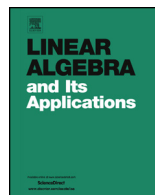




Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laaMaximum spread of $K_{2,t}$ -minor-free graphsWilliam Linz^{a,*}, Linyuan Lu^{a,1}, Zhiyu Wang^b^a University of South Carolina, 1523 Greene St., Columbia, SC, 29208, USA^b Georgia Institute of Technology, 686 Cherry St. NW, Atlanta, GA, 30332, USA

ARTICLE INFO

Article history:

Received 27 January 2023

Received in revised form 20 July 2023

Accepted 21 July 2023

Available online 27 July 2023

Submitted by M. Tait

MSC:

05C50

05C35

Keywords:

Spread

Minor-free graphs

Spectral Turán-type problem

ABSTRACT

The spread of a graph G is the difference between the largest and smallest eigenvalues of the adjacency matrix of G . In this paper, we consider the family of graphs which contain no $K_{2,t}$ -minor. We show that for any $t \geq 2$, there is an integer ξ_t such that the maximum spread of an n -vertex $K_{2,t}$ -minor-free graph is achieved by the graph obtained by joining a vertex to the disjoint union of $\lfloor \frac{2n+\xi_t}{3t} \rfloor$ copies of K_t and $n - 1 - t \lfloor \frac{2n+\xi_t}{3t} \rfloor$ isolated vertices. The extremal graph is unique, except when $t \equiv 4 \pmod{12}$ and $\frac{2n+\xi_t}{3t}$ is an integer, in which case the other extremal graph is the graph obtained by joining a vertex to the disjoint union of $\lfloor \frac{2n+\xi_t}{3t} \rfloor - 1$ copies of K_t and $n - 1 - t(\lfloor \frac{2n+\xi_t}{3t} \rfloor - 1)$ isolated vertices. Furthermore, we give an explicit formula for ξ_t .

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1. Introduction

Given a square matrix M , the *spread* of M , denoted by $S(M)$, is defined as $S(M) := \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of M . In other

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words, $S(M)$ is the diameter of the spectrum of M . Given a graph $G = (V, E)$ on n vertices, the *spread* of G , denoted by $S(G)$, is defined as the spread of the adjacency matrix $A(G)$ of G . Let $\lambda_1(G) \geq \dots \geq \lambda_n(G)$ be the eigenvalues of $A(G)$. Here λ_1 is called the *spectral radius* of G . Since $A(G)$ is a real symmetric matrix, we have that the λ_i s are all real numbers. Thus $S(G) = \lambda_1 - \lambda_n$.

The systematic study of the spread of graphs was initiated by Gregory, Hershkovitz, and Kirkland [10]. One of the central focuses of this area is to find the maximum or minimum spread over a fixed family of graphs and characterize the extremal graphs. Problems of such extremal flavor have been investigated for trees [1], graphs with few cycles [8,16,23], the family of all n -vertex graphs [2,3,17–19,21], the family of bipartite graphs [3], graphs with a given matching number [12], girth [22], or size [11], and very recently for the families of outerplanar graphs [9,13] and planar graphs [13]. We note that the spreads of other matrices associated with a graph have also been extensively studied (see e.g. references in [9,5,7]).

Given two graphs G and H , the *join* of G and H , denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H by connecting every vertex of G with every vertex of H . Let P_k denote the path on k vertices. Given two graphs G and H , let $G \cup H$ denote the disjoint union of G and H . Given a graph G and a positive integer k , we use kG to denote the disjoint union of k copies of G . Given $v \subseteq V(G)$, let $N_G(v)$ denote the set of neighbors of v in G , and let $d_G(v)$ denote the degree of v in G , i.e., $d_G(v) = |N(v)|$. Given $S \subseteq V(G)$, define $N_G(S)$ as $N_G(S) = \cup_{v \in S} (N_G(v) \setminus S)$. Given a graph G and disjoint vertex subsets $S, T \subseteq V(G)$, we use $E_G(S)$ to denote the set of edges in $E(G[S])$, and use $E_G(S, T)$ to denote the set of edges with one endpoint in S and the other endpoint in T . For all above definitions, we may omit the subscript G when there is no ambiguity. A graph H is called a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A graph G is called *H -minor-free* if H is not a minor of G .

There has been extensive work on finding the maximum spectral radius of $K_{s,t}$ -minor-free graphs. Nikiforov [15] showed that every sufficiently large n -vertex $K_{2,t}$ -minor-free graph G satisfies $\lambda_1(G) \leq (t-1)/2 + \sqrt{n + (t^2 - 2t - 3)/4}$, with equality if and only if $n \equiv 1 \pmod{t}$ and G is $K_1 \vee \lfloor n/t \rfloor K_t$. Tait [20] extended Nikiforov's result to $K_{s,t}$ -minor-free graphs by giving an upper bound on the maximum spectral radius of a sufficiently large n -vertex $K_{s,t}$ -minor-free graph G , and showed that the upper bound is tight if and only if $n \equiv s-1 \pmod{t}$ and G is $K_{s-1} \vee \lfloor (n-s+1)/t \rfloor K_t$. In the same paper, Tait conjectured that for all $t \geq s \geq 2$, the maximum spectral radius of a sufficiently large n -vertex $K_{s,t}$ -minor-free graph is attained by $K_{s-1} \vee (pK_t \cup K_q)$, where p, q satisfy that $n-s+1 = pt+q$ and $q \in [t]$. Very recently, the $K_{s,t}$ -minor-free graphs with maximum spectral radius were determined for $t \geq s \geq 2$ by Zhai and Lin [24].

In this paper, we determine the maximum-spread $K_{2,t}$ -minor-free graphs on n vertices for sufficiently large n and for all $t \geq 2$.

Table 1The values of ξ_t for $2 \leq t \leq 20$.

t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ξ_t	0	1	2	1	0	0	-2	-3	-6	-8	-12	-15	-20	-24	-28	-34	-40	-46	-54

Theorem 1. For $t \geq 2$ and n sufficiently large, the graph that maximizes the spread over the family of $K_{2,t}$ -minor-free graphs on n vertices is

$$K_1 \vee \left(\left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor K_t \cup \left(n - 1 - t \left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor \right) P_1 \right)$$

where

$$\xi_t = \begin{cases} 2 \left\lfloor \frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right\rfloor & \text{if } t \text{ is even,} \\ \left\lfloor \frac{3t}{2} - 2 - \frac{2(t-1)^2}{9} \right\rfloor & \text{if } t \geq 3, \text{ and } t \text{ is odd.} \end{cases}$$

The extremal graph is unique unless $t \equiv 4 \pmod{12}$ and $\frac{2n + \xi_t}{3t}$ is an integer. In this special case, the maximum spread is achieved by two extremal graphs

$$K_1 \vee \left(\left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor K_t \cup \left(n - 1 - t \left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor \right) P_1 \right)$$

and

$$K_1 \vee \left(\left(\left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor - 1 \right) K_t \cup \left(n - 1 - t \left(\left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor - 1 \right) \right) P_1 \right).$$

We give a list of values of ξ_t for small t in Table 1.

Our paper is organized as follows. In Section 2, we recall some useful lemmas and prove that in any maximum-spread $K_{2,t}$ -minor-free graph G , there is a vertex u_0 which is adjacent to all other vertices in G . In Section 3, we show that $G - u_0$ is a disjoint union of cliques on t vertices and isolated vertices and complete the proof of Theorem 1.

2. Notations and lemmas

We first recall a result of Chudnovsky, Reed and Seymour [6] on the maximum number of edges of a $K_{2,t}$ -minor-free graph, which extends an earlier result of Myers [14].

Theorem 2. [6] Let $t \geq 2$ be a positive integer, and G be a graph on $n > 0$ vertices with no $K_{2,t}$ minor. Then

$$|E(G)| \leq \frac{1}{2}(t+1)(n-1).$$

Let G be a graph which attains the maximum spread among all n -vertex $K_{2,t}$ -minor-free graphs. As a first step towards proving Theorem 1, we want to show that G must contain a vertex of degree $n - 1$.

Recall the result of Nikiforov [15] on the maximum spectral radius of $K_{2,t}$ -minor-free graphs.

Theorem 3. [15] *Let $t \geq 3$ and G be a graph of order n with no $K_{2,t}$ minor. If $n \geq 400t^6$, then the spectral radius $\lambda_1(G)$ satisfies*

$$\lambda_1(G) \leq \frac{t-1}{2} + \sqrt{n + \frac{t^2 - 2t - 3}{4}},$$

with equality if and only if $n \equiv 1 \pmod{t}$ and $G = K_1 \vee \lfloor n/t \rfloor K_t$.

We first give some upper and lower bounds on $\lambda_1(G)$ and $|\lambda_n(G)|$ when n is sufficiently large. We use known expressions for the eigenvalues of a join of two regular graphs [4, pg.19].

Lemma 1. [4] *Let G and H be regular graphs with degrees k and ℓ respectively. Suppose that $|V(G)| = m$ and $|V(H)| = n$. Then, the characteristic polynomial of $G \vee H$ is $p_{G \vee H}(t) = ((t-k)(t-\ell) - mn) \frac{p_G(t)p_H(t)}{(t-k)(t-\ell)}$. In particular, if the eigenvalues of G are $k = \lambda_1 \geq \dots \geq \lambda_m$ and the eigenvalues of H are $\ell = \mu_1 \geq \dots \geq \mu_n$, then the eigenvalues of $G \vee H$ are $\{\lambda_i : 2 \leq i \leq m\} \cup \{\mu_j : 2 \leq j \leq n\} \cup \{x : (x-k)(x-\ell) - mn = 0\}$.*

We will apply Lemma 1 to the graph $K_1 \vee qK_t$ to obtain a lower bound on $S(G)$.

Lemma 2. *Let G be a graph which attains the maximum spread among all n -vertex $K_{2,t}$ -minor-free graphs. Then*

$$\sqrt{n-1} - \frac{t-1}{2} - O\left(\frac{1}{\sqrt{n}}\right) \leq -\lambda_n(G) \leq \lambda_1(G) \leq \sqrt{n-1} + \frac{t-1}{2} + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. The upper bound of $\lambda_1(G)$ is due to Theorem 3. Now let us prove the lower bound. We will compute $S(K_1 \vee qK_t)$, where $q = \lfloor (n-1)/t \rfloor$. Note that $K_1 \vee qK_t$ is $K_{2,t}$ -minor-free. Hence, we can lower bound $S(G)$ by $S(K_1 \vee qK_t)$. By Lemma 1, both $\lambda_1(K_1 \vee qK_t)$ and $\lambda_n(K_1 \vee qK_t)$ satisfy the equation

$$\lambda(\lambda - (t-1)) - qt = 0.$$

Thus, we have

$$\begin{aligned} \lambda_1(K_1 \vee qK_t) &= \frac{t-1}{2} + \sqrt{qt + \frac{t^2 - 2t + 1}{4}}, \\ \lambda_n(K_1 \vee qK_t) &= \frac{t-1}{2} - \sqrt{qt + \frac{t^2 - 2t + 1}{4}}. \end{aligned}$$

Thus $S(K_1 \vee qK_t) = \sqrt{4qt + t^2 - 2t + 1}$. Since $q = \lfloor (n-1)/t \rfloor$, we then have

$$\begin{aligned} S(G) &\geq \sqrt{4qt + t^2 - 2t + 1} \geq \sqrt{4(n-t) + t^2 - 2t + 1} = \sqrt{4n + t^2 - 6t + 1} \\ &= 2\sqrt{n-1} + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} -\lambda_n(G) &= S(G) - \lambda_1(G) \\ &\geq 2\sqrt{n-1} + O\left(\frac{1}{\sqrt{n}}\right) - \left(\sqrt{n-1} + \frac{t-1}{2} + O\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \sqrt{n-1} - \frac{t-1}{2} - O\left(\frac{1}{\sqrt{n}}\right). \quad \square \end{aligned}$$

For the rest of this paper, let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix $A(G)$ of G . Given a vector $\mathbf{w} \in \mathbb{R}^n$, let $\mathbf{w}'$ denotes its transpose, and for each $i \in [n]$, let $\mathbf{w}_i$ denote the i -th coordinate of $\mathbf{w}$. Using the Rayleigh quotient of symmetric matrices, we have the following equalities for λ_1 and λ_n :

$$\lambda_1 = \max_{\substack{\mathbf{w} \in \mathbb{R}^n \\ \mathbf{w} \neq 0}} \frac{\mathbf{w}' A(G) \mathbf{w}}{\mathbf{w}' \mathbf{w}} = \max_{\substack{\mathbf{w} \in \mathbb{R}^n \\ \mathbf{w} \neq 0}} \frac{2 \sum_{ij \in E(G)} \mathbf{w}_i \mathbf{w}_j}{\mathbf{w}' \mathbf{w}}, \quad (1)$$

$$\lambda_n = \min_{\substack{\mathbf{w} \in \mathbb{R}^n \\ \mathbf{w} \neq 0}} \frac{\mathbf{w}' A(G) \mathbf{w}}{\mathbf{w}' \mathbf{w}} = \min_{\substack{\mathbf{w} \in \mathbb{R}^n \\ \mathbf{w} \neq 0}} \frac{2 \sum_{ij \in E(G)} \mathbf{w}_i \mathbf{w}_j}{\mathbf{w}' \mathbf{w}}. \quad (2)$$

Let $\mathbf{x}$ and $\mathbf{z}$ be the eigenvectors of $A(G)$ corresponding to the eigenvalues λ_1 and λ_n respectively. For convenience, let $\mathbf{x}$ and $\mathbf{z}$ be indexed by the vertices of G . By the Perron-Frobenius theorem, we may assume that all entries of $\mathbf{x}$ are positive. We also assume that $\mathbf{x}$ and $\mathbf{z}$ are normalized so that the maximum absolute values of the entries of $\mathbf{x}$ and $\mathbf{z}$ are equal to 1, and so there are vertices u_0 and w_0 with $\mathbf{x}_{u_0} = \mathbf{z}_{w_0} = 1$.

Let $V_+ = \{v: \mathbf{z}_v > 0\}$, $V_0 = \{v: \mathbf{z}_v = 0\}$, and $V_- = \{v: \mathbf{z}_v < 0\}$. Since $\mathbf{z}$ is a non-zero vector, at least one of V_+ and V_- is non-empty. By considering the eigen-equations of $\lambda_n \sum_{v \in V_+} \mathbf{z}_v$ or $\lambda_n \sum_{v \in V_-} \mathbf{z}_v$, we obtain that both V_+ and V_- are non-empty. For any vertex subset S , we define the *volume* of S , denoted by $\text{Vol}(S)$, as $\text{Vol}(S) = \sum_{v \in S} |\mathbf{z}_v|$. In the following lemmas, we use the bounds of λ_n to deduce some information on V_+ , V_- and V_0 .

Lemma 3. *We have*

$$\text{Vol}(V(G)) = O(\sqrt{n}).$$

Proof. For any vertex $v \in V(G)$, we have

$$d(v) \geq \left| \sum_{y \in N(v)} z_y \right| = |\lambda_n| |z_v|.$$

Applying Theorem 2, we have

$$(t+1)n \geq \sum_{v \in V} d(v) \geq \sum_{v \in V(G)} |\lambda_n| |z_v| = |\lambda_n| \text{Vol}(V).$$

By Lemma 2, $|\lambda_n| \geq \sqrt{(n-1)} - \frac{t-1}{2} - O\left(\frac{1}{\sqrt{n}}\right)$. We thus have $\text{Vol}(V) = O(\sqrt{n})$. $\square$

Lemma 4. *There exists some constant C_1 such that for all n sufficiently large, we have*

1. $d(w_0) \geq n - C_1 \sqrt{n}$.
2. For any vertex $u \neq w_0$, $d(u) \leq 2C_1 \sqrt{n}$ and $|z_u| = O(\frac{1}{\sqrt{n}})$.

Proof. For any $u \in V_+$, we have

$$|\lambda_n| z_u = -\lambda_n z_u = - \sum_{v \in N(u)} z_v \leq \sum_{v \in N(u) \cap V_-} |z_v|.$$

Therefore, for any $u \in V_+$,

$$\begin{aligned} |\lambda_n|^2 z_u &\leq \sum_{v \in N(u) \cap V_-} |\lambda_n| |z_v| = \sum_{v \in N(u) \cap V_-} \lambda_n z_v \\ &\leq \sum_{v \in N(u) \cap V_-} \sum_{y \in N(v) \cap V_+} z_y \\ &\leq d(u) z_u + \sum_{y \in V_+ \setminus \{u\}} z_y |N(y) \cap N(u) \cap V_-| \\ &\leq d(u) z_u + \sum_{y \in V_+ \setminus \{u\}} z_y (t-1) \quad \text{since } G \text{ is } K_{2,t}\text{-minor-free} \\ &\leq d(u) z_u + (t-1) \text{Vol}(V_+). \end{aligned}$$

Similarly, if $u \in V_-$, we have

$$|\lambda_n|^2 |z_u| \leq d(u) |z_u| + (t-1) \text{Vol}(V_-).$$

Setting $u = w_0$, we get

$$|\lambda_n|^2 - d(w_0) \leq (t-1) \text{Vol}(V) = O(\sqrt{n}).$$

Hence,

$$d(w_0) \geq n - O(\sqrt{n}) \geq n - C_1\sqrt{n}, \text{ for some } C_1 > 0.$$

Now we show $d(u) \leq 2C_1\sqrt{n}$ for any vertex u other than w_0 . Otherwise, if $d(u) \geq 2C_1\sqrt{n}$, then u and w_0 have at least $C_1\sqrt{n} \geq t$ neighbors (when n is sufficiently large). Thus G contains the subgraph $K_{2,t}$, contradicting that G is $K_{2,t}$ -minor-free. It then follows that for all $u \neq w_0$, we have

$$|z_u| \leq \frac{(t-1)\text{Vol}(V)}{|\lambda_n|^2 - d(u)} = O\left(\frac{1}{\sqrt{n}}\right). \quad \square$$

Lemma 5. *We have*

- (i) $u_0 = w_0$.
- (ii) *For any vertex $v \neq w_0$, $\mathbf{x}_v = O\left(\frac{1}{\sqrt{n}}\right)$.*

Proof. We will prove (ii) first. For any $v \in V(G) \setminus \{w_0\}$, we have

$$\begin{aligned} \lambda_1^2 x_v &= \lambda_1 \sum_{s \in N(v)} \mathbf{x}_s \\ &\leq \lambda_1 \left(\mathbf{x}_{w_0} + \sum_{s \in N(v) \setminus \{w_0\}} \mathbf{x}_s \right) \\ &\leq \lambda_1 + \sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s)} \mathbf{x}_t \\ &\leq \lambda_1 + \sum_{s \in N(v) \setminus \{w_0\}} \left(\mathbf{x}_{w_0} + \sum_{t \in N(s) \setminus \{w_0\}} \mathbf{x}_t \right) \\ &\leq \lambda_1 + (2C_1\sqrt{n})\mathbf{x}_{w_0} + \sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s) \setminus \{w_0\}} \mathbf{x}_t \end{aligned} \quad (3)$$

Claim 1. *For any $v \in V(G) \setminus \{w_0\}$, we have $\sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s) \setminus \{w_0\}} \mathbf{x}_t = O(\sqrt{n})$.*

Proof. Observe that

$$\begin{aligned} \sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s) \setminus \{w_0\}} \mathbf{x}_t &\leq \sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s) \setminus \{w_0\}} 1 \\ &= |\{(s, t) \in V(G)^2 : s \in N(v) \setminus \{w_0\}, t \in N(s) \setminus \{w_0\}\}| \\ &\leq 2|E_{G-w_0}(N(v))| + |E_{G-w_0}(N(v), V(G) \setminus N(v))| \\ &\leq 2|E_{G-w_0}(N(v))| + |E_{G-w_0}(N(v), N_G(w_0) \setminus N(v))| + \\ &\quad |E_{G-w_0}(N(v), V(G) \setminus (N_G(w_0) \cup N(v)))| \end{aligned} \quad (4)$$

By Theorem 2 and Lemma 4,

$$2|E_{G-w_0}(N(v))| \leq (t+1)2C_1\sqrt{n}.$$

Since G is $K_{2,t}$ -minor-free, the bipartite graph induced by $E_{G-w_0}(N(v), N_G(w_0) \setminus N(v))$ is $K_{1,t}$ -free. Hence every vertex in $N(v)$ has at most $t-1$ neighbors in $N_G(w_0) \setminus N(v)$. It follows that

$$|E_{G-w_0}(N(v), N_G(w_0) \setminus N(v))| \leq (t-1)|N(v)| \leq 2(t-1)C_1\sqrt{n}.$$

Similarly, every vertex in $V(G) \setminus (N_G(w_0) \cup N(v))$ has at most $t-1$ neighbors in $N(v)$. It follows that

$$|E_{G-w_0}(N(v), V(G) \setminus (N_G(w_0) \cup N(v)))| \leq (t-1)|V(G) \setminus (N_G(w_0) \cup N(v))| \leq (t-1)C_1\sqrt{n}.$$

Hence by (4),

$$\sum_{s \in N(v) \setminus \{w_0\}} \sum_{t \in N(s) \setminus \{w_0\}} \mathbf{x}_t \leq (t+1)2C_1\sqrt{n} + 2(t-1)C_1\sqrt{n} + (t-1)C_1\sqrt{n} = O(\sqrt{n}). \quad \square$$

Now by the claim above and (3), we have that

$$\lambda_1^2 x_v \leq \lambda_1 + 2C_1\sqrt{n} + O(\sqrt{n}) = O(\sqrt{n}).$$

Using the fact that $|\lambda_1| \geq \sqrt{(n-1)} - \frac{t-1}{2} - O\left(\frac{1}{\sqrt{n}}\right)$, we have that

$$x_v = O\left(\frac{1}{\sqrt{n}}\right).$$

It follows that $w_0 = u_0$. $\square$

Lemma 6. *We have that $d(u_0) = n-1$.*

Proof. Suppose for contradiction that $d(u_0) < n-1$. Let $S = V(G) \setminus (N(u_0) \cup \{u_0\})$. Then $S \neq \emptyset$. By Lemma 4, $|S| \leq C_1\sqrt{n}$. Note that $G[S]$ is also $K_{2,t}$ -minor-free. Hence by Theorem 2, $|E(G[S])| \leq \frac{1}{2}(t+1)|S|$. It follows that there exists a vertex $v \in S$ such that $d_S(v) \leq t+1$. Moreover, since G is $K_{2,t}$ -minor-free, we have that $d_{N(u_0)}(v) \leq t-1$. Hence $d_G(v) \leq t+1 + (t-1) = 2t$. Let G' be obtained from G by removing all the edges of G incident with v and adding the edge vu_0 .

We claim that $\lambda_n(G') < \lambda_n(G)$. Indeed, consider the vector $\tilde{\mathbf{z}}$ such that $\tilde{\mathbf{z}}_u = \mathbf{z}_u$ for $u \neq v$ and $\tilde{\mathbf{z}}_v = -|\mathbf{z}_v|$. Then for sufficiently large n , we have

$$\begin{aligned}
\tilde{\mathbf{z}}' A(G') \tilde{\mathbf{z}} &\leq \mathbf{z}' A(G) \mathbf{z} + 2 \sum_{y \sim v} |\mathbf{z}_y \mathbf{z}_v| - 2 |\mathbf{z}_v| z_{u_0} \\
&\leq \mathbf{z}' A(G) \mathbf{z} + 2 \cdot 2t \cdot O\left(\frac{1}{\sqrt{n}}\right) \cdot |\mathbf{z}_v| - 2 |\mathbf{z}_v| \\
&< \mathbf{z}' A(G) \mathbf{z}.
\end{aligned}$$

By the Rayleigh quotient, we have

$$\lambda_n(G') \leq \frac{\tilde{\mathbf{z}}' A(G') \tilde{\mathbf{z}}}{\tilde{\mathbf{z}}' \tilde{\mathbf{z}}} < \frac{\mathbf{z}' A(G) \mathbf{z}}{\mathbf{z}' \mathbf{z}} = \lambda_n(G).$$

Similarly, we claim that $\lambda_1(G') > \lambda_1(G)$. Indeed,

$$\begin{aligned}
\mathbf{x}' A(G') \mathbf{x} &= \mathbf{x}' A(G) \mathbf{x} - 2 \sum_{y \sim v} \mathbf{x}_y \mathbf{x}_v + 2 \mathbf{x}_v \mathbf{x}_{u_0} \\
&\geq \mathbf{x}' \lambda_1(G) \mathbf{x} - 2 \cdot 2t \cdot O\left(\frac{1}{\sqrt{n}}\right) \cdot \mathbf{x}_v + 2 \mathbf{x}_v \\
&> \mathbf{x}' A(G) \mathbf{x}.
\end{aligned}$$

Using the Rayleigh quotient again,

$$\lambda_1(G') \geq \frac{\mathbf{x}' A(G') \mathbf{x}}{\mathbf{x}' \mathbf{x}} > \frac{\mathbf{x}' A(G) \mathbf{x}}{\mathbf{x}' \mathbf{x}} = \lambda_1(G).$$

Therefore, we have $S(G') = \lambda_1(G') - \lambda_n(G') > \lambda_1(G) - \lambda_n(G) = S(G)$, giving a contradiction. $\square$

3. Proof of Theorem 1

By Lemma 6, a maximum-spread $K_{2,t}$ -minor-free graph G has a vertex u_0 with degree $n-1$. Let α be a normalized eigenvector corresponding to an eigenvalue λ of the adjacency matrix of G so that $\alpha_{u_0} = 1$. Let $H = G - u_0$ and A_H be the adjacency matrix of H . Note that H is $K_{1,t}$ -minor-free since G is $K_{2,t}$ -minor-free. Let I denote the identity matrix of dimension $n-1$ and let $\mathbf{1}$ denote the all one vector of dimension $n-1$. Moreover, let $\mathbf{x}$ denote the restriction of α to the vertices of H . The following lemma computes the vector $\mathbf{x}$.

Lemma 7. *We have*

$$\mathbf{x} = \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_H^k \mathbf{1}. \quad (5)$$

Proof. Since H is $K_{1,t}$ -minor-free, the maximum degree of H is at most $t-1$. For sufficiently large n , both $\lambda_1(G)$ and $|\lambda_n(G)|$ are greater than $t-1$. Each vertex $v \neq u_0$

is adjacent to u_0 and $\alpha_{u_0} = 1$. Hence when restricting the coordinates of $A(G)\alpha$ to $V(G) \setminus \{u_0\}$, we have that

$$A_H \mathbf{x} + \mathbf{1} = \lambda \mathbf{x}. \quad (6)$$

It then follows that

$$\begin{aligned} \mathbf{x} &= (\lambda I - A_H)^{-1} \mathbf{1} \\ &= \lambda^{-1} (I - \lambda^{-1} A_H)^{-1} \mathbf{1} \\ &= \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1} A_H)^k \mathbf{1} \\ &= \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_H^k \mathbf{1}. \end{aligned} \quad (7)$$

Here we use the assumption that $|\lambda| > t - 1 \geq \lambda_1(A_H)$ so that the infinite series converges. $\square$

Lemma 8. Both λ_1 and λ_n satisfy the following equation.

$$\lambda^2 = (n - 1) + \sum_{k=1}^{\infty} \lambda^{-k} \mathbf{1}' A_H^k \mathbf{1}. \quad (8)$$

Proof. The eigen-equation at u_0 gives

$$\lambda = \lambda \mathbf{x}_{u_0} = \sum_{v \in V(H)} \mathbf{x}_v. \quad (9)$$

Applying Lemma 7, we get

$$\begin{aligned} \sum_{v \in V(H)} \mathbf{x}_v &= \mathbf{1}' \cdot \mathbf{x} \\ &= \mathbf{1}' \cdot \sum_{k=0}^{\infty} \lambda^{-(k+1)} A_H^k \mathbf{1} \\ &= \sum_{k=0}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_H^k \mathbf{1}. \end{aligned}$$

Plugging it to Equation (9), we have

$$\lambda = (n - 1) \frac{1}{\lambda} + \sum_{k=1}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_H^k \mathbf{1}. \quad (10)$$

Multiplying by λ on both sides, we get Equation (8). $\square$

For $k = 1, 2, 3, \dots$, let $a_k = \mathbf{1}' A_H^k \mathbf{1}$. In particular, $a_1 = \mathbf{1}' A_H \mathbf{1} = \sum_{v \in V(H)} d_H(v) = 2|E(H)|$; $a_2 = \mathbf{1}' A_H^2 \mathbf{1} = \sum_{v \in V(H)} d_H(v)^2$.

Lemma 9. *We have the following estimation of the spread of G :*

$$S(G) = 2\sqrt{n-1} + \frac{2c_2}{\sqrt{n-1}} + \frac{2c_4}{(n-1)^{3/2}} + \frac{2c_6}{(n-1)^{5/2}} + O\left(n^{-7/2}\right). \quad (11)$$

Here

$$c_2 = -\frac{3}{8} \left(\frac{a_1}{n-1} \right)^2 + \frac{1}{2} \frac{a_2}{n-1}, \quad (12)$$

$$c_4 = -\frac{105}{128} \left(\frac{a_1}{n-1} \right)^4 + \frac{35}{16} \left(\frac{a_1}{n-1} \right)^2 \frac{a_2}{n-1} - \frac{5}{8} \left(\frac{a_2}{n-1} \right)^2 - \frac{5}{4} \frac{a_1}{n-1} \frac{a_3}{n-1} + \frac{1}{2} \frac{a_4}{n-1} \quad (13)$$

$$c_6 = -\frac{3003}{1024} \left(\frac{a_1}{n-1} \right)^6 + \frac{3003}{256} \left(\frac{a_1}{n-1} \right)^4 \frac{a_2}{n-1} - \frac{693}{64} \left(\frac{a_1}{n-1} \right)^2 \left(\frac{a_2}{n-1} \right)^2 + \frac{21}{16} \left(\frac{a_2}{n-1} \right)^3 - \frac{21}{32} \left(11 \left(\frac{a_1}{n-1} \right)^3 - 12 \left(\frac{a_1}{n-1} \right) \left(\frac{a_2}{n-1} \right) \right) \left(\frac{a_3}{n-1} \right) - \frac{7}{8} \left(\frac{a_3}{n-1} \right)^2 + \frac{7}{16} \left(9 \left(\frac{a_1}{n-1} \right)^2 - 4 \frac{a_2}{n-1} \right) \frac{a_4}{n-1} - \frac{7}{4} \frac{a_1}{n-1} \frac{a_5}{n-1} + \frac{1}{2} \frac{a_6}{n-1}. \quad (14)$$

Proof. Recall that by (8), we have that for $\lambda \in \{\lambda_1, \lambda_n\}$,

$$\lambda = (n-1) \frac{1}{\lambda} + \sum_{k=1}^{\infty} \lambda^{-(k+1)} \mathbf{1}' A_H^k \mathbf{1}.$$

Multiplying by λ on both sides, we have that

$$\lambda^2 = (n-1) + \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}. \quad (15)$$

By similar logic in the main lemma of the appendix in [13], λ has the following series expansion:

$$\lambda_1 = \sqrt{(n-1)} + c_1 + \frac{c_2}{\sqrt{n-1}} + \frac{c_3}{n-1} + \frac{c_4}{(n-1)^{\frac{3}{2}}} + \frac{c_5}{(n-1)^2} + \frac{c_6}{(n-1)^{\frac{5}{2}}} + O\left(n^{-7/2}\right).$$

Similarly,

$$\lambda_n = -\sqrt{(n-1)} + c_1 - \frac{c_2}{\sqrt{n-1}} + \frac{c_3}{n-1} - \frac{c_4}{(n-1)^{\frac{3}{2}}} + \frac{c_5}{(n-1)^2} - \frac{c_6}{(n-1)^{\frac{5}{2}}} + O\left(n^{-7/2}\right).$$

Using SageMath (computation available at https://github.com/wzy3210/graph_spreads), we get that c_2, c_4, c_6 are the values in Equations (12), (13), (14) respectively. It follows that

$$S(G) = \lambda_1 - \lambda_n = 2\sqrt{(n-1)} + \frac{2c_2}{\sqrt{n-1}} + \frac{2c_4}{(n-1)^{\frac{3}{2}}} + \frac{2c_6}{(n-1)^{\frac{5}{2}}} + O\left(n^{-7/2}\right). \quad \square$$

Lemma 10. *For sufficiently large n , a maximum-spread $K_{2,t}$ -minor-free n -vertex graph G must be of the form*

$$K_1 \vee (\ell K_t \cap (n-1-\ell t)P_1).$$

Proof of Lemma 10. By Lemma 6, there exists a vertex $u_0 \in V(G)$ of degree $n-1$. Let $H = G - u_0$. Since G is $K_{2,t}$ -minor-free, every vertex in H has at most $t-1$ neighbors in H . Thus $\Delta(H) \leq t-1$, and it follows that

$$a_2 = \mathbf{1}' A_H^2 \mathbf{1} = \sum_{v \in V(H)} d_H(v)^2 \leq (t-1) \sum_{v \in V(H)} d_H(v) = (t-1)a_1.$$

Note that

$$a_1 = 2|E(H)| \leq \Delta(H)|V(H)| \leq (t-1)(n-1).$$

It follows that $a_i \leq (t-1)^i(n-1)$ for all $i \geq 2$. By Lemma 9, we have the following estimation of the spread of G :

$$S(G) = 2\sqrt{n-1} + \frac{2c_2}{\sqrt{n-1}} + \frac{2c_4}{(n-1)^{3/2}} + \frac{2c_6}{(n-1)^{5/2}} + O\left(n^{-7/2}\right), \quad (16)$$

where c_2, c_4, c_6 are computed in Lemma 9, and all c_i s are bounded by constants depending on t . Note

$$\begin{aligned} c_2 &= -\frac{3}{8} \left(\frac{a_1}{n-1} \right)^2 + \frac{1}{2} \frac{a_2}{n-1} \\ &\leq -\frac{3}{8} \left(\frac{a_1}{n-1} \right)^2 + \frac{1}{2} \frac{(t-1)a_1}{n-1} \\ &= \frac{(t-1)^2}{6} - \frac{3}{8} \left(\frac{a_1}{n-1} - \frac{2}{3}(t-1) \right)^2 \\ &\leq \frac{(t-1)^2}{6}, \end{aligned}$$

where in the last inequality, the equality is only achieved when $a_1 = \frac{2}{3}(t-1)(n-1)$. For $G_0 = K_1 \vee \left(\left\lfloor \frac{2n+\xi_t}{3t} \right\rfloor K_t \cup \left(n-1-t \left\lfloor \frac{2n+\xi_t}{3t} \right\rfloor \right) P_1 \right)$, we have $\frac{a_1}{n-1} = \frac{2}{3}(t-1) + O\left(\frac{1}{n}\right)$. Thus

$$S(G_0) = 2\sqrt{n-1} + \frac{(t-1)^2}{3\sqrt{n-1}} + O\left(\frac{1}{n^{3/2}}\right).$$

Claim 2. *There exists a constant $C > 0$ such that the value of a_1 that maximizes $S(G)$ lies in the interval $(\frac{2}{3}(t-1)(n-1) - Cn^{1/2}, \frac{2}{3}(t-1)(n-1) + Cn^{1/2})$.*

Proof. Let C be a sufficiently large constant chosen later. Suppose for contradiction that a_1 is not contained in the interval above. Then, we must have that

$$c_2 \leq \frac{(t-1)^2}{6} - \frac{3C^2n}{8(n-1)^2}.$$

This implies that

$$S(G) \leq 2\sqrt{n-1} + 2 \cdot \frac{\frac{(t-1)^2}{6} - \frac{3C^2n}{8(n-1)^2}}{\sqrt{n-1}} + O\left(\frac{1}{(n-1)^{3/2}}\right) < S(G_0),$$

when C is chosen to be large enough such that

$$-\frac{2 \cdot 3C^2}{8(n-1)^2} \frac{n}{\sqrt{n-1}} + O\left(\frac{1}{(n-1)^{3/2}}\right) < 0.$$

This gives us a contradiction since G is assumed to be an extremal graph that maximizes the spread over all $K_{2,t}$ -minor-free graphs. $\square$

From now on, we assume that $a_1 \in (\frac{2}{3}(t-1)(n-1) - Cn^{1/2}, \frac{2}{3}(t-1)(n-1) + Cn^{1/2})$ for some constant $C > 0$.

Claim 3. *There is a constant C_2 such that the value of a_2 lies in the interval $[(t-1)a_1 - C_2, (t-1)a_1]$.*

Proof. Let C_2 be a sufficiently large constant chosen later. Suppose for contradiction that $a_2 < (t-1)a_1 - C_2$. We then have that

$$S(G) \leq 2\sqrt{n-1} + \frac{(t-1)^2}{3\sqrt{n-1}} - \frac{C_2}{(n-1)^{3/2}} + O\left(\frac{1}{n^{3/2}}\right) < S(G_0),$$

if we choose C_2 large enough, giving a contradiction. $\square$

Claim 4. *For $i \geq 2$, we have $a_i \in [(t-1)^{i-1}(a_1 - (i-1)C_2), a_1(t-1)^{i-1}]$.*

Proof. We will show this claim by inducting on $i \geq 2$. Note that by Claim 3, we have that $a_2 \geq (t-1)a_1 - C_2$. Moreover, $a_2 \leq (t-1)a_1$ since $\Delta(H) \leq t-1$. Hence the base case holds. Moreover, we also obtain from above that $C_2 \geq (t-1)a_1 - a_2$.

Let H' be the set of vertices in H such that its degree is in the interval $[1, t-2]$. We have

$$C_2 \geq (t-1)a_1 - a_2 = \sum_{v \in H'} (t-1-d(v))d(v) \geq (t-2)|H'|.$$

This implies

$$|H'| \leq \frac{C_2}{t-2}.$$

For a vertex $v \in H$ and non-negative integer k , let $w_k(v)$ denote the number of walks of length k in H starting at v . Observe that

$$\begin{aligned} (t-1)a_{i-1} - a_i &= (t-1) \sum_{v \in V(H)} w_{i-1}(v) - \sum_{v \in V(H)} w_i(v) \\ &\leq \sum_{v \in H'} ((t-1) - d_H(v)) (t-1)^{i-1} \\ &\leq |H'| (t-2)(t-1)^{i-1} \\ &\leq C_2 (t-1)^{i-1} \end{aligned}$$

Thus,

$$\begin{aligned} a_i &\geq (t-1)a_{i-1} - C_2(t-1)^{i-1} \\ &\geq (t-1)((t-1)a_{i-2} - C_2(t-1)^{i-2}) - C_2(t-1)^{i-1} && \text{by induction} \\ &= (t-1)^2 a_{i-2} - 2C_2(t-1)^{i-1} \\ &\geq (t-1)^{i-1} a_1 - (i-1)C_2(t-1)^{i-1}, \end{aligned}$$

where the last inequality is obtained by repeatedly applying induction. $\square$

Claim 5. $a_2 = (t-1)a_1$.

Proof. Assume that $a_1 = \frac{2}{3}(t-1)(n-1) + A$, and $a_2 = (t-1)a_1 - B$, where $A \in [-Cn^{1/2}, Cn^{1/2}]$ and $0 \leq B \leq C_2$. For $i \geq 2$, let $c_i(G), c_i(G_0)$ denote the c_i values of G and G_0 respectively. Observe that

$$\begin{aligned} c_2(G) &= -\frac{3}{8} \left(\frac{a_1}{n-1} \right)^2 + \frac{1}{2} \frac{a_2}{n-1} \\ &= \frac{(t-1)^2}{6} - \frac{3A^2}{8(n-1)^2} - \frac{B}{2(n-1)}. \end{aligned}$$

It follows that

$$c_2(G) - c_2(G_0) = -\frac{3A^2}{8(n-1)^2} - \frac{B}{2(n-1)} + O(n^{-2}).$$

Moreover, by Claim 4, for all $i \geq 4$, we have that

$$c_i(G) - c_i(G_0) = O(n^{-1/2}).$$

Thus

$$\begin{aligned} S(G) - S(G_0) &= 2 \cdot \frac{c_2(G) - c_2(G_0)}{\sqrt{n-1}} + 2 \cdot \frac{c_4(G) - c_4(G_0)}{(n-1)^{3/2}} + O((n-1)^{-5/2}) \\ &\leq 2 \cdot \frac{O(n^{-2}) - \frac{3A^2}{8(n-1)^2} - \frac{B}{2(n-1)}}{\sqrt{n-1}} + 2 \cdot \frac{O(n^{-1/2})}{(n-1)^{3/2}} + O((n-1)^{-5/2}). \end{aligned}$$

Since $S(G) \geq S(G_0)$, this implies that $A = O(n^{1/4})$, $B = 0$ and thus $a_2 = (t-1)a_1$. $\square$

Claim 6. H is the union of vertex disjoint K_t s and isolated vertices.

Proof. Recall that $a_1 = \mathbf{1}'A_H\mathbf{1} = \sum_{v \in V(H)} d_H(v) = 2|E(H)|$, and $a_2 = \mathbf{1}'A_H^2\mathbf{1} = \sum_{v \in V(H)} d_H(v)^2$. By Claim 5, we have that

$$\sum_{v \in V(H)} d_H(v)^2 = (t-1) \sum_{v \in V(H)} d_H(v).$$

Since $d_H(v) \leq t-1$ for every $v \in V(H)$, it follows that H is the disjoint union of $(t-1)$ -regular graphs and isolated vertices. Let K be an arbitrary non-trivial component of H . We will show that K is a clique on t vertices.

We first claim that for any $u, v \in V(K)$, $N(u) \cap N(v) \neq \emptyset$. Otherwise, pick a shortest path P between u and v in P . Observe that $|V(P) \cap N(u)| = |V(P) \cap N(v)| = 1$. Contract uPv into one vertex x (call the new graph G'). Note that x and $N(u) \cup N(v)$ form a $K_{1,t}$ in G' . Together with u_0 which is adjacent to every vertex in K , we have a $K_{2,t}$ minor in G , giving a contradiction.

Next, we claim that for any $u, v \in V(K)$ with $uv \notin E(K)$, $|N(u) \cap N(v)| \geq t-2$. Otherwise, $|N(u) \setminus N(v)| \geq 2$ and $|N(v) \setminus N(u)| \geq 2$. Similar to before, pick an arbitrary vertex $w \in N(u) \cap N(v)$ and contract the path uwv , we then obtain a $K_{1,t}$ -minor in K , and thus a $K_{2,t}$ -minor in G . Similarly, for any $u, v \in V(K)$ with $uv \in E(K)$, we have $|N(u) \cap N(v)| \geq t-3$. Moreover, note that for any $u, v \in V(K)$, $|N(u) \cap N(v)| \leq t-2$, since otherwise $\{u, v\}$ and $(N(u) \cap N(v)) \cup \{u_0\}$ forms a $K_{2,t}$ in G , giving a contradiction. Hence, we have that for any $u, v \in V(K)$ with $uv \notin E(K)$, $|N(u) \cap N(v)| = t-2$.

Now if K is not a clique on t vertices, then let $u, v \in V(K)$ be two vertices in K such that $uv \notin E(K)$. By the above claim, there exists $u', v' \in V(K)$ such that $u' \in N(u) \setminus N(v)$ and $v' \in N(v) \setminus N(u)$.

We claim that $u'v' \notin E(K)$. Indeed, if $u'v' \in E(K)$, contract $v'u'$ into w' . Then $\{u, v\} \cup (\{w', u_0\} \cup (N(u) \cap N(v)))$ is a $K_{2,t}$ minor in G , giving a contradiction. Now note that since $u'v \notin E(K)$, we have $|N(u') \cap N(v)| = t - 2$. It follows that $N(u') \cap N(v) = N(u) \cap N(v)$. Similarly, $N(v') \cap N(u) = N(u) \cap N(v)$.

We claim that each vertex in $N(u) \cap N(v)$ has exactly one non-neighbor in $N(u) \cap N(v)$. Indeed, let w be an arbitrary vertex in $N(u) \cap N(v)$. Note that w cannot be adjacent to all other vertices in $N(u) \cap N(v)$; otherwise since w is adjacent to u', u , and v' , we then have $d(w) \geq (t - 3) + 3 = t$, contradicting that K is $(t - 1)$ -regular. On the other hand, suppose w has at least two non-neighbors in $N(u) \cap N(v)$. Then it follows that $|N(w) \cap (N(u) \cap N(v))| \leq t - 2 - 3 = t - 5$. Now observe that $N(u) \cap N(w) = (N(w) \cap N(u) \cap N(v)) \cup \{u'\}$. It follows that $|N(u) \cap N(w)| \leq t - 4$, contradicting our claim before that any two adjacent vertices must have at least $t - 3$ common neighbors. Hence w has exactly one non-neighbor in $N(u) \cap N(v)$, say w' . But now observe that

$$N(w) \cap N(w') \supseteq (N(u) \cap N(v) \setminus \{w, w'\}) \cup \{u, u', v, v'\},$$

which implies that $|N(w) \cap N(w')| \geq t - 4 + 4 = t$, contradicting that K is $(t - 1)$ -regular. Hence by contradiction, K is a clique on t vertices. $\square$

This completes the proof of Lemma 10. $\square$

Proof of Theorem 1 . For sufficiently large n , let G be an extremal graph attaining the maximum spread among all n -vertex $K_{2,t}$ -minor-free graphs. By Lemma 10, we only need to consider graphs in the form of $G_\ell = K_1 \vee (\ell K_t \cup (n - 1 - \ell t)P_1)$. It also follows from Lemma 10 that for $i \geq 1$,

$$a_i = \ell t(t - 1)^i. \quad (17)$$

For each $i \geq 2$, let $c_i(\ell)$ denote the c_i value of G_ℓ . Plugging a_i 's into Equations (12), (13), and (14), we get

$$c_2(\ell) = -\frac{3}{8} \frac{t^2(t - 1)^2}{(n - 1)^2} \ell^2 + \frac{1}{2} \frac{t(t - 1)^2}{(n - 1)} \ell, \quad (18)$$

$$c_4(\ell) = -\frac{105}{128} \frac{t^4(t - 1)^4}{(n - 1)^4} \ell^4 + \frac{35}{16} \frac{t^3(t - 1)^4}{(n - 1)^3} \ell^3 - \frac{15}{8} \frac{t^2(t - 1)^4}{(n - 1)^2} \ell^2 + \frac{1}{2} \frac{t(t - 1)^4}{(n - 1)} \ell, \quad (19)$$

$$\begin{aligned} c_6(\ell) = & -\frac{3003}{1024} \frac{t^6(t - 1)^6}{(n - 1)^6} \ell^6 + \frac{3003}{256} \frac{t^5(t - 1)^6}{(n - 1)^5} \ell^5 - \frac{1155}{64} \frac{t^4(t - 1)^6}{(n - 1)^4} \ell^4 \\ & + \frac{105}{8} \frac{t^3(t - 1)^6}{(n - 1)^3} \ell^3 - \frac{35}{8} \frac{t^2(t - 1)^6}{(n - 1)^2} \ell^2 + \frac{1}{2} \frac{t(t - 1)^6}{(n - 1)} \ell. \end{aligned} \quad (20)$$

Let $\ell_1 = \frac{2(n-1)}{3t}$, which is the (possibly real) argmax value of $c_2(\ell)$. Let $\ell_0 = \lfloor \frac{2n+\xi_t}{3t} \rfloor$ be the target maximum integer point of $S(G_\ell)$. By Claim 2, we assume that $\ell \in (\ell_1 - C\sqrt{n-1}, \ell_1 + C\sqrt{n-1})$. Let us compute $S(G_{\ell+1}) - S(G_\ell)$. We have

$$\begin{aligned}
c_2(\ell+1) - c_2(\ell) &= -\frac{3}{8} \frac{t^2(t-1)^2}{(n-1)^2} (2\ell+1) + \frac{1}{2} \frac{t(t-1)^2}{(n-1)}, \\
c_4(\ell+1) - c_4(\ell) &= -\frac{105}{128} \frac{t^4(t-1)^4}{(n-1)^4} (4\ell^3 + 6\ell^2 + 4\ell + 1) + \frac{35}{16} \frac{t^3(t-1)^4}{(n-1)^3} (3\ell^2 + 3\ell + 1) \\
&\quad - \frac{15}{8} \frac{t^2(t-1)^4}{(n-1)^2} (2\ell+1) + \frac{1}{2} \frac{t(t-1)^4}{(n-1)}, \\
c_6(\ell+1) - c_6(\ell) &= O\left(\frac{1}{n-1}\right).
\end{aligned}$$

Plugging $\ell = \ell_1 \cdot \left(1 + O\left(\frac{1}{\sqrt{n-1}}\right)\right)$ into $c_4(\ell+1) - c_4(\ell)$, we have

$$c_4(\ell+1) - c_4(\ell) = -\frac{1}{18} \frac{t(t-1)^4}{n-1} + O\left(\frac{1}{(n-1)^{3/2}}\right).$$

Therefore, we have

$$\begin{aligned}
S(G_{\ell+1}) - S(G_\ell) &= \frac{2(c_2(\ell+1) - c_2(\ell))}{\sqrt{n-1}} + \frac{2(c_4(\ell+1) - c_4(\ell))}{(n-1)^{3/2}} + \frac{2(c_6(\ell+1) - c_6(\ell))}{(n-1)^{5/2}} \\
&\quad + O\left(\frac{1}{(n-1)^3}\right) \\
&= \frac{2t(t-1)^2}{(n-1)^{5/2}} \left(-\frac{3}{8}t(2\ell+1) + \frac{1}{2}(n-1) - \frac{(t-1)^2}{18}\right) + O\left(\frac{1}{(n-1)^3}\right) \\
&= -\frac{3t^2(t-1)^2}{2(n-1)^{5/2}} \left(\ell + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t}\right) + O\left(\frac{1}{(n-1)^3}\right).
\end{aligned} \tag{21}$$

Case a: $t \geq 3$ and t is odd. Recall that in this case we let

$$\ell_0 = \left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor$$

where

$$\xi_t = \left\lfloor \frac{3t}{2} - 2 - \frac{2(t-1)^2}{9} \right\rfloor.$$

For $\ell \geq \ell_0$, we have

$$\begin{aligned}
\ell + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} &\geq \ell_0 + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} \\
&\geq \frac{2n + \xi_t}{3t} - \left(1 - \frac{1}{3t}\right) + \frac{1}{2} - \frac{2(n-1)}{3t} + \frac{2(t-1)^2}{27t}
\end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{3t} \left(\xi_t + 1 - \left(\frac{3t}{2} - 2 - \frac{2(t-1)^2}{9} \right) \right) \\ &> 0. \end{aligned}$$

Plugging it into Equation (21), we have that for $\ell \geq \ell_0$,

$$S(G_{\ell+1}) - S(G_\ell) \leq -\frac{t(t-1)^2 \left(\xi_t + 1 - \left(\frac{3t}{2} - 2 - \frac{2(t-1)^2}{9} \right) \right)}{2(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) < 0.$$

When $\ell \leq \ell_0 - 1$, we have

$$\begin{aligned} \ell + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} &\leq \ell_0 - 1 + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} \\ &\leq \frac{2n + \xi_t}{3t} - 1 + \frac{1}{2} - \frac{2(n-1)}{3t} + \frac{2(t-1)^2}{27t} \\ &\leq \frac{1}{3t} \left(\xi_t - \left(\frac{3t}{2} - 2 - \frac{2(t-1)^2}{9} \right) \right) \\ &< 0. \end{aligned}$$

At the last step, we observe that $\frac{3t}{2} - 2 - \frac{2(t-1)^2}{9}$ is not an integer for odd t . Thus, the inequality is strict. Therefore, for $\ell \leq \ell_0 - 1$,

$$S(G_{\ell+1}) - S(G_\ell) \geq \frac{t(t-1)^2 \left(-\eta_t + \left(\frac{3t}{4} - 2 - \frac{2(t-1)^2}{9} \right) \right)}{2(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) > 0.$$

Therefore, $S(G_\ell)$ reaches the unique maximum at ℓ_0 for sufficiently large n . This completes the case for odd t .

Case b: $t \geq 2$ even. Let

$$\ell_0 = \left\lfloor \frac{n + \eta_t}{3t/2} \right\rfloor$$

where

$$\eta_t = \left\lfloor \frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right\rfloor.$$

For $\ell \geq \ell_0$, we have

$$\begin{aligned} \ell + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} &\geq \ell_0 + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} \\ &\geq \frac{n + \eta_t}{3t/2} - \left(1 - \frac{1}{3t/2} \right) + \frac{1}{2} - \frac{2(n-1)}{3t} + \frac{2(t-1)^2}{27t} \end{aligned}$$

$$\begin{aligned} &\geq \frac{2}{3t} \left(\eta_t + 1 - \left(\frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right) \right) \\ &> 0. \end{aligned}$$

Plugging it into Equation (21), we have that for $\ell \geq \ell_0$,

$$S(G_{\ell+1}) - S(G_\ell) \leq -\frac{t(t-1)^2 \left(\eta_t + 1 - \left(\frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right) \right)}{2(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) < 0.$$

When $\ell \leq \ell_0 - 1$, we have

$$\begin{aligned} \ell + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} &\leq \ell_0 - 1 + \frac{1}{2} - \frac{2}{3t}(n-1) + \frac{2(t-1)^2}{27t} \\ &\leq \frac{\eta_t}{3t/2} - 1 + \frac{1}{2} + \frac{2}{3t} + \frac{2(t-1)^2}{27t} \\ &\leq \frac{2}{3t} \left(\eta_t - \left(\frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right) \right) \\ &\leq 0. \end{aligned} \tag{22}$$

If $\frac{2}{3t}(n + \eta_t)$ is not an integer, we have

$$S(G_{\ell+1}) - S(G_\ell) > \frac{t(t-1)^2 \left(-\eta_t + \left(\frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right) \right)}{2(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) \geq 0.$$

Therefore ℓ_0 is the unique maximum point of $S(G_\ell)$.

Now we assume $\frac{2}{3t}(n + \eta_t)$ is an integer. Observe that $\frac{3t}{4} - 1 - \frac{(t-1)^2}{9}$ is an integer if and only if t is divisible by 4 and $t-1$ is divisible by 3. Therefore, the inequality (22) is strict except for the case when $t \equiv 4 \pmod{12}$ and $\ell = \ell_0 - 1$. It implies that for $t \not\equiv 4 \pmod{12}$,

$$S(G_{\ell+1}) - S(G_\ell) \geq \frac{t(t-1)^2 \left(-\eta_t + \left(\frac{3t}{4} - 1 - \frac{(t-1)^2}{9} \right) \right)}{2(n-1)^{5/2}} + O\left(\frac{1}{(n-1)^3}\right) > 0.$$

Thus for $t \not\equiv 4 \pmod{12}$, $S(G_\ell)$ reaches the unique maximum at $\ell_0 = \left\lfloor \frac{2n + \xi_t}{3t} \right\rfloor$ for sufficiently large n .

Now we consider the remaining case that $t \equiv 4 \pmod{12}$ and $\frac{2}{3t}(n + \eta_t)$ is an integer. In this case, $S(G_\ell)$ can only achieve the maximum at ℓ_0 , or $\ell_0 - 1$, or both. In fact, we claim both of them are maximum points.

Let $k = (t-4)/12$ and $\ell_0 = \frac{2}{3t}(n + \eta_t)$. Note that by our assumption k and ℓ_0 are both integers. Rearranging the terms, we have

$$t = 12k + 4, \quad (23)$$

$$\eta_t = 1 + k - 16k^2, \quad (24)$$

$$n = 6(3k + 1)\ell_0 + 16k^2 - k - 1. \quad (25)$$

Now we compute the spread of G_ℓ where $\ell = \ell_0$ or $\ell_0 - 1$. By Lemma 8, λ_1 and λ_n of G_ℓ satisfies the equation

$$\begin{aligned} \lambda^2 &= (n - 1) + \sum_{k=1}^{\infty} \lambda^{-k} \mathbf{1}' A_H^k \mathbf{1} \\ &= (n - 1) + \sum_{k=1}^{\infty} \lambda^{-k} \ell t (t - 1)^k \\ &= (n - 1) + \ell t \frac{(t - 1)/\lambda}{1 - (t - 1)/\lambda} \\ &= (n - 1) + \frac{\ell t (t - 1)}{\lambda - (t - 1)}. \end{aligned}$$

Simplifying it, we get

$$\lambda^3 - (t - 1)\lambda^2 - (n - 1)\lambda + (t - 1)(n - 1 - \ell t) = 0. \quad (26)$$

Let us define the *spread of a polynomial* ϕ , denoted by $S(\phi)$, as the difference of largest root and the smallest root. Thus, we have

$$S(G_\ell) = S(\phi_\ell),$$

where ϕ_ℓ is defined by the left hand side of Equation (26). Let $\lambda = x + \frac{t-1}{3}$. The cubic equation (26) can be written as

$$x^3 - \frac{1}{3}(n + t^2 - 2t - 2)x + \frac{1}{27}(-27\ell t^2 - 2t^3 + 27\ell t + 18nt + 6t^2 - 18n - 24t + 20) = 0. \quad (27)$$

Now plugging $\ell = \ell_0$, t as in Equation (23), and n as in Equation (25), into Equation (27), we get

$$x^3 - (6(3k + 1)\ell_0 + 64k^2 + 23k + 1)x - (72k^2 + 42k + 6) = 0. \quad (28)$$

Similarly, plugging $\ell = \ell_0 - 1$, t as in Equation (23), and n as in Equation (25), into Equation (27), we get

$$x^3 - (6(3k + 1)\ell_0 + 64k^2 + 23k + 1)x + (72k^2 + 42k + 6) = 0. \quad (29)$$

Let the ϕ_1 (or ϕ_2) denote the cubic polynomial in the left hand of Equation (28) (or Equation (29) respectively). Observe that $\phi_2(x) = -\phi_1(-x)$. If ϕ_1 has three real roots $x_1 \leq x_2 \leq x_3$, then ϕ_2 has three real roots $-x_3 \leq -x_2 \leq -x_1$. Thus

$$S(\phi_1) = x_3 - x_1 = (-x_1) - (-x_3) = S(\phi_2).$$

It then follows that

$$S(G_{\ell_0}) = S(\phi_{\ell_0}) = S(\phi_1) = S(\phi_2) = S(\phi_{\ell_0-1}) = S(G_{\ell_0-1}).$$

Therefore both G_{ℓ_0} and G_{ℓ_0-1} are extremal graphs for this special case. This completes the proof of Theorem 1. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

We thank the anonymous referee for a number of useful comments which improved the presentation of the manuscript.

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