K-1) \times m$ box; its parts are given by $\bar{\nu}_j = K - \nu_{m-j+1} - 1$, for each $j \in [1, m]$. Denote $\mathfrak{T}(\bar{\nu}) = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_m)$ so, for each $j \in [1, m]$,

$$\mathfrak{n}_j = \bar{\nu}_j + m - j + 1 = K - \nu_{m-j+1} + m - j. \tag{16.1.5}$$

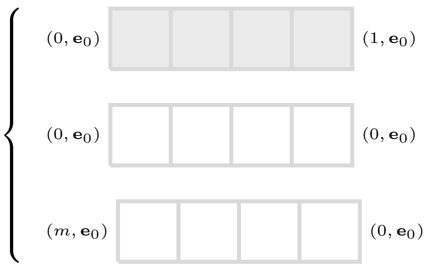
The coefficients (16.1.1) *are given by the partition function*

(16.1.6) $$f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \frac{(-1)^{|\bar{\nu}|}}{b_\nu(q)(q;q)_m} \times$$



consisting of $\mathfrak{n}_1 + 1$ *rows, where the* i*-th row of the lattice (counted from the top to bottom, starting at* $i = 0$*) takes the form*

(16.1.7)



$i \in \mathfrak{T}(\bar{\nu})$,

$i \notin \mathfrak{T}(\bar{\nu}), \; i \neq 0$,

$i = 0$.

The constant $b_\nu(q)$ *appearing in* (16.1.6) *is given by* $b_\nu(q) = \prod_{j=1}^{\infty} (q;q)_{m_j(\nu)}$.

16.2. Expansion Coefficients for Modified Macdonald Polynomials

Fix complex numbers $q, t \in \mathbb{C}$; let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ denote a (possibly infinite) set of variables; and let $\lambda \in \mathrm{Sign}$ denote a signature. Recalling the modified Macdonald polynomial $\widetilde{J}_\lambda(\mathbf{x}; q, t)$ from (15.1.5), we next provide combinatorial formulas for the coefficients $g_\lambda^\nu(q, t)$ in the expansion

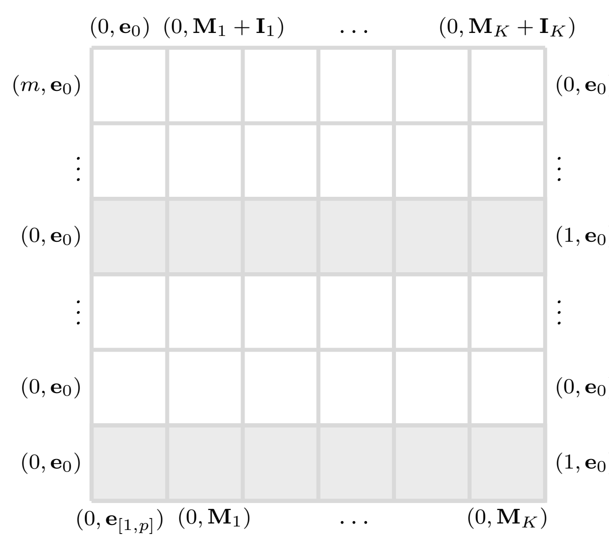
$$\widetilde{J}_\lambda(\mathbf{x}; q, t) = \sum_\nu g_\lambda^\nu(q, t) Q'_\nu(\mathbf{x}; t), \tag{16.2.1}$$

where $Q'_\nu(\mathbf{x}; t)$ denotes a modified Hall–Littlewood polynomial (with its parameter q replaced by t). Since Corollary 15.1.4 expands $\widetilde{J}_\lambda$ over the LLT polynomials, this follows as a quick corollary of Theorem 16.1.3.

Corollary 16.2.1. *Fix integers* $p, m \in [1, n]$*; an anti-dominant composition* λ *of length* p*; a partition* ν *of length* m*; and an integer* $K \geqslant \max\{\lambda_p, \nu_1 + 1\}$*. For any integers* $i \in [1, p]$ *and* $j \geqslant 1$*, set* $w_{i,j} = q^{\lambda_i - j} \mathbf{1}_{\lambda_i > j}$*. Then, the coefficient* $g_{\lambda^+}^\nu(q, t)$ *is given by*

(16.2.2)

$$g_{\lambda^+}^\nu(q, t) = \frac{(-1)^{|\bar{\nu}|}}{b_\nu(t)(t;t)_m} \sum_{\mathcal{M}} \prod_{i=1}^{p} \prod_{j=1}^{K} w_{i,j}^{M_{i,j}} \times$$



where the rows of the partition function (16.2.2) *are specified by* (16.1.5) *and* (16.1.7)*, using the same weights as in Definition 16.1.1 and Definition 16.1.2 but with the q there replaced by t here. The sum is over all sequences $\mathcal{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_K)$ of elements in $\{0,1\}^n$, with $\mathbf{M}_j = (M_{1,j}, M_{2,j}, \ldots, M_{n,j})$, such that $M_{i,k} = 0$ for $i > p$ and $\mathbf{I}_k(\lambda) + \mathbf{M}_k \in \{0,1\}^n$ for each $k \in [1, K]$. Here, we abbreviated $\mathbf{I}_k = \mathbf{I}_k(\lambda)$ for each k, and we recalled $b_\nu(t)$ from Theorem 16.1.3.*

Proof. Together (8.4.1), (7.1.3), and Remark 8.4.4 imply the diagrammatic equality

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0) = \begin{array}{c|c|c|c|c|c|c|c} & \mathbf{e}_0 & \mathbf{S}_1(\boldsymbol{\lambda}) & \cdots & \cdots & & \mathbf{S}_K(\boldsymbol{\lambda}) & \\ \hline \mathbf{e}_0 & x_n & & \cdots & \cdots & & x_n & \mathbf{e}_0 \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline \mathbf{e}_0 & x_2 & & \cdots & \cdots & & x_2 & \mathbf{e}_0 \\ \hline \mathbf{e}_0 & x_1 & & \cdots & \cdots & & x_1 & \mathbf{e}_0 \\ \hline & \mathbf{e}_{[1,p]} & \mathbf{S}_1(\boldsymbol{\mu}) & \cdots & \cdots & & \mathbf{S}_K(\boldsymbol{\mu}) & \end{array} \tag{16.2.3}$$

where the weights in the j-th row (from the bottom) are given by $\mathcal{W}_{x_j}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0) = x_j^{|\mathbf{D}|} q^{\varphi(\mathbf{D}, \mathbf{C}+\mathbf{D})} \mathbf{1}_{|\mathbf{V}|=0} \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}$. Combining this with (15.1.8) gives

$$\widetilde{J}_{\lambda^+}(\mathbf{x}; q, t) = \sum_{\mathcal{M}} \prod_{i=1}^{p} \prod_{j=1}^{N} w_{i,j}^{M_{i,j}} \times \begin{array}{c|c|c|c|c|c|c|c} & \mathbf{e}_0 & \mathbf{M}_1 + \mathbf{I}_1 & \cdots & \cdots & & \mathbf{M}_K + \mathbf{I}_K & \\ \hline 0 & x_n & & \cdots & \cdots & & x_n & 0 \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline \vdots & \vdots & & & & & \vdots & \vdots \\ \hline 0 & x_2 & & \cdots & \cdots & & x_2 & 0 \\ \hline 0 & x_1 & & \cdots & \cdots & & x_1 & 0 \\ \hline & \mathbf{e}_{[1,p]} & \mathbf{M}_1 & \cdots & \cdots & & \mathbf{M}_K & \end{array}$$

where the vertices in the j-row are again taken with respect to the weights $\mathcal{W}_x(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D} \mid \infty, 0)$. This, together with (16.2.3) and Theorem 16.1.3, implies the corollary. $\square$

16.3. Examples

Before moving on to the proof of Theorem 16.1.3, we present some explicit examples of the combinatorial formula (16.1.6). Naively one would expect (16.1.6) to be a huge sum, but in practice it consists of just a handful of non-zero lattice configurations, as the following examples indicate.

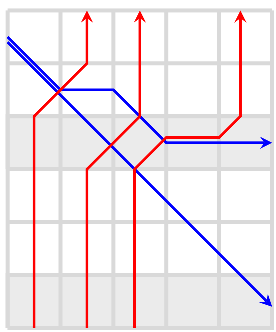
All examples in this section were generated by a `Mathematica` implementation of (16.1.6); the code is available from the authors upon request.

In what follows, we draw bosonic paths in the color blue. Unlike in previous diagrams, to simplify the figures, we depict paths as sometimes making diagonal steps, which are equivalent to making one step up or down and one step right.

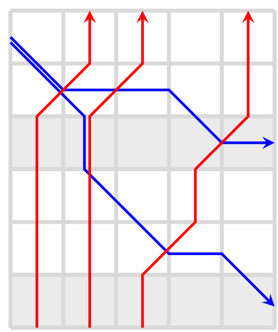
Example 16.3.1. Let $n = 1$, $M = 3$, $\boldsymbol{\lambda} = \big(\lambda^{(1)}\big)$, and $\boldsymbol{\mu} = \big(\mu^{(1)}\big)$ with $\lambda^{(1)} = (2,1,1)$ and $\mu^{(1)} = (0,0,0)$. Let $\nu = (3,1)$ and choose $K = 5$, so that $\bar{\nu} = (3,1)$ and $\mathfrak{T}(\bar{\nu}) = (5,2)$. Given that $|\bar{\nu}| = 4$, $m = 2$, and $b_\nu(q) = (1-q)^2$, we have

$$\frac{(-1)^{|\bar{\nu}|}}{b_\nu(q)(q;q)_m} = \frac{1}{(1-q)^3(1-q^2)}. \tag{16.3.1}$$

The formula (16.1.6) provides two non-vanishing lattice configurations, indicated below with their weights, where we have multiplied by the overall factor (16.3.1).



$-q^2$



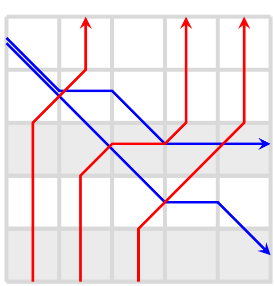
$-q(1-q)$

We therefore find that $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = -q^2 - q(1-q) = -q$.

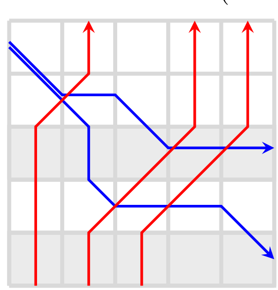
Example 16.3.2. Let $n = 1$, $M = 3$, $\boldsymbol{\lambda} = \big(\lambda^{(1)}\big)$, and $\boldsymbol{\mu} = \big(\mu^{(1)}\big)$ with $\lambda^{(1)} = (2,2,1)$ and $\mu^{(1)} = (0,0,0)$. Let $\nu = (3,2)$ and choose $K = 5$, so that $\bar{\nu} = (2,1)$ and $\mathfrak{T}(\bar{\nu}) = (4,2)$. Given that $|\bar{\nu}| = 3$, $m = 2$, and $b_\nu(q) = (1-q)^2$, we have

$$\frac{(-1)^{|\bar{\nu}|}}{b_\nu(q)(q;q)_m} = \frac{-1}{(1-q)^3(1-q^2)}. \tag{16.3.2}$$

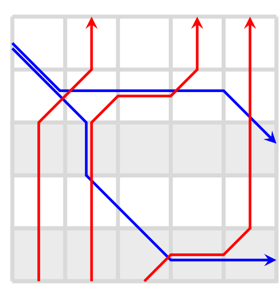
The formula (16.1.6) provides three non-vanishing lattice configurations, indicated below with their weights, where we have multiplied by the overall factor (16.3.2).



$-q^2(1-q)$



$q^2(1-q)$



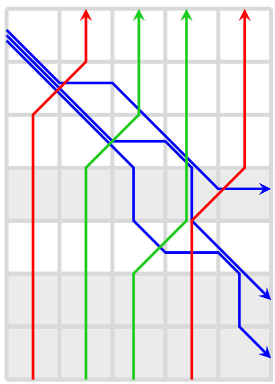
$-q$

We therefore find that $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = -q^2(1-q) + q^2(1-q) - q = -q$.

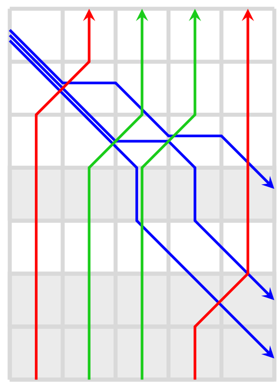
Example 16.3.3. Let $n = 2$, $M = 2$, $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \lambda^{(2)}\right)$, and $\boldsymbol{\mu} = \left(\mu^{(1)}, \mu^{(2)}\right)$ with $\lambda^{(1)} = (3,1)$, $\lambda^{(2)} = (2,2)$, $\mu^{(1)} = (2,0)$, and $\mu^{(2)} = (1,1)$. Let $\nu = (2,1,1)$ and choose $K = 5$, so that $\bar{\nu} = (3,3,2)$ and $\mathfrak{T}(\bar{\nu}) = (6,5,3)$. Given that $|\bar{\nu}| = 8$, $m = 3$, and $b_\nu(q) = (1-q)^2(1-q^2)$, we have

$$\frac{(-1)^{|\bar{\nu}|}}{b_\nu(q)(q;q)_m} = \frac{1}{(1-q)^3(1-q^2)^2(1-q^3)}. \tag{16.3.3}$$

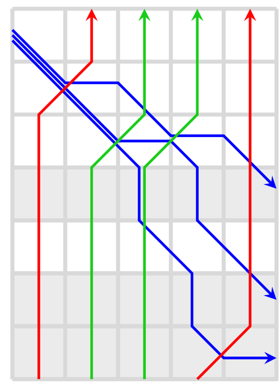
The formula (16.1.6) provides five non-vanishing lattice configurations, indicated below with their weights, where we have multiplied by the overall factor (16.3.3).



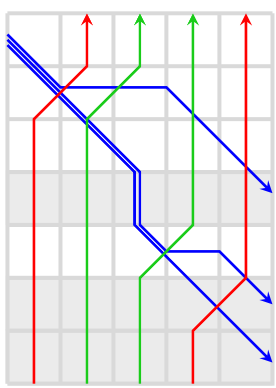
$q(1-q^2)$



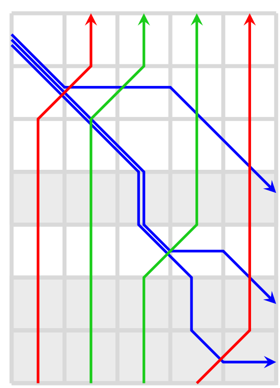
$\frac{-q^3}{1+q}$



$\frac{-q^2}{1+q}$



$\frac{q(1-q)}{1+q}$



$\frac{1-q}{1+q}$

We therefore find that $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = q(1-q^2) - \frac{q^3}{1+q} - \frac{q^2}{1+q} + \frac{q(1-q)}{1+q} + \frac{1-q}{1+q} = 1 - q^2 - q^3$.

16.4. Pre-fused LLT and Hall–Littlewood Polynomials

Before beginning with the proof of Theorem 16.1.3, in this section we make a slight detour and give an alternative definition of the coefficients $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$, which will be useful in that proof. This is done via a "pre-fused[1] formulation" (16.1.1), which at the level of the participating symmetric functions means applying the plethysm $X \mapsto (1-q)X$ to (16.1.1), where $X = \sum_{x \in \mathbf{x}} x$ (recall Section 9.5 for our notation on plethysms). By Proposition 9.5.4, this converts $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ into the *pre-fused LLT function* $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0)$, and $Q'_\nu(\mathbf{x})$ into an ordinary Hall–Littlewood function $Q_\nu(\mathbf{x})$. This plethysm does not affect the value of $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$.

Lemma 16.4.1. *Fix an integer $M \geqslant 0$, and let $\mathbf{x} = (x_1, x_2, \ldots)$ denote an infinite set of variables. For any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$ and partition ν, we have that $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ is equal to the coefficient of Q_ν in the Hall–Littlewood expansion of a pre-fused LLT function $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0)$, Explicitly,*

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0) = \sum_{\nu} f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) Q_\nu(\mathbf{x}), \tag{16.4.1}$$

with the sum taken over all partitions ν.

Proof. This follows from applying the plethystic substitution $X \mapsto (1-q)X$ to both sides of (16.1.1). Indeed, (9.5.2) and (9.3.3) together imply that this substitution maps $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; \infty \mid 0; 0)$ to $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0)$ and, by the definition (from Section 9.6) of the modified Hall–Littlewood polynomials, it also maps $Q'_\nu(\mathbf{x})$ to $Q_\nu(\mathbf{x})$. $\square$

This pre-fused formulation of the $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ coefficients turns out to be valuable in the proof presented in Chapter 17 below, as the pre-fused functions $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0)$ and $Q_\nu(\mathbf{x})$ have a simpler algebraic structure than their fused counterparts. Indeed, (16.4.1) will be our starting point in the proof of Theorem 16.1.3; one of its main advantages is that it allows us to write an integral formula for the coefficients $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$, provided by the following proposition and corollary.

Proposition 16.4.2. *Let $\mathbf{x} = (x_1, x_2, \ldots)$ denote an infinite set of variables, and let $F \in \Lambda(\mathbf{x})$ be a symmetric function with the expansion $F = \sum_\nu c_\nu(q) Q_\nu$ over the Hall–Littlewood basis. Write $F(x_1, \ldots, x_N)$ for the symmetric polynomial obtained by setting $x_i = 0$, for all $i > N$. For any partition $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$, we have*

$$c_\nu(q) = \frac{1}{b_\nu(q)} \frac{1}{(2\pi \mathbf{i})^N} \oint_C \cdots \oint_C F(x_1^{-1}, \ldots, x_N^{-1}) \prod_{1 \leqslant i < j \leqslant N} \frac{x_j - x_i}{x_j - q x_i} \prod_{i=1}^{N} x_i^{\nu_i - 1} dx_i, \tag{16.4.2}$$

where the contour C is a positively oriented circle centered on the origin, and where we assume that qC is contained in C (equivalently, $|q| < 1$). Here, we recall $b_\nu(q) = \prod_{j=1}^{\infty} (q;q)_{m_j(\nu)}$.

Proof. The Hall–Littlewood polynomials P_λ (see Chapter 3.2 of [**66**]) have the following orthogonality property, valid for any $\lambda, \nu \in \mathrm{Sign}_N$:

$$\frac{1}{(2\pi \mathbf{i})^N} \oint_C \cdots \oint_C P_\lambda(x_1^{-1}, \ldots, x_N^{-1}) \prod_{1 \leqslant i < j \leqslant N} \frac{x_j - x_i}{x_j - q x_i} \prod_{i=1}^{N} x_i^{\nu_i - 1} dx_i = \mathbf{1}_{\lambda = \nu}. \tag{16.4.3}$$

The relation (16.4.3) can be deduced as a consequence of equation (3.2.15) of [**66**]. Indeed, the latter states for any sequence of $M \geqslant N$ complex numbers $\mathbf{y} = (y_1, y_2, \ldots, y_M)$ that

$$Q_\nu(y_1, \ldots, y_M) = \mathrm{Coeff}\left[\prod_{1 \leqslant i < j \leqslant N} \left(\frac{1 - x_i x_j^{-1}}{1 - q x_i x_j^{-1}} \right) \prod_{i=1}^{N} \prod_{j=1}^{M} \left(\frac{1 - q y_j x_i^{-1}}{1 - y_j x_i^{-1}} \right); \prod_{i=1}^{N} x_i^{-\nu_i} \right], \tag{16.4.4}$$

[1] Our reason for this terminology is that the associated vertex models under this transformation will only be partially fused, in that multiple arrows can exist along a vertical edge, but only one can exist along a horizontal edge.

where the right hand side calls for the coefficient of $\prod_{i=1}^{N} x_i^{-\nu_i}$ in the Laurent series expansion (about $x_1 = \cdots = x_N = 0$) of the indicated product, and where this Laurent series is computed under the assumption that

$$|y_j x_i^{-1}| < 1, \quad \text{for all } 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant M,$$
$$|q x_i x_j^{-1}| < 1; \quad \text{for all } 1 \leqslant i < j \leqslant N.$$

Casting (16.4.4) in terms of integrals, it becomes

$$Q_\nu(y_1, \ldots, y_M) = \frac{1}{(2\pi\mathbf{i})^N} \oint_C \cdots \oint_C \prod_{1 \leqslant i < j \leqslant N} \frac{x_j - x_i}{x_j - q x_i} \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 - q y_j x_i^{-1}}{1 - y_j x_i^{-1}} \prod_{i=1}^{N} x_i^{\nu_i - 1} dx_i,$$

where the contour C is a positively oriented circle centered on the origin, such that qC and $\{y_1, \ldots, y_M\}$ are contained in C. Finally, we use the Cauchy identity for Hall–Littlewood polynomials (see equation (3.4.4) [**66**]) in the preceding integral, which yields

$$Q_\nu(y_1, \ldots, y_M) = \frac{1}{(2\pi\mathbf{i})^N} \oint_C \cdots \oint_C \sum_{\lambda \in \mathrm{Sign}_N} P_\lambda(x_1^{-1}, \ldots, x_N^{-1}) Q_\lambda(y_1, \ldots, y_M)$$
$$\times \prod_{1 \leqslant i < j \leqslant N} \frac{x_j - x_i}{x_j - q x_i} \prod_{i=1}^{N} x_i^{\nu_i - 1} dx_i,$$

and (16.4.3) follows from the fact that for $\lambda \in \mathrm{Sign}_N$ the polynomials $Q_\lambda(y_1, \ldots, y_M)$ are linearly independent as elements in the ring of symmetric polynomials in the $M \geqslant N$ variables $(y_1, \ldots, y_M)$.

Now starting from the symmetric function identity $F = \sum_\lambda c_\lambda(q) Q_\lambda$, we set $x_i = 0$ for all $i > N$, which yields

$$F(x_1, \ldots, x_N) = \sum_{\lambda : \ell(\lambda) \leqslant N} c_\lambda(q) Q_\lambda(x_1, \ldots, x_N). \tag{16.4.5}$$

First reciprocating all variables in equation (16.4.5); then multiplying by $\prod_{i=1}^{N} x_i^{\nu_i - 1} \prod_{1 \leqslant i < j \leqslant N} (x_j - x_i)(x_j - q x_i)^{-1}$, and next performing n integrations as in (16.4.3), the result (16.4.2) then follows, recalling that $Q_\lambda = b_\lambda(q) P_\lambda$. ☐

Corollary 16.4.3. *Fix integers $M \geqslant 0$ and $N \geqslant 1$; signature sequences $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathrm{SeqSign}_{n;M}$; and a partition ν. Recalling $b_\nu(q)$ from Proposition 16.4.2, the coefficient $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$ is given by*

$$f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q) = \frac{1}{b_\nu(q)} \frac{1}{(2\pi\mathbf{i})^N} \cdot \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}^{-1}; r^{-1/2} \mid 0; 0) \prod_{1 \leqslant i < j \leqslant N} \frac{x_j - x_i}{x_j - q x_i} \prod_{i=1}^{N} x_i^{\nu_i - 1} dx_i, \tag{16.4.6}$$

where we have denoted $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, and the contour C is a positively oriented circle centered on the origin, and where we assume that qC is contained in C (equivalently, $|q| < 1$).

Proof. This follows from Lemma 16.4.1 and Proposition 16.4.2. ☐

CHAPTER 17

Proof of Theorem 16.1.3

In this chapter we provide the proof of Theorem 16.1.3, which will proceed as follows. First, in Section 17.1, we write down vertex weights w_x that are slightly more general than the L_x weights from (13.2.1); see equations (17.1.1)–(17.1.2). At the level of the underlying quantized affine Lie algebras, the L_x weights are based on $U_q(\widehat{\mathfrak{sl}}(1|n))$, whereas the w_x weights that we provide in Section 17.1 are based on $U_q(\widehat{\mathfrak{sl}}(2|n))$. This is the first time in our text that we make use of a model of mixed boson-fermion nature for computational purposes. At a conceptual level, one may think of the n fermionic species in (17.1.1)–(17.1.2) as the progenitors of a pre-fused LLT polynomial, while the single bosonic family present in (17.1.1)–(17.1.2) is the forerunner of a Hall–Littlewood polynomial; the model (17.1.1)–(17.1.2) thus provides us with the necessary algebraic framework for studying expansions of the form (16.4.1).

Second, in Section 17.2, we construct a certain family of partition functions $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ using the weights of the model (17.1.1)–(17.1.2). The Yang–Baxter equation provides us with exchange relations on the family $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ which, combined with an explicit initial condition, suffice to determine the family uniquely; see Section 17.3. In Section 17.4 we solve these exchange relations and obtain an integral formula for $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

The integral formula obtained in Section 17.4 bears obvious resemblance to the right hand side of (16.4.6), and can in fact be brought identically to that form, via a certain limiting procedure; the details of this procedure are the content of Section 17.6. Finally, one needs to examine what happens to the partition function $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ under the same limiting procedure; this involves some delicate calculations at the level of the fully-fused $U_q(\widehat{\mathfrak{sl}}(2|n))$ model, and is the subject of Section 17.7. The final result of these computations is the combinatorial formula quoted in Theorem 16.1.3, which is then matched with $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ courtesy of (16.4.6); this completes the proof of Theorem 16.1.3.

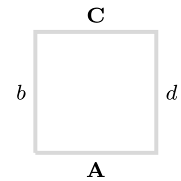
Here, for any $\boldsymbol{\lambda} \in \mathrm{SeqSign}_n$, we recall the sequence $\mathcal{S}(\boldsymbol{\lambda}) = \big(\mathbf{S}_1(\boldsymbol{\lambda}), \mathbf{S}_2(\boldsymbol{\lambda}), \dots\big)$ of elements in $\{0,1\}^n$ from Section 7.1.

17.1. $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ Model

We begin with the following definition for the weights of the vertex model we will use here.

Definition 17.1.1. Fix two n-tuples of integers $\mathbf{A} = (A_1, A_2, \dots, A_{n+1}) \in \mathbb{Z}^{n+1}_{\geqslant 0}$ and $\mathbf{C} = (C_1, C_2, \dots, C_{n+1}) \in \mathbb{Z}^{n+1}_{\geqslant 0}$ such that $A_k, B_k \in \{0,1\}$ for each $k \in [2, n+1]$ and $A_1, C_1 \geqslant 0$.

Further fix two integers $b, d \in [0, n+1]$. For any complex number $x \in \mathbb{C}$, define the vertex weights

(17.1.1)  $= w_x(\mathbf{A}, b; \mathbf{C}, d)$

where the values of $w_x(\mathbf{A}, b; \mathbf{C}, d)$ are tabulated below (recalling $\mathbf{A}_i^+, \mathbf{A}_i^-, \mathbf{A}_{ij}^{+-}$ from (8.1.1)), assuming that $1 \leqslant i < j \leqslant n+1$:

(17.1.2)

$w_x(\mathbf{A}, 0; \mathbf{A}, 0)$ $= 1$	$w_x(\mathbf{A}, i; \mathbf{A}, i)$ $= x(-q)^{\mathbf{1}_{i \geqslant 2} \cdot A_i} q^{\mathbf{A}_{[i+1,n+1]}}$
$w_x(\mathbf{A}, 0; \mathbf{A}_i^-, i)$ $= x(1 - q^{A_i}) q^{\mathbf{A}_{[i+1,n+1]}}$	$w_x(\mathbf{A}, i; \mathbf{A}_i^+, 0)$ $= \mathbf{1}_{i=1} + \mathbf{1}_{i \geqslant 2} \mathbf{1}_{A_i = 0}$
$w_x(\mathbf{A}, i; \mathbf{A}_{ij}^{+-}, j)$ $= x(1 - q^{A_j}) q^{\mathbf{A}_{[j+1,n+1]}} \Big(\mathbf{1}_{i=1} + \mathbf{1}_{i \geqslant 2} \mathbf{1}_{A_i = 0} \Big)$	$w_x(\mathbf{A}, j; \mathbf{A}_{ji}^{+-}, i)$ $= 0$

with $w_x(\mathbf{A}, b; \mathbf{C}, d) = 0$ for any $(\mathbf{A}, b; \mathbf{C}, d)$ not listed above.

We once again view $\mathbf{A}$, b, $\mathbf{C}$, and d as indexing the colors of the paths vertically entering, horizontally entering, vertically exiting, and horizontally exiting through a vertex, respectively. Observe here that, since we impose $A_k, C_k \in \{0, 1\}$ for each $k \in [2, n+1]$, the colors $\{2, 3, \ldots, n+1\}$ are all fermionic. However, since the integers A_1 and C_1 are permitted to be of unbounded size, the color 1 is bosonic. Thus, the model (17.1.1)–(17.1.2) is of boson-fermion nature. In fact, the $w_x(\mathbf{A}, b; \mathbf{C}, d)$ are special limits of the $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ weights $\mathcal{R}_{x,y}^{(2;n)}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$ introduced and evaluated in Definition 3.2.1 and Theorem 4.3.2, respectively; we will see this more explicitly in Section 17.7.2 below.

Remark 17.1.2. The $w_x(\mathbf{A}, b; \mathbf{C}, d)$ weights reduce to the $L_x(\mathbf{A}, b; \mathbf{C}, d)$ ones from Definition 13.2.1 by choosing $A_1 = C_1 = 0$; restricting b, d to values in the set $\{2, \ldots, n+1\}$; shifting all color labels downwards by one unit; and replacing q with t. This can be verified directly, using the explicit forms for the w_x and L_x weights. It can also be viewed as a consequence of the facts that these w_x and L_x can both be obtained as the same limit of the $\mathcal{R}^{(2;n)}$ and $\mathcal{R}^{(1;n)}$ fused weights (from Definition 3.2.1), respectively, and that the former degenerate to the latter in the absence of any color 1 arrows.

Next, similarly to as in Section 6.2, we will define a certain vector space and a family of operators on it. To that end, let $\mathfrak{A}(n)$ denote the set of all $(n+1)$-tuples $(A_1, A_2, \ldots, A_{n+1}) \in \mathbb{Z}_{\geqslant 0}^{n+1}$ such that

$A_k \in \{0,1\}$, for each $k \in [2, n+1]$. We associate with this a vector space $U(n)$ (over $\mathbb{C}$) as follows:

$$U(n) := \operatorname{Span}_{\mathbb{C}} \big\{ |\mathbf{A}\rangle \big\}_{\mathbf{A} \in \mathfrak{A}(n)},$$

and its dual

$$U(n)^* := \operatorname{Span}_{\mathbb{C}} \big\{ \langle \mathbf{A}| \big\}_{\mathbf{A} \in \mathfrak{A}(n)}.$$

We introduce an inner product between $U(n)$ and $U(n)^*$ by imposing that $\langle \mathbf{C} | \mathbf{A} \rangle = \mathbf{1}_{\mathbf{A} = \mathbf{C}}$.

Next, fix a nonnegative integer K and consider the $(K+1)$-fold tensor products of such spaces,

$$\begin{aligned} \mathbb{U}^K = \mathbb{U}^K(n) &:= U_0(n) \otimes U_1(n) \otimes \cdots \otimes U_K(n); \\ (\mathbb{U}^K)^* = \mathbb{U}^K(n)^* &:= U_0(n)^* \otimes U_1(n)^* \otimes \cdots \otimes U_K(n)^*, \end{aligned}$$

where each $U_i(n)$ (and $U_i(n)^*$) denotes a copy of $U(n)$ (and $U(n)^*$, respectively). The inner product between $U(n)$ and $U(n)^*$ induces one between $\mathbb{U}^K$ and $(\mathbb{U}^K)^*$.

For a $(K+1)$-tuple $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K)$ of basis vectors in $\mathfrak{A}(n)$, denote the elements

$$|\mathcal{A}\rangle = |\mathbf{A}_0\rangle \otimes |\mathbf{A}_1\rangle \otimes \cdots \otimes |\mathbf{A}_K\rangle \in \mathbb{U}^K; \qquad \langle \mathcal{A}| = \langle \mathbf{A}_0| \otimes \langle \mathbf{A}_1| \otimes \cdots \otimes \langle \mathbf{A}_K| \in (\mathbb{U}^K)^*.$$

Then, $\mathbb{U}^K$ and $(\mathbb{U}^K)^*$ are spanned by such vectors of the form $|\mathcal{A}\rangle$ and $\langle \mathcal{A}|$, respectively.

We next define a family of transfer operators on $\mathbb{U}^K$ and $(\mathbb{U}^K)^*$, similarly to in Definition 13.2.3 and Definition 13.2.4.

Definition 17.1.3. Fix a complex number $x \in \mathbb{C}$; two $(K+1)$-tuples $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K)$ and $\mathcal{C} = (\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_K)$ of basis vectors in $\mathfrak{A}(n)$; and two indices $b, d \in [0, n+1]$. Define

$$w_x(\mathcal{A}, b; \mathcal{C}, d) = \sum_{\mathfrak{J}} \prod_{i=0}^{K} w_x(\mathbf{A}_i, j_i; \mathbf{C}_i, j_{i+1}),$$

where we sum over all sequences $\mathfrak{J} = (j_0, j_1, \ldots, j_K)$ of indices in $[0, n+1]$ such that $j_0 = b$ and $j_{K+1} = d$. By arrow conservation, this sum is supported on at most one term.

Next, for any index $0 \in [1, n+1]$, we introduce the *transfer operator* $\mathfrak{B}_i(x) : \mathbb{U}^K \to \mathbb{U}^K$ by, for any $(K+1)$-tuple $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K)$ of basis vectors in $\mathfrak{A}(n)$, setting

$$\mathfrak{B}_i(x) |\mathcal{A}\rangle = \sum_{\mathcal{C}} w_x(\mathcal{A}, 0; \mathcal{C}, i) |\mathcal{C}\rangle,$$

where we sum over all $(K+1)$-tuples $\mathcal{C} = (\mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_K)$ of basis vectors in $\mathfrak{A}(n)$. This induces a dual action of $\mathfrak{B}_i(x)$ on $(\mathbb{U}^K)^*$, defined by

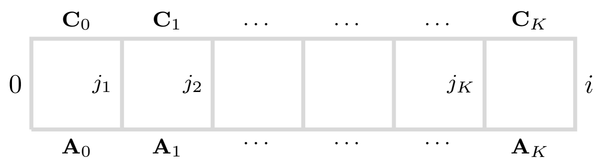
$$\langle \mathcal{C} | \mathfrak{B}_i(x) = \sum_{\mathcal{A}} w_x(\mathcal{A}, 0; \mathcal{C}, i) \langle \mathcal{A}|,$$

where now we sum over all $(K+1)$-tuples $\mathcal{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K)$ of basis vectors in $\mathfrak{A}(n)$.

The operators $\mathfrak{B}_i(x)$ can also be expressed in diagrammatic notation, as follows:

$$\mathfrak{B}_i(x) |\mathcal{A}\rangle = \sum_{\mathcal{C}} \left(\begin{array}{c|c|c|c|c|c|c|c} & \mathbf{C}_0 & \mathbf{C}_1 & \cdots & \cdots & \cdots & \mathbf{C}_K & \\ \hline 0 & & & & & & & i \\ \hline & \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \cdots & \cdots & \mathbf{A}_K & \end{array} \right) |\mathcal{C}\rangle. \tag{17.1.3}$$

Note that each face weight appearing on the right-hand side of (17.1.3) depends on the same parameter, x. The quantity



is a one-row partition function in the model (17.1.2), and can be calculated by multiplying the weights of each face from left to right, noting that the integer values j_i prescribed to all internal vertical edges are fixed by arrow conservation.

The following exchange relation for the transfer operators $\mathfrak{B}_i(x)$ will be useful for us. For $K = 0$, it can either be verified directly or deduced from the Yang–Baxter equation Proposition 3.2.4 for the $\mathcal{R}_{x,1}^{(2;n)}$ weights (since the content of Section 17.7.2 below implies that the weights w_x (17.1.2) are limits of these fused weights). For $K \geqslant 1$, it then follows from $K + 1$ applications of the Yang–Baxter equation. We omit further details of this proof, since it is very similar to that of Lemma 6.1.3.

Proposition 17.1.4. *Fix complex numbers $x, y \in \mathbb{C}$; an integer $K \geqslant 0$; and indices $0 \leqslant i < j \leqslant n + 1$. The transfer operators (17.1.3) on $\mathbb{U}^K$ satisfy the exchange relations*

$$\mathfrak{B}_i(x)\mathfrak{B}_j(y) = \frac{(1-q)y}{y-qx}\mathfrak{B}_i(y)\mathfrak{B}_j(x) + \frac{y-x}{y-qx}\mathfrak{B}_j(y)\mathfrak{B}_i(x). \tag{17.1.4}$$

17.2. $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ Partition Functions

In this section we define certain partition functions for the vertex model with weights w_x from Definition 17.1.1, which will be useful for combinatorially interpreting the integral appearing in (16.4.6). To that end, we first require some notation.

Fix an integer $M \geqslant 0$ and signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$. Further let K be any integer such that $K \geqslant \max_{i \in [1,n]} \max \mathfrak{T}\big(\boldsymbol{\lambda}^{(i)}\big)$. Additionally fix two integers $m, N \geqslant 1$ as well as a set $\mathcal{A} = \{a_1 < \cdots < a_m\} \subset [1, N+m]$, and denote its complement by $\bar{\mathcal{A}} = \{\bar{a}_1 < \cdots < \bar{a}_N\} = [1, N+m] \setminus \mathcal{A}$. From these sets, for all $1 \leqslant i \leqslant N + m$ we define

$$b_i(\mathcal{A}) = \begin{cases} 0, & i = a_j, \quad j \in [1, m], \\ 1, & i = \bar{a}_j, \quad j \in [1, N]. \end{cases}$$

We introduce a family[1] of partition functions $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m})$ in the model (17.1.1)–(17.1.2) as follows:

$$
(17.2.1)\qquad Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) =
\begin{array}{c|c|c|c|c|c|c|c}
 & (0,\mathbf{e}_0) & (0,\mathbf{S}_1(\boldsymbol{\lambda})) & \cdots & \cdots & & (0,\mathbf{S}_K(\boldsymbol{\lambda})) & \\
\hline
0 & x_{N+m} & & \cdots & \cdots & & x_{N+m} & b_{N+m}(\mathcal{A}) \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
\vdots & \vdots & & & & & \vdots & \vdots \\
\hline
0 & x_2 & & \cdots & \cdots & & x_2 & b_2(\mathcal{A}) \\
\hline
0 & x_1 & & \cdots & \cdots & & x_1 & b_1(\mathcal{A}) \\
\hline
 & (N,\mathbf{e}_0) & (0,\mathbf{S}_1(\boldsymbol{\mu})) & \cdots & \cdots & & (0,\mathbf{S}_K(\boldsymbol{\mu})) &
\end{array}
$$

where the boundary conditions prescribed to the leftmost column inject N bosonic arrows at the base of the lattice; all other all columns have analogous boundary conditions to those used to define the $G_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ functions from Definition 7.1.3 (see also the left side of Figure 7.1), namely, they inject fermionic arrows at the base and eject fermionic arrows at the top of the lattice. Note that exactly N of the integers $b_i(\mathcal{A})$ for $i \in [1, N+m]$ are equal to 1; the remaining m are equal to 0. Hence the number of bosonic arrows injected into the lattice (17.2.1) is equal to the number ejected from it, and the partition function (17.2.1) is generically nonzero.

We may use the row operators (17.1.3) to write the partition function (17.2.1) algebraically as

$$
(17.2.2)\qquad Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) = \Big\langle 0,\boldsymbol{\lambda}\Big|\mathfrak{B}_{b_{N+m}(\mathcal{A})}(x_{N+m})\cdots\mathfrak{B}_{b_1(\mathcal{A})}(x_1)\Big|N,\boldsymbol{\mu}\Big\rangle,
$$

with row operators given by

$$
\mathfrak{B}_{b_i(\mathcal{A})}(x) = \begin{cases} \mathfrak{B}_0(x), & i = a_j, \quad j \in [1,m], \\ \\ \mathfrak{B}_1(x), & i = \bar{a}_j, \quad j \in [1,N]. \end{cases}
$$

The vectors in (17.2.2) are defined as

$$
\Big\langle 0,\boldsymbol{\lambda}\Big| = \Big\langle 0,\mathbf{e}_0\Big| \otimes \bigotimes_{j=1}^{K} \Big\langle 0,\mathbf{S}_j(\boldsymbol{\lambda})\Big| \in \mathbb{U}^K(n)^*; \qquad \Big|N,\boldsymbol{\mu}\Big\rangle = \Big|N,\mathbf{e}_0\Big\rangle \otimes \bigotimes_{j=1}^{K} \Big|0,\mathbf{S}_j(\boldsymbol{\mu})\Big\rangle \in \mathbb{U}^K(n).
$$

17.3. Properties of $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$

The partition functions (17.2.1) satisfy two basic properties which are easily seen to characterize them. We derive these properties in the following two propositions. In what follows, only the

[1]We use the word *family* in the sense of varying the set $\mathcal{A} \subset [1, N+m]$ over all $\binom{N+m}{m}$ possibilities.

$r = q^{-1/2}$ case of the function $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; r \mid 0; 0)$ will be useful for us. So, to ease notation, we will abbreviate in the remainder of this chapter

$$\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}; q^{-1/2} \mid 0; 0). \tag{17.3.1}$$

As previously, we refer to $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x})$ as a pre-fused LLT function.

Proposition 17.3.1. *Fix an integer $M \geqslant 1$ and signature sequences $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{SeqSign}_{n;M}$. When $\mathcal{A} = \{N+1, \dots, N+m\}$, we have*

$$Z^{\{N+1,\dots,N+m\}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \dots, x_{N+m}) = q^{nMN}(q;q)_N \prod_{j=1}^{N} x_j^{K+1} \cdot \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_{N+1}, \dots, x_{N+m}), \tag{17.3.2}$$

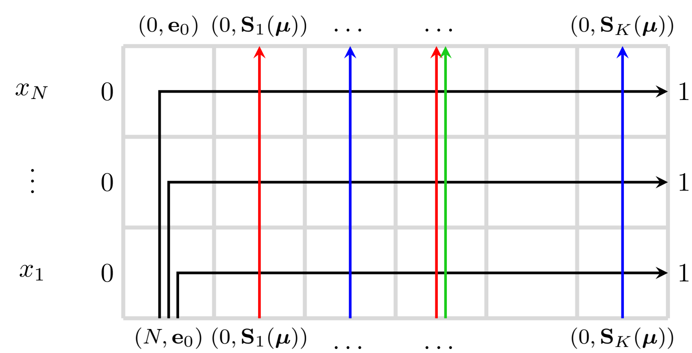
where $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ denotes a pre-fused LLT polynomial (17.3.1).

Proof. In the case $\mathcal{A} = \{N+1, \dots, N+m\}$, (17.2.2) becomes

$$Z^{\{N+1,\dots,N+m\}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \dots, x_{N+m}) = \Big\langle 0, \boldsymbol{\lambda} \Big| \mathfrak{B}_0(x_{N+m}) \cdots \mathfrak{B}_0(x_{N+1}) \mathfrak{B}_1(x_N) \cdots \mathfrak{B}_1(x_1) \Big| N, \boldsymbol{\mu} \Big\rangle.$$

Now consider the partition function

(17.3.3)



which consists of a unique configuration with weight $q^{nMN}(q;q)_N \prod_{j=1}^{N} x_j^{K+1}$. Indeed, the N bosonic paths which enter via the base of the leftmost column turn right into each of the N rows, and completely saturate all edges in those rows. This in turn forces all nM fermionic paths which enter at the base of the remaining columns to propagate in vertical straight lines to the top of the lattice. It is then clear that the partition function has a unique configuration and its weight is readily computed from (17.1.1)–(17.1.2).

Matching each of the N rows in (17.3.3) with their algebraic equivalents (17.1.3), we obtain

$$\mathfrak{B}_1(x_N) \cdots \mathfrak{B}_1(x_1) \Big| N, \boldsymbol{\mu} \Big\rangle = q^{nMN}(q;q)_N \prod_{j=1}^{N} x_j^{K+1} \cdot \Big| 0, \boldsymbol{\mu} \Big\rangle.$$

Thus, to establish (17.3.2), it suffices to verify

$$\Big\langle 0, \boldsymbol{\lambda} \Big| \mathfrak{B}_0(x_{N+m}) \cdots \mathfrak{B}_0(x_{N+1}) \Big| 0, \boldsymbol{\mu} \Big\rangle = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_{N+1}, \dots, x_{N+m}). \tag{17.3.4}$$

To that end, recall the transfer operators $\mathsf{D}(x)$ from Definition 13.2.4 for the L_x weights from Definition 13.2.1. By Remark 17.1.2, Definition 17.1.3, Definition 13.2.3, and Definition 13.2.4, we have

$$\left\langle 0, \boldsymbol{\lambda} \middle| \mathfrak{B}_0(x_{N+m}) \cdots \mathfrak{B}_0(x_{N+1}) \middle| 0, \boldsymbol{\mu} \right\rangle = \left\langle \boldsymbol{\lambda} \middle| \mathsf{D}(x_{N+m}) \cdots \mathsf{D}(x_{N+1}) \middle| \boldsymbol{\mu} \right\rangle. \tag{17.3.5}$$

Moreover, Definition 13.2.4, (13.2.1), the last statement of (8.4.3), and Definition 7.1.1 together imply

$$\left\langle \boldsymbol{\lambda} \middle| \mathsf{D}(x_{N+m}) \cdots \mathsf{D}(x_{N+1}) \middle| \boldsymbol{\mu} \right\rangle = \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_{N+1}, \ldots, x_{N+m}). \tag{17.3.6}$$

Together, (17.3.5) and (17.3.6) establish (17.3.4) and thus the proposition. $\square$

Proposition 17.3.2. *Fix a set $\mathcal{A} = \{a_1 < \cdots < a_m\} \subset [1, N+m]$ and an integer $\alpha \in [1, N+m-1]$ such that $\alpha \in \mathcal{A}$ and $\alpha + 1 \notin \mathcal{A}$. Define the shifted set*

$$\mathcal{A}(\alpha \to \alpha+1) = \{a_1 < \cdots < a_{\ell-1} < \alpha+1 < a_{\ell+1} < \cdots < a_m\},$$

where $\ell \in [1, n]$ is the integer such that $a_\ell = \alpha$ (necessarily $a_{\ell+1} > \alpha+1$, since $\alpha+1 \notin \mathcal{A}$). We have the recursion relation

$$Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}} = \frac{x_{\alpha+1} - q x_\alpha}{x_{\alpha+1} - x_\alpha} \cdot \mathfrak{s}_\alpha \left(Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}(\alpha \to \alpha+1)} \right) + \frac{(1-q) x_{\alpha+1}}{x_\alpha - x_{\alpha+1}} \cdot Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}(\alpha \to \alpha+1)}, \tag{17.3.7}$$

where $\mathfrak{s}_\alpha$ denotes the transposition of the variables $(x_\alpha, x_{\alpha+1})$.

Proof. Recall the commutation relation (17.1.4). After choosing $i = 0$, $j = 1$, $x = x_\alpha$, $y = x_{\alpha+1}$, and rearranging, (17.1.4) becomes

$$\mathfrak{B}_1(x_{\alpha+1}) \mathfrak{B}_0(x_\alpha) = \frac{x_{\alpha+1} - q x_\alpha}{x_{\alpha+1} - x_\alpha} \mathfrak{B}_0(x_\alpha) \mathfrak{B}_1(x_{\alpha+1}) + \frac{(1-q) x_{\alpha+1}}{x_\alpha - x_{\alpha+1}} \mathfrak{B}_0(x_{\alpha+1}) \mathfrak{B}_1(x_\alpha). \tag{17.3.8}$$

We use this relation in the algebraic formula (17.2.2) for $Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}(x_1, \ldots, x_{N+m})$; it affects the operators standing at positions α and $\alpha+1$ in the operator product, namely, $\mathfrak{B}_1(x_{\alpha+1}) \mathfrak{B}_0(x_\alpha)$. One sees that the two terms on the right hand side of (17.3.8) correspond to the respective two terms on the right hand side of (17.3.7), and the proof is completed. $\square$

Remark 17.3.3. Proposition 17.3.2 allows us to incrementally increase the elements of the set $\mathcal{A}$ which indexes $Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}$, at the expense of generating two terms for each application of the rule (17.3.7). At the end of this ordering procedure, one obtains a sum over partition functions of the form $Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\{N+1, \ldots, N+m\}}$; these are known explicitly by Proposition 17.3.1. Proposition 17.3.2 and Proposition 17.3.1 thus completely determine the family (17.2.2).[2]

17.4. Evaluation of the Partition Function $Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}$

In this section we solve for $Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}$ that satisfies the two defining relations (17.3.2) and (17.3.7). Before we do so, it is convenient to define a version of the partition function (17.2.1) with reciprocated variables:

$$\bar{Z}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}(x_1, \ldots, x_{N+m}) = \prod_{i=1}^{N+m} x_i^{K+1} \cdot Z_{\boldsymbol{\lambda}/\boldsymbol{\mu}}^{\mathcal{A}}(x_1^{-1}, \ldots, x_{N+m}^{-1}). \tag{17.4.1}$$

[2] In fact, these propositions overdetermine the family (17.2.2), since there are multiple ways of performing the ordering $\mathcal{A} \to \{N+1, \ldots, N+m\}$. The fact that all ways of doing this ordering lead to the same answer is a consequence of the Yang–Baxter equation.

In terms of these new conventions, (17.3.2) and (17.3.7) translate to

$$(17.4.2)\qquad \bar{Z}^{\{N+1,\dots,N+m\}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) = q^{nMN}(q;q)_N \prod_{i=1}^{m}(x_{N+i})^{K+1}\cdot \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_{N+1}^{-1},\dots,x_{N+m}^{-1}),$$

and

$$(17.4.3)\qquad \bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \frac{x_\alpha - qx_{\alpha+1}}{x_\alpha - x_{\alpha+1}}\cdot \mathfrak{s}_\alpha\left(\bar{Z}^{\mathcal{A}(\alpha\to\alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\right) + \frac{(1-q)x_\alpha}{x_{\alpha+1}-x_\alpha}\cdot \bar{Z}^{\mathcal{A}(\alpha\to\alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}.$$

We may also write (17.4.3) as

$$(17.4.4)\qquad \bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}} = T_\alpha\left(\bar{Z}^{\mathcal{A}(\alpha\to\alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}\right), \qquad T_\alpha = q - \frac{x_\alpha - qx_{\alpha+1}}{x_\alpha - x_{\alpha+1}}(1-\mathfrak{s}_\alpha),$$

where T_α denotes a Hecke algebra generator in its polynomial representation (which we recall from Section 13.1).

Now we can state the following contour integral formula for $\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. In the below, we recall the pre-fused LLT polynomial $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ from (17.3.1).

Theorem 17.4.1. *The partition function* (17.2.1) *is given by the following multiple integral formula:*

(17.4.5)
$$\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) = \frac{q^{nMN}}{(2\pi\mathbf{i})^m}(q;q)_N \prod_{i=1}^{m} x_{a_i} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m}\frac{w_i-w_j}{w_i-qw_j}$$
$$\times \prod_{i=1}^{m}\prod_{j=a_i+1}^{N+m}\frac{w_i-qx_j}{w_i-x_j}\cdot\prod_{i=1}^{m}\frac{w_i^K dw_i}{w_i-x_{a_i}},$$

where C is a positively oriented contour that surrounds all points in qC and in the set $\{0\}\cup\{x_1,\dots,x_{N+m}\}$.

Proof. By Remark 17.3.3, it suffices to check that (17.4.5) satisfies the properties (17.4.2) and (17.4.4).

Property (17.4.2). Setting $a_i = N+i$ for all $i\in[1,m]$, formula (17.4.5) becomes

(17.4.6)
$$\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) = \frac{q^{nMN}}{(2\pi\mathbf{i})^m}(q;q)_N \prod_{i=1}^{m} x_{N+i} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m}\frac{w_i-w_j}{w_i-qw_j}$$
$$\times \prod_{i=1}^{m}\prod_{j=N+i+1}^{N+m}\frac{w_i-qx_j}{w_i-x_j}\cdot\prod_{i=1}^{m}\frac{w_i^K dw_i}{w_i-x_{N+i}}.$$

The integrals in (17.4.6) may now be performed sequentially, starting with the integral over w_m, and working through to w_1. We see that the integrand of (17.4.6) picks up a simple pole at $w_m = x_{N+m}$, and has no other singularity in that variable. Indeed, by virtue of the fact that C is contained in $q^{-1}C$, the points $q^{-1}w_i$ for $i\in[1,m-1]$ are not enclosed by the w_m integration contour; furthermore, the integrand is non-singular at $w_m=0$, since $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1})$ is at most of degree K in w_m^{-1} (which can can be deduced directly from the vertex model interpretation for $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, w_2^{-1}, \ldots, w_m^{-1})$). Evaluating the residue of the pole at $w_m = x_{N+m}$, we find that

$$\begin{aligned}\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = \frac{q^{nMN}}{(2\pi\mathbf{i})^{m-1}}(q;q)_N \prod_{i=1}^{m-1} x_{N+i} \cdot (x_{N+m})^{K+1} \\ \times \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_{n-1}^{-1}, x_{N+m}^{-1}) \prod_{1 \leqslant i < j \leqslant m-1} \frac{w_i - w_j}{w_i - qw_j} \\ \times \prod_{i=1}^{m-1} \prod_{j=N+i+1}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \cdot \prod_{i=1}^{m-1} \frac{w_i - x_{N+m}}{w_i - qx_{N+m}} \frac{w_i^K dw_i}{w_i - x_{N+i}}.\end{aligned}$$

After some cancellation of factors within the integrand, this simplifies to

$$\begin{aligned}\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = \frac{q^{nMN}}{(2\pi\mathbf{i})^{m}}(q;q)_N \prod_{i=1}^{m-1} x_{N+i} \cdot (x_{N+m})^{K+1} \\ \times \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_{n-1}^{-1}, x_{N+m}^{-1}) \prod_{1 \leqslant i < j \leqslant m-1} \frac{w_i - w_j}{w_i - qw_j} \\ \times \prod_{i=1}^{m-1} \prod_{j=N+i+1}^{N+m-1} \frac{w_i - qx_j}{w_i - x_j} \cdot \prod_{i=1}^{m-1} \frac{w_i^K dw_i}{w_i - x_{N+i}}.\end{aligned}$$

We are left with an $(m-1)$-fold integral that has the same structure as that which we started with. Indeed, the w_{m-1} variable develops a unique simple pole at $w_{n-1} = x_{N+m-1}$; this integration may be explicitly performed, similarly to above. Iterating this procedure all the way down to w_1, we arrive at the desired result, (17.4.2).

Property (17.4.4). The proof of (17.4.4) makes use of the elementary identity

(17.4.7)

$$T_\alpha\left(\frac{x_{\alpha+1}}{w - x_{\alpha+1}}\right) = \frac{x_\alpha - qx_{\alpha+1}}{x_\alpha - x_{\alpha+1}} \cdot \frac{x_\alpha}{w - x_\alpha} + \frac{(1-q)x_\alpha}{x_{\alpha+1} - x_\alpha} \cdot \frac{x_{\alpha+1}}{w - x_{\alpha+1}} = \frac{x_\alpha(w - qx_{\alpha+1})}{(w - x_\alpha)(w - x_{\alpha+1})},$$

as well as the fact that

$$T_\alpha\Big(f(x_\alpha, x_{\alpha+1})g(x_\alpha, x_{\alpha+1})\Big) = f(x_\alpha, x_{\alpha+1}) \cdot T_\alpha\Big(g(x_\alpha, x_{\alpha+1})\Big) \tag{17.4.8}$$

for functions $f(x_\alpha, x_{\alpha+1})$ which are symmetric in $(x_\alpha, x_{\alpha+1})$. Choose $\alpha \in [1, N+m-1]$ such that $\alpha \in \mathcal{A}$ and $\alpha + 1 \notin \mathcal{A}$. Further, fix $\ell \in [1, m]$ such that $a_\ell = \alpha$ (and note that $a_{\ell+1} > \alpha + 1$). We act on $\bar{Z}^{\mathcal{A}(\alpha \to \alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, as given by (17.4.5), with the operator T_α. Denoting $\tilde{a}_i = a_i$ for each $i \in [1, m] \setminus \{\ell\}$ and $\tilde{a}_\ell = a_\ell + 1 = \alpha + 1$, the product

$$\prod_{i \neq \ell} x_{a_i} \cdot \prod_{i=1}^{m} \prod_{j=\tilde{a}_i+1}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \cdot \prod_{i \neq \ell} \frac{1}{w_i - x_{a_i}}$$

is symmetric in $(x_\alpha, x_{\alpha+1})$; by virtue of (17.4.8), we therefore have

$$\begin{aligned}&T_\alpha\left(\bar{Z}^{\mathcal{A}(\alpha \to \alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})\right) \\ &\quad = \frac{q^{nMN}}{(2\pi\mathbf{i})^m}(q;q)_N \prod_{i \neq \ell} x_{a_i} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_m^{-1}) \prod_{1 \leqslant i < j \leqslant m} \frac{w_i - w_j}{w_i - qw_j}\end{aligned}$$

$$\times \prod_{i=1}^{m} \prod_{j=\tilde{a}_i+1}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \cdot \prod_{i \neq \ell} \frac{1}{w_i - x_{a_i}} \cdot T_\alpha \left(\frac{x_{\alpha+1}}{w_\ell - x_{\alpha+1}} \right) \prod_{i=1}^{m} w_i^K dw_i.$$

Using the relation (17.4.7), we conclude that

$$\begin{aligned}
&T_\alpha \left(\bar{Z}^{\mathcal{A}(\alpha\to\alpha+1)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) \right) \\
&= \frac{q^{nMN}}{(2\pi\mathbf{i})^m} (q;q)_N \prod_{i\neq \ell} x_{a_i} \cdot x_\alpha \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i - w_j}{w_i - qw_j} \\
&\qquad \times \prod_{i=1}^{m} \prod_{j=\tilde{a}_i+1}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \cdot \frac{w_\ell - qx_{\alpha+1}}{w_\ell - x_{\alpha+1}} \cdot \prod_{i\neq\ell} \frac{1}{w_i - x_{a_i}} \cdot \frac{1}{w_\ell - x_\alpha} \prod_{i=1}^{m} w_i^K dw_i \\
&= \frac{q^{nMN}}{(2\pi\mathbf{i})^m} (q;q)_N \prod_{i=1}^{m} x_{a_i} \oint_C 1 \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i - w_j}{w_i - qw_j} \\
&\qquad \times \prod_{i=1}^{m} \prod_{j=a_i+1}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \cdot \prod_{i=1}^{m} \frac{1}{w_i - x_{a_i}} \prod_{i=1}^{m} w_i^K dw_i,
\end{aligned}$$

where the final expression is recognized as $\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})$. This validates (17.4.4). $\square$

17.5. Degree Counting

In this section we pause to determine the degree of $\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, as a polynomial in $(x_1, \ldots, x_{N+m})$.

Proposition 17.5.1. *$\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})$ is a homogeneous polynomial in $(x_1, \ldots, x_{N+m})$ of degree $m(K+1) - |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$, and is divisible by x_{a_i} for all $i \in [1, m]$.*

PROOF. Examining the weights (17.1.2), we see that vertices (or faces, which we use to diagrammatically represent them) whose right vertical edge is occupied by a path (of any color) receive a weight of x; conversely, vertices in which this edge is unoccupied have degree zero in x. Working out the degree of $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})$ is then a matter of counting the total number of right vertical edges which are occupied by a path; this number is invariant across all lattice configurations, which immediately yields the homogeneity claim. The number of right vertical edges occupied by fermionic colors $\{2, \ldots, n+1\}$ is equal to $|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$. The number of right vertical edges occupied by the bosonic color 1 is equal to $N(K+1)$. It follows that

$$\deg Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}| + N(K+1),$$

and accordingly (by (17.4.1)),

$$\deg \bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = (N+m)(K+1) - |\boldsymbol{\lambda}| + |\boldsymbol{\mu}| - N(K+1) = m(K+1) - |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|.$$

For the divisibility claim, note that the effect of reciprocating all rapidities and multiplying the partition function by $\prod_{i=1}^{N+m} x_i^{K+1}$ (this is exactly one x_i factor for every vertex in row i, for all $i \in [1, N+m]$) is to reverse the above rule; namely, vertices in (17.1.2) with an empty right vertical edge will receive a weight of x, while those with an occupied right edge have degree zero in x. As a result, any row of the partition function $\bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})$ that has an empty right vertical edge will be divisible by the rapidity variable associated to that row; the rows which have this property are precisely those encoded by $\mathcal{A} = \{a_1 < \cdots < a_m\}$. $\square$

17.6. Limiting Procedure

The integral formula (17.4.5) for $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ depends on a set of variables $(x_1, \ldots, x_{N+m})$, which up to this point were indeterminates. We now provide a certain limiting procedure on $(x_1, \ldots, x_{N+m})$ which transforms (17.4.5) to match directly with (16.4.6).

Let us define

$$\mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = q^{-nMN}(q;q)_N^{-1} \prod_{i=1}^{m} x_{a_i}^{-1} \cdot \bar{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}). \tag{17.6.1}$$

17.6.1. Sending $x_{a_1}, \ldots, x_{a_m}$ to 0. We begin by taking each of the variables $x_{a_1}, \ldots, x_{a_m}$ in (17.6.1) to zero; using (17.4.5), this limit is computed as

$$\begin{aligned}\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) &= \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i - w_j}{w_i - qw_j} \\ &\quad \times \prod_{i=1}^{m} \prod_{\substack{j=a_i+1 \\ j\neq a_{i+1},\ldots,a_m}}^{N+m} \frac{w_i - qx_j}{w_i - x_j} \prod_{i=1}^{m} w_i^{K-1} dw_i.\end{aligned} \tag{17.6.2}$$

17.6.2. Principal Specializations. For the remaining variables x_b with $b \notin \mathcal{A}$, we make the following choice. We let $a_1 = 1$, and write $a_{m+1} = N + m + 1$ by agreement. We further assume for each $i \in [1, m]$ that $a_{i+1} - a_i - 1 = Lg_i$, where $L \geqslant 1$ and $g_i \geqslant 0$ are fixed integers. Then for each $i \in [1, m]$ we set

$$\left(x_{a_i+1}, \ldots, x_{a_{i+1}-1}\right) = \left(y_1^{(1)}, \ldots, y_L^{(1)}\right) \cup \cdots \cup \left(y_1^{(g_i)}, \ldots, y_L^{(g_i)}\right) \tag{17.6.3}$$

with

$$y_c^{(j)} = yq^{L-c}, \qquad \text{for all } j \in [1, g_i] \text{ and } c \in [1, L]. \tag{17.6.4}$$

This amounts to splitting the group of variables x_b for $b \in (a_i, a_{i+1})$ into g_i bundles each of cardinality L, and principally specializing within the bundles. Note that we use the same indeterminate, y, as base of every geometric progression. In the equations that follow, we denote change of variables (17.6.3)–(17.6.4) by the symbol †. Applying the change of variables (17.6.3)–(17.6.4) to (17.6.2), we obtain

$$\begin{aligned}&\left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})\right]^{\dagger} \\ &\qquad = \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1}, \ldots, w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i - w_j}{w_i - qw_j} \\ &\qquad\quad \times \prod_{1\leqslant c\leqslant d\leqslant m} \left(\frac{w_c - yq^L}{w_c - y}\right)^{g_d} \prod_{i=1}^{m} w_i^{K-1} dw_i.\end{aligned} \tag{17.6.5}$$

Due to Proposition 17.5.1 and the fact that we divide by $x_{a_1}, \ldots, x_{a_m}$ in the definition (17.6.1), when (17.6.5) is viewed as a polynomial in y it must take the form

$$\left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})\right]^{\dagger} = y^{mK-|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|} \cdot C_{\{g_1,\ldots,g_m\}}(q^L) \tag{17.6.6}$$

where $C_{\{g_1,\dots,g_m\}}$ is independent of y, but depends polynomially on q^L.

17.6.3. Sending $q^L \to \infty$. Using (17.6.5), we compute

$$(17.6.7)\quad (-y)^{-\sum_{i=1}^m ig_i} \lim_{q^L\to\infty} q^{-L\sum_{i=1}^m ig_i} \left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) \right]^{\dagger}$$
$$= \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i-w_j}{w_i-qw_j} \prod_{1\leqslant c\leqslant d\leqslant m} \frac{1}{(w_c-y)^{g_d}} \prod_{i=1}^m w_i^{K-1} dw_i.$$

Now let us specify the nonnegative integers $\{g_1,\dots,g_m\}$. Fix a partition $\nu = (\nu_1,\nu_2,\cdots,\nu_m)$ such that $|\nu| = |\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$ and let $\bar{\nu} = (\bar{\nu}_1,\bar{\nu}_2,\cdots,\bar{\nu}_m)$ denote the partition obtained from ν by complementation in an $m\times(K-1)$ box; namely, $\bar{\nu}_i = K-\nu_{m-i+1}-1$, for all $i\in[1,m]$. We define $g_i = \bar{\nu}_i - \bar{\nu}_{i+1}$, for all $i\in[1,m]$, where $\bar{\nu}_{m+1} = -1$ by agreement. This yields

$$\sum_{i=1}^m ig_i = \sum_{i=1}^m i(\bar{\nu}_i - \bar{\nu}_{i+1}) = \sum_{i=1}^m \bar{\nu}_i + m = mK - |\nu| = mK - |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|.$$

Comparing (17.6.6) and (17.6.7), we see that the integral on the right hand side of (17.6.7) is necessarily independent of y. Noting that the contour C has been chosen such that 0 and y are contained in C, we are at liberty to take $y=0$ in the right hand side of (17.6.7):

$$(17.6.8)\quad (-y)^{-\sum_{i=1}^m ig_i} \lim_{q^L\to\infty} q^{-L\sum_{i=1}^m ig_i} \left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) \right]^{\dagger}$$
$$= \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i-w_j}{w_i-qw_j} \prod_{i=1}^m w_i^{K-\bar{\nu}_i-2} dw_i$$
$$= \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_i-w_j}{w_i-qw_j} \prod_{i=1}^m w_i^{\nu_{m-i+1}-1} dw_i,$$

where we have used the fact that $\sum_{i\leqslant d\leqslant m} g_d = \bar{\nu}_i - \bar{\nu}_{m+1} = \bar{\nu}_i + 1$.

17.6.4. Match With $f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q)$. After relabelling integration variables in (17.6.8) by replacing w_i with w_{m-i+1} for all $i\in[1,m]$, and recalling that $\mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ is symmetric in its arguments, we obtain

$$(17.6.9)\quad (-y)^{-\sum_{i=1}^m ig_i} \lim_{q^L\to\infty} q^{-L\sum_{i=1}^m ig_i} \left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) \right]^{\dagger}$$
$$= \frac{1}{(2\pi\mathbf{i})^m} \oint_C \cdots \oint_C \mathcal{G}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(w_1^{-1},\dots,w_m^{-1}) \prod_{1\leqslant i<j\leqslant m} \frac{w_j-w_i}{w_j-qw_i} \prod_{i=1}^m w_i^{\nu_i-1} dw_i.$$

Matching with (16.4.6), we have proved that

(17.6.10)
$$(-y)^{-\sum_{i=1}^m ig_i} \lim_{q^L\to\infty} q^{-L\sum_{i=1}^m ig_i} \left[\lim_{x_{a_1}\to 0} \cdots \lim_{x_{a_m}\to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m}) \right]^{\dagger} = b_\nu(q) f^{\nu}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(q);$$

the expansion coefficients of (16.4.1) thus emerge as a direct limit of the partition function (17.2.1).

17.7. Limiting Procedure Applied to $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$

We begin with some preliminaries regarding vertex weights in Sections 17.7.1–17.7.4, before turning to the explicit computation of the limits (17.6.10) applied to the the partition function (17.6.1) in Sections 17.7.5–17.7.6.

The analysis in this section will make use of the $(m, n) = (2, n)$ cases of the $\mathcal{R}^{(m;n)}_{x,y}$ fused weights introduced in Definition 3.2.1 and provided explicitly by Theorem 4.3.2. It will be convenient for us to alter several aspects of the notation for these weights in this section. First, these fused weights implicitly depend on two additional parameters (L, M); here, we will relabel them to (L, M), in order to disambiguate them from the parameters used in previous sections of this chapter. Second, the argument for $\mathcal{R}^{(m;n)}_{x,y}$ is a quadruple of $(n+2)$-tuples $\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}' \in \mathbb{Z}^{n+2}_{\geqslant 0}$, with coordinates indexed by $[0, n+1]$, such that $|\mathbf{A}'| = \mathsf{M} = |\mathbf{C}'|$ and $|\mathbf{B}'| = \mathsf{L} = |\mathbf{D}'|$. Here, we omit the 0-th coordinate of these $(n+2)$-tuples from consideration. In particular, denoting $\mathbf{X}' = \big(\mathsf{M} - |\mathbf{X}|, \mathbf{X}\big)$ for $X \in \{A, C\}$ and $\mathbf{X}' = \big(\mathsf{L} - |\mathbf{X}|, \mathbf{X}\big)$ for $X \in \{B, D\}$, we write[3]

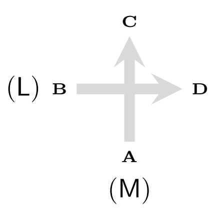
$$\mathcal{R}_{x/y}(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}) = \mathcal{R}^{(2;n)}_{x,y}(\mathbf{A}', \mathbf{B}'; \mathbf{C}', \mathbf{D}').$$

17.7.1. Doubly-Fused $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ Model. Fix four $(n+1)$-tuples $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbb{Z}^{n+1}_{\geqslant 0}$, with coordinates indexed by $[1, n+1]$, such that

$$A_1, B_1, C_1, D_1 \geqslant 0, \qquad \text{and} \qquad \max\{A_j, B_j, C_j, D_j\} \leqslant 1, \qquad \text{for} \qquad j \in [2, n+1].$$

Construct the n-tuple $\mathbf{V} = (V_1, V_2, \ldots, V_{n+1})$, where $V_1 = 0$ and $V_j = \min\{A_j, B_j, C_j, D_j\}$ for $j \in [2, n+1]$, and write $v = \sum_{i=1}^{n+1} V_i$. By Theorem 4.3.2, the doubly-fused $U_q\big(\widehat{\mathfrak{sl}}(2|n)\big)$ weights read

(17.7.1)



$$= \mathcal{R}_z(\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D})$$

$$\begin{aligned} &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{|\mathbf{A}|\leqslant \mathsf{M}} \mathbf{1}_{|\mathbf{C}|\leqslant \mathsf{M}} \mathbf{1}_{|\mathbf{B}|\leqslant \mathsf{L}} \mathbf{1}_{|\mathbf{D}|\leqslant \mathsf{L}} \\ &\quad \times (-1)^v q^{\varphi(\mathbf{V}, \mathbf{A}) - v\mathsf{M}} \frac{(q^{\mathsf{L}-v+1} z; q)_v}{(q^{\mathsf{L}-\mathsf{M}-v} z; q)_v} \omega_z^{(\mathsf{L}-v, \mathsf{M})} (\mathbf{A}, \mathbf{B} - \mathbf{V}; \mathbf{C}, \mathbf{D} - \mathbf{V}). \end{aligned}$$

Here, we used the fact that $\sum_{h: V_h = 1} A_{[h+1, n+1]} = \varphi(\mathbf{V}, \mathbf{A})$ and we recalled from (4.1.2) that

$$\begin{aligned} \omega_z^{(\mathsf{L}-v, \mathsf{M})} (\mathbf{A}, \mathbf{B} - \mathbf{V}; \mathbf{C}, \mathbf{D} - \mathbf{V}) &= z^{|\mathbf{D}| - |\mathbf{B}|} q^{|\mathbf{A}|\mathsf{L} - |\mathbf{D}|\mathsf{M} + v\mathsf{M} - v|\mathbf{A}|} \\ &\quad \times \sum_{\mathbf{P}} \Phi(\mathbf{C} - \mathbf{P}, \mathbf{C} + \mathbf{D} - \mathbf{V} - \mathbf{P}; q^{\mathsf{L}-v-\mathsf{M}} z, q^{-\mathsf{M}} z) \\ &\quad \times \Phi(\mathbf{P}, \mathbf{B} - \mathbf{V}; q^{-\mathsf{L}+v} z^{-1}, q^{-\mathsf{L}+v}), \end{aligned} \tag{17.7.2}$$

with the sum over all $\mathbf{P} = (P_1, P_2, \ldots, P_{n+1}) \in \mathbb{Z}^{n+1}_{\geqslant 0}$ such that $P_i \leqslant \min\{C_i, B_i - V_i\}$ for all $i \in [1, n+1]$, with the function Φ given by (4.1.1).

[3]Recall from Theorem 4.3.2 that these weights only depend on (x, y) through xy^{-1}.

Now fix complex numbers $s, x \in \mathbb{C}$. Inserting (17.7.2) into (17.7.1), analytically continuing in q^{M}, and setting $q^{-\mathsf{M}} = s^2$ and $z = s^{-1}x$ gives

(17.7.3)

$$
\begin{aligned}
&(-s)^{|\mathbf{D}|}\mathcal{R}_{x/s}(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) \\
&\quad = \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{|\mathbf{B}|\leqslant\mathsf{L}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathsf{L}}\cdot(-1)^{|\mathbf{D}|+v}q^{\varphi(\mathbf{V},\mathbf{A})}s^{|\mathbf{B}|}x^{|\mathbf{D}|-|\mathbf{B}|}q^{(|\mathbf{A}|-|\mathbf{C}|)(\mathsf{L}-v)}\frac{(s^{-1}q^{\mathsf{L}-v+1}x;q)_v}{(sq^{\mathsf{L}-v}x;q)_v} \\
&\qquad \times \sum_{\mathbf{P}} \frac{(sq^{\mathsf{L}-v}x;q)_{|\mathbf{C}|-|\mathbf{P}|}(q^{v-\mathsf{L}};q)_{|\mathbf{D}|-v}}{(sq^{\mathsf{L}-v}x;q)_{|\mathbf{C}|+|\mathbf{D}|-|\mathbf{P}|-v}}\frac{(sq^{v-\mathsf{L}}x^{-1};q)_{|\mathbf{P}|}(s^{-1}x;q)_{|\mathbf{B}|-|\mathbf{P}|-v}}{(q^{v-\mathsf{L}};q)_{|\mathbf{B}|-v}}(s^{-1}x)^{|\mathbf{P}|} \\
&\qquad \times q^{(\mathsf{L}-v)|\mathbf{P}|+\varphi(\mathbf{D}-\mathbf{V},\mathbf{C}-\mathbf{P})+\varphi(\mathbf{B}-\mathbf{V}-\mathbf{P},\mathbf{P})}\prod_{i=1}^{n+1}\frac{(q;q)_{C_i+D_i-V_i-P_i}}{(q;q)_{C_i-P_i}(q;q)_{D_i-V_i}}\frac{(q;q)_{B_i-V_i}}{(q;q)_{P_i}(q;q)_{B_i-P_i}}.
\end{aligned}
$$

17.7.2. Recovering the Model (17.1.1)–(17.1.2)**.** We consider the limits of the weights (17.7.3) as s tends to 0. Let us denote this limit by

$$
\mathbb{R}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \lim_{s\to 0}\left((-s)^{-|\mathbf{D}|}\mathcal{R}_z(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D})\Big|_{q^{\mathsf{M}}\mapsto s^{-2},\ z\mapsto x/s}\right).
$$

Inserting the limits

$$
\begin{aligned}
&\lim_{s\to 0} s^v(s^{-1}q^{\mathsf{L}-v+1}x;q)_v = (-x)^v q^{(\mathsf{L}-v+1)v+\binom{v}{2}}; \\
&\lim_{s\to 0} s^{|\mathbf{B}|-|\mathbf{P}|-v}(s^{-1}x;q)_{|\mathbf{B}|-|\mathbf{P}|-v} = (-x)^{|\mathbf{B}|-|\mathbf{P}|-v}q^{\binom{|\mathbf{B}|-|\mathbf{P}|-v}{2}},
\end{aligned}
$$

into (17.7.3), we obtain

$$
\begin{aligned}
\mathbb{R}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{|\mathbf{B}|\leqslant\mathsf{L}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathsf{L}}\cdot(-1)^{|\mathbf{D}|-|\mathbf{B}|+v}q^{\varphi(\mathbf{V},\mathbf{A})}x^{|\mathbf{D}|}q^{(|\mathbf{A}|-|\mathbf{C}|)(\mathsf{L}-v)} \\
&\quad \times q^{(\mathsf{L}-v+1)v+\binom{v}{2}}q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})}\frac{(q^{v-\mathsf{L}};q)_{|\mathbf{D}|-v}}{(q^{v-\mathsf{L}};q)_{|\mathbf{B}|-v}} \\
&\quad \times \sum_{\mathbf{P}}(-1)^{|\mathbf{P}|}q^{(\mathsf{L}-v)|\mathbf{P}|}q^{-\varphi(\mathbf{D}-\mathbf{V},\mathbf{P})}q^{\binom{|\mathbf{B}|-v-|\mathbf{P}|}{2}}q^{\varphi(\mathbf{B}-\mathbf{V}-\mathbf{P},\mathbf{P})} \\
&\quad \times \prod_{i=1}^{n+1}\frac{(q;q)_{C_i+D_i-V_i-P_i}}{(q;q)_{C_i-P_i}(q;q)_{D_i-V_i}}\frac{(q;q)_{B_i-V_i}}{(q;q)_{P_i}(q;q)_{B_i-V_i-P_i}}.
\end{aligned}
\tag{17.7.4}
$$

Using the bilinearity of φ, we deduce

$$
\begin{aligned}
&(\mathsf{L}-v+1)v+\binom{v}{2}+\binom{|\mathbf{B}|-v-|\mathbf{P}|}{2}+\varphi(\mathbf{B}-\mathbf{V}-\mathbf{P},\mathbf{P}) \\
&\quad = \big(\mathsf{L}-|\mathbf{B}|+1\big)v+\binom{|\mathbf{B}|}{2}+\binom{|\mathbf{P}|+1}{2}-\varphi(\mathbf{P},\mathbf{P})+|\mathbf{P}|\big(v-|\mathbf{B}|\big)+\varphi(\mathbf{B}-\mathbf{V},\mathbf{P}).
\end{aligned}
$$

Inserting this, together with the fact that

$$
\frac{|\mathbf{P}|^2}{2}-\varphi(\mathbf{P},\mathbf{P}) = \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}P_iP_j - \sum_{1\leqslant i<j\leqslant n+1}P_iP_j = \frac{1}{2}\sum_{i=1}^{n+1}P_i^2.
$$

into (17.7.4), it becomes

$$
\begin{aligned}
\mathbb{R}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{|\mathbf{B}|\leqslant \mathsf{L}} \mathbf{1}_{|\mathbf{D}|\leqslant \mathsf{L}} \cdot (-1)^{|\mathbf{D}|-|\mathbf{B}|+v} q^{\varphi(\mathbf{V},\mathbf{A})} x^{|\mathbf{D}|} q^{(|\mathbf{A}|-|\mathbf{C}|)(\mathsf{L}-v)} \\
\times q^{(\mathsf{L}-|\mathbf{B}|+1)v+\binom{|\mathbf{B}|}{2}} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})} \frac{(q^{v-\mathsf{L}};q)_{|\mathbf{D}|-v}}{(q^{v-\mathsf{L}};q)_{|\mathbf{B}|-v}} \\
\times \sum_{\mathbf{P}} (-1)^{|\mathbf{P}|} q^{\mathsf{L}|\mathbf{P}|+\varphi(\mathbf{B}-\mathbf{D},\mathbf{P})-|\mathbf{P}||\mathbf{B}|} \\
\times \prod_{i=1}^{n+1} q^{\binom{P_i+1}{2}} \frac{(q;q)_{C_i+D_i-V_i-P_i}}{(q;q)_{C_i-P_i}(q;q)_{D_i-V_i}} \frac{(q;q)_{B_i-V_i}}{(q;q)_{P_i}(q;q)_{B_i-V_i-P_i}}.
\end{aligned}
\tag{17.7.5}
$$

The sum on the right side of (17.7.5) factors, thereby yielding

$$
\begin{aligned}
\mathbb{R}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{|\mathbf{B}|\leqslant \mathsf{L}} \mathbf{1}_{|\mathbf{D}|\leqslant \mathsf{L}} \cdot (-1)^{|\mathbf{D}|-|\mathbf{B}|+v} q^{\varphi(\mathbf{V},\mathbf{A})} x^{|\mathbf{D}|} q^{(|\mathbf{A}|-|\mathbf{C}|)(\mathsf{L}-v)} \\
\times q^{v(\mathsf{L}-|\mathbf{B}|+1)+\binom{|\mathbf{B}|}{2}} q^{\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})} \frac{(q^{v-\mathsf{L}};q)_{|\mathbf{D}|-v}}{(q^{v-\mathsf{L}};q)_{|\mathbf{B}|-v}} \\
\times \prod_{i=1}^{n+1} \sum_{P_i \leqslant \min\{C_i, B_i-V_i\}} (-1)^{P_i} q^{\binom{P_i+1}{2}} q^{P_i(\mathsf{L}-B_{[i,n+1]}-D_{[1,i-1]})} \\
\times \frac{(q;q)_{C_i+D_i-V_i-P_i}}{(q;q)_{C_i-P_i}(q;q)_{D_i-V_i}} \frac{(q;q)_{B_i-V_i}}{(q;q)_{P_i}(q;q)_{B_i-V_i-P_i}}.
\end{aligned}
\tag{17.7.6}
$$

Of the $n+1$ sums in (17.7.6), only the first one is non-trivial; the sums over $(P_2,\ldots,P_{n+1})$ each consist of either one or two nonzero summands. Explicitly computing those final n sums, we arrive at the expression

$$
\begin{aligned}
\mathbb{R}_x(\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{|\mathbf{B}|\leqslant \mathsf{L}} \mathbf{1}_{|\mathbf{D}|\leqslant \mathsf{L}} \cdot (-1)^{|\mathbf{D}|-|\mathbf{B}|+v} x^{|\mathbf{D}|} q^{\varphi(\mathbf{V},\mathbf{A})+(|\mathbf{A}|-|\mathbf{C}|)(\mathsf{L}-v)} \\
\times q^{v(\mathsf{L}-|\mathbf{B}|+1)+\binom{|\mathbf{B}|}{2}+\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})} \frac{(q^{v-\mathsf{L}};q)_{|\mathbf{D}|-v}}{(q^{v-\mathsf{L}};q)_{|\mathbf{B}|-v}} \prod_{\substack{i\in[2,n+1]\\ B_i-D_i=1}} \big(1-q^{\mathsf{L}-B_{[i+1,n+1]}-D_{[1,i-1]}}\big) \\
\times \sum_{P_1\leqslant \min\{C_1,B_1\}} (-1)^{P_1} q^{\binom{P_1+1}{2}} q^{P_1(\mathsf{L}-|\mathbf{B}|)} \frac{(q;q)_{C_1+D_1-P_1}}{(q;q)_{C_1-P_1}(q;q)_{D_1}} \frac{(q;q)_{B_1}}{(q;q)_{P_1}(q;q)_{B_1-P_1}}.
\end{aligned}
\tag{17.7.7}
$$

Observe the similarity between these weights and those given by Proposition 8.2.5, in that they both involve a product of terms of the form $1-uq^{-B_{[j+1,n]}-D_{[1,j-1]}}$ for u either equal to q^{L} or r^{-2}. Indeed, it can be shown that the former degenerate to the latter under the correspondence described in Remark 17.1.2.

Proposition 17.7.1. *When* $\mathsf{L}=1$*, the model* (17.7.7) *reduces to the model* (17.1.1)–(17.1.2)*; namely*

$$
\mathbb{R}_x(\mathbf{A},\mathrm{e}_b;\mathbf{C},\mathrm{e}_d)\Big|_{\mathsf{L}=1} = w_x(\mathbf{A},b;\mathbf{C},d),
$$

for all $(n+1)$*-tuples* $\mathbf{A}$*,* $\mathbf{C}\in\mathbb{Z}_{\geqslant 0}^{n+1}$ *and integers* $b,d\in[0,n+1]$.

Proof. This follows from inserting $\mathsf{L}=1$ in (17.7.7). $\square$

17.7.3. Two Saturated Horizontal Edges. We now examine what happens to the formula (17.7.7) when the states assigned to the horizontal edges of vertices, namely $\mathbf{B}$ and $\mathbf{D}$, are almost saturated by bosonic arrows (which have color index 1). More precisely, we fix integers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \geqslant 0$ and choose four $(n+1)$-tuples of integers $\widetilde{\mathbf{A}} = (\mathfrak{a}, \mathbf{A})$, $\widetilde{\mathbf{B}} = (\mathsf{L} - \mathfrak{b}, B_2, \ldots, B_{n+1})$, $\widetilde{\mathbf{C}} = (\mathfrak{c}, \mathbf{C})$, and $\widetilde{\mathbf{D}} = (\mathsf{L} - \mathfrak{d}, \mathbf{D})$, where for each $X \in \{A, B, C, D\}$ we write $\mathbf{X} = (X_2, X_3, \ldots, X_{n+1}) \in \mathbb{Z}_{\geqslant 0}^{n+1}$. Then,

$$|\mathbf{A}| = \sum_{i=2}^{n+1} A_i, \qquad |\mathbf{B}| = \sum_{i=2}^{n+1} B_i, \qquad |\mathbf{C}| = \sum_{i=2}^{n+1} C_i, \qquad |\mathbf{D}| = \sum_{i=2}^{n+1} D_i. \tag{17.7.8}$$

We continue to write $\mathbf{V} = (0, V_2, \ldots, V_{n+1})$ with $V_j = \min\{A_j, B_j, C_j, D_j\}$ for $j \in [2, n+1]$. By arrow conservation, we must have $\mathfrak{a} - \mathfrak{b} = \mathfrak{c} - \mathfrak{d}$. By the finite capacity L of the horizontal line, we also necessarily have $\mathfrak{b} \geqslant |\mathbf{B}|$ and $\mathfrak{d} \geqslant |\mathbf{D}|$.

For reasons that will become clear later, we let $x = y^{-1} q^{-\mathsf{L}+1}$ in (17.7.7) and multiply our weights by $y^{\mathsf{L}} q^{\binom{\mathsf{L}}{2}}$. Since we have

$$\begin{aligned}
\binom{\mathsf{L}}{2} &+ (1 - \mathsf{L})\big(\mathsf{L} - \mathfrak{d} + |\mathbf{D}|\big) + \big(|\mathbf{A}| + \mathfrak{a} - |\mathbf{C}| - \mathfrak{c}\big)(\mathsf{L} - v) + v(\mathfrak{b} - \mathbf{B} + 1) + \binom{L - \mathfrak{b} + |\mathbf{B}|}{2} \\
&= \mathsf{L}\big(|\mathbf{A}| + |\mathbf{B}| - |\mathbf{C}| - |\mathbf{D}| + \mathfrak{a} - \mathfrak{b} - \mathfrak{c} + \mathfrak{d}\big) + \big(|\mathbf{C}| + \mathfrak{c} - |\mathbf{A}| - \mathfrak{a} - |\mathbf{B}| + \mathfrak{b} + 1\big) v \\
&\quad + |\mathbf{D}| - \mathfrak{d} + \binom{\mathfrak{b} - |\mathbf{B}| + 1}{2} = |\mathbf{D}| - \mathfrak{d} + \binom{\mathfrak{b} - |\mathbf{B}| + 1}{2} + \big(\mathfrak{d} - |\mathbf{D}| + 1\big) v,
\end{aligned}$$

if $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$ and $\mathfrak{a} - \mathfrak{b} = \mathfrak{c} - \mathfrak{d}$, and since we also have

$$\varphi(\widetilde{\mathbf{D}}, \widetilde{\mathbf{C}}) = \varphi(\mathbf{D}, \mathbf{C}) + (\mathsf{L} - \mathfrak{d}) |\mathbf{C}|,$$

the result of all of these choices is the following expression:

$$\begin{aligned}
y^{\mathsf{L}} q^{\binom{\mathsf{L}}{2}} &\mathbb{R}_{y^{-1} q^{-\mathsf{L}+1}} \big(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}; \widetilde{\mathbf{C}}, \widetilde{\mathbf{D}}\big) \\
&= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}} \mathbf{1}_{\mathfrak{a}-\mathfrak{b}=\mathfrak{c}-\mathfrak{d}} \mathbf{1}_{|\mathbf{B}| \leqslant \mathfrak{b}} \mathbf{1}_{|\mathbf{D}| \leqslant \mathfrak{d}} \\
&\quad \times (-1)^{|\mathbf{D}| - |\mathbf{B}| - \mathfrak{d} + \mathfrak{b} + v} y^{\mathfrak{d} - |\mathbf{D}|} q^{\varphi(\mathbf{V}, \mathbf{A}) + |\mathbf{D}| - \mathfrak{d} + \binom{\mathfrak{b} - |\mathbf{B}| + 1}{2} + (\mathfrak{d} - |\mathbf{D}| + 1) v} \\
&\quad \times q^{(\mathsf{L} - \mathfrak{d})|\mathbf{C}| + \varphi(\mathbf{D} - \mathbf{V}, \mathbf{C})} \frac{(q^{v - \mathsf{L}}; q)_{\mathsf{L} - \mathfrak{d} + |\mathbf{D}| - v}}{(q^{v - \mathsf{L}}; q)_{\mathsf{L} - \mathfrak{b} + |\mathbf{B}| - v}} \prod_{\substack{i \in [2, n+1] \\ B_i - D_i = 1}} \big(1 - q^{\mathfrak{d} - B_{[i+1, n+1]} - D_{[2, i-1]}}\big) \\
&\quad \times \sum_{p=0}^{\mathfrak{c}} (-1)^p q^{\binom{p+1}{2}} q^{p(\mathfrak{b} - |\mathbf{B}|)} \frac{(q; q)_{\mathfrak{c} - \mathfrak{d} + \mathsf{L} - p}}{(q; q)_{\mathfrak{c} - p} (q; q)_{\mathsf{L} - \mathfrak{d}}} \frac{(q; q)_{\mathsf{L} - \mathfrak{b}}}{(q; q)_p (q; q)_{\mathsf{L} - \mathfrak{b} - p}}.
\end{aligned} \tag{17.7.9}$$

Our goal is to bring (17.7.9) into a form whereby all L dependence occurs in the combination q^{L}. Manipulating the q-Pochhammer and q-binomial functions in (17.7.9), we have

$$\frac{(q^{v - \mathsf{L}}; q)_{\mathsf{L} - \mathfrak{d} + |\mathbf{D}| - v}}{(q^{v - \mathsf{L}}; q)_{\mathsf{L} - \mathfrak{b} + |\mathbf{B}| - v}} = \frac{(q^{-1}; q^{-1})_{\mathfrak{b} - |\mathbf{B}|}}{(q^{-1}; q^{-1})_{\mathfrak{d} - |\mathbf{D}|}},$$

$$\frac{(q; q)_{\mathfrak{c} - \mathfrak{d} + \mathsf{L} - p}}{(q; q)_{\mathfrak{c} - p} (q; q)_{\mathsf{L} - \mathfrak{d}}} = \frac{\prod_{i=1}^{\mathfrak{c} - p} (1 - q^{i + \mathsf{L} - \mathfrak{d}})}{(q; q)_{\mathfrak{c} - p}}, \qquad \frac{(q; q)_{\mathsf{L} - \mathfrak{b}}}{(q; q)_p (q; q)_{\mathsf{L} - \mathfrak{b}}} = \frac{\prod_{i=1}^{p} (1 - q^{i + \mathsf{L} - \mathfrak{b} - p})}{(q; q)_p}.$$

Making these substitutions in (17.7.9), we obtain

(17.7.10)

$$\begin{aligned}
y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big) &= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}-\mathfrak{b}=\mathfrak{c}-\mathfrak{d}}\mathbf{1}_{|\mathbf{B}|\leqslant\mathfrak{b}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}(-1)^{|\mathbf{D}|-|\mathbf{B}|-\mathfrak{d}+\mathfrak{b}+v}y^{\mathfrak{d}-|\mathbf{D}|}\\
&\quad\times q^{\mathsf{L}(\mathfrak{c}+|\mathbf{C}|)+\varphi(\mathbf{V},\mathbf{A})+|\mathbf{D}|-\mathfrak{d}+\binom{\mathfrak{b}-|\mathbf{B}|+1}{2}+(\mathfrak{d}-|\mathbf{D}|+1)v-\mathfrak{d}|\mathbf{C}|+\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})}\\
&\quad\times\frac{(q^{-1};q^{-1})_{\mathfrak{b}-|\mathbf{B}|}}{(q^{-1};q^{-1})_{\mathfrak{d}-|\mathbf{D}|}}\prod_{\substack{i\in[2,n+1]\\B_i-D_i=1}}\big(1-q^{\mathfrak{d}-B_{[i+1,n+1]}-D_{[2,i-1]}}\big)\\
&\quad\times\sum_{p=0}^{\mathfrak{c}}(-1)^p q^{\binom{p+1}{2}}q^{p(\mathfrak{b}-|\mathbf{B}|)}\frac{\prod_{i=1}^{\mathfrak{c}-p}(q^{-\mathsf{L}}-q^{i-\mathfrak{d}})}{(q;q)_{\mathfrak{c}-p}}\frac{\prod_{i=1}^{p}(q^{-\mathsf{L}}-q^{i-\mathfrak{b}-p})}{(q;q)_p},
\end{aligned}$$

and indeed our expression depends rationally on q^{L}, with no other L dependence. Treating q^{L} as an arbitrary complex parameter (by analytic continuation) we conclude, by computing the limit $q^{\mathsf{L}}\to\infty$:

$$\begin{aligned}
&\lim_{q^{\mathsf{L}}\to\infty}\Big[q^{-\mathsf{L}(\mathfrak{c}+|\mathbf{C}|)}y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big)\Big]\\
&\qquad= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}-\mathfrak{b}=\mathfrak{c}-\mathfrak{d}}\mathbf{1}_{|\mathbf{B}|\leqslant\mathfrak{b}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}\cdot(-1)^{|\mathbf{D}|-|\mathbf{B}|-\mathfrak{d}+\mathfrak{b}+v}y^{\mathfrak{d}-|\mathbf{D}|}\\
&\qquad\quad\times q^{\varphi(\mathbf{V},\mathbf{A})+|\mathbf{D}|-\mathfrak{d}+\binom{\mathfrak{b}-|\mathbf{B}|+1}{2}+(\mathfrak{d}-|\mathbf{D}|+1)v-\mathfrak{d}|\mathbf{C}|+\varphi(\mathbf{D}-\mathbf{V},\mathbf{C})}\\
&\qquad\quad\times\frac{(q^{-1};q^{-1})_{\mathfrak{b}-|\mathbf{B}|}}{(q^{-1};q^{-1})_{\mathfrak{d}-|\mathbf{D}|}}\prod_{\substack{i\in[2,n+1]\\B_i-D_i=1}}\big(1-q^{\mathfrak{d}-B_{[i+1,n+1]}-D_{[2,i-1]}}\big)\\
&\qquad\quad\times(-1)^{\mathfrak{c}}q^{\binom{\mathfrak{c}+1}{2}-\mathfrak{c}\mathfrak{d}}\sum_{p=0}^{c}(-1)^p q^{\binom{p+1}{2}}q^{p(\mathfrak{b}-|\mathbf{B}|-\mathfrak{a})}\frac{1}{(q;q)_{\mathfrak{c}-p}(q;q)_p},
\end{aligned}\tag{17.7.11}$$

where we have used the fact that for $\mathfrak{a}-\mathfrak{b}=\mathfrak{c}-\mathfrak{d}$ we have

$$\begin{aligned}
\sum_{i=1}^{\mathfrak{c}-p}(i-\mathfrak{d})+\sum_{i=1}^{p}(i-\mathfrak{b}-p) &= \binom{\mathfrak{c}-p+1}{2}-\mathfrak{d}(\mathfrak{c}-p)+\binom{p+1}{2}-p(\mathfrak{b}+p)\\
&= \binom{\mathfrak{c}+1}{2}-\mathfrak{c}\mathfrak{d}+p(\mathfrak{d}-\mathfrak{b}-\mathfrak{c})=\binom{\mathfrak{c}+1}{2}-\mathfrak{c}\mathfrak{d}-p\mathfrak{a}.
\end{aligned}$$

The sum on the final line of (17.7.11) may now be computed explicitly, using the q-binomial theorem:

$$\sum_{p=0}^{\mathfrak{c}}(-1)^p q^{\binom{p+1}{2}}q^{p(\mathfrak{b}-|\mathbf{B}|-\mathfrak{a})}\frac{1}{(q;q)_{\mathfrak{c}-p}(q;q)_p}=\frac{(q^{\mathfrak{b}-|\mathbf{B}|-\mathfrak{a}+1};q)_{\mathfrak{c}}}{(q;q)_{\mathfrak{c}}}.$$

Using this in (17.7.11), together with the fact that

$$\frac{(q^{-1};q^{-1})_{\mathfrak{b}-|\mathbf{B}|}}{(q^{-1};q^{-1})_{\mathfrak{d}-|\mathbf{D}|}}=(-1)^{\mathfrak{b}-|\mathbf{B}|-\mathfrak{d}+|\mathbf{D}|}q^{\binom{\mathfrak{d}-|\mathbf{D}|+1}{2}-\binom{\mathfrak{b}-|\mathbf{B}|+1}{2}}\frac{(q;q)_{\mathfrak{b}-|\mathbf{B}|}}{(q;q)_{\mathfrak{d}-|\mathbf{D}|}},$$

we arrive at our final expression

$$\begin{aligned}
&\lim_{q^{\mathsf{L}}\to\infty}\Big[q^{-\mathsf{L}(\mathfrak{c}+|\mathbf{C}|)}y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big)\Big]\\
&\qquad= \mathbf{1}_{\mathbf{A}+\mathbf{B}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}-\mathfrak{b}=\mathfrak{c}-\mathfrak{d}}\mathbf{1}_{|\mathbf{B}|\leqslant\mathfrak{b}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}\cdot(-1)^{\mathfrak{c}+v}y^{\mathfrak{d}-|\mathbf{D}|}q^{\chi}
\end{aligned}$$

$$\times \frac{(q;q)_{\mathfrak{b}-|\mathbf{B}|}}{(q;q)_{\mathfrak{d}-|\mathbf{D}|}} \cdot \frac{(q^{\mathfrak{b}-|\mathbf{B}|-\mathfrak{a}+1};q)_{\mathfrak{c}}}{(q;q)_{\mathfrak{c}}} \prod_{\substack{i\in[2,n+1]\\ B_i-D_i=1}} \left(1-q^{\mathfrak{d}-B_{[i+1,n+1]}-D_{[2,i-1]}}\right),$$

where the exponent χ is given by

$$\chi = \varphi(\mathbf{V},\mathbf{A}) + \binom{\mathfrak{d}-|\mathbf{D}|}{2} + v(\mathfrak{d}-|\mathbf{D}|+1) + \binom{\mathfrak{c}+1}{2} - (\mathfrak{c}+|\mathbf{C}|)\mathfrak{d} + \varphi(\mathbf{D}-\mathbf{V},\mathbf{C}).$$

17.7.4. One Saturated Horizontal Edge. Let us consider a further case of the formula (17.7.7), when one horizontal edge state ($\widetilde{\mathbf{D}}$) is almost saturated by bosonic arrows, while the other horizontal edge state ($\widetilde{\mathbf{B}}$) is empty. Specifically, we fix integers $\mathfrak{a},\mathfrak{c},\mathfrak{d}\geqslant 0$ and $N>\mathsf{L}$, and choose $(n+1)$-tuples $\widetilde{\mathbf{A}}=(N-\mathfrak{a},\mathbf{A})$, $\widetilde{\mathbf{B}}=\mathbf{e}_0$, $\widetilde{\mathbf{C}}=(N-\mathsf{L}-\mathfrak{c},\mathbf{C})$, $\mathbf{D}=(\mathsf{L}-\mathfrak{d},\mathbf{D})$, with the coordinates of $\mathbf{A},\mathbf{C},\mathbf{D}\in\{0,1\}^n$ indexed by $[2,n+1]$.

In this case, given that $\mathbf{B}=\mathbf{e}_0$, we have $\mathbf{V}=\mathbf{e}_0$. By arrow conservation, we must have $\mathfrak{a}=\mathfrak{c}+\mathfrak{d}$. By the finite capacity L of the horizontal line, we also necessarily have $\mathfrak{d}\geqslant|\mathbf{D}|$. As we did previously, we let $x=y^{-1}q^{-\mathsf{L}+1}$ in (17.7.7) and multiply our weights by $y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}$, which results in the following expression:

(17.7.12)
$$y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}\big(\widetilde{\mathbf{A}},\mathbf{e}_0;\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big) = \mathbf{1}_{\mathbf{A}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}=\mathfrak{c}+\mathfrak{d}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}(-1)^{\mathsf{L}-\mathfrak{d}+|\mathbf{D}|}y^{\mathfrak{d}-|\mathbf{D}|}$$
$$\times q^{\binom{\mathsf{L}+1}{2}+\mathsf{L}|\mathbf{C}|+|\mathbf{D}|-\mathfrak{d}-|\mathbf{C}|\mathfrak{d}+\varphi(\mathbf{D},\mathbf{C})}\frac{(q;q)_{N-\mathfrak{a}}(q^{-\mathsf{L}};q)_{\mathsf{L}-\mathfrak{d}+|\mathbf{D}|}}{(q;q)_{N-\mathsf{L}-\mathfrak{c}}(q;q)_{\mathsf{L}+\mathfrak{c}-\mathfrak{a}}}.$$

Once again, we wish to rearrange so that all L dependence occurs via q^{L}. We do this making use of the relations

$$(q^{-\mathsf{L}};q)_{\mathsf{L}-\mathfrak{d}+|\mathbf{D}|} = \frac{(q^{-\mathsf{L}};q)_{\mathsf{L}}}{(q^{|\mathbf{D}|-\mathfrak{d}};q)_{\mathfrak{d}-|\mathbf{D}|}} = q^{-\binom{\mathsf{L}+1}{2}}(-1)^{\mathsf{L}}\cdot\frac{(q;q)_{\mathsf{L}}}{(q^{|\mathbf{D}|-\mathfrak{d}};q)_{\mathfrak{d}-|\mathbf{D}|}};$$
$$\frac{(q;q)_{N-\mathfrak{a}}}{(q;q)_{N-\mathsf{L}-\mathfrak{c}}(q;q)_{\mathsf{L}-\mathfrak{d}}} = \frac{(q;q)_{N-\mathfrak{a}}}{(q;q)_{N-\mathsf{L}-\mathfrak{c}}}\cdot\frac{(q^{\mathsf{L}-\mathfrak{d}+1};q)_{\mathfrak{d}}}{(q;q)_{\mathsf{L}}},$$

which when substituted into (17.7.12) produce the formula

(17.7.13)
$$y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}(\widetilde{\mathbf{A}},\mathbf{e}_0;\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}) = \mathbf{1}_{\mathbf{A}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}=\mathfrak{c}+\mathfrak{d}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}\cdot(-1)^{\mathfrak{d}-|\mathbf{D}|}y^{\mathfrak{d}-|\mathbf{D}|}$$
$$\times q^{\mathsf{L}|\mathbf{C}|+|\mathbf{D}|-\mathfrak{d}-|\mathbf{C}|\mathfrak{d}+\varphi(\mathbf{D},\mathbf{C})}\frac{(q^{\mathsf{L}-\mathfrak{d}+1};q)_{\mathfrak{d}}}{(q^{|\mathbf{D}|-\mathfrak{d}};q)_{\mathfrak{d}-|\mathbf{D}|}}\cdot\frac{(q;q)_{N-\mathfrak{a}}}{(q;q)_{N-\mathsf{L}-\mathfrak{c}}}$$
$$= \mathbf{1}_{\mathbf{A}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}=\mathfrak{c}+\mathfrak{d}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}\cdot q^{\mathsf{L}(|\mathbf{C}|+\mathfrak{d})}\frac{(q;q)_{N-\mathfrak{a}}}{(q;q)_{N-\mathsf{L}-\mathfrak{c}}}$$
$$\times(-1)^{\mathfrak{d}-|\mathbf{D}|}y^{\mathfrak{d}-|\mathbf{D}|}q^{|\mathbf{D}|-\mathfrak{d}-|\mathbf{C}|\mathfrak{d}+\varphi(\mathbf{D},\mathbf{C})}\frac{\prod_{i=1}^{\mathfrak{d}}(q^{-\mathsf{L}}-q^{-\mathfrak{d}+i})}{(q^{|\mathbf{D}|-\mathfrak{d}};q)_{\mathfrak{d}-|\mathbf{D}|}}.$$

We conclude by taking the limit $q^{\mathsf{L}}\to\infty$:

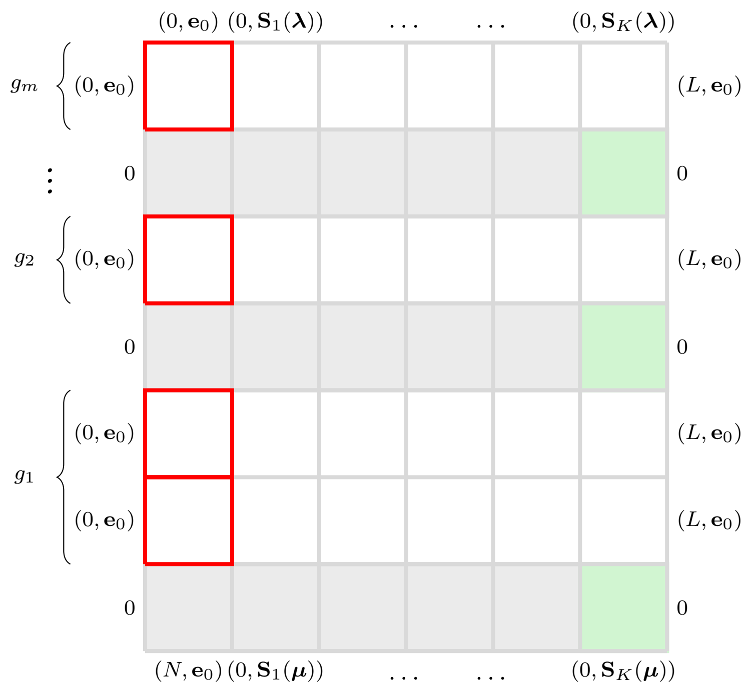
(17.7.14)
$$\lim_{q^{\mathsf{L}}\to\infty}\left[q^{-\mathsf{L}(|\mathbf{C}|+\mathfrak{d})}\frac{(q;q)_{N-\mathsf{L}-\mathfrak{c}}}{(q;q)_{N-\mathfrak{a}}}y^{\mathsf{L}}q^{\binom{\mathsf{L}}{2}}\mathbb{R}_{y^{-1}q^{-\mathsf{L}+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big)\right]$$
$$= \mathbf{1}_{\mathbf{A}=\mathbf{C}+\mathbf{D}}\mathbf{1}_{\mathfrak{a}=\mathfrak{c}+\mathfrak{d}}\mathbf{1}_{|\mathbf{D}|\leqslant\mathfrak{d}}\cdot(-1)^{|\mathbf{D}|}y^{\mathfrak{d}-|\mathbf{D}|}q^{|\mathbf{D}|-|\mathbf{C}|d-\binom{\mathfrak{d}+1}{2}+\varphi(\mathbf{D},\mathbf{C})}\frac{1}{(q^{|\mathbf{D}|-\mathfrak{d}};q)_{\mathfrak{d}-|\mathbf{D}|}}.$$

17.7.5. Partition Function Limits (1).

Proposition 17.7.2. *Computing the same limits and change of variables as in* (17.6.5), *we have*

$$\left[\lim_{x_{a_1}\to 0}\cdots\lim_{x_{a_m}\to 0}\mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1,\dots,x_{N+m})\right]^{\dagger} = \tag{17.7.15}$$

$$q^{-nMN}(q;q)_N^{-1}\times$$



where we use different vertex weights throughout the lattice, as indicated by different colorings/shadings:

- $\begin{array}{ccc} & \mathbf{C} & \\ b & \square & d \\ & \mathbf{A} & \end{array}$ *is given by* (17.1.1)–(17.1.2) *with* $w_x(\mathbf{A},b;\mathbf{C},d)\mapsto \lim_{x\to 0} x^{\mathbf{1}_{b>0}-\mathbf{1}_{d>0}+1}w_{1/x}(\mathbf{A},b;\mathbf{C},d)$;
- $\begin{array}{ccc} & \mathbf{C} & \\ b & \square & 0 \\ & \mathbf{A} & \end{array}$ *is given by* (17.1.1)–(17.1.2);
- $\begin{array}{ccc} & \widetilde{\mathbf{C}} & \\ \widetilde{\mathbf{B}} & \square & \widetilde{\mathbf{D}} \\ & \widetilde{\mathbf{A}} & \end{array}$ *is given by* $y^Lq^{\binom{L}{2}}\cdot\mathbb{R}_{y^{-1}q^{-L+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big)$ *as in* (17.7.10);
- $\begin{array}{ccc} & \widetilde{\mathbf{C}} & \\ \widetilde{\mathbf{B}} & \square & \widetilde{\mathbf{D}} \\ & \widetilde{\mathbf{A}} & \end{array}$ *is given by* $y^Lq^{\binom{L}{2}}\cdot\mathbb{R}_{y^{-1}q^{-L+1}}\big(\widetilde{\mathbf{A}},\widetilde{\mathbf{B}};\widetilde{\mathbf{C}},\widetilde{\mathbf{D}}\big)$ *as in* (17.7.13).

PROOF. Recall the definition (17.6.1) of $\mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m})$. It may be rewritten as

$$\mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) = q^{-nMN}(q;q)_N^{-1} \prod_{i=1}^{m} x_{a_i}^{-1} \prod_{i=1}^{N+m} x_i^{K+1} \cdot Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1^{-1}, \ldots, x_{N+m}^{-1}) \tag{17.7.16}$$

where the partition function $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1^{-1}, \ldots, x_{N+m}^{-1})$ is given by (17.2.1). The q-dependent factors within (17.7.16) match with those on the right hand side of (17.7.15), so we may neglect them in the remainder of the proof.

Taking the limits as $x_{a_1}, \ldots, x_{a_m}$ tend to 0 affects only the weights of rows $a_1, \ldots, a_m$ within $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. We compute these limits by distributing the factor $x_{a_i}^K$ across the weights of the leftmost K vertices of row a_i, prior to taking x_{a_i} to 0; we do this for each $i \in [1, m]$. After performing the limits the effective weights in row a_i are given by (17.1.1)–(17.1.2) with $w_x(\mathbf{A}, b; \mathbf{C}, d) \mapsto \lim_{x \to 0} \big(x^{\mathbf{1}_{b>0} - \mathbf{1}_{d>0} + 1} \cdot w_{1/x}(\mathbf{A}, b; \mathbf{C}, d) \big)$ for any vertex not in the rightmost column of the lattice, and by (17.1.1)–(17.1.2) for the rightmost vertex within the row. The resulting rows $a_1, \ldots, a_m$ ultimately become the shaded rows of (17.7.15).

Next we study the effect of specializing variables, as in (17.6.3)–(17.6.4), within $\mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. Bearing in mind that the alphabet $(x_1, \ldots, x_{N+m})$ is reciprocated within $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1^{-1}, \ldots, x_{N+m}^{-1})$, the change of variables (17.6.4) instigates fusion (recall Chapter 3, in particular Definition 3.2.1, Definition 3.1.3, and Definition 3.1.1) of the rows $a_i + 1, \ldots, a_{i+1} - 1$ of the lattice, for all $i \in [1, m]$; these rows get replaced by g_i rows within the model (17.7.7). The parameter x appearing in (17.7.7) is replaced by $y^{-1} q^{-L+1}$, which is the base of the geometric progression (17.6.4) modulo the aforementioned reciprocation of variables. The weights (17.7.7) also need to be multiplied by $y^{\mathsf{L}} q^{\binom{\mathsf{L}}{2}}$, which is an artifact of the factor x_j^{K+1} appearing in (17.7.16), once that factor is distributed over the $K+1$ vertices within row j of $Z^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, for each $j \in (a_i, a_{i+1})$. These considerations lead to the unshaded rows of (17.7.15).

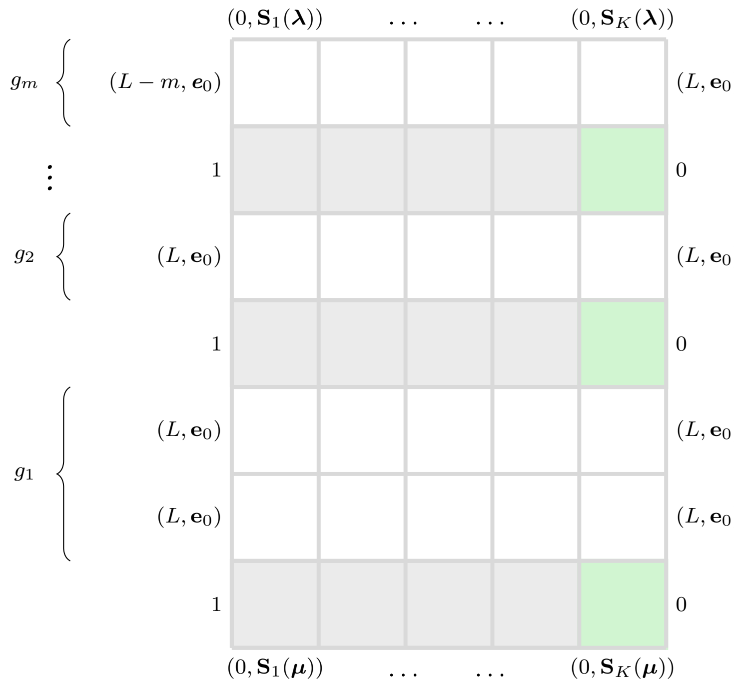
Finally, after performing fusion each of the unshaded rows in (17.7.15) has a right edge state $(L, \mathbf{e}_0)$; that is, it consists of L bosonic arrows and no fermionic arrows. It is therefore convenient to work in terms of saturated left/right edge states as in Sections 17.7.3 and 17.7.4. □

17.7.6. Partition Function Limits (2).

Proposition 17.7.3. *Computing the same limits and change of variables as in* (17.6.7)*, we have*

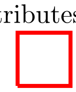
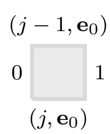
$$(-y)^{-\sum_{i=1}^m i g_i} \lim_{q^{\mathsf{L}} \to \infty} q^{-L \sum_{i=1}^m i g_i} \bigg[\lim_{x_{a_1} \to 0} \cdots \lim_{x_{a_m} \to 0} \mathfrak{Z}^{\mathcal{A}}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_{N+m}) \bigg]^{\dagger} = \tag{17.7.17}$$

$$\frac{(-1)^{|\bar{\nu}|}}{(q;q)_m} \times$$



where the vertices in (17.7.17) *are the same as those used in Proposition 17.7.2, but with* $y = 1$.

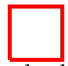
Proof. We study the limit as q^{L} tends to ∞ of the partition function (17.7.15). The first thing to observe is the cancellation of the factors q^{-nMN} and $(q;q)_N^{-1}$ multiplying (17.7.15) with certain terms in the weights of the vertices.

The factor q^{-nMN} cancels perfectly with $q^{\mathsf{L}|\mathbf{C}|}$ present in (17.7.10) and (17.7.13). Indeed, the factor $q^{\mathsf{L}|\mathbf{C}|}$ contributes one power of q^{L} for every fermionic path which passes through the top of a vertex  or  . This leads to nM powers of q^{L} per unshaded row, and there are $\sum_{i=1}^{n} g_i$ such rows; the total contribution from the $q^{\mathsf{L}|\mathbf{C}|}$ factors is thus $q^{nM \cdot \mathsf{L} \sum_{j=1}^{n} g_j} = q^{nMN}$ as claimed. The factor $(q;q)_N^{-1}$ cancels perfectly with $(q;q)_{N-\mathfrak{a}}(q;q)_{N-\mathsf{L}-\mathfrak{c}}^{-1}$ present in (17.7.13) and the factor $(1-q^j)$ arising from vertices 

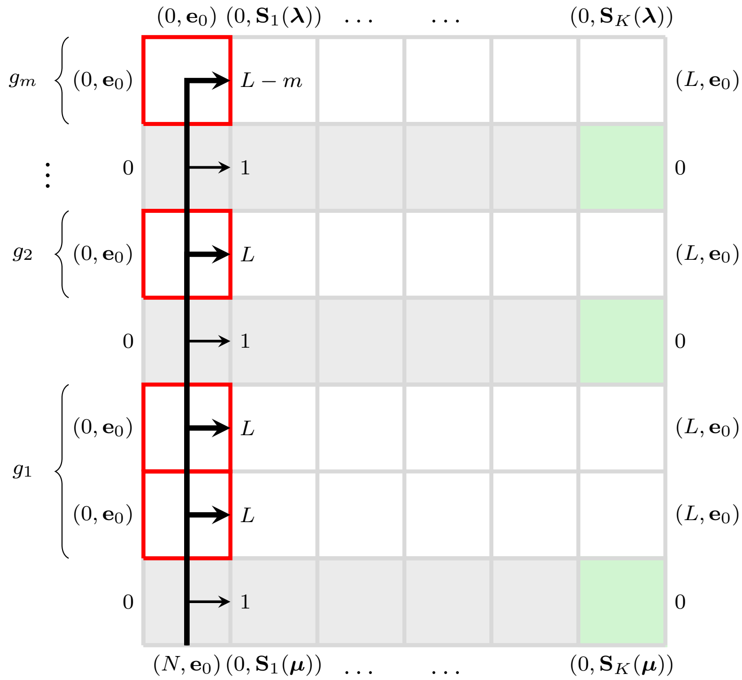
in the leftmost column of (17.7.15). Indeed, in every configuration of the leftmost column, these factors must always telescope to yield $(q;q)_N$, in view of the top/bottom boundary conditions of this column.

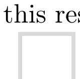
After cancelling $q^{\mathsf{L}|\mathbf{C}|}$ from (17.7.10), the renormalized weights are polynomials in q^{L} of degree $\mathfrak{c}$. Similarly, after cancelling $q^{\mathsf{L}|\mathbf{C}|}(q;q)_{N-\mathfrak{a}}(q;q)_{N-\mathsf{L}-\mathfrak{c}}^{-1}$ from (17.7.13), the renormalized weights are polynomial in q^{L} of degree $\mathfrak{d}$. Since we wish to take the limit $q^{\mathsf{L}} \to \infty$, we now examine which lattice configurations contribute maximal degree in q^{L}.

The leftmost column can contribute at most degree m in q^L; this occurs when a bosonic path turns right into each of the m shaded rows of (17.7.15), leaving exactly m "vacancies" distributed across the unshaded rows, and results from taking the top degree term of the renormalized weights

(17.7.13) for every vertex  in the column. If we choose the configuration with all m vacancies occurring in the highest unshaded row, we obtain the following picture:

(17.7.18)



It is straightforward to show that the remaining columns in (17.7.18) contribute degree $\sum_{i=1}^{m} ig_i - m$ in q^L (again, this results from taking the top degree term of the renormalized weights (17.7.10) for every vertex  in the lattice). Furthermore, it easy to show that any other configuration of the leftmost column would lead to smaller overall degree in q^L. It follows that when taking the limit $q^L \to \infty$, only the configuration (17.7.18) needs to be considered, and it has degree $(\sum_{i=1}^{m} ig_i - m) + m = \sum_{i=1}^{n} ig_i$ in q^L.

Multiplying (17.7.15) by $q^{-L\sum_{i=1}^{m} ig_i}(-y)^{-\sum_{i=1}^{m} ig_i}$ and letting q^L tend to ∞, we thus obtain (17.7.17). The factor of $(-1)^{|\bar{\nu}|+m}$ is due to $(-1)^{-\sum_{i=1}^{m} ig_i}$ that we multiply by, and the factor of $\frac{(-1)^m}{(q;q)_m}$ is the result of computing the weight of the top vertex in the leftmost column of (17.7.18) (all other vertices in the leftmost column of (17.7.18) can be considered to have weight 1, in view of previously cancelled-off factors). $\square$

17.7.7. Final Matching. Given the analysis in the previous sections, we can now quickly establish Theorem 16.1.3.

Proof of Theorem 16.1.3. Comparing (17.6.10) and (17.7.17) now yields equation (16.1.6). In producing the match we perform a complementation by L of bosonic states that live on left and right edges of vertices; namely, such a state $(L-\mathfrak{b},\mathbf{B})$ in (17.7.17) gets replaced by $(\mathfrak{b},\mathbf{B})$ in (16.1.6). We also implement an analogous complementation for labelling of left and right edges of shaded tiles in going from (17.7.17) to (16.1.6) (namely, we replace an arrow configuration $(\mathbf{A},b;\mathbf{C},d)$ with $(\mathbf{A},1-b;\mathbf{C},1-d)$). It is quickly verified under this complementation that the white tiles of

Proposition 17.7.2 transform at $y = 1$ into the unshaded ones of Definition 16.1.1 and that both the green and gray tiles of Proposition 17.7.2 transform into the shaded oned ones of Definition 16.1.2.

Shaded rows in (17.7.17) occur at row $m-i+\sum_{j=i}^{m} g_j$ for each $i \in [1, m]$, where rows are counted top to bottom, starting at zero. This matches with the vector $\mathfrak{n}$ in (16.1.6), since $g_j = \bar{\nu}_j - \bar{\nu}_{j+1}$ and $\bar{\nu}_{m+1} = -1$ together imply $\sum_{j=i}^{m} g_j = \bar{\nu}_i + 1$. $\square$

Index

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