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Entanglement capacity of fermionic Gaussian states

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Abstract

We study the capacity of entanglement as an alternative to entanglement entropies in estimating the degree of entanglement of quantum bipartite systems over fermionic Gaussian states. In particular, we derive the exact and asymptotic formulas of average capacity of two different cases—with and without particle number constraints. For the later case, the obtained formulas generalize some partial results of average capacity in the literature. The key ingredient in deriving the results is a set of new tools for simplifying finite summations developed very recently in the study of entanglement entropy of fermionic Gaussian states.

Keywords: quantum entanglement, entanglement capacity, fermionic Gaussian states, random matrix theory, orthogonal polynomials, special functions

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement is a fundamental feature of quantum mechanics and it is also the resource that enables quantum information processing as an emerging technology. The understanding of entanglement is crucial to a successful exploitation of advances of the quantum revolution. In the past decades, there has been considerable progress in estimating the degree of entanglement over different models of generic states, where one of the most extensively studied area is the entropy based estimations using, for example, von Neumann entropy [1–11], quantum purity [7, 12–16], and Tsallis entropy [17, 18] as entanglement indicators. These results mainly focus on the statistical behavior of entanglement entropies over generic state models, such

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as the well-known Hilbert–Schmidt ensemble [1–6, 8, 11, 12, 15, 17, 18], the Bures–Hall ensemble [7, 9, 10, 13, 14, 16, 19], and the fermionic Gaussian ensemble [20–23].

Besides entropies, there is a growing interest in understanding the capacity of entanglement as another entanglement quantifier. Similarly to entanglement entropy as an analogy to the thermal entropy, the entanglement capacity introduced in [24] serves as an analogy to thermal heat capacity. In the time evolution of quantum systems, capacity is observed to detect the presence of entanglement at earlier times than entropies could capture. It is also identified as a critical value to distinguish integrable systems from chaotic ones [25]. In the literature, different properties of entanglement capacity have been numerically studied in [25, 26]. Moreover, exact formulas of the average capacity of finite subsystem dimensions are recently obtained for the Hilbert–Schmidt ensemble [27–29] and the Bures–Hall ensemble [29]. For the fermionic Gaussian ensemble without particle number constraint, the average capacity of equal subsystem dimensions is derived in [21, 30], whereas the corresponding finite-size formula in the general case of unequal dimensions remains open. Knowing the finite-size formula of average capacity allows the comparison to the finite-size results of entanglement entropy obtained recently in [20–23], leading to a more comprehensive understanding of the properties of entanglement capacity. For noisy intermediate-scale quantum systems [31], where the qubits number is limited to a few dozens, finite-size estimations of degree of entanglement including the finite-size average capacity become crucial. Furthermore, knowing the finite-size formula of average capacity is useful in constructing simple Gaussian approximations to the distribution of entanglement capacity.

In this work, we compute the exact average entanglement capacity valid for any subsystem dimensions of fermionic Gaussian states for the cases of with and without particle number constraints. A key ingredient in obtaining the results is the set of tools for simplifying finite summations developed very recently [23] in the study of von Neumann entropy of the fermionic Gaussian ensemble. Our exact results also lead to the limiting values of average capacity when the subsystem dimensions approach infinity with a fixed dimension difference. Simulations are performed to numerically verify the derived results.

The rest of the paper is organized as follows. In section 2, we first outline the problem formulation before presenting our main results of the exact mean capacity of fixed particle numbers and arbitrary particle numbers in propositions 1 and 2, respectively. The corresponding asymptotic capacity formulas are given in corollary 1. Proofs to the results are provided in section 3. In appendix A, we list summation representations of the integrals involved in the proofs. Summation identities utilized in the simplification are listed in appendix B. The coefficients of some intermediate results appeared in the derivation are provided in appendix C.

2. Problem formulation and main results

2.1. Problem formulation

We first introduce the formulation that leads to the entanglement capacity of fermionic Gaussian states with and without particle number constraints as well as the corresponding statistical ensembles.

A system of N fermionic degree of freedom can be formulated in terms of a set of fermionic creation and annihilation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$, $i=1,\ldots,N$, which obey the canonical anti-commutation relation,

$$\left\{\hat{a}_i,\hat{a}_j^\dagger\right\}=\delta_{ij}\mathbb{I},\qquad \{\hat{a}_i,\hat{a}_j\}=0=\left\{\hat{a}_i^\dagger,\hat{a}_j^\dagger\right\}, \tag{1}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the anti-commutation relation and $\mathbb{I}$ is an identity operator. These fermionic modes can be equivalently described via the Majorana operators γ_l, $l = 1, \ldots, 2N$, and

$$\hat{\gamma}_{2i-1} = \frac{\hat{a}_i^\dagger + \hat{a}_i}{\sqrt{2}}, \qquad \hat{\gamma}_{2i} = \imath \frac{\hat{a}_i^\dagger + \hat{a}_i}{\sqrt{2}} \tag{2}$$

with $\imath = \sqrt{-1}$ denoting the imaginary unit. The Majorana operators also satisfy the anti-commutation relation

$$\{\hat{\gamma}_l, \hat{\gamma}_k\} = \delta_{lk}\mathbb{I}. \tag{3}$$

By collecting the Majorana operators into a $2N$ dimensional operator-valued column vector $\gamma = (\hat{\gamma}_1, \ldots, \hat{\gamma}_{2N})^\dagger$, a system of fermionic Gaussian state is then characterized by the density operator of the form [22, 32]

$$\rho(\gamma) = \frac{\mathrm{e}^{-\gamma^\dagger Q \gamma}}{\mathrm{tr}\left(\mathrm{e}^{-\gamma^\dagger Q \gamma}\right)}, \tag{4}$$

where the coefficient matrix Q is a $2N \times 2N$ imaginary anti-symmetric matrix as the consequence of the anti-communication relation (3).

2.2. *Entanglement capacity over fermionic Gaussian states without particle number constraint*

There always exists an orthogonal matrix M that diagnoses the coefficient matrix Q by transforming γ into another Majorana basis $\mu = (\hat{\mu}_1, \ldots, \hat{\mu}_{2N})^\dagger = M\gamma$. A fermionic Gaussian state of arbitrary particle numbers is determined by the anti-symmetric covariance matrix [22]

$$J = -\imath \tanh(Q) = M^T J_0 M, \tag{5}$$

where $\tanh(x)$ denotes the hyperbolic tangent function [33], the matrix J_0 takes the block diagonal form

$$J_0 = \begin{pmatrix} \tanh(\lambda_1)\mathbb{A} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \tanh(\lambda_N)\mathbb{A} \end{pmatrix}, \tag{6}$$

and

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7}$$

In the setting of the quantum bipartite model [34], the system of N fermionic degree of freedoms can be decomposed into two subsystems A and B of dimension m and n, respectively, with $m + n = N$. We assume $m \leqslant n$ without loss of generality. By restricting the matrix J to the entries from subsystem A, the restricted covariance matrix J_A is the $2m \times 2m$ left-upper block of J. The entanglement capacity can be represented via the real positive eigenvalues x_i, $i = 1, \ldots, m$ of $\imath J_A$ as [25, 26, 30]

$$C = \sum_{i=1}^{m} u(x_i) \tag{8}$$

with

$$u(x)=\frac{1-x^2}{4}\ln^2\frac{1+x}{1-x}. \tag{9}$$

The resulting joint probability density of the eigenvalues x_i, $i=1,\ldots,m$ is proportional to [20]

$$\prod_{1\leqslant i<j\leqslant m}\left(x_i^2-x_j^2\right)^2\prod_{i=1}^{m}\left(1-x_i^2\right)^{n-m}, \qquad x_i\in[0,1], \tag{10}$$

which is obtained by recursively applying the result in [35, proposition A.2].

2.3. Entanglement capacity over fermionic Gaussian states with particle number constraint

For a fermionic Gaussian state $|F\rangle$ with a fixed particle number p, it is more convenient to formulate it with the fermionic creation and annihilation operators, and the corresponding covariance matrix can be expressed as [22, 36, 37]

$$H_{ij}=-\imath\langle F|\hat{a}_i^\dagger\hat{a}_j-\hat{a}_j\hat{a}_i^\dagger|F\rangle. \tag{11}$$

Using the anti-commutation relation (1), the entries of the matrix H then become

$$H_{ij}=-2\imath G_{ij}+\imath\delta_{ij}, \tag{12}$$

where $G_{ij}=\langle F|\hat{a}_i^\dagger\hat{a}_j|F\rangle$ denotes the entries of an $N\times N$ matrix G. There always exists a unitary transformation U that diagonalizes G. In the resulting diagonal form, the first p elements are equal to 1 and the rest are 0. Therefore, one can write

$$G=U_{N\times p}U_{N\times p}^\dagger. \tag{13}$$

Denoting y_i, $i=1,\ldots,m$ the eigenvalues of the restricted matrix $G_A=U_{m\times p}U_{m\times p}^\dagger$, the entanglement capacity can be represented as the function of y_i as [26]

$$C=-\sum_{i=1}^{m}u(2y_i-1), \qquad y_i\in[0,1]. \tag{14}$$

The eigenvalue distribution of the random matrix $U_{m\times p}U_{m\times p}^\dagger$ is the well-known Jacobi unitary ensemble [38, 39]. Here, it is more convenient to use the eigenvalues of matrix $\imath H$. Denote x_i, $i=1,\ldots,m$, as the eigenvalues of the $m\times m$ upper-left block of the matrix $\imath H$, the change of variables $x_i=2y_i-1$ in (14) leads to the entanglement capacity (8) for the case of fixed particle number. The resulting joint probability density of the eigenvalues $x_i, i=1,\ldots,m$, is proportional to [40]

$$\prod_{1\leqslant i<j\leqslant m}(x_i-x_j)^2\prod_{i=1}^{m}(1+x_i)^{p-m}(1-x_i)^{n-p}, \qquad x_i\in[-1,1]. \tag{15}$$

It has been introduced in [23] that the joint probability densities (10) and (15) can be compactly represented by a single joint density as

$$f_{\mathrm{FG}}(x)\propto\prod_{1\leqslant i<j\leqslant m}\left(x_i^\gamma-x_j^\gamma\right)^2\prod_{i=1}^{m}(1-x_i)^a(1+x_i)^b. \tag{16}$$

The two considered scenarios of fermionic Gaussian states can now be conveniently identified by the above density (16), where we have

$$\gamma=1,\quad a=n-p\geqslant 0,\quad b=p-m\geqslant 0,\quad x\in[-1,1] \tag{17}$$

for fermionic Gaussian states with an arbitrary number of particles, and

$$\gamma=2,\quad a=b=n-m\geqslant 0,\quad x\in[0,1] \tag{18}$$

for fermionic Gaussian states with a fixed number of particles. Note that computing the average capacity for the two cases will be performed separately below since the computation for an arbitrary γ in (16) appears difficult. We omit the normalization constants in (16) as they will not be utilized in the calculation.

2.4. Main results

We now present our main results on the exact and asymptotic average capacity of the fermionic Gaussian states for the cases of fixed and arbitrary number of particles.

Proposition 1. *Denote the summation $\Phi_{c,d}$ as*

$$\Phi_{c,d}=\frac{c!}{(c+d)!}\sum_{k=1}^{c}\frac{(c+d-k)!}{(c-k)!}\frac{1}{k^2},\qquad c,d\in\mathbb{Z}^+, \tag{19}$$

and the function $F(a,b)$ as

$$\begin{aligned}F(a,b)=&\,\alpha_0\big(2\Phi_{a+m,b}+2\Phi_{m,a}+\psi_1(a+b+m+1)+\psi_1(a+m+1)+(\psi_0(a+m+1)\\&-\psi_0(a+b+m+1))^2-\psi_1(1)\big)+\alpha_1\psi_0(a+m+1)+\alpha_2\psi_0(a+1)+\alpha_3,\end{aligned} \tag{20}$$

where the coefficients α_i are

$$\alpha_0=\frac{m(a+m)(b+m)(a+b+m)}{(a+b+2m-1)_3} \tag{21}$$

$$\alpha_1=\frac{(a+b)(a+m-1)(a+m)}{(a+b+2m-1)_2} \tag{22}$$

$$\alpha_2=-\frac{a\left(a^2+ab+2am-a+2bm-b+2m^2-2m\right)}{(a+b+2m-1)_2} \tag{23}$$

$$\alpha_3=\frac{m(a+m-1)}{(a+b+2m-1)_2}-\frac{m}{2}. \tag{24}$$

Then, for any subsystem dimensions $m\leqslant n$, the mean value of entanglement capacity (8) of fermionic Gaussian states with a fixed particle number p as in (17) is given by

$$\mathbb{E}[C]=F(p-m,n-p)+F(n-p,p-m). \tag{25}$$

In proposition 1,

$$\psi_0(x) = \frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x} \tag{26}$$

and

$$\psi_1(x) = \frac{\mathrm{d}^2\ln\Gamma(x)}{\mathrm{d}^2 x} \tag{27}$$

denote respectively the digamma and trigamma functions, and

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{28}$$

denotes the Pochhammer symbol. The proof of proposition 1 can be found in section 3.1. Note that the summation $\Phi_{c,d}$ in (19) does not in general admit a closed-form representation for arbitrary c and d. On the other hand, the sum $\Phi_{c,d}$ may be further simplified in some special cases as discussed in the following remark.

Remark 1. Substituting $i\to k$, $m\to c$, $n\to c+d$ in the identity (B.12), the summation $\Phi_{c,d}$ in (19) admits an alternative form

$$\sum_{k=1}^{c}\frac{\psi_0(k+d)}{k} + \mathrm{CF}, \tag{29}$$

where CF denotes the closed-form terms in the bracket of (B.12). The sum in (29) may not be summable into a closed-form expression and is referred to as an unsimplifiable basis [6, 8, 10, 11, 21, 23, 29]. However, in the special cases of a given integer d, it permits closed-form representation as a result of the identity (B.3). This corresponds to the case of fixed differences $a = n-p$, $b = p-m$, where the average capacity (25) admits more explicit expressions. The cases $a = b = 0, 1, 2$ are provided respectively in below as examples

$$\mathbb{E}[C] = -\frac{2m^3}{(2m-1)(2m+1)}\left(\psi_1(m+1) - \frac{\pi^2}{4}\right) - \frac{2m^2-2m+1}{2m-1} \tag{30}$$

$$\mathbb{E}[C] = -\frac{2m(m+1)(m+2)}{(2m+1)(2m+3)}\left(\psi_1(m+1) - \frac{\pi^2}{4}\right) - \frac{m(2m(m+3)+5)}{(m+1)(2m+3)} \tag{31}$$

$$\begin{aligned}\mathbb{E}[C] = &-\frac{2m(m+2)(m+4)}{(2m+3)(2m+5)}\left(\psi_1(m+1) - \frac{\pi^2}{4}\right) + \frac{4}{(m+1)(m+3)} \\ &\times(\psi_0(m+1) - \psi_0(1)) - \frac{m\left(m^2+4m+5\right)\left(4m^3+30m^2+72m+57\right)}{(2m+3)(2m+5)(m+1)_3}.\end{aligned} \tag{32}$$

Remark 2. By using the limiting behavior of polygamma functions

$$\psi_0(x) = \ln(x) - \frac{1}{2x} - \sum_{l=1}^{\infty}\frac{B_{2l}}{2lx^{2l}}, \qquad x\to\infty, \tag{33}$$

$$\psi_1(x) = \frac{1+2x}{2x^2} + \sum_{l=1}^{\infty}\frac{B_{2l}}{x^{2l+1}}, \qquad x\to\infty, \tag{34}$$

where B_k is the kth Bernoulli number [33], the finite-size formulas (30)–(32) respectively give the following asymptotic results for the cases $a=b=0,1,2$

$$\frac{\mathbb{E}[C]}{m}=\frac{\pi^2}{8}-1+\frac{\pi^2}{32m^2}+o\left(\frac{1}{m^3}\right) \tag{35}$$

$$\frac{\mathbb{E}[C]}{m}=\frac{\pi^2}{8}-1+\left(\frac{\pi^2}{8}-1\right)\frac{1}{m}-\frac{3\pi^2}{32m^2}+o\left(\frac{1}{m^3}\right) \tag{36}$$

$$\frac{\mathbb{E}[C]}{m}=\frac{\pi^2}{8}-1+\left(\frac{\pi^2}{4}-2\right)\frac{1}{m}-\frac{15\pi^2}{32m^2}+o\left(\frac{1}{m^3}\right). \tag{37}$$

Proposition 2. *For any subsystem dimensions $m\leqslant n$, the mean value of entanglement capacity* (8) *of fermionic Gaussian states with an arbitrary particle number as in* (18) *is given by*

$$\begin{aligned}\mathbb{E}[C]=&\,\beta\left(\Phi_{2m-1,n-m}+\Phi_{m+n-1,n-m}\right)+\frac{1}{4}\left(\Phi_{m-1,n}+\Phi_{m-1,n-m}\right)+\left(\frac{\beta}{2}+\frac{1}{8}\right)\\&\times\psi_1(m+n)+\frac{1}{8}\psi_1(n)+\frac{\beta}{2}\left(\left(\psi_0(2n)-\psi_0(m+n)\right)^2+\psi_1(2n)-\psi_1(1)\right)\\&+\frac{1}{8}\left(\psi_0(n)-\psi_0(m+n)\right)^2+\frac{n-m}{2}\left(\psi_0(m+n)-\psi_0(n-m)\right)-m,\end{aligned} \tag{38}$$

where $\Phi_{a,b}$ is defined in (19) *and the coefficient β is given by*

$$\beta=\frac{(2m-1)(2n-1)}{4m+4n-2}. \tag{39}$$

Proposition 2 is proved in section 3.2. It is important to point out that in deriving the results (25) and (38), we make use of the lemmas 1–4 in [23] as will also be discussed in section 3.1.2. The four lemmas are examples of a new simplification framework recently developed in [23] when studying the exact variance of von Neumann entropy. This new framework consists of a set of novel tools useful in simplifying the summations involved, including (A.3), (A.4), and (A.7) in appendix A. These summations do not permit further simplifications when using the existing simplification tools for the computation over Hilbert–Schmidt ensemble [6, 8, 11] or the Bures–Hall ensemble [9, 10, 16]. For proposition 2, we also have the following remark.

Remark 3. For the same reason as in remark 1, the result (38) admits closed-form representations for the special cases when the subsystem dimension difference $a=n-m$ is fixed. For example, by fixing $a=0,1,2,3$ in (38), we recover the recently obtained mean capacity values in [21, equations (27)–(30)]. We also list below the limiting behavior of average capacity for the cases $a=0,1,2,3$

$$\frac{\mathbb{E}[C]}{m}=\frac{\pi^2}{8}-1-\left(\frac{\pi^2}{32}-\frac{1}{4}\right)\frac{1}{m}+\frac{\pi^2}{128m^2}+o\left(\frac{1}{m^3}\right) \tag{40}$$

$$\frac{\mathbb{E}[C]}{m}=\frac{\pi^2}{8}-1+\left(\frac{\pi^2}{32}-\frac{1}{4}\right)\frac{1}{m}-\frac{3\pi^2}{128m^2}+o\left(\frac{1}{m^3}\right) \tag{41}$$

$$\frac{\mathbb{E}[C]}{m} = \frac{\pi^2}{8} - 1 + \left(\frac{3\pi^2}{32} - \frac{3}{4}\right)\frac{1}{m} - \frac{15\pi^2}{128m^2} + o\left(\frac{1}{m^3}\right) \tag{42}$$

$$\frac{\mathbb{E}[C]}{m} = \frac{\pi^2}{8} - 1 + \left(\frac{5\pi^2}{32} - \frac{5}{4}\right)\frac{1}{m} - \frac{35\pi^2}{128m^2} + o\left(\frac{1}{m^3}\right). \tag{43}$$

Based on the two propositions, the limiting behavior of the average capacity for any fixed subsystem dimension can now be obtained. The results are summarized in corollary 1 below, and the corresponding proof can be found in section 3.3.

Corollary 1. *For any subsystem dimensions $m \leqslant n$ in the asymptotic regime*

$$m \to \infty, \quad n \to \infty, \quad \text{with a fixed } n-m, \tag{44}$$

the average entanglement capacity of fermionic Gaussian states with a fixed particle number (25) *and with an arbitrary particle number* (38) *approach to the same limit*

$$\frac{\mathbb{E}[C]}{m} \longrightarrow \frac{\pi^2}{8} - 1. \tag{45}$$

In corollary 1, we note that for the case of fixed particle number, the particle number p also goes to infinity of the same rate as m and n in the limit (44). For the case of arbitrary number of particles, the limiting value (45), also known as the leading volume-law coefficient, was first obtained in [30] for equal subsystem dimensions. Here, we have extended it rigorously to a more general regime (44) starting from our explicit result (38). We also observe the interesting fact that the limiting value (45) is the same for the cases (17) and (18) despite the fundamental difference of the two underlying models.

Note that the asymptotic capacity over the fermionic Gaussian ensemble is different than the Hilbert–Schmidt ensemble or the Bures–Hall ensemble [29]. This is due to the difference between the entanglement structures of the ensembles. In particular, the interaction parts of the Hilbert–Schmidt ensemble and the Bures–Hall ensemble are different. Moreover, the observation of different asymptotic values can be understood from the variance of the modular Hamiltonian interpretation of the capacity [29]. The fact that the asymptotic average of entanglement capacity over Bures–Hall ensemble attains a larger value than the Hilbert–Schmidt ensemble implies that the width of the spectrum of Bures–Hall ensemble is on average wider than that of the Hilbert–Schmidt ensemble. On the other hand, the capacity behavior is expected to be model dependent, and the choice of the appropriate model is crucial for an accurate estimation.

To illustrate the obtained results, we plot in figure 1 the exact formulas (25) and (38) per dimension m for fixed subsystem dimension differences $n-m = 0,\ 4,\ 8$, along with the asymptotic value (45). The left-hand side figure corresponds to the case of a fixed particle number $p = (m+n)/2$, and the right-hand side corresponds to the case of an arbitrary particle number. We also plot the simulated values of mean capacity in figure 1, which match well with the analytical results. It is observed that as the dimension difference $n-m$ increases, the average capacity (25) and (38) approach to the limiting value (45) more slowly. This fact indicates that the finite-size capacity formulas are more useful when the dimension difference $n-m$ is large, see [29], and otherwise the asymptotic value (45) serves as a reasonably accurate approximation in the regime (44).

On the other hand, for the regimes where the dimension differences $n-m$ are increasing with the subsystem dimension m, the true average capacity starts to deviate from the limiting value (45). For example, in figure 2, we plot the exact formulas (25) and (38) per dimension

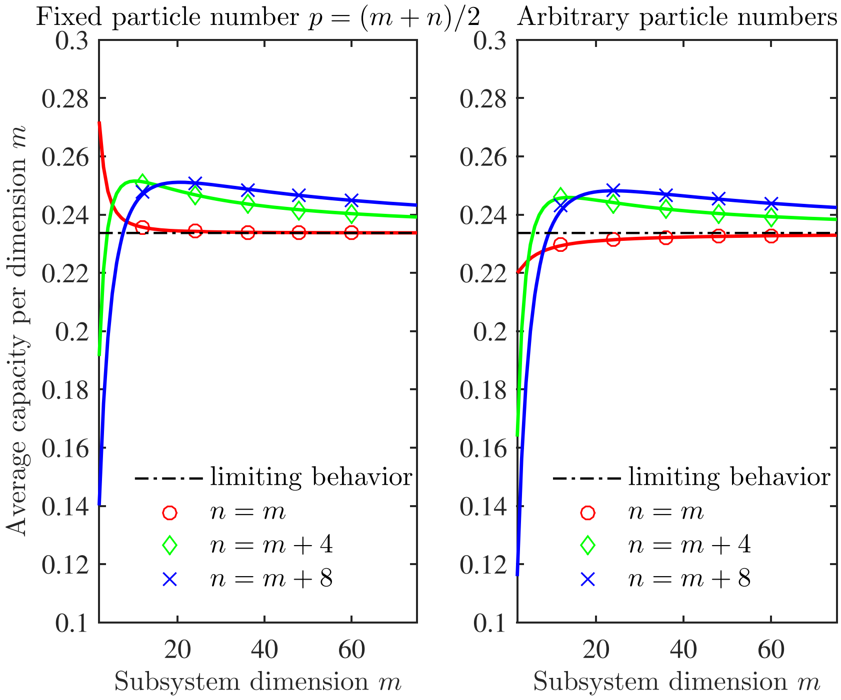

Figure 1. Average of entanglement capacity (per dimension) of fermionic Gaussian states with and without particle number constraints: analytical results versus simulations. The solid lines are drawn by the exact capacity formulas (25) and (38), while the dash-dot horizontal lines represent the limiting behaviors of average capacity (45). The corresponding scatters in the symbols of circle, diamond, and asterisk are obtained from numerical simulations.

m for the cases $n-m=0.5\sqrt{m},\ \sqrt{m},\ 2\sqrt{m},\ 4\sqrt{m}$, as compared to the limiting value (45). It turns out that in the regime where the dimension differences are of order $\sqrt{m}$, the limiting value (45) still provides a moderately accurate estimation for a relatively slow increasing rate of dimension differences, yet become less accurate for a higher rate.

Larger deviations to the limiting value (45) are observed in figure 3, where we plot the exact formulas (25) and (38) per dimension m for dimension differences $n-m=1.1m,\ 1.5m,\ 2m,\ 3m$, in comparison to the limiting value (45). It is also observed in figure 3 that the values of average capacity outside of the regime (44) seem to oscillate around the asymptotic value within the regime. Note that both the figures 2 and 3 are plotted by the numerical results that are outside of the regime (44), where the scaling limit of average capacity appears robust to the case within the asymptotic regime (44).

3. Computation of average capacity

In this section, we prove the results presented in the previous section. The mean formula of entanglement capacity for fermionic Gaussian states with a fixed particle number in proposition 1 is calculated in section 3.1. The computation for the case of an arbitrary particle number in proposition 2 is performed in section 3.2. The limiting value of the average capacity in corollary 1 is proved in section 3.3.

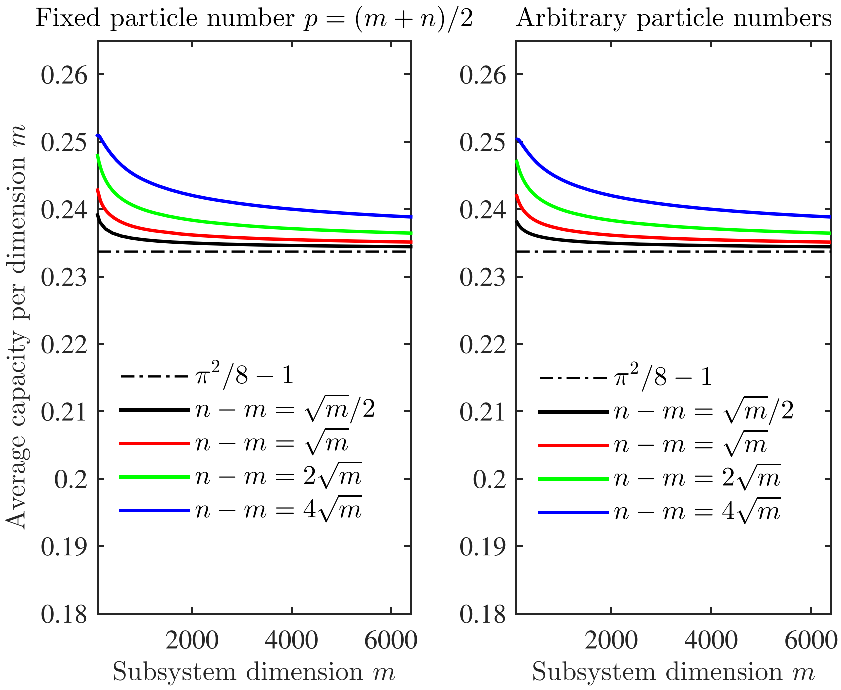

Figure 2. Average of entanglement capacity (per dimension) of fermionic Gaussian states with and without particle number constraints: analytical results of order $\sqrt{m}$ subsystem dimension differences versus the asymptotic results of fixed subsystem dimension differences. The solid lines are drawn by the exact capacity formulas (25) and (38), while the dashed line represents the limiting behaviors of average capacity (45) in the regime (44).

3.1. Average capacity over fermionic Gaussian states with particle number constraint

Here, we compute the mean value of entanglement capacity (8) over fermionic Gaussian states with particle number constraint (17). The computation mainly consists of two parts. The first part is to obtain a summation representation of the average capacity as shown in section 3.1.1. In section 3.1.2, we then simplify the summations in arriving at the desired result (25) in proposition 1.

3.1.1. Correlation functions and integral calculations. Recall the definition (8) of entanglement capacity

$$C = \sum_{i=1}^{m} u(x_i) \tag{46}$$

with

$$u(x) = \frac{1-x^2}{4} \ln^2 \frac{1+x}{1-x}, \tag{47}$$

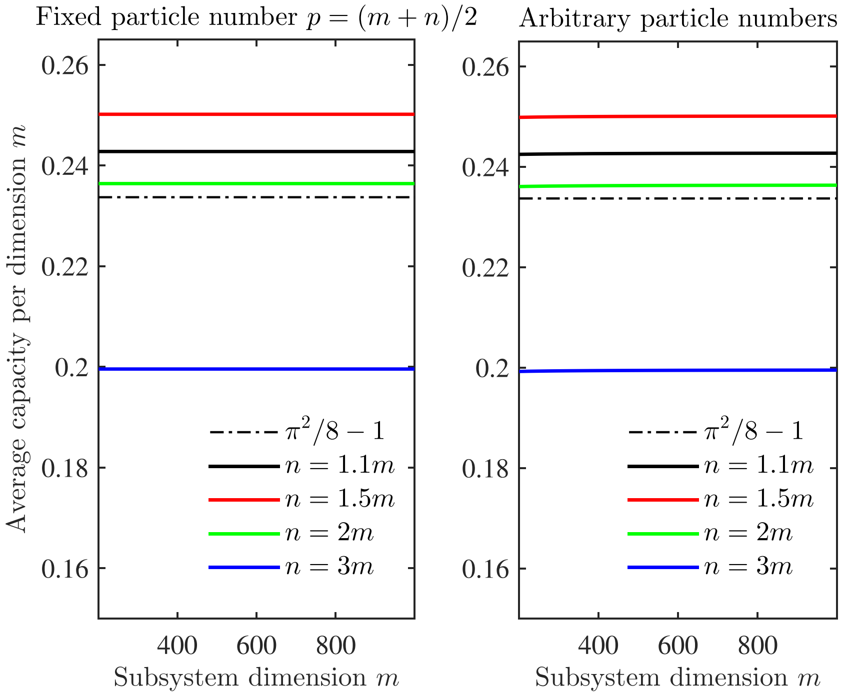

Figure 3. Average of entanglement capacity (per dimension) of fermionic Gaussian states with and without particle number constraints: analytical results of order m subsystem dimension differences versus the asymptotic results of fixed subsystem dimension differences. The solid lines are drawn by the exact capacity formulas (25) and (38), while the dashed line represents the limiting behaviors of average capacity (45) in the regime (44).

computing its average requires the probability density function of one arbitrary eigenvalue of the fermionic Gaussian ensemble. Denoting $g_l(x_1,\ldots,x_l)$ as the joint density of l arbitrary eigenvalues, the average capacity is written as

$$\mathbb{E}[C]=m\int_{-1}^{1}u(x)\,g_1(x)\,\mathrm{d}x. \tag{48}$$

When $\gamma=1$, the ensemble (16) is the well-known Jacobi unitary ensemble. In this case, the joint density $g_l(x_1,\ldots,x_l)$ can be written in terms of an $l\times l$ determinant as [38, 39]

$$g_l(x_1,\ldots,x_l)=\frac{(m-l)!}{m!}\det\left(K(x_i,x_j)\right)_{i,j=1}^{l}. \tag{49}$$

The determinant in (49) is known as the l-point correlation function [39], where

$$K(x,y)=\sqrt{w(x)\,w(y)}\sum_{k=0}^{m-1}\frac{J_k^{(a,b)}(x)\,J_k^{(a,b)}(y)}{h_k} \tag{50}$$

is the correlation kernel with the weight function

$$w(x)=\left(\frac{1-x}{2}\right)^{a}\left(\frac{1+x}{2}\right)^{b}. \tag{51}$$

In (50), the polynomial $J_k^{(a,b)}(x)$ is the Jacobi polynomial supported in $x \in [-1,1]$, and

$$h_k = \frac{2\Gamma(k+a+1)\Gamma(k+b+1)}{(2k+a+b+1)\Gamma(k+1)\Gamma(k+a+b+1)} \tag{52}$$

is the normalization constant, which is obtained by the orthogonality relation of Jacobi polynomials [39]

$$\begin{aligned}&\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{a}\left(\frac{1+x}{2}\right)^{b}J_k^{(a,b)}(x)J_l^{(a,b)}(x)\,\mathrm{d}x\\&\quad=\frac{2\Gamma(k+a+1)\Gamma(k+b+1)}{(2k+a+b+1)\Gamma(k+1)\Gamma(k+a+b+1)}\delta_{kl},\quad \Re(a,b)>-1.\end{aligned} \tag{53}$$

By rewriting the function $u(x)$ in (9) as

$$\begin{aligned}u(x)=&\frac{1+x}{2}\ln^2\frac{1+x}{2}+\frac{1-x}{2}\ln^2\frac{1-x}{2}\\&-\left(\frac{1+x}{2}\ln\frac{1+x}{2}+\frac{1-x}{2}\ln\frac{1-x}{2}\right)^2,\end{aligned} \tag{54}$$

the average capacity (48) boils down to computing two integrals involving the one-point correlation function, see [21], as

$$\mathbb{E}[C] = \mathrm{I}_{\mathcal{C}} - \mathrm{I}_{\mathcal{A}}, \tag{55}$$

where

$$\mathrm{I}_{\mathcal{C}} = \int_{-1}^{1}\left(\frac{1+x}{2}\ln^2\frac{1+x}{2}+\frac{1-x}{2}\ln^2\frac{1-x}{2}\right)K(x,x)\,\mathrm{d}x \tag{56}$$

$$\mathrm{I}_{\mathcal{A}} = \int_{-1}^{1}\left(\frac{1+x}{2}\ln\frac{1+x}{2}+\frac{1-x}{2}\ln\frac{1-x}{2}\right)^2 K(x,x)\,\mathrm{d}x. \tag{57}$$

By the definition of the correlation kernel (50), the integral $\mathrm{I}_{\mathcal{C}}$ in (56) is further written as

$$\begin{aligned}\mathrm{I}_{\mathcal{C}} =& \sum_{k=0}^{m-1}\frac{1}{h_k}\int_{-1}^{1}\left(\frac{1+x}{2}\ln^2\frac{1+x}{2}+\frac{1-x}{2}\ln^2\frac{1-x}{2}\right)\\&\times\left(\frac{1-x}{2}\right)^{a}\left(\frac{1+x}{2}\right)^{b}J_k^{(a,b)}(x)^2\,\mathrm{d}x.\end{aligned} \tag{58}$$

Similarly, the integral $\mathrm{I}_{\mathcal{A}}$ in (57) now consists of two parts

$$\mathrm{I}_{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2, \tag{59}$$

where

$$\begin{aligned}\mathcal{A}_1 =& \sum_{k=0}^{m-1}\frac{1}{h_k}\int_{-1}^{1}\left(\left(\frac{1+x}{2}\right)^2\ln^2\frac{1+x}{2}+\left(\frac{1-x}{2}\right)^2\ln^2\frac{1-x}{2}\right)\\&\times\left(\frac{1-x}{2}\right)^{a}\left(\frac{1+x}{2}\right)^{b}J_k^{(a,b)}(x)^2\,\mathrm{d}x\end{aligned} \tag{60}$$

$$\mathcal{A}_2 = \sum_{k=0}^{m-1} \frac{2}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a+1} \left(\frac{1+x}{2}\right)^{b+1} \ln\frac{1-x}{2} \ln\frac{1+x}{2} J_k^{(a,b)}(x)^2 \, \mathrm{d}x. \tag{61}$$

Here, we recall that $a = n - p \geqslant 0$ and $b = p - m \geqslant 0$ in (17). Due to the parity property of Jacobi polynomials [41]

$$J_k^{(a,b)}(-x) = (-1)^k J_k^{(b,a)}(x), \tag{62}$$

the integrals $\mathrm{I}_{\mathcal{C}}$ and $\mathcal{A}_1$ admit the following symmetric structures

$$\mathrm{I}_{\mathcal{C}} = \mathrm{I}_{\mathcal{C}}{}^{(a,b)} + \mathrm{I}_{\mathcal{C}}{}^{(b,a)} \tag{63}$$

$$\mathcal{A}_1 = \mathcal{A}_1^{(a,b)} + \mathcal{A}_1^{(b,a)}, \tag{64}$$

where

$$\mathrm{I}_{\mathcal{C}}{}^{(a,b)} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{b+1} \ln^2\frac{1+x}{2} J_k^{(a,b)}(x)^2 \, \mathrm{d}x \tag{65}$$

$$\mathcal{A}_1^{(a,b)} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{b+2} \ln^2\frac{1+x}{2} J_k^{(a,b)}(x)^2 \, \mathrm{d}x. \tag{66}$$

The summations in (61), (65) and (66) can be evaluated by using the confluent form of Christoffel–Darboux formula [39]

$$\sum_{k=0}^{m-1} \frac{J_k^{(a,b)}(x)^2}{h_k} = \alpha_1 J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x) - \alpha_2 J_{m-2}^{(a+1,b+1)}(x) J_m^{(a,b)}(x), \tag{67}$$

where

$$\alpha_1 = \frac{m(a+b+m)(a+b+m+1)}{h_{m-1}(a+b+2m-1)_2} \tag{68}$$

$$\alpha_2 = \frac{m(a+b+m)^2}{h_{m-1}(a+b+2m-1)_2}. \tag{69}$$

Consequently, we have

$$\begin{aligned}\mathrm{I}_{\mathcal{C}}{}^{(a,b)} = {} & \alpha_1 \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{b+1} \ln^2\frac{1+x}{2} J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x) \, \mathrm{d}x \\ & - \alpha_2 \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{b+1} \ln^2\frac{1+x}{2} J_{m-2}^{(a+1,b+1)}(x) J_m^{(a,b)}(x) \, \mathrm{d}x\end{aligned} \tag{70}$$

$$\mathcal{A}_1^{(a,b)} = \alpha_1 \int_{-1}^{1} \left(\frac{1-x}{2}\right)^a \left(\frac{1+x}{2}\right)^{b+2} \ln^2 \frac{1+x}{2} J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x)\, \mathrm{d}x$$
$$- \alpha_2 \int_{-1}^{1} \left(\frac{1-x}{2}\right)^a \left(\frac{1+x}{2}\right)^{b+2} \ln^2 \frac{1+x}{2} J_{m-2}^{(a+1,b+1)}(x) J_{m}^{(a,b)}(x)\, \mathrm{d}x \tag{71}$$

and

$$\mathcal{A}_2 = 2\alpha_1 \mathcal{A}_2(m-1,m-1) - 2\alpha_2 \mathcal{A}_2(m-2,m), \tag{72}$$

where

$$\mathcal{A}_2(m-1,m-1) = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a+1} \left(\frac{1+x}{2}\right)^{b+1}$$
$$\times \ln \frac{1-x}{2} \ln \frac{1+x}{2} J_{m-1}^{(a+1,b+1)}(x) J_{m-1}^{(a,b)}(x)\, \mathrm{d}x \tag{73}$$

$$\mathcal{A}_2(m-2,m) = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a+1} \left(\frac{1+x}{2}\right)^{b+1}$$
$$\times \ln \frac{1-x}{2} \ln \frac{1+x}{2} J_{m-2}^{(a+1,b+1)}(x) J_{m}^{(a,b)}(x)\, \mathrm{d}x. \tag{74}$$

The above integrals $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$, $\mathcal{A}_1^{(a,b)}$, and $\mathcal{A}_2$ in (70)–(72) are computed by using the following two integral identities

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a_1} \left(\frac{1+x}{2}\right)^{c} J_{k_1}^{(a_1,b_1)}(x) J_{k_2}^{(a_2,b_2)}(x)\, \mathrm{d}x$$
$$= \frac{2(k_1+1)_{a_1}}{(b_2+k_2+1)_{a_2}} \sum_{i=0}^{k_2} \frac{(-1)^{i+k_2}(i+1)_c (i+b_2+1)_{a_2+k_2}}{\Gamma(k_2-i+1)\Gamma(a_1+c+i+k_1+2)}$$
$$\times (c+i-b_1-k_1+1)_{k_1}, \quad \Re(a_1,a_2,b_1,b_2,c) > -1, \tag{75}$$

and

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{d} \left(\frac{1+x}{2}\right)^{c} J_{k_1}^{(a_1,b_1)} J_{k_2}^{(a_2,b_2)}(x)\, \mathrm{d}x$$
$$= \frac{2\Gamma(a_2+k_2+1)\Gamma(b_2+k_2+1)}{\Gamma(c+d+k_1+k_2+2)} \sum_{i=0}^{k_2} \frac{(-1)^i \Gamma(d-a_1+i+1)}{\Gamma(i+1)\Gamma(a_2+i+1)}$$
$$\times \frac{\Gamma(c-b_1-i+k_2+1)}{\Gamma(k_2-i+1)\Gamma(b_2-i+k_2+1)} \sum_{j=0}^{k_1} \frac{(-1)^j (k_1-j+1)_{d+i}}{\Gamma(j+1)}$$
$$\times \frac{(c-i+j-b_1-k_1+k_2+1)_{b_1+k_1}}{\Gamma(d-a_1+i-j+1)} \quad \Re(a_1,a_2,b_1,b_2,c,d) > -1. \tag{76}$$

The proofs of the two identities (75) and (76) can be found in [23, section 2.1].

Computing $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$ and $\mathcal{A}_1^{(a,b)}$ will require the identity (75). In (75), by specializing

$$a_1 = a, \quad a_2 = a+1, \quad b_1 = b, \quad b_2 = b+1, \quad k_1 = k_2 = m-1 \tag{77}$$

so that

$$J_{k_1}^{(a_1,b_1)}(x) \to J_{m-1}^{(a,b)}(x), \qquad J_{k_2}^{(a_2,b_2)}(x) \to J_{m-1}^{(a+1,b+1)}(x), \tag{78}$$

the first integral in (70) can now be computed by taking twice derivatives with respect to the parameter c of the specialized identity (75) before setting $c = b+1$. Other integrals in (70)–(71) are calculated in the same manner.

To compute the integral $\mathcal{A}_2$ in (72), one will need the integral identity (76). The two integrals (73) and (74) in $\mathcal{A}_2$ are calculated by taking derivatives of c and d of identity (76) with the specialization (77) and the specialization

$$a_1 = a, \quad b_1 = b, \quad a_2 = a+1, \quad b_2 = b+1, \quad k_1 = m, \quad k_2 = m-2, \tag{79}$$

respectively, before setting $c = b+1$, $d = a+1$.

In writing down the summation forms of $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$, $\mathcal{A}_1^{(a,b)}$, and $\mathcal{A}_2^{(a,b)}$, one will also have to resolve the indeterminacy by using the following asymptotic expansions of gamma and polygamma functions of negative arguments [33] when $\epsilon \to 0$,

$$\Gamma(-l+\epsilon) = \frac{(-1)^l}{l!\epsilon}\left(1 + \psi_0(l+1)\epsilon + o\left(\epsilon^2\right)\right) \tag{80}$$

$$\psi_0(-l+\epsilon) = -\frac{1}{\epsilon} + \psi_0(l+1) + (2\psi_1(1) - \psi_1(l+1))\epsilon + o\left(\epsilon^2\right) \tag{81}$$

$$\psi_1(-l+\epsilon) = \frac{1}{\epsilon^2} - \psi_1(l+1) + \psi_1(1) + \zeta(2) + o(\epsilon), \tag{82}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{83}$$

is the Riemann zeta function. The resulting summation forms of $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$, $\mathcal{A}_1^{(a,b)}$, and $\mathcal{A}_2$ are summarized in (A.1)–(A.4) in appendix A.1.

3.1.2. Simplification of summations.

The remaining task in computing the average capacity

$$\mathbb{E}[C] = \mathrm{I}_{\mathcal{C}} - \mathrm{I}_{\mathcal{A}}, \tag{84}$$

is to simplify the summations in (A.1)–(A.4). In the subsequent calculation, we first simplify the summation (A.1) in obtaining $\mathrm{I}_{\mathcal{C}}$, whereas $\mathrm{I}_{\mathcal{A}}$ is obtained by simplifying the summations (A.2)–(A.4).

We first simplify the summations in (A.1). Note that the first two sums in (A.1) are single sums consisting of polygamma and rational functions, and the last sum can be directly reduced to a closed-form expression. The two single summations are simplified, by using the identities (B.1)–(B.8) while keeping in mind the symmetric structure (63)

$$\mathrm{I}_{\mathcal{C}} = \mathrm{I}_{\mathcal{C}}^{(\mathrm{a,b})} + \mathrm{I}_{\mathcal{C}}^{(\mathrm{b,a})}, \tag{85}$$

as

$$
\begin{aligned}
\mathrm{I}_{\mathcal{C}}^{(\mathrm{a,b})} = & \, a_0 \sum_{k=1}^{m} \frac{\psi_0\left(a+b+k+m\right)}{b+k} - a_1 \sum_{k=1}^{m} \frac{\psi_0\left(a+b+k+m\right)}{k} + a_1 \sum_{k=1}^{m} \frac{\psi_0\left(b+k\right)}{k} + a_2 \\
& \times \left(\psi_0^2\left(a+b+2m\right) - \psi_0\left(a+b+m\right)\psi_0\left(a+b+2m\right) - \psi_0\left(a+b+2m\right)\right. \\
& \left.\times \psi_0\left(b+m\right)\right) + a_0\psi_0\left(b\right)\psi_0\left(a+b+m\right) + \frac{a_1}{2}\left(\psi_1\left(b\right) - \psi_1\left(a+b+m\right)\right. \\
& + \psi_0\left(a+b+m\right)\left(\psi_0\left(a+b+m\right) + 2\psi_0\left(m\right) - 2\psi_0\left(1\right)\right) + 2\psi_0\left(b\right)\left(\psi_0\left(b+m\right)\right. \\
& \left.\left. - \psi_0\left(m\right) + \psi_0\left(1\right)\right) - \psi_0^2\left(b\right)\right) + a_3\psi_0(a+b+2m) + a_4\psi_0(a+b+m) \\
& + a_5\psi_0(b+m) + a_6\psi_0(b) + a_7,
\end{aligned} \tag{86}
$$

where the coefficients a_i are summarized in (C.1)–(C.8) of appendix C.1.

We now simplify the summations (A.2)–(A.4) in obtaining $\mathrm{I}_{\mathcal{A}}$. The summation (A.2) is simplified into a similar form as the result (86) by using the identities (B.1)–(B.8). The integral $\mathcal{A}_1$ is then obtained by adding the result of (A.2) and its symmetric form according to (64). Continue to simplify the summations (A.3) and (A.4) will require the following four lemmas.

Lemma 1. *For any complex numbers $a, b, c \notin \mathbb{Z}^-$, we have*

$$
\begin{aligned}
& \sum_{i=1}^{m} \frac{1}{\Gamma\left(i\right)\Gamma\left(a+i\right)\Gamma\left(m+1-i\right)\Gamma\left(m+b+1-i\right)\left(c+i\right)} \\
& \quad = \frac{1}{\Gamma\left(b+m\right)\Gamma\left(c+m+1\right)\Gamma\left(a+b+m\right)} \sum_{i=1}^{m} \frac{\Gamma\left(c-i+m+1\right)\Gamma\left(a+b-i+2m\right)}{\Gamma\left(m-i+1\right)\Gamma\left(a-i+m+1\right)}.
\end{aligned} \tag{87}
$$

Lemma 2. *For any complex numbers $a, b \notin \mathbb{Z}^-$, and any $c \in \mathbb{Z}^+$, we have*

$$
\begin{aligned}
& \sum_{i=1}^{m} \frac{1}{\Gamma\left(c+i\right)\Gamma\left(a+i\right)\Gamma\left(m+1-i\right)\Gamma\left(m+b+1-i\right)} \\
& \quad = \frac{1}{\Gamma\left(m+b\right)\Gamma\left(m+a+b\right)\Gamma\left(c\right)\Gamma\left(m+c\right)} \sum_{i=1}^{m} \frac{\Gamma\left(m+a+b+i-1\right)\Gamma\left(m+c-i\right)}{\Gamma\left(a+i\right)\Gamma\left(m-i+1\right)}.
\end{aligned} \tag{88}
$$

Lemma 3. *For any complex numbers $a, b \notin \mathbb{Z}^-$, and any $c \in \mathbb{Z}^+$, we have*

$$
\begin{aligned}
& \sum_{i=1}^{m} \frac{1}{\Gamma\left(c+i\right)\Gamma\left(a+i\right)\Gamma\left(m-i+1\right)\Gamma\left(b-i+m+1\right)i} \\
& \quad = \frac{1}{\Gamma\left(a\right)\Gamma\left(a+m\right)\Gamma\left(1+b+m\right)\Gamma\left(b+c+m\right)} \sum_{i=1}^{m} \frac{\Gamma\left(a-i+m\right)\Gamma\left(b+c+i+m\right)}{\Gamma\left(c+i\right)\Gamma\left(m-i+1\right)i} \\
& \qquad + \frac{\psi_0\left(a\right) - \psi_0\left(a+m\right)}{\Gamma\left(a\right)\Gamma\left(c\right)\Gamma\left(m+1\right)\Gamma\left(b+m+1\right)}.
\end{aligned} \tag{89}
$$

Lemma 4. *For any complex numbers $a,b \notin \mathbb{Z}^-$, and any $c,d \in \mathbb{Z}^+$, we have*

$$\begin{aligned}&\sum_{i=1}^{m}\frac{1}{\Gamma(c+i)\Gamma(a+i)\Gamma(d+m-i+1)\Gamma(b+m-i+1)}\\&=\frac{1}{\Gamma(d)\Gamma(a+m)\Gamma(a+b+m)\Gamma(c+d+m)}\sum_{i=1}^{m}\frac{\Gamma(c+d+i-1)\Gamma(a+b-i+2m)}{\Gamma(c+i)\Gamma(b-i+m+1)}\\&\quad+\frac{1}{\Gamma(c)\Gamma(b+m)\Gamma(a+b+m)\Gamma(c+d+m)}\sum_{i=1}^{m}\frac{\Gamma(c+d+i-1)\Gamma(a+b-i+2m)}{\Gamma(d+i)\Gamma(a-i+m+1)}.\end{aligned} \tag{90}$$

Proofs to the above four lemmas can be found in [23, section 2.2.2], where a new simplification framework is utilized. Equipped with these tools, the summations (A.3) and (A.4) can now be simplified. In the following, we will first present the simplification of (A.4), whereas (A.3) is simplified in the same manner.

Note that (A.4) consists of one single summation and two double summations. To proceed with the single summation

$$\begin{aligned}&\sum_{i=1}^{m-1}\frac{1}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)}\left(\left(\psi_0(a+b+2m+2)-\psi_0(a+m+1)\right.\right.\\&\quad\left.-\psi_0(i+1)+\psi_0(1)\right)\left(\psi_0(m-i+1)-\psi_0(a+b+2m+2)+\psi_0(b+m+1)-\psi_0(1)\right)\\&\quad\left.+\psi_1(a+b+2m+2)\right),\end{aligned} \tag{91}$$

we first rewrite it as

$$\begin{aligned}(91)=&\,(s_0-s_1s_2)\sum_{i=1}^{m-1}\frac{1}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)}\\&+\left(s_1-\frac{1}{m}\right)\sum_{i=1}^{m-1}\frac{1}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i+1)\Gamma(b-i+m+1)}\\&+\left(s_2-\frac{1}{m}\right)\sum_{i=1}^{m-1}\frac{1}{\Gamma(i+1)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)}\\&+s_1\sum_{i=1}^{m-1}\frac{\psi_0(i)}{\Gamma(i)\Gamma(b+i+1)\Gamma(m-i)\Gamma(a-i+m+1)}\\&-\sum_{i=1}^{m-1}\frac{\psi_0(i)}{\Gamma(i)\Gamma(b+i+1)\Gamma(m-i+1)\Gamma(a-i+m+1)}\\&-\sum_{i=1}^{m-1}\frac{\psi_0(i)}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i+1)\Gamma(b-i+m+1)}\\&+s_2\sum_{i=1}^{m-1}\frac{\psi_0(i)}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)}\\&-\sum_{i=1}^{m-1}\frac{\psi_0(i)\psi_0(m-i)}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)},\end{aligned} \tag{92}$$

where

$$s_0 = \psi_1(a+b+2m+2) \tag{93}$$

$$s_1 = \psi_0(a+b+2m+2) - \psi_0(a+m+1) + \psi_0(1) \tag{94}$$

$$s_2 = \psi_0(a+b+2m+2) - \psi_0(b+m+1) + \psi_0(1). \tag{95}$$

The summations in (92) are then simplified into single sums of the forms

$$\sum_{j=1}^{m} \frac{\Gamma(a+b-j+2m-1)}{\Gamma(a-j+m)j} \tag{96}$$

$$\sum_{j=1}^{m} \frac{\Gamma(a+b-j+2m-1)}{\Gamma(a-j+m)j^2} \tag{97}$$

by using lemmas 2, 4, and the closed-form identity [42]

$$\begin{aligned}&\sum_{i=1}^{m} \frac{1}{\Gamma(i)\Gamma(a+i)\Gamma(m-i+1)\Gamma(m+b+1-i)}\\ &\quad= \frac{\Gamma(a+b+2m-1)}{\Gamma(m)\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m)}.\end{aligned} \tag{98}$$

More specifically, the first three summations in (92) are simplified into closed-form expressions by using the identity (98), and the next four summations are simplified by taking derivative of c of the identity (88) in lemma 2 before setting $c=0$. The last summation in (92) is simplified by taking derivatives of c and d of the identity (90) in lemma 4 before setting $c=d=0$.

We now move on to the double summations in (A.4), which are

$$\begin{aligned}&\sum_{i=1}^{m-1} \frac{i(m-i)}{\Gamma(b+i+1)\Gamma(a-i+m+1)} \sum_{j=1}^{m-i} \frac{\Gamma(a+j+m+1)\Gamma(b-j+m+1)}{j\Gamma(i+j+1)\Gamma(m-i-j+1)}\\ &\quad\times(\psi_0(a+j+m+1) - \psi_0(a+b+2m+2) + \psi_0(m-i+1) - \psi_0(j+1))\end{aligned} \tag{99}$$

and

$$\begin{aligned}&\sum_{i=1}^{m-1} \frac{i(m-i)}{\Gamma(a+i+1)\Gamma(b-i+m+1)} \sum_{j=1}^{m-i} \frac{\Gamma(a-j+m+1)\Gamma(b+j+m+1)}{j\Gamma(i+j+1)\Gamma(m-i-j+1)}\\ &\quad\times(\psi_0(b+j+m+1) - \psi_0(a+b+2m+2) + \psi_0(m-i+1) - \psi_0(j+1)).\end{aligned} \tag{100}$$

The two summations (99) and (100) admit a similar symmetric structure as (63)–(64). Therefore, by simplifying the summation (99), the summation (100) can be directly obtained by switching a and b. We start with the summation (99) by dividing it into two parts

$$\sum_{i=1}^{m-1}\frac{i\,(m-i)}{\Gamma\,(b+i+1)\,\Gamma\,(a-i+m+1)}\sum_{j=1}^{m-i}\frac{\Gamma\,(a+j+m+1)\,\Gamma\,(b-j+m+1)}{j\Gamma\,(i+j+1)\,\Gamma\,(m-i-j+1)}$$
$$\times\left(-\psi_0\,(a+b+2m+2)-\psi_0\,(j+1)\right) \tag{101}$$

and

$$\sum_{i=1}^{m-1}\frac{i\,(m-i)}{\Gamma\,(b+i+1)\,\Gamma\,(a-i+m+1)}\sum_{j=1}^{m-i}\frac{\Gamma\,(a+j+m+1)\,\Gamma\,(b-j+m+1)}{j\Gamma\,(i+j+1)\,\Gamma\,(m-i-j+1)}$$
$$\times\left(\psi_0\,(m-i+1)+\psi_0\,(a+j+m+1)\right). \tag{102}$$

In (101), after changing the summation order as

$$(101)=\sum_{j=1}^{m-1}\frac{\Gamma\,(a+j+m+1)\,\Gamma\,(b-j+m+1)}{j}\left(-\psi_0\,(a+b+2m+2)-\psi_0\,(j+1)\right)$$
$$\times\sum_{i=1}^{m-j}\frac{i\,(m-i)}{\Gamma\,(b+i+1)\,\Gamma\,(i+j+1)\,\Gamma\,(a-i+m+1)\,\Gamma\,(m-i-j+1)}, \tag{103}$$

we evaluate the sum over i by using lemma 2. The double becomes

$$\frac{1}{\Gamma\,(b)\,\Gamma\,(a+m)}\Bigg((1-a-m)\sum_{j=1}^{m-1}\frac{(a+j+m)\,(b-j+m)}{j}\,(\psi_0\,(a+b+2m+2)$$
$$+\psi_0\,(j+1))\times\sum_{i=1}^{m-j}\frac{\Gamma\,(b+i-1)\,\Gamma\,(a-i+2m)}{\Gamma\,(i)\,\Gamma\,(m-i+2)}+\frac{a\,(a+m)}{a+b+m}\sum_{j=1}^{m-1}\frac{b-j+m}{j}\,(\psi_0\,(j+1)$$
$$+\psi_0\,(a+b+2m+2))\sum_{i=1}^{m-j}\frac{\Gamma\,(b+i-1)\,\Gamma\,(a-i+2m+1)}{\Gamma\,(i)\,\Gamma\,(m-i+2)}+\frac{(a+m-1)\,(b+m)}{a+b+m}$$
$$\times\sum_{j=1}^{m-1}\frac{(a+j+m)}{j}(\psi_0(a+b+2m+2)+\psi_0(j+1))\sum_{i=1}^{m-j}\frac{\Gamma(b+i)\Gamma(a-i+2m)}{\Gamma(i)\Gamma(m-i+2)}\Bigg), \tag{104}$$

where the sums over j can be further simplified into closed-form expressions by using the identity (B.3). As a result, the remaining summations only involve single sums as in (92) that are simplified similarly.

The sum (102) is simplified by first using lemma 3 along with its derivative with respect to b to evaluate the inner sum over j. As a result, the remaining sums are reduced to single sums after computing the sum over i except for the sum

$$\sum_{j=1}^{m}\frac{1}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)}\times\sum_{i=1}^{m-j+1}\left(\frac{\psi_0(a+i+j)}{i}+\frac{\psi_0(i+j)}{i}\right). \tag{105}$$

To proceed with (105), we first use the identity (B.9) to compute the inner sum

$$\sum_{i=1}^{m-j+1}\frac{\psi_0(a+i+j)}{i} \tag{106}$$

into

$$\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i}-\sum_{l=1}^{a}\frac{\psi_0(l+m+1)}{j+l-1}+\frac{1}{2}\left((\psi_0(a+j)-\psi_0(j))\right.$$
$$\left.\times(\psi_0(a+j)+2\psi_0(m-j+2)+\psi_0(j)-2\psi_0(1))-\psi_1(a+j)+\psi_1(j)\right). \tag{107}$$

Inserting the result (107) into (105), the double sum in (105) now boils down to simplifying the three sums

$$\frac{1}{2}\sum_{j=1}^{m}\frac{1}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)}\left(\psi_1(j)+(\psi_0(a+j)-\psi_0(j))\right.$$
$$\left.\times(\psi_0(a+j)+2\psi_0(m-j+2)+\psi_0(j)-2\psi_0(1))-\psi_1(a+j)\right), \tag{108}$$

$$\sum_{l=1}^{a}\psi_0(l+m+1)\sum_{j=1}^{m}\frac{-1}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)(j+l-1)}, \tag{109}$$

and

$$\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)}\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i}. \tag{110}$$

The single sum (108) is simplified in the same manner as (92). For the double summation in (109), after evaluating the inner sum over j by using lemma 1, we arrive at

$$(109)=-\frac{1}{\Gamma(b+m)\Gamma(a+b+m+1)}\sum_{l=1}^{a}\frac{\psi_0(l+m+1)}{\Gamma(l+m)}\times\sum_{j=1}^{m-1}\frac{\Gamma(m-j+l)\Gamma(a+b-j+2m)}{\Gamma(m-j)\Gamma(a-j+m+1)}. \tag{111}$$

The above sum (111) can now be simplified into single sums by using the identities (B.13) and (B.14) to evaluate the sum over l, where the remaining single sums are

$$\sum_{j=1}^{m}\frac{\Gamma(a+b-j+2m-1)}{\Gamma(a-j+m)j}\psi_0(a+b-j+2m-1), \tag{112}$$

and

$$\sum_{j=1}^{m}\frac{\psi_0(a+b+j+m)}{j}. \tag{113}$$

So far, the only part that remains to be simplified in (99) is the double sum (110). We first point out that the sum (110) has to be treated together with its symmetric part in (100), which is

$$\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(b+j)\Gamma(m-j+1)\Gamma(a-j+m+2)}\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i}. \tag{114}$$

The two summations (110) and (114) may not be further simplified individually. However, we observe cancellations among the two sums by adding them up, where the key ingredient is the identity (B.9). Specifically, we evaluate the inner summation

$$\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i} \tag{115}$$

in (110) by the identity (B.9) with the specialization

$$a\to j,\qquad b\to 0,\qquad m\to m-j+1. \tag{116}$$

The sum (110) becomes

$$\begin{aligned}(110)=\;&-\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b+m-j+2)}\sum_{i=1}^{j-1}\frac{\psi_0(m-j+i+2)}{i}\\&+\sum_{j=1}^{m}\frac{1}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b+m-j+2)}\left((\psi_0(m-j+2)+\psi_0(j))\right.\\&\left.\times(\psi_0(m-j+2)+\psi_0(j)-2\psi_0(1))-\psi_1(m-j+2)-\psi_1(j)+2\psi_1(1)\right).\end{aligned} \tag{117}$$

Shifting the index $j\to m+2-j$ of the double sum in (117) as

$$\begin{aligned}&-\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b+m-j+2)}\sum_{i=1}^{j-1}\frac{\psi_0(m-j+i+2)}{i}\\&=-\sum_{j=2}^{m+1}\frac{2}{\Gamma(j-1)\Gamma(b+j)\Gamma(m-j+1)\Gamma(a-j+m+2)}\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i},\end{aligned} \tag{118}$$

which is now the same form as (114). Inserting the result (118) into (110) before adding up (114), we obtain

$$\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)}\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i}$$
$$+\sum_{j=1}^{m}\frac{2}{\Gamma(j-1)\Gamma(b+j)\Gamma(m-j+1)\Gamma(a-j+m+2)}\sum_{i=1}^{m-j+1}\frac{\psi_0(i+j)}{i}$$
$$=\sum_{j=1}^{m}\frac{1}{\Gamma(j-1)\Gamma(a+j)\Gamma(m-j+1)\Gamma(b-j+m+2)}\left((\psi_0(m-j+2)+\psi_0(j))\right.$$
$$\left.\times(\psi_0(m-j+2)+\psi_0(j)-2\psi_0(1))-\psi_1(m-j+2)-\psi_1(j)+2\psi_1(1)\right), \tag{119}$$

which is simplified into single sums of the forms (96), (97), (112), and

$$\sum_{j=1}^{m}\frac{\Gamma(a+b-j+2m-1)}{\Gamma(a-j+m)j}\psi_0(j), \tag{120}$$

by using lemmas 2 and 4, and their derivatives with respect to c. After inserting the simplified results of (91), (99), and (100) into (A.4), we observe complete cancellations of the single sums (96), (97), (112) and (120). The sum $\mathcal{A}_2(m-2,m)$ is simplified to

$$\mathcal{A}_2(m-2,m)=\frac{4\Gamma(a+m+1)\Gamma(b+m+1)}{\Gamma(m-1)\Gamma(a+b+m+1)(a+b+2m-1)_3}$$
$$\times\sum_{j=1}^{m}\frac{\psi_0(a+b+j+m)}{j}+\mathrm{CF}, \tag{121}$$

where the shorthand notation CF, different in each use, denotes some closed-form terms omitted due to the length. Using the same approach, one is able to simplify (A.3) into a similar form, which completes the simplification of $\mathcal{A}_2$ as per (72).

Now inserting the resulting forms of $\mathcal{A}_1$ and $\mathcal{A}_2$ into (59), $\mathrm{I}_{\mathcal{A}}$ is finally obtained as

$$\mathrm{I}_{\mathcal{A}}=f_{\mathcal{A}}(a,b)+f_{\mathcal{A}}(b,a), \tag{122}$$

where

$$f_{\mathcal{A}}(a,b)=b_0\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k}-m\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{k}+b_1\sum_{k=1}^{m}\frac{\psi_0(a+k)}{k}$$
$$+\frac{m}{2}\left(\psi_0^2(a+b+m)-\psi_1(a+b+m)\right)+b_2(\psi_0(a+b+2m)-\psi_0(a+b+m))$$
$$\times\psi_0(a+b+2m)+b_3\psi_0(a+m)\psi_0(a+b+2m)+b_4\psi_0(a+b+2m)$$
$$+b_0\psi_0(a)\psi_0(a+b+m)+m(\psi_0(m)-\psi_0(1))\psi_0(a+b+m)+b_5\psi_0(a+m)$$
$$\times(2\psi_0(a+b+m)+\psi_0(b+m))+b_6\psi_0(a+b+m)+\frac{b_1}{2}(2\psi_0(a)\psi_0(a+m)$$
$$-2\psi_0(a)\psi_0(m)-\psi_0^2(a)+2\psi_0(1)\psi_0(a)+\psi_1(a))+b_7\psi_0(a+m)+b_8\psi_0(a)$$
$$+b_9. \tag{123}$$

The coefficients b_i in (123) are summarized in (C.9)–(C.18) in appendix C.2.

By inserting (86) and (123) into (55), we obtain

$$\mathbb{E}[C] = \frac{2m(a+m)(b+m)(a+b+m)}{(a+b+2m-1)_3}\left(\sum_{k=1}^{m}\frac{\psi_0(a+k)}{k}+\sum_{k=1}^{m}\frac{\psi_0(b+k)}{k}\right.$$
$$\left.+\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k}+\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{b+k}\right)+\mathrm{CF}. \tag{124}$$

The remaining task in obtaining (25) is to represent the single summations

$$\sum_{k=1}^{m}\frac{\psi_0(a+k)}{k} \tag{125}$$

$$\sum_{k=1}^{m}\frac{\psi_0(b+k)}{k} \tag{126}$$

$$\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k} \tag{127}$$

$$\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{b+k} \tag{128}$$

in (124) into (19) as reproduced below

$$\Phi_{c,d} = \frac{c!}{(c+d)!}\sum_{k=1}^{c}\frac{(c+d-k)!}{(c-k)!}\frac{1}{k^2}, \qquad c,d\in\mathbb{Z}^{+}. \tag{129}$$

By utilizing the identity (B.12), the summations in (125) and (126) are respectively computed into the summations $\Phi_{m,a}$ and $\Phi_{m,b}$ as

$$\sum_{k=1}^{m}\frac{\psi_0(a+k)}{k} = \Phi_{m,a}+\mathrm{CF} \tag{130}$$

$$\sum_{k=1}^{m}\frac{\psi_0(b+k)}{k} = \Phi_{m,b}+\mathrm{CF}. \tag{131}$$

To proceed with the summations (127) and (128), we have to consider their combination

$$\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k}+\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{b+k}. \tag{132}$$

We first rewrite (127) as

$$\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k} = \sum_{k=1}^{m}\frac{\psi_0(b+k)}{a+k}+\sum_{k=1}^{m}\frac{1}{a+k}\sum_{l=0}^{a+m-1}\frac{1}{b+k+l}, \tag{133}$$

where we have used the finite sum form of digamma function [43]

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k} \tag{134}$$

to replace

$$\psi_0(a+b+k+m) \tag{135}$$

by

$$\psi_0(b+k) + \sum_{l=0}^{a+m-1} \frac{1}{b+k+l}. \tag{136}$$

We then change the order of summation of the double sum in (133) to evaluate the sum over k first, where the remaining sums are further evaluated by the identity (B.3), leading to

$$\begin{aligned}\sum_{k=1}^{m} \frac{\psi_0(a+b+k+m)}{b+k} = & \sum_{k=1}^{a+m-1} \frac{\psi_0(b+k+1)}{k} - \sum_{k=1}^{a+m-1} \frac{\psi_0(b+k+m+1)}{k} \\ & + \frac{1}{2}\left((2\psi_0(1) - 2\psi_0(a+m) - \psi_0(b+m+1) - \psi_0(b+1)\right) \\ & \times (\psi_0(b+1) - \psi_0(b+m+1)) - \psi_1(b+m+1) + \psi_1(b+1)).\end{aligned} \tag{137}$$

Similarly, one has (128) manipulated to

$$\sum_{k=1}^{m} \frac{\psi_0(a+b+k+m)}{a+k} = \sum_{k=1}^{b+m-1} \frac{\psi_0(b+k+1)}{k} - \sum_{k=1}^{b+m-1} \frac{\psi_0(a+k+m+1)}{k} + \mathrm{CF}. \tag{138}$$

Here, we also need the result

$$\begin{aligned}& \sum_{k=1}^{a+m-1} \frac{\psi_0(b+k+m+1)}{k} + \sum_{k=1}^{b+m-1} \frac{\psi_0(a+k+m+1)}{k} \\ & = -\frac{1}{2}(\psi_1(a+m+1) + \psi_1(b+m+1)) - \frac{(a+b+2m)\psi_0(a+b+2m)+1}{(a+m)(b+m)} \\ & \quad - \frac{1}{2}(2\psi_0(1) - \psi_0(a+m+1) - \psi_0(b+m+1))(\psi_0(a+m+1) + \psi_0(b+m+1)) \\ & \quad + \psi_1(1),\end{aligned} \tag{139}$$

which is obtained by evaluating the summation

$$\sum_{k=1}^{a+m-1} \frac{\psi_0(b+k+m+1)}{k} = \sum_{k=1}^{a+m-1} \frac{\psi_0(k)}{k} + \sum_{k=1}^{a+m-1} \frac{1}{k} \sum_{l=0}^{b+m} \frac{1}{(k+l)} \tag{140}$$

in the same manner as we have processed (133). Finally, by adding (137) and (138) before using (139), we obtain

$$\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{a+k}+\sum_{k=1}^{m}\frac{\psi_0(a+b+k+m)}{b+k}$$
$$=\sum_{k=1}^{b+m}\frac{\psi_0(a+k)}{k}+\sum_{k=1}^{a+m}\frac{\psi_0(b+k)}{k}+\mathrm{CF}. \tag{141}$$
$$=\Phi_{b+m,a}+\Phi_{a+m,b}+\mathrm{CF}, \tag{142}$$

where the last equality (142) is obtained by using the identity (B.12). Inserting the results (130), (131) and (142) into (124), we complete the proof of proposition 1.

3.2. *Average capacity over fermionic Gaussian states without particle number constraint*

In this section, we compute the mean value of entanglement capacity (8) over fermionic Gaussian states without particle number constraint (18) in proving proposition 2. The same as the previous section, we first discuss the computation that leads to the summation representation in section 3.2.1. Simplification of the summations is performed in section 3.2.2.

3.2.1. *Correlation functions and integral calculations.*

For fermionic Gaussian states of arbitrary number of particles, by definition the average capacity is given by the integral

$$\mathbb{E}[C]=m\int_0^1 u(x)\,g_1(x)\,\mathrm{d}x, \tag{143}$$

where $g_1(x_1,\ldots,x_l)$ denotes the joint probability density of l arbitrary eigenvalues. Similar to the previous case, the density $g_1(x_1,\ldots,x_l)$ can be written in terms of the l-point correlation function as

$$g_l(x_1,\ldots,x_l)=\frac{(m-l)!}{m!}\det\left(K(x_i,x_j)\right)_{i,j=1}^{l}, \tag{144}$$

where

$$K(x,y)=\sqrt{w(x)\,w(y)}\sum_{k=0}^{m-1}\frac{J_{2k}^{(a,a)}(x)\,J_{2k}^{(a,a)}(y)}{h_k} \tag{145}$$

with the weight function being

$$w(x)=\left(\frac{1-x}{2}\right)^a\left(\frac{1+x}{2}\right)^a. \tag{146}$$

By rewriting the orthogonality relation (53) as

$$\int_0^1\left(\frac{1-x}{2}\right)^a\left(\frac{1+x}{2}\right)^a J_{2k}^{(a,a)}(x)\,J_{2l}^{(a,a)}(x)\,\mathrm{d}x$$
$$=\frac{\Gamma(2k+a+1)\,\Gamma(2k+a+1)}{(4k+2a+1)\,\Gamma(2k+1)\,\Gamma(2k+2a+1)}\delta_{kl},\quad \Re(a)>-1, \tag{147}$$

we obtain the normalization constant h_k of the polynomials $J_{2k}^{(a,a)}(x)$

$$h_k=\frac{\Gamma(2k+a+1)\,\Gamma(2k+a+1)}{(4k+2a+1)\,\Gamma(2k+1)\,\Gamma(2k+2a+1)}. \tag{148}$$

By using (54) and (145), the computation of the average capacity (143) boils down to computing two integrals

$$\mathbb{E}[C] = \mathrm{I_C} - \mathrm{I_A}, \tag{149}$$

where

$$\mathrm{I_C} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{a+2} \ln^2 \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 \, \mathrm{d}x \tag{150}$$

$$\mathrm{I_A} = \mathrm{A_1} + \mathrm{A_2} \tag{151}$$

with

$$\mathrm{A_1} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a} \left(\frac{1+x}{2}\right)^{a+2} \ln^2 \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 \, \mathrm{d}x \tag{152}$$

$$\mathrm{A_2} = \sum_{k=0}^{m-1} \frac{1}{h_k} \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{a+1} \left(\frac{1+x}{2}\right)^{a+1} \ln \frac{1-x}{2} \ln \frac{1+x}{2} J_{2k}^{(a,a)}(x)^2 \, \mathrm{d}x. \tag{153}$$

The integral in $\mathrm{I_C}$ is calculated by applying the identity (75), where we need to assign

$$a_1 = b_1 = a_2 = b_2 = a, \qquad k_1 = k_2 = 2k, \tag{154}$$

and take twice derivatives of c before setting $c = a+1$. Under the same specialization (154), the integral in $\mathrm{A_1}$ is calculated by taking twice derivatives of c of the identity (75) before setting $c = a+1$, whereas the integral in $\mathrm{A_2}$ is calculated by taking derivatives of both c and d of the identity (76) before setting $c = d = a+1$. After resolving the indeterminacy of gamma and polygamma functions by using (80)–(82), one arrives at the summation representations (A.5)–(A.7) of the above integrals as listed in appendix A.2.

3.2.2. Simplification of summations. The remaining task in computing the mean value (149) is to simplify the summation representations (A.5)–(A.7) of the integrals $\mathrm{I_C}$ and $\mathrm{I_A}$.

We first compute $\mathrm{I_C}$ by simplifying the summations in (A.5). Note that (A.5) consists of two double summations. The first double summation is readily reduced to a single sum by evaluating the inner sum over j. The resulting single sum is further simplified by using the identities (B.1)–(B.8) similarly to the simplification of (A.1). Here, one will also need the results

$$\psi_0(mk) = \ln m + \frac{1}{m} \sum_{i=0}^{m-1} \psi_0\left(k + \frac{i}{m}\right), \qquad m \in \mathbb{Z}^+ \tag{155}$$

$$\psi_1(mk) = \frac{1}{m^2} \sum_{i=0}^{m-1} \psi_1\left(\frac{i}{m} + k\right), \qquad m \in \mathbb{Z}^+ \tag{156}$$

to evaluate the sums involving polygamma functions with even argument. In (A.5), the second double sum is

$$\sum_{k=1}^{m-1} 2\left(2a+4k+1\right)\sum_{j=0}^{2k-2}\frac{2\left(j+1\right)\left(a+j+1\right)}{\left(2k-j-1\right)_2\left(2a+j+2k+1\right)_2}$$
$$\times\left(\psi_0\left(a+j+2\right)-\psi_0\left(2a+j+2k+3\right)-\psi_0\left(2k-j-1\right)+\psi_0\left(j+2\right)\right). \tag{157}$$

By the partial fraction decomposition

$$\frac{2\left(j+1\right)\left(a+j+1\right)}{\left(2k-j-1\right)_2\left(2a+j+2k+1\right)_2}$$
$$=\frac{1}{2a+4k+1}\left(\frac{-2a-2k-1}{2a+j+2k+2}+\frac{2\left(a+k\right)}{2a+j+2k+1}-\frac{2k}{j-2k+1}+\frac{2k+1}{j-2k}\right), \tag{158}$$

we rewrite (157) as the sum of the following five double summations (159)–(163),

$$2\sum_{k=1}^{m-1}\sum_{j=0}^{2k-2}\left(\frac{2a+2k+1}{2a+j+2k+2}-\frac{2\left(a+k\right)}{2a+j+2k+1}\right)\psi_0\left(2a+j+2k+3\right) \tag{159}$$

$$2\sum_{k=1}^{m-1}\sum_{j=0}^{2k-2}\left(\frac{2k}{j-2k+1}-\frac{2k+1}{j-2k}\right)\psi_0\left(2k-j-1\right) \tag{160}$$

$$2\sum_{k=1}^{m-1}\sum_{j=0}^{2k-2}\left(\frac{2k}{2k-j-1}-\frac{2k+1}{2k-j}\right)\psi_0\left(2a+j+2k+3\right) \tag{161}$$

$$2\sum_{k=1}^{m-1}\sum_{j=0}^{2k-2}\left(\frac{2a+2k+1}{2a+j+2k+2}-\frac{2\left(a+k\right)}{2a+j+2k+1}\right)\psi_0\left(2k-j-1\right) \tag{162}$$

$$2\sum_{k=1}^{m-1}\sum_{j=0}^{2k-2}\left(\frac{-2a-2k-1}{2a+j+2k+2}+\frac{2\left(a+k\right)}{2a+j+2k+1}-\frac{2k}{j-2k+1}+\frac{2k+1}{j-2k}\right)$$
$$\times\left(\psi_0\left(a+j+2\right)+\psi_0\left(j+2\right)\right). \tag{163}$$

We now simplify each of the summations (159)–(163) into single sums. Specifically, the summation (159) is simplified by using the identity (B.3) to evaluate the sum over j. The summation (160) is simplified similarly after shifting the index $j\to 2k-2-j$. The summation (161) is simplified by using the identity (B.1) to evaluate the sum over k after shifting the index $j\to 2k-2-j$ and changing the summation order as

$$2\sum_{j=0}^{m-1}\sum_{k=j+1}^{m-1}\left(\frac{2k}{2j+1}-\frac{2k+1}{2j+2}\right)\psi_0\left(2a-2j+4k+1\right)$$
$$+2\sum_{j=0}^{m-1}\sum_{k=j+1}^{m-1}\left(\frac{2k}{2j+2}-\frac{2k+1}{2j+3}\right)\psi_0\left(2a-2j+4k\right), \tag{164}$$

where one has divided the summation over j into even and odd ones. The remaining two sums (162) and (163) are simplified in a similar approach as (161). For (162), one needs to shift the index $j \to 2k-2-j$ before changing the summation order to evaluate the sum over k. For (163), one directly evaluates the sum over k by changing the summation order.

Putting together the results of (159)–(163), the summation (A.5) now consists of single sums, cf (A.1), which are further simplified by the identities (B.1)–(B.8). This leads to

$$\begin{aligned}
\mathrm{I_C} = {} & \sum_{k=1}^{m-1}\left(\left(-\frac{1}{4k}-\frac{1}{4k+2}\right)\psi_0(a+k)+\left(\frac{4m-3}{4k+2}+\frac{4m+1}{4k}\right)\psi_0(a+2k)\right. \\
& +\left(\frac{4a+4m-1}{2a+4k+2}+\frac{4a+4m-1}{2a+4k}-\frac{2a}{2k+1}+\frac{1-2a}{2k}+\frac{1}{2(a+k)}\right)\psi_0(2a+2k) \\
& +\left(\frac{2a-1}{2k}+\frac{2a+1}{2k+1}+\frac{-2a-1}{2a+2k}+\frac{1-2a}{2a+2k+1}\right)\psi_0(2a+4k) \\
& \left.+\left(\frac{1}{4k+2}\frac{1}{4k}\right)\psi_0(a+k+m)+\left(\frac{-2a-2m+1}{2k}-\frac{2a+2m}{2k+1}\right)\psi_0(2a+2k+2m)\right) \\
& +c_0\psi_1(2a+2m)-\frac{1}{4}\psi_1(a+m)+c_1\left(\psi_1(2a)-\psi_0^2(2a)\right)+c_2\psi_1(a) \\
& +c_3\psi_0(2a+4m)\left(\psi_0(a+2m)+\psi_0(2a+2m)-\psi_0(2a+4m)\right)-2c_0\psi_0(2a+2m) \\
& \times\left(\psi_0(a)+\psi_0(2m)-\psi_0(1)\right)-c_0\psi_0^2(2a+2m)+c_4\psi_0^2(a+2m)+c_5\psi_0(a) \\
& \times\psi_0(a+2m)+\frac{1}{2}\left(\psi_0(a)+\psi_0(2m)-\psi_0(1)\right)\psi_0(a+m)-\frac{1}{2}\psi_0(a)\psi_0(m) \\
& +c_6\psi_0(a)\psi_0(2m)+c_7\psi_0^2(a)+c_8\psi_0(2a+4m)+c_9\psi_0(2a+2m)+c_{10}\psi_0(a+2m) \\
& +c_{11}\psi_0(a+m)+c_{12}\psi_0(2a)+c_{13}\psi_0(1)\psi_0(a)+c_{14}\psi_0(a) \\
& +c_{15}\left(\psi_0\left(\frac{a}{2}+m+\frac{1}{4}\right)-\psi_0\left(\frac{a}{2}+\frac{1}{4}\right)\right)+c_{16}\psi_0\left(\frac{a}{2}+m\right) \\
& +c_{17}\left(\psi_0(m)-2\psi_0(2m)+\psi_0(1)\right)+c_{18}\psi_0\left(\frac{a}{2}\right)-2m,
\end{aligned} \tag{165}$$

where the coefficients c_i are listed in (C.19)–(C.37) in appendix C.3.

The simplification of (A.6) and (A.7) in computing $\mathrm{I_A}$ is parallel to that of (A.5) and (A.4), respectively, where much of details are omitted here. However, we note that when first evaluating the inner summations over i and j in (A.7), the resulting sum simply becomes

$$-r(k)\frac{2\left(2a^2+4ak+a+4k^2+2k-1\right)}{(2a+4k-1)(2a+4k+3)}\sum_{j=1}^{2k}\frac{\psi_0(2a+j+2k)}{j}+\mathrm{CF}, \tag{166}$$

where the term

$$r(k)=\frac{\Gamma(2a+4k+4)}{(2a+4k+1)\,\Gamma(2k+1)\,\Gamma(2a+2k+1)} \tag{167}$$

cancels completely with that in (A.7). The remaining sums now only consist of rational functions and polygamma functions, which are readily simplifiable. Inserting the resulting forms of (A.6) and (A.7) into (151), we obtain

$$
\begin{aligned}
\mathrm{I_A} = &\sum_{k=1}^{m-1}\Bigg(\left(-\frac{1}{2\left(2k+1\right)}-\frac{1}{4k}\right)\psi_0\left(a+k\right)+\Bigg(\frac{2am-2a+6m^2-6m+1}{\left(2k+1\right)\left(2a+4m-1\right)}\\
&+\frac{1}{4\left(a+k\right)}+\frac{4am+2a+12m^2-1}{4k\left(2a+4m-1\right)}\Bigg)\psi_0\left(a+2k\right)+\Bigg(\frac{1-2a}{2k}+\frac{1}{2\left(a+k\right)}-\frac{2a}{2k+1}\\
&+\frac{2\left(2a^2+5am-a+3m^2-m\right)}{2a+4m-1}\left(\frac{1}{a+2k+1}+\frac{1}{a+2k}\right)\Bigg)\psi_0\left(2a+2k\right)\\
&+\left(\frac{2a-1}{2k}+\frac{-2a-1}{2\left(a+k\right)}+\frac{2a+1}{2k+1}+\frac{1-2a}{2a+2k+1}\right)\psi_0\left(2a+4k\right)+\left(\frac{1}{2\left(2k+1\right)}\frac{1}{4k}\right)\\
&-\times\psi_0\left(a+k+m\right)+\left(\frac{-2a-2m+1}{2k}-\frac{2a+2m}{2k+1}\right)\psi_0\left(2a+2k+2m\right)\Bigg)\\
&+d_0\psi_1\left(2a+2m\right)+d_1\left(\psi_1(2a)-\psi_0^2(2a)\right)+d_2\psi_1(a)-\frac{1}{4}\psi_1(a+m)\\
&+d_3\left(\psi_0(a+2m)+\psi_0(2a+2m)-\psi_0(2a+4m)\right)\psi_0(2a+4m)+d_0\psi_0(2a+2m)\\
&\times\left(-\psi_0(2a+2m)-2\psi_0(2m)+2\psi_0(1)\right)+d_4\psi_0(a+2m)\psi_0(2a+2m)\\
&+d_5\psi_0^2(a+2m)+d_6\psi_0(a)\psi_0(2a+2m)+a\psi_0(a)\left(\psi_0(a)-2\psi_0(a+2m)\right)\\
&+\frac{1}{4}\left(\psi_0(a)+2\psi_0(2m)-2\psi_0(1)\right)\psi_0(a+m)+d_7\psi_0(a)\psi_0(2m)-\frac{1}{4}\psi_0(a)\psi_0(m)\\
&+d_8\psi_0(2a+4m)+d_9\psi_0(2a+2m)+d_{10}\psi_0(a+2m)+d_{11}\psi_0(a+m)\\
&+d_{12}\left(\psi_0(m)-2\psi_0(2m)+\psi_0(1)\right)+d_{13}\psi_0(2a)+d_{14}\psi_0(1)\psi_0(a)+d_{15}\psi_0(a)+d_{16}\\
&\times\left(\psi_0\left(\frac{a}{2}+m+\frac{1}{4}\right)-\psi_0\left(\frac{a}{2}+\frac{1}{4}\right)\right)+d_{17}\left(\psi_0\left(\frac{a}{2}\right)-\psi_0\left(\frac{a}{2}+m\right)\right)-m,
\end{aligned}
\tag{168}
$$

where the coefficients d_i are listed in (C.38)–(C.55) in appendix C.4.

Inserting the results (165) and (168) into (149), the mean capacity becomes

$$
\begin{aligned}
\mathbb{E}[C] = &\frac{m\left(a+m\right)}{2a+4m-1}\sum_{k=1}^{m-1}\frac{\psi_0\left(a+2k\right)}{k}-\frac{1}{4}\sum_{k=1}^{m-1}\frac{\psi_0\left(a+2k\right)}{a+k}+\frac{\left(2m-1\right)\left(2a+2m-1\right)}{2\left(2a+4m-1\right)}\\
&\times\sum_{k=1}^{m-1}\left(\frac{\psi_0\left(a+2k+1\right)}{2k+1}+\frac{\psi_0\left(2a+2k\right)}{a+2k}+\frac{\psi_0\left(2a+2k+1\right)}{a+2k+1}\right)+\mathrm{CF},
\end{aligned}
\tag{169}
$$

where we recall that the shorthand notation CF denotes the closed-form terms omitted. In the above result (169), we rewrite the single summations

$$
\sum_{k=1}^{m-1}\frac{\psi_0\left(a+2k+1\right)}{2k+1}
\tag{170}
$$

and

$$
\sum_{k=1}^{m-1}\frac{\psi_0\left(2a+2k+1\right)}{a+2k+1}
\tag{171}
$$

as

$$
\sum_{k=1}^{m-1}\frac{\psi_0\left(a+2k+1\right)}{2k+1}=\sum_{k=2}^{2m}\frac{\psi_0\left(a+k\right)}{k}-\frac{1}{2}\sum_{k=1}^{m}\frac{\psi_0\left(a+2k\right)}{k}
\tag{172}
$$

and

$$\sum_{k=1}^{m-1}\frac{\psi_0(a+2k+1)}{2k+1}=\sum_{k=2}^{2m}\frac{\psi_0(2a+k)}{a+k}-\sum_{k=1}^{m}\frac{\psi_0(2a+2k)}{a+2k} \tag{173}$$

$$=\sum_{k=1}^{a+2m}\frac{\psi_0(a+k)}{k}-\sum_{k=1}^{m}\frac{\psi_0(2a+2k)}{a+2k}+\mathrm{CF}, \tag{174}$$

respectively. Here, the equality (174) is obtained by shifting the summation index as

$$\sum_{k=2}^{2m}\frac{\psi_0(2a+k)}{a+k}=\sum_{k=2+a}^{2m+a}\frac{\psi_0(a+k)}{k}=\sum_{k=1}^{2m+a}\frac{\psi_0(a+k)}{k}-\sum_{k=1}^{a+1}\frac{\psi_0(a+k)}{k}, \tag{175}$$

before evaluating the last sum by the identity (B.5). Moreover, for the summation

$$\sum_{k=1}^{m-1}\frac{\psi_0(a+2k)}{k}, \tag{176}$$

we have

$$\sum_{k=1}^{m-1}\frac{\psi_0(a+2k)}{k}=\sum_{k=1}^{m-1}\left(\frac{\psi_0(a+k+m)}{k}+\frac{\psi_0(a+k)}{k}+\frac{\psi_0(a+2k)}{a+k}\right)+\mathrm{CF}, \tag{177}$$

which is obtained by the fact that

$$\sum_{k=1}^{m-1}\frac{\psi_0(a+2k)}{a+k}=\sum_{k=1}^{m-1}\sum_{l=0}^{k-1}\frac{1}{(a+k)(a+k+l)}+\sum_{k=1}^{m-1}\frac{\psi_0(a+k)}{a+k} \tag{178}$$

similarly to the identity (133). By substituting in (169) the sums (170), (171) and (176) with their equivalent forms (172), (174) and (177), respectively, we arrive at

$$\begin{aligned}\mathbb{E}[C]=&\frac{(2m-1)(2a+2m-1)}{4a+8m-2}\left(\sum_{k=1}^{2m-1}\frac{\psi_0(a+k)}{k}+\sum_{k=1}^{2m+a-1}\frac{\psi_0(a+k)}{k}\right)\\&+\frac{1}{4}\left(\sum_{k=1}^{m-1}\frac{\psi_0(a+k+m)}{k}+\sum_{k=1}^{m-1}\frac{\psi_0(a+k)}{k}\right)+\mathrm{CF}.\end{aligned} \tag{179}$$

Finally, replacing the single sums in (179) by the short-hand notation $\Phi_{c,d}$ defined in (19) the claimed result (38) is obtained. This completes the proof of proposition 2.

3.3. *Asymptotic capacity*

In this section, we compute the limiting average capacity in corollary 1. Note that the limiting average capacity can be obtained by using the limiting level density of the Jacobi unitary ensemble instead of the general form (49) that we have utilized in section 3 for the finite-size computation. On the other hand, computing the limiting capacity is straightforward when the corresponding finite-size formulas are available. Specifically, the limiting values in (45) are obtained by computing the limits of the exact capacity (25) and (38) in the regime (44). To this end, the following asymptotic results are needed. The first one is the limiting behavior of polygamma functions (33) and (34). The second one is the fact that in the asymptotic regime

$$c\to\infty,\qquad \text{with a fixed } d, \tag{180}$$

one has

$$\Phi_{c,d} \longrightarrow \psi_1(1) = \frac{\pi^2}{6}. \tag{181}$$

For the exact capacity formula (25) of fermionic Gaussian states with fixed particle number (17), we now have in the limit (44),

$$\frac{\alpha_0}{m} = \frac{1}{8} + o\left(\frac{1}{m}\right) \tag{182}$$

$$\frac{\alpha_1}{m} = o\left(\frac{1}{m}\right) \tag{183}$$

$$\frac{\alpha_2}{m} = o\left(\frac{1}{m}\right) \tag{184}$$

$$\frac{\alpha_3}{m} = -\frac{1}{2} + o\left(\frac{1}{m}\right), \tag{185}$$

and

$$\psi_1(a+b+m+1) + \psi_1(a+m+1) = o\left(\frac{1}{m}\right) \tag{186}$$

$$\psi_0(a+m+1) - \psi_0(a+b+m+1) = o\left(\frac{1}{m}\right) \tag{187}$$

$$\psi_0(a+m+1) = o(\ln m), \tag{188}$$

where we recall $a = n - p$ and $b = p - m$. Consequently, we obtain

$$\begin{aligned}\mathbb{E}[C] = {} & 2\left(\frac{1}{8} + o\left(\frac{1}{m}\right)\right)\left(\frac{\pi^2}{2} + o\left(\frac{1}{m}\right)\right) \\ & + 2o\left(\frac{1}{m}\right) o(\ln m) - 1 + o\left(\frac{1}{m}\right),\end{aligned} \tag{189}$$

where, by using the fact that

$$\lim_{m\to\infty} \frac{\ln m}{m} = 0, \tag{190}$$

one arrives at the claimed asymptotic result

$$\mathbb{E}[C] \xrightarrow{(44)} \frac{\pi^2}{8} - 1. \tag{191}$$

For the exact capacity (38) of fermionic Gaussian states with arbitrary particle number (18), similarly we have in the limit (44),

$$\psi_1(m+n) = o\left(\frac{1}{m}\right) \tag{192}$$

$$\psi_1(n) = o\left(\frac{1}{m}\right) \tag{193}$$

$$\psi_0(2n) - \psi_0(m+n) = o\left(\frac{1}{m}\right) \tag{194}$$

$$\psi_0(m+n) - \psi_0(n) = \ln 2 + o\left(\frac{1}{m}\right) \tag{195}$$

$$\psi_0(m+n) - \psi_0(n-m) = -\psi_0(n-m) + \ln 2 + o(\ln m). \tag{196}$$

As a result, we have

$$\begin{aligned}\mathbb{E}[C] = &\frac{1}{3}\pi^2\left(o\left(\frac{1}{m}\right)+\frac{1}{2}\right)+\left(o\left(\frac{1}{m}\right)+\frac{1}{4}\right)\left(o\left(\frac{1}{m}\right)-\frac{\pi^2}{6}\right)\\ &+o\left(\frac{1}{m}\right)o(\ln m)+o\left(\frac{1}{m}\right)-1,\end{aligned} \tag{197}$$

which leads to the claimed result

$$\mathbb{E}[C] \xrightarrow{(44)} \frac{\pi^2}{8} - 1. \tag{198}$$

This completes the proof of corollary 1.

4. Conclusion

In this work, we derived the exact and asymptotic average capacity formulas of fermionic Gaussian states with and without particle number constraints. The derivation of the results relies on tools from random matrix theory and, more importantly, recent progress in simplifying finite summations involving special functions. The obtained analytical formulas provide insights into the statistical behavior of entanglement as measured by entanglement capacity. Future works include computing higher-order statistics, such as the variance, of entanglement capacity. In particular, by obtaining the finite-size variance formulas, a simple Gaussian approximation to the distribution of entanglement capacity can be constructed.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Summation representations of integrals

In this appendix, we list the summation representations of the integrals $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$, $\mathcal{A}_1^{(a,b)}$, $\mathcal{A}_2$ in (70)–(72) and I_C, A_1, A_2 in (150), (152) and (153) in the computation of average entanglement capacity in section 3.

A.1. Summation representations of integrals $\mathrm{I}_{\mathcal{C}}{}^{(a,b)}$, $\mathcal{A}_1^{(a,b)}$, $\mathcal{A}_2(m-1,m-1)$, *and* $\mathcal{A}_2(m-2,m)$

$$\begin{aligned}
\mathrm{I}_{\mathcal{C}}{}^{(a,b)} &= \frac{2m(b+m)}{a+b+2m}\sum_{i=1}^{m-2}\frac{i}{(m-i-1)_2}\left(\psi_0(b+i+1)-\psi_0(m-i-1)+\psi_0(i+1)\right.\\
&\quad\left.-\psi_0(a+b+i+m+1)\right)-\frac{(a+m)(a+b+m)}{a+b+2m}\sum_{i=1}^{m-1}\frac{2i}{(a+b+i+m)_2}\\
&\quad\times\left(\psi_0(b+i+1)-\psi_0(m-i)+\psi_0(i+1)-\psi_0(a+b+i+m+2)\right)\\
&\quad+\frac{m(b+m)}{a+b+2m}\sum_{i=m}^{m+1}\frac{(i-1)(-1)^{i+m-1}}{\Gamma(i-m+1)\Gamma(m-i+2)}\Big(\psi_1(b+i)-\psi_1(i-m+1)\\
&\quad+\psi_1(i)-\psi_1(a+b+i+m)+(\psi_0(b+i)-\psi_0(i-m+1)+\psi_0(i)\\
&\quad-\psi_0(a+b+i+m))^2\Big)
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
\mathcal{A}_1^{(a,b)} &= -\frac{2m(b+m)}{a+b+2m}\sum_{i=1}^{m-3}\frac{(b+i+1)(i)_2}{(m-i-2)_3(a+b+i+m+1)}\left(\psi_0(b+i+2)+\psi_0(i+2)\right.\\
&\quad\left.-\psi_0(a+b+i+m+2)-\psi_0(m-i-2)\right)+\frac{2(a+m)(a+b+m)}{a+b+2m}\\
&\quad\times\sum_{i=1}^{m-2}\frac{(b+i+1)(i)_2}{(m-i-1)(a+b+i+m)_3}\left(\psi_0(b+i+2)-\psi_0(a+b+i+m+3)\right.\\
&\quad\left.-\psi_0(m-i-1)+\psi_0(i+2)\right)-\frac{m(b+m)}{a+b+2m}\sum_{i=m-3}^{m-1}\frac{(b+i+2)(-1)^{i+m}(i+1)_2}{\Gamma(m-i)\Gamma(i-m+4)}\\
&\quad\times\frac{1}{a+b+i+m+2}\left(\psi_1(b+i+3)-\psi_1(i-m+4)-\psi_1(a+b+i+m+3)\right.\\
&\quad\left.+(\psi_0(i+3)-\psi_0(a+b+i+m+3)-\psi_0(i-m+4)+\psi_0(b+i+3))^2\right)\\
&\quad+\psi_1(i+3)-\frac{(a+m)(a+b+m)(b+m)(m-1)_2}{(a+b+2m)(a+b+2m-1)_3}\Big(-\psi_1(a+b+2m+2)\\
&\quad+\psi_1(b+m+1)+\psi_1(m+1)-\psi_1(1)+\psi_0^2(1)+(\psi_0(b+m+1)+\psi_0(m+1)\\
&\quad-\psi_0(a+b+2m+2))(\psi_0(b+m+1)+\psi_0(m+1)-\psi_0(a+b+2m+2)\\
&\quad-2\psi_0(1))\Big)
\end{aligned}\tag{A.2}$$

$$
\begin{aligned}
&\mathcal{A}_2(m-1,m-1)\\
&=\frac{2\Gamma(a+m+1)\Gamma(b+m+1)}{\Gamma(a+b+2m+2)}\left(\sum_{i=1}^{m}\frac{i\,(m-i+1)\,(-1)^i}{\Gamma(a+i+1)\,\Gamma(b-i+m+2)}\sum_{j=i-2}^{i}(-1)^j\right.\\
&\quad\times\frac{\Gamma(a+i-j+m)\,\Gamma(b-i+j+m+2)}{\Gamma(j+1)\,\Gamma(i-j+1)\,\Gamma(j-i+3)\,\Gamma(m-j)}\Big(\psi_1(a+b+2m+2)\\
&\quad+(\psi_0(a+i-j+m)-\psi_0(i-j+1)+\psi_0(i+1)-\psi_0(a+b+2m+2))\\
&\quad\times\Big(\psi_0(a+b+2m+2)-\psi_0(b-i+j+m+2)+\psi_0(j-i+3)\\
&\quad-\psi_0(m-i+2)\Big)\Big)+\sum_{i=1}^{m-2}\frac{i(m-i+1)}{\Gamma(b+i+1)\Gamma(a-i+m+2)}\sum_{j=1}^{m-i-1}\frac{\Gamma(b-j+m)}{\Gamma(m-i-j)}\\
&\quad\times\frac{\Gamma(a+j+m+2)}{(j)_3\Gamma(i+j+1)}(\psi_0(a+j+m+2)+\psi_0(m-i+2)-\psi_0(j+3)\\
&\quad-\psi_0(a+b+2m+2))+\sum_{i=1}^{m-2}\frac{i\,(m-i+1)}{\Gamma(a+i+1)\Gamma(b-i+m+2)}\\
&\quad\times\sum_{j=1}^{m-i-1}\frac{\Gamma(a-j+m)\Gamma(b+j+m+2)}{(j)_3\Gamma(i+j+1)\Gamma(m-i-j)}(\psi_0(b+j+m+2)+\psi_0(m-i+2)\\
&\quad\left.-\psi_0(j+3)-\psi_0(a+b+2m+2))\right)
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&\mathcal{A}_2(m-2,m)\\
&=\frac{2\Gamma(a+m)\Gamma(b+m)}{\Gamma(a+b+2m+2)}\left(\sum_{i=1}^{m-1}\frac{\Gamma(a+m+1)\Gamma(b+m+1)}{\Gamma(i)\Gamma(a+i+1)\Gamma(m-i)\Gamma(b-i+m+1)}\right.\\
&\quad\times\Big((\psi_0(a+b+2m+2)-\psi_0(a+m+1)-\psi_0(i+1)+\psi_0(1))(\psi_0(m-i+1)\\
&\quad-\psi_0(a+b+2m+2)+\psi_0(b+m+1)-\psi_0(1))+\psi_1(a+b+2m+2)\Big)\\
&\quad+\sum_{i=1}^{m-1}\frac{i\,(m-i)}{\Gamma(b+i+1)\Gamma(a-i+m+1)}\sum_{j=1}^{m-i}\frac{\Gamma(a+j+m+1)\Gamma(b-j+m+1)}{j\Gamma(i+j+1)\Gamma(m-i-j+1)}\\
&\quad\times(\psi_0(a+j+m+1)-\psi_0(a+b+2m+2)+\psi_0(m-i+1)-\psi_0(j+1))\\
&\quad+\sum_{i=1}^{m-1}\frac{i\,(m-i)}{\Gamma(a+i+1)\Gamma(b-i+m+1)}\sum_{j=1}^{m-i}\frac{\Gamma(a-j+m+1)\Gamma(b+j+m+1)}{j\Gamma(i+j+1)\Gamma(m-i-j+1)}\\
&\quad\left.\times(\psi_0(b+j+m+1)-\psi_0(a+b+2m+2)+\psi_0(m-i+1)-\psi_0(j+1))\right)
\end{aligned}
\tag{A.4}
$$

A.2. Summation representations of integrals $\mathrm{I_C}$, $\mathrm{A_1}$, *and* $\mathrm{A_2}$

$$
\begin{aligned}
\mathrm{I_C} = {} & (\psi_0(a+2) - \psi_0(2a+3))^2 + \psi_1(a+2) - \psi_1(2a+3) + \sum_{k=1}^{m-1} 2(2a+4k+1) \\
& \times \left(\sum_{j=2k-1}^{2k} \frac{(-1)^j (j+1)(a+j+1)}{(2a+j+2k+1)_2} \Big((\psi_0(j+2) - \psi_0(2a+j+2k+3) \right. \\
& + \psi_0(a+j+2) - \psi_0(j-2k+2))^2 + \psi_1(a+j+2) - \psi_1(2a+j+2k+3) \\
& + \psi_1(j+2) - \psi_1(j-2k+2) \Big) + \sum_{j=0}^{2k-2} \frac{2(j+1)(a+j+1)}{(2k-j-1)_2 (2a+j+2k+1)_2} \\
& \left. \times (\psi_0(a+j+2) - \psi_0(2a+j+2k+3) - \psi_0(2k-j-1) + \psi_0(j+2)) \right)
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\mathrm{A_1} = {} & \sum_{k=0}^{m-1} 2\,(2a+4k+1) \left(\sum_{j=2k-2}^{2k} \frac{(-1)^j\,(j+1)_2\,(a+j+1)_2}{\Gamma(2k-j+1)\,\Gamma(j-2k+3)\,(2a+j+2k+1)_3} \right. \\
& \times \Big((\psi_0\,(a+j+3) - \psi_0\,(2a+j+2k+4) - \psi_0\,(j-2k+3) + \psi_0\,(j+3))^2 \\
& - \psi_1\,(2a+j+2k+4) + \psi_1\,(a+j+3) - \psi_1\,(j-2k+3) + \psi_1\,(j+3) \Big) \\
& + \sum_{j=0}^{2k-3} \frac{2\,(j+1)_2\,(a+j+1)_2}{(2k-j-2)_3\,(2a+j+2k+1)_3} \,(\psi_0\,(2a+j+2k+4) - \psi_0\,(a+j+3) \\
& \left. + \,\psi_0\,(2k-j-2) - \psi_0(j+3)) \right)
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\mathrm{A_2} = {} & \sum_{k=0}^{m-1} \frac{(2a+4k+1)\Gamma(2k+1)\Gamma(2a+2k+1)}{\Gamma(2a+4k+4)} \left(\sum_{i=0}^{2k} \frac{2(i+1)(2k-i+1)}{\Gamma(i+1)\Gamma(a+i+1)} \right. \\
& \times \frac{\Gamma^2(a+2k+2)}{\Gamma(2k-i+1)\Gamma(a+2k-i+1)} ((\psi_0(a+2k+2) - \psi_0(2a+4k+4) - \psi_0(2) \\
& + \psi_0(2k-i+2))(\psi_0(a+2k+2) - \psi_0(2a+4k+4) + \psi_0(i+2) - \psi_0(2)) \\
& - \psi_1(2a+4k+4)) - \sum_{j=0}^{2k} \frac{(j+1)\Gamma(a+2k+1)\Gamma(a+2k+3)}{\Gamma(j)\Gamma(a+j+1)\Gamma(2k-j+1)\Gamma(a-j+2k+1)} \\
& \times ((\psi_0(a+2k+1) - \psi_0(2a+4k+4) + \psi_0(2k-j+2) - \psi_0(1))(\psi_0(a+2k+3) \\
& - \psi_0(2a+4k+4) + \psi_0(j+2) - \psi_0(3)) - \psi_1(2a+4k+4)) - \sum_{j=0}^{2k} \frac{(2k-j+1)}{\Gamma(j+1)} \\
& \times \frac{\Gamma(a+2k+1)\Gamma(a+2k+3)}{\Gamma(a+j+1)\Gamma(2k-j)\Gamma(2k-j+a+1)} ((\psi_0(a+2k+3) - \psi_0(2a+4k+4) \\
& + \psi_0(2k-j+2) - \psi_0(3))(\psi_0(a+2k+1) - \psi_0(2a+4k+4) + \psi_0(j+2)
\end{aligned}
$$

$$
\begin{aligned}
&-\psi_0(1)) - \psi_1(2a+4k+4)) + 4\sum_{j=0}^{2k}\frac{\Gamma(a-j+2k)\Gamma(a+j+2k+4)}{(j+1)_3} \\
&\times \sum_{i=0}^{2k-j-2}\frac{(2k-i-j-1)(i+j+3)}{\Gamma(i+1)\Gamma(2k-i+1)\Gamma(a+i+j+3)\Gamma(a-i-j+2k-1)} \\
&\times \left(\psi_0(a+j+2k+4) - \psi_0(2a+4k+4) + \psi_0(i+j+4) - \psi_0(j+4)\right)\Bigg). \qquad \text{(A.7)}
\end{aligned}
$$

Appendix B. List of summation identities

In this appendix, we list the finite sum identities useful in simplifying the summations in appendix A. Here, it is sufficient to assume $a,b \geq 0, a \neq b$ in identities (B.1)–(B.3), (B.6) and (B.7), $a > m$ in (B.8), and $a,b \geqslant 0$, $n > m$ in (B.9)–(B.14)

$$
\sum_{i=1}^{m}\psi_0(i+a) = (m+a)\psi_0(m+a+1) - a\psi_0(a+1) - m \qquad \text{(B.1)}
$$

$$
\sum_{i=1}^{m}\psi_1(i+a) = (m+a)\psi_1(m+a+1) - a\psi_1(a+1) + \psi_0(m+a+1) - \psi_0(a+1) \qquad \text{(B.2)}
$$

$$
\sum_{i=1}^{m}\frac{\psi_0(i+a)}{i+a} = \frac{1}{2}\left(\psi_1(m+a+1) - \psi_1(a+1) + \psi_0^2(m+a+1) - \psi_0^2(a+1)\right) \qquad \text{(B.3)}
$$

$$
\sum_{i=1}^{m}\frac{\psi_0(m+1-i)}{i} = \psi_0^2(m+1) - \psi_0(1)\psi_0(m+1) + \psi_1(m+1) - \psi_1(1) \qquad \text{(B.4)}
$$

$$
\sum_{i=1}^{m}\frac{\psi_0(m+1+i)}{i} = \psi_0^2(m+1) - \psi_0(1)\psi_0(m+1) - \frac{1}{2}\psi_1(m+1) + \frac{\psi_1(1)}{2} \qquad \text{(B.5)}
$$

$$
\begin{aligned}
\sum_{i=1}^{m}\psi_0(i+a)\psi_0(i+b) = (b-a)&\sum_{i=1}^{m-1}\frac{\psi_0(a+i)}{b+i} + (m+a)\psi_0(m+a)\psi_0(m+b) - a \\
&\times \psi_0(a+1)\psi_0(b+1) - (m+a-1)\psi_0(m+a) + a\psi_0(a+1) \\
&-(m+b)\psi_0(m+b) + (b+1)\psi_0(b+1) + 2m - 2 \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{m}\frac{\psi_0(i+b)}{i+a} = &-\sum_{i=1}^{m}\frac{\psi_0(i+a)}{i+b} + \psi_0(m+a+1)\psi_0(m+b+1) - \psi_0(a+1) \\
&\times \psi_0(b+1) + \frac{1}{a-b}\left(\psi_0(m+a+1) - \psi_0(m+b+1) - \psi_0(a+1)\right. \\
&\left.+\psi_0(b+1)\right) \qquad \text{(B.7)}
\end{aligned}
$$

$$\sum_{i=1}^{m}\frac{\psi_0(a+1-i)}{i}=-\sum_{i=1}^{m}\frac{\psi_0(i+a-m)}{i}+\left(\psi_0(a-m)+\psi_0(a+1)\right)\left(\psi_0(m+1)\right.$$
$$\left.-\psi_0(1)\right)+\frac{1}{2}\Big(\left(\psi_0(a-m)-\psi_0(a+1)\right)^2+\psi_1(a+1)-\psi_1(a-m)\Big) \tag{B.8}$$

$$\sum_{i=1}^{m}\frac{\psi_0(a+b+i)}{i}=\sum_{i=1}^{m}\frac{\psi_0(b+i)}{i}-\sum_{i=1}^{a}\frac{\psi_0(b+i+m)}{b+i-1}+\frac{1}{2}\left(\psi_1(b)+\left(\psi_0(a+b)-\psi_0(b)\right)\right.$$
$$\left.\times\left(\psi_0(a+b)+\psi_0(b)+2\left(\psi_0(m+1)-\psi_0(1)\right)\right)-\psi_1(a+b)\right) \tag{B.9}$$

$$\sum_{i=1}^{m}\frac{(n-i)!}{(m-i)!}=\frac{n!}{(m-1)!\,(n-m+1)} \tag{B.10}$$

$$\sum_{i=1}^{m}\frac{(n-i)!}{(m-i)!i}=\frac{n!}{m!}\left(\psi_0(n+1)-\psi_0(n-m+1)\right) \tag{B.11}$$

$$\sum_{i=1}^{m}\frac{(n-i)!}{(m-i)!i^2}=\frac{n!}{m!}\left(\sum_{i=1}^{m}\frac{\psi_0(i+n-m)}{i}+\frac{1}{2}\big(\psi_1(n-m+1)-\psi_1(n+1)-\psi_0^2(n+1)\right.$$
$$+\psi_0^2(n-m+1)\big)+\psi_0(n-m)(\psi_0(n+1)-\psi_0(n-m+1)$$
$$\left.-\psi_0(m+1)+\psi_0(1))\right) \tag{B.12}$$

$$\sum_{i=1}^{m}\frac{(n-i)!}{(m+a-i)!}=\frac{1}{n-m-a+1}\left(\frac{n!}{(a+m-1)!}-\frac{(n-m)!}{(a-1)!}\right) \tag{B.13}$$

$$\sum_{i=1}^{m}\frac{(n-i)!}{(m+a-i)!}\psi_0(m+a-i+1)$$
$$=\frac{1}{1-a-m+n}\left(\frac{n!}{(a+m-1)!}\left(\psi_0(a+m)-\frac{1}{1-a-m+n}\right)-\frac{(n-m)!}{(a-1)!}\right.$$
$$\left.\times\left(\psi_0(a)-\frac{1}{1-a-m+n}\right)\right) \tag{B.14}$$

Proofs to the above identities (B.1)–(B.14) can be found, for example, in [6, 8, 10, 11, 21, 23, 44]. For convenience, we summarize in the following the main strategies in obtaining these identities. Specifically, the main idea in deriving the identities (B.1)–(B.9) is to change the summation orders and make use of the obtained lower order summation formulas in a recursive manner. For example, by using the finite sum form of the digamma function

$$\psi_0(l) = -\gamma + \sum_{k=1}^{l-1} \frac{1}{k}, \tag{B.15}$$

the summation

$$\sum_{i=1}^{m} \psi_0(i+a)\,\psi_0(i+b) \tag{B.16}$$

can be rewritten as

$$\psi_0(b) \sum_{i=1}^{m} \psi_0(i+a) + \sum_{j=1}^{m} \frac{1}{b+j-1} \sum_{i=j}^{m} \psi_0(i+a), \tag{B.17}$$

where we have changed the summation order of the double sum. The remaining sums can be simplified by using the lower order identity (B.1), leading to the result in (B.6).

The identitiy (B.10) is a special case of the Chu-Vandermonde identity [42], which can be utilized to derive the identities (B.11) and (B.12) recursively. For example, in (B.11), the summation

$$S(m,n) = \sum_{i=1}^{m} \frac{(n-i)!}{(m-i)!} \frac{1}{i} \tag{B.18}$$

is computed by first obtaining the recurrence relation

$$S(m,n) = \frac{n}{m} S(m-1,n-1) + \frac{n-m}{m} \sum_{i=1}^{m} \frac{(n-1-i)!}{(m-i)!}. \tag{B.19}$$

After recurring m times, and using the existing result (B.10), one obtains the closed-form result in (B.11). The identity (B.12) is derived in a similar manner, where one needs to utilize the result (B.11).

For the identity (B.13), it is obtained by first considering

$$\sum_{i=1}^{m} \frac{(n-i)!}{(m+a-i)!} = \sum_{i=1}^{a+m} \frac{(n-i)!}{(m+a-i)!} - \sum_{i=1}^{a} \frac{(n-m-i)!}{(a-i)!} \tag{B.20}$$

before applying (B.10). The identity (B.13) is analytically continued to any complex number a. Taking a derivative of a in (B.13) gives (B.14).

Appendix C. Coefficients of results in section 3

In this appendix, we list the coefficients in the results (86), (123), (165) and (168).

C.1. Coefficients in (86)

$$a_0 = \frac{2(a+m)(a+b+m)}{a+b+2m} \tag{C.1}$$

$$a_1 = \frac{2m(b+m)}{a+b+2m} \tag{C.2}$$

$$a_2 = \frac{2\left(a^2+b\left(a+2m\right)+2am+2m^2\right)}{a+b+2m} \tag{C.3}$$

$$\begin{aligned} a_3 = &-\frac{2}{\left(b+m\right)\left(a+b+2m\right)^2}\left(b^2\left(a^2+8am+a+10m^2\right)+2a^2m\left(m+2\right)+a^3+b^4\right. \\ &+b^3\left(2a+5m\right)+b\left(a^2\left(3m+2\right)+6am\left(2m+1\right)+2m^2\left(5m+1\right)\right)+6am^2\left(m+1\right) \\ &\left.+2m^3\left(2m+1\right)\right) \end{aligned} \tag{C.4}$$

$$\begin{aligned} a_4 = &\frac{2}{b\left(a+b+2m\right)^2}\left(b^2\left(a^2+a\left(5m+2\right)+m\left(5m+3\right)\right)+b\left(m+2\right)\left(a^2+3am+2m^2\right)\right. \\ &\left.+b^3\left(2a+4m+1\right)+\left(a+m\right)^2\left(a+2m\right)+b^4\right) \end{aligned} \tag{C.5}$$

$$a_5 = \frac{2\left(b+m\right)\left(a^2+b\left(2a+3m\right)+3am+b^2+m\left(2m-1\right)\right)}{\left(a+b+2m\right)^2} \tag{C.6}$$

$$a_6 = -\frac{2b\left(a+b+2m+1\right)}{a+b+2m} \tag{C.7}$$

$$a_7 = -\frac{2m\left(a+m\right)\left(a^2+2ab+4am+a+b^2+4bm+b+4m^2+2m+1\right)}{\left(a+b+2m\right)^3}. \tag{C.8}$$

C.2. Coefficients in (123)

$$\begin{aligned} b_0 = &\frac{2\left(b+m\right)}{\left(a+b+2m-1\right)_3}\left(a^2\left(3b+4m\right)+a^3+a\left(3b^2+9bm+6m^2-1\right)+5b^2m+b^3\right. \\ &\left.+7bm^2-b+3m^3-m\right) \end{aligned} \tag{C.9}$$

$$b_1 = \frac{2m\left(a+m\right)\left(a^2+a\left(b+3m\right)+2bm+3m^2-1\right)}{\left(a+b+2m-1\right)_3} \tag{C.10}$$

$$b_2 = 2\left(a+m\right) \tag{C.11}$$

$$b_3 = -\frac{2\left(a\left(b+2m\right)+b^2+2bm+2m^2\right)}{a+b+2m} \tag{C.12}$$

$$b_4 = \frac{m-m^2}{2\left(a+b+2m-1\right)}+\frac{m^2+m}{2\left(a+b+2m+1\right)}-\frac{2b}{a+m}-\frac{2m}{a+b+2m}-2a-2m \tag{C.13}$$

$$b_5 = \frac{m(a+m)(b+m)(a+b+m)}{(a+b+2m-1)_3} \tag{C.14}$$

$$\begin{aligned} b_6 = \frac{1}{8a}\Bigg(& \frac{-2a^2b^2+a^4+4a^2+b^4+4b^2}{a+b+2m} - \frac{1}{2}(a-b-1)(a-b+1)(a+b-1)(a+b+1) \\ & \times \left(\frac{1}{a+b+2m+1} + \frac{1}{a+b+2m-1}\right) + 16ab+8am+7a+11b+6m\Bigg) \end{aligned} \tag{C.15}$$

$$\begin{aligned} b_7 = & \frac{b}{a+b+2m} + \frac{(a-b)(a+b)}{2(a+b+2m)^2} + \frac{1}{4}(5a+b-1)+m+\frac{1}{8}((a-b-1)(a-b+1) \\ & \times (a+b+1))\left(\frac{1}{a+b+2m+1} - \frac{1}{a+b+2m-1}\right) \end{aligned} \tag{C.16}$$

$$b_8 = -\frac{a\left(a^2+3a(b+2m+1)+2b^2+b(6m+3)+6m^2+6m+2\right)}{(a+b+2m)(a+b+2m+1)} \tag{C.17}$$

$$b_9 = -\frac{m}{(a+b+2m)^2} - \frac{m}{2(a+b+2m+1)} - \frac{m}{2}. \tag{C.18}$$

C.3. Coefficients in (165)

$$c_0 = -\frac{1}{2}(2a+2m-1) \tag{C.19}$$

$$c_1 = \frac{1}{2}(2a-1) \tag{C.20}$$

$$c_2 = \frac{1}{4}(4m+1) \tag{C.21}$$

$$c_3 = -2a-4m+1 \tag{C.22}$$

$$c_4 = \frac{1}{4}(4a+4m-1) \tag{C.23}$$

$$c_5 = -2a \tag{C.24}$$

$$c_6 = 1-2m \tag{C.25}$$

$$c_7 = \frac{1}{4}(4a-1) \tag{C.26}$$

$$c_8 = -2(a+2m) \tag{C.27}$$

$$c_9 = -\frac{-12a^3-6a^2+4a+1}{4a^3+6a^2+2a} \tag{C.28}$$

$$c_{10} = \frac{a^2(4m-1)+4a^3+a-1}{2(a-1)a} \tag{C.29}$$

$$c_{11} = \frac{a^4(12-8m)+a^3(3-8m)+a^2(2m-13)-12a^5+2am+1}{2(4a^5-5a^3+a)} \tag{C.30}$$

$$c_{12} = \frac{a\left(8\left(a+1\right)m+2a+3\right)+2m}{a\left(a+1\right)\left(2a+1\right)} \tag{C.31}$$

$$c_{13} = \frac{1}{2}\left(4m-1\right) \tag{C.32}$$

$$c_{14} = \frac{-4m-3}{4\left(a-1\right)} - 2a - \frac{3}{4\left(a+1\right)} - \frac{1}{2a-1} - \frac{1}{2a+1} - \frac{1}{a} + \frac{1}{2}\left(4m-3\right) \tag{C.33}$$

$$c_{15} = -\frac{4a^2+1}{4a^2-1} \tag{C.34}$$

$$c_{16} = -\frac{a^2\left(4m+1\right)+4a^3+a\left(4m-1\right)-1}{2a\left(a^2-1\right)} \tag{C.35}$$

$$c_{17} = \frac{m}{1-a} \tag{C.36}$$

$$c_{18} = \frac{8m+3}{4\left(a-1\right)} + \frac{1}{2a} - \frac{3}{4\left(a+1\right)} + 2. \tag{C.37}$$

C.4. Coefficients in (168)

$$d_0 = -\frac{1}{2}\left(2a+2m-1\right) \tag{C.38}$$

$$d_1 = a - \frac{1}{2} \tag{C.39}$$

$$d_2 = \frac{4a^2-1}{16\left(2a+4m-1\right)} - \frac{a}{8} + \frac{3m}{4} + \frac{3}{16} \tag{C.40}$$

$$d_3 = -\left(2a+4m-1\right) \tag{C.41}$$

$$d_4 = \frac{\left(2m-1\right)\left(2a+2m-1\right)}{4a+8m-2} \tag{C.42}$$

$$d_5 = \frac{1}{4}\left(4a+4m-1\right) \tag{C.43}$$

$$d_6 = \frac{\left(2a+2m-1\right)\left(4a+6m-1\right)}{2\left(2a+4m-1\right)} \tag{C.44}$$

$$d_7 = \frac{1-4a^2}{8\left(2a+4m-1\right)} + \frac{a}{4} + \frac{1}{8}\left(5-12m\right) \tag{C.45}$$

$$d_8 = -2\left(a+2m\right) \tag{C.46}$$

$$d_9 = -\frac{-12a^3-6a^2+4a+1}{4a^3+6a^2+2a} \tag{C.47}$$

$$d_{10} = \frac{4a^2\left(4m^2-2m+1\right)+a^3\left(20m-3\right)+6a^4-4a\left(m-1\right)^2-4m+1}{2\left(a-1\right)a\left(2a+4m-1\right)} \tag{C.48}$$

$$d_{11} = \frac{88a^5 - 76a^4 - 10a^3 + 83a^2 - 3a - 4}{16(1-a)a(a+1)(2a-1)(2a+1)} - \frac{(2a-1)(2a+1)}{16(a-1)(2a+4m-1)} - \frac{3m}{4(a-1)} \tag{C.49}$$

$$d_{12} = -\frac{a(4m+2) + 12m^2 - 1}{4(a-1)(2a+4m-1)} \tag{C.50}$$

$$d_{13} = \frac{8a^3 + 4a^2 - 2a - 1}{8a(a+1)(2a+4m-1)} + \frac{3(4m-1)}{8(a+1)} + \frac{3(4m+1)}{8a} + \frac{4}{2a+1} - \frac{1}{2} \tag{C.51}$$

$$d_{14} = \frac{1-4a^2}{-16a-32m+8} - \frac{a}{4} + \frac{1}{8}(12m-3) \tag{C.52}$$

$$d_{15} = \frac{8a^3 - 12a^2 - 2a + 3}{16(a-1)(2a+4m-1)} - \frac{36a^3 + 24a^2 - 5a - 2}{4a(a+1)(2a-1)(2a+1)} - \frac{12m+13}{16(a-1)} - \frac{7a}{4} + \frac{1}{4}(6m-4) \tag{C.53}$$

$$d_{16} = \frac{1+4a^2}{1-4a^2} \tag{C.54}$$

$$d_{17} = \frac{4a^2-1}{8(a-1)(2a+4m-1)} + \frac{12m+7}{8(a-1)} + \frac{1}{4a} - \frac{3}{4(a+1)} + \frac{7}{4}. \tag{C.55}$$

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References

[1] Page D N 1993 Average entropy of a subsystem *Phys. Rev. Lett.* **71** 1291–4
[2] Foong S K and Kanno S 1994 Proof of Page's conjecture on the average entropy of a subsystem *Phys. Rev. Lett.* **72** 1148–51
[3] Sánchez-Ruiz J 1995 Simple proof of Page's conjecture on the average entropy of a subsystem *Phys. Rev.* E **52** 5653–5
[4] Hayden P, Leung D and Winter A 2006 Aspects of generic entanglement *Commun. Math. Phys.* **265** 95–117
[5] Vivo P, Pato M P and Oshanin G 2016 Random pure states: quantifying bipartite entanglement beyond the linear statistics *Phys. Rev.* E **93** 052106
[6] Wei L 2017 Proof of Vivo-Pato-Oshanin's conjecture on the fluctuation of von Neumann entropy *Phys. Rev.* E **96** 022106
[7] Sarkar A and Kumar S 2019 Bures-Hall ensemble: spectral densities and average entropies *J. Phys. A: Math. Theor.* **52** 295203
[8] Wei L 2020 Skewness of von Neumann entanglement entropy *J. Phys. A: Math. Theor.* **53** 075302

[9] Wei L 2020 Proof of Sarkar-Kumar conjectures on average entanglement entropies over the Bures-Hall ensemble *J. Phys. A: Math. Theor.* **53** 235203
[10] Wei L 2020 Exact variance of von Neumann entanglement entropy over the Bures-Hall measure *Phys. Rev.* E **102** 062128
[11] Huang Y, Wei L and Collaku B 2021 Kurtosis of von Neumann entanglement entropy *J. Phys. A: Math. Theor.* **54** 504003
[12] Lubkin E 1978 Entropy of an *n*-system from its correlation with a *k*-reservoir *J. Math. Phys.* **19** 1028–31
[13] Sommers H-J and Życzkowski K 2004 Statistical properties of random density matrices *J. Phys. A: Math. Gen.* **37** 35
[14] Osipov V, Sommers H-J and Życzkowski K 2010 Random Bures mixed states and the distribution of their purity *J. Phys. A: Math. Theor.* **43** 055302
[15] Giraud O 2007 Distribution of bipartite entanglement for random pure states *J. Phys. A: Math. Theor.* **40** 2793
[16] Li S-H and Wei L 2021 Moments of quantum purity and biorthogonal polynomial recurrence *J. Phys. A: Math. Theor.* **54** 445204
[17] Malacarne L C, Mendes R S and Lenzi E K 2002 Average entropy of a subsystem from its average Tsallis entropy *Phys. Rev.* E **65** 046131
[18] Wei L 2019 On the exact variance of Tsallis entanglement entropy in a random pure state *Entropy* **21** 539
[19] Borot G and Nadal C 2012 Purity distribution for generalized random Bures mixed states *J. Phys. A: Math. Theor.* **45** 075209
[20] Bianchi E, Hackl L and Kieburg M 2021 The Page curve for fermionic Gaussian states *Phys. Rev.* B **103** L241118
[21] Huang Y and Wei L 2022 Second-order statistics of fermionic Gaussian states *J. Phys. A: Math. Theor.* **55** 105201
[22] Bianchi E, Hackl L, Kieburg M, Rigol M and Vidmar L 2022 Volume-law entanglement entropy of typical pure quantum states *PRX Quantum* **3** 030201
[23] Huang Y and Wei L 2023 Entropy fluctuation formulas of fermionic Gaussian states *Ann. Henri Poincaré* (https://doi.org/10.1007/s00023-023-01342-w)
[24] Yao H and Qi X-L 2010 Entanglement entropy and entanglement spectrum of the Kitaev model *Phys. Rev. Lett.* **105** 080501
[25] Nandy P 2021 Capacity of entanglement in local operators *J. High Energy Phys.* JHEP07(2021)019
[26] Arias R, Di Giulio G, Keski-Vakkuri E and Tonni E 2023 Probing RG flows, symmetry resolution and quench dynamics through the capacity of entanglement *J. High Energy Phys.* JHEP03(2023)175
[27] de Boer J, Järvelä J and Keski-Vakkuri E 2019 Aspects of capacity of entanglement *Phys. Rev.* D **99** 066012
[28] Okuyama K 2021 Capacity of entanglement in random pure state *Phys. Lett.* B **820** 136600
[29] Wei L 2023 Average capacity of quantum entanglement *J. Phys. A: Math. Theor.* **56** 015302
[30] Bhattacharjee B, Nandy P and Pathak T 2021 Eigenstate capacity and Page curve in fermionic Gaussian states *Phys. Rev.* B **104** 214306
[31] Preskill J 2018 Quantum computing in the NISQ era and beyond *Quantum* **2** 79
[32] Surace J and Tagliacozzo L 2021 Fermionic Gaussian states: an introduction to numerical approaches (arXiv:2111.08343)
[33] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Dover)
[34] Bengtsson I and Życzkowski K 2017 *Geometry of Quantum States: An Introduction to Quantum Entanglement* 2nd edn (Cambridge University Press)
[35] Kieburg M, Forrester P and Ipsen J R 2019 Multiplicative convolution of real asymmetric and real anti-symmetric matrices *Adv. Pure Appl. Math.* **10** 467
[36] Lydżba P, Rigol M and Vidmar L 2020 Eigenstate entanglement entropy in random quadratic Hamiltonians *Phys. Rev. Lett.* **125** 180604
[37] Lydżba P, Rigol M and Vidmar L 2021 Entanglement in many-body eigenstates of quantum-chaotic quadratic Hamiltonians *Phys. Rev.* B **103** 104206
[38] Mehta M L 2004 *Random Matrices* 3rd edn (Elsevier)
[39] Forrester P 2010 *Log-Gases and Random Matrices* (Princeton University Press)

[40] Bernard D and Piroli L 2021 Entanglement distribution in the quantum symmetric simple exclusion process *Phys. Rev.* E **104** 014146

[41] Szegő G 1975 *Orthogonal Polynomials* (American Mathematical Society)

[42] Luke Y L 1969 *The Special Functions and Their Approximations* vol 1 (Academic)

[43] Brychkov Y A 2008 *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas* (CRC Press)

[44] Milgram M 2017 On some sums of digamma and polygamma functions (arXiv:0406338v3)