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Alternate Computation of Gravitational Effects from a Single Loop of Inflationary Scalars

S. P. Miao*

Department of Physics, National Cheng Kung University, Tainan City 70101, TAIWAN

N. C. Tsamis†

Institute of Theoretical & Computational Physics, and Department of Physics, University of Crete GR-700 13 Heraklion, HELLAS

R. P. Woodard#

Department of Physics, University of Florida Gainesville, FL 32611, UNITED STATES

ABSTRACT

We present a new computation of the renormalized graviton self-energy induced by a loop of massless, minimally coupled scalars on de Sitter background. Our result takes account of the need to include a finite renormalization of the cosmological constant, which was not included in the first analysis. We also avoid preconceptions concerning structure functions and instead express the result as a linear combination of 21 tensor differential operators. By using our result to quantum-correct the linearized effective field equation we derive logarithmic corrections to both the electric components of the Weyl tensor for gravitational radiation and to the two potentials which quantify the gravitational response to a static point mass.

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* *e-mail:* spmiao5@phys.ncku.edu.tw
† *e-mail:* tsamis@physics.uoc.gr
# *e-mail:* woodard@phys.ufl.edu

1 Prologue

A wide variety of observational evidence points to the very early universe having experienced a phase of accelerated expansion, or inflation [1]. Cosmological spacetimes are described by the scale factor $a(t)$ and its two first time derivatives $H(t)$ (Hubble parameter) and $\epsilon(t)$ ($1st$ slow-roll parameter):

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x} \cdot d\mathbf{x} \quad , \quad H(t) \equiv \frac{\dot{a}}{a} \quad , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} \quad . \tag{1}$$

Inflation is characterized by the positivity of both derivatives of $a(t)$:
$H(t) > 0 \;\; \& \;\; 0 \leq \epsilon(t) < 1$. The standard inflationary cosmology is that of the maximally symmetric de Sitter spacetime: [1]

$$ds^2 \;=\; -dt^2 + a^2(t) d\mathbf{x} \cdot d\mathbf{x} = a^2(\eta) \Big[- d\eta^2 + a^2(\eta) d\mathbf{x} \cdot d\mathbf{x} \Big] \quad , \tag{2}$$

$$a(t) \;=\; e^{Ht} = -\frac{1}{H\eta} = a(\eta) \quad . \tag{3}$$

During inflation, quantum physics implies the production of real particles out of the vacuum as long as they are effectively massless, possess classically non-conformally invariant free Lagrangians, and have adequately large wavelength. The carrier of the gravitational force, the graviton, is such a particle and inflationary evolution eventually will produce a dense ensemble of infrared gravitons [2, 3].

Can the universally attractive nature of the gravitational interaction alter cosmological parameters, kinematical parameters and long-range forces? There are perturbative indications for such changes from many loop computations which show a time dependence of powers of $\ln a(t)$ [4–13]. Because the dimensionless coupling constant of quantum gravity is GH^2, at some time the secular increase by powers of $\ln a(t)$ will overwhelm GH^2 causing perturbation theory to break.

It is always a formidable affair to decipher the dynamics of a theory after the perturbative analysis has become invalid. While it is easy to state what is needed - a re-summation technique for these leading logarithms - it is very hard to realize it. Usually one tries to first understand what happens in an

[1] Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has spacelike signature $(-+++)$ and our curvature tensor equals $R^{\alpha}{}_{\beta\mu\nu} \equiv \Gamma^{\alpha}{}_{\nu\beta,\mu} + \Gamma^{\alpha}{}_{\mu\rho} \Gamma^{\rho}{}_{\nu\beta} - (\mu \leftrightarrow \nu)$. Co-moving time is denoted by t and conformal time by η.

analogous situation in a simpler theory that retains the essential interaction structure of gravity. The latter feature is satisfied by non-linear σ-models since they possess the same kind of derivative interactions as gravity; they lack the tensor structure and gauge fixing dependence of quantum gravity. Of these, it is obviously the gauge issue that is the strongest simplification because a true physical effect is by definition independent of the gauge fixing functional.

Recently a particular non-linear σ-model - the AB model - has been perturbatively analyzed and the required re-summation techniques have been indicated: curvature-dependent variants of the stochastic technique [14, 15] *and* of the renormalization group [16–18].

Yet another theory that could be similarly analyzed before facing the full quantum gravity, is that of a massless minimally coupled (MMC) scalar in a de Sitter spacetime which we describe in Section 2. The contribution of a loop of MMC scalars to the graviton self-energy has been computed before [19], however, it was not then realized that a finite renormalization of the cosmological constant is required in order to make the graviton self-energy conserved [20]. This mistake was compounded by choosing to represent the non-conserved result as a linear combination of automatically conserved tensor differential operators acting on structure functions [19, 21]. These erroneous representations were then used to solve for 1-loop corrections to the graviton mode function [22] and to the two scalar potentials which represent the gravitational response to a static point mass [23]. Four conclusions were reached:

- The graviton mode function experiences no secular enhancement;
- Both potentials experience secular enhancement by fractional corrections of the form $GH^2 \ln(a)$, and a logarithmic running in space by fractional corrections of the form $GH^2 \ln(Hr)$;
- The coefficients of the temporal and spatial logarithms differ; and
- There was a huge gravitational slip.

The two errors described above might have canceled out, but it would be foolhardy to attempt resummation before checking. That is the purpose of this paper; we also wish to determine leading late time correction to the graviton mode function. We will show that the two errors do not cancel, and that this changes some of the four conclusions. The result for the 1-loop MMC correction to the 2-point gravitational function is found in Section 3, including the finite renormalization of the cosmological constant. We also represent the result, without any preconceptions, as the sum of a complete set of 21 tensor differential operators. Its effect on the gravitational mode functions and gravitational force are displayed in Sections 4 and 5, respectively. Our conclusions are given in Section 6. One appendix contains the relevant Feynman diagrams as well as a tabulation of the primitive results. A second appendix describes the various integrals needed to solve the 1-loop effective field equations.

2 The MMC theory

The dynamics of a MMC scalar in a de Sitter background are defined by:

$$\mathcal{L} = \frac{[R-(D-2)\Lambda]\sqrt{-g}}{16\pi G} - \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,g^{\mu\nu}\sqrt{-g} \quad , \quad \Lambda \equiv (D-1)H^2 \quad , \tag{4}$$

leading to the following gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \frac{1}{2}(D-2)\Lambda g_{\mu\nu} = 8\pi G\Big\{\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi\Big\} \quad . \tag{5}$$

The graviton field $h_{\mu\nu}(x)$ is defined by conformally rescaling the full metric with the scale factor:

$$g_{\mu\nu} \equiv a^2\,\widetilde{g}_{\mu\nu} \equiv a^2\,(\eta_{\mu\nu} + \kappa h_{\mu\nu}) \qquad , \qquad \kappa^2 \equiv 16\pi G \ . \tag{6}$$

Here κ is the loop-counting parameter of quantum gravity. Our notation throughout is that indices are raised and lowered with the Minkowski metric, for instance, $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$ and $\partial^\mu \equiv \eta^{\mu\rho}\partial_\rho$. Furthermore, parenthesized indices are symmetrized.

- *The MMC Model: Counterterms*

Our model (4) is not renormalizable, but the divergences of any theory can be removed by BPHZ (Bogoliubov, Parasiuk [24], Hepp [25] and Zimmermann [26, 27] counterterms. Those of our model (4) were the first ever studied using dimensional regularization [28]. At 1-loop order they consist of Eddington and Weyl terms:

$$\Delta\mathcal{L} = c_1 R^2\sqrt{-g} + c_2\,C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\sqrt{-g} \ , \tag{7}$$

where the coefficients are [19]:

$$c_1 = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\frac{(D-2)}{(D-1)^2(D-3)(D-4)} \quad , \tag{8}$$

$$c_2 = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\frac{2}{(D+1)(D-1)(D-3)^2(D-4)} \quad . \tag{9}$$

We can decompose R^2 into three pieces as follows:

$$R^2 = \Bigl(R - D\Lambda\Bigr)^2 + 2D\Lambda\Bigl[R - (D-2)\Lambda\Bigr] + D(D-4)\Lambda^2 \quad , \tag{10}$$

so that the Eddington counterterm becomes the sum of three contributions:

$$\Delta\mathcal{L}_{1a} \equiv c_1\Bigl(R - D\Lambda\Bigr)^2\sqrt{-g} \quad , \tag{11}$$

$$\Delta\mathcal{L}_{1b} \equiv 2Dc_1\Lambda\Bigl[R - (D-2)\Lambda\Bigr]\sqrt{-g} \quad , \tag{12}$$

$$\Delta\mathcal{L}_{1c} \equiv D(D-4)c_1\Lambda^2\sqrt{-g} \quad . \tag{13}$$

- There is also a finite renormalization of the cosmological constant which is necessary to make the graviton self-energy conserved [20]:

$$\Delta\mathcal{L}_3 = c_3\sqrt{-g} = c_3\, a^D\sqrt{-\widetilde{g}} \quad . \tag{14}$$

• *The MMC Model: Conservation*

Stress-energy conservation has been discussed in detail in [20]. The starting point of the analysis is the Ward identity which follows from stress-energy conservation for a matter loop contribution to the graviton self-energy:

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times (-i)\left[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm total}\right](x;x') = 0 \quad , \tag{15}$$

with the Ward operator defined thusly:

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \equiv \delta^{\mu}{}_{(\alpha}\partial_{\beta)} + aH\delta^{\mu}{}_{0}\,\eta_{\alpha\beta} \quad . \tag{16}$$

When the scalar obeys its equation of motion the Ward operator annihilates the graviton variation:

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times \frac{i\delta S[\phi,0]}{\delta h_{\alpha\beta}(x)} = \frac{\kappa}{2}\,\partial^{\mu}\phi(x) \times \frac{i\delta S[\phi,0]}{\delta\phi(x)} \quad . \tag{17}$$

- It can be shown [20] that in de Sitter spacetime and before renormalization we have:

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times (-i)\Big[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{\rm prim}\Big](x;x') \; = \; \frac{i\xi\kappa^2}{2}\Bigg\{-\eta^{\mu(\rho}\partial^{\sigma)}\Big[a^D\delta^D(x-x')\Big] + \frac{1}{2}a^{D-2}\eta^{\rho\sigma}\partial^{\mu}\Big[a^2\delta^D(x-x')\Big]\Bigg\} \; , \tag{18}$$

where $\xi \equiv \frac{(D-2)D-1)}{2D}H^2k$. This equation exhibits the obstacle to achieving conservation albeit when only the primitive form of the 1-loop graviton self-energy is taken into account.

- Upon renormalizing the 1-loop graviton self-energy the contributions from the counterterms (7) are considered. We employ the decomposition of the Eddington counterterm into three pieces (11-13) and have sequentially computed the action of the Ward operator on the various pieces. It turns out [20] that except for the counterterm (13) the Ward operator annihilates their contribution to the graviton self-energy:

$$\begin{aligned}\mathcal{W}^{\mu}{}_{\alpha\beta} \times (-i)\left[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{2}\right](x;x') \; &= \; \mathcal{W}^{\mu}{}_{\alpha\beta} \times (-i)\left[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{1a}\right](x;x') \\ = \; \mathcal{W}^{\mu}{}_{\alpha\beta} \times (-i)\left[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{1b}\right](x;x') \; &= \; 0 \; ,\end{aligned} \tag{19}$$

The only non-zero contributions come from:

(i) the cosmological constant counterterm (14) with coefficient c_3, and

(ii) the cosmological constant like counterterm (13) emanating from the Eddington decomposition (10) with coefficient $\gamma \equiv D(D-1)^2(D-4)H^4c_1$.

The results are [20]:

$$\mathcal{W}^{\mu}{}_{\alpha\beta} \times -i\left[{}^{\alpha\beta}\Sigma^{\rho\sigma}_{3+1c}\right](x;x') \; = \; \frac{i(c_3+\gamma)\kappa^2}{2}\Bigg\{-\eta^{\mu(\rho}\partial^{\sigma)}\Big[a^D\delta^D(x-x')\Big] + \frac{1}{2}a^{D-2}\eta^{\rho\sigma}\partial^{\mu}\Big[a^2\delta^D(x-x')\Big]\Bigg\} \; , \tag{20}$$

and because the functional form of the objection (20) is identical to that of the primitive objection (18), it is possible to arrange their coefficients so that they cancel against each other:

$$\xi = -(c_3+\gamma) \; , \tag{21}$$

and achieve the desired conservation. Substituting into (21) the values of k, ξ and γ we see that this corresponds to a finite renormalization of the cosmological constant in $D\!=\!4$:

$$c_3 = -\frac{\mu^{D-4} H^4}{2^7 \pi^{\frac{D}{2}}} \frac{(D\!-\!2)\Gamma(\frac{D}{2}\!+\!1)}{D\!-\!3} - \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{(D\!-\!2)\Gamma(D)}{4\Gamma(\frac{D}{2}\!+\!1)} \longrightarrow -\frac{H^4}{8\pi^2} \; . \qquad (22)$$

3 The 1-loop QFT results: Self-Energy

As discussed previously, the first step in our project involving (4) is its perturbative contribution to the 1-loop graviton self-energy given by:

$$\begin{aligned} -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') \;\; = \;\; & \Big\langle \Omega \Big| T^* \Bigg[\frac{i\delta S[\varphi, g]}{\delta h_{\mu\nu}(x)}_{\varphi\varphi} \times \frac{i\delta S[\varphi, g]}{\delta h_{\rho\sigma}(x')}_{\varphi\varphi} \\ & + \frac{i\delta^2 S[\varphi, g]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')}_{\varphi\varphi} + \frac{i\delta^2 \Delta S[g]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')}_{1} \Bigg] \Big| \Omega \Big\rangle \; , \end{aligned} \qquad (23)$$

where the subscripts indicate how many weak fields contribute, the T^* symbol stands for time-ordering with any derivatives taken outside, and $\Delta S[g]$ denotes the 1-loop counterterm action. [2] The 1st and 2nd variations required for (23) are: [3]

$$\frac{i\delta S[\varphi, g]}{\delta h_{\mu\nu}(x)}_{\varphi\varphi} = \frac{i\kappa}{2} a^{D-2} \Big[\partial^\mu \varphi \, \partial^\nu \varphi - \frac{1}{2} \eta^{\mu\nu} \partial^\alpha \varphi \, \partial_\alpha \varphi \Big] \; , \qquad (24)$$

$$\begin{aligned} \frac{i\delta^2 S[\varphi, g]}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')}_{\varphi\varphi} = \; & \frac{\kappa^2}{2} a^{D-2} \Bigg[-\eta^{\mu(\rho} \partial^{\sigma)} \varphi \, \partial^\nu \varphi - \eta^{\nu(\rho} \partial^{\sigma)} \varphi \, \partial^\mu \varphi \\ & + \frac{1}{2} \eta^{\mu\nu} \partial^\rho \varphi \, \partial^\sigma \varphi + \frac{1}{2} \eta^{\rho\sigma} \partial^\mu \varphi \, \partial^\nu \varphi + \frac{1}{2} \Big(\eta^{\mu(\rho} \eta^{\sigma)\nu} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \Big) \partial^\alpha \varphi \, \partial_\alpha \varphi \Bigg] \\ & \times i\delta^D(x\!-\!x') \; . \end{aligned} \qquad (25)$$

- *The 4-point Contribution*

[2]The Feynman diagrams corresponding to the 3 terms in (23) can be seen in Fig. 1-3.

[3]The variation with respect to the graviton is related to the variation with respect to the metric as $\frac{\delta}{\delta h_{\mu\nu}(x)} = \kappa a^2 \frac{\delta}{\delta g_{\mu\nu}(x)}$ due to (6).

The 4-point contribution (Fig. 2) is the expectation value of (25):

$$-i\Big[{}^{\mu\nu}\Sigma_4^{\rho\sigma}\Big](x;x') \;=\; \frac{\kappa^2}{2}a^{D-2}\Bigg[-\eta^{\mu(\rho}\partial'^{\sigma)}\partial^{\nu}\,i\Delta(x;x') - \eta^{\nu(\rho}\partial'^{\sigma)}\partial^{\mu}\,i\Delta(x;x') + \frac{1}{2}\eta^{\mu\nu}\partial^{\rho}\partial'^{\sigma}\,i\Delta(x;x') + \frac{1}{2}\eta^{\rho\sigma}\partial^{\mu}\partial'^{\nu}\,i\Delta(x;x') + \frac{1}{2}\Big(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\Big)\partial^{\alpha}\partial'_{\alpha}\,i\Delta(x;x')\Bigg]i\delta^D(x-x') \;, \quad (26)$$

which simplifies to: [4]

$$-i\Big[{}^{\mu\nu}\Sigma_4^{\rho\sigma}\Big](x;x') = \frac{(D-1)(D-4)}{4D}\kappa^2 kH^2 a^D\Big(\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu(\rho}\eta^{\sigma)\nu}\Big)i\delta^D(x-x') \;, \quad (27)$$

and hence vanishes in $D=4$ spacetime dimensions.

[4]Due to the identity: $\partial_\alpha\partial'_\beta\,i\Delta(x;x')\Big|_{x'=x} = -\Big(\frac{D-1}{D}\Big)kH^2 g_{\alpha\beta} \quad , \quad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}$.

• *The 3-point Contribution*

The expectation value of (24) is the 3-point contribution to the 1-loop self-energy (Fig. 1):

$$-i\Big[{}^{\mu\nu}\Sigma_3^{\rho\sigma}\Big](x;x') \;=\; \Big(\frac{i\kappa}{2}\Big)^2 (aa')^{D-2} \Bigg\{ 2\partial^\mu \partial'^{(\rho}\, i\Delta(x;x')\; \partial'^{\sigma)} \partial^\nu\, i\Delta(x;x')$$
$$-\eta^{\mu\nu} \partial^\alpha \partial'^\rho\, i\Delta(x;x')\; \partial_\alpha \partial'^\sigma\, i\Delta(x;x') - \eta^{\rho\sigma} \partial^\mu \partial'^\beta\, i\Delta(x;x')\; \partial^\nu \partial'_\beta i\Delta(x;x')$$
$$+\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} \partial^\alpha \partial'^\beta\, i\Delta(x;x')\; \partial_\alpha \partial'_\beta\, i\Delta(x;x') \Bigg\}$$
$$\equiv -i\Sigma_{3i} - i\Sigma_{3ii} - i\Sigma_{3iii} - i\Sigma_{3iv} \;\; . \qquad (28)$$

Notice that the first term $-i\Sigma_{3i}$, besides being the most difficult, recovers all three remaining terms by suitable contractions:

$$-i\Big[{}^{\mu\nu}\Sigma_3^{\rho\sigma}\Big](x;x') = \Big[\delta^\mu{}_\alpha \delta^\nu{}_\beta - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}\Big] \Big[\delta^\rho{}_\gamma \delta^\sigma{}_\delta - \frac{1}{2}\eta^{\rho\sigma}\eta_{\gamma\delta}\Big]$$
$$\times\, 2\Big(\frac{i\kappa}{2}\Big)^2 (aa')^{D-2}\, \partial^\alpha \partial'^{(\gamma} i\Delta(x;x')\, \partial'^{\delta)} \partial^\beta i\Delta(x;x') \;\; . \qquad (29)$$

All terms are quartically divergent, whereas the product of two undifferentiated propagators is logarithmically divergent. In view of the two derivatives on each propagator, we must therefore retain three terms in the expansion of each propagator [4, 29]:

$$i\Delta(x;x') = \frac{1}{4\pi^{\frac{D}{2}}} \Bigg\{ \frac{2\Gamma(\frac{D}{2})}{D-2} \frac{1}{(aa'\Delta x^2)^{\frac{D}{2}-1}}$$
$$+ \frac{\Gamma(\frac{D}{2}+1)}{2(D-4)} \frac{H^2}{(aa'\Delta x^2)^{\frac{D}{2}-2}} + \frac{\Gamma(\frac{D}{2}+2)}{16(D-6)} \frac{H^4}{(aa'\Delta x^2)^{\frac{D}{2}-3}} + \dots \Bigg\}$$
$$+k \Bigg\{ -\pi \cot\Big(\frac{D\pi}{2}\Big) + \ln(aa') + \Big(\frac{D-1}{2D}\Big) H^2 aa' \Delta x^2 + \dots \Bigg\} \;\; . \qquad (30)$$

Coordinate differences are indicated throughout by a Δ:

$$\Delta x^\mu \equiv (x-x')^\mu \quad , \quad \Delta\eta \equiv \eta - \eta' \quad , \quad \Delta r \equiv \|\vec{x} - \vec{x}'\| \; . \qquad (31)$$

Taking two derivatives of the propagator gives:

$$\partial^\mu \partial'^\rho i\Delta(x;x') = \frac{\delta^\mu{}_0 \delta^\rho{}_0 i\delta^D(x-x')}{a^{D-2}} + \frac{\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}(aa')^{\frac{D}{2}-1}} \Biggl\{ \frac{\eta^{\mu\rho}}{\Delta x^D} - \frac{D\Delta x^\mu \Delta x^\rho}{\Delta x^{D+2}}$$
$$+ \frac{(D-2)[aH\delta^\mu{}_0 \Delta x^\rho - \Delta x^\mu a' H \delta^\rho{}_0]}{2\Delta x^D} + \frac{(D-2)aa'H^2 \delta^\mu{}_0 \delta^\rho{}_0}{4\Delta x^{D-2}} + \frac{D}{8} aa'H^2$$
$$\times \Biggl[\frac{\eta^{\mu\rho}}{\Delta x^{D-2}} - \frac{(D-2)\Delta x^\mu \Delta x^\rho}{\Delta x^D} + \frac{(D-4)[aH\delta^\mu{}_0 \Delta x^\rho - \Delta x^\mu a' H \delta^\rho{}_0]}{2\Delta x^{D-2}}$$
$$+ \frac{(D-4)aa'H^2 \delta^\mu{}_0 \delta^\rho{}_0}{4\Delta x^{D-4}} \Biggr] + \frac{D(D+2)}{128} a^2 a'^2 H^4 \Biggl[\frac{\eta^{\mu\rho}}{\Delta x^{D-4}} - \frac{(D-4)\Delta x^\mu \Delta x^\rho}{\Delta x^{D-2}}$$
$$+ \frac{(D-6)[aH\delta^\mu{}_0 \Delta x^\rho - \Delta x^\mu a' H \delta^\rho{}_0]}{2\Delta x^{D-4}} + \frac{(D-6)aa'H^2 \delta^\mu{}_0 \delta^\rho{}_0}{4\Delta x^{D-6}} \Biggr] + \dots \Biggr\}$$
$$+ k \Biggl\{ 0 - \Bigl(\frac{D-1}{D}\Bigr) aa'H^2 \Bigl[\eta^{\mu\rho} - aH\delta^\mu{}_0 \Delta x^\rho + a'H \Delta x^\mu \delta^\rho{}_0$$
$$- \frac{1}{2} aa'H^2 \Delta x^2 \delta^\mu{}_0 \delta^\rho{}_0 \Bigr] + \dots \Biggr\} \ . \tag{32}$$

Now the product of the two doubly-differentiated propagators in (29) consists of two local terms plus a product of two nonlocal terms:

$$2\Bigl(\frac{i\kappa}{2}\Bigr)^2 (aa')^{D-2} \, \partial^\mu \partial'^{(\rho} i\Delta(x;x') \, \partial'^{\sigma)} \partial^\nu i\Delta(x;x') =$$
$$+ \Bigl(\frac{D-1}{D}\Bigr) \kappa^2 k H^2 a^D \delta^{(\mu}{}_0 \eta^{\nu)(\rho} \delta^{\sigma)}{}_0 \, i\delta^D(x-x')$$
$$- \frac{\kappa^2 \Gamma^2(\frac{D}{2})}{8\pi^D} \Biggl\{ \frac{\eta^{\mu(\rho}}{\Delta x^D} - \frac{D\Delta x^\mu \Delta x^{(\rho}}{\Delta x^{D+2}} + \dots \Biggr\} \Biggl\{ \frac{\eta^{\sigma)\nu}}{\Delta x^D} - \frac{D\Delta x^{\sigma)} \Delta x^\nu}{\Delta x^{D+2}} + \dots \Biggr\} \tag{33}$$

The first 8 terms in each of the curly bracketed expressions can make a non-zero contribution and are shown in Table 8. Because of symmetrization there are a total of 36 independent products of these 8 terms.

We shall not present the very complicated procedure of reducing the aforementioned products to isolate the divergences and the non-local finite parts they contain. It should be sufficient to incorporate the reuslts in the analysis that follows.

• *The 3-point Contribution: Tensor Basis*

A covariant generalization of [30] provides an appropriate tensor basis for the primitive divergent contributions:

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm prim}\Big](x;x') = \mathcal{K} \times \sum_{i=1}^{21} T^i(x;x') \times \Big[{}^{\mu\nu} D^{\rho\sigma}_i\Big] \times i\delta^D(x-x') \;\; , \qquad (34)$$

where the divergent prefactor $\mathcal{K}$ is:

$$\mathcal{K} \equiv \frac{\kappa^2\Gamma^2(\frac{D}{2})}{8\pi^D} \times K = \frac{\kappa^2\mu^{D-4}\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}(D-3)(D-4)} \;\; . \qquad (35)$$

The tensor differential operators $[{}^{\mu\nu} D^{\rho\sigma}_i]$ are listed in the table below:

i	$[{}^{\mu\nu} D^{\rho\sigma}_i]$	i	$[{}^{\mu\nu} D^{\rho\sigma}_i]$	i	$[{}^{\mu\nu} D^{\rho\sigma}_i]$
1	$\eta^{\mu\nu}\eta^{\rho\sigma}$	8	$\partial^\mu\partial^\nu\eta^{\rho\sigma}$	15	$\delta^{(\mu}{}_0\partial^{\nu)}\delta^\rho{}_0\delta^\sigma{}_0$
2	$\eta^{\mu(\rho}\eta^{\sigma)\nu}$	9	$\delta^{(\mu}{}_0\eta^{\nu)(\rho}\delta^{\sigma)}{}_0$	16	$\delta^\mu{}_0\delta^\nu{}_0\partial^\rho\partial^\sigma$
3	$\eta^{\mu\nu}\delta^\rho{}_0\delta^\sigma{}_0$	10	$\delta^{(\mu}{}_0\eta^{\nu)(\rho}\partial^{\sigma)}$	17	$\partial^\mu\partial^\nu\delta^\rho{}_0\delta^\sigma{}_0$
4	$\delta^\mu{}_0\delta^\nu{}_0\eta^{\rho\sigma}$	11	$\partial^{(\mu}\eta^{\nu)(\rho}\delta^{\sigma)}{}_0$	18	$\delta^{(\mu}{}_0\partial^{\nu)}\delta^{(\rho}{}_0\partial^{\sigma)}$
5	$\eta^{\mu\nu}\delta^{(\rho}{}_0\partial^{\sigma)}$	12	$\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}$	19	$\delta^{(\mu}{}_0\partial^{\nu)}\partial^\rho\partial^\sigma$
6	$\delta^{(\mu}{}_0\partial^{\nu)}\eta^{\rho\sigma}$	13	$\delta^\mu{}_0\delta^\nu{}_0\delta^\rho{}_0\delta^\sigma{}_0$	20	$\partial^\mu\partial^\nu\delta^{(\rho}{}_0\partial^{\sigma)}$
7	$\eta^{\mu\nu}\partial^\rho\partial^\sigma$	14	$\delta^\mu{}_0\delta^\nu{}_0\delta^{(\rho}{}_0\partial^{\sigma)}$	21	$\partial^\mu\partial^\nu\partial^\rho\partial^\sigma$

Table 1: The 21 basis tensors used in expression (34). The pairs $(3,4)$, $(5,6)$, $(7,8)$, $(10,11)$, $(14,15)$, $(16,17)$ and $(19,20)$ are related by reflection.

The same tensor basis is appropriate for the counterterm contributions (Fig. 3):

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm count}\Big](x;x') = \mathcal{K} \times \sum_{i=1}^{21} \Delta T^i(x;x') \times \Big[{}^{\mu\nu} D^{\rho\sigma}_i\Big] \times i\delta^D(x-x') \;\; , \qquad (36)$$

• *The 3-point Contribution: Primitive Divergences*

The primitive divergences coming only from the first term $-i\Sigma_{3i}$ provide contributions of the same form as (34), but with different coefficients that we shall call $\mathcal{T}^i(x;x')$. Including the three trace terms according to (29) gives complicated relations for those T^i's whose $[{}^{\mu\nu}D^{\rho\sigma}]$'s contain either $\eta^{\mu\nu}$ or $\eta^{\rho\sigma}$. Hence, a subset of the T^i's will be equal to the $\mathcal{T}^i$'s:

$$T^2 = \mathcal{T}^2 \qquad , \qquad T^i = \mathcal{T}^i \quad \forall i \geq 9 \ , \tag{37}$$

while the remaining T^i's will be linear combinations of the $\mathcal{T}^i$'s. Of these, T^1 satisfies:

$$\begin{aligned} T^1 \;=\; & \frac{(D-2)^2}{4}\mathcal{T}^1 + \frac{(D-4)}{4}\mathcal{T}^2 - \frac{(D-2)}{4}(\mathcal{T}^3 + \mathcal{T}^4) + \frac{(D-2)}{4}(\mathcal{T}^5 + \mathcal{T}^6)\partial_0 \\ & + \frac{(D-2)}{4}(\mathcal{T}^7 + \mathcal{T}^8)\partial^2 - \frac{1}{4}\mathcal{T}^9 + \frac{1}{4}(\mathcal{T}^{10} + \mathcal{T}^{11})\partial_0 + \frac{1}{4}\mathcal{T}^{12}\partial^2 + \frac{1}{4}\mathcal{T}^{13} \\ & - \frac{1}{4}(\mathcal{T}^{14} + \mathcal{T}^{15})\partial_0 - \frac{1}{4}(\mathcal{T}^{16} + \mathcal{T}^{17})\partial^2 + \frac{1}{4}\mathcal{T}^{18}\partial_0^2 \\ & + \frac{1}{4}(\mathcal{T}^{19} + \mathcal{T}^{20})\partial^2\partial_0 + \frac{1}{4}\mathcal{T}^{21}\partial^4 \ , \end{aligned} \tag{38}$$

while the remaining ones are simpler and are presented in Table 9.

The $\mathcal{T}^i(x;x')$ have been laboriously computed and are presented in Table 10. For the cases $i = 2$ and $i \geq 9$ they are identical with $T^i(x;x')$. The most complicated case when $T^i \neq \mathcal{T}^i$ is (38): [5]

$$\begin{aligned} T^1(x;x') \;=\; & \frac{(D^2-2D-2)\partial^4}{32(D+1)(D-1)(D-2)} + \frac{(3D^3-18D^2+24D-16)aa'H^2\partial^2}{512(D-1)} \\ & - \frac{(D-2)(D-3)aa'H^2\partial_0^2}{64(D-1)} + \frac{(D-2)(D-4)(D^4-48D+64)a^2a'^2H^4}{4096(D-1)} \ . \end{aligned} \tag{39}$$

The remaining cases for which $T^i \neq \mathcal{T}^i$ are given in Table 11.

- *The 3-point Contribution: Weyl Counterterm*

A simple computation shows that the contribution of the Weyl counterterm to the graviton self-energy equals:

$$\frac{\delta^2\, i\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h_{\mu\nu}=0} = 2\kappa^2 c_2\, \mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}\Big[a^{D-4}\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma} i\delta^D(x-x')\Big] \ , \tag{40}$$

[5]Note that differences of scale factors combine to give $a - a' = aa'H\Delta\eta$. The resulting factors of $\Delta\eta$ acting on $\delta^D(x-x')$ can be reduced using: $\Delta\eta\partial_0 \longrightarrow -1$, $\Delta\eta^2\partial^2 \longrightarrow -2$, $\Delta\eta\partial_0\partial^2 \longrightarrow -\partial^2 + 2\partial_0^2$.

where the tensor differential operator $\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}$ is defined via the linearized, conformally rescaled Weyl tensor thusly:

$$\widetilde{C}_{\alpha\beta\gamma\delta} = \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu} \times \kappa h_{\mu\nu} + O(\kappa^2 h^2) \quad . \tag{41}$$

Its explicit form is:

$$\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu} = \mathcal{D}_{\alpha\beta\gamma\delta}{}^{\mu\nu} - \frac{1}{D-2}\Big[\eta_{\alpha\gamma}\mathcal{D}_{\beta\delta}{}^{\mu\nu} - \eta_{\gamma\beta}\mathcal{D}_{\delta\alpha}{}^{\mu\nu} + \eta_{\beta\delta}\mathcal{D}_{\alpha\gamma}{}^{\mu\nu} - \eta_{\delta\alpha}\mathcal{D}_{\gamma\beta}{}^{\mu\nu}\Big] + \frac{(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})\mathcal{D}^{\mu\nu}}{(D-1)(D-2)} \quad , \tag{42}$$

where the various derivatives are:

$$\mathcal{D}_{\alpha\beta\gamma\delta}{}^{\mu\nu} \equiv -\frac{1}{2}\Big[\delta^{(\mu}{}_{\alpha}\delta^{\nu)}{}_{\gamma}\,\partial_\beta\partial_\delta - \delta^{(\mu}{}_{\gamma}\delta^{\nu)}{}_{\beta}\,\partial_\delta\partial_\alpha + \delta^{(\mu}{}_{\beta}\delta^{\nu)}{}_{\delta}\,\partial_\alpha\partial_\gamma - \delta^{(\mu}{}_{\delta}\delta^{\nu)}{}_{\alpha}\,\partial_\gamma\partial_\beta\Big] \quad , \tag{43}$$

$$\mathcal{D}_{\beta\delta}{}^{\mu\nu} \equiv \eta^{\alpha\gamma}\mathcal{D}_{\alpha\beta\gamma\delta}{}^{\mu\nu} = -\frac{1}{2}\Big[\eta^{\mu\nu}\partial_\beta\partial_\delta - 2\partial^{(\mu}\delta^{\nu)}{}_{(\beta}\partial_{\delta)} + \delta^{(\mu}{}_{\beta}\delta^{\nu)}{}_{\delta)}\partial^2\Big] \quad , \tag{44}$$

$$\mathcal{D}^{\mu\nu} \equiv \eta^{\beta\delta}\mathcal{D}_{\beta\delta}{}^{\mu\nu} = \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2 = \Pi^{\mu\nu} \quad . \tag{45}$$

Consequently, the explicit form of (40) becomes: [6]

$$\frac{\delta^2\, i\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h_{\mu\nu}=0} = 2\kappa^2 c_2\Big(\frac{D-3}{D-2}\Big)\Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Big] i\delta^D(x-x') + \frac{\kappa^2}{2^6\cdot 3\cdot 5\cdot \pi^2}\,\mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}\Big[\ln(a)\,\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma}\, i\delta^4(x-x')\Big] + O(D-4) \quad , \tag{46}$$

with the prefactor of the divergence equaling:

$$2\kappa^2\, c_2\Big(\frac{D-3}{D-2}\Big) = \mathcal{K} \times \frac{1}{32(D+1)(D-1)(D-2)} \quad . \tag{47}$$

The divergent contribution to each $T^i(x;x')$ is given in Table 12.

- *The 3-point Contribution: Eddington Counterterm*

In (10), the Eddington counterterm R^2 was decomposed into three pieces. We shall analyze them in reverse order.

[6] Taking also into account the usual expansion of the measure factor, $a^{D-4} = 1 + (D-4)\ln(a) + O[(D-4)^2]$.

- The last of these is the counterterm (13) which is the same as a cosmological constant and finite:

$$\frac{\delta^2 \, i\Delta S_{1c}}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h_{\mu\nu}=0} = \frac{\kappa^2 H^4}{64\pi^2}\Big[-\eta^{\mu(\rho}\eta^{\sigma)\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\Big] a^4 \, i\delta^4(x-x') \quad . \tag{48}$$

- The second of these is the Einstein counterterm (12) whose contribution to the graviton self-energy is:

$$\frac{\delta^2 \, i\Delta S_{1b}}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h_{\mu\nu}=0} = D(D-1)c_1\kappa^2 H^2 \Big\{ \Big[\eta^{\mu(\rho}\eta^{\sigma)\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}\Big]\mathcal{D}\, i\delta^D(x-x')$$
$$+\Big[2\partial'^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)} + \eta^{\mu\nu}\partial^\rho\partial^\sigma + \partial'^\mu\partial'^\nu\eta^{\rho\sigma}\Big]\Big[a^{D-2}\, i\delta^D(x-x')\Big]\Big\} \quad , \tag{49}$$

where $\mathcal{D} \equiv \partial^\mu a^{D-2}\partial_\mu$. By employing the delta function to express the scale factor a^{D-2} as half primed and half unprimed so that:

$$a^{D-2} \quad \longrightarrow \quad (aa')^{\frac{D}{2}-1} = aa'\Big(1 - (\tfrac{D}{2}-2)\ln(aa') + O[(D-4)^2]\Big) \quad , \tag{50}$$

by moving all scale factors to the left, for example:

$$\mathcal{D}\, i\delta^D(x-x') = (aa')^{\frac{D}{2}-1}\Big[\partial^2 + \tfrac{D}{2}(\tfrac{D}{2}-1)aa'H^2\Big] i\delta^D(x-x') \quad , \tag{51}$$

by extracting a factor of $\mathcal{K}$ from the multiplicative prefactor:

$$D(D-1)c_1\kappa^2 H^2 = \mathcal{K} \times \frac{D(D-2)H^2}{128(D-1)} \quad , \tag{52}$$

we derive the divergences shown in Table 13.

- Finally, we consider the divergences associated with $\Delta\mathcal{L}_{1a}$ of (11). We follow the same procedure as for the Weyl countertem and express $R - D\Lambda$ as a tensor differential operator acting on a single graviton:

$$R - D\Lambda = \frac{1}{a^2} \times \overline{\mathcal{F}}^{\mu\nu} \times \kappa h_{\mu\nu} + O(\kappa^2 h^2) \quad , \tag{53}$$

where the tensor differential operator $\overline{\mathcal{F}}^{\mu\nu}$ is:

$$\overline{\mathcal{F}}^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\Big[\partial^2 - (D-1)aH\partial_0\Big] - 2(D-1)aH\delta^{(\mu}{}_0\partial^{\nu)}$$
$$+D(D-1)a^2H^2\delta^\mu{}_0\delta^\nu{}_0 \quad . \tag{54}$$

The operator we actually need is obtained by partially integrating $\overline{\mathcal{F}}^{\mu\nu}$ to obtain:

$$\mathcal{F}^{\mu\nu} \equiv \partial^\mu \partial^\nu - \eta^{\mu\nu}\Big[\partial^2 + (D\!-\!1) a H \partial_0 + (D\!-\!1) a^2 H^2\Big] + 2(D\!-\!1) a H \delta^{(\mu}{}_0 \partial^{\nu)} \\ +(D\!-\!2)(D\!-\!1) a^2 H^2 \delta^\mu{}_0 \delta^\nu{}_0 \;\; , \tag{55}$$

The resulting second variation of S_{1a} is:

$$\frac{\delta^2 \, i\Delta S_{1a}}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')}\Big\vert_{h_{\mu\nu}=0} = 2\kappa^2 c_1 \, \mathcal{F}^{\mu\nu}\Big[a^{D-4} \mathcal{F}^{\rho\sigma} i\delta^D(x\!-\!x')\Big] \tag{56}$$

$$= 2\kappa^2 c_1 \mathcal{F}^{\mu\nu} \mathcal{F}^{\rho\sigma} i\delta^D(x\!-\!x') + \frac{\kappa^2}{2^6 \!\cdot\! 3^2 \!\cdot\! \pi^2} \, \mathcal{F}^{\mu\nu} \Big[\ln(a) \, \mathcal{F}^{\rho\sigma} i\delta^4(x\!-\!x')\Big] + \dots \tag{57}$$

The multiplicative prefactor is:

$$2\kappa^2 c_1 = \mathcal{K} \times \frac{(D\!-\!2)}{64(D\!-\!1)^2} \;\; . \tag{58}$$

Table 14 gives the divergences contributed by S_{1a}.

- *The 3-point Contribution: Primitive + Counterterm Results*

The primitive divergences of the 1-loop graviton self-energy that have just been evaluated, can be summarized in Table 15 which lists the basis coefficients $T^i(x;x')$.

Similarly, the counterterm divergences of the 1-loop graviton self-energy that have just been evaluated, can be summarized in Table 16 which lists the basis coefficients $\Delta T^i(x;x')$.

By adding Tables 15 and 16 we arrive at the residuals shown in Table 17. A few remarks are in order:

- The residuals actually vanish for tensor factors $[^{\mu\nu} D^{\rho\sigma}_i]$ with $13 \leq i \leq 21$.

- For $i = 1$, $i = 2$ and $i = 12$ the residual vanishes in $D = 4$.

- For $3 \leq i \leq 11$ there is a more complicated cancellation scheme based on the observation:

$$\Delta\eta \partial^\mu = \partial^\mu \Delta\eta + \delta^\mu_{\ 0} \;\; . \tag{59}$$

- Sometimes (for instance, $i = 7$ and $i = 8$) this must be done twice before the $\Delta\eta$ acts on the $i\delta^D(x - x')$ and vanishes. The clusters of tensor factors which cancel in this way are:

1. The case of $i = 10$ and $i = 11$, which combine to cancel $i = 9$.

2. The case of $i = 8$, which contributes to $i = 6$ to produce a $\Delta\eta$ term that cancels $i = 4$.

3. The case of $i = 7$, which contributes to $i = 5$ to produce a $\Delta\eta$ term that cancels $i = 3$.

The final result is displayed in Table 18 with each tensor factor being proportional to at least one factor of $(D - 4)$. The final step, reported in Table 19, multiplies by the factor of $\frac{1}{D-4}$ in $\mathcal{K}$:

$$\mathcal{K} = \frac{\kappa^2}{2\pi^2} \times \frac{1}{D-4} + O[(D-4)] \ , \tag{60}$$

and then takes the limit of $D = 4$.

• *The 3-point Contribution: Finite Local Contributions*

There are six sources of finite, local contributions to the graviton 1-loop self-energy:

- The local terms arising from the action of two derivatives on the scalar propagator which is given in equation (32). Upon substituting the local contribution from (32) into equation (29) and setting $D = 4$ gives:

$$\frac{3\kappa^2 H^4}{32\pi^2}\Big\{\delta^{(\mu}{}_0\eta^{\nu)(\rho}\delta^{\sigma)}{}_0 - \frac{1}{2}\delta^{\mu}{}_0\delta^{\nu}{}_0\eta^{\rho\sigma} - \frac{1}{2}\eta^{\mu\nu}\delta^{\rho}{}_0\delta^{\sigma}{}_0 - \eta^{\mu\nu}\eta^{\rho\sigma}\Big\}a^4 i\delta^4(x-x') \ . \tag{61}$$

- The local terms arising from adding the primitive divergences to the counterterms; they can be found in Table 19:

$$\begin{aligned}\frac{aa'\kappa^2 H^2}{96\pi^2}\Bigg\{ & -\Big(\frac{5}{4}\partial^2 + 2aa'H^2\Big)\eta^{\mu\nu}\eta^{\rho\sigma} + \frac{1}{2}\partial^2\eta^{\mu(\rho}\eta^{\sigma)\nu} - a'H\eta^{\mu\nu}\delta^{(\rho}{}_0\partial^{\sigma)} \\ & + a\delta^{(\mu}{}_0\partial^{\nu)}\eta^{\rho\sigma} + \eta^{\mu\nu}\partial^\rho\partial^\sigma + \partial^\mu\partial^\nu\eta^{\rho\sigma} - 3aa'H^2\delta^{(\mu}{}_0\eta^{\nu)(\rho}\delta^{\sigma)}{}_0 \\ & - a'H\delta^{(\mu}{}_0\eta^{\nu)(\rho}\partial^{\sigma)} + aH\partial^{(\mu}\eta^{\nu)(\rho}\delta^{\sigma)}{}_0 - \partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}\Bigg\} i\delta^4(x-x') \ .\end{aligned} \tag{62}$$

- The local logarithm terms we found in equation (46) from the Weyl counterterm:

$$\frac{\kappa^2}{2^6\!\cdot\!3\!\cdot\!5\!\cdot\!\pi^2}\,\mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}\Big[\ln(a)\,\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma}\, i\delta^4(x-x')\Big] \ . \tag{63}$$

- The local terms we found in equation (48) from the finite renormalization of the cosmological constant:

$$\frac{\kappa^2 H^4 a^4}{64\pi^2}\Big[-\eta^{\mu(\rho}\eta^{\sigma)\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\Big] i\delta^4(x-x') \ . \tag{64}$$

- The local logarithm terms we found in equation (49) from the Einstein counterterm:

$$\frac{\kappa^2 H^2}{192\pi^2}\, aa' \ln(aa') \Bigg\{ \Big[\eta^{\mu(\rho}\eta^{\sigma)\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}\Big] \Big[\partial^2 + 2aa'H^2\Big]$$
$$+2\Big[-\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)} - a'H\delta^{(\mu}{}_0\eta^{\nu)(\rho}\partial^{\sigma)} + aH\partial^{(\mu}\eta^{\nu)(\rho}\delta^{\sigma)}{}_0 + aa'H^2\delta^{(\mu}{}_0\eta^{\nu)(\rho}\delta^{\sigma)}{}_0\Big]$$
$$+\eta^{\mu\nu}\Big[\partial^\rho\partial^\sigma - 2aH\delta^{(\rho}{}_0\partial^{\sigma)} + 2a^2H^2\delta^\rho{}_0\delta^\sigma{}_0\Big]$$
$$+\eta^{\rho\sigma}\Big[\partial^\mu\partial^\nu - 2aH\delta^{(\mu}{}_0\partial^{\nu)} + 2a^2H^2\delta^\mu{}_0\delta^\nu{}_0\Big]\Bigg\}\, i\delta^4(x-x') \ . \qquad (65)$$

- The local logarithm terms we found in equation (57) from the Eddington counterterm:

$$\frac{\kappa^2}{2^6\cdot 3^2\cdot \pi^2}\, \mathcal{F}^{\mu\nu}\Big[\ln(a)\, \mathcal{F}^{\rho\sigma} i\delta^4(x-x')\Big] \ . \qquad (66)$$

- *The 3-point Contribution: Finite Non-local Contributions*

  The finite, non-local contributions are of the form:

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm nonloc}\Big](x;x') \;=\; -\frac{\kappa^2}{8\pi^4} \times \sum_{i=1}^{21}\Bigg\{ T^i_A(a,a',\partial) \times \Big[{}^{\mu\nu}D^{\rho\sigma}_i\Big] \times \ln(\mu^2\Delta x^2)$$
$$+T^i_B(a,a',\partial) \times \Big[{}^{\mu\nu}D^{\rho\sigma}_i\Big] \times \partial^2\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]\Bigg\} \ , \qquad (67)$$

where the tensor differential operators in (67) have been defined before in Table 1. As before, the finite non-local contributions coming from the first term $-i\Sigma_{3i}$ in (28) have coefficients $\mathcal{T}^i$. Including the three trace terms $-i\Sigma_{3ii,\,3iii,\,3iv}$ gives the full coefficients T^i. The relations between these two sets of coefficients were displayed in (37) and in Table 9; here, since we only condider finite contributions we can set $D=4$ in the latter relations.

Our final results for $\mathcal{T}^i_A$ are given in Table 20, with the trace terms included to give T^i_A in Table 21. Similarly, Table 22 gives $\mathcal{T}^i_B$, and Table 23 presents T^i_B.

- *The 3-point Contribution: Summary*

  We conclude this Section by summarizing the various contributions to the 1-loop renormalized graviton self-energy. We organize these according to the tensor operators they contain.

- First we gather the contributions coming from expressing the linearized Weyl tensor and Ricci scalar as 2nd order tensor differential operators contracted into the graviton field - (41) and (53). The resulting contributions are (63) and (66):

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') \;=\; \frac{\kappa^2}{2^6\!\cdot\!3\!\cdot\!5\!\cdot\!\pi^2}\Big[\ln(a)\,\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma}\,i\delta^4(x\!-\!x')\Big] + \frac{\kappa^2}{2^6\!\cdot\!3^2\!\cdot\!\pi^2}\,\mathcal{F}^{\mu\nu}\Big[\ln(a)\,\mathcal{F}^{\rho\sigma} i\delta^4(x\!-\!x')\Big] \;. \qquad (68)$$

- Next we gather the contributions to the 1-loop graviton self-energy involving the 21 tensor differential operators of Table 1. There are local contributions that can be expressed as (34):

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm loc}\Big](x;x') = \frac{\kappa^2 H^2}{192\pi^2}\sum_{i=1}^{21} T^i(a,a',\partial)\times\Big[{}^{\mu\nu}D^{\rho\sigma}_i\Big]\,i\delta^4(x\!-\!x') \;, \qquad (69)$$

and are shown in Table 2; as well as non-local contributions that can be expressed as (67):

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm nonloc}\Big](x;x') \;=\; -\frac{\kappa^2}{8\pi^4}\times\sum_{i=1}^{21}\Big\{T^i_A(a,a',\partial)\times\Big[{}^{\mu\nu}D^{\rho\sigma}_i\Big]\times\ln(\mu^2\Delta x^2) + T^i_B(a,a',\partial)\times\Big[{}^{\mu\nu}D^{\rho\sigma}_i\Big]\times\partial^2\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]\Big\} \;, \qquad (70)$$

with the explicit results for the coefficients $T^i_A(a,a',\partial)$, $T^i_B(a,a',\partial)$ shown in Tables 21,23 respectively.

We conclude with a description of the origin of the various entries in Table 2. In the 1st column under the name "Residuals" we have included the contributions of (62) from the renormalization residuals of Table 19, in the 2nd column under the name "$\ln(aa')\times$ Einstein" the logarithmic contributions of (65) coming from the Einstein term, in the 3rd column under the name "$\Delta\Lambda$" the contributions of (64) from the cosmological constant like term and of (14) with c_3 given by (22) from the cosmological counterterm, and in the 4th column under the name "Marginal" the contributions of (61) emanating from the action of the two derivatives on the scalar propagator.

i	Residuals	$\ln(aa')\times$ Einstein	$\Delta\Lambda$	Marginal
1	$-\frac{5}{2}aa'\partial^2 - 4a^2a'^2H^2$	$-aa'\partial^2 - 2a^2a'^2H^2$	$-\frac{9}{2}a^2a'^2H^2$	$-3a^2a'^2H^2$
2	$aa'\partial^2$	$aa'\partial^2 + 2a^2a'^2H^2$	$9a^2a'^2H^2$	$-6a^2a'^2H^2$
3	0	$2a^3a'H^2$	0	0
4	0	$2aa'^3H^2$	0	0
5	$-2aa'^2H$	$-2a^2a'H$	0	0
6	$2a^2a'H$	$2aa'^2H$	0	0
7	$2aa'$	aa'	0	0
8	$2aa'$	aa'	0	0
9	$-6a^2a'^2H^2$	$2a^2a'^2H^2$	0	0
10	$-2aa'^2H$	$-2aa'^2H$	0	0
11	$2a^2a'^2H$	$2a^2a'H$	0	0
12	$-2aa'$	$-2aa'$	0	0

Table 2: Local contributions to each $T^i(a, a', \partial)$ from the various sources.

4 Solving the Effective Field Equations

The linearized effective field equation for the graviton field $h_{\mu\nu}(x)$ (6) is:

$$\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) = 8\pi G T^{\mu\nu}(x) + \int d^4x' \, [{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x') \, \kappa h_{\rho\sigma}(x') \ , \qquad (71)$$

where $[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ is the graviton self-energy in the $''in-in''$ formalism [31–39], $T^{\mu\nu}(x)$ is minus the variation of the matter action with respect to $h_{\mu\nu}(x)$ and $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator on de Sitter background:

$$\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma} = \frac{1}{2}a^2\Big[\partial^2 h^{\mu\nu} - \eta^{\mu\nu}\partial^2 h + \eta^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} + \partial^\mu\partial^\nu h - 2\partial^\rho\partial^{(\mu}h^{\nu)}{}_\rho\Big] \\ + Ha^3\Big[\eta^{\mu\nu}\partial_0 h - \partial_0 h^{\mu\nu} - 2\eta^{\mu\nu}\partial^\rho h_{\rho 0} + 2\partial^{(\mu}h^{\nu)}{}_0\Big] + 3H^2a^4\eta^{\mu\nu}h_{00} \ . \qquad (72)$$

The purpose of this Section is to use the 1-loop scalar contribution to the graviton self-energy to solve (71) for 1-loop corrections to plane wave gravitational radiation and for the gravitational response to a static point mass. The Section begins by giving the $''in-in''$ form of the 1-loop scalar contribution to $[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$. Then, we explain generally how equation (71) can be solved for 1-loop corrections to $h_{\mu\nu}(x)$. Setting $T^{\mu\nu}(x)=0$ gives dynamical gravitons, and setting $T^{\mu\nu}(x)=-\delta^\mu{}_0\delta^\nu{}_0 Ma\,\delta^3(\vec{x})$ gives the Newtonian potential.

4.1 The Graviton Self-Energy

After making the simple conversion from the $''in$–out'' formalism of Section 3 to the $''in$–in'' formalism [39], the 1-loop scalar contribution to the graviton self-energy $[^{\mu\nu}\Sigma^{\rho\sigma}_1](x;x')$ can be written:

$$
\begin{aligned}
[^{\mu\nu}\Sigma^{\rho\sigma}_1](x;x') = & -\frac{\kappa^2\mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}}{960\pi^2}\Big[\ln(a)\mathcal{C}'_{\alpha\beta\gamma\delta}{}^{\rho\sigma}\delta^4(\Delta x)\Big] \\
& -\frac{\kappa^2\mathcal{F}^{\mu\nu}}{576\pi^2}\Big[\ln(a)\mathcal{F}'^{\rho\sigma}\delta^4(\Delta x)\Big]-\frac{\kappa^2H^2}{192\pi^2}\sum_{i=1}^{21}\widehat{T}^i(a,a',\partial)[^{\mu\nu}D^{\rho\sigma}_i]\delta^4(\Delta x) \\
& +\frac{\kappa^2H^2}{384\pi^3}\sum_{i=1}^{21}\Bigg\{\widehat{T}^i_A(a,a',\partial)[^{\mu\nu}D^{\rho\sigma}_i]\times\theta(\Delta\eta-\Delta r) \\
& \qquad +\widehat{T}^i_B(a,a',\partial)\,[^{\mu\nu}D^{\rho\sigma}_i]\,\partial^4\Big[\theta(\Delta\eta-\Delta r)\Big(\ln[\mu^2(\Delta\eta^2-\Delta r^2)]-1\Big)\Big]\Bigg\}\ . \qquad (73)
\end{aligned}
$$

The notation in expression (73) requires explanation:
- The tensor differential operators $[^{\mu\nu}D^{\rho\sigma}_i]$ are given in Table 1.
- The coefficient functions $\widehat{T}^i(a,a',\partial)$ and $\widehat{T}^i_A(a,a',\partial)$ are listed in Table 3, while the $\widehat{T}^i_B(a,a',\partial)$ are listed in Table 4.
- The tensor differential operators $\mathcal{F}^{\mu\nu}$ and $\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}$ were defined in (41) and (53-55) respectively by expanding the Weyl tensor and Ricci scalar in powers of the graviton field $h_{\mu\nu}$.

i	$\widehat{T}^i(a,a',\partial)$	$\widehat{T}^i_A(a,a',\partial)$
1	$-aa'\ln(aa')[\partial^2+2aa'H^2]$ $-3aa'\partial^2-aa'\partial_0^2-6a^2a'^2H^2$	$-\frac{1}{4}a^2a'^2H^2\partial^2\partial_0^2$
2	$aa'\ln(aa')[\partial^2+2aa'H^2]$ $+aa'\partial^2+3a^2a'^2H^2$	0
3	$2a^3a'H^2[\ln(aa')+\frac{9}{2}]$	0
4	$2aa'^3H^2[\ln(aa')+\frac{9}{2}]$	0
5	$-2a^2a'H[\ln(aa')+4]$	$\frac{1}{2}a^2a'^2H^2\partial_0\partial^2$
6	$2aa'^2H[\ln(aa')+4]$	$\frac{1}{2}a^2a'^2H^2\partial_0\partial^2$
7	$aa'[\ln(aa')+3]$	$-\frac{1}{2}a^2a'H\partial_0\partial^2+\frac{1}{2}a^2a'^2H^2\partial^2$
8	$aa'[\ln(aa')+3]$	$\frac{1}{2}aa'^2H\partial_0\partial^2+\frac{1}{2}a^2a'^2H^2\partial^2$
9	$2a^2a'^2H^2[\ln(aa')+1]$	0
10	$-2aa'^2H[\ln(aa')+2]$	0
11	$2a^2a'H[\ln(aa')+2]$	0
12	$-2aa'[\ln(aa')+1]$	$-a^2a'^2H^2\partial^2$
18	0	$-3a^2a'^2H^2\partial^2$
19	0	$a^2a'H\partial^2$
20	0	$-aa'^2H\partial^2$
21	0	$\frac{1}{2}aa'\partial^2-a^2a'^2H^2$

Table 3: Coefficients $\widehat{T}^i(a,a',\partial)$ and $\widehat{T}^i_A(a,a',\partial)$ which appear in expression (73).

i	$\widehat{T}^i_B(a,a',\partial)$	i	$\widehat{T}^i_B(a,a',\partial)$
1	$-\frac{3\partial^4}{80H^2}+\frac{aa'\partial^2}{8}+\frac{aa'\partial_0^2}{8}$	12	$\frac{\partial^2}{40H^2}+\frac{aa'}{2}$
2	$-\frac{\partial^4}{80H^2}-\frac{aa'\partial^2}{4}-\frac{a^2a'^2H^2}{2}$	13	$-\frac{3a^2a'^2H^2}{2}$
3	$\frac{a'^2\partial^2}{4}+\frac{3aa'^2H\partial_0}{4}+\frac{a^2a'^2H^2}{2}$	14	$\frac{3a^2a'H}{2}$
4	$\frac{a^2\partial^2}{4}-\frac{3a^2a'H\partial_0}{4}+\frac{a^2a'^2H^2}{2}$	15	$-\frac{3aa'^2H}{2}$
5	$-\frac{a'\partial^2}{4H}-\frac{3aa'\partial_0}{4}+\frac{aa'^2H}{4}$	16	$-\frac{a^2}{4}$
6	$\frac{a\partial^2}{4H}-\frac{3aa'\partial_0}{4}-\frac{a^2a'H}{4}$	17	$-\frac{a'^2}{4}$
7	$\frac{3\partial^2}{80H^2}+\frac{a\partial_0}{8H}-\frac{a^2}{8}$	18	$\frac{3aa'}{2}$
8	$\frac{3\partial^2}{80H^2}-\frac{a'\partial_0}{8H}-\frac{a'^2}{8}$	19	$-\frac{a}{4H}$
9	$-\frac{3a^2a'^2H^2}{2}$	20	$\frac{a'}{4H}$
10	$\frac{a^2a'H}{2}$	21	$-\frac{1}{20H^2}$
11	$-\frac{aa'^2H}{2}$		

Table 4: Coefficient functions $\widehat{T}^i_B(a,a',\partial)$ which appear in expression (73).

The terms in expression (73) involving summation have the generic form of coefficient functions of $(a,a',\partial))$ multiplying tensor differential operators $[^{\mu\nu}D_i^{\rho\sigma}]$, all acting on three different functions of $(x-x')^\mu$:

$$\delta^4(x-x') \qquad , \qquad \theta(\Delta\eta-\Delta r)$$
$$f_B(x;x') \equiv \partial^4\Big[\theta(\Delta\eta-\Delta r)\Big(\ln[\mu^2(\Delta\eta^2-\Delta r^2)]-1\Big)\Big] \quad . \tag{74}$$

Important relations convert the three functions into one another:

$$\partial^4\,\theta(\Delta\eta-\Delta r) = 8\pi\delta^4(x-x') \quad , \tag{75}$$
$$\Delta\eta\, f_B(x;x') = -2\partial_0\partial^2\,\theta(\Delta\eta-\Delta r) \quad . \tag{76}$$

The middle function also obeys the relation:

$$(\Delta\eta\partial^2+2\partial_0)\,\theta(\Delta\eta-\Delta r) = 0 \quad . \tag{77}$$

4.2 Perturbative Solution

We possess only the single scalar loop contribution to the graviton self-energy so it is only possible to solve equation (71) perturbatively:

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \kappa^2 h^{(1)}_{\mu\nu} + \kappa^4 h^{(2)}_{\mu\nu} + \dots \tag{78}$$

We consider the stress tensor to be 0th order so $h^{(0)}_{\mu\nu}$ obeys the equation:

$$\mathcal{D}^{\mu\nu\rho\sigma} \kappa h^{(0)}_{\rho\sigma}(x) = 8\pi G\, T^{\mu\nu}(x) \ . \tag{79}$$

The 1-loop correction we seek obeys:

$$\mathcal{D}^{\mu\nu\rho\sigma}\, \kappa^3 h^{(1)}_{\rho\sigma}(x) = \int d^4x' \, \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_1\right](x;x')\, \kappa h^{(0)}_{\rho\sigma}(x') \ . \tag{80}$$

The $D = 4$ contributions from the first two terms of $[{}^{\mu\nu}\Sigma^{\rho\sigma}_1](x;x')$ in expression (73) can be written in terms of linearized curvatures. Relation (41) implies that the first term is:

$$-\frac{\kappa^2 \mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}}{960\pi^2}\left[\ln(a)\int d^4x'\, \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\rho\sigma}\, \delta^4(x-x') \times \kappa h^{(0)}_{\rho\sigma}(x')\right]$$
$$= -\frac{\kappa^2\, \mathcal{C}^{\alpha\beta\gamma\delta\mu\nu}}{960\pi^2}\left[\ln(a)\, C^{(0)}_{\alpha\beta\gamma\delta}\right] = \frac{\kappa^2 \partial_\rho \partial_\sigma}{480\pi^2}\left[\ln(a)\, C^{(0)\,\rho\mu\sigma\nu}\right] \ , \tag{81}$$

and relation (53) implies a similar form for the second term:

$$-\frac{\kappa^2 \mathcal{F}^{\mu\nu}}{576\pi^2}\left[\ln(a)\int d^4x'\, \mathcal{F}^{\rho\sigma} \delta^4(x-x') \times \kappa h^{(0)}_{\rho\sigma}(x')\right] = -\frac{\kappa^2 \mathcal{F}^{\mu\nu}}{576\pi^2}\left[\ln(a)\, a^2 R^{(0)}\right] . \tag{82}$$

4.3 Dynamical Gravitons

Dynamical gravitons are characterized by their 3-momenta $\vec{k}$ and polarization λ. The graviton field for a dynamical graviton takes the form:

$$\kappa h_{\mu\nu}(x) = \epsilon_{\mu\nu}(\vec{k},\lambda) e^{i\vec{k}\cdot\vec{x}} u(t,k) \ , \tag{83}$$

where $u(t,k)$ is the graviton mode function and the polarization tensors take the same form in cosmology that they do in flat space. In particular, their temporal components vanish, they are transverse and traceless:

$$\epsilon_{0\mu}(\vec{k},\lambda) = 0 \qquad , \qquad k_i \epsilon_{i\mu}(\vec{k},\lambda) = 0 \qquad , \qquad \epsilon_{ii}(\vec{k},\lambda) = 0 \ . \tag{84}$$

The action of the Lichnerowicz operator (72) on such a field is:

$$\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) = \epsilon^{\mu\nu}(\vec{k},\lambda) e^{i\vec{k}\cdot\vec{x}} \times \Bigl(-\frac{a^2}{2}\Bigr) \Bigl[\partial_0^2 + 2aH\partial_0 + k^2\Bigr] u(t,k) \quad . \tag{85}$$

The general perturbative expansion (78) implies a similar expansion for the graviton mode function:

$$u(t,k) = u_0(t,k) + \kappa^2 u_1(t,k) + \kappa^4 u_2(t,k) + \ldots \tag{86}$$

The point of this sub-section is to compute the 1st order solution $u_1(t,k)$. The canonically normalized 0th order solution is well known:

$$u_0(t,k) = \frac{H}{\sqrt{2k^3}} \Bigl[1 - \frac{ik}{Ha}\Bigr] \exp\Bigl[\frac{ik}{Ha}\Bigr] = \frac{H}{\sqrt{2k^3}} \, (1 + ik\eta) e^{-ik\eta} \quad . \tag{87}$$

Its first two conformal time derivatives are:

$$\partial_0 u_0 = \frac{H}{\sqrt{2k^3}} \Bigl[-\frac{k^2}{Ha}\Bigr] \exp\Bigl[\frac{ik}{Ha}\Bigr] \quad , \quad \partial_0^2 u_0 = \frac{H}{\sqrt{2k^3}} \Bigl[k^2 + \frac{ik^3}{Ha}\Bigr] \exp\Bigl[\frac{ik}{Ha}\Bigr] \quad . \tag{88}$$

Relations (88) imply four useful identities:

$$(\partial_0^2 - k^2) u_0 = -2ik\partial_0 u_0 \quad , \quad (\partial_0^2 - k^2) u_0 = \tfrac{1}{2}(\partial_0 - ik)^2 u_0 \; , \tag{89}$$

$$(\partial_0^2 + k^2) u_0 = -2Ha\partial_0 u_0 \quad , \quad (\partial_0^2 + k^2)^2 u_0 = 0 \; . \tag{90}$$

The right hand side of the effective field equation (71) for dynamical gravitons (83-84) consists of the two local contributions (81-82) and a series of local and non-local terms proportional to the 21 tensor differential operators of Table 1.

- *Contributions from the $\mathcal{F}$-term (82):* Because dynamical gravitons are source-free solutions, the linearized Ricci scalar vanishes: $R^{(0)} = 0$. Hence, there are no such contributions.

- *Contributions from the $\mathcal{C}$-term (81):* To evaluate (81) we first make a $3+1$ expansion of the derivatives:

$$\begin{aligned} \frac{\kappa^2 \partial_\rho \partial_\sigma}{480\pi^2} \Bigl[\ln(a) \, C^{(0)\,\rho\mu\sigma\nu}\Bigr] \; &= \; \frac{\kappa^2 \partial_0^2}{480\pi^2} \Bigl[\ln(a) \, C^{(0)\,0\mu 0\nu}\Bigr] - \frac{2\kappa^2 \partial_0 \partial_i}{480\pi^2} \Bigl[\ln(a) \, C^{(0)\,0(\mu\nu)i}\Bigr] \\ &\quad + \frac{\kappa^2 \partial_i \partial_j}{480\pi^2} \Bigl[\ln(a) \, C^{(0)\,i\mu j\nu}\Bigr] \quad . \end{aligned} \tag{91}$$

Now note that, for dynamical gravitons, the non-zero components of the linearized Weyl tensor are:

$$C^{(0)}_{0i0j} = e^{i\vec{k}\cdot\vec{x}}\epsilon_{ij} \times (-\tfrac{1}{4})(\partial_0^2 - k^2)\, u_0 \quad , \tag{92}$$

$$C^{(0)}_{0ijk} = e^{i\vec{k}\cdot\vec{x}}\Big(\epsilon_{ij} k_k - \epsilon_{ik} k_j\Big) \times \tfrac{i}{2}\partial_0 u_0 \quad , \tag{93}$$

$$\begin{aligned} C^{(0)}_{ijk\ell} &= e^{i\vec{k}\cdot\vec{x}}\Big(\epsilon_{ik}\delta_{j\ell} - \epsilon_{kj}\delta_{\ell i} + \epsilon_{j\ell}\delta_{ik} - \epsilon_{\ell i}\delta_{kj}\Big) \times (-\tfrac{1}{4})(\partial_0^2 + k^2)\, u_0 \\ &\quad + e^{i\vec{k}\cdot\vec{x}}\Big(\epsilon_{ik} k_j k_\ell - \epsilon_{kj} k_\ell k_i + \epsilon_{j\ell} k_i k_k - \epsilon_{\ell i} k_k k_j\Big) \times \tfrac{1}{2} u_0 \quad . \end{aligned} \tag{94}$$

Substituting (92-94) into (91) and exploiting the properties (84) of the polarization tensor and the identities (89) obeyed by the mode function implies:

$$\begin{aligned} \frac{\kappa^2 \partial_\rho \partial_\sigma}{480\pi^2}\Big[\ln(a)\, C^{(0)\,\rho\mu\sigma\nu}\Big] &= \frac{\kappa^2 e^{i\vec{k}\cdot\vec{x}}\epsilon^{\mu\nu}}{480\pi^2}\Big\{-\tfrac{1}{8}(\partial_0 + ik)^2\Big[\ln(a)(\partial_0 - ik)^2 u_0\Big]\Big\} \\ &= \frac{\kappa^2 e^{i\vec{k}\cdot\vec{x}}\epsilon^{\mu\nu}}{480\pi^2} \times (-\tfrac{i}{2}) k H^2 a^2 \partial_0 u_0 \quad . \end{aligned} \tag{95}$$

- *Contributions from the "Summation" terms:* The remaining terms on the right hand side of equation (71) all involve sums of the tensor differential operators $[^{\mu\nu}D_i^{\rho\sigma}]$ of Table 1. These can be partially integrated to act on the factor of $h_{\rho\sigma}(x')$, whereupon most of the $[^{\mu\nu}D_i^{\rho\sigma}]$ give zero because they access a temporal component, or a divergence, or a trace. Only the case of $[^{\mu\nu}D_2^{\rho\sigma}] = \eta^{\mu(\rho}\eta^{\sigma)\nu}$ contributes, and a further simplification is that the coefficient function $\widehat{T}^2_A(a, a', \partial)$ vanishes. Each surviving term has a common factor of $e^{i\vec{k}\cdot\vec{x}}\epsilon^{\mu\nu}$ which can be canceled out from equations (85) and (95) to give a scalar equation for $u_1(t, k)$:

$$\begin{aligned} -\frac{a^2}{2}\Big[\partial_0^2 + 2aH\partial_0 + k^2\Big]\kappa^2 u_1 &= \frac{\kappa^2 H^2 a^2}{480\pi^2} \times (-i) k \partial_0 u_0 \\ &\quad - \frac{\kappa^2 H^2}{192\pi^2}\int d^4x'\, T^2(a, a', \partial)\, \delta^4(x - x') \times e^{-i\vec{k}\cdot\Delta\vec{x}} u_0(t', k) \\ &\quad + \frac{\kappa^2 H^2}{384\pi^3}\int d^4x'\, T^2_B(a, a', \partial)\, f_B(x; x') \times e^{-i\vec{k}\cdot\Delta\vec{x}} u_0(t', k) \quad . \end{aligned} \tag{96}$$

(i) For the term in (96) involving the coefficient $\widehat{T}^2(a, a', \partial)$, it is possible to obtain an exact result:

$$\widehat{T}^2(a, a', \partial) = aa' \ln(aa')\Big[\partial^2 + 2aa'H^2\Big] + aa'\partial^2 + 3a^2 a'^2 H^2 \quad . \tag{97}$$

Performing the delta function integral, and using relations (89-90) gives:

$$
\begin{aligned}
-\frac{\kappa^2 H^2}{192\pi^2} & \int d^4x' \, \widehat{T}^2(a,a',\partial) \, \delta^4(x-x') \times e^{-i\vec{k}\cdot\Delta\vec{x}} u_0(t',k) \\
&= -\frac{\kappa^2 H^2}{192\pi^2} \Big\{ -a\ln(a)(\partial_0^2+k^2)[a u_0] + 4a^4\ln(a) H^2 u_0 \\
& \qquad -a(\partial_0^2+k^2)[a\ln(a) u_0] - a(\partial_0^2+k^2)[a u_0] + 3a^4 H^2 u_0 \Big\} \qquad (98) \\
&= \frac{\kappa^2 H^2}{96\pi^2} \Big\{ a^4 H^2 u_0 + a^3 H \partial_0 u_0 \Big\} \ . \qquad (99)
\end{aligned}
$$

Expression (87) implies that $u_0(t,k)$ approaches a constant at late times:

$$
\lim_{t\to\infty} u_0(t,k) = \frac{H}{\sqrt{2k^3}} \equiv u_\infty \ . \qquad (100)
$$

Expression (88) implies that $\partial_0 u_0$ falls off like $1/a$, so (99) goes like a^4.

(ii) For the non-local part of (96) which involves $\widehat{T}^2_B(a,a',\partial)$:

$$
\widehat{T}^2_B(a,a',\partial) = -\frac{\partial^4}{80H^2} - \tfrac{1}{4} a a' \partial^2 - \tfrac{1}{2} a^2 a'^2 H^2 \ , \qquad (101)
$$

upon substituting (101) in the final term of (96), reflecting the derivatives and partially integrating to act on $u_0(t',k)$ we get:

$$
\begin{aligned}
\frac{\kappa^2 H^2}{384\pi^3} & \int d^4x' \, f_B(x;x') \times e^{-i\vec{k}\cdot\Delta\vec{x}} \\
& \times \Big\{ -\frac{(\partial_0'^2+k^2)^2}{80H^2} - \tfrac{1}{4} a(\partial_0'^2+k^2) a' - \tfrac{1}{2} a^2 a'^2 H^2 \Big\} u_0(t',k) \ . \qquad (102)
\end{aligned}
$$

The mode function identities (90) serve to eliminate the derivatives on the 2nd line of (102):

$$
\Big\{ -\frac{(\partial_0'^2+k^2)^2}{80H^2} - \tfrac{1}{4} a(\partial_0'^2+k^2) a' - \tfrac{1}{2} a^2 a'^2 H^2 \Big\} u_0(t',k) = -\tfrac{1}{2} a^2 a'^3 H^3 \Delta\eta \, u_0(t',k) \ . \qquad (103)
$$

At this stage we can exploit relation (76) to simplify the $\widehat{T}^i_B$ contribution to:

$$
\begin{aligned}
\frac{\kappa^2 H^2}{384\pi^3} & \int d^4x' \, \partial_0 \partial^2 \Big[\theta(\Delta\eta - \Delta r) \Big] \times e^{-i\vec{k}\cdot\Delta\vec{x}} \times a^2 a'^3 H^3 u_0(t',k) \\
&= -\frac{\kappa^2 H^2}{96\pi^2} \, a^2 H^3 \partial_0 (\partial_0^2 + k^2) \int_{\eta_i}^{\eta} d\eta' \, a'^3 u_0(t',k) \int_0^{\Delta\eta} dr \, r^2 \, \frac{\sin(kr)}{kr} \qquad (104) \\
&= -\frac{\kappa^2 H^2}{96\pi^2} \, a^2 H^3 \int_{\eta_i}^{\eta} d\eta' \, a'^3 u_0(t',k) \cos(k\Delta\eta) \ . \qquad (105)
\end{aligned}
$$

The integrand in (105) is a total derivative:

$$a'^3 u_0(t',k)\cos(k\Delta\eta) = \frac{u_\infty}{4H}\frac{\partial}{\partial\eta'}\Bigl\{a'^2 e^{ik\eta-2ik\eta'} + \Bigl[a'^2 - \frac{2ika'}{H}\Bigr]e^{-ik\eta}\Bigr\} \; . \quad (106)$$

If we keep only the upper limit contribution the result is:

$$-\frac{\kappa^2 H^2}{96\pi^2}\, a^2 H^3 \int_{\eta_i}^{\eta} d\eta'\, a'^3 u_0(t',k)\cos(k\Delta\eta) \;\longrightarrow\; -\frac{\kappa^2 H^2}{96\pi^2} \times a^4 H^2 u_0(t,k) \; . \quad (107)$$

- *The Total Contribution:* Combining expressions (95), (99) and (107) gives:

$$-\frac{a^2}{2}\Bigl[\partial_0^2 + 2aH\partial_0 + k^2\Bigr]\kappa^2 u_1$$
$$= \frac{\kappa^2 H^2}{96\pi^2}\Bigl\{-\tfrac{ik}{10aH}\times a^3 H\partial_0 u_0 + a^4 H^2 u_0 + a^3 H\partial_0 u_0 - a^4 H^2 u_0\Bigr\} \quad (108)$$
$$= -\frac{a^2}{2}\times\frac{\kappa^2 H^2 k^2}{48\pi^2}\Bigl\{1 + \Bigl[1 - \tfrac{1}{10}\Bigr]\frac{ik}{aH} + \dots\Bigr\} \; . \quad (109)$$

Equation (109) is easy to solve at late times:

$$\kappa^2 u_1(t,k) \;\longrightarrow\; \frac{\kappa^2 H^2}{48\pi^2}\times u_\infty\Bigl\{1 - \tfrac{3i}{10}(\tfrac{k}{aH})^3 \ln(a) + \dots\Bigr\} \; . \quad (110)$$

The fact that there is no growing contribution agrees with previous analyses [21,22]. It is useful to combine the 1-loop correction (110) with the tree order result and express both in terms of the "electric" components of the Weyl tensor:

$$C_{0i0j}(t,\vec{x}) \;\longrightarrow\; C^{(0)}_{0i0j}(t,\vec{x})\Bigl\{1 - \tfrac{3\kappa^2 H^2}{160\pi^2}\ln(a) + O(\kappa^4)\Bigr\} \quad (111)$$

The functional form, although not the sign, resembles the logarithmic enhancement induced by inflationary gravitons in the electric field strength of electromagnetic radiation [11].

4.4 Response to A Point Mass

The symmetries of cosmology are homogeneity and isotropy. Four components of the metric are scalar under these symmetries, of which any two can be gauged away. We choose the two non-zero scalar potentials to be:

$$\kappa h_{00} = 2\Psi(t,r) \qquad , \qquad \kappa h_{ij} = -2\Phi(t,r)\delta_{ij} \; , \quad (112)$$

so that the invariant element in conformal coordinates is:

$$ds^2 = -a^2(1-2\Psi)d\eta^2 + a^2(1-2\Phi)d\vec{x}\cdot d\vec{x} \ . \tag{113}$$

In the spacetime geometry (112-113) the left hand side of the effective field equation (71):

$$E^{\mu\nu} \equiv \mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma} \ , \tag{114}$$

can be $3+1$ decomposed to give:

$$E^{00} = a^2\Big[-6a^2H^2\Psi + (-2\nabla^2{+}6aH\partial_0)\Phi\Big] \ , \tag{115}$$

$$E^{0i} = a^2\partial^i\Big[-2aH\Psi + 2\partial_0\Phi\Big] \ , \tag{116}$$

$$\begin{aligned} E^{ij} = {} & a^2\partial^i\partial^j\Big[-\Psi-\Phi\Big] + a^2\delta^{ij}\Big\{(\nabla^2{+}2aH\partial_0{+}6a^2H^2)\Psi \\ & +(\nabla^2{-}4aH\partial_0{-}2\partial_0^2)\Phi\Big\} \ . \end{aligned} \tag{117}$$

Relations (115-117) suggest that we identify four scalar components:

$$E^{00} \equiv \mathcal{E}_1 \quad , \quad E^{0i} \equiv \partial^i\mathcal{E}_2 \quad , \quad E^{ij} \equiv \partial^i\partial^j\mathcal{E}_3 + \delta^{ij}\mathcal{E}_4 \ . \tag{118}$$

Conservation implies two relations between them:

$$\partial_\nu E^{\mu\nu} + aH\delta^\mu_{\ 0}E^\rho_{\ \rho} = 0 \tag{119}$$

$$\Longrightarrow \quad \begin{Bmatrix} \partial_0\mathcal{E}_1 + \nabla^2\mathcal{E}_2 + aH(-\mathcal{E}_1{+}\nabla^2\mathcal{E}_3{+}3\mathcal{E}_4) = 0 \\ \partial_0\mathcal{E}_2 + \nabla^2\mathcal{E}_3 + \mathcal{E}_4 = 0 \end{Bmatrix} \ . \tag{120}$$

Hence, up to integration constants, any two of the four components (118) determine the other two.

In the geometry (112-113) the right hand side of equation (71):

$$S^{\mu\nu}(x) \equiv 8\pi G T^{\mu\nu} + \int d^4x' \, [{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')\,\kappa h_{\rho\sigma}(x') \ , \tag{121}$$

must obviously have the same tensor structure (118) as the left hand side:

$$S^{00} \equiv \mathcal{S}_1 \quad , \quad S^{0i} \equiv \partial^i\mathcal{S}_2 \quad , \quad S^{ij} \equiv \partial^i\partial^j\mathcal{S}_3 + \delta^{ij}\mathcal{S}_4 \ . \tag{122}$$

Therefore we can solve any two of the four scalar equations $\mathcal{E}_i = \mathcal{S}_i$. The simplest choice is obviously the combination of $i=2$ and $i=3$, which implies first order equations for Ψ and Φ:

$$2a\partial_0(a\Psi) = -\mathcal{S}_2 - 2(\partial_0{-}2aH)\mathcal{S}_3 \quad , \quad 2a\partial_0(a\Phi) = \mathcal{S}_2 - 2aH\mathcal{S}_3 \ . \tag{123}$$

However, one must bear in mind that $\mathcal{S}_2$ and $\mathcal{S}_3$ alone leave Ψ and Φ unfixed up to a free function $f(r)$:

$$\Delta\Psi(t,r) = \frac{f(r)}{a} = -\Delta\Phi(t,r) \quad . \tag{124}$$

Because the $i = 4$ equation $\mathcal{E}_4 = \mathcal{S}_4$ vanishes for (124), this ambiguity must be fixed by appealing to the $i = 1$ equation, $\mathcal{E}_1 = \mathcal{S}_1$:

$$\Delta\mathcal{E}_1 = 2a\nabla^2 f(r) \quad . \tag{125}$$

For a static point mass M in an expanding universe we find:

$$T^{\mu\nu}(x) \equiv -2\frac{\delta}{\delta\kappa h_{\mu\nu}(x)}\left\{-M\int d\tau\sqrt{-g_{\alpha\beta}(\chi(\tau))\dot{\chi}^\alpha(\tau)\dot{\chi}^\beta(\tau)}\right\}_{\substack{h_{\alpha\beta}=0 \\ \chi^\mu=(\tau,\vec{x})}} \tag{126}$$

$$= -\delta^\mu{}_0\delta^\nu{}_0 Ma\delta^3(\vec{x}) \quad . \tag{127}$$

This is one of those cases for which the $i = 1$ equation $\mathcal{E}_1 = \mathcal{S}_1$ must be employed to determine the full 0th order solutions:

$$\Psi_0(t,r) = \frac{GM}{ar} = -\Phi_0(t,r) \quad . \tag{128}$$

There are three derivatives of $\Psi_0(t,r)$ which shall be important in the analysis that follows:

$$\partial_0\Psi_0 = -aH\Psi_0 \quad , \quad \partial_0^2\Psi_0 = 0 \quad , \quad \nabla^2\Psi_0 = -\frac{4\pi GM\delta^3(\vec{x})}{a} \quad . \tag{129}$$

- *Contributions from the $\mathcal{F}$-term (82) and $\mathcal{C}$-term (81):* The linearized Ricci scalar and Weyl tensor are:

$$R^{(0)} = -\frac{2\nabla^2\Psi_0}{a^2} \quad , \tag{130}$$

$$C^{(0)}_{0i0j} = (-\partial_i\partial_j + \tfrac{1}{3}\delta_{ij}\nabla^2)\Psi_0 \quad , \tag{131}$$

$$C^{(0)}_{0ijk} = 0 \quad , \tag{132}$$

$$C^{(0)}_{ijk\ell} = \Big[\tfrac{2}{3}(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk})\nabla^2-\delta_{ik}\partial_j\partial_\ell+\delta_{kj}\partial_\ell\partial_i-\delta_{j\ell}\partial_i\partial_k+\delta_{\ell i}\partial_k\partial_j\Big]\Psi_0 \quad . \tag{133}$$

Although the Ricci scalar is proportional to a δ-function, the δ-function contributions to the Weyl tensor (131-133) all cancel. Substituting into expressions (81-82) and segregating appropriate components yields the local contributions to $\mathcal{S}_{1-3}$:

$$\begin{aligned}
\mathcal{S}_{1\mathcal{FC}} &= -\frac{\kappa^2(\nabla^2 - 3aH\partial_0 + 9a^2H^2)}{576\pi^2}\Big[-2\ln(a)\,\nabla^2\Psi_0\Big] \\
&\quad + \frac{\kappa^2\nabla^2}{480\pi^2}\Big[-\tfrac{2}{3}\ln(a)\,\nabla^2\Psi_0\Big] \ , \qquad (134) \\
\mathcal{S}_{2\mathcal{FC}} &= -\frac{\kappa^2(-\partial_0 + 3aH)}{576\pi^2}\Big[-2\ln(a)\,\nabla^2\Psi_0\Big] + \frac{\kappa^2\partial_0}{480\pi^2}\Big[\tfrac{2}{3}\ln(a)\,\nabla^2\Psi_0\Big] \ , \qquad (135) \\
\mathcal{S}_{3\mathcal{FC}} &= -\frac{\kappa^2}{576\pi^2}\Big[-2\ln(a)\,\nabla^2\Psi_0\Big] + \frac{\kappa^2}{480\pi^2}\Big[(\tfrac{1}{3}\nabla^2 - \partial_0^2)\ln(a)\,\Psi_0\Big] \ . \qquad (136)
\end{aligned}$$

Of these the only contribution that does not vanish away from the origin comes from $\mathcal{S}_3$: [7]

$$\mathcal{S}_{3\mathcal{FC}}\Big|_{\vec{x}\neq 0} = \frac{\kappa^2 H^2 a^2}{480\pi^2} \times \Psi_0(t,r) \ . \qquad (137)$$

i	$[^{\mu\nu}D_i^{\rho\sigma}]\times h_{\rho\sigma}(x')$	i	$[^{\mu\nu}D_i^{\rho\sigma}]\times h_{\rho\sigma}(x')$	i	$[^{\mu\nu}D_i^{\rho\sigma}]\times h_{\rho\sigma}(x')$
1	$\eta^{\mu\nu}h(x')$	8	$\partial^\mu\partial^\nu \times h(x')$	15	$\delta^{(\mu}{}_0\partial^{\nu)}\times h_{00}(x')$
2	$h^{\mu\nu}(x')$	9	$\delta^{(\mu}{}_0 h^{\nu)}{}_0(x')$	16	$\delta^\mu{}_0\delta^\nu{}_0\partial^\rho\partial^\sigma \times h_{\rho\sigma}(x')$
3	$\eta^{\mu\nu}h_{00}(x')$	10	$\delta^{(\mu}{}_0\partial_\rho \times h^{\nu)\rho}(x')$	17	$\partial^\mu\partial^\nu \times h_{00}(x')$
4	$\delta^\mu{}_0\delta^\nu{}_0 h(x')$	11	$\partial^{(\mu}\times h^{\nu)}{}_0(x')$	18	$\delta^{(\mu}{}_0\partial^{\nu)}\partial^\rho \times h_{\rho 0}(x')$
5	$\eta^{\mu\nu}\partial^\rho \times h_{\rho 0}(x')$	12	$\partial^{(\mu}\partial_\rho \times h^{\nu)\rho}(x')$	19	$\delta^{(\mu}{}_0\partial^{\nu)}\partial^\rho\partial^\sigma \times h_{\rho\sigma}(x')$
6	$\delta^{(\mu}{}_0\partial^{\nu)}\times h(x')$	13	$\delta^\mu{}_0\delta^\nu{}_0 h_{00}(x')$	20	$\partial^\mu\partial^\nu\partial^\rho \times h_{\rho 0}(x')$
7	$\eta^{\mu\nu}\partial^\rho\partial^\sigma \times h_{\rho\sigma}(x')$	14	$\delta^\mu{}_0\delta^\nu{}_0\partial^\rho \times h_{\rho 0}(x')$	21	$\partial^\mu\partial^\nu\partial^\rho\partial^\sigma \times h_{\rho\sigma}(x')$

Table 5: Contraction of $h_{\rho\sigma}(x')$ into the 21 tensor differential operators given in Table 1, which act on the functions (74).

[7] This is because $\nabla^2\Psi_0 = -\frac{4\pi GM}{a}\delta^3(\vec{x})$.

- *Contributions from the "Summation" terms:* The other three contributions to the graviton self-energy (73) involve sums over the 21 tensor differential operators $[^{\mu\nu}D_i^{\rho\sigma}]$ listed in Table 1, acting on the three functions of $(x-x')^\mu$ given in expression (74), and contracted into $\kappa h_{\rho\sigma}^{(0)}(x')$. The results of these contractions are presented in Table 5. It remains to substitute $\kappa h_{00}^{(0)}(x') = 2\Psi_0(t',r')$ and $\kappa h_{ij}^{(0)}(x') = 2\Psi_0(t',r')\delta_{ij}$, and then identify contributions to each of the three sources $\mathcal{S}_{1-3}$. Note the relevant contractions:

$$\kappa h^{(0)}(x') = 4\Psi_0(t',r') \quad , \quad \partial^\rho \times \kappa h_{\rho 0}^{(0)}(x') = -2\partial_0 \times \Psi_0(t',r') \ ,$$
$$\partial^\rho \partial^\sigma \times \kappa h_{\rho\sigma}^{(0)} \ = \ 2(\partial_0^2 + \nabla^2) \times \Psi_0(t',r') \ . \qquad (138)$$

Recall that each of the contractions $[^{\mu\nu}D_i^{\rho\sigma}] \times \kappa h_{\rho\sigma}^{(0)}(x')$ consists of a tensor differential operator acting on x^μ and multiplied by the same function $\Psi_0(t',r')$. Therefore, we need only keep track of the factor $\mathcal{F}_{1-3}^i$ which acts on the functions (74) for each of the three sources. Table 6 gives these factors for the sources $\mathcal{S}_3$ and $\mathcal{S}_2$, while Table 7 for the source $\mathcal{S}_1$. Because partially integrating temporal derivatives would produce unwanted surface terms, whereas there are none for spatial derivatives, we have eliminated second time derivatives:

$$\partial_0^2 = -\partial^2 + \nabla^2 \ . \qquad (139)$$

i	$\mathcal{F}_3^i$	i	$\mathcal{F}_2^i$	i	$\mathcal{F}_2^i$
8	4	6	2	17	$-2\partial_0$
12	2	8	$-4\partial_0$	18	$-\partial_0$
17	2	10	1	19	$-\partial^2 + 2\nabla^2$
20	$-2\partial_0$	11	-1	20	$-2\partial^2 + 2\nabla^2$
21	$-2\partial^2 + 4\nabla^2$	15	1	21	$2\partial_0\partial^2 - 4\partial_0\nabla^2$

Table 6: The first two columns give the non-zero factors contributing to the source $\mathcal{S}_3$. The last four columns present the nonzero factors contributing to the source $\mathcal{S}_2$.

i	$\mathcal{F}_1^i$	i	$\mathcal{F}_1^i$	i	$\mathcal{F}_1^i$
1	-4	8	$-4\partial^2+4\nabla^2$	15	$-2\partial_0$
2	2	9	-2	16	$-2\partial^2+4\nabla^2$
3	-2	10	$2\partial_0$	17	$-2\partial^2+2\nabla^2$
4	4	11	$2\partial_0$	18	$-2\partial^2+2\nabla^2$
5	$2\partial_0$	12	$2\partial^2-2\nabla^2$	19	$2\partial_0\partial^2-4\partial_0\nabla^2$
6	$-4\partial_0$	13	2	20	$2\partial_0\partial^2-2\partial_0\nabla^2$
7	$2\partial^2-4\nabla^2$	14	$-2\partial_0$	21	$2\partial^4-2\partial^2\nabla^2+4\partial_0^2\nabla^2$

Table 7: Source $\mathcal{S}_1$ factors arising from the contractions $[{}^{00}D_i^{\rho\sigma}]\times\kappa h_{\rho\sigma}^{(0)}(x')$.

- *The Source $\mathcal{S}_3$:* The simplest source is $\mathcal{S}_3$, which receives contributions only from $i=8,12,17,20,21$. Combining information from Table 3 for $\widehat{T}^i(a,a'\partial)$ and for $[{}^{\mu\nu}D_i^{\rho\sigma}]\times\kappa h_{\rho\sigma}^{(0)}$ from Table 6 gives:

$$\mathcal{S}_{3T} \equiv -\frac{\kappa^2H^2}{192\pi^2}\sum_{i=1}^{21}\int d^4x'\,\widehat{T}^i(a,a',\partial)\times\mathcal{F}_3^i\times\delta^4(x-x')\times\Psi_0(t',r') \qquad (140)$$

$$= -\frac{\kappa^2H^2a^2}{192\pi^2}\int d^4x'\,8aa'\,\delta^4(x-x')\times\Psi_0(t',r') \;\; . \qquad (141)$$

The same two tables give the initial contribution from $\widehat{T}_A^i(a,a',\partial)$:

$$\mathcal{S}_{3T_A} \equiv \frac{\kappa^2H^2}{384\pi^3}\sum_{i=1}^{21}\int d^4x'\,\widehat{T}_A^i(a,a',\partial)\times\mathcal{F}_3^i\times\theta(\Delta\eta-\Delta r)\times\Psi_0(t',r') \qquad (142)$$

$$= \frac{\kappa^2H^2}{384\pi^3}\int d^4x'\Big\{-aa'\partial^4+4aa'^2H\partial_0\partial^2+2a^2a'^2H^2\partial^2$$

$$+\Big[2aa'^2\partial^2-4a^2a'^2H^2\Big]\nabla^2\Big\}\,\theta(\Delta\eta-\Delta r)\times\Psi_0(t',r') \;\; , \qquad (143)$$

as well as the initial contribution from $\widehat{T}_B^i(a,a',\partial)$:

$$\mathcal{S}_{3T_B} \equiv \frac{\kappa^2H^2}{384\pi^3}\sum_{i=1}^{21}\int d^4x'\,\widehat{T}_B^i(a,a',\partial)\times\mathcal{F}_3^i\times f_B(x;x')\times\Psi_0(t',r') \qquad (144)$$

$$= \frac{\kappa^2H^2}{384\pi^3}\int d^4x'\Big\{\frac{3\partial^2}{10H^2}-\frac{a'\partial_0}{H}+aa'^2H\Delta\eta-\frac{\nabla^2}{5H^2}\Big\}\,f_B(x;x')\times\Psi_0(t',r') \qquad (145)$$

There is some ambiguity in how we describe the three contributions (141), (143) and (145). For instance, one can exploit relation (75) to convert the $-aa'\partial^4$ in (143) into an additional $+4aa'$ in (141). Also the factor of $+aa'^2 H\Delta\eta$ in (145) can be converted by expression (76) into an additional $-2aa'^2 H\partial_0\partial^2$ in (143). When these simplifications are made the result can be written in terms of particular cases of the four generic integrals which are defined and evaluated in the Appendix:

$$\begin{aligned}
&\mathcal{S}_{3T} + \mathcal{S}_{3T_A} + \mathcal{S}_{3T_B} = \\
&\quad -\frac{\kappa^2 H^2}{192\pi^2} \times 12a^2\Psi_0 + \frac{\kappa^2 H^2}{384\pi^3} \Big\{ (2aH\partial_0\partial^2 + 2a^2H^2\partial^2) \times I_A^2 \\
&\quad +2a\partial^2 \times I_{A\delta}^1 - 4a^2H^2 \times I_{A\delta}^2 + \frac{3\partial^2}{10H^2} \times I_B^0 - \frac{\partial_0}{H} \times I_B^1 - \frac{1}{5H^2} \times I_{B\delta}^0 \Big\} \ . \quad (146)
\end{aligned}$$

Substituting the relevant results from the Appendix in (146) gives the final contribution from $\mathcal{S}_3$:

$$\mathcal{S}_{3T} + \mathcal{S}_{3T_A} + \mathcal{S}_{3T_B} = a^2\Psi_0 \Big\{ -\tfrac{\kappa^2}{240\pi^2 a^2 r^2} - \tfrac{\kappa^2 H^2}{48\pi^2} - \tfrac{\kappa^2 H^3 ar}{24\pi^2} \Big\} \ . \qquad (147)$$

- *The Source* $\mathcal{S}_2$*:* The contributions to $\mathcal{S}_2$ from $\widehat{T}^i(a, a', \partial)$ and $\widehat{T}_A^i(a, a', \partial)$ can be obtained from Tables 3 and 6:

$$\begin{aligned}
\mathcal{S}_{2T} &\equiv -\frac{\kappa^2 H^2}{192\pi^2} \sum_{i=1}^{21} \int d^4x' \, \widehat{T}^i(a, a', \partial) \times \mathcal{F}_2^i \times \delta^4(x-x') \times \Psi_0(t', r') \qquad (148) \\
&= -\frac{\kappa^2 H^2 a^2}{192\pi^2} \int d^4x' \Big\{ \ln(aa') \Big[-aa'\partial_0 - 2a^2a'^2H^2\Delta\eta \Big] - 12aa'\partial_0 \\
&\qquad -4a^2a'H + 12aa'^2H \Big\} \, \delta^4(x-x') \times \Psi_0(t', r') \ , \qquad (149)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_{2T_A} &\equiv \frac{\kappa^2 H^2}{384\pi^3} \sum_{i=1}^{21} \int d^4x' \, \widehat{T}_A^i(a, a', \partial) \times \mathcal{F}_2^i \times \theta(\Delta\eta - \Delta r) \times \Psi_0(t', r') \qquad (150) \\
&= \frac{\kappa^2 H^2}{384\pi^3} \int d^4x' \Big\{ \Big[aa'\partial_0 - a^2a'H + 4aa'^2H \Big] \partial^4 + \Big[-2aa'\partial_0\partial^2 + 2a^2a'H^2\partial^2 \\
&\qquad -4aa'^2H\partial^2 + 4a^2a'^2H^2\partial_0 \Big] \nabla^2 \Big\} \, \theta(\Delta\eta - \Delta r) \times \Psi_0(t', r') \ . \qquad (151)
\end{aligned}$$

Similarly, Tables 4 and 6 give the contribution from $\widehat{T}^i_B(a,a',\partial)$:

$$\mathcal{S}_{2T_B} \equiv \frac{\kappa^2 H^2}{384\pi^3} \sum_{i=1}^{21} \int d^4x' \, \widehat{T}^i_B(a,a',\partial) \times \mathcal{F}^i_2 \times f_B(x;x') \times \Psi_0(t',r') \tag{152}$$

$$= \frac{\kappa^2 H^2}{384\pi^3} \int d^4x' \Big\{ -\frac{\partial_0 \partial^2}{4H^2} + \Big[\frac{3a}{4H} - \frac{a'}{H}\Big] \partial^2 + \Big[-3aa' + a'^2\Big] \partial_0 - aa'^2 H + \Big[\frac{\partial_0}{5H^2} - \frac{a}{2H} + \frac{a'}{H}\Big] \nabla^2 \Big\} \times f_B(x;x') \times \Psi_0(t',r') \ . \tag{153}$$

To simplify expressions (149), (151) and (153) we use the identities (75-76) and we reduce their total to a sum of the integrals evaluated in the Appendix:

$$\mathcal{S}_{2T} + \mathcal{S}_{2T_A} + \mathcal{S}_{2T_B} = \frac{\kappa^2 H^2}{384\pi^3} \Big\{ 2a^2 H \partial^2 \times I^1_{A\delta} - (2aH\partial^2 - 4a^2 H^2 \partial_0) \times I^2_{A\delta} - \Big(\frac{\partial_0 \partial^2}{4H^2} + \frac{a\partial^2}{4H}\Big) \times I^0_B + \Big(\frac{\partial_0}{5H^2} + \frac{a}{2H}\Big) \times I^0_{B\delta} \Big\} \ . \tag{154}$$

The final answer is:

$$\mathcal{S}_{2T} + \mathcal{S}_{2T_A} + \mathcal{S}_{2T_B} = a^3 H \Psi_0 \Big\{ -\tfrac{\kappa^2}{80\pi^2 a^2 r^2} \Big\} \ . \tag{155}$$

- *The Source* $\mathcal{S}_1$*:* The factors $\mathcal{F}^i_1$ needed for the $\mathcal{S}_1$ source can be seen in Table 7. [8] The results are:

$$\mathcal{S}_{1T} \equiv -\frac{\kappa^2 H^2}{192\pi^2} \sum_{i=1}^{21} \int d^4x' \, \widehat{T}^i(a,a',\partial) \times \mathcal{F}^i_1 \times \delta^4(x-x') \times \Psi_0(t',r') \tag{156}$$

$$= -\frac{\kappa^2 H^2 a^2}{192\pi^2} \int d^4x' \, \Big\{ \ln(aa') \Big[-aa'^2 H \partial_0 + 12a^2 a'^2 H^2\Big] - \Big[8a^2 a' + 40aa'^2\Big] H \partial_0 + 44a^2 a'^2 H^2 \Big\} \, \delta^4(x-x') \times \Psi_0(t',r') \ , \tag{157}$$

$$\mathcal{S}_{1T_A} \equiv \frac{\kappa^2 H^2}{384\pi^3} \sum_{i=1}^{21} \int d^4x' \, \widehat{T}^i_A(a,a',\partial) \times \mathcal{F}^i_1 \times \theta(\Delta\eta - \Delta r) \times \Psi_0(t',r') \tag{158}$$

$$= \frac{\kappa^2 H^2}{384\pi^3} \int d^4x' \Big\{ \Big[aa'\partial^2 + (a^2 a' - 4aa'^2) H \partial_0 + a^2 a'^2 H^2\Big] \partial^4 + \Big[-3aa'\partial^4 + 2aa'\nabla^2\partial^2 - (2a^2 a' - 4aa'^2) H \partial_0 \partial^2 + 2a^2 a'^2 H^2 \partial^2 - 4a^2 a'^2 H^2 \nabla^2\Big] \nabla^2 \Big\} \, \theta(\Delta\eta - \Delta r) \times \Psi_0(t',r') \ , \tag{159}$$

[8]The factor of $\delta^4(x - x')$ in $\mathcal{S}_{1T}$ has been used to consolidate factors of a and a', and we have suppressed contributions proportional to $\delta^3(\vec{x})$.

$$
\begin{aligned}
\mathcal{S}_{1T_B} &\equiv \frac{\kappa^2 H^2}{384\pi^3}\sum_{i=1}^{21}\int d^4x'\, \widehat{T}^i_B(a,a',\partial)\times\mathcal{F}^i_1\times f_B(x;x')\times\Psi_0(t',r') \\
&= \frac{\kappa^2 H^2}{384\pi^3}\int d^4x'\Big\{\Big[-\frac{5a}{4H}+\frac{a'}{2H}\Big]\partial_0\partial^2+\Big[\frac{5}{4}a^2-4aa'+\frac{1}{2}a'^2\Big]\partial^2 \\
&-\Big[4a^2a'-aa'^2\Big]H\partial_0+\Big[\frac{\partial^2}{4H^2}-\frac{\nabla^2}{4H^2}+\Big(\frac{a}{2H}-\frac{a}{H}\Big)\partial_0-\frac{1}{2}a^2+3aa'-a'^2\Big]\nabla^2\Big\} \\
&\quad\times f_B(x;x')\times\Psi_0(t',r') \ . \qquad (160)
\end{aligned}
$$

We next simplify using the identities (75) and (76), express the result in terms of the generic integrals evaluated in the Appendix:

$$
\begin{aligned}
&\mathcal{S}_{1T}+\mathcal{S}_{1T_A}+\mathcal{S}_{1T_B} = \\
&\quad \frac{\kappa^2 H^2}{384\pi^3}\Big\{-a^2H\partial_0\partial^2\times I^1_{A\delta}+\Big[2aH\partial_0\partial^2+2a^2H^2\partial^2-4a^2H^2\nabla^2\Big]\times I^2_{A\delta} \\
&\quad -\frac{3a\partial_0\partial^2}{4H}\times I^0_B-\Big[\frac{3}{4}a\partial^2+\frac{3}{2}a^2H\partial_0\Big]\times I^1_B \\
&\quad +\Big[\frac{\partial^2}{4H^2}-\frac{\nabla^2}{5H^2}-\frac{a\partial_0}{2H}\Big]\times I^0_{B\delta}-\frac{1}{2}a\times I^1_{B\delta}\Big\} \ , \qquad (161)
\end{aligned}
$$

and conclude:

$$
\mathcal{S}_{1T}+\mathcal{S}_{1T_A}+\mathcal{S}_{1T_B} = a^4H^2\Psi_0\left\{-\tfrac{\kappa^2H^2}{80\pi^2a^4H^4r^4}\right\} \ . \qquad (162)
$$

- *Corrections to the potentials* Φ *and* Ψ*:* The 3rd and final term in expression (147) is a spatial constant and would therefore be annihilated by the prefactor of $\overline{\partial}^\mu\overline{\partial}^\nu$ which $\mathcal{S}_3$ carries as seen from (122) . If we drop this term and include the Weyl contribution (137) to $\mathcal{S}_3$, the $\mathcal{E}_3=\mathcal{S}_3$ equation reads:

$$
-a^2(\Psi_1+\Phi_1) = a^2\Psi_0\Big\{-\tfrac{\kappa^2}{240\pi^2a^2r^2}-\tfrac{3\kappa^2H^2}{160\pi^2}\Big\} \ . \qquad (163)
$$

Solving for Ψ_1 gives:

$$
\Psi_1 = -\Phi_1+\Psi_0\Big\{\tfrac{\kappa^2}{240\pi^2a^2r^2}+\tfrac{3\kappa^2H^2}{160\pi^2}\Big\} \ . \qquad (164)
$$

Substituting (164) in the $\mathcal{E}_2=\mathcal{S}_2$ equation, and making some simple manipulations implies:

$$
\partial_0(a\Phi_1) = a^2H\Psi_0\Big\{-\tfrac{\kappa^2}{480\pi^2a^2r^2}-\tfrac{3\kappa^2H^2}{160\pi^2}\Big\} \ . \qquad (165)
$$

The solution, up to a function $f(r)$, takes the form:

$$\Phi_1 = \Psi_0 \Bigl\{ \tfrac{\kappa^2}{960\pi^2 a^2 r^2} + \tfrac{3\kappa^2 H^2}{160\pi^2} \times \ln(a) \Bigr\} + \frac{f(r)}{a} \quad . \tag{166}$$

We can determine the function $f(r)$ by combining the $\mathcal{E}_1 = \mathcal{S}_1$ and $\mathcal{E}_2 = \mathcal{S}_2$ equations to find:

$$\nabla^2 \Phi_1 = \Psi_0 \Bigl\{ \tfrac{\kappa^2}{160\pi^2 a^2 r^4} - \tfrac{3\kappa^2 H^2}{160\pi^2 r^2} \Bigr\} \quad . \tag{167}$$

If we choose the integration constant so that Ψ_1 agrees with the flat space limit at the initial time, the two 1-loop potentials are:

$$\Psi_1 = \Psi_0 \Bigl\{ \tfrac{\kappa^2}{320\pi^2 a^2 r^2} - \tfrac{3\kappa^2 H^2}{160\pi^2} \times \ln(aHr) \Bigr\} \quad , \tag{168}$$

$$\Phi_1 = \Psi_0 \Bigl\{ \tfrac{\kappa^2}{960\pi^2 a^2 r^2} + \tfrac{3\kappa^2 H^2}{160\pi^2} \Bigl[\ln(aHr) + 1 \Bigr] \Bigr\} \quad . \tag{169}$$

5 Epilogue

This re-computation of the MMC scalar loop contribution to the graviton self-energy was prompted by the recent discovery that a finite renormalization of the cosmological constant is needed to make $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ conserved [40]. The absence of this renormalization in the original computation [19] was compounded by the decision to express the result as a sum of conserved tensor differential operators acting on structure functions [21]. While it was possible that the two mistakes might have canceled one another, it was obviously necessary to check. In addition to including the missing renormalization we have expressed the fully renormalized, $''in\!-\!in''$ result (73) without any preconceptions about its tensor structure. Our solution of the effective field equations in Section 4 confirms the original finding of no secular 1-loop corrections to graviton mode function [21,22], however, our results (168-169) for the response to a point mass differ from the previous calculation [23] in two ways:

1. The coefficient of the $\ln(a)$ correction changed to agree with that of the old $\ln(Hr)$ correction so that the two add to give $\ln(aHr)$; and

2. We concluded that the large gravitational slip originally reported [23] is not present.

We also pushed the calculation of the graviton mode function to include falling corrections which nonetheless cause the electric components of the Weyl tensor (111) to experience secular growth compared to the classical result.

It is significant that all three of our secular 1-loop corrections (111) and (168-169) have the same coefficient of $-\frac{3\kappa^2 H^2}{160\pi^2}$. This is reminiscent of the secular 1-graviton loop corrections to the electric field strength of plane wave photons [11] and to the Coulomb potential [10]. Those effects both had a renormalization group explanation [40] and we will show, in a follow-up work [41], that the same applies to (111) and (168-169). Support for this comes from the observation that all three of these secular corrections derive from the factors of $\ln(a)$ which were induced by the incomplete cancellation of primitive divergences and counterterms:

$$\frac{(2H)^{D-4}}{D-4} - \frac{(\mu a)^{D-4}}{D-4} = -\ln\Bigl(\frac{\mu a}{2H}\Bigr) + O(D-4) \; . \qquad (170)$$

for the Einstein counterterm (12) and for the ∂_0^2 part of the Weyl counterterm.

We should also comment on the sign of the three secular 1-loop effects. In each case they reduce the classical result. We believe this might arise from the inflationary production of scalars sucking energy from the gravitational sector. Additional support for this supposition comes from the fact that the scalar loop induces a *negative* cosmological constant, which is what required a positive renormalization of the cosmological constant [20], both in order to make the graviton self-energy conserved and so that the constant "H" corresponds to the true Hubble parameter.

Finally, a major reason for making this computation was to facilitate the analysis of graviton loops. It seems inevitable that a finite renormalization of the cosmological constant is also necessary for them. The fact that we have seen here that this matters for the scalar loop contribution points to the need for re-computing the effects of gravitons [12, 13].

Acknowledgements

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6 Appendix: Figures & Tables

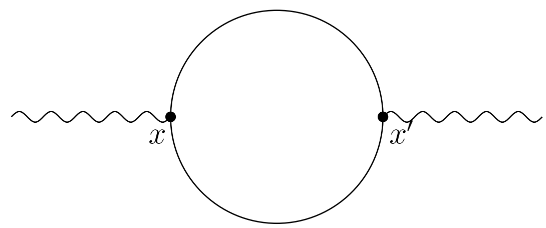

Fig. 1: Contribution from the two 3-point vertices; graviton lines are wavy, scalar lines are solid.

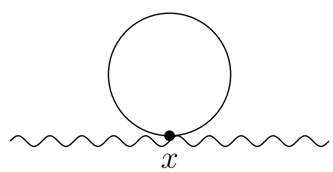

Fig. 2: Contribution from the 4-point vertex; graviton lines are wavy, scalar lines are solid.

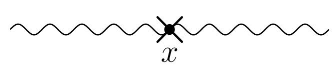

Fig. 3: Contribution from the counterterms; graviton lines are wavy.

i	V^i
1	$\frac{\eta^{\mu\rho}}{\Delta x^D}$
2	$-\frac{D\Delta x^{\mu}\Delta x^{\rho}}{\Delta x^{D+2}}$
3	$\frac{(D-2)[aH\delta^{\mu}{}_0\Delta x^{\rho}-\Delta x^{\mu}a'H\delta^{\rho}{}_0]}{2\Delta x^D}$
4	$\frac{(D-2)aa'H^2\delta^{\mu}{}_0\delta^{\rho}{}_0}{4\Delta x^{D-2}}$
5	$\frac{Daa'H^2\eta^{\mu\rho}}{8\Delta x^{D-2}}$
6	$-\frac{(D-2)Daa'H^2\Delta x^{\mu}\Delta x^{\rho}}{8\Delta x^D}$
7	$\frac{(D-4)Daa'H^2[aH\delta^{\mu}{}_0\Delta x^{\rho}-\Delta x^{\mu}a'H\delta^{\rho}{}_0]}{16\Delta x^{D-2}}$
8	$\frac{(D-4)Da^2a'^2H^4\delta^{\mu}{}_0\delta^{\rho}{}_0}{32\Delta x^{D-4}}$

Table 8: Contributing terms from the doubly-differentiated propagator.

i	T^i
3	$-\frac{(D-2)}{2}\mathcal{T}^3 - \frac{1}{2}\mathcal{T}^9 + \frac{1}{2}\mathcal{T}^{13} - \frac{1}{2}\mathcal{T}^{15}\partial_0 - \frac{1}{2}\mathcal{T}^{17}\partial^2$
4	$-\frac{(D-2)}{2}\mathcal{T}^4 - \frac{1}{2}\mathcal{T}^9 + \frac{1}{2}\mathcal{T}^{13} - \frac{1}{2}\mathcal{T}^{14}\partial_0 - \frac{1}{2}\mathcal{T}^{16}\partial^2$
5	$-\frac{(D-2)}{2}\mathcal{T}^5 - \frac{1}{2}(\mathcal{T}^{10} + \mathcal{T}^{11}) + \frac{1}{2}\mathcal{T}^{14} - \frac{1}{2}\mathcal{T}^{18}\partial_0 - \frac{1}{2}\mathcal{T}^{20}\partial^2$
6	$-\frac{(D-2)}{2}\mathcal{T}^6 - \frac{1}{2}(\mathcal{T}^{10} + \mathcal{T}^{11}) + \frac{1}{2}\mathcal{T}^{15} - \frac{1}{2}\mathcal{T}^{18}\partial_0 - \frac{1}{2}\mathcal{T}^{19}\partial^2$
7	$-\frac{(D-2)}{2}\mathcal{T}^7 - \frac{1}{2}\mathcal{T}^{12} + \frac{1}{2}\mathcal{T}^{16} - \frac{1}{2}\mathcal{T}^{19}\partial_0 - \frac{1}{2}\mathcal{T}^{21}\partial^2$
8	$-\frac{(D-2)}{2}\mathcal{T}^8 - \frac{1}{2}\mathcal{T}^{12} + \frac{1}{2}\mathcal{T}^{17} - \frac{1}{2}\mathcal{T}^{20}\partial_0 - \frac{1}{2}\mathcal{T}^{21}\partial^2$

Table 9: The coefficients T^i as linear combinations of the $\mathcal{T}^i$'s.

i	$\mathcal{T}^i(x;x')$
1	$\frac{\partial^4}{64(D+1)(D-1)(D-2)} + \frac{Daa'H^2\partial^2}{128(D-1)} + \frac{D^2(D-2)^2a^2a'^2H^4}{1024(D-1)}$
2	$\frac{\partial^4}{32(D+1)(D-1)(D-2)} + \frac{Daa'H^2\partial^2}{64(D-1)} + \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
3	$\frac{(D-2)a'^2H^2\partial^2}{64(D-1)}$
4	$\frac{(D-2)a^2H^2\partial^2}{64(D-1)}$
5	$-\frac{a'H\partial^2}{32(D-1)} - \frac{D(D-2)^2aa'^2H^3}{128(D-1)}$
6	$\frac{aH\partial^2}{32(D-1)} + \frac{D(D-2)^2a^2a'H^3}{128(D-1)}$
7	$\frac{D\partial^2}{64(D+1)(D-1)(D-2)} + \frac{D(D-2)aa'H^2}{128(D-1)}$
8	$\frac{D\partial^2}{64(D+1)(D-1)(D-2)} + \frac{D(D-2)aa'H^2}{128(D-1)}$
9	$\frac{D(D-2)^2a^2a'^2H^4}{128}$
10	$-\frac{D(D-2)a^2a'H^3}{64(D-1)}$
11	$\frac{D(D-2)aa'^2H^3}{64(D-1)}$
12	$-\frac{\partial^2}{16(D+1)(D-1)(D-2)} - \frac{Daa'H^2}{32(D-1)}$
13	$\frac{(D-2)^3a^2a'^2H^4}{64}$
14	$-\frac{(D-2)^2a^2a'H^3}{32}$
15	$\frac{(D-2)^2aa'^2H^3}{32}$
16	$\frac{(D-2)^2a^2H^2}{64(D-1)}$
17	$\frac{(D-2)^2a'^2H^2}{64(D-1)}$
18	$-\frac{(D-2)aa'H^2}{16}$
19	$\frac{(D-2)aH}{32(D-1)}$
20	$-\frac{(D-2)a'H}{32(D-1)}$
21	$\frac{D}{64(D+1)(D-1)}$

Table 10: Primitive divergences before including the three trace terms. Act each $\mathcal{T}^i$ on $-\mathcal{K} \times [^{\mu\nu}D_i^{\rho\sigma}] \times i\delta^D(x-x')$.

i	$T^i(x;x')$
3	$-\frac{(D-2)^2 a'^2 H^2 \partial^2}{64(D-1)} - \frac{(D-2)^2 a a'^2 H^3 \partial_0}{64} + \frac{(D-2)^2 (D-4) a^2 a'^2 H^4}{256}$
4	$-\frac{(D-2)^2 a^2 H^2 \partial^2}{64(D-1)} + \frac{(D-2)^2 a^2 a' H^3 \partial_0}{64} + \frac{(D-2)^2 (D-4) a^2 a'^2 H^4}{256}$
5	$\frac{(D-2) a' H \partial^2}{32(D-1)} + \frac{(D-2) a a' H^2 \partial_0}{32} - \frac{(D-2)^2 a^2 a' H^3}{64} + \frac{D(D-2)^3 a a'^2 H^3}{256(D-1)} + \frac{D(D-2) a^2 a'^2 H^4 \Delta\eta}{128(D-1)}$
6	$-\frac{(D-2) a H \partial^2}{32(D-1)} + \frac{(D-2) a a' H^2 \partial_0}{32} + \frac{(D-2)^2 a a'^2 H^3}{64} - \frac{D(D-2)^3 a^2 a' H^3}{256(D-1)} + \frac{D(D-2) a^2 a'^2 H^4 \Delta\eta}{128(D-1)}$
7	$-\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)} - \frac{(D-2) a H \partial_0}{64(D-1)} - \frac{D^2 (D-4) a a' H^2}{256(D-1)} + \frac{(D-2)^2 a^2 H^2}{128(D-1)}$
8	$-\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)} + \frac{(D-2) a' H \partial_0}{64(D-1)} - \frac{D^2 (D-4) a a' H^2}{256(D-1)} + \frac{(D-2)^2 a'^2 H^2}{128(D-1)}$

Table 11: Primitive divergences after including the three trace terms.
Act each T^i on $-\mathcal{K} \times [{}^{\mu\nu} D_i^{\rho\sigma}] \times i\delta^D(x-x')$.

i	$\Delta T_2^i(x;x')$
1	$\frac{\partial^4}{32(D+1)(D-1)^2(D-2)}$
2	$-\frac{\partial^4}{32(D+1)(D-1)(D-2)}$
7	$-\frac{\partial^2}{32(D+1)(D-1)^2(D-2)}$
8	$-\frac{\partial^2}{32(D+1)(D-1)^2(D-2)}$
12	$\frac{\partial^2}{16(D+1)(D-1)(D-2)}$
21	$-\frac{1}{32(D+1)(D-1)^2}$

Table 12: Divergent contributions from the Weyl counterterm (46).
Act each ΔT_2^i on $-\mathcal{K} \times [{}^{\mu\nu} D_i^{\rho\sigma}] \times i\delta^D(x-x')$.

i	$\Delta T^i_{1b}(x;x')$
1	$\frac{D(D-2)aa'H^2\partial^2}{128(D-1)} + \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
2	$-\frac{D(D-2)aa'H^2\partial^2}{128(D-1)} - \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
3	$-\frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
4	$-\frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
5	$\frac{D(D-2)^2a^2a'H^3}{128(D-1)}$
6	$-\frac{D(D-2)^2aa'^2H^3}{128(D-1)}$
7	$-\frac{D(D-2)aa'H^2}{128(D-1)}$
8	$-\frac{D(D-2)aa'H^2}{128(D-1)}$
9	$-\frac{D(D-2)^3a^2a'^2H^4}{256(D-1)}$
10	$\frac{D(D-2)^2aa'^2H^3}{128(D-1)}$
11	$-\frac{D(D-2)^2a^2a'H^3}{128(D-1)}$
12	$\frac{D(D-2)aa'H^2}{64(D-1)}$

Table 13: Divergent contributions from the Einstein counterterm (49). Act each ΔT^i_{1b} on $-\mathcal{K} \times [^{\mu\nu}D^{\rho\sigma}_i] \times i\delta^D(x-x')$.

i	$\Delta T^i_{1a}(x;x')$
1	$-\frac{(D-2)\mathcal{D}_1\mathcal{D}'_1}{64(D-1)^2}$
3	$\frac{(D-2)^2 a'^2 H^2 \mathcal{D}_1}{64(D-1)}$
4	$\frac{(D-2)^2 a^2 H^2 \mathcal{D}'_1}{64(D-1)}$
5	$-\frac{(D-2)a' H \mathcal{D}_1}{32(D-1)}$
6	$\frac{(D-2)aH\mathcal{D}'_1}{32(D-1)}$
7	$\frac{(D-2)\mathcal{D}_1}{64(D-1)^2}$
8	$\frac{(D-2)\mathcal{D}'_1}{64(D-1)^2}$
13	$-\frac{(D-2)^3 a^2 a'^2 H^4}{64}$
14	$\frac{(D-2)^2 a^2 a' H^3}{32}$
15	$-\frac{(D-2)^2 a a'^2 H^3}{32}$
16	$-\frac{(D-2)^2 a^2 H^2}{64(D-1)}$
17	$-\frac{(D-2)^2 a'^2 H^2}{64(D-1)}$
18	$\frac{(D-2)aa'H^2}{16}$
19	$-\frac{(D-2)aH}{32(D-1)}$
20	$\frac{(D-2)a'H}{32(D-1)}$
21	$-\frac{(D-2)}{64(D-1)^2}$

Table 14: Divergent contributions from the Eddington counterterm (57). Act each ΔT^i_{1a} on $-\mathcal{K} \times [^{\mu\nu}D^{\rho\sigma}_i] \times i\delta^D(x-x')$.

i	$T^i(x;x')$
1	$\frac{(D^2-2D-2)\partial^4}{32(D+1)(D-1)(D-2)} + \frac{(3D^3-18D^2+24D-16)aa'H^2\partial^2}{512(D-1)}$ $-\frac{(D-2)(D-3)aa'H^2\partial_0^2}{64(D-1)} + \frac{(D-2)(D-4)(D^4-48D+64)a^2a'^2H^4}{4096(D-1)}$
2	$\frac{\partial^4}{32(D+1)(D-1)(D-2)} + \frac{Daa'H^2\partial^2}{64(D-1)} + \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
3	$-\frac{(D-2)^2a'^2H^2\partial^2}{64(D-1)} - \frac{(D-2)^2aa'^2H^3\partial_0}{64} + \frac{(D-2)^2(D-4)a^2a'^2H^4}{256}$
4	$-\frac{(D-2)^2a^2H^2\partial^2}{64(D-1)} + \frac{(D-2)^2a^2a'H^3\partial_0}{64} + \frac{(D-2)^2(D-4)a^2a'^2H^4}{256}$
5	$\frac{(D-2)a'H\partial^2}{32(D-1)} + \frac{(D-2)aa'H^2\partial_0}{32} - \frac{(D-2)^2a^2a'H^3}{64} + \frac{D(D-2)^3aa'^2H^3}{256(D-1)} + \frac{D(D-2)a^2a'^2H^4\Delta\eta}{128(D-1)}$
6	$-\frac{(D-2)aH\partial^2}{32(D-1)} + \frac{(D-2)aa'H^2\partial_0}{32} + \frac{(D-2)^2aa'^2H^3}{64} - \frac{D(D-2)^3a^2a'H^3}{256(D-1)} + \frac{D(D-2)a^2a'^2H^4\Delta\eta}{128(D-1)}$
7	$-\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)} - \frac{(D-2)aH\partial_0}{64(D-1)} - \frac{D^2(D-4)aa'H^2}{256(D-1)} + \frac{(D-2)^2a^2H^2}{128(D-1)}$
8	$-\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)} + \frac{(D-2)a'H\partial_0}{64(D-1)} - \frac{D^2(D-4)aa'H^2}{256(D-1)} + \frac{(D-2)^2a'^2H^2}{128(D-1)}$
9	$\frac{D(D-2)^2a^2a'^2H^4}{128}$
10	$-\frac{D(D-2)a^2a'H^3}{64(D-1)}$
11	$\frac{D(D-2)aa'^2H^3}{64(D-1)}$
12	$-\frac{\partial^2}{16(D+1)(D-1)(D-2)} - \frac{Daa'H^2}{32(D-1)}$
13	$\frac{(D-2)^3a^2a'^2H^4}{64}$
14	$-\frac{(D-2)^2a^2a'H^3}{32}$
15	$\frac{(D-2)^2aa'^2H^3}{32}$
16	$\frac{(D-2)^2a^2H^2}{64(D-1)}$
17	$\frac{(D-2)^2a'^2H^2}{64(D-1)}$
18	$-\frac{(D-2)aa'H^2}{16}$
19	$\frac{(D-2)aH}{32(D-1)}$
20	$-\frac{(D-2)a'H}{32(D-1)}$
21	$\frac{D}{64(D+1)(D-1)}$

Table 15: Coefficients T^i of primitive divergences in (34). Each term acts on $-\mathcal{K} \times [^{\mu\nu}D_i^{\rho\sigma}] \times i\delta^D(x-x')$.

i	$\Delta T^i(x;x')$
1	$-\frac{(D^2-2D-2)\partial^4}{(D+1)(D-1)(D-2)}+\frac{(D-2)^2aa'H^2\partial^2}{128(D-1)}+\frac{(D-2)(D-3)aa'H^2\partial_0^2}{64(D-1)}+\frac{(D+4)(D-2)^2(D-4)a^2a'^2H^4}{512(D-1)}$
2	$-\frac{\partial^4}{32(D+1)(D-1)(D-2)}-\frac{D(D-2)aa'H^2\partial^2}{128(D-1)}-\frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
3	$\frac{(D-2)^2a'^2H^2\partial^2}{64(D-1)}+\frac{(D-2)^2aa'^2H^3\partial_0}{64}+\frac{(D-2)^2a^2a'^2H^4}{64}-\frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
4	$\frac{(D-2)^2a^2H^2\partial^2}{64(D-1)}-\frac{(D-2)^2a^2a'H^3\partial_0}{64}+\frac{(D-2)^2a^2a'^2H^4}{64}-\frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
5	$-\frac{(D-2)a'H\partial^2}{32(D-1)}-\frac{(D-2)aa'H^2\partial_0}{32}-\frac{(D-2)a^2a'H^3}{32}+\frac{D(D-2)^2a^2a'H^3}{128(D-1)}$
6	$\frac{(D-2)aH\partial^2}{32(D-1)}-\frac{(D-2)aa'H^2\partial_0}{32}+\frac{(D-2)aa'^2H^3}{32}-\frac{D(D-2)^2aa'^2H^3}{128(D-1)}$
7	$\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)}+\frac{(D-2)aH\partial_0}{64(D-1)}+\frac{(D-2)a^2H^2}{64(D-1)}-\frac{D(D-2)aa'H^2}{128(D-1)}$
8	$\frac{(D^2-2D-2)\partial^2}{64(D+1)(D-1)(D-2)}-\frac{(D-2)a'H\partial_0}{64(D-1)}+\frac{(D-2)a'^2H^2}{64(D-1)}-\frac{D(D-2)aa'H^2}{128(D-1)}$
9	$-\frac{D(D-2)^3a^2a'^2H^4}{256(D-1)}$
10	$\frac{D(D-2)^2aa'^2H^3}{128(D-1)}$
11	$-\frac{D(D-2)^2a^2a'H^3}{128(D-1)}$
12	$\frac{\partial^2}{16(D+1)(D-1)(D-2)}+\frac{D(D-2)aa'H^2}{64(D-1)}$
13	$-\frac{(D-2)^3a^2a'^2H^4}{64}$
14	$\frac{(D-2)^2a^2a'H^3}{32}$
15	$-\frac{(D-2)^2aa'^2H^3}{32}$
16	$-\frac{(D-2)^2a^2H^2}{64(D-1)}$
17	$-\frac{(D-2)^2a'^2H^2}{64(D-1)}$
18	$\frac{(D-2)aa'H^2}{16}$
19	$-\frac{(D-2)aH}{32(D-1)}$
20	$\frac{(D-2)a'H}{32(D-1)}$
21	$-\frac{D}{64D+1)(D-1)}$

Table 16: Divergent coefficients ΔT^i of the counterterms in (36). Each term acts on $-\mathcal{K}\times[^{\mu\nu}D_i^{\rho\sigma}]\times i\delta^D(x-x')$.

i	$T^i(x;x') + \Delta T^i(x;x')$
1	$\frac{D(D-4)(3D-2)aa'H^2\partial^2}{512(D-1)} + \frac{D(D-2)(D-4)(D^3+8D-32)a^2a'^2H^4}{4096(D-1)}$
2	$-\frac{D(D-4)aa'H^2\partial^2}{128(D-1)}$
3	$\frac{(D-2)^2(D-4)a^2a'^2H^4}{256} + \frac{(D-2)^2a^2a'^2H^4}{64} - \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
4	$\frac{(D-2)^2(D-4)a^2a'^2H^4}{256} + \frac{(D-2)^2a^2a'^2H^4}{64} - \frac{D^2(D-2)^2a^2a'^2H^4}{512(D-1)}$
5	$-\frac{D^2(D-2)a^2a'H^3}{128(D-1)} + \frac{D(D-2)^3aa'^2H^3}{256(D-1)} + \frac{D(D-2)a^2a'^2H^4\Delta\eta}{128(D-1)}$
6	$\frac{D^2(D-2)aa'^2H^3}{128(D-1)} - \frac{D(D-2)^3a^2a'H^3}{256(D-1)} + \frac{D(D-2)a^2a'^2H^4\Delta\eta}{128(D-1)}$
7	$-\frac{D^2(D-4)aa'H^2}{256(D-1)} + \frac{D(D-2)a^2a'H^3\Delta\eta}{128(D-1)}$
8	$-\frac{D^2(D-4)aa'H^2}{256(D-1)} - \frac{D(D-2)aa'^2H^3\Delta\eta}{128(D-1)}$
9	$\frac{D^2(D-2)^2a^2a'^2H^4}{256(D-1)}$
10	$\frac{D(D-2)(D-4)aa'^2H^3}{128(D-1)} - \frac{D(D-2)a^2a'^2H^4\Delta\eta}{64(D-1)}$
11	$-\frac{D(D-2)(D-4)a^2a'H^3}{128(D-1)} - \frac{D(D-2)a^2a'^2H^4\Delta\eta}{64(D-1)}$
12	$\frac{D(D-4)aa'H^2}{64(D-1)}$
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	0
21	0

Table 17: Sum of Tables 15 and 16. Each term multiplies $-\mathcal{K} \times [{}^{\mu\nu}D_i^{\rho\sigma}] \times i\delta^D(x-x')$.

i	$T^i(x;x') + \Delta T^i(x;x')$
1	$\frac{D(D-4)(3D-2)aa'H^2\partial^2}{512(D-1)} + \frac{D(D-2)(D-4)(D^3+8D-32)a^2a'^2H^4}{4096(D-1)}$
2	$-\frac{D(D-4)aa'H^2\partial^2}{128(D-1)}$
3	$\frac{D(D-2)(D-4)^2a^2a'^2H^4}{512(D-1)}$
4	$\frac{D(D-2)(D-4)^2a^2a'^2H^4}{512(D-1)}$
5	$\frac{D(D-2)^2(D-4)aa'^2H^3}{256(D-1)}$
6	$-\frac{D(D-2)^2(D-4)a^2a'H^3}{256(D-1)}$
7	$-\frac{D^2(D-4)aa'H^2}{256(D-1)}$
8	$-\frac{D^2(D-4)aa'H^2}{256(D-1)}$
9	$\frac{(D+2)D(D-2)(D-4)a^2a'^2H^4}{256(D-1)}$
10	$\frac{D(D-2)(D-4)aa'^2H^3}{128(D-1)}$
11	$-\frac{D(D-2)(D-4)a^2a'H^3}{128(D-1)}$
12	$\frac{D(D-4)aa'H^2}{64(D-1)}$
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	0
21	0

Table 18: Reduction of Table 17 using $\Delta\eta\partial^\mu = \partial^\mu\Delta\eta + \delta^\mu_{\ 0}$.

i	$\lim_{D=4} \frac{1}{D-4}[T^i(x;x') + \Delta T^i(x;x')]$
1	$\frac{5aa'H^2\partial^2}{192} + \frac{a^2a'^2H^4}{24}$
2	$-\frac{aa'H^2\partial^2}{96}$
3	0
4	0
5	$\frac{aa'^2H^3}{48}$
6	$-\frac{a^2a'H^3}{48}$
7	$-\frac{aa'H^2}{48}$
8	$-\frac{aa'H^2}{48}$
9	$\frac{a^2a'^2H^4}{16}$
10	$\frac{aa'^2H^3}{48}$
11	$-\frac{a^2a'H^3}{48}$
12	$\frac{aa'H^2}{48}$
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	0
21	0

Table 19: Final finite residuals. Act each factor on $-\frac{\kappa^2}{2\pi^2} \times [{}^{\mu\nu}D_i^{\rho\sigma}] \times i\delta^4(x-x')$.

i	$\mathcal{T}_A^i(a,a',\partial)$
3	$\frac{aa'^3H^4\partial^4}{64}$
4	$\frac{a^3a'H^4\partial^4}{64}$
5	$-\frac{aa'^2H^3\partial^4}{96}$
6	$\frac{a^2a'H^3\partial^4}{96}$
7	$\frac{a^2a'^2H^4\partial^2}{192}$
8	$\frac{a^2a'^2H^4\partial^2}{192}$
9	$-\frac{a^2a'^2H^4\partial^4}{32}$
10	$\frac{a^2a'H^3\partial^4}{192}$
11	$-\frac{aa'^2H^3\partial^4}{192}$
12	$-\frac{a^2a'^2H^4\partial^2}{96}$
18	$-\frac{a^2a'^2H^4\partial^2}{32}$
19	$\frac{a^2a'H^3\partial^2}{96}$
20	$-\frac{aa'^2H^3\partial^2}{96}$
21	$\frac{aa'H^2\partial^2}{192}-\frac{a^2a'^2H^4}{96}$

Table 20: Non-local contributions acting on $\ln(\mu^2\Delta x^2)$ before including the trace terms.

i	$T^i_A(a,a',\partial)$
1	$\frac{aa'H^2\partial^6}{768} - \frac{a^2a'^2H^4\partial^4}{128} - \frac{a^2a'^2H^4\partial_0^2\partial^2}{128} + \frac{7a^2a'^2H^4\Delta\eta\partial_0\partial^4}{768} - \frac{a^3a'^3H^6\Delta\eta^2\partial^4}{128}$
3	$\frac{a^2a'^3H^5\Delta\eta\partial^4}{64}$
4	$-\frac{a^3a'^2H^5\Delta\eta\partial^4}{64}$
5	$\frac{aa'^2H^3\partial^4}{64} - \frac{a^2a'^2H^4\Delta\eta\partial^4}{384} + \frac{a^2a'^2H^4\partial_0\partial^2}{64}$
6	$-\frac{a^2a'H^3\partial^4}{64} - \frac{a^2a'^2H^4\Delta\eta\partial^4}{384} + \frac{a^2a'^2H^4\partial_0\partial^2}{64}$
7	$-\frac{aa'H^2\partial^4}{384} - \frac{a^2a'H^3\partial_0\partial^2}{192} + \frac{a^2a'^2H^4\partial^2}{192}$
8	$-\frac{aa'H^2\partial^4}{384} + \frac{aa'^2H^3\partial_0\partial^2}{192} + \frac{a^2a'^2H^4\partial^2}{192}$
9	$-\frac{a^2a'^2H^4\partial^4}{32}$
10	$\frac{a^2a'H^3\partial^4}{192}$
11	$-\frac{aa'^2H^3\partial^4}{192}$
12	$-\frac{a^2a'^2H^4\partial^2}{96}$
18	$-\frac{a^2a'^2H^4\partial^2}{32}$
19	$\frac{a^2a'H^3\partial^2}{96}$
20	$-\frac{aa'^2H^3\partial^2}{96}$
21	$\frac{aa'H^2\partial^2}{192} - \frac{a^2a'^2H^4}{96}$

Table 21: Non-local contributions acting on $\ln(\mu^2\Delta x^2)$ in expression (67).

i	$\mathcal{T}_B^i(a,a',\partial)$
1	$-\frac{\partial^4}{3840} - \frac{aa'H^2\partial^2}{192} - \frac{a^2a'^2H^4}{96}$
2	$-\frac{\partial^4}{1920} - \frac{aa'H^2\partial^2}{96} - \frac{a^2a'^2H^4}{48}$
3	$-\frac{a'^2H^2\partial^2}{192}$
4	$-\frac{a^2H^2\partial^2}{192}$
5	$\frac{a'H\partial^2}{192} + \frac{aa'^2H^3}{48}$
6	$-\frac{aH\partial^2}{192} - \frac{a^2a'H^3}{48}$
7	$-\frac{\partial^2}{960} - \frac{aa'H^2}{96}$
8	$-\frac{\partial^2}{960} - \frac{aa'H^2}{96}$
9	$-\frac{a^2a'^2H^4}{16}$
10	$\frac{a^2a'H^3}{48}$
11	$-\frac{aa'^2H^3}{48}$
12	$\frac{\partial^2}{960} + \frac{aa'H^2}{48}$
13	$-\frac{a^2a'^2H^4}{16}$
14	$\frac{a^2a'H^3}{16}$
15	$-\frac{aa'^2H^3}{16}$
16	$-\frac{a^2H^2}{96}$
17	$-\frac{a'^2H^2}{96}$
18	$\frac{aa'H^2}{16}$
19	$-\frac{aH}{96}$
20	$\frac{a'H}{96}$
21	$-\frac{1}{480}$

Table 22: Nonlocal contributions acting on $\partial^2[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}]$ before including the trace terms.

i	$T_B^i(a,a',\partial)$
1	$-\frac{\partial^4}{640}+\frac{aa'H^2\partial_0^2}{64}-\frac{aa'H^2\Delta\eta\partial_0\partial^2}{192}+\frac{a^2a'^2H^4\Delta\eta^2\partial^2}{192}-\frac{a^2a'^2H^4\Delta\eta\partial_0}{48}-\frac{a^2a'^2H^4}{96}$
2	$-\frac{\partial^4}{1920}-\frac{aa'H^2\partial^2}{96}-\frac{a^2a'^2H^4}{48}$
3	$\frac{a'^2H^2\partial^2}{96}+\frac{aa'^2H^3\partial_0}{32}$
4	$\frac{a^2H^2\partial^2}{96}-\frac{a^2a'H^3\partial_0}{32}$
5	$-\frac{a'H\partial^2}{96}-\frac{aa'H^2\partial_0}{32}+\frac{a^2a'H^3}{48}-\frac{aa'^2H^3}{96}$
6	$\frac{aH\partial^2}{96}-\frac{aa'H^2\partial_0}{32}+\frac{a^2a'H^3}{96}-\frac{aa'^2H^3}{48}$
7	$\frac{\partial^2}{640}+\frac{aH\partial_0}{192}-\frac{a^2H^2}{192}$
8	$\frac{\partial^2}{640}-\frac{a'H\partial_0}{192}-\frac{a'^2H^2}{192}$
9	$-\frac{a^2a'^2H^4}{16}$
10	$\frac{a^2a'H^3}{48}$
11	$-\frac{aa'^2H^3}{48}$
12	$\frac{\partial^2}{960}+\frac{aa'H^2}{48}$
13	$-\frac{a^2a'^2H^4}{16}$
14	$\frac{a^2a'H^3}{16}$
15	$-\frac{aa'^2H^3}{16}$
16	$-\frac{a^2H^2}{96}$
17	$-\frac{a'^2H^2}{96}$
18	$\frac{aa'H^2}{16}$
19	$-\frac{aH}{96}$
20	$\frac{a'H}{96}$
21	$-\frac{1}{480}$

Table 23: Non-local contributions acting on $\partial^2[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}]$ in expression (67).

7 Appendix: Integrals for the Potential

The purpose of this section is to evaluate the integrations needed for Section 4.4. These have the generic form:

$$I^J_A(t,r) \equiv \int d^4x' \, \theta(\Delta\eta - \Delta r) \times a'^J \, \Psi_0(t',r') \;\; , \tag{171}$$

$$I^J_B(t,r) \equiv \partial^4 \int d^4x' \, \Bigl\{ \theta(\Delta\eta - \Delta r) \Bigl(\ln[\mu^2(\Delta\eta^2 - \Delta r^2)] - 1 \Bigr) \Bigr\} \times a'^J \, \Psi_0(t',r') \;\; . \tag{172}$$

Relations (75) and (77) constrain I^J_A:

$$\partial^4 I^J_A = 8\pi a^J \Psi_0(t,r) \qquad , \qquad \Bigl(\frac{\partial^2}{a} - 2H\partial_0 \Bigr) I^J_A = \partial^2 I^{J-1}_A \;\; . \tag{173}$$

Similarly, expression (76) implies a relation between I^J_A and I^J_B:

$$-2H\partial_0 \partial^2 I^J_A = I^{J-1}_B - \tfrac{1}{a} I^J_B \; . \tag{174}$$

Both the A-Type and B-Type integrand factors:

$$\theta(\Delta\eta - \Delta r) \qquad , \qquad \partial^4 \Bigl\{ \theta(\Delta\eta - \Delta r) \Bigl(\ln[\mu^2(\Delta\eta^2 - \Delta r^2)] - 1 \Bigr) \Bigr\} \;\; . \tag{175}$$

depend only on the conformal coordinate difference $(x - x')^\mu$, so derivatives can be reflected $\partial_\mu \rightarrow -\partial'_\mu$. Partially integrating with respect to time produces surface terms which we will always avoid. On the other hand, causality compels the integrands to vanish at spatial infinity, so we will partially integrate factors of ∇^2 and take advantage of the simplification:

$$\nabla'^2 \Psi_0(t',r') = -\frac{4\pi G M \delta^3(\vec{x}')}{a'} \; . \tag{176}$$

This implies two additional generic integrations:

$$I^J_{A\delta} \equiv \int d^4x' \, \theta(\Delta\eta - \Delta r) \times a'^J \, \nabla'^2 \Psi_0(t',r') \; , \tag{177}$$

$$I^J_{B\delta} \equiv \partial^4 \int d^4x' \Bigl\{ \theta(\Delta\eta - \Delta r) \Bigl(\ln[\mu^2(\Delta\eta^2 - \Delta r^2)] - 1 \Bigr) \Bigr\} \times a'^J \, \nabla'^2 \Psi_0(t',r') \;\; , \tag{178}$$

obviously related to (171-172):

$$I^J_{A\delta}(t,r) = \nabla^2 I^J_A(t,r) \qquad , \qquad I^J_{B\delta}(t,r) = \nabla^2 I^J_B(t,r) \ . \tag{179}$$

The temporal integrations begin at $\eta_i = -H^{-1}$, and we will assume that the coordinate radius from source to observer obeys:

$$Hr < 1 - \frac{1}{a} \ . \tag{180}$$

- *Evaluation of the I^J_A integral:*
We begin by making the change of variable $\vec{x}' = \vec{x} - \vec{y}$ and then performing the angular integrations:

$$I^J_A = GM \int_{\eta_i}^{\eta} d\eta' \, a'^{J-1} \int d^3y \, \frac{\theta(\Delta\eta - y)}{\|\vec{x} - \vec{y}\|} \tag{181}$$

$$= \frac{2\pi GM}{r} \int_{\eta_i}^{\eta} d\eta' \, a'^{J-1} \int_0^{\Delta\eta} dy \, y \Big[r + y - |r - y| \Big] \ . \tag{182}$$

We now make use of (180) to decompose the η' and y integrations of (182) into regions for which the absolute value $|r - y|$ is either $r - y$ or $y - r$:

$$I^J_A = \frac{4\pi GM}{r} \left\{ \int_{\eta_i}^{\eta - r} d\eta' \, a'^{J-1} \Big[\int_0^r dy \, y^2 + \int_r^{\Delta\eta} dy \, yr \Big] + \int_{\eta - r}^{\eta} d\eta' \, a'^{J-1} \int_0^{\Delta\eta} dy \, y^2 \right\} \tag{183}$$

$$= \frac{4\pi GM}{r} \left\{ \int_{\eta_i}^{\eta - r} d\eta' \, a'^{J-1} \Big[\tfrac{1}{2} r \, \Delta\eta^2 - \tfrac{1}{6} r^3 \Big] + \int_{\eta - r}^{\eta} d\eta' \, a'^{J-1} \, \tfrac{1}{3} \Delta\eta^3 \right\} \ . \tag{184}$$

In acting ∂^2 it is useful to exploit the identity:

$$\partial^2 f(\eta, r) = \frac{1}{r} (\partial_r - \partial_0)(\partial_r + \partial_0) [r f(\eta, r)] \ . \tag{185}$$

It follows that:

$$\partial^2 I^J_A = -\frac{8\pi GM}{r} \left\{ r \int_{\eta_i}^{\eta - r} d\eta' \, a'^{J-1} + \int_{\eta - r}^{\eta} d\eta' \, a'^{J-1} \Delta\eta \right\} \ . \tag{186}$$

Proceeding similarly, it is straightforward to check relations (173).

- *Evaluation of the I_B^J integral:*
The initial reduction of $I_B^J(t,r)$ is the same as that of I_A^J, except for the logarithm and the external derivatives:

$$I_B^J = \frac{4\pi GM}{r}(\partial_r^2 - \partial_0^2)^2 \times \Bigg\{ \int_{\eta_i}^{\eta-r} d\eta'\, a'^{J-1} \Big[\int_0^r dy\, y^2 \Big(\ln[\mu^2(\Delta\eta^2 - y^2)] - 1 \Big) + \int_r^{\Delta\eta} dy\, yr \times \Big(\ln[\mu^2(\Delta\eta^2 - y^2)] - 1 \Big) \Big] + \int_{\eta-r}^{\eta} d\eta'\, a'^{J-1} \int_0^{\Delta\eta} dy\, y^2 \Big(\ln[\mu^2(\Delta\eta^2 - y^2)] - 1 \Big) \Bigg\} \; , \quad (187)$$

so that we get:

$$I_B^J = \frac{4\pi GM}{r}(\partial_r^2 - \partial_0^2)^2 \times \Bigg\{ \int_{\eta_i}^{\eta-r} d\eta'\, a'^{J-1} \Big[-\tfrac{1}{3}(\Delta\eta^3 - r^3)\ln[\mu(\Delta\eta - r)] + \tfrac{1}{3}(\Delta\eta^3 + r^3)\ln[\mu(\Delta\eta + r)] - \tfrac{2}{3}\Delta\eta^2 r - \tfrac{5}{9}r^3 \Big] + \int_{\eta_i}^{\eta-r} d\eta'\, a'^{J-1} r(\Delta\eta^2 - r^2)\Big(\tfrac{1}{2}\ln[\mu^2(\Delta\eta^2 - r^2)] - 1 \Big) + \int_{\eta-r}^{\eta} d\eta'\, a'^{J-1} \Delta\eta^3 \Big[\tfrac{2}{3}\ln(2\mu\Delta\eta) - \tfrac{11}{9} \Big] \Bigg\} \; . \quad (188)$$

As a result, I_B^J equals:

$$I_B^J = -\frac{16\pi GM}{r}\Big\{ -2a_r^{J-1}\ln(2\mu r) + (\partial_r - \partial_0)\int_{\eta-r}^{\eta} d\eta'\, a'^{J-1}\ln(2\mu\Delta\eta) \Big\} \; . \quad (189)$$

Here $a_r = (Hr + \frac{1}{a})^{-1}$ is the scale factor evaluated at $\eta' = \eta - r$.

- *Evaluation of the $I^J_{A\delta}$ and $I^J_{B\delta}$ integrals:*
The δ-function integrals (177-178) are rather simple. For the A-type generic integral we have:

$$I^J_{A\delta} = -4\pi GM \int_{\eta_i}^{\eta-r} d\eta' \, a'^{J-1} \quad , \tag{190}$$

while the generic B-type integral is:

$$I^J_{B\delta} = -\frac{4\pi GM}{r}(\partial_r^2 - \partial_0^2)^2 \left\{ r \int_{\eta_i}^{\eta-r} d\eta' \, a'^{J-1} \Bigl(\ln[\mu^2(\Delta\eta^2 - r^2)] - 1 \Bigr) \right\} \tag{191}$$

$$= -\frac{16\pi GM}{r} \left\{ \frac{a_r^{J-1}}{r^2} + \frac{(J-1) H a_r^J}{r} \right\} \quad . \tag{192}$$

- *Particular integrals required:*
We display here the list of the specific integrals needed for the computation of the gravitationally induced potentials in Section 4.4.

(i) For I^J_A we require only $J = 2$, acted on by either ∂^2 or $\partial_0 \partial^2$:

$$\partial^2 I^2_A = -\frac{8\pi GM}{r} \left\{ \tfrac{r}{H} \ln(a_r) + \tfrac{r}{H} - \tfrac{1}{H^2 a} \ln(\tfrac{a}{a_r}) \right\} \quad , \tag{193}$$

$$\partial_0 \, \partial^2 I^2_A = -\frac{8\pi GM}{r} \left\{ \tfrac{1}{H} \ln(\tfrac{a}{a_r}) \right\} \quad . \tag{194}$$

(ii) For I^J_B we only need two values of J:

$$I^0_B = \frac{16\pi GM}{r} \Bigl\{ Hr + \tfrac{1}{a} \ln(2\mu r) \Bigr\} \quad , \quad I^1_B = \frac{16\pi GM}{r} \Bigl\{ \ln(2\mu r) \Bigr\} \, . \tag{195}$$

(iii) For $I^J_{A\delta}$ the only cases we need are $J = 1$ and $J = 2$:

$$I^1_{A\delta} = -\frac{4\pi GM}{r} \times \frac{r}{H} \Bigl[1 - \frac{1}{a_r} \Bigr] \qquad , \qquad I^2_{A\delta} = -\frac{4\pi GM}{r} \times \frac{r}{H} \ln(a_r) \quad . \tag{196}$$

(iv) For $I^J_{B\delta}$ only the cases of $J = 0$ and $J = 1$ are required:

$$I^0_{B\delta} = -\frac{16\pi GM}{r} \times \frac{1}{a r^2} \qquad , \qquad I^1_{B\delta} = -\frac{16\pi GM}{r} \times \frac{1}{r^2} \quad . \tag{197}$$

References

[1] C. L. Bennett, et.al., Astrophys. J. Suppl. **208** (2013) 20B, arXiv:1212.5225; G. F. Hinshaw, et.al., Astrophys. J. Suppl. **208** (2013) 19H, arXiv:1212.5226; P. A. R. Ade, et. al., arXiv:1403.4302

[2] L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825-838 (1974)

[3] A. A. Starobinsky, JETP Lett. **30**, 682-685 (1979)

[4] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. **19**, 4607 (2002) doi:10.1088/0264-9381/19/17/311 [arXiv:gr-qc/0204065 [gr-qc]].

[5] T. Prokopec, O. Tornkvist and R. P. Woodard, Annals Phys. **303**, 251-274 (2003) doi:10.1016/S0003-4916(03)00004-6 [arXiv:gr-qc/0205130 [gr-qc]].

[6] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 044019 (2006) doi:10.1103/PhysRevD.74.044019 [arXiv:gr-qc/0602110 [gr-qc]].

[7] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 024021 (2006) doi:10.1103/PhysRevD.74.024021 [arXiv:gr-qc/0603135 [gr-qc]].

[8] E. O. Kahya and V. K. Onemli, Phys. Rev. D **76**, 043512 (2007) doi:10.1103/PhysRevD.76.043512 [arXiv:gr-qc/0612026 [gr-qc]].

[9] T. Prokopec, N. C. Tsamis and R. P. Woodard, Phys. Rev. D **78**, 043523 (2008) doi:10.1103/PhysRevD.78.043523 [arXiv:0802.3673 [gr-qc]].

[10] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. **31**, 175002 (2014) doi:10.1088/0264-9381/31/17/175002 [arXiv:1308.3453 [gr-qc]].

[11] C. L. Wang and R. P. Woodard, Phys. Rev. D **91**, no.12, 124054 (2015) doi:10.1103/PhysRevD.91.124054 [arXiv:1408.1448 [gr-qc]].

[12] L. Tan, N. C. Tsamis and R. P. Woodard, Phil. Trans. Roy. Soc. Lond. A **380**, 0187 (2021) doi:10.1098/rsta.2021.0187 [arXiv:2107.13905 [gr-qc]].

[13] L. Tan, N. C. Tsamis and R. P. Woodard, Universe **8**, no.7, 376 (2022) doi:10.3390/universe8070376 [arXiv:2206.11467 [gr-qc]].

[14] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357-6368 (1994) doi:10.1103/PhysRevD.50.6357 [arXiv:astro-ph/9407016 [astro-ph]].

[15] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **724**, 295-328 (2005) doi:10.1016/j.nuclphysb.2005.06.031 [arXiv:gr-qc/0505115 [gr-qc]].

[16] S. P. Miao, N. C. Tsamis and R. P. Woodard, JHEP **03**, 069 (2022) doi:10.1007/JHEP03(2022)069 [arXiv:2110.08715 [gr-qc]].

[17] R. P. Woodard and B. Yesilyurt, JHEP **06**, 206 (2023) doi:10.1007/JHEP06(2023)206 [arXiv:2302.11528 [gr-qc]].

[18] C. Litos, R. P. Woodard and B. Yesilyurt, Phys. Rev. D **108**, no.6, 065001 (2023) doi:10.1103/PhysRevD.108.065001 [arXiv:2306.15486 [gr-qc]].

[19] S. Park and R. P. Woodard, Phys. Rev. D **83**, 084049 (2011) doi:10.1103/PhysRevD.83.084049 [arXiv:1101.5804 [gr-qc]].

[20] N. C. Tsamis, R. P. Woodard and B. Yesilyurt, Phys. Lett. B **849**, 138472 (2024) doi:10.1016/j.physletb.2024.138472 [arXiv:2312.15913 [gr-qc]].

[21] K. E. Leonard, S. Park, T. Prokopec and R. P. Woodard, Phys. Rev. D **90**, no.2, 024032 (2014) doi:10.1103/PhysRevD.90.024032 [arXiv:1403.0896 [gr-qc]].

[22] S. Park and R. P. Woodard, Phys. Rev. D **84**, 124058 (2011) doi:10.1103/PhysRevD.84.124058 [arXiv:1109.4187 [gr-qc]].

[23] S. Park, T. Prokopec and R. P. Woodard, JHEP **01**, 074 (2016) doi:10.1007/JHEP01(2016)074 [arXiv:1510.03352 [gr-qc]].

[24] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97**, 227-266 (1957) doi:10.1007/BF02392399

[25] K. Hepp, Commun. Math. Phys. **2**, 301-326 (1966) doi:10.1007/BF01773358

[26] W. Zimmermann, Commun. Math. Phys. **11**, 1-8 (1968) doi:10.1007/BF01654298

[27] W. Zimmermann, Commun. Math. Phys. **15**, 208-234 (1969) doi:10.1007/BF01645676

[28] G. 't Hooft and M. J. G. Veltman, Ann. Inst. H. Poincare Phys. Theor. A **20**, 69-94 (1974)

[29] V. K. Onemli and R. P. Woodard, Phys. Rev. D **70**, 107301 (2004) doi:10.1103/PhysRevD.70.107301 [arXiv:gr-qc/0406098 [gr-qc]].

[30] L. Tan, N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. **38**, no.14, 145024 (2021) doi:10.1088/1361-6382/ac0233 [arXiv:2103.08547 [gr-qc]].

[31] J. S. Schwinger, J. Math. Phys. **2**, 407-432 (1961) doi:10.1063/1.1703727

[32] K. T. Mahanthappa, Phys. Rev. **126**, 329-340 (1962) doi:10.1103/PhysRev.126.329

[33] P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4**, 1-11 (1963) doi:10.1063/1.1703883

[34] P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4**, 12-16 (1963) doi:10.1063/1.1703879

[35] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515-1527 (1964)

[36] K. c. Chou, Z. b. Su, B. l. Hao and L. Yu, Phys. Rept. **118**, 1-131 (1985) doi:10.1016/0370-1573(85)90136-X

[37] R. D. Jordan, Phys. Rev. D **33**, 444-454 (1986) doi:10.1103/PhysRevD.33.444

[38] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987) doi:10.1103/PhysRevD.35.495

[39] L. H. Ford and R. P. Woodard, Class. Quant. Grav. **22**, 1637-1647 (2005) doi:10.1088/0264-9381/22/9/011 [arXiv:gr-qc/0411003 [gr-qc]].

[40] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, JHEP **03**, 129 (2024) doi:10.1007/JHEP03(2024)129 [arXiv:2402.05452 [hep-th]].

[41] S. P. Miao, N. C. Tsamis and R. P. Woodard, [arXiv:2405.01024 [gr-qc]].