# EQUIDISTRIBUTION AND PARTITION POLYNOMIALS

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Abstract. Using equidistribution criteria, we establish divisibility by cyclotomic polynomials of several partition polynomials of interest, including *spt*-crank, overpartition pairs, and $t$-core partitions. As corollaries, we obtain new proofs of various Ramanujan-type congruences for associated partition functions. Moreover, using results of Erdös and Turán, we establish the equidistribution of roots of partition polynomials on the unit circle including those for the rank, crank, *spt*, and unimodal sequences. Our results complement earlier work on this topic by Stanley, Boyer-Goh, and others. We explain how our methods may be used to establish similar results for other partition polynomials of interest, and offer many related open questions and examples.

## 1. Introduction

The theory of ranks and cranks was initiated in order to study congruences for the partition function $p(n)$, which counts the number of integer partitions of $n$, that is, the number of ways to write a non-negative integer $n$ as a non-increasing sum of positive integers (called "parts"). For example, $p(5) = 7$, as

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

The definitions of the rank and crank of a partition $\lambda$ may appear artificial at first inspection:

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda,$$

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if 1 is not a part of } \lambda, \\ \mu(\lambda) - o(\lambda) & \text{if 1 is a part of } \lambda, \end{cases} \tag{1}$$

where $\mu(\lambda)$ denotes the number of parts of $\lambda$ strictly larger than the number of 1s in $\lambda$, and $o(\lambda)$ denotes the number of 1s in $\lambda$. However, these two perhaps seemingly peculiar partition statistics play significant roles in understanding partition numbers $p(n)$. After Ramanujan conjectured his famous congruences modulo $5, 7$ and $11$ (for all $n \in \mathbb{N}_0$)

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned} \tag{2}$$

while still an undergraduate in 1944, Dyson [26] defined the rank of a partition in order to try and combinatorially explain them. The following table illustrates how Dyson's rank divides the

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partitions of 5 into 7 groups of equal size:

| partition | rank | rank (mod 7) |
|---|---|---|
| $5$ | $5-1$ | $4$ |
| $4+1$ | $4-2$ | $2$ |
| $3+2$ | $3-2$ | $1$ |
| $3+1+1$ | $3-3$ | $0$ |
| $2+2+1$ | $2-3$ | $6$ |
| $2+1+1+1$ | $2-4$ | $5$ |
| $1+1+1+1+1$ | $1-5$ | $3.$ |

Further investigations led Dyson to conjecture that the rank would always divide the partitions of $7n+5$ (resp. $5n+4$) for any $n \in \mathbb{N}_0$ into 7 (resp. 5) groups of equal size when reduced mod 7 (resp. mod 5), thereby explaining Ramanujan's congruences mod 7 and mod 5. This was proved by Atkin and Swinnerton-Dyer in 1954 [8], and has led to further important related results in the literature. A quick calculation reveals that the rank fails to explain Ramanujan's congruences mod 11 in the same way as we now know it does for the moduli 5 and 7, and this led Dyson to conjecture the existence of another partition statistic which would simultaneously explain all three of Ramanujan's congruences in (2). Over four decades after Dyson's paper, important work of Garvan and Andrews [5, 29] led them to the definition of the crank statistic in (1), resolving Dyson's question.

Also playing key roles in understanding partitions, ranks, and cranks, are their generating functions. To describe this, we let $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$ be the usual $q$-Pochhammer symbol ($n \in \mathbb{N}_0 \cup \{\infty\}$). Then we have that

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N(m,n)w^m q^n =: \sum_{n\geq 0} \mathrm{rank}_n(w)q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \tag{3}$$

and

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} M(m,n)w^m q^n =: \sum_{n\geq 0} \mathrm{crank}_n(w)q^n = \prod_{n\geq 1} \frac{(1-q^n)}{(1-wq^n)(1-w^{-1}q^n)}, \tag{4}$$

where $N(m,n)$ (resp. $M(m,n)$) denotes the number of partitions of $n$ with rank (resp. crank) $m$. For example, when $w=1$ in (3) we have that the partition generating function

$$\sum_{n\geq 0} p(n)q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \prod_{n\geq 1} \frac{1}{(1-q^n)}$$

is essentially the reciprocal of the weight $1/2$ modular form

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n),$$

with $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$, the upper-half of the complex plane. (Here we have also used Euler's product identity for the partition generating function, see e.g. [1].) In general, when viewed as a two-variable function in $(z,\tau) \in \mathbb{C}\times\mathbb{H}$ (additionally letting $w = e^{2\pi iz}$), the rank generating funciton in (3) does not posses the same strict modular properties as it does at $w=1$, but thanks to influential work of Zwegers [54], Bringmann-Ono [22], and Zagier [53], we now know that (after a minor normalization) it is a mock Jacobi form. The crank generating function in (4) too possesses modular properties, and is (up to a minor normalization) a (true) Jacobi form. Such connections between partition generating functions and modular(-type) forms have led to significant advances in our understanding in both the theory of partitions and in modular forms (broadly speaking). For example, work of Hardy-Ramanujan and Rademacher established an exact formula for $p(n)$ using the modularity of its generating function; we also now know to look at families of partition generating functions as potential explicit sources of holomorphic parts of harmonic Maass forms. (For more on these topics, see e.g. [16].)

When expanded as a $q$-series, the polynomial (in $w$) coefficients $\mathrm{rank}_n(w)$ and $\mathrm{crank}_n(w)$ of the generating functions (3) and (4) clearly carry important combinatorial information – which it turns out may be revealed algebraically. To describe this, we review work of Stanton, and Bringmann et al, and after Stanton define the modified rank and crank polynomials by

$$\mathrm{rank}^*_{\ell,n}(w) := \mathrm{rank}_{\ell n+\beta}(w) + w^{\ell n+\beta-2} - w^{\ell n+\beta-1} + w^{2-\ell n-\beta} - w^{1-\ell n-\beta},$$
$$\mathrm{crank}^*_{\ell,n}(w) := \mathrm{crank}_{\ell n+\beta}(w) + w^{\ell n+\beta-\ell} - w^{\ell n+\beta} + w^{\ell-\ell n-\beta} - w^{-\ell n-\beta},$$

where $\beta := \ell - (\ell^2-1)/24$. In unpublished notes, Stanton made some related conjectures on divisibility by the cyclotomic polynomials $\Phi_\ell(w)$, including the following [50].

**Conjecture** (Stanton [50])**.** *For $n \in \mathbb{N}_0$, the following are Laurent polynomials with non-negative coefficients:*

$$\frac{\mathrm{rank}^*_{5,n}(w)}{\Phi_5(w)}, \frac{\mathrm{rank}^*_{7,n}(w)}{\Phi_7(w)},$$

*and*

$$\frac{\mathrm{crank}^*_{5,n}(w)}{\Phi_5(w)}, \frac{\mathrm{crank}^*_{7,n}(w)}{\Phi_7(w)}, \frac{\mathrm{crank}^*_{11,n}(w)}{\Phi_{11}(w)}.$$

The connection between these conjectured – and now proved in [17, 21] – algebraic properties of polynomials and the combinatorial divisibilities prescribed by Ramanujan's congruences (2) is given by Lemma 2.1 [17, Lemma 2.4]: in short, divisibility by cyclotomic polynomials is equivalent to an equidistribution result on coefficients of (rank and crank) polynomials. As explained in [17], Stanton's modified rank and crank polynomials are designed to ensure positivity, with the eventual goal of uncovering a combinatorial interpretation of what these positive coefficients count, thereby leading to a map between rank or crank congruence classes. Lemma 2.2 [17, Lemma 3.1] reveals that such positivity is related to unimodality of coefficients.

In the sections that follow, using equidistribution criteria, we establish divisibility by cyclotomic polynomials of several partition polynomials of interest, including *spt*-crank, overpartition pairs, and $t$-core partitions (see Theorem 3.1, Theorem 3.6, and Theorem 3.9). As corollaries, we obtain new proofs of various Ramanujan-type congruences for associated partition functions (see Corollary 3.2, Corollary 3.7, and Corollary 3.10). Moreover, using results of Erdös and Turán, we establish the equidistribution of roots of partition polynomials on the unit circle including those for the rank, crank, *spt*, and unimodal sequences (see Theorem 4.2, Theorem 4.3, and Theorem 4.4). Our results complement earlier work on this topic by Stanley, Boyer-Goh [11, 12], and Boyer-Parry [13]. We also explain how our methods may be used to establish similar results for other partition polynomials of interest, and offer many related open questions and examples in the ensuing sections.

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## 2. Preliminaries

We use the following lemmas of Bringmann–Gomez–Rolen–Tripp [17]. To state these, recall that the $\ell$th cyclotomic polynomial $\Phi_\ell(w)$ is the unique irreducible polynomial in $\mathbb{Z}[w]$ which divides $w^\ell - 1$ but which does not divide $w^j - 1$ for any $1 \le j < \ell$ ($\ell \in \mathbb{N}, \ell \ge 2$), and $\Phi_1(w) := w-1$. We also define (for $(r, \ell) \in \mathbb{Z} \times \mathbb{N}$)

$$\widehat{f}_{r,\ell} := \sum_{j \equiv r \pmod{\ell}} [w^j] f(w).$$

Here, by $[w^j]f(w)$ we mean the coefficient of $w^j$ in (the series expansion of) $f(w)$. We also have the notion of divisibility in $\mathbb{Q}[w^{-1}, w]$, where we say that $\Phi_\ell(w)$ divides $f(w)$ if there exists some $g(w) \in \mathbb{Q}[w^{-1}, w]$ such that $f(w) = g(w)\Phi_\ell(w)$. The following lemma relates divisibility of polynomials to an equidistribution criterion over arithmetic progressions.

**Lemma 2.1** (Lemma 2.4 of [17]). *Let $f(w)$ be a Laurent polynomial in $\mathbb{Q}[w^{-1}, w]$ and $\ell$ a prime. Then $\Phi_\ell(w)$ divides $f(w)$ in $\mathbb{Q}[w^{-1}, w]$ if and only if*

$$\widehat{f}_{a,\ell} = \widehat{f}_{b,\ell}$$

*for all $a, b$.*

In particular, by showing divisibility by the $\ell$-th cyclotomic polynomial, we immediately obtain equidistribution modulo $\ell$ and vice-versa. The next result gives non-negativity of the coefficients of $f(w)/\Phi(w)$ under certain conditions.

**Lemma 2.2** (Lemma 3.1 of [17]). *Let $f(w)$ be a symmetric unimodal Laurent polynomial that is divisible by $\Phi_\ell(w)$ for an odd prime $\ell$. Then the coefficients of $\frac{f(w)}{\Phi_\ell(w)}$ are non-negative. Moreover, if $f(w)$ is strictly unimodal then the coefficients of $\frac{f(w)}{\Phi_\ell(w)}$ are positive.*

## 3. Divisibility of (c)rank polynomials

In the following three subsections, we give three examples of combinatorial objects that are well-studied in the literature. We establish divisibility properties of their two-variable generating functions, and establish new proofs of Ramanujan-type congruences. We also offer many related open questions of interest.

3.1. **The spt-crank.** In [2], Andrews introduced the function $\mathrm{spt}(n)$, which counts the number of smallest parts among the integer partitions of $n$. For example, the smallest parts among the partitions of three are underlined here:

$$\underline{3}, 2+\underline{1}, \underline{1}+\underline{1}+\underline{1}$$

and hence we see that $\mathrm{spt}(3) = 5$. Among the many interesting properties of spt now known to be true, Andrews [2] proved that the spt function satisfies three beautiful Ramanujan-type congruences

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5, \qquad \mathrm{spt}(7n+5) \equiv 0 \pmod 7, \qquad \mathrm{spt}(13n+6) \equiv 0 \pmod{13}, \tag{5}$$

for $n \in \mathbb{N}_0$, in analogy to the celebrated Ramanujan partition congruences for the partition function $p(n)$ modulo $5, 7$, and $11$ in (2). As the partition crank function was famously found to combinatorially explain Ramanujan's partition congruences (see Section 1), it is natural to search for an spt-crank function that combinatorially explains Andrews's spt-congruences above. To this end, Andrews, Garvan and Liang [6] defined an spt-crank which explains Andrews' spt-congruences modulo 5 and 7 in (5). In order to prove their results, Andrews, Garvan, and Liang first introduced the set of vector partitions $V$, defined by the Cartesian product

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P},$$

where $\mathcal{D}$ is the set of partitions into distinct parts and $\mathcal{P}$ is the set of partitions. Each vector partition $\vec{\pi} \in V$ comes equipped with a crank, defined by

$$\mathrm{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).$$

For a partition $\lambda$, let $s(\lambda)$ denote the smallest part, and define $s(-) = \infty$ for the empty partition. Then a central subset of $V$ in [6] is given by

$$S \coloneqq \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \le s(\pi_1) < \infty \text{ and } s(\pi_1) \le \min(s(\pi_2), s(\pi_3))\}.$$

For $\vec{\pi} \in S$, its weight is defined by $\omega_1(\vec{\pi}) \coloneqq (-1)^{\#(\pi_1)-1}$. Then the number of vector partitions of $n$ in $S$ with crank equal to $m$ counted in accordance with the weight $\omega_1$ is denoted by

$$N_S(m,n) \coloneqq \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}| = n \\ \mathrm{crank}(\vec{\pi}) = m}} \omega_1(\vec{\pi}).$$

Importantly, it turns out that

$$\sum_{m\in\mathbb{Z}} N_S(m,n) = \mathrm{spt}(n).$$

Before stating our results, define the spt-crank polynomials $\text{spt-crank}_n(w)$ ($n \in \mathbb{N}_0$) to be the $q$-series coefficients of the two-variable generating function for $N_S(m,n)$. That is (see [6]),

$$\sum_{n\geq 1}\sum_{m\in\mathbb{Z}} N_S(m,n)w^m q^n =: \sum_{n\geq 1} \text{spt-crank}_n(w)q^n = \sum_{n\geq 1} \frac{q^n(q^{n+1};q)_\infty}{(wq^n;q)_\infty(w^{-1}q^n;q)_\infty}. \tag{6}$$

Our first result, Theorem 3.1 below, explains spt-congruences another way, in terms of cyclotomic polynomial divisibility properties. This leads to a new proof of Andrews' spt-congruences modulo 5 and 7; see Corollary 3.2, its proof, and Remark 3.3.

**Theorem 3.1.** *We have that* $\Phi_5(w)$ *divides* $\text{spt-crank}_{5n+4}(w)$ *and* $\Phi_7(w)$ *divides* $\text{spt-crank}_{7n+5}(w)$ *in* $\mathbb{Z}[w,w^{-1}]$.

*Moreover, the coefficients of both* $\frac{\text{spt-crank}_{5n+4}(w)}{\Phi_5(w)}$ *and* $\frac{\text{spt-crank}_{7n+5}(w)}{\Phi_7(w)}$ *are non-negative.*

**Corollary 3.2.** *For* $n \in \mathbb{N}_0$, *we have that*

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5, \qquad \mathrm{spt}(7n+5) \equiv 0 \pmod 7.$$

*Remark* 3.3. Due to Theorem 3.1 and Lemma 2.1, we in fact have an equidistribution result on the coefficients of the $\text{spt-crank}_{5n+4}(w)$ and $\text{spt}_{7n+5}(w)$ polynomials. While this implies Corollary 3.2, we establish this result more directly from Theorem 3.1 below for simplicity.

*Proof of Theorem 3.1.* The divisibility of the relevant polynomials follows from (6) and Lemma 2.1, using the fact that the spt-crank is equidistributed in these cases (see [6]). To see that the coefficients of each of $\frac{\text{spt-crank}_{5n+4}(w)}{\Phi_5(w)}$ and $\frac{\text{spt-crank}_{7n+5}(w)}{\Phi_7(w)}$ are non-negative it is enough to note that $N_S(m,n)$ is symmetric and unimodal in $m$ for each fixed $n$ by results of Chen–Ji–Zang [23]. Applying Lemma 2.2 finishes the proof. □

*Proof of Corollary 3.2.* By Theorem 3.1 with $w = 1$, we have that $\Phi_5(1)$ divides $\text{spt-crank}_{5n+4}(1)$ and $\Phi_7(1)$ divides $\text{spt-crank}_{7n+5}(1)$. By (6), we have that for $n \in \mathbb{N}_0$, $\text{spt-crank}_n(1) = \mathrm{spt}(n)$, and by definition, we have that $\Phi_5(1) = 5$ and $\Phi_7(1) = 7$. The corollary now follows. □

Of course, since Lemma 2.1 is an if-and-only-if statement, one could also prove the first part of the result in the reverse direction, by showing that the relevant cyclotomic polynomial divides the generating function on the given arithmetic progression.

**Question 1.** *Is there a simple way to show that the relevant cyclotomic polynomial divides the* spt-crank *on the given arithmetic progressions in Theorem 3.1 without using Lemma 2.1 and unimodality?*

*Remark* 3.4. In [6, Theorem 4.1], Andrews, Garvan, and Liang show that the generating function for the spt-crank can be written (up to a multiplicative factor) as the difference between the ordinary – unmodified – crank and rank generating functions. This may provide a starting point towards answering Question 1.

While the non-negativity of the coefficients after dividing by the relevant cyclotomic polynomial is an easy consequence of Lemma 2.2, the coefficients themselves remain mysterious. It would be extremely instructive to determine a combinatorial description for them.

**Question 2.** *Do the coefficients of* $\frac{\text{spt-crank}_{5n+4}(w)}{\Phi_5(w)}$ *and* $\frac{\text{spt-crank}_{7n+5}(w)}{\Phi_7(w)}$ *have a combinatorial interpretation?*

Moreover, Theorem 3.1 only deals with the cases of arithmetic progressions modulo 5 and 7, since here the spt-crank explains the corresponding Ramanujan-type congruences. It is well-known that the spt-crank does not combinatorially explain the congruence $\mathrm{spt}(13n+6) \equiv 0 \pmod{13}$.

**Question 3.** *Can one determine a combinatorial explanation for the* spt *congruence modulo* 13 *– and by establishing results as in this paper modulo* 5 *and* 7*?*

Partition and smallest parts congruences modulo other primes and residue classes are of interest, including modulo the smallest primes 2 and 3; see for example [27], or [45, 46] by Ono and Radu, which notably prove Subbarao's conjecture on the partition function modulo 2 and 3, or [28] on spt-congruences modulo 2 and 3.

**Question 4.** *Is the* spt-crank *equidistributed on certain arithmetic progressions modulo* 2 *and* 3*?*

*Remark* 3.5. While the spt function is known to be almost always even (see [7, Theorem 1.3]), the possible Ramanujan-type congruence modulo 2 may not hold with the exceptions captured by those terms given in [7, Theorem 1.3].

As noted in Section 1, it is well-known that the partition generating function is essentially a modular form. Since the time of Hardy, Ramanujan, and Rademacher and their influential related work, mathematicians have produced a wealth of literature on modularity and partition functions – broadly construed to also include Maass forms, mock theta functions, quantum modular forms, and other modular-type functions. For example, in [14], Bringmann constructs a harmonic Maass form associated to the spt-generating function, and it is of interest to investigate and establish the modular properties of further spt-generating functions as well as its applications. As explained in Remark 3.4, the generating function for spt-crank is essentially a difference between the rank and crank generating functions, thereby implying mock Jacobi properties.

**Question 5.** *What are the precise modular properties of the* spt-crank *generating function given in* (6) *and can they be used to determine further congruences and asymptotic properties of the* spt-crank*?*

3.2. **The rank of overpartition pairs.** An *overpartition* is a partition in which the first occurrence of a part may be overlined. For example, there are four overpartitions of 2, namely $1+1, \overline{1}+1, 2$, and $\overline{2}$. Corteel and Lovejoy helped pioneer the modern-day study of overpartitions and established several important results in [24], noting that related combinatorial and $q$-hypergeometric results trace back to older work of MacMahon [44], Joichi-Stanton [42], and others. It is known that there are no Ramanujan-type congruences for overpartitions, see [25, Theorem 1.2], but for *overpartition pairs*, they do exist. To explain this, an overpartition pair of $n$ is a pair of overpartitions $(\mu, \lambda)$ where the sum of all of the parts is equal to $n$. For example, there are 12 overpartition pairs of $n = 2$ (noting that the definition allows the empty overpartition to be used for $\mu$ or $\lambda$). Overpartition pairs have been of importance as associated to Ramanujan's ${}_1\psi_1$ summation, the $q$-Gauss identity, and other $q$-hypergeometric series [43]. In [19], Bringmann and Lovejoy establish the following overpartition pair congruence for the overparition pair function $\overline{pp}(n)$ which the number of overpartition pairs of $n$ ($n \in \mathbb{N}_0$):

$$\overline{pp}(3n+2) \equiv 0 \pmod 3. \tag{7}$$

In the same way that the partition rank function can be used to combinatorially explain Ramanujan's partition congruences modulo 5 and 7, Bringmann and Lovejoy [20, Theorem 1.2] use the overpartition pair rank function to explain (7) by splitting overpartition pairs into three equinumerous classes sorted by ranks. To define this rank function, we let $\ell(\cdot)$ denote the largest part of a partition, and let $n(\cdot)$ denote the number of parts; their overlined counterparts count only parts which are overlined. With this, the rank of an overpartition pair $(\lambda, \mu)$ is defined by

$$\ell((\lambda,\mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda,\mu)),$$

where $\chi((\lambda,\mu))$ is defined to be 1 if the largest part of $(\lambda,\mu)$ is non-overlined and in $\mu$, and 0 otherwise.

Our next set of results are parallel to Theorem 3.1 and Corollary 3.2. To state them, we define the overpartition pair rank polynomials o-rank$_n(w)$ ($n \in \mathbb{N}_0$) to be the $q$-series coefficients of the

two-variable generating function for $\overline{NN}(m,n)$, the number of overpartition pairs of $n$ with rank $m$. Explicitly [20, Proposition 2.1], we have that

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \overline{NN}(m,n)w^m q^n =: \sum_{n\geq 0} \text{o-rank}_n(w)q^n = \sum_{n\geq 0} \frac{(-1;q)_n^2 q^n}{(wq;q)_n(w^{-1}q;q)_n}. \tag{8}$$

Our first result on overpartition pair rank polynomials is the following.

**Theorem 3.6.** *We have that* $\Phi_3(w)$ *divides* $\text{o-rank}_{3n+2}(w)$ *in* $\mathbb{Z}[w,w^{-1}]$.

*Proof.* By Lemma 2.1 it is enough to have that the o-rank is equidistributed on the arithmetic progression $3n+2$. This is shown by Bringmann and Lovejoy [20, Theorem 1.2] who showed that the overpartition pair rank splits overpartition pairs into three equinumerous classes. □

As a corollary, we obtain a new proof of the overpartition rank congruence modulo 3 of Bringmann and Lovejoy discussed above.

**Corollary 3.7.** *For* $n\in\mathbb{N}_0$, *we have that*

$$\overline{pp}(3n+2)\equiv 0 \pmod 3.$$

*Remark* 3.8. The contents of Remark 3.3 also apply here to $\text{o-rank}_{3n+2}(w)$ and $\overline{pp}(3n+2) \pmod 3$ in a similar manner.

*Proof of Corollary 3.7.* By Theorem 3.6 with $w=1$, we have that $\Phi_3(1)$ divides $\text{o-rank}_{3n+2}(1)$. By (8), we have that for $n\in\mathbb{N}_0$, $\text{o-rank}_n(1)=\overline{pp}(n)$, and by definition, we have that $\Phi_3(1)=3$. The corollary now follows. □

Similar to the case of the spt-crank, the (non-)unimodality of $\overline{NN}(m,n)$ in the $m$-aspect appears to be unknown in the literature. It is clear that $\overline{NN}(-m,n)=\overline{NN}(m,n)$, and numerical tests (using SageMath [48]) suggest that $\overline{NN}(m,n)$ is not unimodal in $m$, although this may be a situation that parallels the ordinary partition rank studied in [17], where the authors introduced a slightly modified definition to ensure unimodality.

**Question 6.** *What are the (non-)unimodality properties of* $\overline{NN}(m,n)$ *in the* $m$*-aspect?*

A choice of crank to explain a particular partition congruence is not unique. For example, very recently Wagner [52] described a vast array of cranks for various partition-theoretic congruence families. In particular, a new crank statistic that explains the congruence $\overline{pp}(3n+2)\equiv 0 \pmod 3$ was introduced on p24 of [52], given by the pleasing infinite product formula

$$\overline{C}_2(w;q) \coloneqq \prod_{n\geq 1} \frac{(1+wq^n)(1+w^{-1}q^n)}{(1-wq^n)(1+w^{-1}q^n)}.$$

One then obtains the analogues of Theorem 3.6 and Corollary 3.7 for this new crank. Moreover, this new crank numerically appears to unimodal, in contrast to the crank of Bringmann–Lovejoy. Unimodality would then lead to the non-negativity of coefficients of this new crank polynomial divided by $\Phi_{3n+2}(w)$. One could ask for a combinatorial interpretation in analogy to Question 2.

**Question 7.** *What are the unimodality properties of* $\overline{C}_2(w;q)$*? Can one extend the ideas presented here to the wide class of crank functions given by Wagner in* [52]*?*

3.3. **$t$-core partitions.** Each partition comes equipped with a statistic called the *hook length* of the partition. To explain this, we recall that every partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)$ has a *Ferrers–Young diagram*

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \ldots & \bullet & \leftarrow & \lambda_1 \text{ many nodes}\\
\bullet & \bullet & \ldots & \bullet & & \leftarrow & \lambda_2 \text{ many nodes}\\
\vdots & \vdots & \vdots & & & & \\
\bullet & \ldots & \bullet & & & \leftarrow & \lambda_m \text{ many nodes,}
\end{array}$$

and each node has an associated *hook length*. The node in row $k$ and column $j$ has hook length given by $h(k,j) := (\lambda_k - k) + (\lambda'_j - j) + 1$, where $\lambda'_j$ is the number of nodes in column $j$. This counts the number of nodes in the diagram directly below the given node plus the number to the right plus one (to count the given node itself). These numbers play many significant roles in combinatorics, number theory, and representation theory. For example, due to the Frame–Robinson–Thrall hook length formula [38, 6.1.19] we know that the counts of hook lengths of partitions control the number of irreducible representations of the symmetric groups $A_n$ and $S_n$.

For positive integers $t$, $t$-core partitions have also been of importance in combinatorial number theory. A $t$-core partition of a positive integer $n$ is a partition of $n$ for which none of its hook lengths are divisible by $t$. Let $c_t(n)$ denote the number of $t$-core partitions of $n$. Granville and Ono [35] showed that there are infinite families of congruences that 5-core, 7-core, and 11-core partitions satisfy, including the Ramanujan-type congruences ($n \in \mathbb{N}_0$)

$$c_5(5n+4) \equiv 0 \pmod 5, \qquad c_7(7n+5) \equiv 0 \pmod 7, \qquad c_{11}(11n+6) \equiv 0 \pmod{11}. \tag{9}$$

In order to combinatorially explain such congruences, Garvan, Kim, and Stanton [31] introduced the $t$-core crank. To define it, we recall Bijection 2 of [31], which states that there is a bijection $\phi_2 \colon P_{\text{t-core}} \to \{\vec{n} = (n_0, n_1, \dots, n_{t-1}) \colon n_i \in \mathbb{Z}, n_0 + n_1 + \cdots + n_{t-1} = 0\}$, where

$$|\tilde{\lambda}| = \frac{t||\vec{n}||^2}{2} + \vec{b} \cdot \vec{n}, \qquad \vec{b} := (0, 1, \dots, t-1).$$

Then the $t$-core crank definition is given algorithmically for a given partition $\lambda$. Choosing $t = 5, 7, 11$ one begins by finding the $t$-core $\tilde{\lambda}$ of $\lambda$. Next, find $\phi_2(\tilde{\lambda}) = \vec{n}$. Then the $t$-core crank is given by the following mod $t$ combination

$$\begin{aligned} 4n_0 + n_1 + n_3 + 4n_4 \qquad & \text{for } t = 5, \\ 4n_0 + 2n_1 + n_2 + n_4 + 2n_5 + 4n_6 \qquad & \text{for } t = 7, \\ 4n_0 + 9n_1 + 5n_2 + 3n_3 + n_4 + n_6 + 3n_7 + 5n_8 + 9n_9 + 4n_{10} \qquad & \text{for } t = 11. \end{aligned}$$

Let $c_t(m,n)$ denote the number of $t$-core partitions of $n$ with $t$-core crank $m$. Then the $t$-core crank polynomials t-core-crank$_n(w)$ are defined for $n \in \mathbb{N}_0$ by

$$\sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} c_t(m,n) w^m q^n =: \sum_{n \geq 0} \text{t-core-crank}_n(w) q^n. \tag{10}$$

Parallel to Theorem 3.1, but for all three moduli 5, 7 and 11, we establish the following result.

**Theorem 3.9.** *We have that* $\Phi_5(w)$ *divides* t-core-crank$_{5n+4}(w)$, $\Phi_7(w)$ *divides* t-core-crank$_{7n+5}(w)$, *and* $\Phi_{11}(w)$ *divides* t-core-crank$_{11n+6}(w)$ *in* $\mathbb{Z}[w, w^{-1}]$.

*Proof.* Garvan, Kim, and Stanton [31] showed that the t-core-crank splits $c_5(5n+4)$, $c_7(7n+5)$, and $c_{11}(11n+6)$ into equinumerous classes (see also [30, Theorem 3.1]). Combined with Lemma 2.1, one immediately obtains the result. □

As a corollary, we obtain a new proof of the Ramanujan-type congruences for $c_t(n)$ modulo 5, 7 and 11 in (9).

**Corollary 3.10.** *For* $n \in \mathbb{N}_0$, *we have that*

$$c_5(5n+4) \equiv 0 \pmod 5, \qquad c_7(7n+5) \equiv 0 \pmod 7, \qquad c_{11}(11n+6) \equiv 0 \pmod{11}.$$

*Remark* 3.11. The contents of Remark 3.3 also apply here to t-core-crank$_{5n+4}(w)$, t-core-crank$_{7n+5}(w)$, t-core-crank$_{11n+6}(w)$, and $c_5(5n+4), c_7(7n+5)$ and $c_{11}(11n+6) \pmod{5, 7, 11}$ (respectively) in a similar manner.

*Proof of Corollary 3.10.* By Theorem 3.9 with $w = 1$, we have that $\Phi_5(1)$ divides t-core-crank$_{5n+4}(1)$, that $\Phi_7(1)$ divides t-core-crank$_{7n+5}(1)$, and that $\Phi_{11}(1)$ divides t-core-crank$_{11n+6}(1)$. By (10), we have that for $n \in \mathbb{N}_0$, t-core-crank$_n(1) = c_t(n)$, and by definition, we have that $\Phi_5(1) = 5$, $\Phi_7(1) = 7$, and $\Phi_{11}(1) = 11$. The corollary now follows. □

To the best of the authors' knowledge, the (non-)unimodality properties of $c_t(m,n)$ in the $m$-aspect are unknown in the literature.

**Question 8.** *What are the (non-)unimodality properties of $c_t(m,n)$ in the $m$-aspect?*

To obtain numerical evidence for this question, it is natural to ask for a $q$-series representation for the two-variable generating function given in (10). To the best of the authors' knowledge, this is unknown in the literature and so we leave this as a question.

**Question 9.** *Is there a (nice) $q$-hypergeometric or product expression (e.g. as in* (3), (4), (6), (8)*) that allows one to answer (or make progress in answering) Question 8?*

We end this section by noting that there are a plethora of other partitions statistics in the literature that one could ask similar questions for. For example, spt residual cranks of overpartitions, unimodal sequences, the rank of overpartition pairs and more general partition pairs [51], and the many crank/rank functions of Garvan and Jennings-Shaffer [32, 33, 39–41].

## 4. Principal Polynomial roots

Refined information on partition statistics can be found by inspecting their so-called principal polynomials (defined explicitly in our cases below). This story begins with Stanley, who investigated the zeros of the partition polynomials

$$F_n(w) \coloneqq \sum_{k=1}^{n} p_k(n) w^n,$$

where $p_k(n)$ denotes the number of partitions into exactly $k$ parts. Stanley plotted the zeros of $F_{200}(w)$ and asked for their limiting behaviour as $n$ tends to $\infty$. This was settled in two beautiful papers of Boyer and Goh [11, 12], who discussed in detail the (rather exotic) zero-attractor of $F_n(x)$ (we refer the reader to these papers for more history and background on this topic). Boyer-Goh also prove similar results for the zero-attractors of several other common objects, including Appell and Euler polynomials [9, 10], and Boyer–Parry proved similar results for traces of plane partitions [13]. In each case, the resulting zero-attractor is rather complicated, and the proofs require technical asymptotics and bounds.

However, in some particular cases the roots of polynomials associated to partition-theoretic objects take a very simple shape - they become equidistributed on the unit circle as $n$ grows. To state this formally, we require classical results of Erdös and Turán, which were conveniently packaged together in our setting by Granville in [34]. Let $f(w) = \sum_{j=0}^{d} a_j w^j$ and

$$L(f) \coloneqq \frac{\sum_{j=0}^{d} |a_j|}{(|a_0||a_d|)^{\frac{1}{2}}}.$$

Let $\nu_{\{|z|=1\}}$ be the Haar measure on the unit circle (that is, equidistribution), and for a polynomial with not necessarily distinct roots $z_1, \dots z_d$ let $\mu_{\{f\}} = \frac{1}{d}\sum_{j=1}^{d} \delta_{z_j}$ where $\delta_z$ is the Dirac delta measure. Then Theorem 1.3 of [34] reads as follows.

**Theorem 4.1.** *Suppose that $f_1, f_2, \dots$ is a sequence of polynomials in $\mathbb{C}[x]$ where $f_d$ has degree $d$ and $f_d(0) \neq 0$. If $L(f_d) = e^{o(d)}$ as $d \to \infty$ then*

$$\lim_{d\to\infty} \mu_{\{f_d(x)\}} = \nu_{\{|z|=1\}}$$

*in the sense of "weak convergence" of measures.*

In general, we consider two situations. First, take a partition family $s(n)$ with partition-theoretic statistic $s(m,n)$ such that $s(-m,n) = s(m,n)$ and $s(m,n) \geq 0$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, with $s(0,n) \neq 0$ for all $n$. Assume that $s(0,n) + 2\sum_{m=1}^{\ell} s(m,n) = s(n)$ for some $\ell \in \mathbb{N}$. Assume that $s(n) \sim e^{o(\ell)}$. Consider the principal polynomial

$$S_n(w) \coloneqq \frac{s(0,n)}{2} + \sum_{1\leq m\leq \ell} s(m,n) w^m.$$

Then using Theorem 4.1 it is not difficult to show that as $n \to \infty$ the roots of $S_n(w)$ tend to equidistribution around the unit circle. Of course, the difficulty is usually in finding the asymptotic behaviour of $s(n)$. The families $s(m,n)$ often occur as ranks and cranks of various objects, which are allowed to be negative.

Similarly, we also consider a partition family $t(n)$ with partition-theoretic statistic $t(m,n)$ which vanishes for $m < 0$, where $t(m,n) \geq 0$ for all $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, with $t(0,n) \neq 0$ for all $n$. Assume that $\sum_{m=0}^{\ell} t(m,n) = t(n)$ for some $\ell \in \mathbb{N}$. Then we consider the principal polynomial

$$T_n(w) \coloneqq \sum_{0 \leq m \leq \ell} t(m,n) w^m. \tag{11}$$

Again, using Theorem 4.1 we see that as $n \to \infty$ the roots of $T_n(w)$ tend to equidistribution around the unit circle. The families $t(m,n)$ often occur from partition statistics which are naturally positive counts, for example partitions with number of parts equal to $m$.

In Section 4.1, following these methods, we establish equidistribution results for some specific partition polynomials of interest. Many other partition-theoretic families can be shown to have principal polynomials whose roots tend to equidistribution on the unit circle in a similar way. We pay special attention to strongly unimodal sequence polynomials in Section 4.2.1 and $t$-hook polynomials in Section 4.2.2, for which equidistribution-type properties are less clear, and offer various computations and open questions.

### 4.1. **Equidistribution of roots of partition polynomials.**

4.1.1. *Rank and crank polynomials.* In [12], the principal polynomials of the rank and crank of ordinary partitions were each shown to have roots whose arguments tend to equidistribution on the unit circle as $n \to \infty$. The authors of [12] go on to conjecture that the zero attractor for these polynomials is the unit circle. To the best of the authors' knowledge, this is not confirmed in the literature, and so we record the result here.

Let $N(m,n)$ and $M(m,n)$ denote the number of partitions of $n$ with rank and crank $m$ respectively. We define the principal polynomials

$$N_n(w) \coloneqq \frac{N(0,n)}{2} + \sum_{1 \leq m < n} N(m,n) w^m, \qquad M_n(w) \coloneqq \frac{M(0,n)}{2} + \sum_{1 \leq m \leq n} M(m,n) w^m,$$

noting that the factor of $\frac{1}{2}$ in the constant term of each is not present in [12]. In particular, recall that

$$\sum_{-n < m < n} N(m,n) = \sum_{-n \leq m \leq n} M(m,n) = p(n).$$

Moreover, we have that $N(0,n) \neq 0$ and $M(0,n) \neq 0$ for $n \geq 1$, along with the well-known asymptotic formula of Hardy and Ramanujan [37]

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \qquad n \to \infty.$$

Thus the assumptions of Theorem 4.1 apply, and we conclude the following theorem.

**Theorem 4.2.** *As $n \to \infty$ the roots of the rank and crank principal polynomials, i.e. $N_n(w)$ and $M_n(w)$, tend to equidistribution on the unit circle.*

4.1.2. *spt-crank polynomials.* Recall the spt-crank generating function given in (6). For fixed $n \in \mathbb{N}$ consider the principal polynomial attached to the spt-crank given by

$$P_n(w) = \tfrac{1}{2} N_S(0,n) + \sum_{1 \leq m < n} N_S(m,n) w^m.$$

We have that $P_n(0) = \frac{1}{2} N_S(0,n) = \frac{1}{2}\,\mathrm{ospt(n)}$ by Section 2 of [4], where ospt counts the difference of first moments of the ordinary crank and rank distributions. Moreover, by [3, Theorem 3] we

have that $\mathrm{ospt}(n) > 0$ for $n > 0$, and so we need only check that

$$\frac{N_S(0,n) + 2\sum_{j=1}^{n-1} N_S(n-1,n)}{(2N_S(0,n)N_S(n-1,n))^{\frac{1}{2}}} = e^{o(n-1)}$$

The numerator is equal to $\mathrm{spt}(n)$ by definition. Using the known asymptotic for $\mathrm{spt}(n)$ given in [14]

$$\mathrm{spt}(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}} e^{\pi\sqrt{\frac{2n}{3}}} = e^{o(n-1)},$$

we may again apply Theorem 4.1 to conclude that the zeros of $Q_n(w)$ are equidistributed on the unit circle as $n \to \infty$. We record this as a theorem.

**Theorem 4.3.** *As $n \to \infty$ the roots of $P_n(w)$ tend to equidistribution on the unit circle.*

4.1.3. *Unimodal sequence polynomials.* A *(weakly) unimodal sequence* of size $n$ is a sequence of positive integers $\{a_j\}_{1\leq j\leq s}$ such that

$$a_1 \leq a_2 \leq \ldots a_{k-1} \leq \overline{a}_k \geq a_{k+1} \geq \cdots \geq a_s, \qquad \sum_{j=1}^{s} a_j = n. \tag{12}$$

We follow the notation of [18] by indicating with the overline on the peak $a_k$ that if the largest part is repeated, the sequences may be further distinguished by specifying the location of the peak. Unimodal sequences are ubiquitous in number theory and in wider mathematics, and we refer the reader to [49] and the references therein for many beautiful examples.

The rank of a unimodal sequence is defined to be the number of parts to the right of the peak minus the number of parts before the peak. If we consider unimodal sequences of size $n$ with rank $m$, denoted by $u(m,n)$, then the generating function is given by (see e.g. [18])

$$U(w;q) \coloneqq \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} u(m,n)w^m q^n = \sum_{n\geq 0} \frac{q^n}{(wq;q)_n (w^{-1}q;q)_n}.$$

For fixed $n \in \mathbb{N}$, we consider the principal part polynomial defined by

$$Q_n(w) \coloneqq \tfrac{1}{2}u(0,n) + \sum_{1\leq m<n} u(m,n)w^m.$$

Then, since $Q_n(0) = \frac{1}{2}u(0,n) \neq 0$, we need only check that

$$\frac{u(0,n) + 2\sum_{j=1}^{n-1} u(j,n)}{(2u(0,n)u(n-1,n))^{\frac{1}{2}}} = e^{o(n-1)}$$

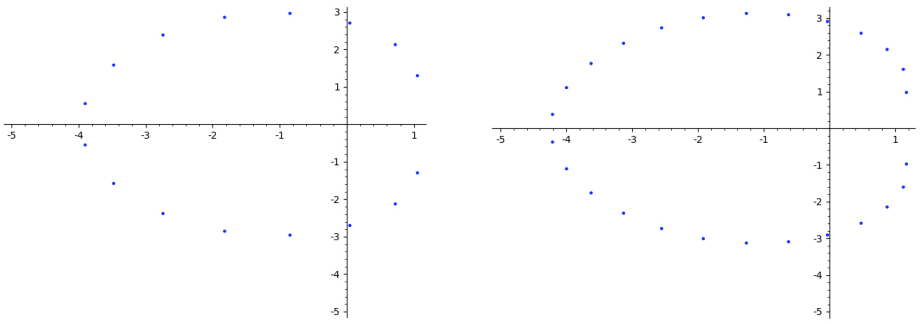
as $n \to \infty$. Using that $u(-m,n) = u(m,n)$, we see that the numerator counts exactly the number of unimodal sequences of size $n$, denoted by $u(n)$. Using [18, Theorem 1.1 (1)] with $k = 0$, which gives

$$u(n) \sim \frac{1}{8\cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{4n}{3}}} = e^{o(n-1)},$$

we apply Theorem 4.1 we conclude that the zeros of $Q_n(w)$ tend toward equidistribution on the unit circle as $n \to \infty$. We record this as a theorem.

**Theorem 4.4.** *As $n \to \infty$ the roots of $Q_n(w)$ tend to equidistribution on the unit circle.*

4.2. **Non-equidistribution of roots of partition polynomials.** In this section we collect some examples of partition polynomials whose roots numerically appear to not be equidistributed on the unit circle as $n$ grows. In each case, we explain why one cannot appeal to Theorem 4.1, and offer some numerical data and open questions.



(A) $n = 200$ (B) $n = 500$

FIGURE 1. Roots of $R_n(w)$ for two choices of $n$.

4.2.1. *Strongly unimodal sequences.* A close relative of unimodal sequences are known as strongly unimodal sequences, where in (12) one requires the inequalities to be strict. We denote the number of strongly unimodal sequences of size $n$ with rank $m$ by $u^*(m,n)$, and the number of strongly unimodal sequences of size $n$ by $u^*(n)$. Numerical experiments (using SageMath [48]) suggest that the roots of the principal polynomial for the rank of strongly unimodal sequences do not tend to equidistribution around the unit circle. Figure 1 plots the roots of the principal polynomial of the rank of strongly unimodal sequences for $n = 200$ and 500.

Let $R_n(w)$ be the principal polynomial attached to strongly unimodal sequences of size $n$. To apply Theorem 4.1, one would need that the growth of strongly unimodal sequences of size $n$ is $o(d(n))$, where $d(n)$ is the degree of $R_n(w)$. By [47] we have

$$u^*(n) \sim \frac{\sqrt{3}}{2(24n-1)^{\frac{3}{4}}} \exp\left(\frac{\pi}{6}\sqrt{24n-1}\right)$$

as $n \to \infty$. However, it is not difficult to see that $d(n)$ is given by taking the index of next triangular number above $n$ and then taking off 1 (which is very roughly $\sqrt{n}$). To see this, simply write out the strongly unimodal sequences diagrammatically. The largest possible rank will be given by a strictly decreasing triangle starting with the largest size in the first column, and a single dot in the final column.

Thus Theorem 4.1 does not apply, and we do not expect the roots of $R_n(w)$ to tend to be equidistributed on the unit disk as $n$ grows (at least, using this method).

**Question 10.** *What is the limiting distribution of the roots of $R_n(w)$?*

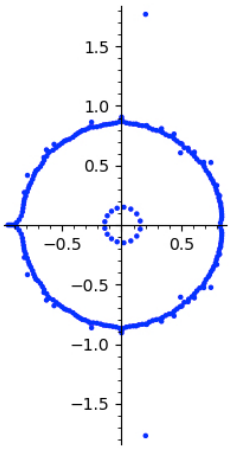
4.2.2. *$t$-hooks.* Recall the hook length of a partition introduced in Section 3.3. The hook lengths which are multiples of a fixed positive integer $t$ are called *$t$-hooks*, and been a central object of focus in several recent papers, including work of Bringmann, Craig, Ono, and the second author [15], who determined an exact formula for the number of $t$-hooks in partitions as well as the non-equidistribution properties over arithmetic progressions.

We also remark that the case of $t = 2, 3$ appears to be similar numerically. However, the interested researcher aiming to answer this question should be aware that their behaviour may be influenced by the fact that there are arithmetic progressions on which the number of 2-hooks and 3-hooks congruent to $a \pmod b$ identically vanish. Moreover, the number of $t$-hooks of partitions is not (in general) equidistributed among congruence classes (see [15]).

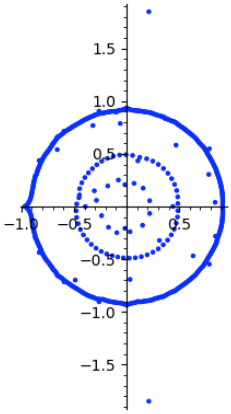
If we let $\mathcal{H}_t(\lambda)$ denote the multiset of $t$-hooks of a partition $\lambda$, then Han [36] proved that the generating function for $t$-hooks in partitions

$$H_t(w;q) := \sum_{\lambda \in \mathcal{P}} w^{\#\mathcal{H}_t(\lambda)} q^{|\lambda|} = \sum_{m,n \geq 0} c_t(m,n) w^m q^n$$

takes the following form.

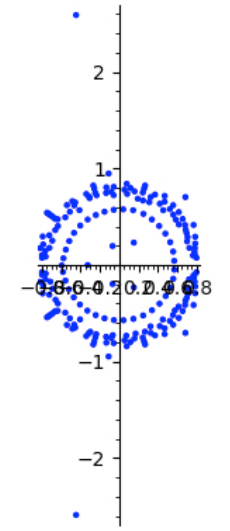



(A) $t = 4, n = 1000$

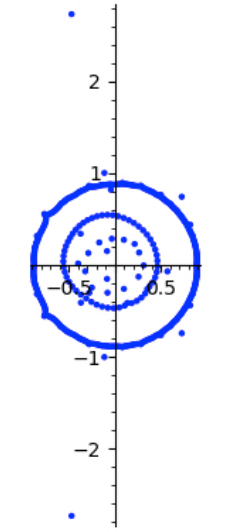



(B) $t = 4, n = 2000$

FIGURE 2. Roots of the 4-hook principal polynomial.




(A) $t = 5, n = 1000$




(B) $t = 5, n = 2000$

FIGURE 3. Roots of the 5-hook principal polynomial.

**Theorem 4.5.** (Corollary 5.1 of [36]) *As formal power series, we have*

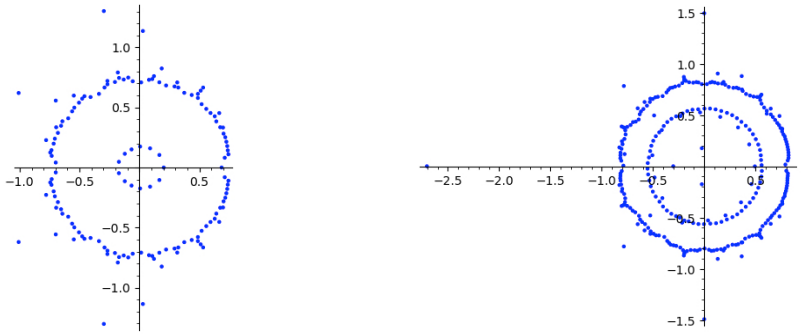
$$H_t(w;q) = \frac{1}{\prod_{n=1}^{\infty}(1-(wq^t)^n)^t} \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

We define the principal polynomial for $t$-hooks by

$$H_{t,n}(w) := \sum_{0 \leq m \leq n} c_t(m,n) w^m.$$

and below we include several figures on the roots of $H_{t,n}(w)$ for several choices of $t$ and $n$. As can be seen, the zeros appear to tend to lie on two distinct circles as $n$ grows although these circles may have radii that tend to a common value of 1. In every case, there are several "sporadic" zeros lying far from any apparent circles outside of the unit circle. For the cases $t = 4, 5$ and $n = 2000$ (see Figures 2 and 3) there appear to be further zeros far from the two apparent circles.

**Question 11.** *What is the limiting distribution of the zeros of $H_{t,n}(w)$?*




(A) $t = 7, n = 1000$ (B) $t = 7, n = 2000$

FIGURE 4. Roots of the 7-hook principal polynomial.

## REFERENCES


[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, no. 2, Cambridge University Press, (1984).

[2] G. E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math. 624 (2008), 133–142.

[3] G. E. Andrews, H. S. Chan, and B. Kim, *The odd moments of ranks and cranks*, J. Combin. Theory Ser. A 120 (2013), no. 1, 77–91.

[4] G. E. Andrews, F. J. Dyson, and R. C. Rhoades, *On the distribution of the spt-crank*, Mathematics, 1(3) (2013), 76–88.

[5] G.E. Andrews and F.G. Garvan, *Dyson's crank of a partition,* Bull. Amer. Math. Soc. (18) no. 2 (1988), 167-171.

[6] G. E. Andrews, F. G. Garvan, and J. Liang, *Combinatorial interpretations of congruences for the spt-function*, Ramanujan J., 29 (2012), no. 1-3, 321–338.

[7] G. E. Andrews, F. G. Garvan, and J. Liang, *Self-conjugate vector partitions and the parity of the spt-function*, Acta Arith. 158 (2013), no. 3, 199–218.

[8] A. O. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. (3) 4 (1954), 84–106.

[9] R.P. Boyer and W.M.Y. Goh, *Appell polynomials and their zero attractors*, Gems in experimental mathematics, 69–96, Contemp. Math., 517, Amer. Math. Soc., Providence, RI, 2010.

[10] R.P. Boyer and W.M.Y. Goh, *On the zero attractor of the Euler polynomials*, Adv. in Appl. Math. 38 (2007), no. 1, 97–132.

[11] R. P. Boyer and W. M. Y. Goh, *Partition polynomials: asymptotics and zeros*, Tapas in experimental mathematics, 99–111, Contemp. Math., 457, Amer. Math. Soc., Providence, RI, 2008.

[12] R.P. Boyer and W.M.Y. Goh, *Polynomials associated with partitions: asymptotics and zeros,* Special functions and orthogonal polynomials, 33–45, Contemp. Math., 471, Amer. Math. Soc., Providence, RI, 2008.

[13] R.P. Boyer and D. Parry, *On the zeros of plane partition polynomials*, Electron. J. Combin. 18 (2011), no. 2, Paper 30, 26 pp.

[14] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J., 144 (2) (2008), pp. 195-233.

[15] K. Bringmann, W. Craig, J. Males, and K. Ono, *Distributions on partitions arising from Hilbert schemes and hook lengths*, Forum. Math. Sigma, 10, E49.

[16] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass forms and mock modular forms: theory and applications,* AMS Colloquium Publications, 64. American Mathematical Society, Providence, RI, 2018. 391pp.

[17] K. Bringmann, K. Gomez, L. Rolen, and Z. Tripp, *Infinite families of crank functions, Stanton-type conjectures, and unimodality*, Res. Math. Sci., 9 (2022), no. 3, Paper No. 37.

[18] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg, *The asymptotic distribution of the rank for unimodal sequences*, J. Number Theory 229 (2021), 444–462.

[19] K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs,* Int. J. Number Theory 4 no. 2 (2008), 303-322.

[20] K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs*, Int. J. Number Theory 4 (2008), no. 2, 303–322.

[21] K. Bringmann, S.H. Man, and L. Rolen, *Unimodality of Ranks and a Proof of Stanton's Conjecture*, preprint, https://arxiv.org/abs/2209.12239.

[22] K. Bringmann and K. Ono *Dyson's ranks and Maass forms*, Ann. of Math. (2) **171** (2010), no. 1, 419–449.
[23] W.Y.C. Chen, K. Q. Ji, and W. Zang, *Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank*, Adv. Math. 270 (2015), 60–96.
[24] S. Corteel and J. Jovejoy, *Overpartitions,* Trans. Amer. Math. Soc. 356 (2004), 1623-1635.
[25] M. Dewar, *Non-existence of Ramanujan congruences in modular forms of level four*, Canad. J. Math. 63 (2011), no. 6, 1284–1306.
[26] F. Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15.
[27] A. Folsom, Z. Kent, and K. Ono, *$\ell$-adic properties of the partition function,* Adv. Math. 229 (2012), 1586–1609.
[28] A. Folsom and K. Ono, *The spt-function of Andrews,* Proc. Natl. Acad. Sci. USA, 105 no. 51 (2008), 20152-20156.
[29] F. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences* mod $5, 7$, *and* $11$, Trans. Amer. Math. Soc (305) no. 1 (1988), 47-77.
[30] F. Garvan, *More cranks and $t$-cores*, Bull. Austral. Math. Soc. 63 (2001), 379–391.
[31] F. Garvan, D. Kim, and D. Stanton, *Cranks and $t$-cores*, Invent. Math. 101 (1990), 1–17.
[32] F. Garvan and C. Jennings-Shaffer, *The spt-crank for overpartitions*, Acta Arith. 166 (2014), no. 2, 141–188.
[33] F. Garvan and C. Jennings-Shaffer, *Exotic Bailey-Slater spt-functions II: Hecke-Rogers-type double sums and Bailey pairs from groups A, C, E*, Adv. Math. 299 (2016), 605–639.
[34] A. Granville, *The distribution of roots of a polynomial*, Equidistribution in number theory, an introduction, NATO Sci. Ser. II Math. Phys. Chem., 237, 93–102, Springer, Dordrecht, 2007.
[35] A. Granville and K. Ono, *Defect zero blocks for finite simple groups*, Trans. Amer. Math. Soc. 348 (1996), 331–347.
[36] G.-N. Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1–29.
[37] G. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. Ser. 2 **17** (1918), 75–115.
[38] G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, Reading, 1979.
[39] C. Jennings-Shaffer, *Another SPT crank for the number of smallest parts in overpartitions with even smallest part*, J. Number Theory 148 (2015), 196–203.
[40] C. Jennings-Shaffer, *Exotic Bailey-Slater spt-functions I: Group A*, Adv. Math. 305 (2017), 479–514.
[41] C. Jennings-Shaffer, *Exotic Bailey-Slater spt-functions III: Bailey pairs from groups B, F, G, and J*, Acta Arith. 173 (2016), no. 4, 317–364.
[42] J.T. Joichi and D. Stanton, *Bijetive proofs of basic hypergeometrics series identities,* Pacific J. Math. 127 (1987), 103-120.
[43] J. Lovejoy, *Overpartition pairs,* Ann. Inst. Fourier 56 no. 3 (2006), 781–794.
[44] P.A. MacMahon, *Combinatory Analysis,* Clelsea, New York, 1960.
[45] K. Ono, *On the Parity of the Partition Function in Arithmetic Progressions,* J. für die Reine und Angew. Math. (Crelle's J.) 472 (1996), 1–15.
[46] S. Radu, A proof of Subbarao's conjecture, J. für die Reine und Angew. Math. (Crelle's J.) 672 (2012), 161-175.
[47] R. C. Rhoades, *Asymptotics for the number of strongly unimodal sequences*, Int. Math. Res. Not. IMRN 2014, no. 3, 700–719.
[48] The Sage developers, *SageMath, the Sage Mathematics Software system,* `https://www.sagemath.org`.
[49] R. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Graph theory and its applications: East and West (Jinan, 1986), 500–535, Ann. New York Acad. Sci., 576, New York Acad. Sci., New York, 1989.
[50] D. Stanton, 2019: The rank and cranks, Open Problems in Algebraic Combinatorics web blog, `https://realopacblog.wordpress.com/2019/09/02/rankcrank/` (accessed Sep. 28 2022).
[51] P. C. Toh, *Ramanujan type identities and congruences for partition pairs*, Discrete Math. 312 (2012), no. 6, 1244–1250.
[52] I. Wagner, *Jacobi forms with CM and applications*, preprint, `https://arxiv.org/abs/2208.01807`.
[53] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Bringmann-Ono)*, Séminaire Bourbaki. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 986, vii-viii, 143–164 (2010).
[54] S.P. Zwegers, *Mock theta functions*, Ph.D. thesis, Universiteit Utrecht, 2002.

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