



# Entropy Fluctuation Formulas of Fermionic Gaussian States

Yuyi Huang and Lu Wei

**Abstract.** We study the statistical behavior of quantum entanglement in bipartite systems over fermionic Gaussian states as measured by von Neumann entropy. The formulas of average von Neumann entropy with and without particle number constraints have been recently obtained, whereas the main results of this work are the exact yet explicit formulas of variances for both cases. For the latter case of no particle number constraint, the results resolve a recent conjecture on the corresponding variance. Different than the existing methods in computing variances over other generic state models, proving the results of this work relies on a new simplification framework. The framework consists of a set of new tools in simplifying finite summations of what we refer to as dummy summation and re-summation techniques. As a by-product, the proposed framework leads to various new transformation formulas of hypergeometric functions.

## Contents

|                                                      |      |
|------------------------------------------------------|------|
| 1. Introduction and Main Results                     | 4284 |
| 1.1. Problem Formulation                             | 4285 |
| Case A: Arbitrary Number of Particles                | 4285 |
| Case B: Fixed Number of Particles                    | 4286 |
| 1.2. Main Results                                    | 4287 |
| 2. Variance Calculation                              | 4292 |
| 2.1. Correlation Functions and Integral Calculations | 4293 |
| Case A: Arbitrary Number of Particles                | 4294 |
| Case B: Fixed Number of Particles                    | 4297 |
| 2.2. Tools for Simplification of Summations          | 4299 |
| 2.2.1. Existing Simplification Framework             | 4299 |
| 2.2.2. New Simplification Framework                  | 4300 |
| 2.3. Simplification of Summations                    | 4314 |
| Case A: Arbitrary Number of Particles                | 4314 |

|                                                                      |      |
|----------------------------------------------------------------------|------|
| Case B: Fixed Number of Particles                                    | 4327 |
| 3. Conclusions                                                       | 4330 |
| Acknowledgements                                                     | 4330 |
| Appendix A: Summation Representations                                | 4330 |
| A.1. Summation Representations of Case A                             | 4330 |
| A.2. Summation Representations of Case B                             | 4332 |
| Appendix B Summation Identities                                      | 4334 |
| B.1. Summation Identities of Existing Simplification Framework       | 4334 |
| B.2. Additional Summation Identities of New Simplification Framework | 4337 |
| Appendix C: Parameters of Correlation Kernel                         | 4339 |
| References                                                           | 4339 |

## 1. Introduction and Main Results

Quantum entanglement is the physical resource that enables quantum computing and other quantum technologies. Understanding the degree of entanglement is crucial in performing any quantum information processing task. Von Neumann entropy is one of the most important quantities to estimate the degree of entanglement of quantum systems. An accurate characterization of the statistical behavior of von Neumann entropy provides a quantitative understanding of the entanglement.

In this work, we study the statistical behavior of entanglement over the fermionic Gaussian states. In the past decades, considerable effort has been devoted to investigating the degree of entanglement as measured by different entanglement entropies over the well-known Hilbert–Schmidt ensemble [12, 16–18, 22, 27, 32, 34, 36, 39–42]. In particular, these studies focus on the statistical behavior of entanglement entropies such as von Neumann entropy [12, 17, 18, 32, 34, 39, 40, 42], quantum purity [16, 22, 36], and Tsallis entropy [27, 41]. Driven by the recent breakthrough in probability theory on the Bures–Hall ensemble [5–7, 14], considerable progress has been made in understanding the von Neumann entropy [21, 35, 43, 44] and quantum purity [8, 21, 31, 36] over the Bures–Hall ensemble. Similar investigations are now being carried out over the fermionic Gaussian ensemble, which is a generic state model relevant for different quantum information processing tasks [2, 3, 24, 25, 30, 37]. Very recently, the mean values of von Neumann entropy with and without particle number constraints over the fermionic Gaussian ensemble are obtained in [2] and [3], respectively. As an important step toward characterizing the statistical distribution of von Neumann entropy, we aim to derive the corresponding variances, which describe the fluctuation of the entropy around their mean values. In particular, we focus on the computation of the exact variance formulas, which are useful in studying quantum systems of finite dimensions. The exact variance of von Neumann entropy over fermionic Gaussian states without particle number constraint has been conjectured in a previous work

of the authors [19]. In the current work, we prove the conjecture and derive a variance formula for the case of fixed particle number.

### 1.1. Problem Formulation

We introduce the formulation that leads to the fermionic Gaussian states with and without particle number constraints. A system of  $N$  fermionic degrees of freedom can be decomposed into two subsystems  $A$  and  $B$  of the dimensions  $m$  and  $n$ , respectively, with  $m + n = N$ . Without loss of generality, we assume  $m \leq n$ . In the present work, we consider two scenarios of fermionic Gaussian states—the fermionic Gaussian states with arbitrary number of particles and the fermionic Gaussian states with a fixed number of particles.

**Case A: Arbitrary Number of Particles.** A system of  $N$  fermionic modes can be formulated in terms of a set of fermionic creation and annihilation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$ ,  $i = 1, \dots, N$ . Since the modes are fermionic, these operators obey the canonical anti-commutation relation [3, 30],

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}\mathbb{I}, \quad \{\hat{a}_i, \hat{a}_j\} = 0 = \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\}, \quad (1)$$

where  $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$  denotes the anti-commutation relation and  $\mathbb{I}$  is an identity operator. Equivalently, one can also describe these fermionic modes in terms of the Majorana operators  $\gamma_l$ ,  $l = 1, \dots, 2N$ , and

$$\hat{\gamma}_{2i-1} = \frac{\hat{a}_i^\dagger + \hat{a}_i}{\sqrt{2}}, \quad \hat{\gamma}_{2i} = \imath \frac{\hat{a}_i^\dagger - \hat{a}_i}{\sqrt{2}} \quad (2)$$

with  $\imath = \sqrt{-1}$  being the imaginary unit. Note that the Majorana operators are Hermitian satisfying the anti-commutation relation

$$\{\hat{\gamma}_l, \hat{\gamma}_k\} = \delta_{lk}\mathbb{I}. \quad (3)$$

By collecting the Majorana operators into a  $2N$  dimensional operator-valued column vector  $\gamma = (\hat{\gamma}_1, \dots, \hat{\gamma}_{2N})^\dagger$ , a fermionic Gaussian state is then written as the density operator of the form [3, 37]

$$\rho(\gamma) = \frac{e^{-\gamma^\dagger Q \gamma}}{\text{tr}(e^{-\gamma^\dagger Q \gamma})}, \quad (4)$$

where the coefficient matrix  $Q$  is a  $2N \times 2N$  imaginary anti-symmetric matrix as the consequence of the anti-commutation relation (3). There always exists an orthogonal matrix  $M$  that diagnoses the coefficient matrix  $Q$  by transforming  $\gamma$  into another Majorana basis  $\mu = (\hat{\mu}_1, \dots, \hat{\mu}_{2N})^\dagger = M\gamma$  [3]. A fermionic Gaussian state is labeled by its anti-symmetric covariance matrix [3]

$$J = -\imath \tanh(Q) = M^T J_0 M, \quad (5)$$

where  $\tanh(x)$  denotes the hyperbolic tangent function [1], and the matrix  $J_0$  takes the block diagonal form

$$J_0 = \begin{pmatrix} \tanh(\lambda_1)\mathbb{A} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \tanh(\lambda_N)\mathbb{A} \end{pmatrix}, \quad (6)$$

and

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (7)$$

We consider the von Neumann entropy as the measure of entanglement between the two subsystems. By restricting the matrix  $J$  to the entries from subsystems  $A$ , the restricted matrix  $J_A$  becomes the  $2m \times 2m$  left-upper block of  $J$ . The von Neumann entropy of a fermionic Gaussian state of case A can be represented in terms of the real positive eigenvalues  $x_i, i = 1, \dots, m$  of  $\imath J_A$  as [2, 3, 19]

$$S = - \sum_{i=1}^m v(x_i), \quad (8)$$

where

$$v(x) = \frac{1-x}{2} \ln \frac{1-x}{2} + \frac{1+x}{2} \ln \frac{1+x}{2}. \quad (9)$$

The resulting joint probability density of the eigenvalues  $x_i, i = 1, \dots, m$  is proportional to [2]

$$\prod_{1 \leq i < j \leq m} (x_i^2 - x_j^2)^2 \prod_{i=1}^m (1 - x_i^2)^{n-m}, \quad x_i \in [0, 1], \quad (10)$$

which is obtained by recursively applying the result in [20, Proposition A.2].

**Case B: Fixed Number of Particles.** For a fermionic Gaussian state  $|F\rangle$  with a fixed particle number  $p$ ,  $m \leq p \leq n$ , the corresponding covariance matrix  $H$  can be expressed via the commutator of fermionic creation and annihilation operators as [3, 24, 25]

$$H_{ij} = -\imath \langle F | \hat{a}_i^\dagger \hat{a}_j - \hat{a}_j \hat{a}_i^\dagger | F \rangle. \quad (11)$$

Recall the canonical anti-commutation relation (1), the entries of the matrix  $H$  are then of the form

$$H_{ij} = -2\imath G_{ij} + \imath \delta_{ij} \mathbb{I}, \quad (12)$$

where  $G_{ij} = \langle F | \hat{a}_i^\dagger \hat{a}_j | F \rangle$  denotes the entries of an  $N \times N$  matrix  $G$  of a fermionic system of  $N$  modes. There exists a unitary transformation  $U$  that diagonalizes  $G$  into the form  $U^\dagger G U$ , where the first  $p$  diagonal elements are equal to 1 and the rest are 0. Therefore, one can write

$$G = U_{N \times p} U_{N \times p}^\dagger. \quad (13)$$

A fermionic Gaussian state of dimension  $N = m + n$  with  $p$  particles can be fully characterized by the matrices  $H$  and  $G$ . The von Neumann entropy of the fermionic system in the case B can be represented as [3, 24, 25]

$$S = - \sum_{i=1}^m v(2y_i - 1), \quad y_i \in [0, 1], \quad (14)$$

where  $y_i, i = 1, \dots, m$  are the eigenvalues of the restricted  $m \times m$  matrix  $G_A = U_{m \times p} U_{m \times p}^\dagger$ . The eigenvalue distribution of the random matrix  $U_{m \times p} U_{m \times p}^\dagger$  is

the well-known Jacobi unitary ensemble [13, 26]. We denote  $x_i$ ,  $i = 1, \dots, m$  the eigenvalues of the  $m \times m$  left-upper block of matrix  $\imath H$ . Changing the variables  $x_i = 2y_i - 1$  in (14) leads to the von Neumann entropy (8) of case B. The resulting joint probability density of the eigenvalues  $x_i$ ,  $i = 1, \dots, m$  is proportional to [4]

$$\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 \prod_{i=1}^m (1 + x_i)^{p-m} (1 - x_i)^{n-p}, \quad x_i \in [-1, 1]. \quad (15)$$

It is important to point out that the joint probability densities (10) and (15) of the considered two cases can be compactly represented by a single joint density as

$$f_{\text{FG}}(x) \propto \prod_{1 \leq i < j \leq m} (x_i^\gamma - x_j^\gamma)^2 \prod_{i=1}^m (1 - x_i)^a (1 + x_i)^b, \quad (16)$$

where for the case A we have

$$\gamma = 2, \quad a = b = n - m \geq 0, \quad x \in [0, 1], \quad (17)$$

and for the case B we have

$$\gamma = 1, \quad a = n - p \geq 0, \quad b = p - m \geq 0, \quad x \in [-1, 1]. \quad (18)$$

We omit the normalizations of the density (16) as they will not be made use of in the subsequent calculations. Note that the variance computation for an arbitrary  $\gamma$  in (16) appears difficult, where one has to consider the case  $\gamma = 2$  in (17) and the case  $\gamma = 1$  in (18) separately.

## 1.2. Main Results

We now introduce the exact mean and variance formulas of von Neumann entropy for both case A and case B. The mean values have been recently computed [2, 3] as summarized in Propositions 1 and 2 for case A and case B, respectively. The corresponding variance formulas are presented in Propositions 3 and 4, which are the main results of the work.

**Proposition 1** ([2]). *For subsystem dimensions  $m \leq n$ , the mean value of the von Neumann entropy (8) of fermionic Gaussian states with arbitrary number of particles (17) is given by*

$$\begin{aligned} \mathbb{E}[S] = & \left(m + n - \frac{1}{2}\right) \psi_0(2m + 2n) + \left(\frac{1}{4} - m\right) \psi_0(m + n) + \left(\frac{1}{2} - n\right) \psi_0(2n) \\ & - \frac{1}{4} \psi_0(n) - m, \end{aligned} \quad (19)$$

where

$$\psi_0(x) = \frac{d \ln \Gamma(x)}{dx} \quad (20)$$

is the digamma function.

**Proposition 2.** [3] For subsystem dimensions  $m \leq n$ , the mean value of the von Neumann entropy (8) of fermionic Gaussian states with a fixed particle number (18) is given by

$$\begin{aligned} \mathbb{E}[S] = & -\frac{m(m+n-p)}{m+n} \psi_0(m+n-p) + (m+n) \psi_0(m+n+1) \\ & - \frac{mp}{m+n} \psi_0(p+1) - n \psi_0(n+1) - m. \end{aligned} \quad (21)$$

**Proposition 3.** For subsystem dimensions  $m \leq n$ , the variance of the von Neumann entropy (8) of fermionic Gaussian states with arbitrary number of particles (17) is given by

$$\begin{aligned} \mathbb{V}[S] = & \left( \frac{1}{2} - m - n \right) \psi_1(2m+2n) + \left( n - \frac{1}{2} \right) \psi_1(2n) + \left( \frac{m(2m+n-1)}{2m+2n-1} - \frac{1}{8} \right) \\ & \times \psi_1(m+n) + \frac{1}{8} \psi_1(n) - \frac{1}{2} (\psi_0(2m+2n) - \psi_0(2n)), \end{aligned} \quad (22)$$

where

$$\psi_1(x) = \frac{d^2 \ln \Gamma(x)}{dx^2} \quad (23)$$

is the trigamma function.

**Proposition 4.** For subsystem dimensions  $m \leq n$ , the variance of the von Neumann entropy (8) of fermionic Gaussian states with a fixed particle number (18) is given by

$$\begin{aligned} \mathbb{V}[S] = & c_0 \psi_1(m+n-p) - (m+n) \psi_1(m+n) + n \psi_1(n) + c_1 \psi_1(p) + c_2 \\ & \times (\psi_0(m+n-p) - \psi_0(p))^2 + c_3 (\psi_0(m+n-p) - \psi_0(p)) - \psi_0(m+n) \\ & + \psi_0(n) + c_4, \end{aligned} \quad (24)$$

where the coefficients  $c_i$  are summarized in Table 1 with  $(a)_n = \Gamma(a+n)/\Gamma(a)$  denoting the Pochhammer symbol.

The proof to Propositions 3 and 4 will be presented in Sect. 2. Note that a special case of equal subsystem dimensions  $m = n$  of Proposition 3 has

TABLE 1. Coefficients of von Neumann entropy variance in Proposition 4

---

|         |                                                |
|---------|------------------------------------------------|
| $c_0 =$ | $\frac{m(m+n-p)(m^2+2mn+n^2-np-1)}{(m+n-1)_3}$ |
| $c_1 =$ | $\frac{mp(m^2+mn+np-1)}{(m+n-1)_3}$            |
| $c_2 =$ | $\frac{mnp(m+n-p)}{(m+n)(m+n-1)_3}$            |
| $c_3 =$ | $-\frac{m(m+1)(m+n-2p)}{(m+n)(m+n)_2}$         |
| $c_4 =$ | $-\frac{m(2m+n+2)}{(m+n)(m+n)_2}$              |

---

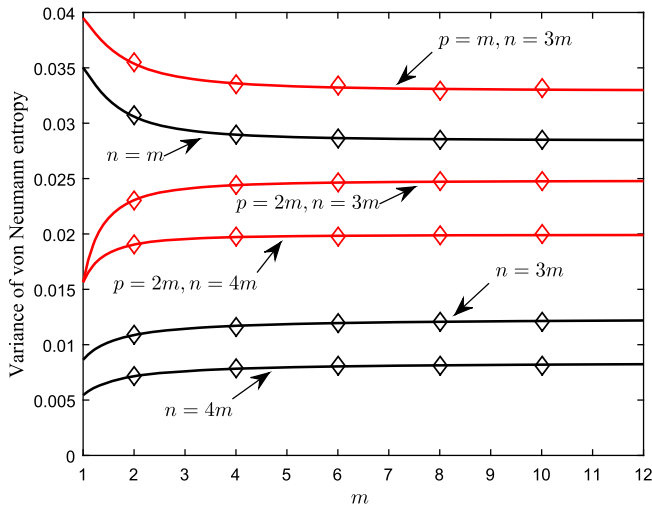


FIGURE 1. Variance of von Neumann entropy: analytical results versus simulations. The black curves represent the obtained analytical result (22) for the cases  $n = m$ ,  $n = 3m$ , and  $n = 4m$ . The red curves are drawn by the result (24) for the cases  $p = m, n = 3m$ ;  $p = 2m, n = 3m$ ; and  $p = 2m, n = 4m$ . The diamond scatters represent numerical simulations

been proved very recently [19] by utilizing an existing simplification framework developed in [18, 19, 21, 40, 42, 44, 45]. However, for the general case of subsystem dimensions  $m \leq n$ , the existing framework is not sufficient to simplify some of the summations in the variance calculation, where a key technical contribution of the work is to develop a new simplification framework. The new simplification framework may also be applied to other fields when performing exact calculations of logarithmic observables.

To illustrate the derived results (22) and (24), we plot in Fig. 1 the exact variance of von Neumann entropy as compared with numerical simulations.<sup>1</sup> We define

$$f_1 = \frac{m}{n+m} \quad (25)$$

$$f_2 = \frac{p}{n+m}, \quad (26)$$

and it is observed in Fig. 1 that the variance in case A approaches to a constant when system dimensions  $m$  and  $n$  increase with a fixed  $f_1$ , while the variance in case B follows the same behavior when the dimensions  $m, n, p$  increase with both  $f_1$  and  $f_2$  kept fixed. This phenomenon can be analytically established by the asymptotic results of variances in the literature. For case A, in the

<sup>1</sup>The simulations performed in Figs. 1, 2, and 3 utilize the Mathematica codes provided by Santosh Kumar based on the log-gas approach as discussed in [35, Appendix B].

asymptotic regime [2]

$$m \rightarrow \infty, \quad n \rightarrow \infty, \quad 0 < f_1 \leq \frac{1}{2}, \quad (27)$$

one has [2]

$$\mathbb{V}[S] = \frac{1}{2} (f_1 + f_1^2 + \ln(1 - f_1)) + o\left(\frac{1}{m+n}\right), \quad (28)$$

whereas for case B, in the asymptotic regime [3]

$$m \rightarrow \infty, \quad p \rightarrow \infty, \quad n \rightarrow \infty, \quad 0 < f_1 \leq f_2 \leq \frac{1}{2}, \quad (29)$$

one has [3]

$$\begin{aligned} \mathbb{V}[S] = & f_1 + f_1^2 + \ln(1 - f_1) + f_1 f_2 (1 - f_1) (1 - f_2) \ln^2 \frac{1 - f_2}{f_2} \\ & + f_1^2 (2f_2 - 1) \ln \frac{1 - f_2}{f_2} + o\left(\frac{1}{(m+n)^2}\right). \end{aligned} \quad (30)$$

The above asymptotic variances (28) and (30) can be directly recovered by the results in Proposition 3 and Proposition 4, respectively. Moreover, the correction terms of any order can be simply obtained from our exact variance formulas upon using the asymptotic behavior of polygamma functions

$$\psi_0(x) = \ln(x) - \frac{1}{2x} - \sum_{l=1}^{\infty} \frac{B_{2l}}{2lx^{2l}}, \quad x \rightarrow \infty, \quad (31)$$

$$\psi_1(x) = \frac{1+2x}{2x^2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{x^{2l+1}}, \quad x \rightarrow \infty, \quad (32)$$

where  $B_k$  is the  $k$ -th Bernoulli number [1]. For example, utilizing the next order of correction, the asymptotic result (30) is refined to

$$\begin{aligned} \mathbb{V}[S] = & f_1^2 + f_1 + \ln(1 - f_1) + f_1 f_2 (1 - f_1) (1 - f_2) \ln^2 \frac{1 - f_2}{f_2} + f_1^2 \\ & \times (2f_2 - 1) \ln \frac{1 - f_2}{f_2} + \frac{1}{12(m+n)^2} \left( \frac{f_1^2}{(f_2 - 1)^2} + \frac{f_1^2}{f_2^2} + 12f_1^2 \right. \\ & - 12f_1 + \frac{1}{(f_1 - 1)^2} + \frac{f_1 - 3f_1^2}{f_2 - 1} + \frac{3f_1^2 - f_1}{f_2} - 1 \\ & + \frac{2(f_1 - 1)f_1(12f_2^3 - 18f_2^2 + 4f_2 + 1)}{(f_2 - 1)f_2} \ln \frac{1 - f_2}{f_2} \\ & \left. + 12(f_1 - 1)f_1(f_2 - 1)f_2 \ln^2 \frac{1 - f_2}{f_2} \right) + o\left(\frac{1}{(m+n)^4}\right). \end{aligned} \quad (33)$$

In the thermodynamic limit, knowing the leading order of variance is relevant to infer whether the average value is typical. On the other hand, for quantum systems with limited number of qubits, such as noisy intermediate-scale quantum (NISQ) systems [33], our finite-size results (22) and (24) become

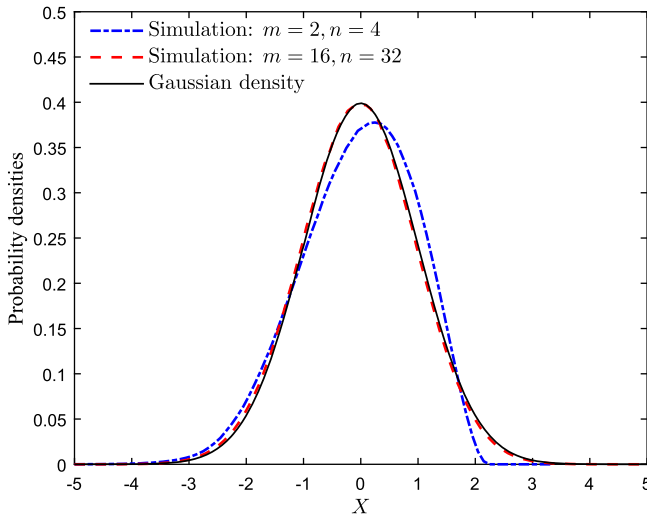


FIGURE 2. Probability densities of standardized von Neumann entropy for case A: a comparison of Gaussian density (35) to simulation results. The dash-dot curve in blue and the dashed curve in red refer to the standardized von Neumann entropy (34) of subsystem dimensions  $m = 2, n = 4$ , and  $m = 16, n = 32$ , respectively. The solid black curve represents the Gaussian density (35) (Color figure online)

more relevant. Moreover, the methodology developed in this work for the exact variance may be also utilized to compute higher-order moments.

To understand the distribution of the von Neumann entropy, simple approximations can now be constructed by using the obtained mean and variance formulas. We first standardize the von Neumann entropy as

$$X = \frac{S - \mathbb{E}[S]}{\sqrt{\mathbb{V}[S]}}, \quad (34)$$

where the random variable  $X$  is of zero mean and unit variance. We now compare the distribution of  $X$  with a standard Gaussian distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad x \in (-\infty, \infty). \quad (35)$$

In Figs. 2 and 3, we plot the simulation results of the standardized von Neumann entropy  $X$  as compared with a standard Gaussian. Specifically, the ratios (25)–(26) are fixed to  $f_1 = 1/3$  for case A in Fig. 2 and  $f_1 = 1/4, f_2 = 1/2$  for case B in Fig. 3. It is observed from the figures that the Gaussian density captures accurately the distribution of the standardized von Neumann entropy  $X$  for moderately large dimensions. We also observe that the true distribution of  $X$  is non-symmetric and appears to be left-skewed when the subsystem dimensions are small as seen from the dash-dot blue curves. In comparison,

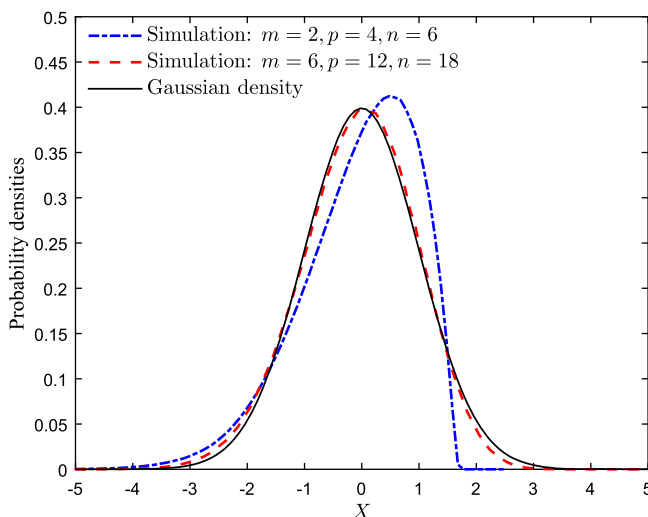


FIGURE 3. Probability densities of standardized von Neumann entropy for case B: a comparison of Gaussian density (35) to simulation results. The dash-dot curve in blue and the dashed curve in red refer to the standardized von Neumann entropy (34) of dimensions  $m = 2$ ,  $p = 4$ ,  $n = 6$ , and  $m = 6$ ,  $p = 12$ ,  $n = 18$ , respectively. The solid black curve represents the Gaussian density (35) (Color figure online)

when the subsystem dimensions become larger, the distribution of  $X$  appears to be closer to the Gaussian distribution. In fact, the Gaussian density as a limiting behavior of von Neumann entropy has been conjectured for different random matrix models of Hilbert–Schmidt ensemble [42], Bures–Hall ensemble [44], and fermionic Gaussian ensemble of an arbitrary number of particles [19]. Here, as motivated by the simulations in Fig. 3, one is also tempted to conjecture that under the asymptotic regime (29), the standardized von Neumann entropy (34) of fermionic Gaussian states with a fixed particle number (18) converges in distribution to a standard Gaussian.

The rest of the paper is organized as follows. The detailed calculations of the main results in Proposition 3 and Proposition 4 are provided in Sect. 2. In Appendix A, we list the summation representations of the integrals involved in the variance computation. Some additional finite sum identities utilized in the simplification are listed in Appendix B.

## 2. Variance Calculation

In this section, we prove the results in Proposition 3 and Proposition 4. In Sect. 2.1, we obtain the summation representations of the variances. Tools in

simplifying these summations are presented in Sect. 2.2, where the new simplification framework consisting of six lemmas is introduced first. The detailed simplification procedures that lead to the claimed results (22) and (24) are discussed in Sect. 2.3.

## 2.1. Correlation Functions and Integral Calculations

Recall the definition (8) of von Neumann entropy

$$S = - \sum_{i=1}^m v(x_i), \quad (36)$$

with

$$v(x) = \frac{1-x}{2} \ln \frac{1-x}{2} + \frac{1+x}{2} \ln \frac{1+x}{2}, \quad (37)$$

computing its variance requires one and two arbitrary eigenvalue densities of the fermionic Gaussian ensemble (16). Denoting  $g_l(x_1, \dots, x_l)$  as the joint density of  $l$  arbitrary eigenvalues, the variance of von Neumann entropy is written as

$$\mathbb{V}[S] = \mathbb{E}[S^2] - \mathbb{E}^2[S], \quad (38)$$

where

$$\mathbb{E}[S^2] = m \int_x v^2(x) g_1(x) dx + m(m-1) \iint_{x,y} v(x)v(y) g_2(x,y) dx dy \quad (39)$$

$$\mathbb{E}[S] = m \int_x v(x) g_1(x) dx. \quad (40)$$

In (39) and (40), the support is  $x, y \in [0, 1]$  for case A, and for case B the support is  $x, y \in [-1, 1]$ .

For the fermionic Gaussian ensemble (16), it is a well-known result in random matrix theory that the joint density  $g_l(x_1, \dots, x_l)$  can be written in terms of an  $l \times l$  determinant as [13, 26]

$$g_l(x_1, \dots, x_l) = \frac{(m-l)!}{m!} \det (K(x_i, x_j))_{i,j=1}^l. \quad (41)$$

The determinant in (41) is known as the  $l$ -point correlation function [13], where

$$K(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{m-1} \frac{J_k^{(a,b)}(x) J_k^{(a,b)}(y)}{h_k} \quad (42)$$

is the correlation kernel with the weight function

$$w(x) = \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b. \quad (43)$$

In (42), the Jacobi polynomial  $J_k^{(a,b)}(x)$  is defined by the following conditions of orthogonality [38]

$$\int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b J_k^{(a,b)}(x) J_l^{(a,b)}(x) dx$$

$$= \frac{2\Gamma(k+a+1)\Gamma(k+b+1)}{(2k+a+b+1)\Gamma(k+1)\Gamma(k+a+b+1)}\delta_{kl}, \quad \Re(a, b) > -1. \quad (44)$$

For convenience, we summarize in Table 3 in Appendix C the parameters  $a$ ,  $b$ , and the degree  $k$  of the polynomials  $J_k^{(a,b)}(x)$  along with the normalization constants  $h_k$  of case A and case B.

By using the joint density (41) and the results (39)–(40), the variance (38) now boils down to computing two integrals involving the 1-point and 2-point correlation functions, cf. [2, 19, 40, 44], as

$$\mathbb{V}[S] = \mathbf{I}_A - \mathbf{I}_B, \quad (45)$$

where

$$\mathbf{I}_A = \int_x v^2(x)K(x, x) \, dx \quad (46)$$

$$\mathbf{I}_B = \iint_{x,y} v(x)v(y)K^2(x, y) \, dx \, dy \quad (47)$$

with  $x, y \in [0, 1]$  for case A and  $x, y \in [-1, 1]$  for case B.

We now compute the above two integrals  $\mathbf{I}_A$  and  $\mathbf{I}_B$  into finite summations. Note that the subsequent calculations of the case A in (17) and case B in (18) are different, which will be performed separately in the following.

**Case A: Arbitrary Number of Particles.** By the definition of the correlation kernel (42) and keeping in mind the parity property of Jacobi polynomials [38]

$$J_k^{(a,b)}(-x) = (-1)^k J_k^{(b,a)}(x), \quad (48)$$

the integral

$$\mathbf{I}_A = \int_0^1 v^2(x)K(x, x) \, dx \quad (49)$$

of fermionic Gaussian states with arbitrary number of particles boils down to computing the two parts

$$\mathbf{I}_A = \mathbf{A}_1 + \mathbf{A}_2, \quad (50)$$

where

$$\mathbf{A}_1 = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^1 \left(\frac{1-x}{2}\right)^a \left(\frac{1+x}{2}\right)^{a+2} \ln^2 \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 \, dx \quad (51)$$

$$\mathbf{A}_2 = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^1 \left(\frac{1-x}{2}\right)^{a+1} \left(\frac{1+x}{2}\right)^{a+1} \ln \frac{1-x}{2} \ln \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 \, dx. \quad (52)$$

Here, we recall that  $a = n - m \geq 0$  denotes the difference of subsystem dimensions.

Similarly, the integral

$$\mathbf{I}_B = \int_0^1 \int_0^1 v(x)v(y)K^2(x, y) \, dx \, dy \quad (53)$$

can be written in terms of the following two integrals

$$I_B = B_1 + B_2, \quad (54)$$

where

$$B_1 = \sum_{k=0}^{m-1} \frac{1}{h_k^2} \left( \int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^{a+1} \ln \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 dx \right)^2 \quad (55)$$

$$B_2 = \sum_{j=1}^{m-1} \sum_{k=0}^{m-j-1} \frac{2}{h_{k+j} h_k} \left( \int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^{a+1} \ln \frac{1+x}{2} \right. \\ \left. \times J_{2k+2j}^{(a,a)}(x) J_{2k}^{(a,a)}(x) dx \right)^2. \quad (56)$$

Computing the above integrals in  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  requires the following two integral identities. The first one is

$$\int_{-1}^1 \left( \frac{1-x}{2} \right)^{a_1} \left( \frac{1+x}{2} \right)^c J_{k_1}^{(a_1,b_1)}(x) J_{k_2}^{(a_2,b_2)}(x) dx \\ = \frac{2(k_1+1)_{a_1}}{(b_2+k_2+1)_{a_2}} \sum_{i=0}^{k_2} \frac{(-1)^{i+k_2} (i+1)_c (i+b_2+1)_{a_2+k_2}}{\Gamma(k_2-i+1) \Gamma(a_1+c+i+k_1+2)} \\ \times (c+i-b_1-k_1+1)_{k_1}, \quad \Re(a_1, a_2, b_1, b_2, c) > -1. \quad (57)$$

To show this identity, we first note that the Jacobi polynomial  $J_k^{(a,b)}(x)$  supported in  $x \in [-1, 1]$  admits different representations [13, 38]

$$J_k^{(a,b)}(x) = \frac{(-1)^k (b+1)_k}{k!} \sum_{i=0}^k \frac{(-k)_i (k+a+b+1)_i}{(b+1)_i \Gamma(i+1)} \left( \frac{1+x}{2} \right)^i \quad (58) \\ = \sum_{i=0}^k \frac{(-1)^i \Gamma(a+k+1) (k+b-i+1)_i}{\Gamma(i+1) \Gamma(a+i+1) \Gamma(k-i+1)} \left( \frac{1-x}{2} \right)^i \left( \frac{1+x}{2} \right)^{k-i}. \quad (59)$$

The identity (57) is then obtained by using the definition (58) for the polynomial  $J_{k_2}^{(a_2,b_2)}$  before applying the well-known integral identity [13, 38]

$$\int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^c J_k^{(a,b)}(x) dx \\ = \frac{2\Gamma(c+1)(k+1)_a (c-b-k+1)_k}{\Gamma(a+c+k+2)}, \quad \Re(a, b, c) > -1. \quad (60)$$

The second integral identity is

$$\int_{-1}^1 \left( \frac{1-x}{2} \right)^d \left( \frac{1+x}{2} \right)^c J_{k_1}^{(a_1,b_1)} J_{k_2}^{(a_2,b_2)}(x) dx \\ = \frac{2\Gamma(a_2+k_2+1) \Gamma(b_2+k_2+1)}{\Gamma(c+d+k_1+k_2+2)} \sum_{i=0}^{k_2} \frac{(-1)^i \Gamma(d-a_1+i+1)}{\Gamma(i+1) \Gamma(a_2+i+1)}$$

$$\begin{aligned} & \times \frac{\Gamma(c - b_1 - i + k_2 + 1)}{\Gamma(k_2 - i + 1)\Gamma(b_2 - i + k_2 + 1)} \sum_{j=0}^{k_1} \frac{(-1)^j (k_1 - j + 1)_{d+i}}{\Gamma(j+1)} \\ & \times \frac{(c - i + j - b_1 - k_1 + k_2 + 1)_{b_1+k_1}}{\Gamma(d - a_1 + i - j + 1)}, \quad \Re(a_1, a_2, b_1, b_2, c, d) > -1, \quad (61) \end{aligned}$$

which is obtained by using the definition (59) for the polynomial  $J_{k_2}^{(a_2, b_2)}$  before applying the identity [19, Equation (62)]

$$\begin{aligned} & \int_{-1}^1 \left(\frac{1-x}{2}\right)^d \left(\frac{1+x}{2}\right)^c J_k^{(a,b)}(x) dx \\ & = \frac{2\Gamma(c-b+1)\Gamma(d-a+1)}{\Gamma(c+d+k+2)} \sum_{i=0}^k \frac{(-1)^i \Gamma(c+i+1)\Gamma(d-i+k+1)}{\Gamma(i+1)\Gamma(k-i+1)} \\ & \quad \times \frac{1}{\Gamma(d-a-i+1)\Gamma(c-b+i-k+1)}, \quad \Re(a, b, c, d) > -1. \quad (62) \end{aligned}$$

The integral in  $A_1$  is now calculated by applying the identity (57), where we need to assign

$$a_1 = b_1 = a_2 = b_2 = a, \quad k_1 = k_2 = 2k \quad (63)$$

and take twice derivatives of  $c$  before setting  $c = a + 2$ . Under the same specialization (63), the integral in  $A_2$  is calculated by taking derivatives of both  $c$  and  $d$  of the identity (61) before setting  $c = d = a + 1$ , while the integral in  $B_1$  is calculated by taking derivative of  $c$  of the identity (57) before setting  $c = a + 1$ . Finally, the integral in  $B_2$  is calculated by specializing

$$a_1 = b_1 = a_2 = b_2 = a, \quad k_1 = 2k + 2j, \quad k_2 = 2k, \quad (64)$$

in the identity (57), and taking derivative of  $c$  before setting  $c = a + 1$ .

In writing down the summation forms of  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$ , one will also have to resolve the indeterminacy by using the following asymptotic expansions of gamma and polygamma functions of negative arguments [1] when  $\epsilon \rightarrow 0$ ,

$$\Gamma(-l + \epsilon) = \frac{(-1)^l}{l! \epsilon} (1 + \psi_0(l+1)\epsilon + o(\epsilon^2)) \quad (65)$$

$$\psi_0(-l + \epsilon) = -\frac{1}{\epsilon} + \psi_0(l+1) + (2\psi_1(1) - \psi_1(l+1))\epsilon + o(\epsilon^2) \quad (66)$$

$$\psi_1(-l + \epsilon) = \frac{1}{\epsilon^2} - \psi_1(l+1) + \psi_1(1) + \zeta(2) + o(\epsilon). \quad (67)$$

The resulting summation forms of  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  are summarized in (A1)–(A4) in Appendix A. It is worth mentioning that a related integral

$$\int_{-1}^1 (1-x^2)^{\lambda+\frac{1}{2}} \ln|x-t| G_m^{(\lambda)}(x)^2 dx \quad (68)$$

is evaluated in [10] by expanding the logarithmic term into Chebyshev polynomials. Here,  $G_m^{(\lambda)}(x)$  denotes the Gegenbauer polynomial, which is a special

case of the Jacobi polynomial as

$$G_m^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} J_m^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x). \quad (69)$$

The approach in [10] may also lead to the summation form of  $B_1$  in (A3).

**Case B: Fixed Number of Particles.** By the definition of correlation kernel (42), the  $I_A$  integral (46) of case B boils down to computing the two parts

$$I_A = \mathcal{A}_1 + \mathcal{A}_2, \quad (70)$$

where

$$\begin{aligned} \mathcal{A}_1 = & \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^1 \left( \left( \frac{1+x}{2} \right)^2 \ln^2 \frac{1+x}{2} + \left( \frac{1-x}{2} \right)^2 \ln^2 \frac{1-x}{2} \right) \\ & \times \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b J_k^{(a,b)}(x)^2 dx \end{aligned} \quad (71)$$

$$\mathcal{A}_2 = \sum_{k=0}^{m-1} \frac{2}{h_k} \int_{-1}^1 \left( \frac{1-x}{2} \right)^{a+1} \left( \frac{1+x}{2} \right)^{b+1} \ln \frac{1-x}{2} \ln \frac{1+x}{2} J_k^{(a,b)}(x)^2 dx. \quad (72)$$

Due to the parity property (48),  $\mathcal{A}_1$  admits a symmetric structure

$$\mathcal{A}_1 = \mathcal{A}_1^{(a,b)} + \mathcal{A}_1^{(b,a)}, \quad (73)$$

where

$$\mathcal{A}_1^{(a,b)} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^{b+2} \ln^2 \frac{1+x}{2} J_k^{(a,b)}(x)^2 dx. \quad (74)$$

The summations in (72) and (74) can be evaluated by using the confluent form of Christoffel–Darboux formula [13]

$$\sum_{k=0}^{m-1} \frac{J_k^{(a,b)}(x)^2}{h_k} = \alpha_1 J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x) - \alpha_2 J_{m-2}^{(a+1,b+1)}(x) J_m^{(a,b)}(x), \quad (75)$$

where

$$\alpha_1 = \frac{m(a+b+m)(a+b+m+1)}{h_{m-1}(a+b+2m-1)_2} \quad (76)$$

$$\alpha_2 = \frac{m(a+b+m)^2}{h_{m-1}(a+b+2m-1)_2}. \quad (77)$$

Consequently, we have

$$\begin{aligned} \mathcal{A}_1^{(a,b)} &= \alpha_1 \int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^{b+2} \ln^2 \frac{1+x}{2} J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x) dx \\ &\quad - \alpha_2 \int_{-1}^1 \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^{b+2} \ln^2 \frac{1+x}{2} J_{m-2}^{(a+1,b+1)}(x) J_m^{(a,b)}(x) dx \end{aligned} \quad (78)$$

$$\begin{aligned} \mathcal{A}_2 &= 2\alpha_1 \int_{-1}^1 \left( \frac{1-x}{2} \right)^{a+1} \left( \frac{1+x}{2} \right)^{b+1} \ln \frac{1-x}{2} \ln \frac{1+x}{2} \\ &\quad \times J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x) dx \\ &\quad - 2\alpha_2 \int_{-1}^1 \left( \frac{1-x}{2} \right)^{a+1} \left( \frac{1+x}{2} \right)^{b+1} \ln \frac{1-x}{2} \ln \frac{1+x}{2} \\ &\quad \times J_{m-2}^{(a+1,b+1)}(x) J_m^{(a,b)}(x) dx. \end{aligned} \quad (79)$$

Similarly, the  $\mathcal{I}_B$  integral (47) can be written in terms of the following two parts

$$\mathcal{I}_B = \mathcal{B}_1 + \mathcal{B}_2, \quad (80)$$

where

$$\begin{aligned} \mathcal{B}_1 &= \sum_{k=0}^{m-1} \frac{1}{h_k^2} \left( \int_{-1}^1 \left( \frac{1-x}{2} \ln \frac{1-x}{2} + \frac{1+x}{2} \ln \frac{1+x}{2} \right) \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b \\ &\quad \times J_k^{(a,b)}(x)^2 dx \right)^2 \end{aligned} \quad (81)$$

$$\begin{aligned} \mathcal{B}_2 &= \sum_{j=1}^{m-1} \sum_{k=0}^{m-j-1} \frac{2}{h_{k+j} h_k} \left( \int_{-1}^1 \left( \frac{1-x}{2} \ln \frac{1-x}{2} + \frac{1+x}{2} \ln \frac{1+x}{2} \right) \right. \\ &\quad \times \left. \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b J_{k+j}^{(a,b)}(x) J_k^{(a,b)}(x) dx \right)^2. \end{aligned} \quad (82)$$

Note that the integrals in  $\mathcal{B}_1$  and  $\mathcal{B}_2$  also admit the same symmetric structure as (73) due to the parity property of Jacobi polynomials. Because of the symmetry, the remaining computation process is similar to that of case A. Specifically, the integrals in  $\mathcal{A}_1^{(a,b)}$  are calculated by taking twice derivative of  $c$  of the identity (57), where one needs to assign

$$a_1 = a, \quad b_1 = b, \quad a_2 = a + 1, \quad b_2 = b + 1, \quad k_1 = k_2 = m - 1, \quad (83)$$

and

$$a_1 = a, \quad b_1 = b, \quad a_2 = a + 1, \quad b_2 = b + 1, \quad k_1 = m, \quad k_2 = m - 2, \quad (84)$$

corresponding to, respectively, the first and second integrals in (78) before setting  $c = b + 2$ . By the result (79), the two integrals in  $\mathcal{A}_2$  are calculated by

taking derivatives of  $c$  and  $d$  of identity (61), respectively, with the specializations (83) and (84), before setting  $c = b + 1$ ,  $d = a + 1$ . The integral

$$\int_{-1}^1 \frac{1+x}{2} \ln \frac{1+x}{2} \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b J_k^{(a,b)}(x)^2 dx \quad (85)$$

in  $\mathcal{B}_1$  is calculated by specializing

$$a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad k_1 = k_2 = k \quad (86)$$

in the identity (57), and taking derivative of  $c$  before setting  $c = b + 1$ . The integral

$$\int_{-1}^1 \frac{1+x}{2} \ln \frac{1+x}{2} \left( \frac{1-x}{2} \right)^a \left( \frac{1+x}{2} \right)^b J_{k+j}^{(a,b)}(x) J_k^{(a,b)}(x) dx \quad (87)$$

in  $\mathcal{B}_2$  is calculated by specializing

$$a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad k_1 = k + j, \quad k_2 = k \quad (88)$$

in the identity (57), and taking derivative of  $c$  before setting  $c = b + 1$ .

After resolving the indeterminacy of gamma and polygamma functions by using (65)–(67), one arrives at the summation representations (A5)–(A10) as listed in Appendix A.

## 2.2. Tools for Simplification of Summations

The major task in obtaining the variance formulas in Propositions 3 and 4 is to simplify the summation representations (A1)–(A10) into the closed-form results (22) and (24). We summarize the necessary simplification tools in this section, where the existing simplification framework is briefly reviewed in Sect. 2.2.1 and the new simplification framework is introduced in Sect. 2.2.2.

**2.2.1. Existing Simplification Framework.** The existing simplification tools have been utilized in various moments calculations over different ensembles, including the Hilbert–Schmidt ensemble [18, 40, 42, 45], the Bures–Hall ensemble [21, 44], and the fermionic Gaussian ensemble [19].

In the existing framework, we have two types of finite sum identities. The first type is of the form

$$\sum_{i=1}^m i^c \psi_{j_1}^{b_1}(i + a_1) \psi_{j_2}^{b_2}(i + a_2) \cdots \psi_{j_m}^{b_m}(i + a_m), \quad (89)$$

where  $a, b, c, j$  are nonnegative integers. The main idea in deriving the identities of this type of sums is to change the summation orders and make use of the obtained lower-order summation formulas in a recursive manner. For example, the summation

$$\sum_{i=1}^m \psi_0^2(i + a) \quad (90)$$

will be derived by using the identity (B1) of the lower-order summation  $\sum_{i=1}^m \psi_0(i+a)$ . Specifically, by using the finite sum form of the digamma function

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \quad (91)$$

the summation (90) can be rewritten as

$$\psi_0(a) \sum_{i=1}^m \psi_0(i+a) + \sum_{j=1}^m \frac{1}{a+j-1} \sum_{i=j}^m \psi_0(i+a), \quad (92)$$

where we have changed the summation order of the double sum. The remaining sums can be simplified by using the lower-order identity (B1), leading to the result in (B6).

The second type of summations is of the form

$$S_f(m, n) = \sum_{i=1}^m \frac{(n-i)!}{(m-i)!} f(i), \quad m \leq n, \quad (93)$$

where  $f(i)$  could be the product of polygamma and rational functions in  $i$ . A well-known result of the second-type summation is the identity

$$\sum_{i=1}^m \frac{(n-i)!}{(m-i)!} = \frac{n!}{(m-1)!(n-m+1)!}, \quad (94)$$

which is a special case of the Chu–Vandermonde identity [23]. In the existing simplification framework, the identity (94) is a fundamental result in obtaining several other second-type summations. For example, when

$$f(i) = \frac{1}{i}, \quad (95)$$

the summation

$$S_{1/i}(m, n) = \sum_{i=1}^m \frac{(n-i)!}{(m-i)!} \frac{1}{i} \quad (96)$$

is computed by first obtaining the recurrence relation

$$S_{1/i}(m, n) = \frac{n}{m} S_{1/i}(m-1, n-1) + \frac{n-m}{m} \sum_{i=1}^m \frac{(n-1-i)!}{(m-i)!}. \quad (97)$$

After recurring  $m$  times, and using the existing result (94), one obtains the closed-form result of the summation (96) as listed in (B19).

**2.2.2. New Simplification Framework.** The existing simplification framework is useful in simplifying the summations in (A1), (A3), (A5), and (A9). However, the framework is insufficient to simplify some of the summations in (A2), (A4), (A7), and (A10). The main technique difficulty is that these summations, after exhausting all the possibilities of changing the order of summations, are not related to the first- or second-type summations of the existing

framework. To convert these sums into the ones of the existing framework, the following new simplification framework is needed.

The first technique in the new framework is what we refer to as “dummy summation.” The idea of the dummy summation is as follows. For a summation  $F$ ,

$$F = \sum_i f_i, \quad (98)$$

where

$$\sum_i = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_a} \quad (99)$$

denotes a finite nested sum of  $a$  indexes  $i = \{i_1, i_2, \dots, i_a\}$ . Each index  $i_k$ ,  $k = 1, 2, \dots, a$  may depend on its previous ones  $i_1, i_2, \dots, i_{k-1}$ . For the summation  $F$  that is not simplifiable under the existing framework by any changes of the order of summations, one may interpret the summand  $f_i$  into an additional nested finite sum over a set of new indexes  $j = \{j_1, j_2, \dots, j_b\}$  as

$$f_i = \sum_j g_{i,j}, \quad (100)$$

such that the resulting representation

$$F = \sum_{i,j} g_{i,j} \quad (101)$$

admits further simplifications by using the existing tools of first- and second-type sums (89) and (93) after appropriately changing some of the summation orders. Note that the indexes in the set  $j$  in (100) may depend on the ones in the set  $i$ .

To illustrate this new technique, we consider the following two examples that will be utilized in the simplifications in Sect. 2.3. The first example is the summation

$$F = \sum_{i_1=1}^m f_{i_1}, \quad (102)$$

where

$$f_{i_1} = i_1 \psi_0(i_1 + a). \quad (103)$$

In this case, we have  $i = \{i_1\}$  in the definition (98). By interpreting  $i_1$  in the summand (103) as a dummy sum

$$i_1 = \underbrace{1 + 1 + \cdots + 1}_{i_1}, \quad (104)$$

the original single sum in (102) now becomes a double sum

$$F = \sum_{i_1=1}^m \sum_{j_1=1}^{i_1} g_{i_1, j_1}, \quad (105)$$

where

$$g_{i_1, j_1} = \psi_0(i_1 + a) \quad (106)$$

with  $j = \{j_1\}$  in the definition (101). After changing the summation order in (105) as

$$F = \sum_{j_1=1}^m \sum_{i_1=j_1}^m \psi_0(i_1 + a), \quad (107)$$

the sum over  $i_1$  can be simplified by using an existing identity (B1), leading to

$$F = \frac{1-a}{2} \sum_{j_1=1}^m \psi_0(j_1 + a) + \frac{m}{4} (2(a+m)\psi_0(a+m+1) - m - 1). \quad (108)$$

Simplifying the single summation in (108) by the identity (B1) directly gives the result listed in (B2). In the above example, the dummy summation (104) converts the original sum (102) into a double sum (105), which appears more complicated but is simplifiable using an available identity (B1) of the existing framework. Note also that the dummy summation technique may not be critical in this example, which can be derived in other ways.

In the second example, we introduce a dummy summation (100) that is crucial as a subroutine in simplifying some of the summations in (A2) and (A7). The essential idea in creating this dummy summation is to interpret a ratio of gamma functions

$$\frac{\Gamma(i)}{\Gamma(c+i)} = \frac{1}{i(i+1)\cdots(i+c-1)}, \quad c, i \in \mathbb{Z}^+ \quad (109)$$

as its partial fraction decomposition

$$a_1 \frac{1}{i} + a_2 \frac{1}{i+1} + \cdots + a_c \frac{1}{i+c-1} \quad (110)$$

with  $a_j, j = 1, 2, \dots, c$  denoting the coefficients of the decomposition. The dummy summation that corresponds to the above interpretation is

$$\frac{\Gamma(i)}{\Gamma(c+i)} = \sum_{j=1}^c a_j \frac{1}{i+j-1}, \quad (111)$$

where

$$a_j = \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)}. \quad (112)$$

The coefficients  $a_j$  are computed by evaluating a unit-argument hypergeometric function [9]

$$\sum_{j=1}^c a_j \frac{1}{i+j-1} = \frac{1}{\Gamma(c+1)} {}_2F_1(-c, i-1; i; 1), \quad (113)$$

where one utilizes the well-known identity [9]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c) > \Re(a+b). \quad (114)$$

The dummy summation in (111) is critical in proving Lemmas 2–4 as discussed later in this section. By utilizing these lemmas, the summations in (A2) and (A7) involving ratios of gamma functions then become simplifiable under the existing simplification framework as will be shown in Sect. 2.3.

The second technique in the new framework is what we refer to as the “re-summation” technique. The idea of this new technique is as follows. For a finite summation  $G_m$  that appears not summable under the existing framework with  $m$  denoting one of its parameters or the finite upper limit of the summation. The re-summation technique aims to find alternative forms of the summation  $G_m$  to reveal the potential cancellations with other sums, such that the remaining terms can be simplified by the existing framework. Specifically, the re-summation of  $G_m$  can be generated by iterating a suitably chosen recurrence relation

$$G_m = c_{m-1}G_{m-1} + r_{m-1}, \quad (115)$$

where  $c_i$  denotes the coefficient and  $r_i$  is the remainder of the recurrence relation. Here, the remainder can contain summations, see the example (122). Each iteration is to replace the term  $G_{m-i}$  with its previous one  $G_{m-i-1}$ . Keep iterating until  $G_{m-i}$  vanishes, we then obtain an alternative form of  $G_m$ , which is considered a useful one if it facilitates the cancellation with other sums.

To illustrate the re-summation technique, we consider the difference of two summations

$$\mathcal{G} = G_m - G'_m, \quad (116)$$

where

$$G_m = \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1)(a-i+2m+1)} \quad (117)$$

$$G'_m = \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1)i}. \quad (118)$$

We first notice that the two summations in (117) and (118) cannot be simplified individually by using the identities of the existing framework. In addition, these two summations do not appear to cancel directly in (116). By iterating  $m$  times the following tailor-made recurrence relations (in terms of the various choices of coefficients and reminders), cf. (115),

$$G_m = c_{m-1}G_{m-1} + r_{m-1} \quad (119)$$

$$G'_m = c_{m-1}G'_{m-1} + r'_{m-1}, \quad (120)$$

where

$$c_{m-1} = \frac{a+m}{m} \quad (121)$$

$$r_{m-1} = -\frac{a}{m} \sum_{i=1}^{m+1} \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+2)} + \frac{a+m}{m} \left( \frac{\Gamma(a+m)}{\Gamma(m)(a+2m-1)} - \frac{\Gamma(a+m+1)}{\Gamma(m+1)(a+2m)} \right) \quad (122)$$

$$r'_{m-1} = \frac{a}{m} \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m)}{\Gamma(m-i+1)}, \quad (123)$$

we obtain the re-summation results

$$G_m = \frac{a\Gamma(a+m+1)}{\Gamma(m+1)} \sum_{i=1}^m \sum_{j=1}^{i+1} \frac{(-1)^{j+1} \Gamma(i) \Gamma(a+i-j+1)}{\Gamma(i-j+2) \Gamma(a+i+1)} + \frac{\Gamma(a+m+1)}{\Gamma(m+1)} \sum_{i=1}^m \left( \frac{1}{a+2i-1} - \frac{a+i}{i(a+2i)} \right) \quad (124)$$

$$G'_m = \frac{a\Gamma(a+m+1)}{\Gamma(m+1)} \sum_{i=1}^m \sum_{j=2}^{i+1} \frac{(-1)^{j+1} \Gamma(i) \Gamma(a+i-j+1)}{\Gamma(i-j+2) \Gamma(a+i+1)}. \quad (125)$$

The relationship between the two summations now becomes obvious. After inserting the results (124) and (125) into (116), the cancellation occurs directly between the two double summations in (124) and (125). The remaining terms give the closed-form result

$$\begin{aligned} & \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1)} \left( \frac{1}{a-i+2m+1} - \frac{1}{i} \right) \\ &= \frac{\Gamma(a+m+1)}{\Gamma(m+1)} (\psi_0(a+2m+1) - \psi_0(a+m+1)), \end{aligned} \quad (126)$$

which is a special case of Lemma 5 when  $c = 0$ . This result is also utilized in simplifying the double summations in (A10). Note that the emphasis here is the inner cancellation among multiple sums when utilizing re-summation technique. On the other hand, this technique concerning only one summation has been studied in [18, 40, 42].

Using the above-discussed two techniques of the new simplification framework, we obtain the following six lemmas. More precisely, proving Lemma 1 utilizes the re-summation technique. Lemmas 2–4 are obtained by using the dummy summation together with the result of Lemma 1. Lemmas 5 and 6 are established by using the re-summation technique. In the six lemmas obtained by the new framework, each single sum is converted to another one, which will be applied in Sect. 2.3 to convert summations involving multiple ratios of gamma functions into the ones that are simplifiable by the existing framework.

**Lemma 1.** *For any complex numbers  $a, b, c \notin \mathbb{Z}^-$ , we have*

$$\sum_{i=1}^m \frac{1}{\Gamma(i) \Gamma(a+i) \Gamma(m+1-i) \Gamma(m+b+1-i) (c+i)}$$

$$= \frac{1}{\Gamma(b+m)\Gamma(c+m+1)\Gamma(a+b+m)} \sum_{i=1}^m \frac{\Gamma(c-i+m+1)\Gamma(a+b-i+2m)}{\Gamma(m-i+1)\Gamma(a-i+m+1)}. \quad (127)$$

*Proof.* Proving Lemma 1 uses the re-summation technique. The left side of the identity (127) can be rewritten as

$$G_m = \sum_{i=1}^m \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i)\Gamma(m+b+1-i)(m-i)(c+i)}. \quad (128)$$

In (128), by performing a partial fraction decomposition

$$\frac{1}{(m-i)(c+i)} = \frac{1}{c+m} \left( \frac{1}{c+i} + \frac{1}{m-i} \right), \quad (129)$$

we obtain the recurrence relation, cf. (115),

$$G_m = c_{m-1}G_{m-1} + r_{m-1}, \quad (130)$$

where

$$c_{m-1} = \frac{1}{c+m} \quad (131)$$

$$r_{m-1} = \frac{1}{c+m} \sum_{i=1}^m \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(m+b+1-i)}. \quad (132)$$

The summation in (132) is related to a hypergeometric function  ${}_2F_1(1-b-m, 1-m; a+1; 1)$  that admits a closed-form representation by using the identity (114). Therefore, we have

$$\begin{aligned} & \sum_{i=1}^m \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(m+b+1-i)} \\ &= \frac{\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)}, \end{aligned} \quad (133)$$

and

$$r_{m-1} = \frac{\Gamma(a+b+2m-1)}{(c+m)\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)}. \quad (134)$$

Iterate  $m$  times the recurrence relation (130), the desired identity (127) is established. This completes the proof of Lemma 1.  $\square$

**Lemma 2.** For any complex numbers  $a, b \notin \mathbb{Z}^-$ , and any  $c \in \mathbb{Z}^+$ , we have

$$\begin{aligned} & \sum_{i=1}^m \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(m+1-i)\Gamma(m+b+1-i)} \\ &= \frac{1}{\Gamma(m+b)\Gamma(m+a+b)\Gamma(c)\Gamma(m+c)} \\ & \quad \times \sum_{i=1}^m \frac{\Gamma(m+a+b+i-1)\Gamma(m+c-i)}{\Gamma(a+i)\Gamma(m-i+1)}. \end{aligned} \quad (135)$$

*Proof.* We prove Lemma 2 by using the dummy summation technique. Denote

$$F = \sum_{i=1}^m f_i \quad (136)$$

with

$$f_i = \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b+m-i+1)}. \quad (137)$$

We first introduce an additional summation in (136) by replacing  $\Gamma(i)/\Gamma(c+i)$  with the summation form in (111). The summation  $F$  now becomes

$$F = \sum_{i=1}^m \sum_{j=1}^c g_{i,j} \quad (138)$$

with

$$g_{i,j} = \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)(i+j-1)} \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b+m-i+1)}. \quad (139)$$

In the double summation (138), we apply Lemma 1 to first evaluate the sum over  $i$ , leading to

$$\begin{aligned} F &= \frac{1}{\Gamma(b+m)\Gamma(a+b+m)} \sum_{i=1}^m \frac{\Gamma(a+b-i+2m)}{\Gamma(m-i+1)\Gamma(a-i+m+1)} \\ &\quad \times \sum_{j=1}^c \frac{(-1)^{j+1}\Gamma(m-i+j)}{\Gamma(j)\Gamma(c-j+1)\Gamma(j+m)}, \end{aligned} \quad (140)$$

where it is now possible to evaluate the sum over  $j$  as

$$\begin{aligned} &\sum_{j=1}^c \frac{(-1)^{j+1}\Gamma(m-i+j)}{\Gamma(j)\Gamma(c-j+1)\Gamma(j+m)} \\ &= \frac{\Gamma(m-i+1)}{\Gamma(c)\Gamma(m+1)} {}_2F_1(1-c, m-i+1; m+1; 1). \end{aligned} \quad (141)$$

$$= \frac{\Gamma(c+i-1)\Gamma(m-i+1)}{\Gamma(c)\Gamma(i)\Gamma(c+m)}. \quad (142)$$

The result (142) is obtained by using the identity (114) to evaluate the unit argument hypergeometric function in (141). Inserting (142) into (140) and shifting the index  $i \rightarrow m-i+1$ , Lemma 2 is proved.  $\square$

**Lemma 3.** For any complex numbers  $a, b \notin \mathbb{Z}^-$ , and any  $c \in \mathbb{Z}^+$ , we have

$$\begin{aligned} &\sum_{i=1}^m \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)i} \\ &= \frac{1}{\Gamma(a)\Gamma(a+m)\Gamma(1+b+m)\Gamma(b+c+m)} \sum_{i=1}^m \frac{\Gamma(a-i+m)\Gamma(b+c+i+m)}{\Gamma(c+i)\Gamma(m-i+1)i} \end{aligned}$$

$$+ \frac{\psi_0(a) - \psi_0(a+m)}{\Gamma(a)\Gamma(c)\Gamma(m+1)\Gamma(b+m+1)}. \quad (143)$$

*Proof.* Similar to the proof of Lemma 2, we introduce an additional summation in

$$F = \sum_{i=1}^m \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)i} \quad (144)$$

by replacing the gamma ratio  $\Gamma(i)/\Gamma(c+i)$  with the dummy summation (111). Consequently, one has

$$F = \sum_{i=1}^m \sum_{j=1}^c g_{i,j} \quad (145)$$

with

$$g_{i,j} = \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)(i+j-1)} \frac{1}{\Gamma(i+1)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)}, \quad (146)$$

which is the same form as the summand in (139). Therefore, one uses the results (138)–(141) to obtain the identity (143), and Lemma 3 is proved.  $\square$

**Lemma 4.** For any complex numbers  $a, b \notin \mathbb{Z}^-$ , and any  $c, d \in \mathbb{Z}^+$ , we have

$$\begin{aligned} & \sum_{i=1}^m \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(d+m-i+1)\Gamma(b+m-i+1)} \\ &= \frac{1}{\Gamma(d)\Gamma(a+m)\Gamma(a+b+m)\Gamma(c+d+m)} \sum_{i=1}^m \frac{\Gamma(c+d+i-1)\Gamma(a+b-i+2m)}{\Gamma(c+i)\Gamma(b-i+m+1)} \\ &+ \frac{1}{\Gamma(c)\Gamma(b+m)\Gamma(a+b+m)\Gamma(c+d+m)} \sum_{i=1}^m \frac{\Gamma(c+d+i-1)\Gamma(a+b-i+2m)}{\Gamma(d+i)\Gamma(a-i+m+1)}. \end{aligned} \quad (147)$$

*Proof.* Proving Lemma 4 also utilizes the dummy summation technique. Denote the left side of the identity (147) as

$$F = \sum_{i=1}^m f_i, \quad (148)$$

with

$$f_i = \frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(d+m-i+1)\Gamma(b+m-i+1)}. \quad (149)$$

By interpreting, respectively, the gamma ratios  $\Gamma(i)/\Gamma(c+i)$  and  $\Gamma(m-i+1)/\Gamma(d+m-i+1)$  as the dummy summations (111) and

$$\sum_{k=1}^d \frac{(-1)^{k+1}}{\Gamma(k)\Gamma(d-k+1)(m-i+k)}, \quad (150)$$

the summation  $F$  can be written as

$$F = \sum_{i=1}^m \sum_{j=1}^c \sum_{k=1}^d g_{i,j,k}, \quad (151)$$

where

$$g_{i,j,k} = \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\ \times \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)(i+j-1)} \frac{(-1)^{k+1}}{\Gamma(k)\Gamma(d-k+1)(m-i+k)}. \quad (152)$$

After taking a partial fraction decomposition

$$\frac{1}{(m-i+k)(i+j-1)} = \frac{1}{(i+j-1)(j+k+m-1)} \\ + \frac{1}{(k+m-i)(j+k+m-1)} \quad (153)$$

in (152), the summation (151) is split into two sums

$$F = F_1 + F_2, \quad (154)$$

where

$$F_1 = \sum_{j=1}^c \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)} \sum_{k=1}^d \frac{(-1)^{k+1}}{\Gamma(k)\Gamma(d-k+1)(j+k+m-1)} \\ \times \sum_{i=1}^m \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)(i+j-1)} \quad (155)$$

$$F_2 = \sum_{k=1}^d \frac{(-1)^{k+1}}{\Gamma(k)\Gamma(d-k+1)} \sum_{j=1}^c \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)(j+k+m-1)} \\ \times \sum_{i=1}^m \frac{1}{\Gamma(i)\Gamma(b+i)\Gamma(m-i+1)\Gamma(a-i+m+1)(i+k-1)}. \quad (156)$$

Specialize  $c = d$  and  $i = j + m$  in the identity (111), one has

$$\sum_{k=1}^d \frac{(-1)^{k+1}}{\Gamma(k)\Gamma(d-k+1)(j+k+m-1)} = \frac{\Gamma(j+m)}{\Gamma(d+j+m)}. \quad (157)$$

Insert the above result into (155), and evaluate the resulting summation over  $i$  by the identity (127), one arrives at

$$F_1 = \frac{1}{\Gamma(b+m)\Gamma(a+b+m)} \sum_{j=1}^c \frac{(-1)^{j+1}}{\Gamma(j)\Gamma(c-j+1)\Gamma(d+j+m)} \\ \times \sum_{i=1}^m \frac{\Gamma(m+j-i)\Gamma(a+b-i+2m)}{\Gamma(m-i+1)\Gamma(a-i+m+1)}, \quad (158)$$

where the summation over  $j$  is simplified, by using the identity (114), to

$$\begin{aligned} & \sum_{j=1}^c \frac{(-1)^{j+1} \Gamma(m-i+j)}{\Gamma(j) \Gamma(c-j+1) \Gamma(d+j+m)} \\ &= \frac{\Gamma(m-i+1)}{\Gamma(c) \Gamma(d+m+1)} {}_2F_1(1-c, m-i+1; d+m+1; 1) \end{aligned} \quad (159)$$

$$= \frac{\Gamma(m-i+1) \Gamma(c+d+i-1)}{\Gamma(c) \Gamma(d+i) \Gamma(c+d+m)}. \quad (160)$$

Consequently, we have

$$\begin{aligned} F_1 &= \frac{1}{\Gamma(c) \Gamma(b+m) \Gamma(a+b+m) \Gamma(c+d+m)} \\ &\quad \times \sum_{i=1}^m \frac{\Gamma(c+d+i-1) \Gamma(a+b-i+2m)}{\Gamma(d+i) \Gamma(a-i+m+1)}. \end{aligned} \quad (161)$$

In the same manner, we obtain

$$\begin{aligned} F_2 &= \frac{1}{\Gamma(d) \Gamma(a+m) \Gamma(a+b+m) \Gamma(c+d+m)} \\ &\quad \times \sum_{i=1}^m \frac{\Gamma(c+d+i-1) \Gamma(a+b-i+2m)}{\Gamma(c+i) \Gamma(b-i+m+1)}. \end{aligned} \quad (162)$$

Insert the results (161)–(162) into (154), we complete the proof of Lemma 4.  $\square$

**Lemma 5.** Denote

$$\Phi_{a,b,c,d}^{(x)} = \frac{\Gamma(x+c+1) \Gamma(x+d+1)}{\Gamma(x+a+1) \Gamma(x+b+1)}, \quad (163)$$

we have

$$\begin{aligned} & \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1)} \left( \frac{1}{a+c-i+2m+1} - \frac{1}{c+i} \right) \\ &= \frac{\Gamma(a+c+m+1)}{\Gamma(c+m+1)} \sum_{i=1}^m \left( \frac{\Phi_{0,a+c,a,c}^{(m-i)}}{a+c-2i+2m+1} + a \Phi_{1,a+c+1,a,c}^{(m-i)} \right. \\ &\quad \left. - \frac{\Phi_{1,a+c,a+1,c}^{(m-i)}}{a+c-2i+2m+2} \right), \end{aligned} \quad (164)$$

where it is sufficient to consider  $\Re(a, c) \geq 0$ .

*Proof.* We prove Lemma 5 by using the re-summation technique. We denote the left side of the identity (164) as

$$\mathcal{G} = G_m - G'_m, \quad (165)$$

where

$$G_m = \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1) (a+c-i+2m+1)} \quad (166)$$

$$G'_m = \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m+1)}{\Gamma(m-i+1)(c+i)}. \quad (167)$$

Here, we choose the recurrence relations

$$G_m = c_{m-1}G_{m-1} + r_{m-1} \quad (168)$$

$$G'_m = c_{m-1}G'_{m-1} + r'_{m-1}, \quad (169)$$

where

$$c_{m-1} = \frac{a+c+m}{c+m} \quad (170)$$

$$\begin{aligned} r_{m-1} &= \frac{a}{c+m} \sum_{i=0}^m \frac{(-1)^i \Gamma(a-i+m)}{\Gamma(m-i+1)} + \frac{(a+c+m)\Gamma(a+m)}{(c+m)\Gamma(m+1)} \\ &\quad \times \left( \frac{m}{a+c+2m-1} - \frac{a+m}{a+c+2m} \right) \end{aligned} \quad (171)$$

$$r'_{m-1} = \frac{a}{c+m} \sum_{i=1}^m \frac{(-1)^i \Gamma(a-i+m)}{\Gamma(m-i+1)}. \quad (172)$$

Iterating  $m$  times the recurrence relations (168) and (169), we obtain

$$\begin{aligned} G_m &= \frac{\Gamma(a+c+m+1)}{\Gamma(c+m+1)} \left( a \sum_{i=1}^m \sum_{j=0}^{m-i+1} (-1)^j \Phi_{a+c+1,1-j,c,a-j}^{(m-i)} \right. \\ &\quad \left. + \sum_{i=1}^m \left( \frac{\Phi_{0,a+c,c,a}^{(m-i)}}{a+c-2i+2m+1} - \frac{\Phi_{1,a+c,c,a+1}^{(m-i)}}{a+c-2i+2m+2} \right) \right) \end{aligned} \quad (173)$$

$$G'_m = \frac{a\Gamma(a+c+m+1)}{\Gamma(c+m+1)} \sum_{i=1}^m \sum_{j=1}^{m-i+1} (-1)^j \Phi_{a+c+1,1-j,c,a-j}^{(m-i)}. \quad (174)$$

It is noticed that the double summation in (173) is the same form as the one in (174). By inserting the re-summations (173) and (174) into (165), we obtain the desired identity (164). This proves Lemma 5.  $\square$

**Lemma 6.** *Using the notation (163), we have*

$$\begin{aligned} &\sum_{i=1}^m \Phi_{0,a,b,a+b}^{(m-i)} \left( \frac{1}{a+b+c-i+2m+1} - \frac{1}{c+i} \right) \\ &= \Phi_{0,a,b,a+b}^{(m+c)} \sum_{i=1}^m \Phi_{0,a,b,a+b}^{(i-1)} \Phi_{b,a+b,0,a}^{(c+i-1)} \left( \frac{1}{a+b+c+2i-1} - \frac{b(a+b)}{ai(a+b+c+i)} \right. \\ &\quad \left. - \frac{b(a-b)}{a(a+i)(b+c+i)} + \frac{(b+i)(a+b+i)}{i(a+i)(a+b+c+2i)} - \frac{a+b+2i-2}{(b+i-1)(a+b+i-1)} \right) \\ &\quad + \frac{1}{b} \Phi_{0,a,b,a+b}^{(m-1)}, \end{aligned} \quad (175)$$

where it is sufficient to consider  $a, b, c, a+b, a+b+c \notin \mathbb{Z}^-$ .

*Proof.* The proof of Lemma 6 uses the re-summation technique. The left side of the identity (175) can be written as

$$\mathcal{G} = G_m - G'_m, \quad (176)$$

where

$$G_m = \sum_{i=1}^m \Phi_{0,a,b,a+b}^{(m-i)} \frac{1}{a+b+c-i+2m+1} \quad (177)$$

$$G'_m = \sum_{i=1}^m \Phi_{0,a,b,a+b}^{(m-i)} \frac{1}{c+i}. \quad (178)$$

In relating the two summations above, we choose the following recurrence relations

$$G_m = c_{m-1}G_{m-1} + r_{m-1} \quad (179)$$

$$G'_m = c_{m-1}G'_{m-1} + r'_{m-1}, \quad (180)$$

where

$$c_{m-1} = \frac{(b+c+m)(a+b+c+m)}{(c+m)(a+c+m)} \quad (181)$$

$$\begin{aligned} r_{m-1} = c_{m-1} & \left( \frac{\Phi_{0,a,b,a+b}^{(m-1)}}{a+b+c+2m-1} + \frac{\Phi_{0,a,b,a+b}^{(m)}}{a+b+c+2m} \right) - \frac{b}{a}c_{m-1} \\ & \times \left( \frac{(a-b)}{b+c+m} \sum_{i=1}^m \Phi_{0,a+1,b,a+b}^{(m-i)} + \frac{(a+b)}{a+b+c+m} \sum_{i=1}^{m+1} \Phi_{1,a,b,a+b}^{(m-i)} \right) \end{aligned} \quad (182)$$

$$r'_{m-1} = \frac{b(a-b)}{a(a+c+m)} \sum_{i=1}^{m-1} \Phi_{0,a+1,b,a+b}^{(m-i-1)} + \frac{b(a+b)}{a(c+m)} \sum_{i=1}^m \Phi_{1,a,b,a+b}^{(m-i-1)}. \quad (183)$$

Iterating  $m$  times the above relations (179) and (180), one obtains

$$\begin{aligned} G_m = & \Phi_{0,a,b,a+b}^{(m+c)} \left( -\frac{b(a+b)}{a} \sum_{i=1}^m \Phi_{1,a,b,a+b}^{(i-2)} \sum_{j=1}^{m-i+1} \Phi_{a+b+1,b,0,a}^{(m-j+c)} - \frac{b(a-b)}{a} \right. \\ & \times \sum_{i=1}^m \Phi_{0,a+1,b,a+b}^{(i-1)} \sum_{j=1}^{m-i+1} \Phi_{b+1,a+b,a,0}^{(m-j+c)} + \sum_{i=1}^m \Phi_{b,a+b,0,a}^{(m-i+c)} \\ & \times \left( \frac{\Phi_{0,a,b,a+b}^{(m-i+1)}}{a+b+c-2i+2m+2} + \frac{\Phi_{0,a,b,a+b}^{(m-i)}}{a+b+c-2i+2m+1} \right. \\ & \left. \left. - \frac{b(a+b)\Phi_{1,a,b,a+b}^{(m-i)}}{a(a+b+c-i+m+1)} \right) \right) \end{aligned} \quad (184)$$

$$G'_m = \Phi_{0,a,b,a+b}^{(m+c)} \left( \frac{b(a+b)}{a} \sum_{i=1}^m \Phi_{1,a,b,a+b}^{(i-2)} \sum_{j=1}^{m-i+1} \Phi_{a+b+1,b+1,0,a+1}^{(m-j+c)} \right.$$

$$+ \frac{b(a-b)}{a} \sum_{i=1}^m \Phi_{0,a+1,b,a+b}^{(i-2)} \sum_{j=1}^{m-i+1} \Phi_{b+1,a+b+1,a,1}^{(m-j+c)} \Bigg). \quad (185)$$

Note that in order to reveal the potential cancellations between the two re-summations above, one will also need the identity

$$\begin{aligned} & (c_1 - b_1) \sum_{j=1}^m \Phi_{a_1,b_1+1,c_1,d_1+1}^{(m-j)} + (d_1 - a_1 + 1) \sum_{j=1}^m \Phi_{a_1,b_1,c_1,d_1}^{(m-j)} \\ &= \Phi_{a_1-1,b_1,c_1,d_1}^{(m)} - \Phi_{a_1-1,b_1,c_1,d_1}^{(0)}, \quad a_1, b_1, c_1, d_1 \notin \mathbb{Z}^-. \end{aligned} \quad (186)$$

This identity is obtained by iterating  $m$  times the following recurrence relation

$$H_{m,a_1,d_1} = H_{m-1,a_1+1,d_1+1} + s_{m-1}, \quad (187)$$

where

$$H_{m,a_1,d_1} = \sum_{j=1}^m \Phi_{a_1,b_1,c_1,d_1}^{(m-j)} \quad (188)$$

$$\begin{aligned} s_{m-1} &= H_{m,a_1,d_1} - H_{m-1,a_1+1,d_1+1} \\ &= (c_1 - b_1) \sum_{j=1}^{m-1} \Phi_{a_1,b_1,c_1-1,d_1}^{(m-j)} + \Phi_{a_1,b_1,c_1,d_1}^{(0)}. \end{aligned} \quad (189)$$

We now use the identity (186) with the specializations

$$\begin{aligned} a_1 &= a + b + c + i, \quad b_1 = b + c + i - 1, \quad c_1 = c + i - 1, \\ d_1 &= a + c + i - 1, \quad m \rightarrow m - i + 1 \end{aligned} \quad (190)$$

and

$$\begin{aligned} a_1 &= b + c + i, \quad b_1 = a + b + c + i - 1, \quad c_1 = a + c + i - 1, \\ d_1 &= c + i - 1, \quad m \rightarrow m - i + 1, \end{aligned} \quad (191)$$

and the result (185) becomes

$$\begin{aligned} G'_m &= \Phi_{0,a,b,a+b}^{(c+m)} \left( -\frac{b(a+b)}{a} \sum_{i=1}^m \Phi_{1,a,b,a+b}^{(i-2)} \sum_{j=1}^{m-i+1} \Phi_{a+b+1,b,0,a}^{(m-j+c)} - \frac{b(a-b)}{a} \right. \\ &\quad \times \sum_{i=1}^m \Phi_{0,a+1,b,a+b}^{(i-2)} \sum_{j=1}^{m-i+1} \Phi_{b+1,a+b,a,0}^{(m-j+c)} + \sum_{i=1}^m \left( \Phi_{a+b,b,0,a}^{(c+i-1)} - \Phi_{a+b,b,0,a}^{(c+m)} \right) \\ &\quad \left. \times \left( \frac{a-b}{a} \Phi_{0,a+1,b,a+b}^{(i-2)} + \frac{a+b}{a} \Phi_{1,a,b,a+b}^{(i-2)} \right) \right), \end{aligned} \quad (192)$$

where the double summations are of the same form as the ones in (184). Therefore, all the double summations are cancelled completely in (176). The remaining terms are the desired result (175). This completes the proof of Lemma 6.  $\square$

Note that the results in Lemmas 2–4 are analytically continued to any complex number  $c$  or  $d$  except for negative integers. This fact allows us to

take derivatives of the identities (135), (143), and (147) in obtaining the identities (B27)–(B33) listed in Appendix B.2. For example, the identity (B31) is established by taking derivatives of the identity (147) with respect to  $c$  and  $d$  before setting  $c = d = 0$ . Lemmas 1–4 along with the identities (B27)–(B33) are useful in simplifying the summations in (A2) and (A7). It is also worth mentioning that the identities in Lemma 5 and Lemma 6 admit closed-form representations for some special cases, for example, when the parameter  $c$  is a fixed nonnegative integer as the gamma functions involved reduce to rational functions. These identities as well as their derivatives with respect to  $c$  are the key tools in simplifying the summations in (A4) and (A10).

Before we move on to the simplification process in the next section, we give an overview on the structure of the relevant expressions in computing the variance and higher-order moments. The variance computation involves one- and two-point correlation functions, which are calculated by using Lemmas 1–4 and Lemmas 5–6, respectively. Moreover, the tools of the new simplification framework in principle work for computing higher-order moments, where one will encounter logarithmic terms of higher powers. Specifically, computing the logarithmic terms over one-point correlation functions may require higher-order derivatives of Lemmas 1–4, leading to the appearance of higher-order polygamma functions. For the computation over arbitrary correlation functions, one would need some new lemmas, which could be obtained in a parallel manner as in Lemmas 5–6 by using the re-summation technique.

By rewriting the summations in Lemmas 1–4 in terms of hypergeometric functions, we obtain relations between hypergeometric functions of unit argument as listed in (193)–(196), which may be of independent interest.

$$\begin{aligned} & {}_3F_2(c+1, 1-m, 1-b-m; a+1, c+2; 1) \\ &= \frac{(c+1)\Gamma(a+1)\Gamma(a+b+2m-1)}{(c+m)\Gamma(a+m)\Gamma(a+b+m)} \\ &\quad \times {}_3F_2(1, 1-m, 1-a-m; 2-a-b-2m, 1-c-m; 1) \end{aligned} \quad (193)$$

$$\begin{aligned} & {}_3F_2(1, 1-m, 1-b-m; a+1, c+1; 1) \\ &= \frac{c}{c+m-1} {}_3F_2(1, 1-m, a+b+m; a+1, 2-c-m; 1) \end{aligned} \quad (194)$$

$$\begin{aligned} & {}_4F_3(1, 1, 1-m, 1-b-m; 2, a+1, c+1; 1) \\ &= \frac{a(b+c+m)}{(a+m-1)(b+m)} {}_4F_3(1, 1, 1-m, b+c+m+1; 2, c+1, 2-a-m; 1) \\ &\quad + \frac{ac(\psi_0(a) - \psi_0(a+m))}{m(b+m)} \end{aligned} \quad (195)$$

$$\begin{aligned} & \frac{{}_3F_2(1, 1-b-m, 1-d-m; a+1, c+1; 1)}{\Gamma(a+1)\Gamma(c+1)\Gamma(b+m)\Gamma(d+m)} \\ &= \frac{{}_3F_2(1, 1-b, 1-d; a+m+1, c+m+1; 1)}{\Gamma(b)\Gamma(d)\Gamma(a+m+1)\Gamma(c+m+1)} - \frac{1}{a+b+m-1} \\ &\quad \times \left( \frac{{}_3F_2(1, 1-a, c+d+m; 2-a-b-m, d+m+1; 1)}{\Gamma(a)\Gamma(c)\Gamma(b+m)\Gamma(d+m+1)} \right) \end{aligned}$$

TABLE 2. Integral and summation forms of  $I_A$  and  $I_B$  for both case A and case B

|                                           | $I_A$ | $I_B$ | Summation forms |
|-------------------------------------------|-------|-------|-----------------|
| Case A: Arbitrary number of particle (17) | (50)  | (54)  | (A1)–(A4)       |
| Case B: Fixed number of particle (18)     | (70)  | (80)  | (A5)–(A10)      |

$$\begin{aligned}
& + \frac{{}_3F_2(1, 1-b, c+d+m; 2-a-b-m, c+m+1; 1)}{\Gamma(b)\Gamma(d)\Gamma(a+m)\Gamma(c+m+1)} \Bigg) \\
& + \frac{\Gamma(c+d)\Gamma(a+b+2m-1)}{\Gamma(c)\Gamma(d)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)\Gamma(c+d+m)} \\
& \times \left( \frac{1}{d} {}_3F_2(1, c+d, 1-a-m; d+1, 2-a-b-2m; 1) \right. \\
& \left. + \frac{1}{c} {}_3F_2(1, c+d, 1-b-m; c+1, 2-a-b-2m; 1) \right). \tag{196}
\end{aligned}$$

Though there are considerable results on the relations of unit argument hypergeometric functions in the special function theory, see [11, 15, 23, 29], for example, the formulas (193)–(196) seem new. On the other hand, there exist some transformation formulas of hypergeometric functions that are related to our results. As an example, the following formula in [11]

$$\begin{aligned}
{}_3F_2(a, b, c; d, e; 1) &= \frac{\Gamma(e)\Gamma(d-a-b+c+e)}{\Gamma(e-c)\Gamma(d-a-b+e)} \\
&\times {}_3F_2(d-a, d-b, c; d, d-a-b+e; 1) \tag{197}
\end{aligned}$$

is similar to the formulas (193)–(194). However, for any possible choice of parameters, the known formula (197) does not lead to any of our results.

### 2.3. Simplification of Summations

In this section, we simplify the obtained summation representations (A1)–(A10) that lead to the claimed variance formulas in Proposition 3 and Proposition 4. For convenience, we summarize in Table 2 the integral and summation forms of  $I_A$  and  $I_B$  of the variance (45) for case A and case B.

**Case A: Arbitrary Number of Particles.** In case A, the variance calculation boils down to simplifying the summation representations of  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  in (A1)–(A4) as listed in Appendix A.1. The corresponding simplification procedures are shown below.

We first discuss the simplifications of summation representations (A1) and (A2) in computing the integral  $I_A$  in (50). The summations in (A1) can be simplified by appropriately changing the order of summations before applying the first-type identities (B1)–(B14) of the existing framework, where the closed-form identities are (B1)–(B9) and the semi-closed-form ones

are (B10)–(B14). The so-defined semi-closed-form identities represent the relation between two single summations. These sums are what we refer to as unsimplifiable basis that will cancel completely in obtaining the final results (22) and (24). Because of the cancellation, whether the considered unsimplifiable basis can be computed into closed-form formulas is not of primary importance.

The simplification of the summations in (A1) is as follows. In (A1), the first double summation is

$$\sum_{k=0}^{m-1} \sum_{j=2k-2}^{2k} \frac{2(-1)^j(2a+4k+1)(j+1)_2(a+j+1)_2}{\Gamma(2k-j+1)\Gamma(j-2k+3)(2a+j+2k+1)_3} \left( (\psi_0(j+3) - \psi_0(2a+j+2k+4) - \psi_0(j-2k+3) + \psi_0(a+j+3))^2 - \psi_1(2a+j+2k+4) + \psi_1(a+j+3) - \psi_1(j-2k+3) + \psi_1(j+3) \right), \quad (198)$$

which is directly reduced to a single sum after evaluating the sum of  $j$ . By using the identities (B1)–(B14) along with the results [1]

$$\psi_0(mk) = \ln m + \frac{1}{m} \sum_{i=0}^{m-1} \psi_0\left(k + \frac{i}{m}\right) \quad (199)$$

$$\psi_1(mk) = \frac{1}{m^2} \sum_{i=0}^{m-1} \psi_1\left(k + \frac{i}{m}\right), \quad m \in \mathbb{Z}^+, \quad (200)$$

the remaining single sum is computed to an unsimplifiable basis of the form

$$\sum_{k=1}^m \frac{\psi_0(k+c)}{k+d}, \quad c \neq d. \quad (201)$$

The other double summation in (A1) is

$$\sum_{k=0}^{m-1} \sum_{j=0}^{2k-3} \frac{4(2a+4k+1)(j+1)_2(a+j+1)_2}{(2k-j-2)_3(2a+j+2k+1)_3} \Psi(j), \quad (202)$$

where

$$\begin{aligned} \Psi(j) &= \psi_0(2a+j+2k+4) - \psi_0(a+j+3) \\ &\quad + \psi_0(2k-j-2) - \psi_0(j+3). \end{aligned} \quad (203)$$

To process the summation (202), we first perform the partial fraction decomposition of the term

$$\begin{aligned} &\frac{4(2a+4k+1)(j+1)_2(a+j+1)_2}{(2k-j-2)_3(2a+j+2k+1)_3} \\ &= -\frac{2(a+k)(2a+2k+1)}{2a+j+2k+2} + \frac{2(a+k)(a+2k-1)(2a+2k-1)}{(2a+4k-1)(2a+j+2k+1)} \\ &\quad + \frac{2(a+k+1)(a+2k+2)(2a+2k+1)}{(2a+4k+3)(2a+j+2k+3)} - \frac{2(k+1)(2k+1)(a+2k+2)}{(2a+4k+3)(j-2k)} \end{aligned}$$

$$-\frac{2k(2k-1)(a+2k-1)}{(2a+4k-1)(j-2k+2)} + \frac{2k(2k+1)}{j-2k+1}. \quad (204)$$

The sum (202) now boils down to summations of the form

$$\sum_{k=0}^{m-1} p_c(k) \sum_{j=0}^{2k-3} \frac{1}{j+2k+2a+c} \Psi(j) \quad (205)$$

$$\sum_{k=0}^{m-1} p'_c(k) \sum_{j=0}^{2k-3} \frac{1}{j-2k-1+c} \Psi(j), \quad (206)$$

where the parameter  $c$  takes the values  $c = 1, 2, 3$ , and  $p_c(k)$ ,  $p'_c(k)$  denote the corresponding rational functions in  $k$  as a result of the partial fraction decomposition (204). When considering  $c = 2$  and the term  $\psi_0(j+3)$  in (203), the double summation

$$-\sum_{k=0}^{m-1} 2(a+k)(2a+2k+1) \sum_{j=0}^{2k-3} \frac{\psi_0(j+3)}{j+2k+2a+2} \quad (207)$$

is simplified to a single sum as

$$\begin{aligned} & \sum_{j=0}^{m-3} \left( \psi_0(2j+3)((j+1)(2j+1)(\psi_0(1) - \psi_0(a+j+m+1)) + (j-m+2) \right. \\ & \quad \times (2a-j+m) + (j+1)(2j+1)(\psi_0(a+2j+3) - \psi_0(1))) + \psi_0(2j+4) \\ & \quad \times \left( (j+1)(2j+3) \left( \psi_0(1) - \psi_0\left(a+j+m+\frac{3}{2}\right) \right) + (j+1)(2j+3) \right. \\ & \quad \times \left. \left. \left( \psi_0\left(a+2j+\frac{7}{2}\right) - \psi_0(1) \right) + (2a-j+m-1)(j-m+2) \right) \right). \end{aligned} \quad (208)$$

The above result is obtained as follows. In (207), it is noticed that directly evaluating the inner summation does not permit further simplifications. Instead, we first have to separate the inner summation that depends on an even index  $2k$  into two as

$$\begin{aligned} & -2 \sum_{k=0}^{m-1} (a+k)(2a+2k+1) \\ & \quad \left( \sum_{j=0}^{k-2} \frac{\psi_0(2j+3)}{2a+2k+2j+2} + \sum_{j=0}^{k-2} \frac{\psi_0(2j+4)}{2a+2k+2j+3} \right), \end{aligned} \quad (209)$$

which allows us to change the summation order as

$$\begin{aligned} & -2 \sum_{j=0}^{m-3} \psi_0(2j+3) \sum_{k=j+2}^{m-1} \frac{(a+k)(2a+2k+1)}{2a+2k+2j+2} \\ & -2 \sum_{j=0}^{m-3} \psi_0(2j+4) \sum_{k=j+2}^{m-1} \frac{(a+k)(2a+2k+1)}{2a+2k+2j+3}. \end{aligned} \quad (210)$$

Evaluating the inner summations over  $k$  in (210) directly gives the result (208). Other double summations in (205) and (206) are similarly simplified into single sums. The resulting single sums, cf. (208), are further manipulated into the unsimplifiable basis of the form (201) by using the first-type identities (B1)–(B14) along with the formula (199).

Different from (A1), simplifying the summations in (A2) requires new simplification tools before the existing framework is applicable. The summations in (A2) include three double sums and one triple sum. We first simplify the inner summations over  $j$  of the three double sums by using the identities (B28) and (B32) obtained from the new simplification framework. Specifically, the inner summation

$$\sum_{j=0}^{2k} \frac{2(j+1)(2k-j+1)\psi_0(j+2)\psi_0(2k-j+2)}{\Gamma(j+1)\Gamma(a+j+1)\Gamma(2k-j+1)\Gamma(a-j+2k+1)} \quad (211)$$

of the first double sum in (A2) is simplified to a semi-closed form that can be compactly written as

$$\begin{aligned} & \frac{1}{\Gamma(2k)\Gamma(a+2k+1)\Gamma(2a+2k+1)} \left( \left( \frac{1-4a^2}{4a+8k-2} + a-8k^2-14k-\frac{15}{2} \right) \right. \\ & \times \sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k)}{\Gamma(a-j+2k)j^2} + \left( \left( \frac{1-4a^2}{4a+8k-2} + a-8k^2-14k-\frac{15}{2} \right) \psi_0(2k) \right. \\ & + \frac{1-4a^2}{2(2a+4k-1)^2} + \frac{3-4a^2}{4a+8k-2} + \frac{4(a-1)}{a+2k} + a-14k-\frac{4}{k}-19 \Big) \\ & \left. \times \sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k)}{\Gamma(a-j+2k)j} \right) + \text{CF}, \end{aligned} \quad (212)$$

where the shorthand notation CF denotes the closed-form terms omitted. The result (212) is obtained by using the identities (133), (B28), and (B32) after one rewrites the summation (211) as

$$\begin{aligned} & 8 \sum_{j=2}^{2k+1} \frac{1}{\Gamma(j)\Gamma(a+j)\Gamma(2k-j+2)\Gamma(a-j+2k+2)} \\ & + 8 \sum_{j=1}^{2k+1} \frac{\psi_0(j)}{\Gamma(j)\Gamma(a+j)\Gamma(2k-j+2)\Gamma(a-j+2k+2)} \\ & + 8 \sum_{j=1}^{2k} \frac{\psi_0(j)}{\Gamma(j)\Gamma(a+j+1)\Gamma(2k-j+1)\Gamma(a-j+2k+1)} \\ & + 2 \sum_{j=1}^{2k+1} \frac{\psi_0(j)\psi_0(2k-j+2)}{\Gamma(j)\Gamma(a+j)\Gamma(2k-j+2)\Gamma(a-j+2k+2)} \\ & + 4 \sum_{j=1}^{2k} \frac{\psi_0(j)\psi_0(2k-j+1)}{\Gamma(j)\Gamma(a+j)\Gamma(2k-j+1)\Gamma(a-j+2k+2)} \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{j=1}^{2k-1} \frac{\psi_0(j)\psi_0(2k-j)}{\Gamma(j)\Gamma(a+j+1)\Gamma(2k-j)\Gamma(a-j+2k+1)} \\
& - \frac{4}{\Gamma(a+1)\Gamma(a+2k+1)} \left( \frac{\psi_0(2k+1)}{\Gamma(2k+1)} + \frac{\psi_0(2k)}{\Gamma(2k)} \right). \quad (213)
\end{aligned}$$

Using the same approach, the other inner summations over  $j$  of the first three double summations in (A2) are simplified to a semi-closed-form representation, cf. (212), where the resulting unsimplifiable basis are

$$\sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k)}{\Gamma(a-j+2k)j} \quad (214)$$

$$\sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k)}{\Gamma(a-j+2k)j^2}. \quad (215)$$

Now, we simplify the inner summations over the indexes  $i$  and  $j$  of the triple sum in (A2). After the partial fraction decomposition

$$\frac{1}{(j)_3} = -\frac{1}{j+1} + \frac{1}{2(j+2)} + \frac{1}{2j}, \quad (216)$$

the corresponding inner summation is written as

$$\begin{aligned}
& \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k+j+3)\Gamma(a+2k-j+1)}{\Gamma(j+i+1)\Gamma(2k-j-i+1)\Gamma(a+i)\Gamma(a+2k-i+2)} \\
& \quad \times (\psi_0(a+2k+j+3) - \psi_0(2a+4k+4) + \psi_0(2k-i+3) - \psi_0(j+3)) \\
& \quad \times \left( -\frac{1}{j+1} + \frac{1}{2(j+2)} + \frac{1}{2j} \right). \quad (217)
\end{aligned}$$

We first consider the simplification of the summation

$$\begin{aligned}
& \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k+j+3)\Gamma(a+2k-j+1)}{(j+1)\Gamma(j+i+1)\Gamma(2k-j-i+1)\Gamma(a+i)\Gamma(a+2k-i+2)} \\
& \quad \times (\psi_0(a+2k+j+3) - \psi_0(2a+4k+4) + \psi_0(2k-i+3) - \psi_0(j+3)), \quad (218)
\end{aligned}$$

which involves the term  $1/(j+1)$  from the decomposition (216), and the remaining parts in (217) can be simplified in the same manner. For convenience, we divide (218) into the three sums

$$\begin{aligned}
& \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k+j+3)\Gamma(a+2k-j+1)}{(j+1)\Gamma(j+i+1)\Gamma(2k-j-i+1)\Gamma(a+i)\Gamma(a+2k-i+2)} \\
& \quad \times (-\psi_0(2a+4k+4) - \psi_0(j+3)) \quad (219)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k+j+3)\Gamma(a+2k-j+1)}{(j+1)\Gamma(j+i+1)\Gamma(2k-j-i+1)\Gamma(a+i)\Gamma(a+2k-i+2)} \\
& \quad \times \psi_0(2k-i+3) \quad (220)
\end{aligned}$$

$$\sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k+j+3)\Gamma(a+2k-j+1)}{(j+1)\Gamma(j+i+1)\Gamma(2k-j-i+1)\Gamma(a+i)\Gamma(a+2k-i+2)} \times \psi_0(a+2k+j+3). \quad (221)$$

The digamma functions in the summation (219) are independent to the index  $i$ . This fact allows us to evaluate the summation over  $i$  by using the identity (135) of Lemma 2. Specifically, we first rewrite (219) as

$$\sum_{j=1}^{2k-1} \frac{\Gamma(a-j+2k+1)\Gamma(a+j+2k+3)(-\psi_0(2a+4k+4)-\psi_0(j+3))}{j+1} \times \sum_{i=1}^{2k-j} \frac{1}{\Gamma(a+i)\Gamma(i+j)\Gamma(a-i+2k+2)\Gamma(2k-i-j)} \frac{4i(2k-i+2)}{(i+j)(2k-j-i)}. \quad (222)$$

We now partial fraction decompose the rational function with respect to  $i$  in (222). The summation (219) becomes

$$\sum_{j=1}^{2k-1} \frac{\Gamma(a-j+2k+1)\Gamma(a+j+2k+3)(-\psi_0(2a+4k+4)-\psi_0(j+3))}{j+1} \times \left( 4 \sum_{i=1}^{2k-j-1} \frac{1}{\Gamma(a+i)\Gamma(i+j)\Gamma(a-i+2k+2)\Gamma(2k-j-i)} - \frac{j+2k+2}{k} \right. \\ \times 2j \sum_{i=1}^{2k-j-1} \frac{1}{\Gamma(a+i)\Gamma(i+j+1)\Gamma(a-i+2k+2)\Gamma(2k-j-i)} + \frac{2k-j}{k} \\ \left. \times 2(j+2) \sum_{i=1}^{2k-j} \frac{1}{\Gamma(a+i)\Gamma(i+j)\Gamma(a-i+2k+2)\Gamma(2k-j+1-i)} \right), \quad (223)$$

where the inner summations over  $i$  can be evaluated by the identity (135) with the specializations

$$a = j, \quad b = a + j + 2, \quad c = a, \quad m = 2k - j - 1 \quad (224)$$

$$a = j + 1, \quad b = a + j + 2, \quad c = a, \quad m = 2k - j - 1 \quad (225)$$

$$a = j, \quad b = a + j + 1, \quad c = a, \quad m = 2k - j. \quad (226)$$

The summation (223) becomes

$$\sum_{j=1}^{2k-1} \left( j+1 - \frac{(a+2k+1)^2}{j+1} \right) \frac{\psi_0(2a+4k+4) + \psi_0(j+3)}{\Gamma(a)\Gamma(a+2k+1)} \times \left( 4(a-j+2k-1)(a+j+2k+1) \sum_{i=1}^{2k-j-1} \frac{\Gamma(a+i-1)\Gamma(a-i+4k)}{\Gamma(i)\Gamma(2k-i)} \right)$$

$$\begin{aligned}
& - \frac{2j(j+2k+2)(a-j+2k-1)}{k} \sum_{i=1}^{2k-j-1} \frac{\Gamma(a+i-1)\Gamma(a-i+4k+1)}{\Gamma(i)\Gamma(2k-i+1)} \\
& + \frac{2(j+2)(2k-j)(a+j+2k+1)}{k} \sum_{i=1}^{2k-j} \frac{\Gamma(a+i-1)\Gamma(a-i+4k+1)}{\Gamma(i)\Gamma(2k-i+1)} \Bigg). \tag{227}
\end{aligned}$$

In (227), we further change the orders of summations to simplify the sums over  $j$ . These summations admit closed-form expressions by using the identities (B2)–(B4) and (B7). The remaining summations only consist of single sums, which can be further simplified by using the identities (B33)–(B34) into the unsimplifiable basis (214) as well as some new basis

$$\sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k-1)\psi_0(j)}{\Gamma(a-j+2k+1)} \tag{228}$$

$$\sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k-1)\psi_0(j)}{\Gamma(a-j+2k)j} \tag{229}$$

$$\sum_{j=1}^{2k} \frac{\Gamma(2a-j+4k)\psi_0(2a-j+4k)}{\Gamma(a-i+2k)j}. \tag{230}$$

We now move on to simplifying the summation (220). In (220), we change the summation order to evaluate the sum over  $j$  by the identity (143) in Lemma 3. The summation then becomes

$$\begin{aligned}
& \Gamma^2(a+2k+2) \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-i} \frac{4(i+j+1)(a+i+j+1)(2k-i-j+1)}{j\Gamma(i+1)\Gamma(a+i+2)\Gamma(2k-i+1)\Gamma(a-i+2k+2)} \\
& \times (a-i-j+2k+1)\psi_0(i+j+2) - \Gamma^2(a+2k+2) \sum_{i=1}^{2k-1} 4i(2k-i+2) \\
& \times \frac{\psi_0(2k-i+3)(\psi_0(a+i+1) - \psi_0(a+2k+2))}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} - \Gamma(a+2k+1) \\
& \times \Gamma(a+2k+3) \sum_{i=1}^{2k-1} \frac{4(2k-i+2)(2k-i+1)\psi_0(2k-i+3)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} + \frac{1}{\Gamma(a+2)} \\
& \times \frac{\Gamma(a+2k+2)}{\Gamma(2k+1)} \sum_{i=1}^{2k-1} \frac{4i(a+i)(2k-i+2)(a-i+2k+2)\psi_0(2k-i+3)}{2k-i+1}. \tag{231}
\end{aligned}$$

We now shift the index  $i \rightarrow 2k-i$  of the three single sums in the above result (231). The first two single sums are simplified into the unsimplifiable basis (214) by using the identities (B28)–(B29). The last single sum in (231) is simplified directly into a closed-form representation by using the identities (B2)–(B4) and (B7). For the double summation in (231), we take the

partial fraction decomposition

$$\begin{aligned} & \frac{4(i+j+1)(a+i+j+1)(2k-i-j+1)(a-i-j+2k+1)}{j} \\ &= \frac{4(i+1)(a+i+1)(2k-i+1)(a-i+2k+1)}{j} + 4j^3 + 16(i-k)j^2 \\ & \quad - 4(a^2 + 2a(k+1) - 6i^2 + 12ik - 4k^2 + 4k + 2)j \\ & \quad + 8(i-k)(-a^2 - 2a(k+1) + 2(i+1)(i-2k-1)). \end{aligned} \quad (232)$$

In (232), the polynomial part can be simplified similarly as in (227), while the rational part is simplified as follows. The summation in question is

$$\sum_{i=1}^{2k-1} \frac{4(i+1)(2k-i+1)\Gamma^2(a+2k+2)}{\Gamma(i+1)\Gamma(a+i+1)\Gamma(2k-i+1)\Gamma(a-i+2k+1)} \sum_{j=1}^{2k-i} \frac{\psi_0(i+j+2)}{j}. \quad (233)$$

To evaluate (233), the sum over  $j$  is computed by using the identity (B15) with the specialization

$$i = j, \quad a = i + 1, \quad b = 1, \quad m = 2k - i, \quad (234)$$

and then we shift the summation index  $i \rightarrow 2k - i$  of the outer sum. Consequently, (233) is simplified to

$$\begin{aligned} & - \sum_{i=1}^{2k-1} \frac{4(i+1)(2k-i+1)\Gamma^2(a+2k+2)}{\Gamma(i+1)\Gamma(a+i+1)\Gamma(2k-i+1)\Gamma(a-i+2k+1)} \\ & \times \left( \sum_{j=1}^{2k-i+1} \frac{\psi_0(i+j+1)}{j} - \frac{1}{2}((\psi_0(2k-i+2) + \psi_0(i+1) - 2\psi_0(1)) \right. \\ & \quad \left. \times (\psi_0(2k-i+2) + \psi_0(i+1)) - \psi_1(2k-i+2) - \psi_1(i+1) + 2\psi_1(1)) \right). \end{aligned} \quad (235)$$

In (235), the double sum is the same form as in (233) but with a negative sign. Therefore, by adding up (233) and (235), and dividing the result by two, we reduce the double summation in (233) to a single sum as

$$\begin{aligned} & \sum_{i=1}^{2k-1} \frac{4(i+1)(2k-i+1)\Gamma^2(a+2k+2)}{\Gamma(i+1)\Gamma(a+i+1)\Gamma(2k-i+1)\Gamma(a-i+2k+1)} \sum_{j=1}^{2k-i} \frac{\psi_0(i+j+2)}{j} \\ &= \sum_{i=1}^{2k-1} \frac{2\Gamma^2(a+2k+2)}{\Gamma(i+1)\Gamma(a+i+1)\Gamma(2k-i+1)\Gamma(a-i+2k+1)} \left( \frac{1}{2}(2k-i+1) \right. \\ & \quad \times (i+1)((\psi_0(2k-i+2) + \psi_0(i+1))(\psi_0(2k-i+2) + \psi_0(i+1) \\ & \quad - 2\psi_0(1)) - \psi_1(2k-i+2) - \psi_1(i+1) + 2\psi_1(1)) + (2k-i+1) \\ & \quad \left. \times (\psi_0(2k-i+2) + \psi_0(i+2) - \psi_0(2k+3) - \psi_0(1)) - (i+1)\psi_0(2k+3) \right), \end{aligned} \quad (236)$$

which is further simplified into a semi-closed-form expression involving the unsimplifiable basis (214)–(215) by using the identities (B28) and (B31).

We now simplify the summation (221) as a last piece in (A2). We first change the summation order in (221) to evaluate the sum over  $j$  by the identity (B35). As a result, (221) is simplified to

$$\begin{aligned}
 & \Gamma^2(a+2k+2) \sum_{i=1}^{2k} 4i(a+i)(2k-i+2)(a-i+2k+2) \\
 & \times \sum_{j=1}^{2k-i+1} \frac{\psi_0(a-i+2k+3) - \psi_0(a-i-j+2k+3) + \psi_0(a+2k+2)}{j\Gamma(i+j)\Gamma(a+i+j+1)\Gamma(2k-i-j+2)\Gamma(a-i-j+2k+3)} \\
 & - \Gamma^2(a+2k+2) \sum_{i=1}^{2k-1} \frac{4i(2k-i+2)\psi_0(a+2k+2)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \\
 & \times (\psi_0(a+i+1) - \psi_0(a+2k+2)) - \Gamma(a+2k+1)\Gamma(a+2k+3) \\
 & \times \sum_{i=1}^{2k-1} \frac{4(-i+2k+2)\psi_0(a+2k+3)}{\Gamma(i)\Gamma(a+i)\Gamma(-i+2k+1)\Gamma(a-i+2k+2)} - 16(a+2)k(a+2k) \\
 & \times \frac{\Gamma(a+2k+2)}{\Gamma(a+2)\Gamma(2k+1)} (\psi_0(a+2k+2) - \psi_0(a+2) + \psi_0(a+3)), \quad (237)
 \end{aligned}$$

where the single summations are simplified similarly to the ones in (231). To simplify the double summation in (237), we shift the summation index  $i \rightarrow 2k-i-j+2$  as

$$\begin{aligned}
 & \sum_{i=1}^{2k} \frac{\Gamma^2(a+2k+2)}{\Gamma(i)\Gamma(a+i+1)\Gamma(2k-i+2)\Gamma(a-i+2k+3)} \\
 & \times \sum_{j=1}^{2k-i+1} \frac{4(i+j)(a+i+j)(2k-i-j+2)(a-i-j+2k+2)}{j} \\
 & \times (\psi_0(a+i+j+1) - \psi_0(a+i+1) + \psi_0(a+2k+2)). \quad (238)
 \end{aligned}$$

In (238), the inner summations over  $j$  are simplified into a closed-form representation except for the sum

$$\sum_{j=1}^{2k-i+1} \frac{\psi_0(a+i+j+1)}{j}. \quad (239)$$

Therefore, the resulting summations consist of several single sums as well as a double sum. These single sums are simplified into closed-form expressions by using the identity (133) and its derivative with respect to  $a$ . The double summation involves the sum in (239) and is written as

$$\sum_{i=1}^{2k} \frac{4i(2k-i+2)\Gamma^2(a+2k+2)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \sum_{j=1}^{2k-i+1} \frac{\psi_0(a+i+j+1)}{j}. \quad (240)$$

To process (240), we first evaluate the inner summation by the identity (B15) with the specialization

$$i = j, \quad a = a + 1, \quad b = i, \quad m = 2k - i + 1, \quad (241)$$

and the sum (240) becomes

$$\begin{aligned} & \sum_{i=1}^{2k} \frac{4i(2k-i+2)\Gamma^2(a+2k+2)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \left( \sum_{j=1}^{2k-i+1} \frac{\psi_0(i+j)}{j} \right. \\ & - \sum_{j=1}^{a+1} \frac{\psi_0(j+2k+1)}{i+j-1} + \frac{1}{2}(-\psi_1(a+i+1) + \psi_1(i) + (\psi_0(a+i+1) - \psi_0(i)) \\ & \times (\psi_0(a+i+1) + 2(\psi_0(2k-i+2) - \psi_0(1)) + \psi_0(i))) \Big). \end{aligned} \quad (242)$$

Comparing to the summations in (235), the summations in (242) can be simplified similarly except for a double sum

$$\sum_{i=1}^{2k} \frac{4i(2k-i+2)\Gamma^2(a+2k+2)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \sum_{j=1}^{a+1} \frac{\psi_0(j+2k+1)}{i+j-1}, \quad (243)$$

where it does not permit further simplifications by directly evaluating the inner sum. To proceed further, we change the summation order in (243) before performing the partial fraction decomposition

$$\frac{4i(2k-i+2)}{i+j-1} = -\frac{4(j-1)(j+2k+1)}{i+j-1} - 4(i-1) + 4(j+2k). \quad (244)$$

The summation (243) is now written as

$$\begin{aligned} & -\Gamma^2(a+2k+2) \sum_{j=1}^{a+1} 4(j-1)(j+2k+1)\psi_0(j+2k+1) \\ & \times \sum_{i=1}^{2k} \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \frac{1}{i+j-1} \\ & - 4 \sum_{j=1}^{a+1} \psi_0(j+2k+1) \sum_{i=2}^{2k} \frac{\Gamma^2(a+2k+2)}{\Gamma(i-1)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \\ & + 4 \sum_{j=1}^{a+1} \psi_0(j+2k+1) \sum_{i=1}^{2k} \frac{(j+2k)\Gamma^2(a+2k+2)}{\Gamma(i)\Gamma(a+i)\Gamma(2k-i+2)\Gamma(a-i+2k+2)}. \end{aligned} \quad (245)$$

In (245), the second and third double summations are simplified directly into closed-form expressions by using the identities (133), and (B1)–(B2). The first double summation in (245) after applying the identity (127) with the specialization

$$b = a, \quad c = j - 1, \quad m = 2k + 1 \quad (246)$$

is simplified to

$$\begin{aligned} & \frac{(a+2k+1)\Gamma(a+2k+2)}{\Gamma(a+1)\Gamma(2k+1)} \sum_{j=1}^{a+1} \frac{4(j-1)(j+2k+1)}{(j+2k)} \psi_0(j+2k+1) \\ & - \frac{(a+2k+1)\Gamma(a+2k+2)}{\Gamma(2a+2k+1)} \sum_{j=1}^{a+1} \sum_{i=1}^{2k+1} 4(j-1)(j+2k+1) \psi_0(j+2k+1) \\ & \times \frac{\Gamma(2a-i+4k+2)}{\Gamma(2k-i+2)\Gamma(a-i+2k+2)} \frac{\Gamma(2k-i+j+1)}{\Gamma(j+2k+1)}. \end{aligned} \quad (247)$$

The single sum over  $j$  in (247) admits a closed-form representation by applying the identity (B1), (B2), and (B7). The double sum in (247) is simplified by first using the identities (B21)–(B22) to compute the sum over  $j$ , before using the identities (B18)–(B20) to evaluate the sum over  $i$ . Consequently, we obtain a semi-closed-form result of (247), and the corresponding unsimplifiable basis are (214) and the basis

$$\sum_{i=1}^{2k} \frac{\psi_0(2a+i+2k)}{i}. \quad (248)$$

We have so far completed the simplification of the inner summations over indexes  $i$  and  $j$  in (A2). Summing up these results, we observe the complete cancellation among the unsimplifiable basis (214)–(215) and (228)–(230). The only surviving term is (248). In the resulting outer sum over index  $k$  in (A2), all the gamma functions are reduced to rational ones. Therefore, (A2) can now be simplified into a similar form as (A1) in terms of the unsimplifiable basis of the form (201).

Inserting the resulting summations of (A1) and (A2) into (50), we obtain

$$\begin{aligned} I_A = & \sum_{k=1}^m \left( \left( \frac{2a-1}{2k} - \frac{2a+1}{2(a+k)} + \frac{2a+1}{2k+1} - \frac{2a-1}{2a+2k+1} \right) \psi_0(2a+4k) \right. \\ & + \left( \frac{2(a+m)(2a+3m-1)}{2a+4m-1} \left( \frac{1}{a+2k+1} + \frac{1}{a+2k} \right) - \frac{2a}{2k+1} \right. \\ & + \left. \left. \frac{1-2a}{2k} + \frac{1}{2(a+k)} \right) \psi_0(2a+2k) + \left( \frac{2am-2a+6m^2-6m+1}{(2k+1)(2a+4m-1)} \right. \right. \\ & + \left. \left. \frac{4am+2a+12m^2-1}{4k(2a+4m-1)} + \frac{1}{4(a+k)} \right) \psi_0(a+2k) - \left( \frac{2a+2m-1}{2(2k+1)} \right. \right. \\ & + \left. \left. \frac{a+m}{2k} \right) \psi_0(a+k+m) - \left( \frac{2a+2m-1}{4k} + \frac{a+m}{2k+1} \right) \right. \\ & \times \left. \psi_0\left(a+k+m+\frac{1}{2}\right) - \left( \frac{1}{2(2k+1)} + \frac{1}{4k} \right) \psi_0(a+k) \right) + \text{CF}. \end{aligned} \quad (249)$$

We remind the readers that the omitted closed-form terms are denoted by the abbreviation CF, which in general is different in each use.

Now, we discuss the simplification of the summations in (A3) and (A4) in computing the integral  $I_B$  in (54). The single summation in (A3) are computed

into the unsimplifiable basis of the form (201) by using the identities (B1)–(B14) along with the result (199). The simplification of the double summation in (A4) will utilize Lemma 6. By partial fraction decomposing the rational functions in  $j$  and shifting the summation index  $j \rightarrow m - k - j$ , (A4) boils down to computing the summations

$$\sum_{k=1}^{m-1} p_{c,\lambda}(k) \frac{\Gamma(2m-2k+1)}{\Gamma(2a-2k+2m+1)} \sum_{j=1}^{m-k} \frac{\Gamma(2m+2a-2j-2k+1)}{\Gamma(2m-2j-2k+1)} \times \left( \frac{1}{(a-j-2k+2m+\frac{1}{2}+c)^\lambda} - \frac{1}{(j+c)^\lambda} \right), \quad (250)$$

where the parameter  $c$  and  $\lambda$  take the values  $c = -1/2, 0, 1/2$ ,  $\lambda = 1, 2$ , and  $p_{c,\lambda}(k)$  denotes rational functions in  $k$ . These summations are simplified by using Lemma 6 to first evaluate the inner sums over  $j$  that reduces the gamma ratio

$$\frac{\Gamma(2m-2k+1)}{\Gamma(2a-2k+2m+1)} \quad (251)$$

into a rational function. Specifically, in the case when  $c = 1/2$  and  $\lambda = 1$  in (250), the corresponding inner summation is

$$\sum_{j=1}^{m-k} \frac{\Gamma(2m+2a-2j-2k+1)}{\Gamma(2m-2j-2k+1)} \left( \frac{1}{1+a-j-2k+2m} - \frac{1}{j+\frac{1}{2}} \right). \quad (252)$$

By using the relation [9]

$$\Gamma(2k) = \frac{2^{2k-1}}{\sqrt{\pi}} \Gamma(k) \Gamma\left(k + \frac{1}{2}\right), \quad (253)$$

the summation (252) is written as

$$2^{2a} \sum_{j=1}^{m-k} \Phi_{0, -\frac{1}{2}, a, a-\frac{1}{2}}^{(m-k-j)} \left( \frac{1}{1+a-j-2k+2m} - \frac{1}{j+\frac{1}{2}} \right), \quad (254)$$

where we recall the notation (163)

$$\Phi_{a,b,c,d}^{(x)} = \frac{\Gamma(x+c+1)\Gamma(x+d+1)}{\Gamma(x+a+1)\Gamma(x+b+1)}. \quad (255)$$

By using Lemma 6 with the specialization

$$a = -\frac{1}{2}, \quad b = a, \quad c = \frac{1}{2}, \quad m \rightarrow m-k, \quad (256)$$

(252) is simplified to

$$\begin{aligned} & \sum_{j=1}^{m-k} \frac{\Gamma(2m+2a-2j-2k+1)}{\Gamma(2m-2j-2k+1)} \left( \frac{1}{1+a-j-2k+2m} - \frac{1}{j+\frac{1}{2}} \right) \\ &= \frac{\Gamma(2a-2k+2m+2)}{\Gamma(2m-2k+2)} \left( \psi_0(a+1) - \psi_0(a-2k+2m+1) \right) \end{aligned}$$

$$- \frac{2(2a+1)}{2a-2k+2m+1} + 2 \Bigg). \quad (257)$$

To simplify the summation (250) for  $c = 1/2$  and  $\lambda = 2$ , we again use the relation (253), and the corresponding inner summation

$$\sum_{j=1}^{m-k} \frac{\Gamma(2m+2a-2j-2k+1)}{\Gamma(2m-2j-2k+1)} \left( \frac{1}{(1+a-j-2k+2m)^2} - \frac{1}{(j+\frac{1}{2})^2} \right) \quad (258)$$

becomes

$$2^{2a} \sum_{j=1}^{m-k} \Phi_{0, -\frac{1}{2}, a, a-\frac{1}{2}}^{(m-k-j)} \left( \frac{1}{(1+a-j-2k+2m)^2} - \frac{1}{(j+\frac{1}{2})^2} \right), \quad (259)$$

which is then evaluated by taking a derivative of the parameter  $c$  of the identity (175) with the specialization

$$a = -\frac{1}{2}, \quad b = a, \quad m \rightarrow m-k, \quad (260)$$

before setting  $c = 1/2$ . As a result, (258) is simplified to

$$\begin{aligned} & \sum_{j=1}^{m-k} \frac{\Gamma(2m+2a-2j-2k+1)}{\Gamma(2m-2j-2k+1)} \left( \frac{1}{(1+a-j-2k+2m)^2} - \frac{1}{(j+\frac{1}{2})^2} \right) \\ &= \frac{\Gamma(2m+2a-2k+2)}{\Gamma(2m-2k+2)} \sum_{j=1}^{m-k} \left( \left( -\frac{4a^2-1}{a+j-1} - \frac{1-4a^2}{a+j} - \frac{4a^2-6a+2}{-2a-2j+3} \right. \right. \\ & \quad - \frac{2a(2a+1)}{2a+2j+1} + \frac{1}{a+2j} - \frac{2-8a}{2a+2j-1} - \frac{1}{-a-2j+1} \Bigg) \left( \psi_0\left(j+\frac{1}{2}\right) \right. \\ & \quad - \psi_0\left(a+j+\frac{1}{2}\right) + \psi_0(a-k+m+1) + \psi_0\left(a-k+m+\frac{3}{2}\right) + \psi_0(j) \\ & \quad - \psi_0(a+j) - \psi_0(m-k+1) - \psi_0\left(m-k+\frac{3}{2}\right) \Bigg) + \frac{a+j}{j(a+2j)^2} \\ & \quad \left. + \frac{1}{2a+2j-1} \left( \frac{a(2a-1)(2j-1)}{j(a+j)^2} - \frac{2a(2a+1)}{(a+j+\frac{1}{2})^2} + \frac{2j-1}{(a+2j-1)^2} \right) \right). \end{aligned} \quad (261)$$

The other cases of combinations of  $c$  and  $\lambda$  are obtained similarly. Inserting the results into (250), the ratio of gamma functions (251) of the outer sum over  $k$  is reduced to a rational function. The resulting summations are further computed into unsimplifiable basis of the form (201) by using the identities (B1)–(B14).

Inserting the resulting summations of (A3) and (A4) into (54), we obtain

$$I_B = \sum_{k=1}^m \left( \left( \frac{2(a+m)(2a+3m-1)}{2a+4m-1} \left( \frac{1}{a+2k+1} + \frac{1}{a+2k} \right) + \frac{m}{k} + \frac{2m-1}{2k+1} \right) \right)$$

$$\begin{aligned}
& \times \psi_0(2a+2k) + \left( \frac{m}{2a+2k+1} + \frac{2m-1}{4(a+k)} - \frac{2m-1}{2(2k+1)} - \frac{m}{2k} \right) \\
& \times \psi_0\left(a+2k+\frac{1}{2}\right) + \left( \frac{m}{2(a+k)} + \frac{m}{2a+2k+1} + \frac{m(2m-1)}{2a+4m-1} \right. \\
& \times \left. \left( \frac{1}{2k+1} + \frac{1}{2k} \right) \right) \psi_0(a+2k) + \text{CF}.
\end{aligned} \tag{262}$$

Now, inserting the  $I_A$  expression (249) and  $I_B$  expression (262) into (45), we finally arrive at

$$\begin{aligned}
\mathbb{V}[S] = & \frac{-2a-2m+1}{4} \Omega_1^{(a, a+\frac{1}{2})} + \frac{-2a-2m+1}{4} \left( \Omega_2^{(a+\frac{1}{2}, 0)} + \Omega_2^{(a+\frac{1}{2}, \frac{1}{2})} \right) \\
& - \frac{a+m}{2} \left( \Omega_2^{(a, 0)} + \Omega_2^{(a, \frac{1}{2})} \right) + \text{CF},
\end{aligned} \tag{263}$$

where

$$\Omega_1^{(a,b)} = \sum_{k=1}^m \left( \frac{\psi_0(a+k)}{b+k} + \frac{\psi_0(b+k)}{a+k} \right) \tag{264}$$

$$\begin{aligned}
\Omega_2^{(a,b)} = & \sum_{k=1}^m \left( \frac{\psi_0(a+b+k+m)}{b+k} + \frac{\psi_0(a+k)}{b+k} + \frac{\psi_0(a+b+2k)}{a+k} \right. \\
& \left. - \frac{\psi_0(a+b+k)}{a+k} - \frac{\psi_0(a+b+2k)}{b+k} \right).
\end{aligned} \tag{265}$$

Simplifying the single sums in (263) by using the closed-form identities (B14) and (B16) directly leads to the desired result (22). This completes the proof of Proposition 3.

**Case B: Fixed Number of Particles.** In case B, the variance calculation boils down to simplifying the summation representations of  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{B}_1$ ,  $\mathcal{B}_2$  as summarized in (A5)–(A10) in Appendix A.2.

Note that the simplification procedure in case A also works for the majority of the summations in case B. The only new summation in case B is the double summation

$$\begin{aligned}
& \sum_{k=1}^{m-2} 4(a+b-2k+2m+1) \frac{\Gamma(m-k+1)}{\Gamma(a+b-k+m+1)} \sum_{j=1}^{m-k-1} \frac{\Gamma(a+b-j-k+m)}{\Gamma(m-k-j)} \\
& \times (-1)^j (2j+2k-2m-a-b+1) \left( \frac{1}{(j)_3(a+b-j-2k+2m-1)_3} \right)^2 \\
& \times ((1-a)j(a+b-j-2k+2m-1) - 2(a-k+m)(a+b-k+m)) \\
& \times ((1-b)j(a+b-j-2k+2m-1) - 2(b-k+m)(a+b-k+m)),
\end{aligned} \tag{266}$$

which is obtained by opening the bracket of the double summation in (A10) and shifting the index  $k \rightarrow m-j-k$ . To process (266), we take the partial

fraction decomposition of the rational functions (from the second to the last line) in  $j$ , the summation (266) now boils down to

$$\sum_{k=1}^{m-2} p_{c,\lambda}(k) \frac{\Gamma(m-k+1)}{\Gamma(a+b-k+m+1)} \sum_{j=1}^{m-k-1} \frac{(-1)^j \Gamma(m-k+a+b-j)}{\Gamma(m-k-j)} \times \left( \frac{1}{(2m-2k-1+a+b-j+c)^\lambda} - \frac{1}{(j+c)^\lambda} \right), \quad (267)$$

where the parameters  $c$  and  $\lambda$  take the values  $c = 0, 1, 2$ ,  $\lambda = 1, 2$ , and  $p_{c,\lambda}(k)$  denotes the rational functions in  $k$ . The simplification of these summations will utilize Lemma 5. For the case when  $c = 0$  and  $\lambda = 1$  in (267), the corresponding inner summation is simplified into a closed-form expression by directly using the result (126), which is a special case of the identity (164) in Lemma 5. When  $c = 0$  and  $\lambda = 2$  in (267), the inner summation

$$\sum_{j=1}^{m-k-1} \frac{(-1)^j \Gamma(a+b-j-k+m)}{\Gamma(m-k-j)} \left( \frac{1}{(a+b-j-2k+2m-1)^2} - \frac{1}{j^2} \right) \quad (268)$$

is simplified to

$$\frac{\Gamma(a+b-k+m)}{\Gamma(m-k)} \sum_{j=1}^{m-k-1} \left( \left( \frac{1}{a+b-j-k+m} - \frac{1}{a+b-2j-2k+2m} - \frac{1}{a+b-2j-2k+2m-1} \right) (\psi_0(m-k) - \psi_0(a+b-j-k+m) - \psi_0(a+b-k+m) - \psi_0(m-k-j)) + \frac{1}{(a+b-2j-2k+2m-1)^2} \right). \quad (269)$$

The result (269) is obtained by taking derivative of  $c$  of the identity (164) with the specialization

$$a \rightarrow a+b, \quad m \rightarrow m-k-1, \quad (270)$$

before setting  $c = 0$ . For other combinations of  $c$  and  $\lambda$ , the corresponding inner summations in (267) can be simplified similarly as the two cases above.

Inserting the simplification results of the inner summations into (266), the resulting sums only consist of polygamma and rational functions, which are further computed into unsimplifiable basis of the form (201) by using the identities (B1)–(B14).

For the integrals  $I_A$  and  $I_B$  in case B, the corresponding summations (A5)–(A10) are now simplified to the results shown below. For  $I_A$ , one has

$$I_A = d_1 \sum_{k=1}^m \frac{\psi_0(a+k)}{k} + d_2 \sum_{k=1}^m \frac{\psi_0(b+k)}{k} - 2m \sum_{k=1}^m \frac{\psi_0(a+b+k+m)}{k}$$

$$+ d_3 \sum_{k=1}^m \frac{\psi_0(a+b+k+m)}{a+k} + (d_3 + d_4) \sum_{k=1}^m \frac{\psi_0(a+b+k+m)}{b+k} + \text{CF}, \quad (271)$$

where the coefficients  $d_i$  are

$$d_1 = \frac{2m(a+m)(a^2 + a(b+3m) + 2bm + 3m^2 - 1)}{(a+b+2m-1)_3} \quad (272)$$

$$d_2 = \frac{2m(b+m)(a(b+2m) + b^2 + 3bm + 3m^2 - 1)}{(a+b+2m-1)_3} \quad (273)$$

$$d_3 = \frac{2(b+m)(a+b+m)(m(3a+4b) + (a+b)^2 + 3m^2 - 1)}{(a+b+2m-1)_3} \quad (274)$$

$$d_4 = \frac{2(a-b)(a+b+m)}{a+b+2m}. \quad (275)$$

For  $I_B$ , one has

$$\begin{aligned} I_B = & d_1 \sum_{k=1}^m \frac{\psi_0(a+k)}{k} + d_2 \sum_{k=1}^m \frac{\psi_0(b+k)}{k} + 2m \sum_{k=1}^m \frac{\psi_0(a+b+k)}{k} \\ & - 2m \sum_{k=1}^m \frac{\psi_0(a+b+2k)}{k} + d_3 \sum_{k=1}^m \frac{\psi_0(a+b+k)}{b+k} + d_4 \sum_{k=1}^m \frac{\psi_0(b+k)}{a+k} \\ & - d_4 \sum_{k=1}^m \frac{\psi_0(a+b+2k)}{a+k} + d_4 \sum_{k=1}^m \frac{\psi_0(a+b+2k)}{b+k} \\ & + (d_3 + d_4) \sum_{k=1}^m \frac{\psi_0(a+b+k)}{a+k} + \text{CF}. \end{aligned} \quad (276)$$

Inserting the  $I_A$  expression (271) and  $I_B$  expression (276) into (45), one arrives at

$$\mathbb{V}[S] = -2m\Omega_2^{(a+b,0)} - d_4\Omega_2^{(b,a)} + (d_3 + d_4)\Omega_3^{(a,b)} + \text{CF}, \quad (277)$$

where the summation  $\Omega_2^{(a,b)}$  has been defined in (265), and we denote

$$\begin{aligned} \Omega_3^{(a,b)} = & \sum_{k=1}^m \left( \frac{\psi_0(a+b+k+m)}{a+k} + \frac{\psi_0(a+b+k+m)}{b+k} - \frac{\psi_0(a+b+k)}{a+k} \right. \\ & \left. - \frac{\psi_0(a+b+k)}{b+k} \right). \end{aligned} \quad (278)$$

By using the identities (B16) and (B17), the result (277) is simplified to the claimed variance formula (24). This completes the proof of Proposition 4.

The fact that all the unsimplifiable basis in (277) are simplified to a closed-form expression is what we refer to as anomaly cancellation. This phenomenon indicates that there are some hidden structures between the logarithmic observables over one and two-point correlation functions. Likewise, there

may also exist some hidden structures for arbitrary moments, the understanding of which will be crucial to develop a more compact and systematic method in computing higher-order moments.

### 3. Conclusions

In this work, we compute the exact yet explicit variance formulas of von Neumann entanglement entropy over fermionic Gaussian states with and without particle number constraints. The obtained formulas provide insights into the fluctuations of von Neumann entropy. An essential ingredient in obtaining the results is a new simplification framework of dummy summation and re-summation techniques. The new framework may also be useful in computing higher-order moments of von Neumann entropy and other entanglement indicators over the fermionic Gaussian ensemble.

### Acknowledgements

The work of Lu Wei is supported in part by the US National Science Foundation (#2306968 and #2150486).

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

## Appendix A: Summation Representations

In this appendix, we list the summation representations of the integrals in  $I_A$  and  $I_B$  as also summarized in Table 2. The summation representations for case A are listed in Appendix A.1, and the ones for case B are listed in Appendix A.2.

### A.1. Summation Representations of Case A

$$A_1 = \sum_{k=0}^{m-1} \sum_{j=2k-2}^{2k} \frac{2(-1)^j (2a+4k+1)(j+1)_2 (a+j+1)_2}{\Gamma(2k-j+1)\Gamma(j-2k+3)(2a+j+2k+1)_3} \left( (\psi_0(j+3) - \psi_0(2a+j+2k+4) - \psi_0(j-2k+3) + \psi_0(a+j+3))^2 - \psi_1(2a+j+2k+4) + \psi_1(a+j+3) - \psi_1(j-2k+3) + \psi_1(j+3) \right)$$

$$\begin{aligned}
& + \sum_{k=0}^{m-1} \sum_{j=0}^{2k-3} \frac{4(2a+4k+1)(j+1)_2(a+j+1)_2}{(2k-j-2)_3(2a+j+2k+1)_3} (\psi_0(2a+j+2k+4) \\
& - \psi_0(a+j+3) + \psi_0(2k-j-2) - \psi_0(j+3)) \quad (A1)
\end{aligned}$$

$$\begin{aligned}
A_2 = & \sum_{k=0}^{m-1} \frac{(2a+4k+1)\Gamma(2k+1)\Gamma(2a+2k+1)}{\Gamma(2a+4k+4)} \left( \sum_{j=0}^{2k} \frac{2(j+1)(2k-j+1)}{\Gamma(j+1)\Gamma(a+j+1)} \right. \\
& \times \frac{\Gamma^2(a+2k+2)}{\Gamma(2k-j+1)\Gamma(a+2k-j+1)} ((\psi_0(a+2k+2) - \psi_0(2a+4k+4) \\
& - \psi_0(2) + \psi_0(2k-j+2))(\psi_0(a+2k+2) - \psi_0(2a+4k+4) + \psi_0(j+2) \\
& - \psi_0(2)) - \psi_1(2a+4k+4)) - \sum_{j=0}^{2k} \frac{(j+1)\Gamma(a+2k+1)\Gamma(a+2k+3)}{\Gamma(j)\Gamma(a+j+1)\Gamma(a-j+2k+1)} \\
& \times \frac{1}{\Gamma(2k-j+1)} ((\psi_0(a+2k+1) - \psi_0(2a+4k+4) + \psi_0(2k-j+2) \\
& - \psi_0(1))(\psi_0(a+2k+3) - \psi_0(2a+4k+4) + \psi_0(j+2) - \psi_0(3)) \\
& - \psi_1(2a+4k+4)) - \sum_{j=0}^{2k} \frac{(2k-j+1)\Gamma(a+2k+1)\Gamma(a+2k+3)}{\Gamma(a+j+1)\Gamma(2k-j)\Gamma(2k-j+a+1)} \\
& \times \frac{1}{\Gamma(j+1)} ((\psi_0(a+2k+3) - \psi_0(2a+4k+4) + \psi_0(2k-j+2) \\
& - \psi_0(3))(\psi_0(a+2k+1) - \psi_0(2a+4k+4) + \psi_0(j+2) - \psi_0(1)) \\
& - \psi_1(2a+4k+4)) + \sum_{j=1}^{2k-1} \sum_{i=1}^{2k-j} \frac{4i(2k-i+2)\Gamma(a+2k-j+1)}{\Gamma(a+i)\Gamma(a+2k-i+2)} \\
& \times \frac{\Gamma(a+2k+j+3)}{(j)_3\Gamma(j+i+1)\Gamma(2k-j-i+1)} (\psi_0(a+2k+j+3) \\
& - \psi_0(2a+4k+4) + \psi_0(2k-i+3) - \psi_0(j+3)) \Big) \quad (A2)
\end{aligned}$$

$$\begin{aligned}
B_1 = & \sum_{k=0}^{m-1} \left( \psi_0(a+2k) + \psi_0(2a+2k) - 2\psi_0(2a+4k) - \frac{1}{2} \left( \frac{a}{a+2k+1} \right. \right. \\
& \left. \left. + \frac{a}{a+2k} + \frac{2}{2a+4k+1} \right) + 1 \right)^2 \quad (A3)
\end{aligned}$$

$$\begin{aligned}
B_2 = & \sum_{k=1}^{m-1} \sum_{j=1}^{m-k} \frac{\Gamma(2a+2k-1)\Gamma(2j+2k-1)}{2(2j-1)^2 j^2 (2j+1)^2 \Gamma(2k-1)\Gamma(2a+2j+2k-1)} \\
& \times \frac{(2a+4k-3)(2a+4j+4k-3)}{(a+j+2k-2)^2 (a+j+2k-1)^2 (2a+2j+4k-3)^2} (a^2(2j+1) \\
& + a(j+1)(2j+4k-3) + 2j^2 + j(4k-3) + 4k^2 - 6k + 2)^2 \quad (A4)
\end{aligned}$$

## A.2. Summation Representations of Case B

$$\mathcal{A}_1 = \mathcal{A}_1^{(a,b)} + \mathcal{A}_1^{(b,a)}, \quad (\text{A5})$$

$$\begin{aligned} \mathcal{A}_1^{(a,b)} = & -\frac{2(m(b+m))}{a+b+2m} \sum_{i=1}^{m-3} \frac{(b+i+1)(i)_2}{(m-i-2)_3(a+b+i+m+1)} (\psi_0(b+i+2) \\ & - \psi_0(a+b+i+m+2) - \psi_0(m-i-2) + \psi_0(i+2)) + \frac{a+b+m}{a+b+2m} \\ & \times 2(a+m) \sum_{i=1}^{m-2} \frac{(b+i+1)(i)_2}{(m-i-1)(a+b+i+m)_3} (-\psi_0(a+b+i+m+3) \\ & + \psi_0(b+i+2) - \psi_0(m-i-1) + \psi_0(i+2)) - \frac{m(b+m)}{a+b+2m} \\ & \sum_{i=m-3}^{m-1} \frac{(b+i+2)(-1)^{i+m}(i+1)_2}{\Gamma(m-i)\Gamma(i-m+4)(a+b+i+m+2)} (\psi_1(b+i+3) \\ & + \psi_1(i+3) - \psi_1(i-m+4) - \psi_1(a+b+i+m+3) + (\psi_0(i+3) \\ & - \psi_0(a+b+i+m+3) - \psi_0(i-m+4) + \psi_0(b+i+3))^2) \\ & - \frac{(a+m)(a+b+m)(b+m)(m-1)_2}{(a+b+2m)(a+b+2m-1)_3} (-\psi_1(a+b+2m+2) \\ & + \psi_1(b+m+1) + \psi_1(m+1) - \psi_1(1) + \psi_0^2(1) + (\psi_0(b+m+1) \\ & - \psi_0(a+b+2m+2) + \psi_0(m+1))(\psi_0(b+m+1) + \psi_0(m+1) \\ & - \psi_0(a+b+2m+2) - 2\psi_0(1))) \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \mathcal{A}_2 = & -\frac{2(a+m)(b+m)\Gamma(m+1)}{(a+b+2m)(a+b+m+2)_m} \sum_{i=0}^{m-1} (-1)^i \frac{(i+1)(m-i)}{\Gamma(a+i+2)} \\ & \times \sum_{j=i-1}^{i+1} (-1)^j \frac{\Gamma(a+i-j+m+1)(b-i+m+1)_j}{\Gamma(j+1)\Gamma(i-j+2)\Gamma(j-i+2)\Gamma(m-j)} \\ & \times (\psi_1(a+b+2m+2) + (\psi_0(a+i-j+m+1) - \psi_0(a+b+2m+2) \\ & - \psi_0(i-j+2) + \psi_0(i+2))(\psi_0(a+b+2m+2) + \psi_0(j-i+2) \\ & - \psi_0(b-i+j+m+1) - \psi_0(m-i+1))) - 2\Gamma(m+1)\Gamma(a+m+1) \\ & \times \frac{(a+b+m)\Gamma(b+m+1)}{(a+b+2m)(a+b+m+1)_{m+1}} \sum_{i=0}^{m-2} \frac{1}{\Gamma(i+1)\Gamma(a+i+2)} \\ & \times \frac{1}{\Gamma(m-i-1)\Gamma(b-i+m)} (\psi_1(a+b+2m+2) + (\psi_0(a+m+1) \\ & - \psi_0(a+b+2m+2) + \psi_0(i+2) - \psi_0(1))(\psi_0(a+b+2m+2) \\ & - \psi_0(b+m+1) - \psi_0(m-i) + \psi_0(1))) + \mathcal{A}_2^{(a,b)} + \mathcal{A}_2^{(b,a)}, \end{aligned} \quad (\text{A7})$$

$$\mathcal{A}_2^{(a,b)} = \frac{2\Gamma(m+1)\Gamma(a+b+m+1)}{\Gamma(a+b+2m+2)} \left( \frac{(a+m)(b+m)(a+b+m+1)}{a+b+2m} \right)$$

$$\begin{aligned}
& \times \sum_{i=1}^{m-2} \frac{i(m-i+1)}{\Gamma(b+i+1)\Gamma(a-i+m+2)} \sum_{j=1}^{m-i-1} \frac{\Gamma(a+j+m+2)}{(j)_3\Gamma(i+j+1)} \\
& \times \frac{\Gamma(b-j+m)}{\Gamma(m-i-j)} (\psi_0(m-i+2) - \psi_0(a+b+2m+2) - \psi_0(j+3) \\
& + \psi_0(a+j+m+2)) - \frac{a+b+m}{a+b+2m} \sum_{i=1}^{m-1} \frac{i(m-i)}{\Gamma(b+i+1)} \\
& \times \frac{1}{\Gamma(a-i+m+1)} \sum_{j=1}^{m-i} \frac{\Gamma(a+j+m+1)\Gamma(b-j+m+1)}{j\Gamma(i+j+1)\Gamma(m-i-j+1)} \\
& \times (\psi_0(a+j+m+1) - \psi_0(a+b+2m+2) + \psi_0(m-i+1) \\
& - \psi_0(j+1)) \Big) \tag{A8}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_1 = & \sum_{k=0}^{m-1} \left( \left( \frac{a^2 - b^2}{4(a+b+2k)} + \frac{b^2 - a^2}{4(a+b+2k+2)} + \frac{1}{2} \right) \psi_0(a+k+1) \right. \\
& + \left( \frac{a^2 - b^2}{4(a+b+2k+2)} + \frac{b^2 - a^2}{4(a+b+2k)} + \frac{1}{2} \right) \psi_0(b+k+1) \\
& + \psi_0(a+b+k+1) - 2\psi_0(a+b+2k+2) - \frac{a+b}{2(a+b+2k)} \\
& \left. - \frac{a+b}{2(a+b+2k+2)} + \frac{1}{a+b+2k+1} + 1 \right)^2 \tag{A9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_2 = & \sum_{k=1}^{m-1} \frac{k(a+b+k)}{2(a+k)(b+k)(a+b+2k)(a+b+2k-1)_3} \left( 2(a+k)(b+k) \right. \\
& \times (\psi_0(b+k+1) - \psi_0(a+k+1)) + \frac{(k-1)(a-b)(a+b+2k+1)}{a+b+k} \Big)^2 \\
& + \sum_{k=1}^{m-2} \sum_{j=1}^{m-k-1} \frac{2(a+b+2k-1)\Gamma(j+k+1)(a+b+2j+2k+1)}{\Gamma(k)\Gamma(a+k)\Gamma(b+k)\Gamma(a+j+k+1)\Gamma(b+j+k+1)} \\
& \times \frac{\Gamma(a+b+k)}{\Gamma(a+b+j+k+1)} \left( \frac{\Gamma(a+k)\Gamma(b+j+k+1)}{(j)_3(a+b+j+2k-1)_3} (a^2(j+2) + a(j+2) \right. \\
& \times (b+j+2k) + j(b+2k+1) + 2k(b+k) + j^2) \\
& - \frac{(-1)^j\Gamma(b+k)\Gamma(a+j+k+1)}{(j)_3(a+b+j+2k-1)_3} (2(k-1)(a+(b+1)(j+2) + 2k-2) \\
& \left. + (b+1)(j+2)(a+b+j+1)) \right)^2 \tag{A10}
\end{aligned}$$

## Appendix B Summation Identities

In this appendix, we list the finite sum identities useful in simplifying the summations (A1)–(A10) performed in Sect. 2.3. The identities of the existing simplification framework are listed in Appendix B.1, and the identities of the new simplification framework are listed in Appendix B.2.

### B.1. Summation Identities of Existing Simplification Framework

We list below the identities obtained from the existing simplification framework. Here, it is sufficient to assume that  $a, b \geq 0, a \neq b$  in identities (B1)–(B7), (B10)–(B12), (B14), (B16),  $a > m$  in (B13), and  $a, b \geq 1, n \geq m$  in (B15), (B17)–(B22).

$$\sum_{i=1}^m \psi_0(i+a) = (m+a)\psi_0(m+a+1) - a\psi_0(a+1) - m \quad (\text{B1})$$

$$\begin{aligned} \sum_{i=1}^m i\psi_0(i+a) &= -\frac{1}{2}(a-m-1)(a+m)\psi_0(a+m+1) \\ &\quad + \frac{1}{2}(a-1)a\psi_0(a+1) - \frac{1}{4}m(-2a+m+3) \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \sum_{i=1}^m i^2\psi_0(i+a) &= \frac{1}{6}(2a^3 - 3a^2 + a + 2m^3 + 3m^2 + m)\psi_0(a+m+1) \\ &\quad - \frac{1}{6}a(2a^2 - 3a + 1)\psi_0(a+1) \\ &\quad - \frac{1}{36}m(12a^2 - 6am - 24a + 4m^2 + 15m + 17) \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \sum_{i=1}^m i^3\psi_0(i+a) &= -\frac{1}{4}(a^4 - 2a^3 + a^2 - m^4 - 2m^3 - m^2)\psi_0(a+m+1) \\ &\quad + \frac{1}{4}(a-1)^2a^2\psi_0(a+1) - \frac{1}{48}m(-12a^3 + 6a^2m + 30a^2 \\ &\quad - 4am^2 - 18am - 26a + 3m^3 + 14m^2 + 21m + 10) \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \sum_{i=1}^m \psi_1(i+a) &= (m+a)\psi_1(m+a+1) - a\psi_1(a+1) \\ &\quad + \psi_0(m+a+1) - \psi_0(a+1) \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \sum_{i=1}^m \psi_0^2(i+a) &= (a+m)\psi_0^2(a+m) - (2a+2m-1)\psi_0(a+m) - a\psi_0^2(a) \\ &\quad + (2a-1)\psi_0(a) + 2m. \end{aligned} \quad (\text{B6})$$

$$\sum_{i=1}^m \frac{\psi_0(i+a)}{i+a} = \frac{1}{2}(\psi_1(m+a+1) - \psi_1(a+1) + \psi_0^2(m+a+1) - \psi_0^2(a+1)) \quad (\text{B7})$$

$$\sum_{i=1}^m \frac{\psi_0(m+1-i)}{i} = \psi_0^2(m+1) - \psi_0(1)\psi_0(m+1) + \psi_1(m+1) - \psi_1(1) \quad (\text{B8})$$

$$\sum_{i=1}^m \frac{\psi_0(m+1+i)}{i} = \psi_0^2(m+1) - \psi_0(1)\psi_0(m+1) - \frac{1}{2}\psi_1(m+1) + \frac{1}{2}\psi_1(1) \quad (\text{B9})$$

$$\begin{aligned}
\sum_{i=1}^m \psi_0(i+a)\psi_0(i+b) &= (b-a) \sum_{i=1}^{m-1} \frac{\psi_0(a+i)}{b+i} \\
&\quad - a\psi_0(a+1)\psi_0(b+1) + (m+a)\psi_0(m+a) \\
&\quad \times \psi_0(m+b) + a\psi_0(a+1) - (m+a-1)\psi_0(m+a) - (m+b)\psi_0(m+b) \\
&\quad + (b+1)\psi_0(b+1) + 2m - 2
\end{aligned} \tag{B10}$$

$$\begin{aligned}
\sum_{i=1}^m i\psi_0(i+a)\psi_0(i+b) &= \frac{1}{2}(b-a+1)(a-b) \sum_{i=1}^{m-1} \frac{\psi_0(a+i)}{b+i} \\
&\quad - \frac{1}{4}a(a+2b-3)\psi_0(a+1) - \frac{1}{4}(b+1) \\
&\quad \times (2a+b-2)\psi_0(b+1) + \frac{1}{2}(a-1)a\psi_0(a+1)\psi_0(b+1) + \frac{1}{4}(a+m-1) \\
&\quad \times (a+2b-m-2)\psi_0(a+m) + \frac{1}{4}(b+m)(2a+b-m-1)\psi_0(b+m) - \frac{1}{2} \\
&\quad \times (a^2 - a - m(m+1))\psi_0(a+m)\psi(b+m) - \frac{1}{4}(m-1)(3a+3b-m-4)
\end{aligned} \tag{B11}$$

$$\begin{aligned}
\sum_{i=1}^m i^2\psi_0(i+a)\psi_0(i+b) &= \frac{1}{6}(a-b)(3a^2+2ab-4a-2b^2-b+1) \sum_{i=1}^{m-1} \frac{\psi_0(a+i)}{b+i} \\
&\quad - \frac{1}{6}(-a^2(5b+2)+3a^3+a(5b-1)-m(2m^2+3m+1))\psi_0 \\
&\quad (a+m)\psi_0(b+m) + \frac{1}{6}(a-1)a \\
&\quad \times (3a-5b+1)\psi_0(a+1)\psi_0(b+1) - \left(-\frac{1}{12}(2b-1)m^2 + \frac{1}{36}m(24a^2 \right. \\
&\quad - 24ab - 24a + 12b^2 + 12b - 1) + \frac{1}{36}(a-1)(28a^2 - 18ab - 5a + 6b \\
&\quad + 12b^2 + 6) + \left.\frac{(a-1)(a-b)}{3(b+m-1)} + \frac{m^3}{9}\right)\psi_0(a+m) - \frac{1}{36}(4m^3 12a^2 b \\
&\quad - 3(2a-1)m^2 + (12a^2 - 12a - 1)m - 30a^2 + 6ab^2 - 12ab + 30a + 4b^3 \\
&\quad - 3b^2 - b)\psi_0(b+m) + \frac{1}{36}a(28a^2 - 9a(2b+1) + 12b^2 - 13)\psi_0(a+1) \\
&\quad + \frac{1}{36}(6a^2(2b-3) + 6a(b^2 - 2b + 2) + 4b^3 - 3b^2 - b + 6)\psi_0(b+1) \\
&\quad + \frac{2m^3}{27} - \frac{5}{36}m^2(a+b-1) + \frac{1}{36}(-40a^2 + 12ab + 51a - 16b^2 + 3b - 16) \\
&\quad + \frac{1}{108}m(120a^2 - 36ab - 138a + 48b^2 + 6b + 25) + \frac{a-1}{3(b+m-1)} \\
&\quad - \frac{a-1}{3(a+m-1)}
\end{aligned} \tag{B12}$$

$$\begin{aligned}
\sum_{i=1}^m \frac{\psi_0(a+1-i)}{i} &= - \sum_{i=1}^m \frac{\psi_0(i+a-m)}{i} + \frac{1}{2}(\psi_1(a+1) - \psi_1(a-m)) \\
&\quad + (\psi_0(a-m) + \psi_0(a+1))(\psi_0(m+1) - \psi_0(1))
\end{aligned}$$

$$+ \frac{1}{2}(\psi_0(a-m) - \psi_0(a+1))^2 \quad (\text{B13})$$

$$\begin{aligned} \sum_{i=1}^m \left( \frac{\psi_0(i+b)}{i+a} + \frac{\psi_0(i+a)}{i+b} \right) &= \psi_0(m+a+1)\psi_0(m+b+1) \\ &- \psi_0(a+1)\psi_0(b+1) + \frac{1}{a-b}(\psi_0(m+a+1) \\ &- \psi_0(m+b+1) - \psi_0(a+1) + \psi_0(b+1)) \end{aligned} \quad (\text{B14})$$

$$\begin{aligned} \sum_{i=1}^m \frac{\psi_0(a+b+i)}{i} &= \sum_{i=1}^m \frac{\psi_0(b+i)}{i} - \sum_{i=1}^a \frac{\psi_0(b+i+m)}{b+i-1} + \frac{1}{2}((\psi_0(a+b) \\ &- \psi_0(b)) \times (\psi_0(a+b) + \psi_0(b) + 2(\psi_0(m+1) - \psi_0(1))) \\ &- \psi_1(a+b) + \psi_1(b)) \end{aligned} \quad (\text{B15})$$

$$\begin{aligned} \sum_{i=1}^m \left( \frac{\psi_0(a+b+i+m)}{b+i} + \frac{\psi_0(a+i)}{b+i} + \frac{\psi_0(a+b+2i)}{a+i} - \frac{\psi_0(a+b+i)}{a+i} \right. \\ \left. - \frac{\psi_0(a+b+2i)}{b+i} \right) \\ = \frac{\psi_0\left(\frac{a}{2} + \frac{b}{2} + m\right)}{b-a} - \frac{(a+b+m)\psi_0(a+b+m)}{b(a+m)} + \frac{\psi_0(a+b+2m)}{a+m} \\ + \psi_0(a+m) \left( \psi_0(b+m+1) - \psi_0(b) - \frac{1}{b-a} \right) + \frac{a\psi_0(a)}{b(b-a)} + \frac{\psi_0(a+b)}{b} \\ - \frac{\psi_0\left(\frac{a}{2} + \frac{b}{2}\right)}{b-a} \end{aligned} \quad (\text{B16})$$

$$\begin{aligned} \sum_{i=1}^m \left( \frac{\psi_0(a+b+i+m)}{a+i} + \frac{\psi_0(a+b+i+m)}{b+i} - \frac{\psi_0(a+b+i)}{a+i} \right. \\ \left. - \frac{\psi_0(a+b+i)}{b+i} \right) \\ = \frac{(a+b+2m)\psi_0(a+b+2m)}{(a+m)(b+m)} - \left( \frac{1}{a+m} + \frac{1}{a} + \frac{1}{b+m} + \frac{1}{b} \right) \psi_0(a+b+m) \\ + \left( \frac{1}{a} + \frac{1}{b} \right) \psi_0(a+b) + (\psi_0(a+m) - \psi_0(a))(\psi_0(b+m) - \psi_0(b)) \end{aligned} \quad (\text{B17})$$

$$\sum_{i=1}^m \frac{(n-i)!}{(m-i)!} = \frac{n!}{(m-1)!(n-m+1)} \quad (\text{B18})$$

$$\sum_{i=1}^m \frac{(n-i)!}{(m-i)!i} = \frac{n!}{m!}(\psi_0(n+1) - \psi_0(n-m+1)) \quad (\text{B19})$$

$$\begin{aligned} \sum_{i=1}^m \frac{(n-i)!}{(m-i)!i^2} &= \frac{n!}{m!} \left( \sum_{i=1}^m \frac{\psi_0(i+n-m)}{i} + \frac{1}{2}(\psi_1(n-m+1) - \psi_1(n+1) \right. \\ &+ \psi_0(n-m+1)^2 - \psi_0(n+1)^2) + \psi_0(n-m) \\ &\times (-\psi_0(n-m+1) + \psi_0(n+1) - \psi_0(m+1) + \psi_0(1)) \Big) \end{aligned} \quad (\text{B20})$$

$$\sum_{i=1}^m \frac{(n-i)!}{(m+a-i)!} = \frac{1}{n-m-a+1} \left( \frac{n!}{(a+m-1)!} - \frac{(n-m)!}{(a-1)!} \right) \quad (\text{B21})$$

$$\begin{aligned}
& \sum_{i=1}^m \frac{(n-i)!}{(m+a-i)!} \psi_0(m+a-i+1) = \frac{1}{1-a-m+n} \\
& \left( \frac{n!}{(a+m-1)!} \left( \psi_0(a+m) - \frac{1}{1-a-m+n} \right) \right. \\
& \left. - \frac{(n-m)!}{(a-1)!} \left( \psi_0(a) - \frac{1}{1-a-m+n} \right) \right) \quad (\text{B22})
\end{aligned}$$

The derivation of the identities (B1)–(B14) and (B18)–(B20) can be found in [18, 19, 28, 40, 42, 44]. In the same manner as obtaining (B6) in Sect. 2.2.1, the three identities (B15)–(B17) are derived by first rewriting, respectively, the summations

$$\sum_{i=1}^m \frac{\psi_0(a+b+i)}{i} = \sum_{i=1}^m \frac{\psi_0(b+i)}{i} + \sum_{j=1}^a \sum_{i=1}^m \frac{1}{i} \frac{1}{b+i+j-1} \quad (\text{B23})$$

$$\sum_{i=1}^m \frac{\psi_0(a+b+2i)}{a+i} = \sum_{i=1}^m \frac{\psi_0(a+b+i)}{a+i} + \sum_{j=1}^m \sum_{i=j}^m \frac{1}{(a+i)(a+b+i+j-1)} \quad (\text{B24})$$

$$\sum_{i=1}^m \frac{\psi_0(a+b+i+m)}{b+i} = \sum_{i=1}^m \frac{\psi_0(a+b+i)}{b+i} + \sum_{j=1}^m \sum_{i=1}^m \frac{1}{(b+i)(a+b+i+j-1)}. \quad (\text{B25})$$

For the identity (B21), it is obtained by first considering

$$\sum_{i=1}^m \frac{(n-i)!}{(m+a-i)!} = \sum_{i=1}^{a+m} \frac{(n-i)!}{(m+a-i)!} - \sum_{i=1}^a \frac{(n-m-i)!}{(a-i)!} \quad (\text{B26})$$

before applying (B18). Note that the identity (B21) is analytically continued to any complex number  $a$ , where, by taking an derivative of  $a$ , the identity (B22) is established.

## B.2. Additional Summation Identities of New Simplification Framework

Here, we list some additional summation identities of the new simplification framework useful in the simplification process in Sect. 2.3. These identities are obtained by taking appropriate derivatives of the formulas in Lemmas 1–4. It is sufficient to assume that  $a, b, c \geq 0$  in (B27)–(B35).

$$\begin{aligned}
& \sum_{i=1}^m \frac{\psi_0^2(a+i) - \psi_1(a+i)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\
& = \frac{\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)} (\psi_1(a+b+2m-1) - \psi_1(a+b+m) \\
& \quad - \psi_1(a+m) + (\psi_0(a+b+m) - \psi_0(a+b+2m-1) + \psi_0(a+m))^2) \quad (\text{B27}) \\
& \sum_{i=1}^m \frac{\psi_0(i)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\Gamma(m)\Gamma(b+m)\Gamma(a+b+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i} \\
&\quad + \frac{\psi_0(m)\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)} \quad (\text{B28})
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^m \frac{\psi_0(i)\psi_0(b-i+m+1)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\
&= \frac{1}{\Gamma(m)\Gamma(b+m)\Gamma(a+b+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i} (-\psi_0(a+b+m) \\
&\quad + \psi_0(a+b-i+2m-1) - \psi_0(b+m)) + \frac{\Gamma(a+b+2m-1)\psi_0(m)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)} \\
&\quad \times \frac{1}{\Gamma(a+b+m)} (\psi_0(a+b+m) - \psi_0(a+b+2m-1) + \psi_0(b+m)) \quad (\text{B29})
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^m \frac{\psi_0(i)\psi_0(a+i)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\
&= \frac{1}{\Gamma(m)\Gamma(b+m)\Gamma(a+b+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i} (-\psi_0(a+b+m) \\
&\quad + \psi_0(a+b-i+2m-1) - \psi_0(a-i+m)) + \frac{\psi_0(m)\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(b+m)\Gamma(a+b+m)} \\
&\quad \times \frac{\psi_0(a+b+m) - \psi_0(a+b+2m-1) + \psi_0(a+m)}{\Gamma(a+m)} \quad (\text{B30})
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^m \frac{\psi_0^2(i) - \psi_1(i)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\
&= \frac{2}{\Gamma(m)\Gamma(b+m)\Gamma(a+b+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i} (\psi_0(i) - \psi_0(m) \\
&\quad - \psi_0(1)) + \frac{\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)} (\psi_0^2(m) - \psi_1(m)) \quad (\text{B31})
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^m \frac{\psi_0(i)\psi_0(m-i+1)}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(b-i+m+1)} \\
&= \frac{1}{\Gamma(m)\Gamma(a+b+m)} \left( -\frac{1}{\Gamma(a+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(b-i+m)i^2} - \frac{1}{\Gamma(b+m)} \right. \\
&\quad \times \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i^2} - \frac{\psi_0(m)}{\Gamma(a+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(b-i+m)i} \\
&\quad \left. - \frac{\psi_0(m)}{\Gamma(b+m)} \sum_{i=1}^{m-1} \frac{\Gamma(a+b-i+2m-1)}{\Gamma(a-i+m)i} \right) + \frac{\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)} \\
&\quad \times \frac{1}{\Gamma(a+b+m)} (\psi_0^2(m) - \psi_1(m) + \psi_1(1)) \quad (\text{B32})
\end{aligned}$$

$$\sum_{i=1}^m \frac{\Gamma(c-i+m)\Gamma(b+i+m)}{\Gamma(i)\Gamma(m-i+1)} \psi_0(i)$$

$$= -\frac{\Gamma(b+1)\Gamma(c)\Gamma(c+m)}{\Gamma(m)\Gamma(b+c+1)} \sum_{i=1}^{m-1} \frac{\Gamma(b+c-i+m)}{\Gamma(c-i+m)i} + \frac{\Gamma(b+1)\Gamma(c)\Gamma(b+c+m)}{\Gamma(m)\Gamma(b+c+1)} \\ \times (\psi_0(b+c+m) - \psi_0(b+c+1) + \psi_0(m)) \quad (\text{B33})$$

$$\sum_{i=1}^m \frac{\Gamma(c-i+m)\Gamma(b+i+m)}{\Gamma(i)\Gamma(m-i+1)} (\psi_0^2(i) - \psi_1(i)) \\ = \frac{2\Gamma(b+1)\Gamma(c)\Gamma(c+m)}{\Gamma(m)\Gamma(b+c+1)} \left( -\sum_{i=1}^{m-1} \frac{\Gamma(b+c-i+m)\psi_0(b+c-i+m)}{\Gamma(c-i+m)i} \right. \\ \left. + \sum_{i=1}^{m-1} \frac{\Gamma(b+c-i+m)\psi_0(i)}{\Gamma(c-i+m)i} - (-\psi_0(b+c+1) + \psi_0(m) + \psi_0(1)) \right. \\ \left. \times \sum_{i=1}^{m-1} \frac{\Gamma(b+c-i+m)}{\Gamma(c-i+m)i} \right) + \frac{\Gamma(b+1)\Gamma(c)\Gamma(b+c+m)}{\Gamma(m)\Gamma(b+c+1)} (\psi_1(b+c+m) \\ - \psi_1(b+c+1) - \psi_1(m) + (\psi_0(b+c+m) - \psi_0(b+c+1) + \psi_0(m))^2) \quad (\text{B34})$$

$$\sum_{i=1}^m \frac{\Gamma(c-i+m)\Gamma(a+b+i+m)}{\Gamma(a+i)\Gamma(m-i+1)i} \psi_0(a+b+i+m) \\ = \Gamma(c+m)\Gamma(a+b+m) \left( \frac{\psi_0(a+b+m)}{\Gamma(a)\Gamma(m+1)} (\psi_0(c+m) - \psi_0(c)) + \Gamma(b+m+1) \right. \\ \left. \times \Gamma(c) \sum_{i=1}^m \frac{\psi_0(a+b+m) - \psi_0(b-i+m+1) + \psi_0(b+m+1)}{\Gamma(a+i)\Gamma(c+i)\Gamma(m-i+1)\Gamma(b-i+m+1)i} \right) \quad (\text{B35})$$

## Appendix C: Parameters of Correlation Kernel

In this appendix, we summarize in the table below the parameters of correlation kernel (42) in Sect. 2.1.

TABLE 3. Parameters of the correlation kernel (42) in case A and case B

|       | Case A: Arbitrary number of particle (17)                            | Case B: Fixed number of particle (18)                                      |
|-------|----------------------------------------------------------------------|----------------------------------------------------------------------------|
| $a$   | $n-m$                                                                | $n-p$                                                                      |
| $b$   | $n-m$                                                                | $p-m$                                                                      |
| $k$   | $2k$                                                                 | $k$                                                                        |
| $h_k$ | $\frac{(4k+2a+1)^{-1}\Gamma^2(2k+a+1)}{\Gamma(2k+2a+1)\Gamma(2k+1)}$ | $\frac{2\Gamma(k+a+1)\Gamma(k+b+1)}{(2k+a+b+1)\Gamma(k+1)\Gamma(k+a+b+1)}$ |

## References

- [1] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York (1972)

- [2] Bianchi, E., Hackl, L., Kieburg, M.: The Page curve for fermionic Gaussian states. *Phys. Rev. B* **103**, L241118 (2021)
- [3] Bianchi, E., Hackl, L., Kieburg, M., Rigol, M., Vidmar, L.: Volume-law entanglement entropy of typical pure quantum states. *PRX Quantum* **3**, 030201 (2022)
- [4] Bernard, D., Piroli, L.: Entanglement distribution in the quantum symmetric simple exclusion process. *Phys. Rev. E* **104**, 014146 (2021)
- [5] Bertola, M., Gekhtman, M., Szmigielski, J.: The Cauchy two-matrix model. *Commun. Math. Phys.* **287**, 983–1014 (2009)
- [6] Bertola, M., Gekhtman, M., Szmigielski, J.: Cauchy biorthogonal polynomials. *J. Approx. Theory* **162**, 832–867 (2010)
- [7] Bertola, M., Gekhtman, M., Szmigielski, J.: Cauchy–Laguerre two-matrix model and the Meijer-G random point field. *Commun. Math. Phys.* **326**, 111–144 (2014)
- [8] Borot, G., Nadal, C.: Purity distribution for generalized random Bures mixed states. *J. Phys. A: Math. Theor.* **45**, 075209 (2012)
- [9] Brychkov, Y.A.: *Handbook of Special Functions: Derivatives, Integrals. Series and Other Formulas*. CRC Press, Boca Raton (2008)
- [10] Dehesa, J.S., Martínez-Finkelshtdein, A., Sánchez-Ruiz, J.: Quantum information entropies and orthogonal polynomials. *J. Comput. Appl. Math.* **133**, 23–46 (2001)
- [11] Exton, H.: A new two-term relation for the  ${}_3F_2$  hypergeometric function of unit argument. *J. Comput. Appl. Math.* **106**, 395–397 (1999)
- [12] Foong, S.K., Kanno, S.: Proof of Page’s conjecture on the average entropy of a subsystem. *Phys. Rev. Lett.* **72**, 1148–1151 (1994)
- [13] Forrester, P.: *Log-Gases and Random Matrices*. Princeton University Press, Princeton (2010)
- [14] Forrester, P., Kieburg, M.: Relating the Bures measure to the Cauchy two-matrix model. *Commun. Math. Phys.* **342**, 151–187 (2016)
- [15] Gasper, G., Rahman, M.: *Basic Hypergeometric Series*. Cambridge University Press, Cambridge (2004)
- [16] Giraud, O.: Distribution of bipartite entanglement for random pure states. *J. Phys. A: Math. Theor.* **40**, 2793 (2007)
- [17] Hayden, P., Leung, D.W., Winter, A.: Aspects of generic entanglement. *Commun. Math. Phys.* **265**, 95–117 (2006)
- [18] Huang, Y., Wei, L., Collaku, B.: Kurtosis of von Neumann entanglement entropy. *J. Phys. A: Math. Theor.* **54**, 504003 (2021)
- [19] Huang, Y., Wei, L.: Second-order statistics of fermionic Gaussian states. *J. Phys. A: Math. Theor.* **55**, 105201 (2022)
- [20] Kieburg, M., Forrester, P., Ipsen, J.R.: Multiplicative convolution of real asymmetric and real anti-symmetric matrices. *Adv. Pure Appl. Math.* **10**, 467 (2019)
- [21] Li, S.-H., Wei, L.: Moments of quantum purity and biorthogonal polynomial recurrence. *J. Phys. A: Math. Theor.* **54**, 445204 (2021)
- [22] Lubkin, E.: Entropy of an n-system from its correlation with a k-reservoir. *J. Math. Phys.* **19**, 1028 (1978)
- [23] Luke, Y.L.: *The Special Functions and Their Approximations*. Academic Press, New York (1969)

- [24] Lydzba, P., Rigol, M., Vidmar, L.: Eigenstate entanglement entropy in random quadratic Hamiltonians. *Phys. Rev. Lett.* **125**, 180604 (2020)
- [25] Lydzba, P., Rigol, M., Vidmar, L.: Entanglement in many-body eigenstates of quantum-chaotic quadratic Hamiltonians. *Phys. Rev. B* **103**, 104206 (2021)
- [26] Mehta, M.L.: *Random Matrices*. Elsevier, Amsterdam (2004)
- [27] Malacarne, L.C., Mendes, R.S., Lenzi, E.K.: Average entropy of a subsystem from its average Tsallis entropy. *Phys. Rev. E* **65**, 046131 (2002)
- [28] Milgram, M.: On some sums of digamma and polygamma functions. Preprint (2017). [arXiv:math/0406338](https://arxiv.org/abs/math/0406338)
- [29] Miller, A.R., Paris, R.B.: Transformation formulas for the generalized hypergeometric function with integral parameter differences. *Rocky Mt. J. Math.* **43**, 291–327 (2013)
- [30] Onuma-Kalu, M., Grimmer, D., Mann, R.B., Martín-Martínez, E.: A classification of Markovian fermionic Gaussian master equations. *J. Phys. A: Math. Theor.* **52**, 435302 (2019)
- [31] Osipov, V., Sommers, H.-J., Życzkowski, K.: Random Bures mixed states and the distribution of their purity. *J. Phys. A: Math. Theor.* **43**, 055302 (2010)
- [32] Page, D.N.: Average entropy of a subsystem. *Phys. Rev. Lett.* **71**, 1291–1294 (1993)
- [33] Preskill, J.: Quantum computing in the NISQ era and beyond. *Quantum* **2**, 79 (2018)
- [34] Sánchez-Ruiz, J.: Simple proof of Page’s conjecture on the average entropy of a subsystem. *Phys. Rev. E* **52**, 5653–5655 (1995)
- [35] Sarkar, A., Kumar, S.: Bures-Hall ensemble: spectral densities and average entropies. *J. Phys. A: Math. Theor.* **52**, 295203 (2019)
- [36] Sommers, H.-J., Życzkowski, K.: Statistical properties of random density matrices. *J. Phys. A: Math. Gen.* **37**, 35 (2004)
- [37] Surace, J., Tagliacozzo, L.: Fermionic Gaussian states: an introduction to numerical approaches. *SciPost Phys. Lect. Notes*, **054** (2022)
- [38] Szegő, G.: *Orthogonal Polynomials*. American Mathematical Society, Providence (1975)
- [39] Vivo, P., Pato, M.P., Oshanin, G.: Random pure states: quantifying bipartite entanglement beyond the linear statistics. *Phys. Rev. E* **83**, 052106 (2016)
- [40] Wei, L.: Proof of Vivo-Pato-Oshanin’s conjecture on the fluctuation of von Neumann entropy. *Phys. Rev. E* **96**, 022106 (2017)
- [41] Wei, L.: On the exact variance of Tsallis entanglement entropy in a random pure state. *Entropy* **21**, 539 (2019)
- [42] Wei, L.: Skewness of von Neumann entanglement entropy. *J. Phys. A: Math. Theor.* **53**, 075302 (2020)
- [43] Wei, L.: Proof of Sarkar–Kumar conjectures on average entanglement entropies over the Bures–Hall ensemble. *J. Phys. A: Math. Theor.* **53**, 235203 (2020)
- [44] Wei, L.: Exact variance of von Neumann entanglement entropy over the Bures–Hall measure. *Phys. Rev. E* **102**, 062128 (2020)
- [45] Wei, L.: Average capacity of quantum entanglement. *J. Phys. A: Math. Theor.* **56**, 015302 (2023)

Youyi Huang and Lu Wei  
Department of Computer Science  
Texas Tech University  
Lubbock TX 79409  
USA  
e-mail: [luwei@ttu.edu](mailto:luwei@ttu.edu)

Communicated by David Pérez-García.

Received: January 31, 2023.

Accepted: June 19, 2023.