Robert Israel "Bob" Jewett (1937–2022)

Walter R. Bloom, Richard J. Gardner, Al Hales, Joel Spencer, Terence Tao, and Benjamin Weiss

Robert Israel "Bob" Jewett was born on December 14, 1937 in Providence, Rhode Island. His father, Abraham, had emigrated from Poland/Ukraine to Canada in 1921 and then to the USA in 1923, while his mother, Mame (Mary) née Katz, was born in Providence to parents from Russia. In 1946, Bob's family, including his older sister Rosalie, moved to Venice, California, a relocation partially motivated by Bob's problems with hay fever. Bob's much younger brother Phil was born in 1948.

Bob attended the local public schools and was very interested, while at Venice High School, not only in physics and mathematics, but also animals and insects. In 1955, he started college at Caltech. There was no zoology major, so he focused on physics and mathematics, eventually the latter. In 1958, he received Honorable Mention for his individual performance in the nationwide Putnam competition in mathematics, helping the Caltech team to place third in the nation. Bob also stood out as a volleyball and track and field star.

Bob graduated from Caltech in 1959 and went to the University of Oregon for graduate work in mathematics. Before this move, however, he worked for the summer at Caltech's Jet Propulsion Laboratory (JPL), in the coding theory section headed by Solomon Golomb. He was also employed there in the summers of 1960 and 1961. During that time he wrote several research papers, one containing the combinatorial Hales–Jewett theorem [HJ63].

Bob received his PhD in 1963, with a thesis on analysis on locally compact abelian groups written under the direction of Karl Stromberg. He spent the next year as a postdoc at the Institute of Advanced Study in Princeton. This was followed by two years teaching at Uppsala University in Sweden. In 1966–69, Bob was an assistant professor at the University of Washington, during which period he wrote his important paper [Jew69] in ergodic theory leading to the Jewett–Krieger theorem. For the academic year 1969–70, Bob taught at the IMPA in Rio de Janiero, returning to the USA in 1970 to take a position at Western Washington University (WWU) in Bellingham.

Figure 1. Bob (center) with the "Diggers" volleyball team, 1968.

Shortly after arriving at WWU, in 1972, Bob shared SIAM's first Pólya Prize in Combinatorics, along with co-honorees Ron Graham, Al Hales, Klaus Leeb, and Bruce Rothschild, for his part in the Hales–Jewett theorem. In 1975, Bob's fundamental paper [Jew75] on convos (a.k.a. topological hypergroups) appeared.

Communicated by Notices *Associate Editor Emilie Purvine.*

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DOI: https://doi.org/10.1090/noti2688

Except for visits to UCLA (Spring 1974), the University of Auckland (1976), the University of Oregon (1982–83), and the University of Wisconsin (1984–85), Bob spent the remainder of his career at WWU, retiring in 2010.

At the suggestion of Ron Graham, in 2016 Árpád Bényi, Steve Butler, Amites Sarkar, and Jozsef Solymosi organized a celebratory meeting at WWU, titled "50 Years of the Hales–Jewett Theorem." This very successful event involved three generations of researchers and attracted distinguished speakers from far and wide: Vitaly Bergelson, Fan Chung, David Conlon, Ron Graham, Neil Hindman, Imre Leader, Dhruv Mubayi, Jaroslav Nesetril, and Gabor Tardos.

In 2014, Bob was hit by a car at a crosswalk. He survived the resulting operation and long rehabilitation with his trademark stoicism and wonderful sense of humor intact. Despite this setback, he was generally healthy until the last few months of his life. He died peacefully on July 30, 2022.

The Hales–Jewett Theorem

Joel Spencer

The Hales–Jewett theorem [HJ63], which appeared in 1963, occupies a central place in the development of Ramsey theory. Indeed, it was this result that turned a collection of Ramsey-type theorems into Ramsey theory; see [GRS90, p. 35]. The mantra: *complete disorder is impossible.*

To take a basic case, Ramsey showed in 1930 that for all $k, l \in \mathbb{N}$, there exists a sufficiently large n such that *every* red/blue coloring of the complete graph K_n contains either a red K_k or a blue K_l. The rediscovery of this result by Erdős and Szekeres and the countless extensions and conjectures of Erdős truly advanced the subject. Several other results are in the same spirit. For example, much earlier, in 1916, Issai Schur showed that if $\mathbb{N}$ is finitely colored, there is a monochromatic solution to the equation $x + y = z$. As with Ramsey, Schur's interests were elsewhere and his result was greatly extended by many, especially Richard Rado and Walter Deuber. In 1950, R.P. Dilworth showed that appropriately large partially ordered sets must contain either large chains or large antichains. The most important result in this telling was van der Waerden's theorem from 1927, stating that for all $k \in \mathbb{N}$, there exists an n such that if $\{1, \dots, n\}$ is k-colored, there must be a monochromatic arithmetic progression of k terms. This was long considered a beautiful number-theoretic result, but (as Al Hales recalls) Bob Jewett felt it could be placed in a more general setting. Their final result was purely combinatorial, removing the algebraic structure. Indeed, it can be formulated in terms of the classic children's game Tic-Tac-Toe!

Define H_t^n, the n-cube over t elements, by

$$H_t^n = \{(x_1, \dots, x_n) : x_i \in \{0, 1, \dots, t-1\}\}.$$

(For $t = 2$, this is the Hamming cube.) By a *combinatorial line* in H_t^n we mean a suitably ordered set of t points $v_0, \dots, v_{t-1}$ in H_t^n such that each coordinate is either constant or runs through $0, 1, \dots, t-1$ in that precise order. For example, in H_4^5,

$$(2,0,3,1,0), (2,1,3,1,1), (2,2,3,1,2), (2,3,3,1,3)$$

form a combinatorial line. Combinatorial lines lie on geometric lines in n-space but the converse is not true. For example, in classic Tic-Tac-Toe played on H_3^2, $(2,0), (1,1), (0,2)$ lie on a geometric line (and represent a winning position in the game) but do not form a combinatorial line.

The Hales–Jewett theorem states that for all $r, t \in \mathbb{N}$, there is an n such that if H_t^n is r-colored, there necessarily exists a monochromatic combinatorial line. For the game enthusiasts: r-player Tic-Tac-Toe with combinatorial lines of length t, played in sufficiently high dimensions, cannot end in a draw!

The Hales–Jewett theorem immediately implies van der Waerden's theorem; just associate $(x_1, \dots, x_n)$ with its base t value $\sum_i x_i t^{n-i}$. But the Hales–Jewett theorem is combinatorial; instead of $0, 1, \dots, t-1$, we may take coordinate values $s_0, \dots, s_{t-1}$ in any space and still get a monochromatic combinatorial line.

Let $HJ(r,t)$ denote the minimal n such that the Hales–Jewett theorem holds. The asymptotic upper bounds on $HJ(r,t)$ as $t \to \infty$ are enormous, even for $r = 2$. The original proof of Hales and Jewett gave an upper bound on $HJ(2,t)$ somewhat like the Ackermann function. In 1988, Shelah [She88] found an upper bound which is primitively recursive. Let $Tower(t)$ denote an exponential tower of twos of height t. Let $WOW(t)$ denote the t-times iterated Tower function, beginning at 1. Shelah's bound was roughly $WOW(t)$. The lower bound is exponential, so there remains a very large gap.

The Hales–Jewett theorem lies at the heart of a group of related results. The general theme is that for any fixed structure V and $r \in \mathbb{N}$, a sufficiently large r-colored structure necessarily contains a monochromatic substructure W. For example, Gallai's theorem from 1943 states that if V is any finite subset of $\mathbb{R}^m$ and $\mathbb{R}^m$ is finitely colored, then there exists a monochromatic W homothetic to V. Indeed, we may find a $W = cV + b$ with $b \in \mathbb{R}^m$ and nonzero

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$c \in \mathbb{Z}$. Further, by compactness, for any r we need only r-color a suitable finite $X \subset \mathbb{R}^m$.

Gallai's theorem is a fairly easy consequence of the Hales–Jewett theorem. Another is the following extended Hales–Jewett theorem, proved by Ron Graham and Bruce Rothschild in 1969: For all $r, s, t \in \mathbb{N}$, there is an n such that if H_t^n is r-colored, there exists a monochromatic combinatorial s-space (a natural generalization of a combinatorial line).

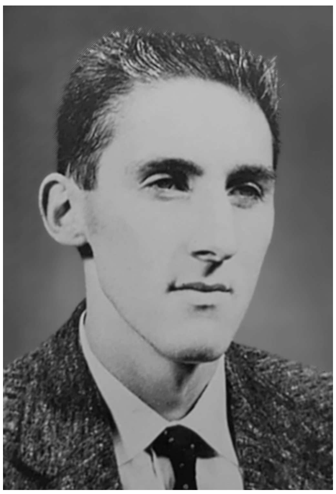
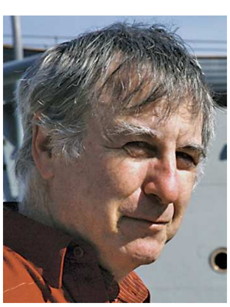


Figure 2. Caltech Big T yearbook photo, 1959.

Gian-Carlo Rota felt that Ramsey's theorem should hold in general lattices and, specifically, conjectured the following vector space Ramsey theorem: For any finite field $\mathbb{F}$ and $k, r, t \in \mathbb{N}$, if n is sufficiently large and the t-dimensional subspaces of $\mathbb{F}^n$ (considered as a vector space) are r-colored, then there exists a k-dimensional subspace W, all of whose t-dimensional subspaces are the same color. This was proved by Graham & Rothschild and independently by Klaus Leeb, a joint paper appearing in 1972.

The Hales–Jewett theorem and its variants have spawned extensions and applications in many directions. For example, the polynomial Hales–Jewett theorem, proved by Bergelson and Leibman [BL99] in 1999, leads to topological dynamics. It is slightly too complicated to state here, but the reader may consult Walters [Wal00] for short combinatorial proofs of it and a consequence, the polynomial van der Waerden theorem. The latter states that if $p_1, p_2, \ldots, p_m$ are polynomials with integer coefficients and no constant term, then whenever $\mathbb{N}$ is finitely colored, there exist $a, d \in \mathbb{N}$ such that a and $a + p_i(d)$, $1 \leq i \leq m$, all have the same color.



Joel Spencer

The Density Hales–Jewett Theorem

Terence Tao

Van der Waerden's theorem can be equivalently phrased as a statement about infinite colorings: whenever the natural numbers are finitely colored, one of the color classes must contain arbitrarily long arithmetic progressions. However, the known proofs of this theorem did not shed much light on *which* color class had this property, and *why*. In 1936, Erdős and Turán conjectured what we would now call a *density* version of the van der Waerden theorem: the assertion that in fact any set of natural numbers of positive (upper) density contains arbitrarily long arithmetic progressions. This of course would imply van der Waerden's theorem, since by the pigeonhole principle whenever one colors the natural numbers into finitely many classes, at least one of them must have positive density; but the claim is far stronger, and formalizes the intuition that it is simply the *size* of a set of natural numbers that forces the existence of patterns such as arithmetic progressions contained inside it.

The conjecture of Erdős and Turán was famously demonstrated in 1975 by Szemerédi [Sze75] in a remarkable *tour de force* of combinatorial reasoning, and the conjecture is now known as Szemerédi's theorem. This theorem and its generalizations have had many applications in combinatorics and number theory; for instance, in 2004 Green and I used it to show that the primes contain arbitrarily long arithmetic progressions (despite having density zero). In 1977, Furstenberg [Fur77] gave a new proof of Szemerédi's theorem that was both conceptual and highly influential, using the tools of *ergodic theory*, developing what is now known as the *Furstenberg correspondence principle* to convert the problem to one of understanding the recurrence properties of dynamical systems. Roughly speaking, the point was that every dense set of integers could be interpreted as the set of *return times* for some dynamical system—the set of times in which a particle traversing some state space returns to a given set of states of positive measure. Furthermore, the dynamics of this system could be made to be measure-preserving, allowing the techniques of ergodic theory to come into play.

Spurred by the success of this ergodic theoretic approach, Furstenberg and his coauthors and students began locating and proving density versions of many of the other

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landmark results of Ramsey theory. Some density versions were false, and others could be established by variants of Furstenberg's methods, but establishing a density version of the Hales–Jewett theorem (which would be a sweeping generalization of Szemerédi's theorem) turned out to be particularly challenging. It was only in 1991 that Furstenberg and Katznelson [FK91] finally managed to establish this result by pushing the methods of ergodic theory to their limits, applying them to systems of families of sets which have only the barest hint of dynamical structure. The proof is fearsomely complicated; for instance, just one step of the argument relies on a Ramsey theorem of Carlson and Simpson which is in turn a significant strengthening of the Hales–Jewett theorem in which one requires a color class to contain an infinite-dimensional combinatorial subspace, rather than simply a combinatorial line. It was also purely *qualitative*: while it does show that a dense subset of a sufficiently high-dimensional cube will necessarily contain a combinatorial line, it does not specify at all the precise relation between the density of the set and the dimension required.

In 2009, inspired in part by his previous work on Szemerédi's theorem, Gowers proposed to locate a purely combinatorial proof of the density Hales–Jewett theorem by a creative new paradigm—an online crowdsourced effort, where dozens of mathematicians, mostly communicating through blogs and wikis, would contribute and debate possible attack strategies. After a very intensive seven-week effort involving thousands of comments by many mathematicians, such a combinatorial proof was finally obtained in 2010, with the results eventually being published in [Pol12] under the pseudonym "D.H.J. Polymath;" the initials here stand for "Density Hales Jewett." The Polymath project continued to solve a number of other problems in the same format; it has kept the same pseudonym ever since, even though the other problems were no longer directly related to the density Hales–Jewett theorem.

The Polymath proof of the density Hales–Jewett theorem was simplified in 2014 in a twelve-page paper of Dodos, Kanellopoulos, and Tyros [DKT14], which as one corollary gives what is arguably the shortest and most elementary proof of Szemerédi's theorem, and also gives the currently best-known quantitative bounds for the density Hales–Jewett theorem. Research in this area is still ongoing; for instance, there is a conjectural "density polynomial Hales–Jewett theorem" (a density version of the polynomial Hales–Jewett theorem of Bergelson and Leibman) that should also be true, and would imply a staggering number of other density Ramsey theorems already known in the literature, but remains open at this time of writing. Gowers [Gow22] provides further information about these developments.

Terence Tao

On the Jewett–Krieger Theorem

Benjamin Weiss

The Jewett–Krieger theorem is one of the fundamental theorems lying on the interface between topological dynamics and ergodic theory. In topological dynamics one studies the properties of the iterations of a homeomorphism T of a compact space X. In classical ergodic theory the setting is that of a probability space $(X, \mathcal{B}, \mu)$ and a measurable invertible mapping T of X that preserves the measure μ. The basic building blocks of measure preserving transformations are those that are indecomposable, in the sense that one cannot find a set $A \in \mathcal{B}$ with $0 < \mu(A) < 1$ such that $T(A) = A$. These are called *ergodic* systems.

It is a basic fact that a homeomorphism T of a compact space X always has at least one invariant measure. Just as the Borel probability measures on X form a compact convex set in the weak* topology, so too the T-invariant probability measures form a compact convex set and it turns out that its extreme points are exactly the invariant ergodic measures for the transformation T. In particular, if there is only one invariant measure, say ν, for T the system (X, ν, T) is ergodic. Such topological dynamical systems are called *uniquely ergodic*. This property has significant topological consequences. Indeed, if X_0 denotes the closed support of this unique measure ν then X_0 is T-invariant and (X_0, T) is a *minimal* system, which means that all orbits $\{T^n(x) : n \in \mathbb{Z}\}$ are dense.

A rotation of the unit circle by an irrational multiple of π is an example of a uniquely ergodic system. Prior to the work of Bob Jewett most of the examples of uniquely ergodic systems were extensions of various kinds of these simple systems. When Kolmogorov defined entropy as a numerical invariant for measure preserving systems it turned out that all the known examples of uniquely ergodic systems had zero entropy. It was only in 1967 that

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Frank Hahn and Yitzhak Katznelson gave an involved construction of a uniquely ergodic system with positive entropy. It came as a complete surprise to all of the experts in the field when Jewett published his result. For its statement, the following definition is needed. The product of a system $(X, \mathcal{B}, \mu, T)$ with itself is the mapping $T \times T$ on the product probability space $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ where $(T \times T)(x, y) = (Tx, Ty)$. A system is called *weakly mixing* if its product is ergodic. Examples of weakly mixing systems are automorphisms of the torus with no eigenvalue that is a root of unity.

Jewett's theorem [Jew69] can be stated as follows.

For every weakly mixing system $(X, \mathcal{B}, \mu, T)$ there is a uniquely ergodic topological system (Y, S), where Y is the Cantor set, with a unique invariant measure ν such that the systems $(X, \mathcal{B}, \mu, T)$ and (Y, ν, S) are isomorphic.

In one stroke the family of uniquely ergodic systems was enlarged to include models of all weakly mixing systems. The natural question arose as to whether one could drop the additional assumption of weak mixing. This was accomplished in the following year by Wolfgang Krieger, who proved the same theorem under only the (necessary) condition of ergodicity. Krieger's result appeared only in 1973 [Kri72] and in the meantime a different proof was given by Georges Hansel and Jean-Pierre Raoult. It then became clear that the property of unique ergodicity was as prevalent as possible. The original proof of Jewett was extended in 1979 to cover the ergodic case by Bellow and Furstenberg [BF79]. Jewett needed a certain lemma which was easy to prove under the hypothesis of weak mixing. Bellow and Furstenberg showed that this property is true for all ergodic systems by a clever use of Neil Hindman's famous combinatorial theorem.

The Jewett–Krieger theorem was extended to the setting of measure preserving actions of the real line; this was the original setting for ergodic theory, which started from questions in statistical mechanics. This was done, first by Konrad Jacobs assuming weak mixing, and then, in 1974, by Denker and Eberlein [DE74] in the general ergodic case.

More generally one can consider actions of any locally compact group G. To begin with one asks when every action of G by homeomorphisms of a compact space fixes some probability measure. The answer is if and only if the group G is *amenable*. This class contains all solvable groups but does not contain the free group on two or more generators. Indeed much of the classical ergodic theory was extended to this class of groups and in particular, the Jewett–Krieger theorem was established for all discrete elementary amenable groups in [Wei85]. In joint work with Alain Rosenthal this was extended to all discrete amenable groups. He wrote a series of papers containing this and many other refinements in the following years.

In that same paper I outlined a different kind of extension. If $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are two systems and there is a measurable mapping $\pi : X \to Y$ such that $\pi\mu = \nu$ and $\pi T = S\pi$ then the second is called a factor of the first. I showed that every uniquely ergodic model of the factor can be continuously extended to a uniquely ergodic model of the larger system. This "relative" version has been applied by Huang, Shao and Ye [HSY19] to prove new results in the study of multiple ergodic averages.

Jewett's result motivated many other results in the spirit of finding topological models for measurable systems with special properties. His work spawned an entire branch in the interplay between measure and topology which is still growing.

Benjamin Weiss

Jewett's Hypergroups

Walter R. Bloom

The concept of a group-like structure where the product of two elements results in a set rather than another element has been around since the first half of the twentieth century, but the tie-up with the topology of the underlying space and Borel measures first appeared in the early 1970s when Charles Dunkl [Dun73], Robert Jewett [Jew75] and René Spector [Spe75] independently created locally compact hypergroups with the view to developing standard harmonic analysis on these spaces. There were also precursors by Jean Delsarte in 1938, Boris Levitan in 1945, and Salomon Bochner in 1956 with the study of generalised translation operators.

There are technical differences between the various definitions, but in the setting of analysis on topological group-like structures, the basis for much subsequent research was Jewett's lengthy paper [Jew75], a remarkable work that developed the main harmonic analysis of what he termed *convos*. It is Jewett's rather extensive axiom scheme that

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has subsequently proved most influential. In a nutshell, we write K for a locally compact Hausdorff space acting as base space, ε_x for the Dirac (point) probability measure at $x \in K$ and $M_b(K)$ for the convolution algebra of bounded complex-valued regular Borel measures on K, where $\varepsilon_x * \varepsilon_y = \varepsilon_{x*y}$ is a probability measure with compact support $x*y$. In the locally compact group case we have $x * y = xy$, but we are in unfamiliar territory when $x * y$ is a subset of K containing more than one point.

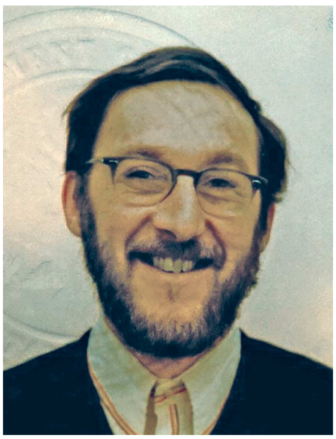

Figure 3. Passport photo, 1982.

We could list all the Jewett axioms and discuss the nuances of their relationships, but this is better left for the readers of his paper. One principal hypergroup axiom is that we have a continuous map $K \times K \ni (x, y) \to \varepsilon_x * \varepsilon_y \in M_b(K)$, with a suitable topology on $M_b(K)$. There is also an involution $x \to x^-$ on K respecting this convolution operation $*$ in the sense that $e \in x * y$ if and only if $y = x^-$. Here, involution takes the place of group inverse and e is the neutral element of K, taking the place of the group identity. We then pass from the sparsely structured base space K to the more richly structured measure algebra $M_b(K)$. For example, the group operation of (left) translation is replaced by the hypergroup operation of generalised (left) translation on suitable functions f on K:

$$(f_x)(y) := f(x * y) := \int_K f(z)\, d\left(\varepsilon_x * \varepsilon_y\right)(z), \; y \in K.$$

At this stage we illustrate the theory with the important case of double coset hypergroups $G \,/\!\!/\, H$, where G is a locally compact group with left Haar measure λ_G and H is a compact subgroup with normalised Haar measure λ_H, and indeed this formed the basis of Jewett's theory [Jew75, Section 8.2]. The double coset space $G \,/\!\!/\, H = \{HgH : g \in G\}$ doesn't inherit a multiplication from G if H isn't normal, but the space of measures on $G \,/\!\!/\, H$ does inherit a convolution from the measure algebra $M_b(G)$.

[Jew75, Theorem 8.2B]: The space $G \,/\!\!/\, H$ with the quotient topology and convolution

$$\varepsilon_{HxH} * \varepsilon_{HyH} := \int_H \varepsilon_{HxtyH}\, d\lambda_H(t)$$

is a hypergroup with neutral element $e = H = H1H$. (This equality of Radon measures and similar equalities below are best understood by evaluating both sides at continuous functions f; the integrand above then simplifies to $f(HxtyH)$.) If $x \in G$ then $(HxH)^- = Hx^{-1}H$. A left Haar measure is given by

$$\lambda_{G/\!\!/H} = \int_G \varepsilon_{HxH}\, d\lambda_G(x).$$

The significance of (left) Haar measure is its invariance under (left) translation.

[Jew75, Theorems 3.3F, 3.3G]: Let $\omega = \lambda_{G/\!\!/H}$. For every σ-finite (with respect to ω) nonnegative Borel-measurable function f on $G \,/\!\!/\, H$ and $x \in G \,/\!\!/\, H$, f_x is also σ-finite,

$$\int_{G/\!\!/H} f_x\, d\omega = \int_{G/\!\!/H} f\, d\omega \text{ and } \mu * \omega = \mu(G \,/\!\!/\, H)\,\omega,$$

where $\mu \in M^+(G \,/\!\!/\, H)$.

The representation theory of hypergroups has been well developed in [Jew75, Chapter 11] and is vastly simplified in the commutative case. Let $(K, *)$ be a commutative hypergroup with neutral element e, involution $^-$ and Haar measure λ_K. Bounded measurable functions $\chi : K \to \mathbb{C}$ are called *characters* when $\chi(e) = 1$, $\chi(x*y) = \chi(x)\chi(y)$ and $\chi(x^-) = \overline{\chi(x)}$ for all $x, y \in K$. The essential difference between characters on groups and characters on hypergroups is that on groups it is easy to "shift around" the character through $\chi(xy) = \chi(x)\chi(y)$. For hypergroups the argument isn't so easy; the problem is that $\chi(x * y)$ is not the evaluation of χ at a point, but rather is in general an integral.

A locally bounded measurable function $\phi : K \to \mathbb{C}$ is said to be *positive definite* if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j \phi(x_i * x_j^-) \geq 0$$

for all choices of $x_1, x_2, \dots, x_n \in K, c_1, c_2, \dots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$. All continuous characters are automatically positive definite and the dual $K^\wedge$ of K is just the set of continuous characters with the compact-open topology in which case $K^\wedge$ must be locally compact.

For $f \in L^1(K, \lambda_K)$ we have the Fourier transform

$$\hat{f}(\chi) := \int_K f \bar{\chi}\, d\lambda_K.$$

[Lev64], [Jew75, Theorem 7.3I] (Levitan–Plancherel theorem): There exists a unique nonnegative measure $\pi_{K^\wedge}$ on $K^\wedge$ such that

$$\int_K |f|^2\, d\lambda_K = \int_{K^\wedge} \left|\hat{f}\right|^2 d\pi_{K^\wedge}$$

for all $f \in L^1(K, \lambda_K) \cap L^2(K, \lambda_K)$, and $C_c(K)^\wedge$ is dense in $L^2(K^\wedge, \pi_{K^\wedge})$.

The inverse Fourier transform of $\tau \in M_b(K^\wedge)$ is given by
$$\overset{\vee}{\tau}(x) := \int_{K^\wedge} \chi(x)\, d\tau(\chi).$$
[Jew75, Theorems 12.3A, 12.3B] (Bochner's theorem): For $\tau \in M^+(K^\wedge)$, $\overset{\vee}{\tau}$ is continuous bounded positive definite on K. For every continuous bounded positive definite function f on K there exists a unique $\tau \in M^+(K^\wedge)$ such that $f = \overset{\vee}{\tau}$.

Back to double coset hypergroups, an interesting example is given by the Naimark hypergroup [BH95, Section 3.5.66], [Jew75, Sections 9.5 and 15.2] which arises from a solution of a particular Sturm–Liouville boundary value problem over $\mathbb{R}^+$ ($\simeq SL(2,\mathbb{C}) /\!/ SU(2)$) or by analysing the geometry of random walks on the hyperbolic plane $\mathbb{H}^2$. We obtain the convolution
$$\varepsilon_x * \varepsilon_y = \frac{1}{2\sinh x \sinh y} \int_{|x-y|}^{x+y} \varepsilon_t \sinh t\, dt$$
whenever $x, y \in \mathbb{R}_+ \backslash \{0\}$, and Haar measure is given by $d\lambda_{\mathbb{R}_+}(x) = \left(\sinh^2 x\right) dx$. Here $\mathbb{R}_+^\wedge \cong \{\chi_a : -1 \le a < \infty\}$ with characters (indexed by $a = b^2$)
$$\chi_a(x) = \begin{cases} \dfrac{\sin bx}{b \sinh x} & \text{if } a \neq 0, \\[2ex] \dfrac{x}{\sinh x} & \text{if } a = 0 \end{cases}$$
for all $x \in \mathbb{R}_+$. Note that $\chi_0 \in \mathcal{C}_0(\mathbb{R}_+)$ and $\chi_{-1} = 1$. The Plancherel measure on $\mathbb{R}_+^\wedge$ is just
$$\int_{\mathbb{R}_+^\wedge} h\, d\pi_{\mathbb{R}_+^\wedge} = \frac{1}{\pi} \int_0^\infty h(\chi_t) \sqrt{t}\, dt$$
and $\operatorname{supp} \pi_{\mathbb{R}_+^\wedge} = \{\chi_t : 0 \le t < \infty\} \subsetneq \mathbb{R}_+^\wedge$.

The mark of an excellent paper is not only that it is well written, novel, and makes a substantial contribution to the field, but also that it lends itself to further developments across several areas. Here, these include harmonic analysis, operator algebras, and differential equations. In particular, we highlight follow-up studies of negative definite functions, the Lévy continuity theorem, the Lévy–Khintchine formula and convolution semigroups, all forming the basis of probability theory on hypergroups and related structures; see [BH95] and the many papers that have appeared since.

Walter R. Bloom

Al Hales

Bob and I were fellow students at Caltech, he a year ahead of me. We met in about 1958, either at the Math Club or on a volleyball court. By 1961 we were both in grad school, he at the University of Oregon and I at Caltech. But during the summers of 1959 and 1960 we were both working at Caltech's Jet Propulsion Lab (JPL) under the supervision of Solomon Golomb. Bob remembered asking me for a good place to read about van der Waerden's theorem, and apparently I suggested Khinchin's "Three Pearls of Number Theory." Later he told me that he thought he could see how to generalize the theorem to structures other than the integers. So he drew me into the project and we started considering possible generalizations. Did we need two operations? Commutativity? Associativity? Units/inverses? Etc. Eventually we decided that an arbitrary semigroup was the right setting. But then it would work for a free semigroup, and this is just the space of sequences! So we realized we had proved that n-dimensional "Tic-Tac-Toe" has no tying positions if n is large compared to the edge length! Using a known result, this meant that the first player has a forced win.

We could not resist considering the dual question, and soon realized that using Hall's theorem on distinct representatives we could show that the second player could force a tie if the edge length was large compared to the dimension n. We were pleased with our dual results, which first appeared in a JPL report, and decided to submit them for publication. But we had no idea of their future.

Bob and I were coauthors of several other reports at JPL, only one of which was published in the open literature: "Recent Results in Comma-free Codes." This appeared under the pseudonym B. H. Jiggs, standing for coauthors Baumert, Hales, Jewett, Golomb, Gordon and Selfridge ("i" being a dummy initial). After that our research directions seemed to drift apart—his in the analytic direction and mine in the algebraic direction. We always

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had plenty of math to talk about, but no further joint papers.

After we received our doctorates Bob went to the East Coast and Europe for several years. Ginny and I were married and spent a year in England and three years in Cambridge, Mass. During this time we met with Bob on several occasions, in Princeton and in the Boston area. Then we both returned to the West Coast. I was at UCLA but took a year's sabbatical at the University of Washington in 1970–71, by which time Bob was at Western Washington University (WWU) in Bellingham. So we had a number of chances to meet with him in the Northwest.

Shortly after Ginny and I returned to UCLA from sabbatical, Bob and I were each pleasantly surprised to receive phone calls from Gian-Carlo Rota telling us that we would be co-recipients (with Graham, Leeb, and Rothschild) of SIAM's first "Pólya Prize in Combinatorics," based on our joint paper. We traveled to Austin, Texas, in late 1972 for this presentation.

Since Bob's family lived in Southern California, he often traveled to our area to visit them, giving us a chance to get together. In addition, I arranged for him to get a visiting position at UCLA for a quarter in 1974.

Twenty years or so later all this changed—directions reversed! I took early retirement from UCLA and we moved to La Jolla for me to take a new position at IDA/CCR. We bought land on Orcas Island in Washington, and then built a small vacation house there. So now we were traveling up to Bob's area every two or three months, and we made a point of meeting him for dinner each time, usually in Bellingham though sometimes on "our" island.

Also, during this period, I gave several colloquium talks at WWU while passing through. And there was the wonderful 2016 conference at WWU on "50 Years of the Hales–Jewett Theorem."

Bob recovered from his 2014 road accident, but his mobility was certainly affected. The series of health problems that followed did not seem to affect his wonderful sense of humor. We continued to see Bob on trips to Orcas until the pandemic began to affect all our travel plans. I think the last time was in May 2021. As should be clear from the above, he was more than a friend and colleague, essentially a member of our extended family. We miss him very much.

Al Hales

Richard J. Gardner

By 2016, when the "50 Years of the Hales–Jewett Theorem" conference was held at WWU (Western Washington University), Bob Jewett had been retired for six years and by choice no longer drove a car, so he was chauffeured to and from his senior living home by volunteers. Still equipped with a sharp mathematical mind, he graciously accepted the attention, but to me seemed slightly bemused by all the fuss.

Before the conference, most of the WWU math faculty knew that Bob had done some fine research, but many were not fully aware of its significance. He was always ready to talk about mathematics, yet almost never mentioned his own work. I only recall Bob giving a single colloquium talk during the twenty years we overlapped at WWU, a beautifully presented and entirely elementary exposition of p-adic addition and multiplication. In fact, although he continued to publish some nice joint work sporadically, Bob's major results, addressed in other articles in this memorial tribute, were all in print by 1975. After the WWU conference, I asked him why, given his obvious talent. He said, "Nothing else turned up."

Behind this reply lies Bob's curious, and to some extent unfathomable, personality. Early adventures described below notwithstanding, he generally preferred not to take any action unless it was necessary, and on occasion even if it was. He was a procrastinator and somewhat forgetful. Letters might remain unopened. Bob sensed that most forms could be ignored, and at some point while still employed even stopped completing his annual tax return, having discovered that the IRS would do it for him. (Dear reader, do not try this at home.) Like G.H. Hardy, he disliked gadgets; he never owned a laptop or cellphone, and never used email or the internet. His office PC was employed solely for exams and lecture notes produced with outdated T^3 software. Telephone was the only means for long-distance communication, but during the thirty years

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Figure 4. Pólya prizewinners at the 2016 Western Washington University conference. Left to right: Ron Graham, Bruce Rothschild, Al Hales, Bob Jewett.

I knew him well and saw him often, Bob phoned me only once.

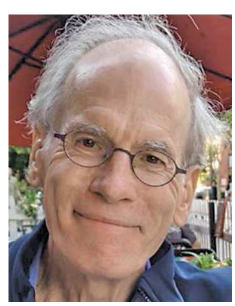
Despite these peculiarities, Bob was an excellent and very popular colleague. He was genial, modest, considerate, witty, and always ready to have a chuckle. By chalking reminders to himself on his office blackboard, he somehow managed to arrive on time for committee meetings, where his intelligence and straightforward good sense were greatly appreciated. He attended most graduate oral exams and colloquium talks, and usually had insightful questions to ask the speaker. Bob's wide knowledge of mathematics was a valuable departmental resource; his insight took him quickly to the heart of the matter, and I do not recall him being proved wrong in a mathematical discussion.

There is no PhD program at WWU, but Bob was very active in teaching master's students. One recalls her classmates referring to Bob as "God," because he seemed to know all mathematics and how the different areas connected. (In fact, Bob's research straddles algebra and analysis, and discrete and continuous.) Others fondly remember his patience and sense of humor. In 2016, I used Bob's notes to teach Math 523, Advanced Calculus of Several Variables. Designed for a quarter-long course, the notes are a masterpiece of efficient and almost error-free exposition, not based on any textbook but developed from scratch. Another set of Bob's class notes, in a similar style, focused on random walks. Bob was also a good undergraduate teacher; some students might have preferred more leniency and less honesty—I don't think Bob was capable of being dishonest—but even they often recognized his brilliance and fundamental kindness. An ex-student I know, now a successful teacher but rather lazy at the time, recalls asking Bob for a letter of recommendation for a PhD program. Bob gently replied, "I would have to tell them that you don't work very hard." To the student, this frank assessment from one of his favorite teachers was a much-appreciated wake-up call. One might think that Bob's exams would be as clever as he was, but on the contrary, Bob always advocated for a completely straightforward approach to testing.

While young, Bob was athletic and adventurous. During his academic years in Sweden and Brazil, he learned the languages well enough to give lectures and exams. In Sweden, he was amused to be able to settle a tax dispute by discovering that the Swedish and American versions of the double-taxation treaty did not agree. At the end of his stay in Sweden, in May 1966, he bought a Volvo, drove it into Eastern Europe (a nontrivial matter at that time), shipped it from Gothenburg to New York, and motored across the Northern USA to Washington state, stopping to hike along the way.

Bob told me there were women he would have married, and those that would have married him, but none in both groups. While I knew him, he lived alone, apparently quite contentedly, in accommodation unadorned by decoration of any kind that always featured a recliner and a desk and tables piled with books and assorted paper. But Bob was not a loner. He welcomed company and could be counted upon to enliven the chat at dinner tables or pubs. More than one faculty spouse told me how much they enjoyed Bob at departmental social gatherings: "At least if Bob was there, there was somebody interesting to talk to."

An observation of Virginia Hales helps reconcile the contradictory aspects of Bob's personality. She wrote, "He seemed to only live in the moment, his mind was never somewhere else when he was with you. . . It may have been this last trait that made him absentminded when it came to doing necessary tasks." I agree. Spending time with Bob was always fun, and it was a privilege and a pleasure to know him.



Richard J. Gardner

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