Stability Analysis of Resolving Pulses of Unknown Shape from Compressive Fourier Measurements

Meghna Kalra and Kiryung Lee
Department of Electrical and Computer Engineering
The Ohio State University
Columbus, OH, USA
Email: kalra.42@osu.edu, kiryung@ece.osu.edu

***Abstract*—We present the problem of resolving pulses of a common shape from noisy Fourier measurements. Specifically, the paper focuses on the challenge of estimating the locations when the pulse shape is unknown. We leverage compressed sensing techniques to implement a larger virtual aperture through random sampling, thereby avoiding the necessity to acquire measurements inversely proportional to minimum separation. Although a larger aperture is beneficial, it introduces a penalty when dealing with unknown non-trivial pulse shapes. We provide theoretical and numerical results to quantify the error and show the efficacy of this approach in accurately resolving pulse locations with acceptable error in the presence of noise and modeling error.**

***Index Terms*—compressed sensing, blind deconvolution, ESPRIT, perturbation analysis**

I. Introduction

We address the problem of estimating a pulse stream with an unknown shape from noisy Fourier measurements of multiple snapshots with varying amplitudes. The challenge arises from the fact that the minimum separation between pulses is minimal, and we seek to resolve the pulse locations using a small number of measurements. As a noisy scenario requires a specific minimum separation between the closest pair of sources according to the Rayleigh resolution, our objective is to determine the model parameters and analyze how the error decays as a function of measurements. In situations where the minimum separation is small, obtaining accurate pulse resolution necessitates a large number of Fourier measurements. This can lead to increased computational costs and induce latency, which is particularly disadvantageous in applications that demand real-time processing. This type of problem emerges in various fields including wireless communications, seismology, sonar imaging, and medical imaging.

A special case where the pulses are the Dirac deltas has been extensively studied in the literature. For example, Prony's method [1], MUSIC (MUltiple SIgnal Classification) [2], ESPRIT (Estimation of Signal Parameters via Rotational Invariant Techniques) [3], and total-variation minimization [4] recover the pulse locations from Fourier measurements on a uniform grid. Recent papers provided non-asymptotic analysis of these methods [5]–[8]. Another line of research studied the atomic norm minimization for the recovery of Diracs from random Fourier samples [9], [10]. Moreover, the atomic norm approach has been developed to retrieve pulses convoluted with partially known point spread functions [11], [12]. These papers assumed that the convolution kernels belong to a known random subspace, which is not suitable for model signals arising in practical applications. A different approach to blind spike deconvolution using ESPRIT has been proposed [13]. This work required that the consecutive samples of the Fourier transform of the pulse shape need to be identical. The analysis has been restricted to exactly satisfy this condition and to the noiseless case.

Our objective is to establish a non-asymptotic theory for estimating a pulse stream of unknown shape from compressive Fourier measurements over multiple snapshots. We aim to derive an analysis of a computationally efficient estimator under mild assumptions on pulse shape and the number of measurements, particularly, without relying critically on the minimum separation. While prior literature has developed relevant work, they have not simultaneously achieved all the expected goals. This paper will accomplish the objectives by establishing a non-asymptotic perturbation analysis of ESPRIT when the pairwise-match condition in [13] is violated. Furthermore, to achieve an accurate estimation by a small number of Fourier measurements, not dictated by the minimal separation, we propose random doublet sampling to design subarrays for ESPRIT. Our approach involves selecting M samples randomly in pairs from a larger aperture pool $\widetilde{M}$, which enables the capture of more information compared to the uniformly spaced M samples. The rationale for selecting random samples in pairs is to satisfy the specific rotational invariant structure, which is a prerequisite for the subsequent application of the ESPRIT [3] used in the later section. In certain applications, the underlying structure of the problem can be highly variable. Hence, selecting M samples randomly from a larger pool allows for a better representation of the variability of the signal.

Once the M random doublets are obtained, we employ the well-established ESPRIT technique to determine both the locations and the shape of the unknown pulse. The contribution of this paper is two folds. First, we provide a robust method to successfully resolve the pulses with trivial pulse shapes that are in proximity to each other with utilizing significantly fewer measurements than the conventionally required. Furthermore,

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in the presence of a non-trivial pulse shape, we provide a quantitative error trade-off between the number of measurements and the model parameters. Second, we provide the theoretical analysis and numerical results to substantiate the effectiveness of our proposed method.

II. Notation

The spectral norm of a matrix $\boldsymbol{A}$ is denoted as $\|\boldsymbol{A}\|$. The k-th eigenvalue or singular value of $\boldsymbol{A}$ is denoted $\lambda_k(\boldsymbol{A})$ or $\sigma_k(\boldsymbol{A})$. The condition number of any matrix $\boldsymbol{A}$ is denoted as $\kappa(\boldsymbol{A})$. The Moore-penrose inverse of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^\dagger$. We use $a \vee b$ to denote $\max(a, b)$. The distance between the column spaces by two matrices $\boldsymbol{U}$ and $\widehat{\boldsymbol{U}}$ is defined by

$$\operatorname{dist}(\widehat{\boldsymbol{U}}, \boldsymbol{U}) := \|\widehat{\boldsymbol{U}}\widehat{\boldsymbol{U}}^{\mathsf{H}} - \boldsymbol{U}\boldsymbol{U}^{\mathsf{H}}\|$$

which corresponds to the sine of the largest principal angle.

III. Perturbation analysis of ESPRIT with unknown pulse shape

We consider an estimation problem in which the observed signal is in the form of

$$y(t) = \sum_{k=1}^{S} x_k g(t - \tau_k)$$

where $g(t)$ is an unknown pulse shape and $y(t)$ consists of S unknown pulses with amplitudes x_k located at $\{\tau_k\}_{k=1}^S$ supported on the interval $[0, T)$ for some $T > 0$.

Suppose that we are collecting this data at L different sensors. The observed signal at the lth sensor is then represented as

$$y_l(t) = \sum_{k=1}^{S} x_{k,l} g(t - \tau_k)$$

for $l = 1, \ldots, L$. Note that the amplitudes vary sensor to sensor, but the locations of the pulses remain the same across the sensors. Let $\widetilde{\Omega}$ denote a uniform grid of frequencies separated by Γ such that

$$\widetilde{\Omega} := \left\{ l\Gamma : l = 0, 1, \ldots, |\widetilde{\Omega}| - 1 \right\}$$

where Γ satisfies $\Gamma \leq 1/T$.

Let $\Omega = \{\omega_i\}_{i=1}^{|\Omega|}$ be a subset of $\widetilde{\Omega}$. Let $\boldsymbol{Y} \in \mathbb{C}^{|\Omega| \times L}$ collect all noise-free Fourier measurements at frequencies given by Ω such that

$$(\boldsymbol{Y})_{i,l} = Y_l(\omega_i) = \sum_{k=1}^{S} x_{k,l} G(\omega_i) e^{-\mathrm{j}2\pi\tau_k\omega_i},\ i \in [|\Omega|],\ l \in [L].$$

The matrix $\boldsymbol{Y}$ is compactly written as

$$\boldsymbol{Y} = \boldsymbol{G}\boldsymbol{\Phi}\boldsymbol{X},$$

where $\boldsymbol{G} \in \mathbb{C}^{|\Omega| \times |\Omega|}$, $\boldsymbol{\Phi} \in \mathbb{C}^{|\Omega| \times S}$, and $\boldsymbol{X} \in \mathbb{C}^{S \times L}$ are defined by

$$\begin{aligned}
(\boldsymbol{G})_{i,j} &= G(\omega_i)\delta_{ij}, \quad i, j \in [|\Omega|], \\
(\boldsymbol{\Phi})_{i,k} &= e^{-\mathrm{j}2\pi\tau_k\omega_i}, \quad i \in [|\Omega|],\ k \in [S], \\
(\boldsymbol{X})_{k,l} &= x_{k,l}, \quad k \in [S],\ l \in [L].
\end{aligned}$$

Suppose that Ω is given as the union of $\Omega_1 = \{\omega_{1,m}\}_{m=1}^{|\Omega_1|}$ and $\Omega_2 = \{\omega_{2,m}\}_{m=1}^{|\Omega_1|}$ such that

$$\omega_{2,m} = \omega_{1,m} + \Gamma, \quad \forall m \in [|\Omega_1|]. \tag{1}$$

Let $\boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2 \in \mathbb{R}^{|\Omega_1| \times |\Omega|}$ be defined by

$$(\boldsymbol{\Pi}_j)_{m,i} = \begin{cases} 1 & \text{if } \omega_i = \omega_{j,m}, \\ 0 & \text{else.} \end{cases}$$

Then by (1) we have

$$\boldsymbol{\Pi}_2\boldsymbol{\Phi} = \boldsymbol{\Pi}_1\boldsymbol{\Phi}\boldsymbol{D}$$

where $\boldsymbol{D} \in \mathbb{C}^{S \times S}$ is a diagonal matrix satisfying $(\boldsymbol{D})_{k,k} = e^{-\mathrm{j}2\pi\Gamma\tau_k}$. Furthermore, if

$$\boldsymbol{\Pi}_1\boldsymbol{G}\boldsymbol{\Pi}_1^\top = \boldsymbol{\Pi}_2\boldsymbol{G}\boldsymbol{\Pi}_2^\top, \tag{2}$$

then we have

$$\boldsymbol{\Pi}_2\boldsymbol{G}\boldsymbol{\Phi} = \boldsymbol{\Pi}_1\boldsymbol{G}\boldsymbol{\Phi}\boldsymbol{D}. \tag{3}$$

Bresler and Delaney [13] showed that the condition in (3) enables to use the ESPRIT algorithm [3] to estimate matrices $\boldsymbol{G}$ and $\boldsymbol{D}$ from a noisy version of $\boldsymbol{Y}$. The ESPRIT algorithm is summarized as follows: The first step estimates an orthonormal basis for the column space of $\boldsymbol{Y}$ from a noisy version $\widehat{\boldsymbol{Y}}$. Let $\widehat{\boldsymbol{U}} \in \mathbb{C}^{|\Omega| \times S}$ span the estimated subspace. Then, the pulse locations are estimated up to a permutation ambiguity by

$$\widehat{\tau}_k = -\frac{\arg(\lambda_k(\widehat{\boldsymbol{U}}_1^\dagger \widehat{\boldsymbol{U}}_2))}{2\pi\Gamma}, \quad k \in [S]$$

where $\widehat{\boldsymbol{U}}_j = \boldsymbol{\Pi}_j\widehat{\boldsymbol{U}}$ for $j = 1, 2$. The eigenvalues of $\widehat{\boldsymbol{U}}_1^\dagger\widehat{\boldsymbol{U}}_2$ denoted by $\{\lambda_k(\widehat{\boldsymbol{U}}_1^\dagger\widehat{\boldsymbol{U}}_2)\}_{k=1}^S$ are not ordered. Therefore, the estimated locations $\widehat{\mathcal{T}} := \{\widehat{\tau}_k\}_{k=1}^S$ are compared to the ground-truth locations $\mathcal{T} := \{\tau_k\}_{k=1}^S$ via the *matching distance* defined by

$$\operatorname{md}(\mathcal{T}, \widehat{\mathcal{T}}) := \min_{\pi \in \operatorname{perm}(S)} \max_{k \in [S]} |\tau_k - \widehat{\tau}_{\pi(k)}|$$

where $\operatorname{perm}(S)$ denotes the set of all possible permutations over $[S]$. Once the pulse locations are estimated, the Fourier measurements of the pulse shape in $\boldsymbol{G}$ can be estimated via least squares.

We consider a more general scenario where the condition in (2) is satisfied only approximately, i.e.

$$G(\omega_{1,m}) \approx G(\omega_{2,m}), \quad m \in [|\Omega_1|]. \tag{4}$$

The following proposition illustrates how the error in (4) propagates to the estimation of the pulse locations.

Proposition III.1. *Let Ω satisfy $|\Omega| \geq S + 1$ and $\boldsymbol{U}_1 = \boldsymbol{\Pi}_1\boldsymbol{U}$ where $\boldsymbol{U} \in \mathbb{C}^{|\Omega| \times S}$ spans the column space of $\boldsymbol{Y}$ and satisfies $\boldsymbol{U}^{\mathsf{H}}\boldsymbol{U} = \boldsymbol{I}_S$. Suppose that $\boldsymbol{X}$ has full row rank and $\widehat{\boldsymbol{U}}$ satisfies*

$$\min_{\boldsymbol{R} \in O_S} \|\widehat{\boldsymbol{U}} - \boldsymbol{U}\boldsymbol{R}\| < \frac{\sigma_S(\boldsymbol{U}_1)}{2}$$

where O_S denotes the set of S-by-S unitary matrices. Then the estimated locations by ESPRIT satisfies

$$\mathrm{md}(\mathcal{T},\widehat{\mathcal{T}}) \le \frac{\kappa(\boldsymbol{\Phi}) G_{\Omega,\max}}{2\Gamma G_{\Omega,\min}} \cdot \left(\frac{3\,\mathrm{dist}(\widehat{\boldsymbol{U}},\boldsymbol{U})}{\sigma_S^2(\boldsymbol{U}_1)} + \frac{\max_{m\in[|\Omega_1|]} |G(\omega_{2,m}) - G(\omega_{1,m})|}{\sigma_S(\boldsymbol{U}_1) G_{\Omega,\min}} \right)$$

where

$$G_{\Omega,\min} := \min_{\omega\in\Omega} |G(\omega)| \quad \textit{and} \quad G_{\Omega,\max} := \max_{\omega\in\Omega} |G(\omega)|.$$

Proposition III.1 provides a non-asymptotic noisy case analysis of ESPRIT with an unknown pulse shape. The analysis of ESPRIT in this scenario has been restricted to the case where the observations are noise-free and the pairwise match condition in (2) is exactly satisfied [13]. We quantify how the violation of (2) propagates to the error in estimating the pulse locations from noisy observations.

IV. ESPRIT WITH RANDOM-DOUBLET SUBARRAYS

In this section, we utilize Proposition III.1 to study the performance of ESPRIT with random doublet sampling. Let $\widetilde{\Omega}$ correspond to a uniform grid of size $2\widetilde{M}$ separated by $\Gamma = \frac{1}{2T}$. Then Ω_1 is constructed as a random subset of $\widetilde{\Omega}$ given by

$$\Omega_1 = \left\{ 2(i-1)\Gamma : i \in [\widetilde{M}],\ \beta_i = 1 \right\}$$

where $\beta_1, \beta_2, \cdots, \beta_{\widetilde{M}/2}$ are independent copies of a Bernoulli random variable β satisfying $\mathbb{P}(\beta = 1) = M/\widetilde{M}$ and $\mathbb{P}(\beta = 0) = 1 - M/\widetilde{M}$. Moreover, Ω_2 is determined by Ω_1 as in (1). Since $\mathbb{E}|\Omega_1| = M$, Chernoff's inequality yields

$$\mathbb{P}(||\Omega_1| - M| > \delta M) \le e^{-c\delta^2 M}$$

for an absolute constant c.

For ESPRIT to estimate the pulse locations from the noisy Fourier measurements on the entire uniform grid in $\widetilde{\Omega}$, it is necessary that the number of Fourier measurements exceeds the inverse of the minimum separation, i.e. $\widetilde{M} > 1/\Delta + 1$ for

$$\Delta := \min_{k\neq j} |\tau_k - \tau_j|_{\mathbb{T}}$$

where the distance is measured modulo the torus on the interval $[0,T)$. Consequently, when the minimum separation is small, a substantial number of measurements are necessary for accurate estimation. Similar to the compressed sensing off-the-grid method [9], we propose to estimate the locations of pulses of unknown shape by ESPRIT from $2M$ Fourier measurements where $M \ll \widetilde{M}$. Reducing the number of measurements is advantageous in certain applications such as real-time analysis, while random sampling from a larger $\widetilde{\Omega}$ effectively captures the variability present in the signal.

The following theorem presents a sufficient condition for the parameter recovery in presence of noise and model error due to the unknown pulse shape

Theorem IV.1. *Let $\widetilde{\Omega}$ correspond to a uniform grid of size $2\widetilde{M}$ separated by $\Gamma = \frac{1}{2T}$ Let $\widetilde{\rho} = \frac{\widetilde{G}_{\max}}{\widetilde{G}_{\min}}$ where $\widetilde{G}_{\max}$ and $\widetilde{G}_{\min}$ respectively denote the maximum and minimum of $|G(\omega)|$ over $\omega \in \widetilde{\Omega}$. Also let $\widehat{\boldsymbol{Y}} = \boldsymbol{Y} + \boldsymbol{Z}$ denote a noisy version of $\boldsymbol{Y} = \boldsymbol{G}\boldsymbol{\Phi}\boldsymbol{X}$ where the entries of $\boldsymbol{Z}$ are i.i.d.* $\mathrm{Normal}(0,\sigma^2)$. *Then there exist absolute constants $C_1, C_2, c_3 > 0$ such that if*

$$\widetilde{M} > S \vee 3\left(\frac{1}{\Delta} + 1\right), \tag{5}$$

$$M \ge C_1 S \ln M, \tag{6}$$

and

$$L \ge S \vee \frac{C_2 \sigma^2 \widetilde{\rho}^2}{\widetilde{G}_{\min}^2 \lambda_S(\boldsymbol{R_X})} \left(\widetilde{\rho}^2 \kappa(\boldsymbol{R_X}) \vee \frac{\sigma^2}{M \widetilde{G}_{\min}^2 \lambda_S(\boldsymbol{R_X})} \right), \tag{7}$$

then it holds with probability $1 - 5e^{-c_3 M} - M^{-1}$ that

$$\mathrm{md}(\mathcal{T},\widehat{\mathcal{T}}) \lesssim \frac{T\widetilde{\rho}^3 \sigma}{\widetilde{G}_{\min} \lambda_S^{1/2}(\boldsymbol{R_X})\sqrt{L}} \cdot \left(\widetilde{\rho}\kappa^{1/2}(\boldsymbol{R_X}) \vee \frac{\sigma}{\widetilde{G}_{\min}\lambda_S^{1/2}(\boldsymbol{R_X})\sqrt{M}} \right) + \widetilde{\rho}^2 \cdot \frac{\sup_{\omega\in\widetilde{\Omega}} |G(\omega + \frac{1}{2T}) - G(\omega)|}{\frac{1}{2T} \cdot \widetilde{G}_{\min}}, \tag{8}$$

Remark IV.2. The requirement on $\widetilde{M} > 3\left(\frac{1}{\Delta} + 1\right)$ in (5) is introduced to simplify the proof. It can be reduced arbitrarily close to $\widetilde{M} > \frac{1}{\Delta} + 1$.

The first term in the error bound in (8) is due to noise and decays as $\frac{\sigma}{\sqrt{L}}$ when $\widetilde{M}$ is sufficiently large. However, the second term prevents ESPRIT to be consistent in this scenario. On the other hand, if $g(t)$ is supported within the interval $[-R/2, R/2)$, by the mean value theorem, the second term in the error bound in (8) simplifies to

$$\frac{|G(\omega+\Gamma) - G(\omega)|}{\Gamma} \le \sup_{\omega\in\mathbb{R}} \left| \frac{dG(\omega)}{d\omega} \right| \le 2\pi R \sup_{\omega\in\mathbb{R}} |G(\omega)|$$

where the second inequality is due to Bernstein's inequality (cf. [14, Theorem 6.7.1]). This implies that the error due to the violation of the pairwise match condition in (2) is small for narrow pulse shapes.

V. NUMERICAL RESULTS

We present numerical results to demonstrate the effectiveness of ESPRIT with the random doublet sampling relative to the uniform sampling using the same number of distinct Fourier measurements. For the uniform sampling, the grid step size is set to $\Gamma = 1/T$.

In the first experiment, we study how the choice of $\widetilde{M}$ affects the estimation error. Fig. 1 presents the result of the median of 100 Monte Carlo trials in case of $g(t) = \delta(t)$. In this scenario, the pairwise match condition in (2) is trivially satisfied; hence, there is no penalty on increasing $\widetilde{M}$. Here, the parameters are set to SNR $= 20$ db, $M = 20$, $\Delta = 0.005$, $S = 7$, $L = 50$. The estimation error is low if $\widetilde{M}$ exceeds a threshold determined by Δ, which is required in Theorem IV.1. In this regime, the random doublet sampling outperforms the

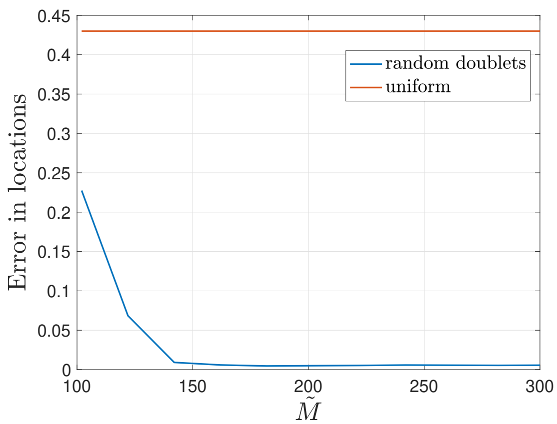

Fig. 1. Estimation error for $g(t) = \delta(t)$ (SNR $= 20$ db, $M = 20$, $\Delta = 0.005$, $S = 7$, $L = 50$).

uniform case. Note that uniform sampling does not change with increasing $\widetilde{M}$.

To model the effect of non-trivial unknown pulse shape, we compare random doublets to uniform sampling for a squared cosine pulse shape given by $g(t) = \mathrm{rect}(20t/\pi)\cos^2(20t)$. The same parameter setting as before is applied. According to Theorem IV.1, the model error predominates due to the violation of the pairwise-match condition (2). Therefore, increasing $\widetilde{M}$ further than the minimum requirement increases the estimation error as the dynamic range between $\widetilde{G}_{min}$ and $\widetilde{G}_{max}$ increases. Fig. 2 illustrates this phenomenon over the median of $1,000$ Monte Carlo trials.

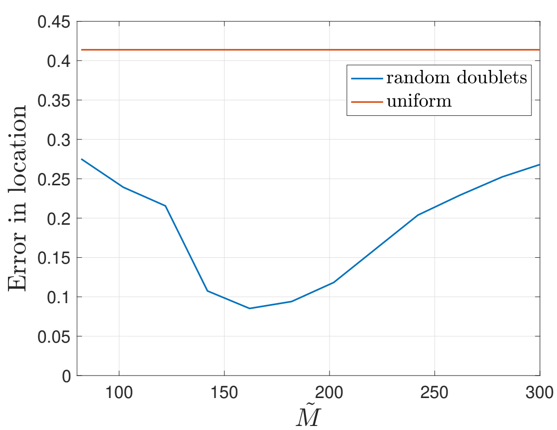

Fig. 2. Estimation error for the squared cosine pulse shape (median SNR $= 28$ db, $M = 20$, $\Delta = 0.005$, $S = 7$, $L = 50$).

Lastly, to demonstrate the sample complexity of the random doublet design, we observe the empirical phase transition of the estimation errors for M versus S. Fig. 3 shows the median error over $1,000$ Monte Carlo trials where a smooth transition can be seen as how the error varies as a function of increasing M and S.

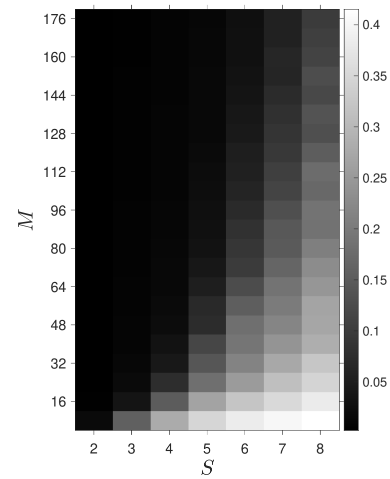

Fig. 3. Empirical phase transition of estimation error for squared cosine pulse shape (median SNR $= 26$ db, $\widetilde{M} = 200$, $\Delta = 0.005$, $L = 50$).

VI. Conclusion

The proposed method formulates a non-asymptotic theory dedicated to the estimation of a stream of unknown pulse shape by utilizing compressive Fourier measurements across multiple snapshots. Our approach utilizes a computationally efficient design with minimal assumptions on pulse shape and the number of measurements, and also without strongly depending on the minimum separation criteria.

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