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ORIGINAL PAPER

A family of Andrews–Curtis trivializations via 4-manifold trisections

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Abstract
An R-link is an n-component link L in S^3 such that Dehn surgery on L yields $\#^n(S^1 \times S^2)$. Every R-link L gives rise to a geometrically simply-connected homotopy 4-sphere X_L, which in turn can be used to produce a balanced presentation of the trivial group. Adapting work of Gompf, Scharlemann, and Thompson, Meier and Zupan produced a family of R-links $L(p, q; c/d)$, where the pairs (p, q) and (c, d) are relatively prime and c is even. Within this family, $L(3, 2; 2n/(2n + 1))$ induces the infamous trivial group presentation $\langle x, y \mid xyx = yxy, x^{n+1} = y^n \rangle$, a popular collection of potential counterexamples to the Andrews–Curtis conjecture for $n \geq 3$. In this paper, we use 4-manifold trisections to show that the group presentations corresponding to a different subfamily, $L(3, 2; 4/d)$, are Andrews–Curtis trivial for all d.

Keywords Andrews–Curtis conjecture · Trisection · R-link

Mathematics Subject Classification (2020) 57K40 · 57R60 · 20F05

1 Introduction

The famous *Andrews–Curtis conjecture* [2] asserts that any balanced presentation

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

of the trivial group can be converted to the trivial presentation $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$ by a finite sequence of the following moves:

1. Replace a relator r_i by r_i^{-1};
2. Replace a relator r_i by $r_i r_j$, where $i \neq j$;
3. Replace a relator r_i by $x_j r_i x_j^{-1}$; and

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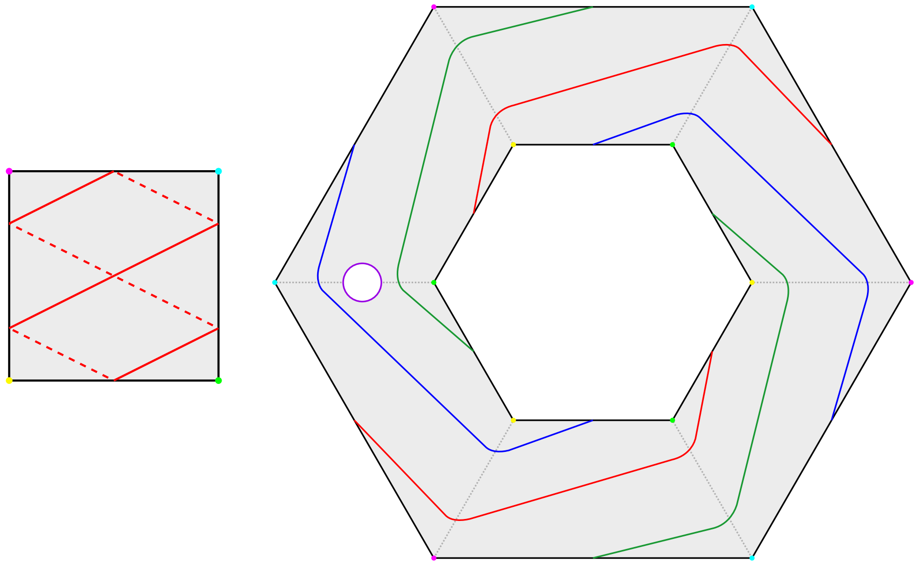


Fig. 1 At left, the curve $\lambda_{2/1}$ contained in S, with cone points at the corners. At right, the lift $\Lambda_{4/1}$ in the fiber $F \subset \widehat{F}$, where opposite edges are identified to form a genus-2 surface with one boundary component. The square knot $Q = \partial F$ is depicted as the small purple circle

4. Add or delete a trivial generator/relator pair x_{n+1} and $r_{n+1} = x_{n+1}$.

A presentation P that admits such a trivialization is called *AC-trivial*. Although the conjecture remains open, there are interesting families of potential counterexamples, many arising from constructions in low-dimensional topology. Perhaps the best known family in this category is the set of presentations

$$P_n = \langle x, y \,|\, xyx = yxy, x^{n+1} = y^n \rangle,$$

coming from a collection H_n of handle decompositions of the 4-sphere, each with two 1-handles and two 2-handles [1, 7]. The presentations P_n are not known to be AC-trivial for $n \geq 3$, and they form a well-studied collection of possible counterexamples to the Andrews–Curtis conjecture (see, for instance, the discussion in [1] or [19]).

A related notion is that of an *R-link*, an n-component link $L \subset S^3$ such that some Dehn surgery on L yields $\#^n(S^1 \times S^2)$. Every R-link L naturally gives rise to a balanced presentation $P(L)$ of the trivial group, and in [8], the authors constructed a family of R-links L_n with the property that $P(L_n) = P_n$, the presentations given above. This construction was generalized by Jeffrey Meier and the second author to produce an R-link $L(p, q; c/d)$ for any co-prime p and q and $c/d \in \mathbb{Q}$ with c even. With these parameters, $L(3, 2; 2n/(2n + 1))$ is stably equivalent (defined below) to the Gompf–Scharlemann–Thompson links L_n [15].

The links $L(p, q; c/d)$ are defined as follows: Let $Q = T_{p,q}\#\overline{T_{p,q}}$ be a *generalized square knot*, with fiber F. The closed fiber $\widehat{F}$ obtained by capping off $\widehat{F}$ with a disk admits a branched covering map ρ to a sphere S with four cone points. Curves in S avoiding the cone points can be parameterized by the extended rational numbers $\overline{\mathbb{Q}}$, and any curve $\lambda_{(c/2)/d}$ with c even lifts to a collection of curves $\Lambda_{c/d} \subset F \subset \widehat{F}$, in turn giving rise to the link $L(p, q; c/d) = Q \cup \Lambda_{c/d}$ in S^3. This construction is described in greater detail in [15], and an example is shown in Figs. 1 and 2.

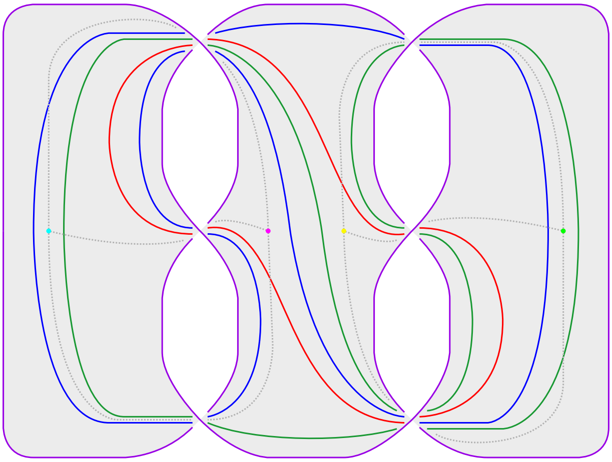


Fig. 2 The link $L(3, 2; 4/1)$ in S^3. Colored vertices correspond to the vertices and gray dotted arcs represent the inner and outer edges at right in Fig. 1

Let $P(p, q; c/d)$ denote the presentation $P(L(p, q; c/d))$ induced by the R-link $L(p, q; c/d)$. We have the following natural question:

Question 1 Which presentations $P(p, q; c/d)$ can be AC-trivialized?

In [8], the authors showed that the link $L(3, 2; 0/1)$ is handle-slide equivalent to the unlink, and in forthcoming work, the second author and collaborators show that for all d, the links $L(3, 2; 2/d)$ have the same property, from which it follows that the presentations $P(3, 2; 0/d)$ and $P(3, 2; 2/d)$ are AC-trivial [10]. The case $c = 4$ is more complicated, and the corresponding question for the links $L(3, 2; 4/d)$ remains open. However, in this paper, we prove

Theorem 1 *Every presentation of the form $P(3, 2; 4/d)$ is AC-trivial.*

The proof breaks into two cases, separated into Proposition 2 (dealing with the case $d = 4n + 1$) and Proposition 3 (dealing with the case $d = 4n + 3$). For both proofs, we use trisections of the closed 4-manifolds $X_{L(3,2;4/d)}$ arising from the R-links $L(3, 2; 4/d)$ in order to construct the presentations $P(3, 2; 4/d)$.

Remark 1 Various sources in the literature differ on whether to allow move (4); in some cases, AC-triviality is defined only with moves (1)–(3), and those sources often use *stable AC-triviality* to allow move (4) as well. In this paper, AC-triviality will always allow moves (1)–(4).

1.1 Organization

In Sect. 2, we establish the necessary background material for the paper. Section 3 deals with the first case of the main theorem, while Sect. 4 deals with the second case. We conclude in Sect. 5 with several questions for further investigation.

2 Preliminaries

We work in the smooth category throughout.

2.1 AC-equivalence and automorphisms

If P and P' are two group presentations related by moves (1)–(4) above, we say that P and P' are *AC-equivalent*, and we write $P \sim P'$. There is an additional move, the transformation move, that we can apply to a group presentation $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$:

(5) For an automorphism ψ of the free group F_n generated by $x_1, \ldots, x_n$, replace every relator r_i with its image $\psi(r_i)$.

Equivalence of presentations allowing moves (1)–(5) is called Q^{**}-*equivalence* [5, 16, 17]. To our knowledge, it remains open whether Q^{**}-equivalence is stronger than AC-equivalence; a detailed discussion can be found in Section 3 of [20]. However, the following is known:

Lemma 1 *[3, 18] If a presentation P can be converted to the trivial presentation via moves (1)–(3) and (5), then P can be converted to the trivial presentation via moves (1)–(3).*

As a corollary, we have

Corollary 1 *If a presentation P is Q^{**}-equivalent to the trivial presentation, then P is AC-trivial.*

Proof Suppose that $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ admits a sequence of moves (1)–(5) converting P to the trivial presentation. Observe that any instances of additions via move (4) can be carried out before any of the other moves, since any automorphism ψ used in move (5) extends by the identity over generators added after the automorphism would have been applied, and instances of deletions via move (4) need not be carried out, because any presentation equivalent to a trivial presentation after a deletion is also equivalent to a trivial presentation of longer length before the deletion. Thus, it follows that for some $m \geq n$, the presentation

$$P' = \langle x_1, \ldots, x_n, x_{n+1} \ldots, x_m \mid r_1, \ldots, r_n, x_{n+1}, \ldots, x_m \rangle$$

can be converted to the trivial presentation by moves of types (1)–(3) and (5). By Lemma 1, P' is AC-trivial, and since P and P' are related by moves of type (4), it follows that P is AC-trivial as well. □

2.2 R-links

As mentioned above, an *R-link* is an n-component link $L \subset S^3$ such that L has a Dehn surgery yielding $\#^n(S^1 \times S^2)$. Every R-link gives rise to a closed 4-manifold X_L built with one 0-handle, n 2-handles, n 3-handles, and one 4-handle, where L is the attaching link for the 2-handles, with framings giving by the Dehn surgery coefficients. In this case, we have $\chi(X_L) = 1+n-n+1 = 2$, and since X is simply-connected, $\beta_1(X_L) = \beta_3(X_L) = 0$, which means that $H_2(X_L; \mathbb{Z}) = \mathbb{Z}^{\beta_2(X_L)} = 0$. It follows from Theorem 1.2.25 of [9] (Whitehead's Theorem) that X_L is a homotopy 4-sphere, as X_L and S^4 have identical intersection forms.

Inverting the handle decomposition of X_L yields one 0-handle, n 1-handles, n 2-handles, and one 4-handle, which can be used to produce a balanced presentation for $\pi_1(X_L)$, the trivial group. In general, an R-link L does not induce a unique such presentation; for instance, a choice of co-cores of the 1-handles in the inverted handle decomposition determines a choice of the generators $x_1, \ldots, x_n$ in the presentation $P(L)$, and a different choice induces an automorphism of the free group F_n, the fundamental group of the union of the 0-handle and the n 1-handles. Nevertheless, we can prove the following:

Lemma 2 *Let L be an R-link, and suppose that P and P' are two different presentations induced by L. Then P and P' are Q^{**}-equivalent.*

Proof Suppose that $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ and $P' = \langle y_1, \ldots y_n \mid s_1, \ldots, s_n \rangle$ are two presentations induced by L. In the context of the AC-moves, we assume that all groups have the same generating set. Thus, let $\iota : \langle y_1, \ldots, y_n \rangle \to \langle x_1, \ldots, x_n \rangle$ be the map $\iota(y_i) = x_i$, and let $P'' = \iota(P') = \langle x_1, \ldots, x_n \mid \iota(s_1), \ldots, \iota(s_n) \rangle$, so that P'' is identical to P' but uses x_i's instead of y_i's.

There are several sets of choices we make to extract P and P' from L: A choice of co-cores of the 1-handles in the inverted handle decomposition, a choice of a base point for X_L, a path from the base point to each component of the attaching link L^* for the 2-handles in the inverted handle decomposition, and an orientation for each component, since we need a place to begin and a direction when using each component of L^* to read off a relator. Different choices of orientations yield relators related by a move of type (1). Likewise, any two choices of base points and paths can be related by conjugating the relators by generators, and as such these relators are related by moves of type (3). Thus, we may assume that P and P' arise from identical choices of base points and orientations of L^*, and in practice, we read off the relators by only considering components of L^*, ignoring the base point and paths.

Regarding the choices of co-cores of the 1-handles, there is a diffeomorphism of the boundary of the union of the 0-handle and 1-handles sending one choice to any other, inducing an automorphism σ of the free group. In other words, each x_i can be expressed as a word in the y_i's, and let $\sigma : \langle x_1, \ldots, x_n \rangle \to \langle y_1, \ldots, y_n \rangle$ be the isomorphism such that $\sigma(x_i)$ is the expression of x_i as this word. Observe that the relators in both presentations are determined by the fixed attaching link L^* with the same base point, paths, and orientations, it follows that (possibly after reindexing), we have $\sigma(r_i) = s_i$.

Finally, define $\psi = \iota \circ \sigma$. Then we have

$$\langle x_1, \ldots, x_n \mid \psi(r_1), \ldots \psi(r_n) \rangle = \langle x_1, \ldots, x_n \mid \iota(s_1), \ldots \iota(s_n) \rangle = P''.$$

We conclude that P and P'' are related by a move of type (5), and thus any two such presentations P and P'' are Q^{**}-equivalent. □

In view of Lemma 2, the Q^{**}-equivalence class of the the presentation induced by the R-link L is well-defined, and so we use $P(L)$ to denote this equivalence class. By Corollary 1, if we can show that some presentation in the equivalence class $P(L)$ is AC-trivial, then every presentation in $P(L)$ is AC-trivial. For this reason and for the purposes of proving AC-triviality, we often blur the distinction and abuse notation to let $P(L)$ denote any representative of the equivalence class $P(L)$.

There are also moves on the R-link L that leave $P(L)$ invariant: If L and L' are related by a sequence of handle-slides, we say L and L' are *handle-slide equivalent*. More generally, if L and L' are R-links (possibly with different numbers of components) and U and U' are unlinks, such that the split links $L \sqcup U$ and $L' \sqcup U'$ are handle-slide equivalent, we say that L and L' are *stably equivalent*. We have the following well-known lemma.

Lemma 3 *If L and L' are stably equivalent, then $P(L) = P(L')$.*

Proof Any handle-slide of L over L' induces a handle-slide of $(L')^*$ over L^*, the dual attaching links in the inverted handle decompositions. As such, the corresponding presentations can be related by moves of type (2) (and possibly other moves corresponding to the choices referenced in the proof of Lemma 2). If L and L' are stably handle-slide equivalent, then there are unlinks U and U' such that $L \sqcup U$ and $L' \sqcup U'$, are handle-slide equivalent, so that $P(L \sqcup U) = P(L' \sqcup U')$. In this case, $P(L)$ and $P(L \sqcup U)$ are related by moves of type (4); thus, $P(L) = P(L \sqcup U)$. Similarly, $P(L') = P(L' \sqcup U')$, completing the proof. □

The generalized Property R conjecture (GPRC) asserts that every R-link L is handle-slide equivalent to an unlink, and the stable version of the GPRC asserts that L is stably handle-slide equivalent to an unlink. Both conjectures, if true, would imply that every presentation $P(L)$ arising in this way is AC-trivial. For a detailed discussion of R-links and the AC-conjecture, the reader is encouraged to refer to [8, 11].

2.3 The family $L(3, 2; c/d)$

In [14], Meier and the second author used work in [8, 21] to introduce the family $L(3, 2; c/d)$ of R-links, and in [15], they extended the construction to $L(p, q; c/d)$. The construction is described in much greater detail in [14, 15] but we briefly summarize here: Let Q be the square knot $3_1\#\overline{3_1}$, and let F be the fiber for Q in S^3, a surface with genus two and one boundary component. Then $S^3_0(Q)$, the closed 3-manifold resulting from 0-surgery on Q, is fibered with fiber $\widehat{F}$, the closed genus two surface obtained by capping off the boundary component of F with a disk. Viewing $\widehat{F}$ as the quotient of an annulus with hexagonal boundary components identified in opposite pairs as shown in Fig. 3 below, the *closed monodromy* $\widehat{\varphi} : \widehat{F} \to \widehat{F}$ associated to $S^3_0(Q)$ is a clockwise rotation of $\pi/3$ radians.

In addition, if S represents the 2-sphere with four cone points of order three, there is a branched covering map $\rho : \widehat{F} \to S$ with the property that $\rho \circ \widehat{\varphi} = \rho$. The map ρ can be understood from Fig. 1; each of the six quadrilateral regions cut out by the dotted lines at right maps to the front or back of the pillowcase at left by identifying the appropriate color-coded vertices. Curves in S that avoid the cone points are parametrized by the extended rational numbers $\overline{\mathbb{Q}}$, and for c even, the curve $\lambda_{(c/2)/d}$ corresponding to the rational number $(c/2)/d$ lifts via ρ to three curves $V_{c/d}$, $V'_{c/d}$, and $V''_{c/d}$ contained in $\widehat{F}$ and permuted by $\widehat{\varphi}$. These curves can be chosen to lie in F, and as such the link $Q \cup V_{c/d} \cup V'_{c/d} \cup V''_{c/d} \subset S^3$ is an R-link, stably handle-slide equivalent to any of its two component sublinks by Lemma 17 of [14]. This lemma is proved by noting that slides in F can be used to convert Q and another component of $V_{c/d} \cup V'_{c/d} \cup V''_{c/d}$ to trivial curves in Q, in addition to observing that since $\widehat{\varphi}$ permutes $V_{c/d}$, $V'_{c/d}$, and $V''_{c/d}$, they are isotopic in $S^3_0(Q)$, and so any two of these three curves can be eliminated by isotopy and slides over Q. In [15], the link $L(3, 2; c/d)$ is defined to be $V_{c/d} \cup V'_{c/d}$, but for the purposes of determining whether $P(3, 2; c/d)$ is AC-trivial, we can use any of these links by Lemmas 2 and 3.

Remark 2 The convention here for c/d agrees with [15] but differs from [14, 21] in that the numerator c is doubled in our setting. This doubling is explained in detail in Remarks 4.1 and 4.12 of [15].

Remark 3 It was proved in [21] that the links $L(3, 2; 2n/(2n + 1))$ are stably handle-slide equivalent to the links L_n appearing in [8], and thus $P(3, 2; 2n/(2n + 1))$ is equivalent to the famous presentation P_n described in the introduction.

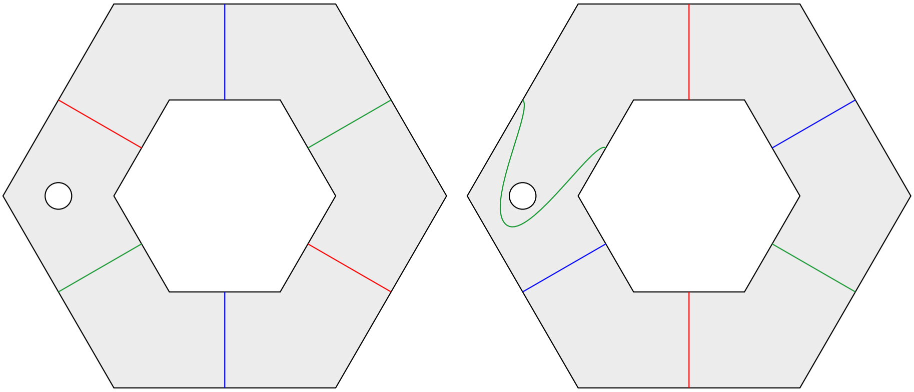


Fig. 3 At left, curves $V_{0/1}$ (red), $V'_{0/1}$ (blue), and $V''_{0/1}$ (green). At right, their images under the monodromy φ. Note that while the closed monodromy $\widehat{\varphi}$ permutes the three curves, the monodromy φ does not. (Color figure online)

2.4 Trisecting $X_{L(3,2;c/d)}$

Gay and Kirby introduced 4-manifold trisections in [6]. A $(g; k_1, k_2, k_3)$-*trisection* $\mathcal{T}$ of a closed, smooth 4-manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ with the properties

1. Each X_i is a 4-dimensional 1-handlebody with $\text{rk}(\pi_1(X_i)) = k_i$;
2. Each $H_i = X_{i-1} \cap X_i$ is a 3-dimensional genus g handlebody; and
3. $\Sigma = X_1 \cap X_2 \cap X_3$ is a genus g surface.

A trisection $\mathcal{T}$ is determined by the union $H_1 \cup H_2 \cup H_3$, which is in turn determined by a collection of three cut systems, α, β, and γ, contained in Σ, called the *central surface* of the trisection. The triple (α, β, γ) is called a *trisection diagram*.

We have already discussed the closed monodromy $\widehat{\varphi} : \widehat{F} \to \widehat{F}$ for the square knot Q, but in this setting, we will need to use the monodromy $\varphi : F \to F$ for Q, which is required to be the identity on $\partial F = Q$. The monodromy φ consists of a $\pi/3$ rotation as before, but this time followed by an isotopy that drags the boundary component of F back to where it started. The curves $V_{0/1}$, $V'_{0/1}$, and $V''_{0,1}$ and their images under φ are shown in Fig. 3. (See also Figure 9 of [21].)

Next, we define three cut systems, which will determine a trisection diagram for a trisection $\mathcal{T}(c/d)$ for $X_{L(3,2;c/d)}$. Define $\Sigma = \partial(F \times I)$, where $\partial F \times I$ has been crushed to the single curve ∂F, so that we may view Σ as $F\#\overline{F}$. For a curve or arc a embedded in F, let $\overline{a}$ denote the mirror image of a contained in $\overline{F}$. Let a_1, a_2, a_3, a_4 be four pairwise disjoint arcs in F cutting F into a disk, and define

$$\alpha = \{\varphi(a_1) \cup \overline{a_1},\ \varphi(a_2) \cup \overline{a_2},\ \varphi(a_3) \cup \overline{a_3},\ \varphi(a_4) \cup \overline{a_4}\}$$
$$\beta = \{a_1 \cup \overline{a_1},\ a_2 \cup \overline{a_2},\ a_3 \cup \overline{a_3},\ a_4 \cup \overline{a_4}\}$$
$$\gamma = \{V_{c/d},\ V'_{c/d},\ \overline{V_{c/d}},\ \overline{V'_{c/d}}\}.$$

Then we have the following, which is Proposition 19 from [14].

Proposition 1 *The triple* (α, β, γ) *is a* $(4; 0, 2, 2)$-*trisection diagram for a trisection* $\mathcal{T}(c/d)$ *of* $X_{L(3,2;c/d)}$.

Finally, we will need a tool which we can use to extract a handle decomposition from a trisection, a restatement of parts of Lemma 13 from [6].

Lemma 4 *Suppose $X = X_1 \cup X_2 \cup X_3$ is a $(g; k_1, k_2, k_3)$-trisection, with $H_i = X_{i-1} \cap X_i$. Then X has a handle decomposition with*

1. *One 0-handle and k_1 1-handles (contained in X_1),*
2. *$g - k_2$ 2-handles (contained in X_2), and*
3. *k_3 3-handles and a 4-handle (contained in X_3).*

In addition, a choice of k_1 pairwise disjoint and mutually nonseparating curves C in Σ bounding disks in both H_1 and H_2 represent the intersections of k_1 co-cores of the 1-handles with Σ. Finally, an attaching link L for the 2-handles can be obtained by choosing $g - k_2$ curves bounding disks in H_3 that are dual to $g - k_2$ curves bounding disks in H_2, and viewing L as a framed link (with framing given by the surface Σ) in the 3-manifold ∂X_1.

The lemma can also be applied by any permutation of $\{1, 2, 3\}$ to the indices of the components X_i. An example that carries out this procedure appears in Subsection 2.6 of [13].

3 The case $d = 4n + 1$

We break the proof of the main theorem into two cases. First, in this section we consider c/d of the form $4/(4n + 1)$. In the next section, we examine c/d of the form $4/(4n + 3)$. The proofs are quite similar but the specific curves and computations are different. For the remainder of this section, we will label curves and arcs in the surface $\overline{F}$ at bottom in Fig. 4 as follows:

1. The red arc is $\overline{a_1}$, and the pink arc is $\overline{a_2}$.
2. The dark blue arc is $\overline{b_1}$, and the light blue arc is $\overline{b_2}$.
3. The dark green curve is $\overline{V_{4/1}}$ and the light green curve is $\overline{V'_{4/1}}$.

The arcs a_1, a_2, b_1, and b_2 in F are the mirror images of $\overline{a_1}$, $\overline{a_2}$, $\overline{b_1}$, and $\overline{b_2}$, respectively. In addition, we let a_3 and a_4 denote the red and pink arcs, respectively, in the top frame of Fig. 4, noting that $a_3 = \varphi(a_1)$, $a_4 = \varphi(a_2)$, and the dark blue and light blue arcs in F are the arcs b_1 and b_2. We also let $\tau : F \to F$ to be the product of right-handed Dehn twists about the pairwise disjoint collection of curves $V_{0/1} \cup V'_{0/1} \cup V''_{0/1}$, so that $\overline{\tau} : \overline{F} \to \overline{F}$ is the product of left-handed Dehn twists about $\overline{V_{0/1}} \cup \overline{V'_{0/1}} \cup \overline{V''_{0/1}}$, shown as dotted purple curves in Fig. 4. Define $V^*_{0/1}$ to be the curve obtained by sliding $V_{0/1}$ over Q (equivalently, $V^*_{0/1}$ is the image of $V''_{0/1}$ under φ as in Fig. 3), and let $\tau_* : F \to F$ be identical to τ except for replacing the Dehn twist about $V_{0/1}$ with a Dehn twist about $V^*_{0/1}$, shown in dotted orange in the same figure. Since $\varphi(V_{0/1} \cup V'_{0/1} \cup V''_{0/1}) = V'_{0/1} \cup V''_{0/1} \cup V^*_{0/1}$, it follows from Section 3.5 of [4] that

$$\varphi \circ \tau = \tau_* \circ \varphi. \tag{1}$$

In Lemmas 4.7 and 4.8 of [15], it was shown that

Lemma 5 *For the curves $V_{4/d}$ and $V'_{4/d}$ in Σ, we have*

$$\tau^n(V_{4/d}) = V_{4/(d+4)} \quad \textit{and} \quad \tau^n(V'_{4/d}) = V'_{4/(d+4)}.$$

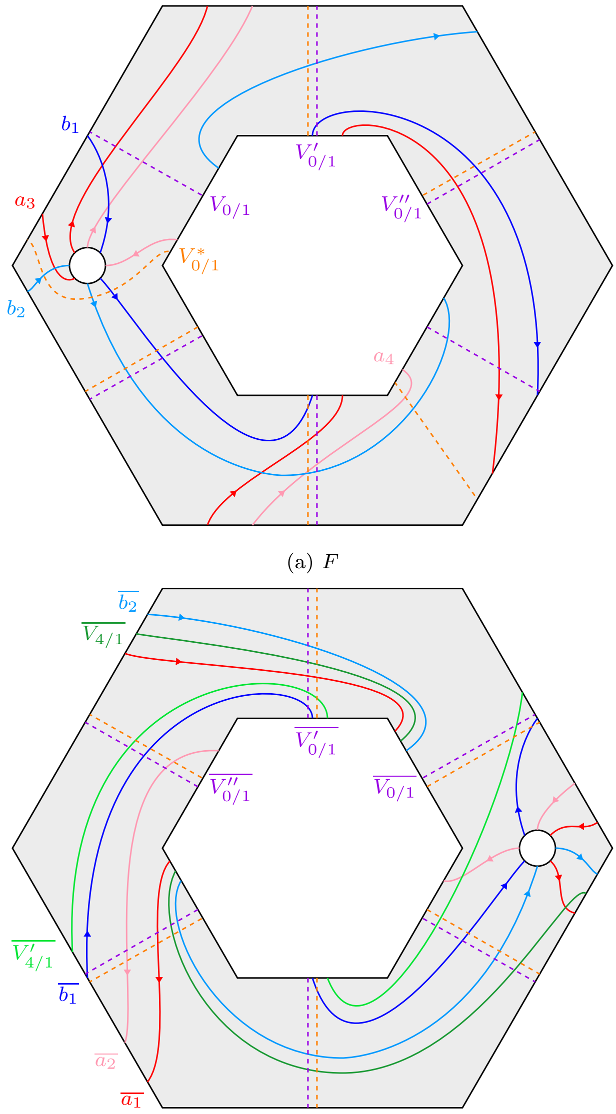


Fig. 4 Curves and arcs in $\Sigma = F\#\overline{F}$ used to compute $P(3, 2; 4/(4n+1))$

Using the symmetry of Σ, it follows that

$$\overline{\tau}^n(\overline{V_{4/d}}) = \overline{V_{4/(d+4)}} \quad \text{and} \quad \overline{\tau}^n(\overline{V'_{4/d}}) = \overline{V'_{4/(d+4)}}$$

as well. This relationship is precisely why we need to address two cases, when $d = 4n+1$ and when $d = 4n+3$. The observant reader may note that our τ differs from τ_0 in [15], in which τ_0 is defined to be a left-handed twist about a six-component multicurve consisting of two copies of each of $V_{0/1}$, $V'_{0/1}$, and $V''_{0/1}$. It follows that $\tau^{-2} = \tau_0$.

Lemma 6 *The trisection $\mathcal{T}(4/(4n+1))$ gives rise to an inverted handle decomposition of $X_{L(3,2;4/(4n+1))}$ with two 1-handles and two 2-handles, where co-cores of the 1-handles meet the central surface Σ in the curves $\tau^n(b_1) \cup \overline{\tau}^n(\overline{b_1})$ and $\tau^n(b_2) \cup \overline{\tau}^n(\overline{b_2})$, while an attaching link for the 2-handles is determined by $\tau_*^n(a_3) \cup \overline{\tau}^n(\overline{a_1})$ and $\tau_*^n(a_4) \cup \overline{\tau}^n(\overline{a_2})$. The corresponding group presentation $P(3, 2; 4/(4n+1))$ is*

$$P(3, 2; 4/(4n+1)) = \langle x, y \mid \overline{x}(y\overline{x})^n\overline{y}(\overline{y}x)^n y, \overline{x}(y\overline{x})^n\overline{y}(x\overline{y})^n \rangle.$$

Proof By Proposition 1, there exists a trisection diagram (α, β, γ) for $\mathcal{T}(4n/(4n+1))$ such that α contains for $i = 1, 2$ the curves $\alpha_i = \varphi(\tau^n(a_i)) \cup \overline{\tau^n(a_i)} = \varphi(\tau^n(a_i)) \cup \overline{\tau}^n(\overline{a_i})$, which bound disks in H_1. Applying Eq. 1 repeatedly, we have

$$\tau_*^n(a_3) \cup \overline{\tau}^n(\overline{a_1}) = \tau_*^n(\varphi(a_1)) \cup \overline{\tau}^n(\overline{a_1}) = \varphi(\tau^n(a_1)) \cup \overline{\tau}^n(\overline{a_1}) = \alpha_1;$$
$$\tau_*^n(a_4) \cup \overline{\tau}^n(\overline{a_2}) = \tau_*^n(\varphi(a_2)) \cup \overline{\tau}^n(\overline{a_2}) = \varphi(\tau^n(a_2)) \cup \overline{\tau}^n(\overline{a_2}) = \alpha_2.$$

Additionally, for $i = 1, 2$, the curves $\beta_i = \tau^n(b_i) \cup \overline{\tau^n(b_i)} = \tau^n(b_i) \cup \overline{\tau}^n(\overline{b_i})$ are curves in β, which bound in H_2, and β_1 and β_2 are disjoint from the curves in $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_3\}$, the curves

$$\{V_{4/(4n+1)}, V'_{4/(4n+1)}, \overline{V_{4/(4n+1)}}, \overline{V_{4/(4n+1)}}\} = \{\tau^n(V_{4/1}), \tau^n(V'_{4/1}), \overline{\tau}^n(\overline{V_{4/1}}), \overline{\tau}^n(\overline{V'_{4/1}})\},$$

as asserted by Lemma 5. Thus, the curves β_1 and β_2 bound disks in the handlebody H_3 as well.

Now, observe that α_1 meets γ_1 once and avoids γ_2, while α_2 meets γ_2 once and avoids γ_1. By Lemma 14 of [6], the handle decomposition of $X_{L(3,2;4/(4n+1))}$ determined by $L(3, 2; 4/(4n+1))$ is compatible with the handle decomposition determined by $\mathcal{T}(4/(4n+1))$, with no 1-handles (contained in the 4-ball X_1), two 2-handles (contained in X_3), and two 3-handles (contained in X_2). It follows that in the inverted handle decomposition, the two 1-handles are in X_2, while the two 2-handles can be viewed in X_3. Since $\partial X_2 = H_2 \cup H_3$, determined by β and γ, the curves β_1 and β_2 above can be chosen as the intersections of co-cores of the 1-handles with Σ. Moreover, the curves α_1 and α_2 are dual to γ_1 and γ_2, so that an attaching link for the 2-handles is $\alpha_1 \cup \alpha_2$ by Lemma 4.

Finally, using Fig. 4 with orientation as shown, we can read off the relators determined by α_1 and α_2. To compute oriented intersections, we find the sign of each point of $\alpha_i \cap \beta_j$, where the direction of α_i is the first element of a standardly oriented ordered basis. Note that the multi-twists τ and τ_* differ only at the curves $V_{0/1}$ (which twists β_i but not α_i) and $V_{0/1}^*$ (which twists α_i but not β_i). We have included an illustration of the twisting in a neighborhood of these curves in Fig. 5 to aid in our computation.

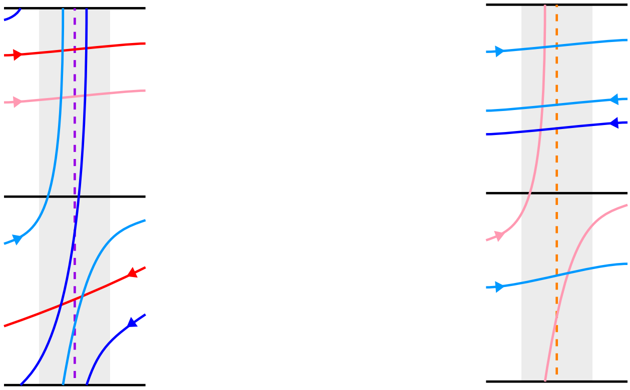
We use the generators x and y for β_1 and β_2, respectively, and we read the relators r and s from α_1 and α_2, respectively, starting in F at Q and following the orientations (note that there are no intersections to be seen in $\overline{F}$). We have

$$r = \overline{x}(y\overline{x})^n\overline{y}(\overline{y}x)^n y;$$
$$s = \overline{x}(y\overline{x})^n\overline{y}(xy\overline{y}\overline{y})^n = \overline{x}(y\overline{x})^n\overline{y}(x\overline{y})^n$$

□

Next, we trivialize our computed presentation.

Proposition 2 *The presentation $P(3, 2; 4/(4n+1))$ is AC-trivial.*



(a) Dehn twisting b_1 and b_2 about $V_{0/1}$ (b) Dehn twisting a_3 and a_4 about $V^*_{0/1}$

Fig. 5 Images of Dehn twists to aid in computations for Lemma 6

Proof Let $(r_0, s_0) = (r, s)$ be the relators from Lemma 6. First, perform a move of type (1), letting $r_1 = r_0^{-1}$ so that

$$r_1 = \overline{y}(\overline{x}y)^n y(x\overline{y})^n x.$$

Next, perform a move of type (2) and let $r_2 = r_1 s_0$, yielding

$$r_2 = (\overline{y}(\overline{x}y)^n y(x\overline{y})^n x)(\overline{x}(y\overline{x})^n \overline{y}(x\overline{y})^n) = \overline{y}(\overline{x}y)^n(x\overline{y})^n.$$

Regroup the terms in s_0 to get

$$s_0 = \overline{x}(y\overline{x})^n \overline{y}(x\overline{y})^n = \overline{x}(y\overline{x})^n(\overline{y}x)^n \overline{y}.$$

Now, use a type (3) move to cyclically permute s_0 to get

$$s_1 = (y\overline{x})^n(\overline{y}x)^n \overline{yx},$$

and then use another type (2) move, letting $r_3 = r_2 s_1$. Thus,

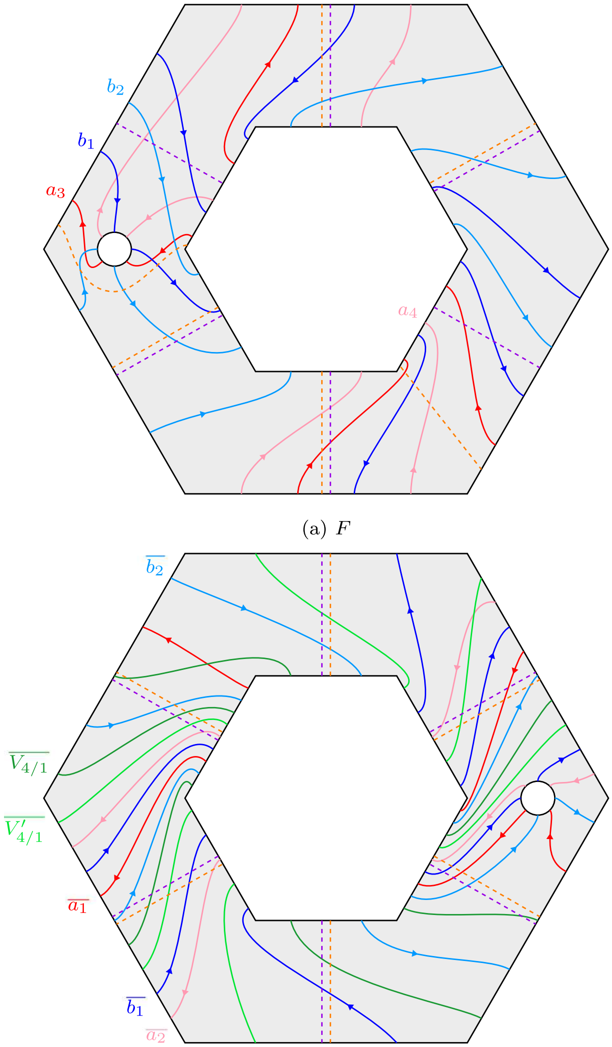
$$r_3 = (\overline{y}(\overline{x}y)^n(x\overline{y})^n)(y\overline{x})^n((\overline{y}x)^n \overline{yx}) = \overline{y}^2\overline{x}.$$

At this point, r_3 is the relator $x = \overline{y}^2$, and we can use a combination of type (3) moves to cyclically permute s_1 so that x or $\overline{x}$ appears as its last term. Then, we use a type (3) move to cyclically permute r_3 and a type (2) move to multiply s_1 by $\overline{xy}^2$, or we use type (1), (2), and (3) moves to multiply s_1 by xy^2. Repeating this process eventually converts s_1 to $s_2 = y$, which in turn lets us use type (1), (2), and (3) moves to convert r_3 to $r_4 = x$, completing the proof. □

4 The case $d = 4n + 3$

We proceed in a manner nearly identical to that of Sect. 3, but starting with different curves and arcs in Σ. For the remainder of this section, we will label curves and arcs in the surface $\overline{F}$ at bottom in Fig. 6 as follows:

1. The red arc is $\overline{a_1}$, and the pink arc is $\overline{a_2}$.



(a) F

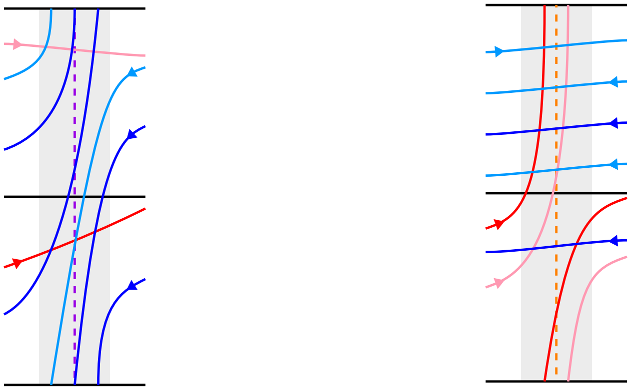
(b) $\overline{F}$

Fig. 6 Curves and arcs in $\Sigma = F\#\overline{F}$ used to compute $P(3, 2; 4/(4n + 3))$

2. The dark blue arc is $\overline{b_1}$, and the light blue arc is $\overline{b_2}$.
3. The dark green curve is $\overline{V_{4/3}}$ and the light green curve is $\overline{V'_{4/3}}$.

In addition, we let a_3 and a_4 denote the red and pink arcs at top in Fig. 6; as before, $a_3 = \varphi(a_1)$, $a_4 = \varphi(a_2)$, and the dark blue and light blue arcs in F are the arcs b_1 and b_2. Recall the definitions of τ and τ_* from Sect. 3.

Lemma 7 *The trisection $\mathcal{T}(4/(4n + 3))$ gives rise to an inverted handle decomposition of $X_{L(3,2;4/(4n+3))}$ with two 1-handles and two 2-handles, where co-cores of the 1-handles meet the central surface Σ in the curves $\tau^n(b_1) \cup \overline{\tau}^n(\overline{b_1})$ and $\tau^n(b_2) \cup \overline{\tau}^n(\overline{b_2})$, while an attaching link for the 2-handles is determined by $\tau_*^n(a_3) \cup \overline{\tau}^n(\overline{a_1})$ and $\tau_*^n(a_4) \cup \overline{\tau}^n(\overline{a_2})$. The*



(a) Dehn twisting b_1 and b_2 about $V_{0/1}$ (b) Dehn twisting a_3 and a_4 about $V^*_{0/1}$

Fig. 7 Images of Dehn twists to aid in computations for Lemma 7

corresponding group presentation $P(3, 2; 4/(4n+3))$ *is*

$$P(3, 2; 4/(4n+3)) = \langle x, y \mid \overline{y}(\overline{xyx})^n(yx^2)^n yx,\ \overline{x}(\overline{yx^2})^n\overline{yxy}(xyx)^n xy\rangle$$

Proof As in the proof of Lemma 6, there is a trisection diagram (α, β, γ) for $\mathcal{T}(4n/(4n+3))$ that includes (for $i = 1, 2$) $\alpha_i = \varphi(\tau^n(a_i)) \cup \overline{\tau}^n(\overline{a_i})$ as curves in α bounding disks in H_1, and by the same argument, we have

$$\tau_*^n(a_3) \cup \overline{\tau}^n(\overline{a_1}) = \alpha_1;$$
$$\tau_*^n(a_4) \cup \overline{\tau}^n(\overline{a_2}) = \alpha_2.$$

For $i = 1, 2$, the curves $\beta_i = \tau^n(b_i) \cup \overline{\tau}^n(\overline{b_i})$ are curves in β, disjoint from the curves in $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_3\}$, the curves

$$\{\tau^n(V_{4/3}), \tau^n(V'_{4/3}), \overline{\tau}^n(\overline{V_{4/3}}), \overline{\tau}^n(\overline{V'_{4/3}})\},$$

as asserted by Lemma 5. Thus, β_1 and β_2 bound disks in the handlebody H_3 as well.

Since α_1 meets γ_1 once and avoids γ_2, while α_2 meets γ_2 once and avoids γ_1, the same argument as in Lemma 6 can be used to show that in the inverted handle composition coming from $L(3, 2; 4/(4n+3))$, co-cores of the 1-handles meet Σ in β_1 and β_2, and an attaching link for the 2-handles consists of α_1 and α_2. Now, using Figs. 6 and 7, we can read off the relators r and s (following orientations and starting in F at Q),

$$r = \overline{y}(\overline{xyx})^n(yxy\overline{y}x)^n yx = \overline{y}(\overline{xyx})^n(yx^2)^n yx$$
$$s = \overline{x}(\overline{yx^2})^n\overline{yxy}(xyxy\overline{y})^n xy = \overline{x}(\overline{yx^2})^n\overline{yxy}(xyx)^n xy$$

□

Proposition 3 *The presentation* $P(3, 2; 4/(4n+3))$ *is AC-trivial.*

Proof Let $(r, s) = (r_0, s_0)$ be the relators from Lemma 7, and note that we can regroup terms to express and r_0 and s_0 as

$$r_0 = \overline{y}(\overline{xyx})^n yx(xyx)^n;$$
$$s_0 = (\overline{xyx})^n\overline{xyxy}(xyx)^n xy.$$

Next, we let r_1 and s_1 be the result of cyclic permutations of r_0 and s_0, respectively, so that

$$r_1 = (\overline{x}\overline{y}\overline{x})^n yx(xyx)^n\overline{y};$$
$$s_1 = xy(\overline{x}\overline{y}\overline{x})^n\overline{x}\overline{y}\overline{x}\overline{y}(xyx)^n.$$

Letting $s_2 = s_1r_1$, we have

$$s_2 = (xy(\overline{x}\overline{y}\overline{x})^n\overline{x}\overline{y}\overline{x}\overline{y}(xyx)^n)((\overline{x}\overline{y}\overline{x})^n yx(xyx)^n\overline{y}) = xy(\overline{x}\overline{y}\overline{x})^n\overline{x}\overline{y}(xyx)^n\overline{y}.$$

Cyclically permuting s_2 to get s_3 yields

$$s_3 = \overline{y}xy(\overline{x}\overline{y}\overline{x})^n\overline{x}\overline{y}(xyx)^n,$$

and letting $s_4 = s_3r_1$,

$$s_4 = (\overline{y}xy(\overline{x}\overline{y}\overline{x})^n\overline{x}\overline{y}(xyx)^n)((\overline{x}\overline{y}\overline{x})^n yx(xyx)^n\overline{y}) = \overline{y}xy(\overline{x}\overline{y}\overline{x})^n(xyx)^n\overline{y} = \overline{y}x.$$

Finally, s_4 is the relator $y = x$, and so we can use AC moves to transform r_1 into

$$r_2 = \overline{x}^{3n}x^2x^{3n}\overline{x} = x.$$

It follows that $P(3, 2; 4/(4n + 3))$ is AC-trivial. □

5 Conclusion

We conclude with a couple of questions to motivate future research in this area. Recall from the introduction that the GPRC and the Andrews–Curtis conjecture are closely connected. We have simplified the group presentations, but the motivation was the following related question about the related links.

Question 2 Are the links $L(3, 2; 4/d)$ handle-slide equivalent or stably equivalent to unlinks?

One way to answer this question would be to show that the trisections $\mathcal{T}(4/d)$ are standard and invoke Theorem 3 from [14] (for a definition of a *standard* trisection, see [14].)

Question 3 Are the trisections $\mathcal{T}(4/d)$ standard?

Notably, it remains open whether there exists a nonstandard trisection of S^4.

Finally, it would be interesting to understand whether these techniques can be applied to other families of group presentations, in particular, because in forthcoming work, Meier and the second author are able to show an number of unexpected equivalences between $P(p_1, q_1; c_1/d_1)$ and $P(p_2, q_2; c_2/d_2)$ for various parameters [12].

Question 4 Can these techniques be extended to show that any presentations of the form $P(5, 2; 4/d)$ are AC trivial? What about presentations of the form $P(p, 2; 4/d)$?

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References

1. Akbulut, S., Kirby, R.: A potential smooth counterexample in dimension 4 to the Poincaré conjecture, the Schoenflies conjecture, and the Andrews–Curtis conjecture. Topology **24**(4), 375–390 (1985). https://doi.org/10.1016/0040-9383(85)90010-2

2. Andrews, J.J., Curtis, M.L.: Free groups and handlebodies. Proc. Amer. Math. Soc. **16**, 192–195 (1965). https://doi.org/10.2307/2033843
3. Burns, R.G., Macedońska, O.: Balanced presentations of the trivial group. Bull. Lond. Math. Soc. **25**(6), 513–526 (1993). https://doi.org/10.1112/blms/25.6.513
4. Farb, B., Margalit, D.: A Primer on Mapping Class Groups, Princeton Mathematical Series, vol. 49. Princeton University Press, Princeton (2012)
5. Fernández, X.: Morse Theory for Group Presentations. arXiv e-prints arXiv:1912.00115 (2019). https://doi.org/10.48550/arXiv.1912.00115
6. Gay, D., Kirby, R.: Trisecting 4-manifolds. Geom. Topol. **20**(6), 3097–3132 (2016). https://doi.org/10.2140/gt.2016.20.3097
7. Gompf, R.E.: Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews–Curtis and Schoenflies problems. Topology **30**(1), 97–115 (1991). https://doi.org/10.1016/0040-9383(91)90036-4
8. Gompf, R.E., Scharlemann, M., Thompson, A.: Fibered knots and potential counterexamples to the property 2R and slice-ribbon conjectures. Geom. Topol. **14**(4), 2305–2347 (2010). https://doi.org/10.2140/gt.2010.14.2305
9. Gompf, R.E., Stipsicz, A.I.: 4-manifolds and Kirby calculus. In: Graduate Studies in Mathematics, vol. 20. American Mathematical Society, Providence (1999). https://doi.org/10.1090/gsm/020. https://doi.org/10.1090/gsm/020
10. Hoffman, H., Nakagawa, K., Potter, R., Zupan, A.: Knots in the fibers of generalized square knots. In preparation
11. Meier, J., Schirmer, T., Zupan, A.: Classification of trisections and the generalized property R conjecture. Proc. Amer. Math. Soc. **144**(11), 4983–4997 (2016). https://doi.org/10.1090/proc/13105
12. Meier, J., Zupan, A.: Presentations of the trivial group and generalized square knots. In preparation
13. Meier, J., Zupan, A.: Bridge trisections of knotted surfaces in S^4. Trans. Amer. Math. Soc. **369**(10), 7343–7386 (2017). https://doi.org/10.1090/tran/6934
14. Meier, J., Zupan, A.: Characterizing Dehn surgeries on links via trisections. Proc. Natl. Acad. Sci. USA **115**(43), 10887–10893 (2018). https://doi.org/10.1073/pnas.1717187115
15. Meier, J., Zupan, A.: Generalized square knots and homotopy 4-spheres. J. Differ. Geom. **122**(1), 69–129 (2022). https://doi.org/10.4310/jdg/1668186788
16. Metzler, W.: Äquivalenzklassen von Gruppenbeschreibungen, Identitäten und einfacher Homotopietyp in niederen Dimensionen. In: Homological Group Theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, pp. 291–326. Cambridge Univ. Press, Cambridge-New York (1979)
17. Metzler, W.: On the Andrews–Curtis conjecture and related problems. In: Combinatorial Methods in Topology and Algebraic Geometry (Rochester, N.Y., 1982), Contemp. Math., vol. 44, pp. 35–50. Amer. Math. Soc., Providence (1985). https://doi.org/10.1090/conm/044/813099. https://doi-org.libproxy.unl.edu/10.1090/conm/044/813099
18. Miasnikov, A.D.: Genetic algorithms and the Andrews–Curtis conjecture. Internat. J. Algebra Comput. **9**(6), 671–686 (1999). https://doi.org/10.1142/S0218196799000370
19. Myasnikov, A.D., Myasnikov, A.G., Shpilrain, V.: On the Andrews–Curtis equivalence. In: Combinatorial and Geometric Group Theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, pp. 183–198. Amer. Math. Soc., Providence (2002). https://doi.org/10.1090/conm/296/05074. https://doi.org/10.1090/conm/296/05074
20. Panteleev, D., Ushakov, A.: Conjugacy search problem and the Andrews–Curtis conjecture. Groups Complex. Cryptol. **11**(1), 43–60 (2019). https://doi.org/10.1515/gcc-2019-2005
21. Scharlemann, M.: Proposed property 2R counterexamples examined. Illinois J. Math. **60**(1), 207–250 (2016)