ANALYSIS & PDE

Volume 17 No. 4 2024

YUNHUI WU, HAOHAO ZHANG AND XUWEN ZHU

DEGENERATING HYPERBOLIC SURFACES AND SPECTRAL GAPS FOR LARGE GENUS

msp

ANALYSIS AND PDE
Vol. 17 (2024), No. 4, pp. 1377–1395
DOI: 10.2140/apde.2024.17.1377

DEGENERATING HYPERBOLIC SURFACES AND SPECTRAL GAPS FOR LARGE GENUS

YUNHUI WU, HAOHAO ZHANG AND XUWEN ZHU

We study the differences of two consecutive eigenvalues $\lambda_i - \lambda_{i-1}$, i up to $2g-2$, for the Laplacian on hyperbolic surfaces of genus g, and show that the supremum of such spectral gaps over the moduli space has infimum limit at least $\frac{1}{4}$ as the genus goes to infinity. A min-max principle for eigenvalues on degenerating hyperbolic surfaces is also established.

1. Introduction

For a closed Riemann surface X_g of genus $g \geq 2$, consider the hyperbolic metric uniquely determined by its complex structure. We study the spectrum of the Laplacian on X_g, which is a discrete subset in $\mathbb{R}^{\geq 0}$ and consists of eigenvalues with finite multiplicities. The eigenvalues, counted with multiplicities, are listed in the following increasing order:

$$0 = \lambda_0(X_g) < \lambda_1(X_g) \leq \lambda_2(X_g) \leq \cdots \to \infty.$$

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, which is an open orbifold of dimension equal to $6g-6$. For each index i, the i-th eigenvalue $\lambda_i(\cdot)$ is a bounded continuous function on $\mathcal{M}_g$. In this paper we study the differences of two consecutive eigenvalues and will focus on the behavior of such spectral gaps when $g \to \infty$.

Definition. For all $i \geq 1$, the *i-th spectral gap* $\mathrm{SpG}_i(\cdot)$ is a bounded continuous function over the moduli space $\mathcal{M}_g$ defined as

$$\mathrm{SpG}_i : \mathcal{M}_g \to \mathbb{R}^{\geq 0}, \quad X_g \mapsto \lambda_i(X_g) - \lambda_{i-1}(X_g). \tag{1}$$

By definition, $\mathrm{SpG}_1(X_g) = \lambda_1(X_g)$. For all $i \geq 1$, the i-th spectral gap $\mathrm{SpG}_i(\cdot)$ can be arbitrarily close to zero (e.g., see Proposition 3.7). In this paper we mainly study the quantity $\sup_{X_g \in \mathcal{M}_g} \mathrm{SpG}_i(X_g)$ for large g and a family of indices i.

The main result of this article is the limiting behavior of the lower bound of the spectral gaps.

Theorem 4.1. *Let $\{\eta(g)\}_{g=2}^{\infty}$ be any sequence of integers with $\eta(g) \in [1, 2g-2]$. Then*

$$\liminf_{g \to \infty} \sup_{X_g \in \mathcal{M}_g} \mathrm{SpG}_{\eta(g)}(X_g) \geq \tfrac{1}{4}.$$

MSC2020: primary 32G15; secondary 58C40.
Keywords: spectral gaps, min-max principle, large genus.

Remark. The sequence $\{\eta(g)\}$ is arbitrary as long as it satisfies the bounds: examples include $\eta(g) \equiv 2$, $\eta(g) = \{2, 3, 2, 3, \dots\}$, and $\eta(g) = 2g - 2$.

On the other hand, by [Cheng 1975, Corollary 2.3], we know that

$$\lambda_i(X_g) \leq \frac{1}{4} + i^2 \cdot \frac{16\pi^2}{\mathrm{Diam}^2(X_g)}.$$

By Gauss–Bonnet, $\mathrm{Area}(X_g) = 4\pi(g-1)$. A simple area argument implies that the diameter satisfies $\mathrm{Diam}(X_g) \geq C \ln(g)$ for some universal constant $C > 0$. So if $\eta(g)$ satisfies

$$\lim_{g\to\infty} \frac{\eta(g)}{\ln(g)} = 0,$$

we have

$$\limsup_{g\to\infty} \sup_{X_g \in \mathcal{M}_g} \mathrm{SpG}_{\eta(g)}(X_g) \leq \tfrac{1}{4}.$$

Together with Theorem 4.1 this yields the following direct consequence.

Corollary 1.1. *If $\eta(g) = o(\ln(g))$, then*

$$\lim_{g\to\infty} \sup_{X_g \in \mathcal{M}_g} \mathrm{SpG}_{\eta(g)}(X_g) = \tfrac{1}{4}.$$

For $\eta(g) = 1$, both Theorem 4.1 and Corollary 1.1 are due to Hide and Magee [2023, Corollary 1.3], who used a probabilistic method to solve the conjecture (e.g., see [Buser 1984; Buser et al. 1988]) that there exists a sequence of closed hyperbolic surfaces with first eigenvalues tending to $\frac{1}{4}$ as the genus goes to infinity.

The following result is important in the proof of Theorem 4.1, which we include for independent interest. The proof is highly motivated by the work of Burger, Buser and Dodziuk [Buser et al. 1988], where they studied the case when the limiting surface is connected (e.g., see Theorem 2.6).

Proposition 3.1 (min-max principle). *Let $X_g(0) \in \partial\mathcal{M}_g$ be the limit of a family of Riemann surfaces $\{X_g(t)\}$ obtained by pinching certain simple closed geodesics such that $X_g(0)$ has k connected components, i.e., $X_g(0) = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_k$, where $k \geq 2$. Let $\lambda_1(Y_1), \dots, \lambda_1(Y_k)$ be the first nonzero eigenvalue of $Y_1, \dots, Y_k$ (if Y_i has no discrete eigenvalues then write $\lambda_1(Y_i) = \infty$) and write $\bar{\lambda}_1(*) = \min\{\lambda_1(*), \frac{1}{4}\}$ for $* = Y_1, \dots, Y_k$. Then*

$$\liminf_{t\to 0} \lambda_k(X_g(t)) \geq \min_{1\leq i\leq k} \{\bar{\lambda}_1(Y_i)\}.$$

Remark. Each component Y_i in the proposition above is a complete open hyperbolic surface of finite volume, whose spectrum consists of possibly discrete eigenvalues and the continuous spectrum $\left[\frac{1}{4}, \infty\right)$. Therefore, in the statement above, $\bar{\lambda}_1(Y_i)$ is the nonzero minimum of the spectrum of Y_i.

Proof sketch of Theorem 4.1. In the proof of Theorem 4.1, we will apply Proposition 3.1 to the case when all the $\bar{\lambda}_1(Y_i)$ are close to $\frac{1}{4}$. The main idea is the following: for each $\eta(g)$ we construct a sequence of genus g surfaces that degenerate into $\eta(g)$ components using only pieces that are known to have the first eigenvalue close to $\frac{1}{4}$. Then by the min-max principle, the $\eta(g)$-th eigenvalue of these surfaces will be close to $\frac{1}{4}$. On the other hand, by a result of Schoen, Wolpert and Yau (see Theorem 2.5), the $(\eta(g)-1)$-th eigenvalue is close to zero. This way we find sequences of surfaces that achieve the spectral gap of $\frac{1}{4}$. For the regime $\eta(g) > g$, the components used in the construction only include the thrice-punctured sphere and a twice-punctured torus. On the other hand, for $\eta(g) \leq g$, the essential components also include a large genus piece that relies on the work of Hide and Magee [2023].

Plan of the paper. Section 2 will first discuss properties of the boundary degeneration of the Riemann moduli spaces; then we will provide a review of the background and recent developments on spectral gaps on hyperbolic surfaces, including a list of punctured surface components with eigenvalue bounds which will be used in the degeneration limits. In Section 3 we will provide a proof for Proposition 3.1 regarding the min-max principle for eigenvalues of degenerating hyperbolic surfaces and a few immediate applications. In Section 4 we will complete the proof of Theorem 4.1.

2. Preliminaries

Boundary of the Riemann moduli spaces. Denote by $\mathcal{M}_{g,n}$ the moduli space of hyperbolic surfaces of genus g with n punctures, and by $\mathcal{M}_g := \mathcal{M}_{g,0}$ the moduli space of compact hyperbolic surfaces with genus g. It is well known that $\dim_{\mathbb{R}}(\mathcal{M}_{g,n}) = 6g+2n-6$. In particular, $\mathcal{M}_{0,3}$ contains only one point represented by the hyperbolic thrice-punctured sphere. The Deligne–Mumford compactification of $\mathcal{M}_{g,n}$ is obtained by adding nodal surfaces into $\mathcal{M}_{g,n}$, which is homeomorphic to the completion of $\mathcal{M}_{g,n}$ endowed with the Weil–Petersson metric. Let $\partial\mathcal{M}_{g,n}$ be the boundary of the Deligne–Mumford compactification of $\mathcal{M}_{g,n}$. Recall that $\partial\mathcal{M}_{g,n}$ is stratified, and each stratum of $\partial\mathcal{M}_{g,n}$ is a product of lower-dimensional moduli spaces. Points in $\partial\mathcal{M}_{g,n}$ are represented by hyperbolic nodal surfaces in $\mathcal{M}_{g,n}$ (see for example [Masur 1976] for more details on the completion of $\mathcal{M}_{g,n}$). Locally the process of pinching a simple closed geodesic into a pair of cusp points can be written with respect to hyperbolic collar coordinates (ρ, θ) with ℓ the length of the central geodesic circle. As $\ell \to 0$, the hyperbolic neck degenerates into a pair of cusps, which can be seen with the choice of the correct coordinates (see for example [Ji 1993; Masur 1976]). Another way to see this would be using the complex “plumbing” coordinates, which we will not discuss. Hyperbolic nodal surfaces are obtained by pinching certain disjoint geodesic circles, and we call such a family of hyperbolic metrics approaching nodal surfaces a degenerating family (see, e.g., [Wolpert 1990], and see Figure 1 for an example).

We also recall the collar lemma on structures of disjoint hyperbolic collars around short geodesics, which will be useful later in decomposing the surfaces.

Lemma 2.1 (collar lemma [Buser 1992, Theorem 4.1.1]). *Let $\gamma_1, \gamma_2, \dots, \gamma_m$ be disjoint simple closed geodesics on a closed hyperbolic Riemann surface X_g, and let $\ell(\gamma_i)$ be the length of γ_i. Then $m \leq 3g-3$, and we can define the collar of γ_i by*

$$T(\gamma_i) = \{x \in X_g : \mathrm{dist}(x, \gamma_i) \leq w(\gamma_i)\},$$

where

$$w(\gamma_i) = \operatorname{arcsinh} \frac{1}{\sinh\big(\frac{1}{2}\ell(\gamma_i)\big)} \tag{2}$$

is the width of the collar.

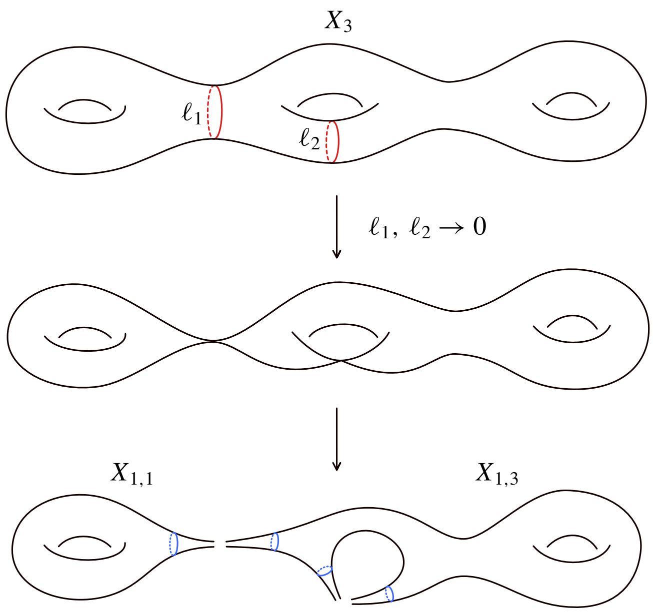

Figure 1. An example of a degenerating family in $\mathcal{M}_3$ whose limit is $X_{1,1} \sqcup X_{1,3}$, which is disconnected.

Then the collars are pairwise disjoint for $i = 1, \dots, m$. *Each* $T(\gamma_i)$ *is isomorphic to a cylinder* $(\rho, \theta) \in [-w(\gamma_i), w(\gamma_i)] \times \mathbb{S}^1$, *where* $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, *with the metric*

$$ds^2 = d\rho^2 + \ell(\gamma_i)^2 \cosh^2 \rho \, d\theta^2. \tag{3}$$

For a point (ρ, θ), *the point* $(0, \theta)$ *is its projection on the geodesic* γ_i, $|\rho|$ *is the distance to* γ_i, *and* θ *is the coordinate on* $\gamma_i \cong \mathbb{S}^1$.

As the length $\ell(\gamma)$ of the central closed geodesic goes to zero, the width $w(\gamma)$ is approximately $\ln(1/\ell(\gamma))$, which tends to infinity. We have the following as an easy corollary.

Corollary 2.2. *For a degenerating family of hyperbolic surfaces* $\{X_g(t)\}$, *the diameter satisfies*

$$\mathrm{Diam}(X_g(t)) \to \infty.$$

The following two lemmas will be useful in the proof of Theorem 4.1.

Lemma 2.3. *For each integer* $\eta(g) \in [g-1, 2g-2]$ *with* $g \geq 2$, *there exist two nonnegative integers* i *and* j *such that*

(1) $i + j = \eta(g)$,

(2) $\underbrace{\mathcal{M}_{0,3} \times \cdots \times \mathcal{M}_{0,3}}_{i\text{ copies}} \times \underbrace{\mathcal{M}_{1,2} \times \cdots \times \mathcal{M}_{1,2}}_{j\text{ copies}} \subset \partial \mathcal{M}_g$.

Remark. Here i and j depend on g and satisfy $i + 2j = 2g - 2$ by the additivity of the Euler characteristic.

Proof. If $\eta(g) = 2g-2$, the conclusion is obvious by choosing $i = 2g-2$ and $j = 0$, which is obtained by pinching $3g-3$ disjoint simple closed curves in a closed surface X_g of genus g.

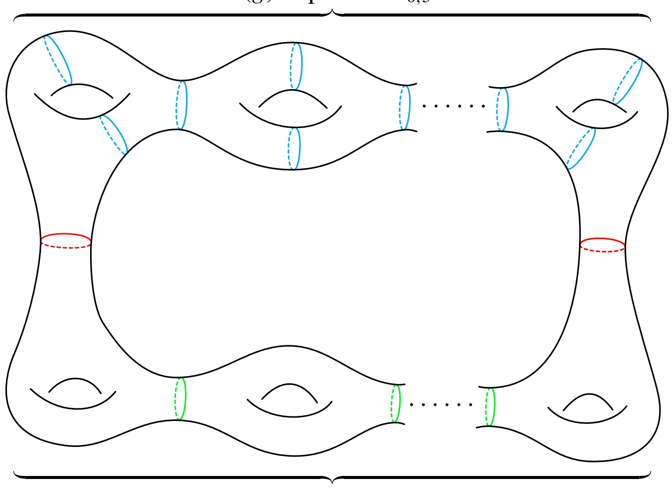

Figure 2. An example of the degeneration of a genus g surface into $i(g)$ copies of $X_{0,3}$ and $j(g)$ copies of $X_{1,2}$ by pinching all the simple geodesics marked in the picture.

Now we assume $g \leq \eta(g) \leq 2g-3$. Given a closed surface X_g of genus g, first one may pinch X_g along two disjoint simple closed curves σ_1 and σ_2 such that $X_g \setminus (\sigma_1 \cup \sigma_2)$ has two connected components $X_{g_1,2} \sqcup X_{g_2,2}$, where g_1 and g_2 are two nonnegative integers satisfying $g_1 + g_2 = g-1$. Here we choose

$$g_1 = (2g-2) - \eta(g) \quad \text{and} \quad g_2 = \eta(g) - (g-1).$$

For the second step, we pinch $X_{g_1,2}$ along $g_1 - 1$ disjoint simple closed curves $\{\gamma_l\}_{1 \leq l \leq g_1 - 1}$ such that the complement decomposes further into g_1 components:

$$X_{g_1,2} \setminus \bigcup_{1 \leq l \leq g_1 - 1} \gamma_l = \underbrace{X_{1,2} \sqcup \cdots \sqcup X_{1,2}}_{g_1 \text{ copies}}.$$

For $X_{g_2,2}$, one may pinch along $3g_2 - 1$ disjoint simple closed curves $\{\gamma'_m\}_{1 \leq m \leq 3g_2 - 1}$ such that the complement decomposes further into $2g_2$ components:

$$X_{g_2,2} \setminus \bigcup_{1 \leq m \leq 3g_2 - 1} \gamma'_m = \underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{2g_2 \text{ copies}}.$$

Pinching all these simple closed curves during cutting above to zero, the conclusion follows since

$$i = 2g_2 = 2\eta(g) - (2g-2) \quad \text{and} \quad j = g_1 = (2g-2) - \eta(g). \tag{4}$$

For an illustration, see Figure 2.

If $\eta(g) = g-1$, we first pinch X_g along a nonseparating simple closed curve to get a surface $X_{g-1,2}$. Then in the same way as with $X_{g_1,2}$ in the previous case, we pinch $X_{g-1,2}$ along $g-2$ disjoint simple closed curves to get $g-1$ copies of $X_{1,2}$. Then the conclusion follows with $i = 0$ and $j = g-1$.

Combining the three cases above, the proof is complete. □

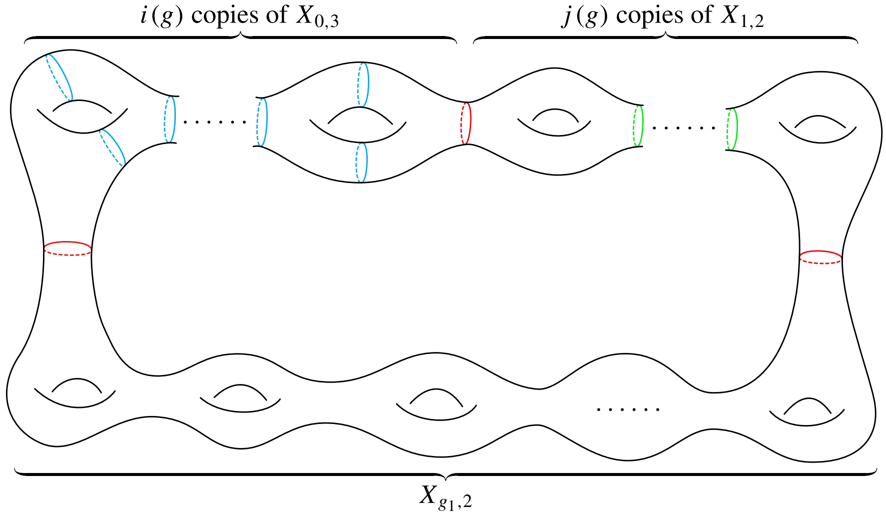

Figure 3. An example of decomposing a surface of genus g into i copies of $X_{0,3}$, j copies of $X_{1,2}$ and a copy of $X_{g_1,2}$, where i, j, and g_1 are given in the proof of Lemma 2.4.

Lemma 2.4. *For each integer* $\eta(g) \in [2, g]$ *with* $g \geq 3$, *there exist three nonnegative integers* g_1, i *and* j *such that*

(1) $2g_1 \geq g - 2$,

(2) $i + j + 1 = \eta(g)$,

(3) $\underbrace{\mathcal{M}_{0,3} \times \cdots \times \mathcal{M}_{0,3}}_{i(g)\ copies} \times \underbrace{\mathcal{M}_{1,2} \times \cdots \times \mathcal{M}_{1,2}}_{j(g)\ copies} \times \mathcal{M}_{g_1,2} \subset \partial\mathcal{M}_g$.

Remark. Similar to the previous lemma, i, j and g_1 depend on g. By calculating the Euler characteristics, these numbers should satisfy $i + 2j + 2g_1 = 2g - 2$.

Proof. Similar to the proof of Lemma 2.3 above, we first decompose X_g as $X_g \setminus (\sigma_1 \cup \sigma_2) = X_{g_1,2} \sqcup X_{g_2,2}$ for two disjoint simple closed curves σ_1 and σ_2, where g_1 and $g_2 := g - 1 - g_1$ will be determined in different cases below. Next we decompose $X_{g_2,2}$ into the disjoint union of i copies of $X_{0,3}$ and j copies of $X_{1,2}$ to obtain the desired properties. For an illustration, see Figure 3.

The proof contains the following three cases.

Case 1: $2 \leq \eta(g) \leq \frac{1}{2}g + 1$. The conclusion follows by choosing

$$i = 0, \quad j = \eta(g) - 1 \quad \text{and} \quad g_1 = g - \eta(g).$$

Case 2: $\frac{1}{2}g + 1 < \eta(g) \leq g$ and $\eta(g)$ is odd. The conclusion follows by choosing

$$i = \eta(g) - 1, \quad j = 0 \quad \text{and} \quad g_1 = g - \tfrac{1}{2}(1 + \eta(g)).$$

Case 3: $\frac{1}{2}g + 1 < \eta(g) \leq g$ and $\eta(g)$ is even. The conclusion follows by choosing

$$i = \eta(g) - 2, \quad j = 1 \quad \text{and} \quad g_1 = g - 1 - \tfrac{1}{2}\eta(g).$$

□

Eigenvalues of hyperbolic surfaces. The study of eigenvalues of the Laplacian on hyperbolic surfaces has a long history and has recently seen much progress. For a compact hyperbolic surface, the eigenvalues are discrete. On the other hand, when the hyperbolic surface degenerates to one with cusps, by [Lax and Phillips 1982] it is known that the spectrum is no longer discrete, rather it consists of a continuous spectrum $\left[\frac{1}{4}, \infty\right)$ and (possibly) additional discrete eigenvalues. The study of spectral degeneration has seen many developments; see [Hejhal 1990; Ji 1993; Ji and Zworski 1993; Wolpert 1987; 1992a; 1992b] for some of the earlier works.

An eigenvalue of a hyperbolic surface is said to be "small" if it is less than $\frac{1}{4}$, where the number $\frac{1}{4}$ shows up as the bottom of the continuous spectrum of a hyperbolic surface with cusps. The questions of existence of eigenvalues less than $\frac{1}{4}$ for both noncompact and compact hyperbolic surfaces not only arise in the field of spectral geometry, but also have deep relations to number theory regarding arithmetic hyperbolic surfaces, dating back to Selberg's famous $\frac{3}{16}$ theorem [1965]. We refer to [Gelbart and Jacquet 1978; Kim 2003; Luo et al. 1995] for more recent developments. Regarding the estimates and multiplicity counting of small eigenvalues, the history goes back to McKean [1972], Randol [1974], and Buser [1982; 1984]. Recently there have been many developments; see [Ballmann et al. 2016; 2017; 2018; Brooks and Makover 2001; Buser 1992; Buser et al. 1988; Mondal 2015; Otal and Rosas 2009; Schoen et al. 1980]. Among these are two classical results of particular relevance to our current work. The first regards bounds of eigenvalues on degenerating hyperbolic surfaces by Schoen, Wolpert and Yau [Schoen et al. 1980]:

Theorem 2.5 [Schoen et al. 1980]. *For any compact hyperbolic surface X_g of genus g and integer $i \in (0, 2g-2)$, the i-th eigenvalue satisfies*

$$\alpha_i(g) \cdot \ell_i \leq \lambda_i \leq \beta_i(g) \cdot \ell_i$$

and

$$\alpha(g) \leq \lambda_{2g-2},$$

where $\alpha_i(g) > 0$ and $\beta_i(g) > 0$ depend only on i and g, $\alpha(g) > 0$ depends only on g, and ℓ_i is the minimal possible sum of the lengths of simple closed geodesics in X_g which cut X_g into $i+1$ connected components.

Dodziuk and Randol [1986] gave an alternative proof of Theorem 2.5, and one may also see Dodziuk, Pignataro, Randol and Sullivan [Dodziuk et al. 1987] on similar results for Riemann surfaces with punctures. It was proved by Otal and Rosas [2009] that the constant $\alpha(g)$ can be optimally chosen to be $\frac{1}{4}$. For large genus g, it was recently proved by the first-named author and Xue [Wu and Xue 2022a; 2022c] that up to multiplication by a universal constant, $\alpha_1(g)$ can be optimally chosen to be $1/g^2$.

The other result that is relevant is [Buser et al. 1988, Theorem 2.1] regarding the first eigenvalue when the limiting degenerating surface is connected:

Theorem 2.6 [Buser et al. 1988]. *Let $\{X_g(t)\} \subset \mathcal{M}_g$ such that $Y = \lim_{t\to 0} X_g(t) \in \partial\mathcal{M}_g$ is connected. Denote by $\lambda_1(Y)$ the first nonzero eigenvalue of Y (if Y has no discrete eigenvalues we write $\lambda_1(Y) = \infty$). Then*

$$\limsup_{t\to 0} \lambda_1(X_g(t)) \geq \bar{\lambda}_1(Y) = \min\left\{\lambda_1(Y), \tfrac{1}{4}\right\}.$$

In Section 3 we will give a similar description of $\lambda_k(X_g(t))$ when the limiting surface has k connected components.

Another related direction in this topic is to understand how the genus of the hyperbolic surface, in particular when $g \to \infty$, affects the eigenvalues via different models of random hyperbolic surfaces. Brooks and Makover [2004] gave a uniform lower bound on the first spectral gap for their combinatorial model of random surfaces by gluing hyperbolic ideal triangles. In terms of Weil–Petersson random closed hyperbolic surfaces, Mirzakhani [2013] showed that the first eigenvalue is greater than 0.0024 with probability one as $g \to \infty$. Recently, the first-named author and Xue [Wu and Xue 2022b] improved this lower bound 0.0024 to be $\frac{3}{16} - \epsilon$, which was also independently obtained by Lipnowski and Wright [2024]. One may also see [Hide 2022] for similar results on Weil–Petersson random punctured hyperbolic surfaces and [Monk 2021] for related results. Recently there have also been many exciting developments in the case of random covers of both compact and noncompact hyperbolic surfaces; see [Magee and Naud 2020; 2021, Magee and Puder 2023,Magee et al. 2022]. For example, Magee, Naud and Puder [Magee et al. 2022] showed that a generic covering of a hyperbolic surface has relative spectral gap of size $\frac{3}{16} - \epsilon$, which was improved to $\frac{1}{4} - \epsilon$ by Hide and Magee [2023] for random covers of punctured hyperbolic surfaces. As an important application, [Hide and Magee 2023] proved that

$$\lim_{g \to \infty} \sup_{X_g \in \mathcal{M}_g} \lambda_1(X_g) = \tfrac{1}{4}.$$

This result provides major inspiration for our current paper.

One major ingredient of our proof is the existence of punctured surfaces with first eigenvalue close to $\frac{1}{4}$. We summarize those components in the two theorems below.

Theorem 2.7. (1) $\lambda_1(X_{0,3}) \geq \frac{1}{4}$;

(2) [Mondal 2015] *There exists a surface* $X_{1,2} \in \mathcal{M}_{1,2}$ *such that* $\lambda_1(X_{1,2}) \geq \frac{1}{4}$.

Proof. The first item is well known; see for example [Otal and Rosas 2009] or [Ballmann et al. 2016]. The existence of the second item was proved by Mondal [2015, Theorem 1.3]. □

The third component is from the recent breakthrough by Hide and Magee [2023]. They use probabilistic methods to show that for any $\epsilon > 0$, there exists an integer $\delta(\epsilon) > 0$ only depending on ϵ such that for all $g > \delta(\epsilon)$ there exists a $2g$-cover $\mathcal{X}$ of $X_{0,3}$ such that

$$\bar{\lambda}_1(\mathcal{X}) = \min\left\{\lambda_1(\mathcal{X}), \tfrac{1}{4}\right\} > \tfrac{1}{4} - \epsilon.$$

It is not hard to see that $\mathcal{X}$ must have an even number of punctures because the Euler characteristic of $\mathcal{X}$ is equal to $-2g$, which is even. Then one may apply the handle lemma of [Buser et al. 1988] (or see [Brooks and Makover 2001, Lemma 1.1]) to get the following.

Theorem 2.8. *For any* $\epsilon > 0$ *and large enough* $g > 0$, *there exists a hyperbolic surface* $\mathcal{X}_{g,2} \in \mathcal{M}_{g,2}$ *such that*

$$\bar{\lambda}_1(\mathcal{X}_{g,2}) = \min\left\{\lambda_1(\mathcal{X}_{g,2}), \tfrac{1}{4}\right\} > \tfrac{1}{4} - \epsilon.$$

Proof. For completeness we sketch an outline of the proof. Suppose by contradiction there exists a constant $\epsilon_0 > 0$ such that

$$\liminf_{g\to\infty} \sup_{X\in\mathcal{M}_{g,2}} \lambda_1(X) \le \tfrac{1}{4} - \epsilon_0. \tag{5}$$

It follows by [Hide and Magee 2023] that, for any $\epsilon > 0$ and large enough g, there exists a $2g$-cover $\mathcal{X}$ of $X_{0,3}$ such that

$$\bar{\lambda}_1(\mathcal{X}) = \min\left\{\lambda_1(\mathcal{X}), \tfrac{1}{4}\right\} > \tfrac{1}{4} - \epsilon.$$

Since the Euler characteristic $\chi(\mathcal{X}) = -2g$ is even, one may assume that $\mathcal{X}$ has an even number of cusps. As in [Buser et al. 1988] we can construct a family of hyperbolic surfaces $\{X_{g,2}(t)\} \subset \mathcal{M}_{g,2}$ such that

$$\lim_{t\to 0} X_{g,2}(t) = \mathcal{X} \in \partial\mathcal{M}_{g,2}.$$

By [Lax and Phillips 1982] we know that, for a hyperbolic surface with cusps, the spectrum below $\frac{1}{4}$ is discrete and only contains eigenvalues. By (5), for some large g one may assume that ϕ_t is the first eigenfunction on $X_{g,2}(t)$ with $\Delta\phi_t = \lambda_1(X_{g,2}(t)) \cdot \phi_t$ on $X_{g,2}(t)$. Then one may apply the handle lemma of [Buser et al. 1988] (or see [Brooks and Makover 2001, Lemma 1.1]) to obtain

$$\limsup_{t\to 0} \lambda_1(X_{g,2}(t)) \ge \bar{\lambda}_1(\mathcal{X}) = \min\left\{\lambda_1(\mathcal{X}), \tfrac{1}{4}\right\} > \tfrac{1}{4} - \epsilon,$$

which is a contradiction to (5) since $\epsilon > 0$ can be chosen to be arbitrarily small. □

3. Eigenvalues on a family of degenerating Riemann surfaces

In this section we will prove the following min-max principle, which was stated earlier.

Proposition 3.1 (min-max principle). *Let $X_g(0) \in \partial\mathcal{M}_g$ be the limit of a family of Riemann surfaces $\{X_g(t)\}$ obtained by pinching certain simple closed geodesics such that $X_g(0)$ has k connected components, i.e., $X_g(0) = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_k$, where $k \ge 2$. Let $\lambda_1(Y_1), \ldots, \lambda_1(Y_k)$ be the first nonzero eigenvalue of $Y_1, \ldots, Y_k$ (if Y_i has no discrete eigenvalues then write $\lambda_1(Y_i) = \infty$) and write $\bar{\lambda}_1(*) = \min\left\{\lambda_1(*), \frac{1}{4}\right\}$ for $* = Y_1, \ldots, Y_k$. Then*

$$\liminf_{t\to 0} \lambda_k(X_g(t)) \ge \min_{1\le i\le k} \{\bar{\lambda}_1(Y_i)\}.$$

To prove the theorem, we will start by discussing the subsequence limits of eigenfunctions. Denote by $\phi_t \in C^\infty(X_g(t))$ (one of) the normalized eigenfunctions corresponding to $\lambda_k(X_g(t))$, i.e.,

$$\Delta_{X_g(t)}\phi_t = \lambda_k(X_g(t)) \cdot \phi_t \quad \text{and} \quad \int_{X_g(t)} |\phi_t|^2 \,\mathrm{dVol}_{X_g(t)} = 1.$$

By [Cheng 1975, Corollary 2.3] we know that for any compact hyperbolic surface X there is an upper bound

$$\lambda_k(X) \le \frac{1}{4} + k^2 \cdot \frac{16\pi^2}{\mathrm{Diam}^2(X)}.$$

Note that $\mathrm{Diam}(X_g(t)) \to \infty$ as $t \to 0$ by Corollary 2.2 for any family of degenerating hyperbolic surfaces $\{X_g(t)\}$ as described in the proposition above. This gives that, for any fixed $k \geq 1$,

$$\liminf_{t\to 0} \lambda_k(X_g(t)) \leq \limsup_{t\to 0} \lambda_k(X_g(t)) \leq \tfrac{1}{4}. \tag{6}$$

On the other hand, by Theorem 2.5 we know that the lowest $k-1$ eigenvalues of $X_g(t)$ go to zero when the degenerating limit has k components, while the k-th eigenvalue $\lambda_k(X_g(t))$ stays bounded away from zero. Therefore

$$\liminf_{t\to 0} \lambda_k(X_g(t)) > 0. \tag{7}$$

Now consider

$$\lambda_k(0) := \liminf_{t\to 0} \lambda_k(X_g(t)). \tag{8}$$

By the discussion above we know that

$$0 < \lambda_k(0) \leq \tfrac{1}{4}. \tag{9}$$

By the collar lemma, Lemma 2.1, each $X_g(t)$ can be decomposed into a number of disjoint degenerating hyperbolic necks and a compact part (which has possibly several connected components). The width of each hyperbolic neck is determined by the central shrinking geodesic γ and can be chosen to be $w(\gamma)-1$, for example, where $w(\gamma)$ is given in (2). For the degenerating family $\{X_g(t)\}$ with N shrinking geodesics $\{\gamma_m(t)\}_{m=1}^N$, we denote the width of each hyperbolic neck by the following N-tuple:

$$\vec{w} := (w(\gamma_1(t))-1, w(\gamma_2(t))-1, \ldots, w(\gamma_N(t))-1).$$

Note that $\vec{w}$ depends on t, and each entry in $\vec{w}$ goes to ∞ as t goes to zero. Geometrically each hyperbolic neck degenerates into a pair of cusps. We remark here that in the definition of $\vec{w}$, the choice $w(\gamma)-1$ is for convenience and can be replaced by $w(\gamma)-c$ for any $c > 0$.

For any $X_g(t)$, we denote the union of all N hyperbolic necks as $C_{\vec{w}}(t)$. In local hyperbolic geodesic coordinates given by $d\rho^2 + \ell^2 \cosh^2 \rho \, d\theta^2$ where ℓ is the length of the central geodesic circle γ_i,

$$C_{\vec{w}}(t) = \bigcup_{m=1}^{N} \{(\rho,\theta) : 0 \leq |\rho| \leq w(\gamma_m(t)) - 1\}. \tag{10}$$

In addition, we also denote the union of all "shells" near the collars by

$$S_{\vec{w}}(t) = \bigcup_{m=1}^{N} \{(\rho,\theta) : w(\gamma_m(t)) - 1 \leq |\rho| \leq w(\gamma_m(t))\}. \tag{11}$$

Then it follows by the collar lemma that all such collar neighborhoods (and shells) are disjoint; see Figure 4 for an illustration of collars and shells.

Denote the compact part by $F_{\vec{w}}(t) = X_g(t) \setminus C_{\vec{w}}(t)$. The compact area and nodal degeneration area are grafted together [Melrose and Zhu 2018; 2019; Wolpert 1990]. For small t, the $F_{\vec{w}}(t)$ are all diffeomorphic. In particular, the metric on $F_{\vec{w}}(t)$ can be written as $e^{2u_t} g_0$, where g_0 is the metric on $F_{\vec{w}}(0)$ and u_t is polyhomogeneous and uniformly bounded in all derivatives [Melrose and Zhu 2019]. That is, we can write the diffeomorphism $D_t : F_{\vec{w}}(t) \to F_{\vec{w}}(0)$ such that $g_t = D_t^* g_0$ and D_t are uniformly bounded. From now on, when we consider the convergence of eigenfunctions $\phi(t)$ on $X_g(t)$, the functions are all defined

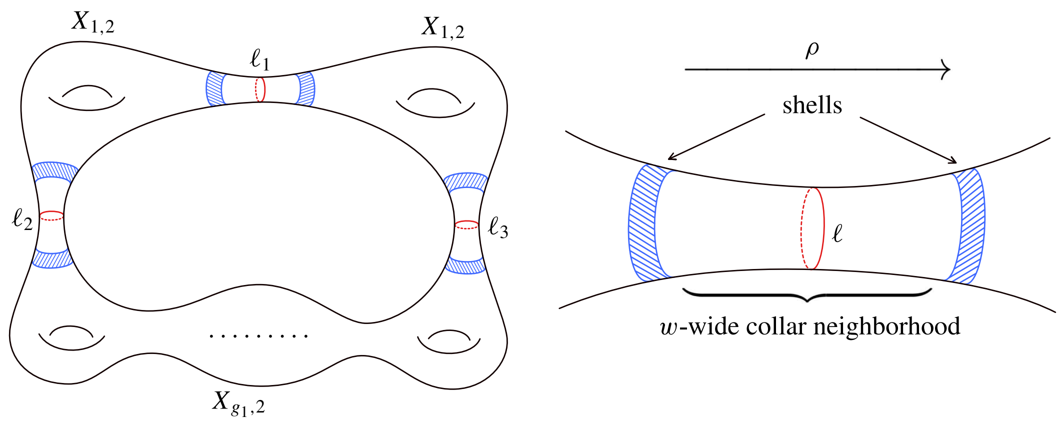

Figure 4. An example of collar neighborhoods and shells.

on $X_g(0)$ via the pullback $(D_t^{-1})^*\phi(t)$; see [Wolpert 1992a; 1992b] for similar approaches. See also another related approach via universal covers in [Buser et al. 1988].

Now take a sequence of metrics such that the corresponding sequence of eigenvalues approaches $\lambda_k(0)$, which is defined in (8). Denote the sequence by $\{X_g(t_i)\}_{i=1}^{\infty}$. By definition,

$$\lim_{i\to\infty} t_i = 0 \quad \text{and} \quad \lim_{i\to\infty} \lambda_k(X_g(t_i)) = \lambda_k(0).$$

Denote the corresponding eigenfunction on $X_g(t_i)$ by ϕ_{t_i}; we discuss the convergence of the sequence of functions $\{\phi_{t_i}\}_{i=1}^{\infty}$ below. One key ingredient is the following Sobolev–Gårding Inequality on the compact part $F_{\vec{w}}(t)$. Denote by $\operatorname{inj}(\cdot)$ the injectivity radius function. Denote by $\nabla^j\phi_{t_i}$ the j-th covariant derivative of ϕ_{t_i}, where $j \in \mathbb{N}$. Then we have the following.

Lemma 3.2. *For any $x \in F_{\vec{w}}(t)$, $j \in \mathbb{N}$ and $r < \operatorname{inj}(F_{\vec{w}}(t))$, there exist a constant $c_{r,j} > 0$ and an integer $N_j > 0$ independent of x such that we have the following pointwise bound for any j-th derivative*:

$$|\nabla^j\phi_t(x)| \le c_{r,j}\sum_{\ell=0}^{N_j}\|\Delta^{\ell}_{X_g(t)}\phi_t\|_{L^2(B_r(x))}. \tag{12}$$

Proof. This equality was shown in [Buser et al. 1988, Theorem 2.1]. The inequality is from the combination of the Sobolev and Gårding inequalities, for example, see [Bers et al. 1964]. □

With the above inequality we have the following uniform bound on $\{\phi_{t_i}\}_{i=1}^{\infty}$ and their derivatives.

Lemma 3.3. *For any $j \in \mathbb{N}$, we have that $\{\nabla^j\phi_{t_i}\}_i$ is uniformly bounded on any compact set of $X_g(0)$.*

Proof. Using (12) in the previous lemma, $\Delta\phi_t = \lambda_k(t)\phi_t$ and $0 < \lambda_k(t) < \frac{1}{3}$, we have

$$|\nabla^j\phi_t(x)| \le c_{r,j}\sum_{\ell=0}^{\infty}\left(\frac{1}{3}\right)^{\ell}\|\phi_t\|_{L^2(X_g(t))} \le 2c_{r,j},$$

where the bound is independent of x. Hence all derivatives of ϕ_t (in particular the sequence $\{\phi_{t_i}\}$) are uniformly bounded. □

Lemma 3.4. *There exists a subsequence of ϕ_{t_i} (denoted by ϕ_i) and $\phi_0 \in H^1(X_g(0))$ such that any derivatives satisfy*

$$\nabla^j \phi_i \to \nabla^j \phi_0$$

uniformly on connected compact set of $X_g(0)$.

Proof. Viewing $\{\phi_t\}$ as functions on F_0 where F_0 is any connected compact set of $X_g(0)$, by the previous lemma we have uniform boundedness of ϕ_t and all their derivatives. Hence by the Arzelà–Ascoli diagonal argument there exists a subsequence ϕ_i such that the function and its derivative converge uniformly on any compact set. □

By the convergence above we have

$$\int_{X_g(0)} |\phi_0|^2 \leq 1, \quad \int_{X_g(0)} |\nabla \phi_0|^2 \leq 1$$

and

$$\Delta_{X_g(0)} \phi_0 = \lambda_k(0) \cdot \phi_0.$$

Now we show the following statement regarding the limit $(\lambda_k(0), \phi_0)$. The argument is similar to [Wu and Xue 2022a, Lemma 9] and [Dodziuk et al. 1987, Lemma 3.3].

Proposition 3.5. *The limit $(\lambda_k(0), \phi_0)$ must satisfy one of the following conditions*:

(1) *ϕ_0 is an eigenfunction of $\Delta_{X_g(0)}$ and also restricts to at least one of the components Y_k as an eigenfunction*; *or*

(2) *$\phi_0 = 0$ everywhere on $X_g(0)$ and $\lambda_k(0) = \frac{1}{4}$.*

Proof. If ϕ_0 is not zero everywhere, then ϕ_0 belongs to $H^1(X_g(0))$ and is an eigenfunction. In particular, it must restrict to a nonzero function on at least one component of $X_g(0)$.

Otherwise suppose $\phi_0 = 0$ everywhere on $X_g(0)$, that is, $\phi_i \to 0$ pointwise everywhere. Then following a similar argument as in [Wu and Xue 2022a, Lemma 9] or [Dodziuk et al. 1987, Lemma 3.3], we can show that $\lambda_k(0) \geq \frac{1}{4}$. For completeness we write out the proof in detail here.

Recall the definitions of collars and shells on hyperbolic necks in (10) and (11). Similar to the definition above, we denote by $C_{\vec{w}}(i)$ the union of $\vec{w}$-wide collar neighborhoods near all degenerating geodesic circles on $X_g(t_i)$, and by $S_{\vec{w}}(i)$ the union of the "shells". To simplify the argument below, we also denote by $C_{i,m}$ and $S_{i,m}$ the individual hyperbolic neck and shell, respectively, with central geodesic circle $\gamma_m(i)$, where $1 \leq m \leq N$, and denote the corresponding width by $w_{i,m} := w(\gamma_m(i)) - 1$. Hence

$$C_{\vec{w}}(i) = \bigcup_{m=1}^{N} C_{i,m} \quad \text{and} \quad S_{\vec{w}}(i) = \bigcup_{m=1}^{N} S_{i,m}.$$

Fix any $\epsilon \in (0, 1)$ and $\delta \in \left(0, \frac{1}{16}\right)$. We write $c = 1 - \epsilon$. Since ϕ_i converges to zero uniformly on any compact set, there exists $N_0 \in \mathbb{N}$ such that for any $i > N_0$ we have

$$\int_{C_{\vec{w}}(i)} |\phi_i|^2 \geq c > 0, \quad \int_{S_{\vec{w}}(i)} |\phi_i|^2 < \delta c \quad \text{and} \quad \int_{S_{\vec{w}}(i)} |\nabla \phi_i|^2 < \delta c.$$

Define a new function on $C_{\vec{w}}(i) \cup S_{\vec{w}}(i)$ as follows:

$$\Phi_i := \begin{cases} \phi_i, & |\rho| \le w_{i,m}, \\ (w_{i,m}+1-|\rho|)\phi_i, & w_{i,m} \le |\rho| \le w_{i,m}+1. \end{cases}$$

Then Φ_i gives a function in $H_0^1(C_{\vec{w}}(i) \cup S_{\vec{w}}(i))$ with $\Phi_i|_{\partial(C_{\vec{w}}(i) \cup S_{\vec{w}}(i))} = 0$. Therefore by applying [Wu and Xue 2022a, Lemma 7] to a union of hyperbolic collars we have

$$\int_{C_{\vec{w}}(i) \cup S_{\vec{w}}(i)} |\nabla \Phi_i|^2 > \frac{1}{4} \int_{C_{\vec{w}}(i) \cup S_{\vec{w}}(i)} |\Phi_i|^2.$$

On the other hand we have

$$\begin{aligned}\int_{S_{\vec{w}}(i)} |\nabla \Phi_i|^2 &= \sum_{m=1}^{N} \int_{S_{i,m}} |\nabla((w_{i,m}+1-|\rho|)\phi_i)|^2 \\ &= \sum_{m=1}^{N} \int_{S_{i,m}} |\nabla(w_{i,m}+1-|\rho|)\cdot\phi_i + (w_{i,m}+1-|\rho|)\cdot\nabla\phi_i|^2 \\ &\le \sum_{m=1}^{N} \int_{S_{i,m}} (|\phi_i| + (w_{i,m}+1-|\rho|)\cdot|\nabla\phi_i|)^2 \le 2\sum_{m=1}^{N} \int_{S_{i,m}} |\phi_i|^2 + 2\sum_{m=1}^{N} \int_{S_{i,m}} |\nabla\phi_i|^2 \le 4\delta c.\end{aligned}$$

Therefore for any $i > N_0$ we have

$$\begin{aligned}\int_{C_{\vec{w}}(i)} |\nabla\phi_i|^2 = \int_{C_{\vec{w}}(i)} |\nabla\Phi_i|^2 &= \int_{C_{\vec{w}}(i)\cup S_{\vec{w}}(i)} |\nabla\Phi_i|^2 - \int_{S_{\vec{w}}(i)} |\nabla\Phi_i|^2 \\ &\ge \frac{1}{4}\int_{C_{\vec{w}}(i)\cup S_{\vec{w}}(i)} |\Phi_i|^2 - \int_{S_{\vec{w}}(i)} |\nabla\Phi_i|^2 \\ &\ge \frac{1}{4}\int_{C_{\vec{w}}(i)} |\phi_i|^2 - \int_{S_{\vec{w}}(i)} |\nabla\Phi_i|^2 \ge \frac{1}{4}c - 4\delta c = \frac{1-16\delta}{4}(1-\epsilon),\end{aligned}$$

which implies

$$\lambda_k(X_g(t_i)) = \frac{\int_{X_g(t_i)} |\nabla\phi_{t_i}|^2}{\int_{X_g(t_i)} |\phi_i|^2} \ge \frac{\int_{C_{\vec{w}}(i)} |\nabla\phi_i|^2}{\int_{X_g(t_i)} |\phi_i|^2} \ge \frac{1-16\delta}{4}(1-\epsilon).$$

Since this argument holds for any $\epsilon \in (0, 1)$ and $\delta \in \left(0, \frac{1}{16}\right)$, we have

$$\lambda_k(0) = \liminf_{i\to\infty} \lambda_k(X_g(t_i)) \ge \tfrac{1}{4}.$$

On the other hand $\lambda_k(0) \le \frac{1}{4}$ by (9), therefore we have $\lambda_k(0) = \frac{1}{4}$. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By the previous proposition, either $\lambda_k(0) = \lambda_1(Y_i)$ for at least one of the components Y_i, or $\lambda_k(0) = \frac{1}{4}$, therefore we obtain

$$\lambda_k(0) \ge \min_{1\le i\le k}\left\{\min\left\{\lambda_1(Y_i), \tfrac{1}{4}\right\}\right\}$$

as desired. □

We enclose in this section the following result, which is an easy application of Proposition 3.1.

Proposition 3.6. *Let $X_g(0) \in \partial\mathcal{M}_g$ be the limit of a family of Riemann surfaces $\{X_g(t)\} \subset \mathcal{M}_g$ by pinching certain simple closed geodesics such that $X_g(0)$ has k connected components, i.e., $X_g(0) = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_k$ for some $k \geq 2$. Assume in addition that $\bar{\lambda}_1(Y_i) = \min\{\lambda_1(Y_i), \frac{1}{4}\} \geq \frac{1}{4}$ for all $1 \leq i \leq k$, where $\lambda_1(Y_i)$ is the first nonzero eigenvalue of Y_i. Then*

$$\lim_{t\to 0} \lambda_k(X_g(t)) = \tfrac{1}{4}.$$

Proof. From (6) we have that

$$\limsup_{t\to 0} \lambda_k(X_g(t)) \leq \tfrac{1}{4}.$$

On the other hand, it follows by Proposition 3.1 that

$$\liminf_{t\to 0} \lambda_k(X_g(t)) \geq \min_{1\leq i\leq k} \bigl\{\min\{\lambda_1(Y_i), \tfrac{1}{4}\}\bigr\} = \tfrac{1}{4}.$$

The conclusion immediately follows. □

We now prove spectral gaps can be arbitrarily close to zero by using this result. Recall that, for all $i \geq 1$ and $X_g \in \mathcal{M}_g$, the i-th spectral gap $\mathrm{SpG}_i(X_g)$ of X is defined as

$$\mathrm{SpG}_i(X_g) := \lambda_i(X_g) - \lambda_{i-1}(X_g).$$

We prove the following.

Proposition 3.7. *For all $i \geq 1$,*

$$\inf_{X_g\in\mathcal{M}_g} \mathrm{SpG}_i(X_g) = 0.$$

Proof. We split the proof into three cases.

Case 1: $1 \leq i \leq 2g-3$. One may choose a closed hyperbolic surface $\mathcal{X}_g \in \mathcal{M}_g$ which is close enough to the maximal nodal surface

$$\underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{2g-2 \text{ copies}} \in \partial\mathcal{M}_g,$$

then $\lambda_i(\mathcal{X}_g)$ is close to zero by Theorem 2.5. So the conclusion follows for this case.

Case 2: $i = 2g-2$. Let $Z_{1,2} \in \mathcal{M}_{1,2}$ such that $\bar{\lambda}_1(Z_{1,2}) = \min\{\frac{1}{4}, \lambda_1(Z_{1,2})\} \geq \frac{1}{4}$ by Theorem 2.7. Recall that $\lambda_1(X_{0,3}) \geq \frac{1}{4}$ from the same theorem. Let $\{X_g(t)\} \subset \mathcal{M}_g$ be a family of hyperbolic surfaces such that

$$\lim_{t\to 0} X_g(t) = \underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{2g-4 \text{ copies}} \sqcup Z_{1,2} \in \partial\mathcal{M}_g.$$

Then it follows from Proposition 3.6 that

$$\lim_{t\to 0} \lambda_{2g-3}(X_g(t)) = \tfrac{1}{4}.$$

Meanwhile, by [Otal and Rosas 2009, Theorem 2], we know that

$$\lambda_{2g-2}(X_g(t)) \geq \tfrac{1}{4}.$$

Since $\mathrm{Diam}(X_g(t)) \to \infty$ as $t \to 0$, by [Cheng 1975, Corollary 2.3] we have that

$$\limsup_{t\to 0} \lambda_{2g-2}(X_g(t)) \le \tfrac{1}{4}.$$

Thus, we have

$$\lim_{t\to 0} \lambda_{2g-2}(X_g(t)) = \tfrac{1}{4}.$$

Then the conclusion also follows for this case because

$$\inf_{X_g\in\mathcal{M}_g} \mathrm{SpG}_{2g-2}(X_g) \le \lim_{t\to 0} \mathrm{SpG}_{2g-2}(X_g(t)) = 0.$$

<u>Case 3</u>: $i > 2g-2$. Let $\{Y_g(t)\} \subset \mathcal{M}_g$ be a family of hyperbolic surfaces such that

$$\lim_{t\to 0} Y_g(t) \in \partial\mathcal{M}_g.$$

Similar to Case 2, by [Otal and Rosas 2009, Theorem 2] and [Cheng 1975, Corollary 2.3], we have

$$\lim_{t\to 0} \lambda_i(Y_g(t)) = \tfrac{1}{4} \quad \text{and} \quad \lim_{t\to 0} \lambda_{i-1}(Y_g(t)) = \tfrac{1}{4}.$$

This implies $\inf_{X_g\in\mathcal{M}_g} \mathrm{SpG}_i(X_g) = 0$ for all $i > 2g-2$. □

4. Proof of Theorem 4.1

Now we are ready to prove Theorem 4.1.

Theorem 4.1. *Let $\{\eta(g)\}_{g=2}^{\infty}$ be any sequence of integers with $\eta(g) \in [1, 2g-2]$. Then*

$$\liminf_{g\to\infty} \sup_{X_g\in\mathcal{M}_g} \mathrm{SpG}_{\eta(g)}(X_g) \ge \tfrac{1}{4}.$$

Proof. We will show that for any $\eta(g)$ with sufficiently large g, one can find a genus g surface X_g with $\mathrm{SpG}_{\eta(g)}(X_g)$ close to $\frac{1}{4}$. To see this, we split the proof into the following four cases.

<u>Case 1</u>: $\eta(g) = 2g-2$. Let $X_g(t) : (0,1) \to \mathcal{M}_g$ be a family of closed hyperbolic surfaces such that

$$\lim_{t\to 0} X_g(t) = \underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{2g-2 \text{ copies}} \in \partial\mathcal{M}_g.$$

First by [Otal and Rosas 2009, Theorem 2], $\lambda_{2g-2}(X_g(t)) \ge \frac{1}{4}$ for all $t \in (0,1)$. Secondly by Theorem 2.5 we know that $\lambda_{2g-3}(X_g(t)) \to 0$ as $t \to 0$. Thus,

$$\sup_{X_g\in\mathcal{M}_g} \mathrm{SpG}_{2g-2}(X_g) \ge \liminf_{t\to 0} \mathrm{SpG}_{2g-2}(X_g(t)) \ge \tfrac{1}{4}.$$

<u>Case 2</u>: $\eta(g) \in [g+1, 2g-3]$. First we choose a hyperbolic surface $Z_{1,2} \in \mathcal{M}_{1,2}$ such that $\bar{\lambda}_1(Z_{1,2}) \ge \frac{1}{4}$ by Theorem 2.7. Recall also that $\lambda_1(X_{0,3}) \ge \frac{1}{4}$. By Lemma 2.3 we can construct $X_g(t) : (0,1) \to \mathcal{M}_g$ as a family of closed hyperbolic surfaces such that

$$\lim_{t\to 0} X_g(t) = \underbrace{X_{0,3} \sqcup \cdots \sqcup X_{0,3}}_{i \text{ copies}} \sqcup \underbrace{Z_{1,2} \sqcup \cdots \sqcup Z_{1,2}}_{j \text{ copies}} \in \partial\mathcal{M}_g,$$

where i and j are two nonnegative integers satisfying $i+j=\eta(g)$. By Theorem 2.5 we know that $\lim_{t\to 0}\lambda_{\eta(g)-1}(X_g(t))=0$. By Proposition 3.6 we have

$$\lim_{t\to 0}\lambda_{\eta(g)}(X_g(t))=\tfrac{1}{4},$$

which implies

$$\sup_{X_g\in\mathcal{M}_g}\mathrm{SpG}_{\eta(g)}(X_g)\geq\lim_{t\to 0}\mathrm{SpG}_{\eta(g)}(X_g(t))=\tfrac{1}{4}.$$

Case 3: $\eta(g)\in[2,g]$. As in Case 2, we choose a hyperbolic surface $Z_{1,2}\in\mathcal{M}_{1,2}$ such that $\bar{\lambda}_1(Z_{1,2})\geq\frac{1}{4}$. Let $g_1>0$ be the integer determined in Lemma 2.4. Note that g_1 tends to ∞ as $g\to\infty$ because $2g_1\geq g-2$. Then by Theorem 2.8 we know that, for any $\epsilon>0$ and large enough $g>0$, one may choose a hyperbolic surface $\mathcal{X}_{g_1,2}\in\mathcal{M}_{g_1,2}$ such that

$$\bar{\lambda}_1(\mathcal{X}_{g_1,2})>\tfrac{1}{4}-\epsilon.$$

Fix any such large g. Then by Lemma 2.4 we construct $X_g(t):(0,1)\to\mathcal{M}_g$ as a family of closed hyperbolic surfaces such that

$$\lim_{t\to 0}X_g(t)=\underbrace{X_{0,3}\sqcup\cdots\sqcup X_{0,3}}_{i\text{ copies}}\sqcup\underbrace{Z_{1,2}\sqcup\cdots\times Z_{1,2}}_{j\text{ copies}}\sqcup\mathcal{X}_{g_1,2}\in\partial\mathcal{M}_g,$$

where i and j are two nonnegative integers satisfying $i+j=\eta(g)-1$. By Theorem 2.5 we know that $\lim_{t\to 0}\lambda_{\eta(g)-1}(X_g(t))=0$. Applying the min-max principle in Proposition 3.1 to this sequence with $k=\eta(g)$ (note that g is a fixed large integer hence $\eta(g)$ is also fixed), we have

$$\liminf_{t\to 0}\lambda_{\eta(g)}(X_g(t))\geq\min\{\bar{\lambda}_1(\mathcal{M}_{0,3}),\ \bar{\lambda}_1(Z_{1,2}),\ \bar{\lambda}_1(\mathcal{X}_{g_1,2})\}\geq\tfrac{1}{4}-\epsilon,$$

which implies

$$\liminf_{g\to\infty}\sup_{X_g\in\mathcal{M}_g}\mathrm{SpG}_{\eta(g)}(X_g)\geq\tfrac{1}{4}-\epsilon$$

because

$$\sup_{X_g\in\mathcal{M}_g}\mathrm{SpG}_{\eta(g)}(X_g)\geq\liminf_{t\to 0}\mathrm{SpG}_{\eta(g)}(X_g(t)).$$

Since $\epsilon>0$ can be arbitrarily small, we have

$$\liminf_{g\to\infty}\sup_{X_g\in\mathcal{M}_g}\mathrm{SpG}_{\eta(g)}(X_g)\geq\tfrac{1}{4}.$$

Case 4: $\eta(g)=1$. This is due to [Hide and Magee 2023, Corollary 1.3] because $\mathrm{SpG}_1(X_g)=\lambda_1(X_g)$.

The four cases above cover all possible $\eta(g)$ and hence complete the proof. □

Remark. The method in this paper works for indices in the range of $[1,2g-2]$ in Theorem 4.1. The restriction comes from the lack of suitable components with λ_1 close to $\frac{1}{4}$ when constructing the degenerating family. It would be interesting to know whether the assumption $\eta(g)\in[1,2g-2]$ can be dropped.

We also note that, together with [Cheng 1975, Corollary 2.3], the proof of Theorem 4.1 above actually gives the following result.

Theorem 4.2. *For any* $0 \leq j < i$ *with* $i = o(\ln(g))$,

$$\lim_{g\to\infty} \sup_{X_g \in \mathcal{M}_g} (\lambda_i(X_g) - \lambda_j(X_g)) = \tfrac{1}{4}.$$

Acknowledgements

The authors would like to thank the referees for their careful reading and valuable comments which improved the quality of this paper. Wu and Zhang would like to thank Yuxin He, Yang Shen and Yuhao Xue for helpful discussions on this project. Wu is partially supported by NSFC grants 12171263 and 12361141813. Zhu is supported by NSF grants DMS-2041823 and DMS-2305363.

References

[Ballmann et al. 2016] W. Ballmann, H. Matthiesen, and S. Mondal, "Small eigenvalues of closed surfaces", *J. Differential Geom.* **103**:1 (2016), 1–13. MR Zbl

[Ballmann et al. 2017] W. Ballmann, H. Matthiesen, and S. Mondal, "On the analytic systole of Riemannian surfaces of finite type", *Geom. Funct. Anal.* **27**:5 (2017), 1070–1105. MR Zbl

[Ballmann et al. 2018] W. Ballmann, H. Matthiesen, and S. Mondal, "Small eigenvalues of surfaces: old and new", *ICCM Not.* **6**:2 (2018), 9–24. MR Zbl

[Bers et al. 1964] L. Bers, F. John, and M. Schechter, *Partial differential equations*, Lectures in Appl. Math. **3**, Interscience, New York, 1964. MR Zbl

[Brooks and Makover 2001] R. Brooks and E. Makover, "Riemann surfaces with large first eigenvalue", *J. Anal. Math.* **83** (2001), 243–258. MR Zbl

[Brooks and Makover 2004] R. Brooks and E. Makover, "Random construction of Riemann surfaces", *J. Differential Geom.* **68**:1 (2004), 121–157. MR Zbl

[Buser 1982] P. Buser, "A note on the isoperimetric constant", *Ann. Sci. École Norm. Sup.* (4) **15**:2 (1982), 213–230. MR Zbl

[Buser 1984] P. Buser, "On the bipartition of graphs", *Discrete Appl. Math.* **9**:1 (1984), 105–109. MR Zbl

[Buser 1992] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progr. Math. **106**, Birkhäuser, Boston, 1992. MR Zbl

[Buser et al. 1988] P. Buser, M. Burger, and J. Dodziuk, "Riemann surfaces of large genus and large λ_1", pp. 54–63 in *Geometry and analysis on manifolds* (Katata/Kyoto, 1987), edited by T. Sunada, Lecture Notes in Math. **1339**, Springer, 1988. MR Zbl

[Cheng 1975] S. Y. Cheng, "Eigenvalue comparison theorems and its geometric applications", *Math. Z.* **143**:3 (1975), 289–297. MR Zbl

[Dodziuk and Randol 1986] J. Dodziuk and B. Randol, "Lower bounds for λ_1 on a finite-volume hyperbolic manifold", *J. Differential Geom.* **24**:1 (1986), 133–139. MR Zbl

[Dodziuk et al. 1987] J. Dodziuk, T. Pignataro, B. Randol, and D. Sullivan, "Estimating small eigenvalues of Riemann surfaces", pp. 93–121 in *The legacy of Sonya Kovalevskaya* (Cambridge/Amherst, MA, 1985), edited by L. Keen, Contemp. Math. **64**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Gelbart and Jacquet 1978] S. Gelbart and H. Jacquet, "A relation between automorphic representations of GL(2) and GL(3)", *Ann. Sci. École Norm. Sup.* (4) **11**:4 (1978), 471–542. MR Zbl

[Hejhal 1990] D. A. Hejhal, *Regular b-groups, degenerating Riemann surfaces, and spectral theory*, Mem. Amer. Math. Soc. **437**, Amer. Math. Soc., Providence, RI, 1990. MR Zbl

[Hide 2022] W. Hide, "Spectral gap for Weil–Petersson random surfaces with cusps", *Int. Math. Res. Not.* **2022** (online publication October 2022). Zbl

[Hide and Magee 2023] W. Hide and M. Magee, "Near optimal spectral gaps for hyperbolic surfaces", *Ann. of Math.* (2) **198**:2 (2023), 791–824. MR Zbl

[Ji 1993] L. Ji, "Spectral degeneration of hyperbolic Riemann surfaces", *J. Differential Geom.* **38**:2 (1993), 263–313. MR Zbl

[Ji and Zworski 1993] L. Ji and M. Zworski, "The remainder estimate in spectral accumulation for degenerating hyperbolic surfaces", *J. Funct. Anal.* **114**:2 (1993), 412–420. MR Zbl

[Kim 2003] H. H. Kim, "Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2", *J. Amer. Math. Soc.* **16**:1 (2003), 139–183. MR Zbl

[Lax and Phillips 1982] P. D. Lax and R. S. Phillips, "The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces", *J. Funct. Anal.* **46**:3 (1982), 280–350. MR Zbl

[Lipnowski and Wright 2024] M. Lipnowski and A. Wright, "Towards optimal spectral gaps in large genus", *Ann. Probab.* **52**:2 (2024), 545–575. MR Zbl

[Luo et al. 1995] W. Luo, Z. Rudnick, and P. Sarnak, "On Selberg's eigenvalue conjecture", *Geom. Funct. Anal.* **5**:2 (1995), 387–401. MR Zbl

[Magee and Naud 2020] M. Magee and F. Naud, "Explicit spectral gaps for random covers of Riemann surfaces", *Publ. Math. Inst. Hautes Études Sci.* **132** (2020), 137–179. MR Zbl

[Magee and Naud 2021] M. Magee and F. Naud, "Extension of Alon's and Friedman's conjectures to Schottky surfaces", preprint, 2021. Zbl arXiv 2106.02555

[Magee and Puder 2023] M. Magee and D. Puder, "The asymptotic statistics of random covering surfaces", *Forum Math. Pi* **11** (2023), art. id. E15. Zbl

[Magee et al. 2022] M. Magee, F. Naud, and D. Puder, "A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16}-\varepsilon$", *Geom. Funct. Anal.* **32**:3 (2022), 595–661. MR Zbl

[Masur 1976] H. Masur, "Extension of the Weil–Petersson metric to the boundary of Teichmuller space", *Duke Math. J.* **43**:3 (1976), 623–635. MR Zbl

[McKean 1972] H. P. McKean, "Selberg's trace formula as applied to a compact Riemann surface", *Comm. Pure Appl. Math.* **25** (1972), 225–246. Correction in **27**:1 (1974), 134. MR Zbl

[Melrose and Zhu 2018] R. Melrose and X. Zhu, "Resolution of the canonical fiber metrics for a Lefschetz fibration", *J. Differential Geom.* **108**:2 (2018), 295–317. MR Zbl

[Melrose and Zhu 2019] R. Melrose and X. Zhu, "Boundary behaviour of Weil–Petersson and fibre metrics for Riemann moduli spaces", *Int. Math. Res. Not.* **2019**:16 (2019), 5012–5065. MR Zbl

[Mirzakhani 2013] M. Mirzakhani, "Growth of Weil–Petersson volumes and random hyperbolic surfaces of large genus", *J. Differential Geom.* **94**:2 (2013), 267–300. MR Zbl

[Mondal 2015] S. Mondal, "On largeness and multiplicity of the first eigenvalue of finite area hyperbolic surfaces", *Math. Z.* **281**:1-2 (2015), 333–348. MR Zbl

[Monk 2021] L. Monk, *Geometry and spectrum of typical hyperbolic surfaces*, Ph.D. thesis, Université de Strasbourg, 2021, available at https://lauramonk.github.io/thesis.pdf. Zbl

[Otal and Rosas 2009] J.-P. Otal and E. Rosas, "Pour toute surface hyperbolique de genre g, $\lambda_{2g-2} > 1/4$", *Duke Math. J.* **150**:1 (2009), 101–115. MR Zbl

[Randol 1974] B. Randol, "Small eigenvalues of the Laplace operator on compact Riemann surfaces", *Bull. Amer. Math. Soc.* **80** (1974), 996–1000. MR Zbl

[Schoen et al. 1980] R. Schoen, S. Wolpert, and S. T. Yau, "Geometric bounds on the low eigenvalues of a compact surface", pp. 279–285 in *Geometry of the Laplace operator* (Honolulu, HI, 1979), edited by R. Osserman and A. Weinstein, Proc. Sympos. Pure Math. **36**, Amer. Math. Soc., Providence, RI, 1980. MR Zbl

[Selberg 1965] A. Selberg, "On the estimation of Fourier coefficients of modular forms", pp. 1–15 in *Theory of numbers* (Pasadena, CA, 1963), edited by A. L. Whiteman, Proc. Sympos. Pure Math. **8**, Amer. Math. Soc., Providence, RI, 1965. MR Zbl

[Wolpert 1987] S. A. Wolpert, "Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces", *Comm. Math. Phys.* **112**:2 (1987), 283–315. MR Zbl

[Wolpert 1990] S. A. Wolpert, "The hyperbolic metric and the geometry of the universal curve", *J. Differential Geom.* **31**:2 (1990), 417–472. MR Zbl

[Wolpert 1992a] S. A. Wolpert, "Spectral limits for hyperbolic surfaces, I", *Invent. Math.* **108**:1 (1992), 67–89. MR Zbl

[Wolpert 1992b] S. A. Wolpert, "Spectral limits for hyperbolic surfaces, II", *Invent. Math.* **108**:1 (1992), 91–129. MR Zbl

[Wu and Xue 2022a] Y. Wu and Y. Xue, "Optimal lower bounds for first eigenvalues of Riemann surfaces for large genus", *Amer. J. Math.* **144**:4 (2022), 1087–1114. MR Zbl

[Wu and Xue 2022b] Y. Wu and Y. Xue, "Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16}-\epsilon$", *Geom. Funct. Anal.* **32**:2 (2022), 340–410. MR Zbl

[Wu and Xue 2022c] Y. Wu and Y. Xue, "Small eigenvalues of closed Riemann surfaces for large genus", *Trans. Amer. Math. Soc.* **375**:5 (2022), 3641–3663. MR Zbl

Received 25 Jan 2022. Revised 29 Jul 2022. Accepted 23 Sep 2022.

YUNHUI WU: yunhui_wu@tsinghua.edu.cn
Yau Mathematical Sciences Center, Tsinghua University, Beijing, China

HAOHAO ZHANG: zhh21@mails.tsinghua.edu.cn
Tsinghua University, Beijing, China

XUWEN ZHU: x.zhu@northeastern.edu
Department of Mathematics, Northeastern University, Boston, MA, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2024 is US $440/year for the electronic version, and $690/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 17 No. 4 2024