Mather classes and conormal spaces of Schubert varieties in cominuscule spaces

Leonardo C. Mihalcea and Rahul Singh

Abstract

Let G/P be a complex cominuscule flag manifold. We prove a type independent formula for the torus equivariant Mather class of a Schubert variety in G/P, and for a Schubert variety pulled back via the natural projection $G/Q \to G/P$. We apply this to find formulae for the local Euler obstructions of Schubert varieties, and for the torus equivariant localizations of the conormal spaces of these Schubert varieties. We conjecture positivity properties for the local Euler obstructions and for the Schubert expansion of Mather classes, and we prove this conjecture in Lie types A,B, and D. We also conjecture that certain 'Mather polynomials' are unimodal in general Lie type, and log concave in type A.

1. Introduction

Let X be a complex, projective manifold and let $Y \subset X$ be a closed irreducible subvariety. The Mather class $c_{\mathrm{Ma}}(Y)$ is a non-homogeneous element in the (Chow) homology $A_*(X)$. Its original definition uses the Nash blowup of X along Y, but in this paper we work with the following equivalent definition, going back to Sabbah [Sab85]; see also [Gin86, AMSS22b]. Let $T^*(X)$ be the cotangent bundle of X, and let $\iota : X \to T^*(X)$ be the zero section embedding. The multiplicative group $\mathbb{C}^*$ acts on $T^*(X)$ by fibrewise dilation with character $\hbar^{-1}$. To the subvariety Y one associates the conormal space $T^*_Y(X) \subset T^*(X)$; this is an irreducible conic Lagrangian cycle in the cotangent bundle. The Mather class $c_{\mathrm{Ma}}(Y)$ is the dehomogenization of the $\mathbb{C}^*$-equivariant class of the conormal space:

$$c_{\mathrm{Ma}}(Y) := (-1)^{\dim Y}(\iota^*[T^*_Y(X)]_{\mathbb{C}^*})_{\hbar=1} \quad \in A_*(X).$$

For example, it follows from definition that if Y is smooth, then $c_{\mathrm{Ma}}(Y)$ is the push-forward of the homology class $c(T_Y) \cap [Y]$ inside $A_*(X)$. If $Y = X$, the degree 0 term equals the topological Euler characteristic, by the well known index formula:

$$\chi(X) = (-1)^{\dim X} \int_X \iota^*[T^*_X(X)].$$

Equivariant Mather classes were defined by Ohmoto [Ohm06], see Section 3.

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Let G be a complex, semisimple Lie group, and fix $T \subset B \subset P \subset G$ a parabolic subgroup P containing a standard Borel subgroup B with a maximal torus T; let $X = G/P$ be the associated flag manifold. The goal of this paper is to study the T-equivariant Mather class $c_{\mathrm{Ma}}^T(Y) \in A_0^{T\times\mathbb{C}^*}(X)$ for a Schubert variety Y in a cominuscule space G/P, or when Y is the pull-back of a Schubert variety along the natural projection $G/Q \to G/P$. For details on cominuscule spaces, see Section 4.

Let W denote the Weyl group and let W^P be the subset of minimal length representatives. For $w \in W^P$, let $X_w^{P,\circ} := BwP/P$ be the Schubert cell, with closure X_w^P, the Schubert variety in G/P; set $X_w^B := \overline{BwB/B}$, the Schubert variety in G/B.

The Mather class of a Schubert variety is related to Chern-Schwartz-MacPherson (CSM) classes of its Schubert cells via the *local Euler obstruction* coefficients $e_{w,v}$:

$$c_{\mathrm{Ma}}(X_w^P) = \sum_v e_{w,v} c_{\mathrm{SM}}(X_v^{P,\circ}). \tag{1}$$

These coefficients were defined by MacPherson [Mac74] and provide a subtle measure of the singularity of X_w^P at v. For instance, consider the parabolic Kazhdan-Lusztig (KL) polynomial $P_{w,v}(q)$; cf. [Deo87]. Then the equalities

$$e_{w,v} = P_{w,v}(1), \quad \forall v \in W^P, \tag{2}$$

hold if and only if the characteristic cycle of the intersection homology (IH) sheaf of the Schubert variety X_w^P is irreducible; see § 8.1. In general, the problem of finding the decomposition of the characteristic cycle of the IH sheaves into irreducible components is open, although some particular cases are known; see e.g. [KL80, KT84, BFL90, BF97, EM99, Bra02, Wil15], and also Section 8 below for more details. We note that since CSM classes of Schubert cells can be explicitly calculated [AM09, AM16, RV18], equation (1) shows that giving an algorithm to calculate Mather classes is equivalent to one for the local Euler obstructions.

To state our results, we need to introduce more notation. For $w \in W^P$, let $I(w)$ denote the inversion set of w, i.e., the set of positive roots α satisfying $w(\alpha) < 0$. For a root α we denote by $\mathfrak{g}_\alpha$ the root subspace of $Lie(G)$ determined by α and by $\mathbb{C}_\alpha$ the one-dimensional B-module of weight α. It follows from [Sin21] (see also [RSW20]) that if G/P is cominuscule and $w \in W^P$ then the vector space $T_w := \oplus_{\alpha \in I(w)} \mathfrak{g}_{-\alpha}$ has a structure of a B-module. Therefore $\mathcal{T}_w := G \times^B T_w$ is a vector bundle over the complete flag variety G/B. Let $c(\mathcal{T}_w)$ denote its total Chern class. The following is the main result of our paper; see Theorem 5.1 and equation (7) below.

Theorem 1.1. *Let G/P be a cominuscule space, let $\pi : G/B \to G/P$ be the projection, and let $w \in W^P$. Then the following hold:*

(a) The Mather class of X_w^P is given by

$$c_{\mathrm{Ma}}(X_w^P) = \pi_*(c(\mathcal{T}_w) \cap [X_w^B]) = \pi_*(\prod_{\alpha \in I(w)} c(G \times^B \mathbb{C}_{-\alpha}) \cap [X_w^B]).$$

(b) Let $Q \subset P$ be any parabolic subgroup, with $\pi_Q : G/Q \to G/P$ the natural projection. The Mather class of the pull-back Schubert variety $\pi_Q^{-1}(X_w^P)$ is

$$c_{\mathrm{Ma}}(\pi_Q^{-1}(X_w^P)) = c(T_{\pi_Q}) \cap \pi_Q^*(c_{\mathrm{Ma}}(X_w^P)),$$

where T_{π_Q} is the relative tangent bundle of the projection π_Q.

(c) The formulae in (a) and (b) hold in the T-equivariant setting.

We encourage the reader to jump directly to Section 5.2 for examples illustrating the formula in part (a) and its equivariant version.

The proof of part (a) exploits the observation that the $\mathbb{C}^*$-equivariant pull-back $\iota^*[T^*_{X^P_w}(G/P)]$ is essentially given by the Segre class of the conormal space $T^*_{X^P_w}(G/P)$; see Section 2 below. To calculate this Segre class, we utilize a desingularization of the conormal space found by Singh [Sin21], together with the birationality of Segre classes.

A different proof of the part (a) of Theorem 1.1 may be obtained using the identification by Richmond, Slofstra and Woo [RSW20, Thm. 2.1] of the Nash blowup of the Schubert varieties in cominuscule spaces. In fact, after this paper was completed, we learned from D. Anderson, L. Escobar, E. Richmond and A. Woo that in ongoing work they also recovered the formula in Theorem 1.1(a). In this paper we emphasize the equivalence between Mather classes and the Segre classes of the conormal spaces, a point of view which we believe it will have further benefits.

Part (b) follows from the Verdier-Riemann-Roch formula proved by Yokura [Yok99], and from the invariance of Euler obstructions under smooth pull-back; cf. Proposition 3.3. All constructions are T-equivariant, and part (c) follows.

We give two applications of Theorem 1.1: an explicit localization formula for the conormal spaces $T^*_{X^P_w}(G/P)$ of Schubert varieties in cominuscule spaces - see Theorem 7.3; and a formula for the local Euler obstructions of Schubert varieties. The proof uses equation (1) and the identification of the Poincaré duals of CSM classes obtained in [AMSS22b]. The resulting formula is given in Theorem 6.1. We propose the following conjecture; see Conjectures 8.1 and 8.4 below.

Conjecture 1.2 Positivity Conjecture. *Let $X = G/P$ be a cominuscule space.*

(a) Consider the Schubert expansion $c_{\mathrm{Ma}}(X^P_w) = \sum a_{w,v}[X^P_v]$. Then $a_{w,v} \geqslant 0$. A positivity property also holds for the equivariant Mather classes.

(b) The local Euler obstruction coefficients are non-negative, i.e. $e_{w,v} \geqslant 0$.

Using equation (1), and the positivity of the non-equivariant CSM classes of Schubert cells [Huh16, AMSS22b], we see that the non-equivariant positivity of part (a) follows from part (b).

Theorem 1.3. *Conjecture 1.2 holds if X is a cominuscule space of classical Lie type (i.e. a Grassmannian, a maximal Orthogonal Grassmannian, a Lagrangian Grassmannian, or a quadric).*

In types A, B, D, the theorem follows from results by Bressler, Finkelberg, and Lunts [BFL90], and by Boe and Fu [BF97], on the IH characteristic cycles of Schubert varieties. After a version of this note appeared on arχiv, LeVan and Raicu [LR22] utilized the results of Boe and Fu to prove non-negativity of the Euler obstructions for Lagrangian Grassmannians (type C). See § 8 below. (Added in proof: using the formulae obtained in this paper, we wrote a Sage program checking that the IH characteristic cycles of Schubert varieties in a cominuscule Grassmannian of Lie type E_6 and E_7 are irreducible. This implies that the non-equivariant statements in Conjecture 1.2 hold in all the remaining cases; details will be given in the upcoming paper [MS22].)

We also conjecture a unimodality property for the *Mather polynomial* of Schubert varieties X^P_w. The Mather polynomial is obtained from the Schubert expansion of the Mather class by replacing each Schubert class $[X^P_v]$ by $x^{\ell(v)}$. We conjecture that the resulting polynomial is unimodal in general, and, for the ordinary Grassmannians, it is also log concave. See Section 8.3 for details and examples.

Formulas for the Mather classes and for the local Euler obstructions have been found by B. Jones [Jon10] in the case of Grassmannians, and in [Rai16, Zha18, PR22, Tim] for certain degeneracy loci. Jones' proof is based on the fact that if $\pi' : Z_w \to X_w^P$ is a small resolution of X_w^P (in the sense of intersection homology) *and* if the characteristic cycle of the IH sheaf of X_w^P is irreducible, then the Mather class satisfies $c_{\mathrm{Ma}}(X_w^P) = \pi'_*(c(T(Z_w)) \cap [Z_w])$. Small resolutions for Grassmannian Schubert varieties were constructed by Zelevinsky [Zel83]; Bressler, Finkelberg and Lunts [BFL90] proved that the characteristic cycles of the IH sheaves of Schubert varieties are irreducible. Outside the type A Grassmannian, Schubert varieties may not admit small resolutions; see [SV94] and also [Per07, Example 7.15].

Boe and Fu [BF97] used delicate techniques from geometric analysis to find formulae for the local Euler obstructions $e_{w,v}$ of cominuscule Schubert varieties in classical types. Using recursive formulae for the KL polynomials, they showed that the identities (2) hold in Lie types A and D, and fail in general for types B, C. We included Examples 6.3 and 8.8, recovering instances of reducible IH sheaves from [BF97] and [KT84], and obtained with the formulae from this paper. In future work, we plan to compare our formula from Theorem 6.1 to the formulae in [BF97].

We work over $\mathbb{C}$. Throughout, we utilize Chow (co)homology theory [Ful84], and its equivariant version from [EG98]. This is related to the ordinary (equivariant) (co)homology via the cycle map - see [Ful84, Ch. 19] and [EG98, § 2.8]; for flag manifolds this map is an isomorphism [Ful84, Ex. 19.1.11].

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2. Segre classes of cones

The treatment in this section follows [Ful84, § 4], see also [BBM87].

2.1 Segre classes of cones

Let $C = Spec(S^\bullet)$ be a cone over a scheme X, where $S^\bullet$ is a sheaf of graded $\mathcal{O}_X$-algebras, $\mathcal{O}_X \to S^0$ is injective, S^1 is coherent, and $S^\bullet$ is generated by S^1. The projective completion $\mathbb{P}(C \oplus 1\!\!1) := \mathrm{Proj}(S^\bullet[z])$ is equipped with the projection $q : \mathbb{P}(C \oplus 1\!\!1) \to X$ and a line bundle $\mathcal{O}_C(-1)$. There are well defined notions of a subcone, and a pull back of a cone. The *Segre class* of C is

$$s(C) := q_*\left(\frac{[\mathbb{P}(C \oplus 1\!\!1)]}{c(\mathcal{O}_C(-1))}\right) = q_*\left(\sum_{i\geqslant 0} c_1(\mathcal{O}_C(1))^i \cap [\mathbb{P}(C \oplus 1\!\!1)]\right) \quad \in A_*(X). \tag{3}$$

If $C = E$ is a vector bundle over X then its Segre class is $s(E) = c(E)^{-1} \cap [X]$. One of the fundamental properties of the Segre classes is their birational invariance:

Lemma 2.1. *Let $f : X' \to X$ be a proper morphism of varieties, and C, C' irreducible cones over X respectively X', such that C' is a subcone of f^*C. Assume that there is a commutative* *diagram*

$$\begin{array}{ccc} C' & \xrightarrow{g} & C \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

where $g : C' \to C$ is proper and birational. Then $f_*(s(C')) = s(C) \in A_*(X)$.

Similar statements may be found in [BBM87] (without proof), and in [Ful84, Proposition 4.2] (for the *normal cone* of a subvariety). For completeness, we include a proof.

Proof. The morphism f induces a morphism between the projective completions, $\mathbb{P}(f^*(C)\oplus \mathbb{1}) \to \mathbb{P}(C \oplus \mathbb{1})$. Let $G : \mathbb{P}(C' \oplus \mathbb{1}) \to \mathbb{P}(C \oplus \mathbb{1})$ be the restriction, giving the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(C' \oplus \mathbb{1}) & \xrightarrow{G} & \mathbb{P}(C \oplus \mathbb{1}) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{\quad f \quad} & X \end{array}$$

The hypothesis on g implies that G is birational, and the subcone hypothesis implies that $G^*(\mathcal{O}_C(-1)) = \mathcal{O}_{C'}(-1)$. From the definition of the Segre class (equation (3)):

$$\begin{aligned} f_*(s(C')) &= f_* q'_* \left(\frac{[\mathbb{P}(C' \oplus \mathbb{1})]}{c(\mathcal{O}_{C'}(-1))} \right) = q_* G_* \Big(\frac{[\mathbb{P}(C' \oplus \mathbb{1})]}{G^* c(\mathcal{O}_C(-1))} \Big) \\ &= q_* \Big(\frac{[\mathbb{P}(C \oplus \mathbb{1})]}{c(\mathcal{O}_C(-1))} \Big) = s(C). \end{aligned}$$

The third equality uses the projection formula and that G is birational. □

Suppose now that C is an irreducible subcone of a vector bundle E, with $\dim C = \operatorname{rank}(E)$. (Later, C will be the conormal space of a subvariety.) In this case, the Segre class of C is related to the dehomogenization of the $\mathbb{C}^*$-equivariant pull back $\iota^*[C]_{\mathbb{C}^*}$ via the zero section $\iota : X \to E$. We recall the relevant facts next.

There is a $\mathbb{C}^*$-action on E by dilation by a character χ, which extends to an action on $E \oplus \mathbb{1}$ by letting $\mathbb{C}^*$ act trivially on the second component. This induces a $\mathbb{C}^*$-action on the projective completion $\mathbb{P}(E \oplus \mathbb{1})$. Both C and its closure are $\mathbb{C}^*$-stable subschemes, and $\mathbb{C}^*$ acts trivially on the base X. The character χ determines a class in the equivariant Chow group $A^1_{\mathbb{C}^*}(pt)$ of degree 1, denoted in the same way. Since $\mathbb{C}^*$ acts trivially on X, a class $a \in A_0^{\mathbb{C}^*}(X)$ is equivalent to a *non-homogeneous* class $a_0 + a_1 + \ldots \in A_*(X)$ ($a_i \in A_i(X)$) obtained by dehomogenizing a. Conversely, if $a = a_0 + a_1 + \ldots$ is a non-homogeneous class, its *χ-homogenization* is the class $a^\chi := a_0 + a_1\chi + a_2\chi^2 + \ldots \in A_0^{\mathbb{C}^*}(X)$. By [AMSS22b, Proposition 2.7 and Eq. (10)],

$$\iota^*[C]_{\mathbb{C}^*} = (c(E) \cap s(C))^\chi \in A_0^{\mathbb{C}^*}(X). \tag{4}$$

All results extend naturally to the case where X is a variety with an action of a torus T, C is a $T \times \mathbb{C}^*$-invariant cone, and the map $C \to X$ is $T \times \mathbb{C}^*$ equivariant (the $\mathbb{C}^*$ acting trivially on X). For instance, in equation (4), the class $\iota^*[C]_{T\times\mathbb{C}^*}$ belongs to $A_0^{T\times\mathbb{C}^*}(X)$, the $T \times \mathbb{C}^*$ equivariant Chow group.

2.2 Conormal spaces

The cones utilized in this note are the conormal spaces of subvarieties, whose definition we recall next. Let X be a smooth, irreducible, complex algebraic variety, and let $Y \subset X$ be a closed irreducible subvariety. Let $Y^{reg} \subset Y$ be any smooth dense subset. The *conormal space* $T^*_Y X$ is the closure of the conormal bundle $T^*_{Y^{reg}} X$ inside the cotangent bundle T^*X. This is a cone, and also a closed subvariety of dimension $\dim X$, contained in the restriction $T^*X_{|Y}$. In particular, it is stable under the $\mathbb{C}^*$-dilation on the fibres of T^*X, and also under any group G leaving Y and X invariant. If one regards T^*X as a symplectic manifold, then the conormal space is an irreducible conic Lagrangian cycle. In fact, any irreducible, conic Lagrangian cycle is the conormal cone of some subvariety; see [HTT08, Thm. E.6] (where it is attributed to Kashiwara) and also [Ken90, § 1].

3. Mather classes and CSM classes

A question with a long and distinguished history is to define analogues of the total Chern class for singular varieties. The Mather classes and the Chern-Schwartz-MacPherson (CSM) classes are among these classes. We recall their definition here.

Let X be a smooth complex algebraic variety with the action of a torus T, and let $Y \subset X$ be a closed irreducible subvariety.

3.1 Mather classes

The *Mather class* of Y is a homology class $c_{\mathrm{Ma}}(Y) \in A_*(Y)$; if Y is smooth then $c_{\mathrm{Ma}}(Y) = c(TY) \cap [Y]$. Its original definition involves the Nash blowup of Y, but here we utilize an alternate construction, given by Sabbah [Sab85] (see also [Gin86, PP01] and especially [AMSS22b, Cor. 4.5, 3.4]) to define it in terms of the conormal space $T^*_Y X$. For the purposes of this paper, we regard classes in $A_*(Y)$ as classes in $A_*(X)$ via the push-forward induced by the closed embedding $X \hookrightarrow Y$. We work with the equivariant generalization of the Mather class, denoted $c^T_{\mathrm{Ma}}(Y) \in A^T_*(X)$, see [Ohm06].

Theorem 3.1 (cf. [AMSS22b]). *Suppose Y is T-stable; let $\hbar^{-1}$ denote the character of the $\mathbb{C}^*$-dilation action on T^*X. The homogenized Chern-Mather class is:*

$$c^T_{\mathrm{Ma}}(Y)^{\hbar} = (-1)^{\dim Y} \iota^*[T^*_Y X]_{T\times\mathbb{C}^*} \quad \in A_0^{T\times\mathbb{C}^*}(X). \tag{5}$$

It will be convenient to work with a dehomogenized variant of this equation. Recall that by equation (4) above, $\iota^*[T^*_Y(X)]_{T\times\mathbb{C}^*} = (c^T(T^*X) \cap s^T(T^*_Y X))^{-\hbar}$, since the $\mathbb{C}^*$ action is induced by $\hbar^{-1}$. By equation (5) this implies that

$$(-1)^{\dim Y} c^T_{\mathrm{Ma}}(Y)^{-\hbar} = (c^T(T^*X) \cap s^T(T^*_Y(X)))^{\hbar}.$$

After dehomogenizing, i.e. setting $\hbar = 1$, we obtain the expression

$$c^{T,\vee}_{\mathrm{Ma}}(Y) = c^T(T^*X) \cap s^T(T^*_Y(X)), \tag{6}$$

where $c^{T,\vee}_{\mathrm{Ma}}(Y) := ((-1)^{\dim Y} c^T_{\mathrm{Ma}}(Y)^{-\hbar})|_{\hbar=1}$. In other words, the class $c^{T,\vee}_{\mathrm{Ma}}(Y)$ is obtained from $c^T_{\mathrm{Ma}}(Y)$ by changing signs of each homogeneous component according to its cohomological degree. This is called the *dual Chern-Mather class*; it appears naturally in relation to characteristic cycles on the cotangent bundle; cf. [Sab85].

3.2 Chern-Schwartz-MacPherson classes

Let $\{S_i\}$ be a Whitney stratification of X by smooth constructible sets. Such a stratification always exists, see [Ver76, Thm. 2.2] (algebraic context), and [Whi65, Thm. 19.2] (analytic context).

Denote by $\mathcal{F}(X)$ the group of constructible functions on X. Its elements are finite sums of the form $\sum a_i 1\!\!1_{W_i}$, where $a_i \in \mathbb{Z}$, the $W_i \subset X$ are constructible subsets, and $1\!\!1_{W_i}$ is the indicator function. If $f : Z \to X$ is a *proper* morphism, we have a push-forward $f_* : \mathcal{F}(Z) \to \mathcal{F}(X)$, given by $f_*(1\!\!1_W)(x) = \chi(f^{-1}(x) \cap W)$, where χ denotes the topological Euler characteristic. Further, for any morphism $f : Z \to X$, we have a pull-back $f^* : \mathcal{F}(X) \to \mathcal{F}(Z)$, given by $f^*(\varphi)(z) = \varphi(f(z))$.

Proving a conjecture of Grothendieck and Deligne, MacPherson [Mac74] defined a transformation $c_* : \mathcal{F}(X) \to H_*(X)$, satisfying $c_*(1\!\!1_X) = c(T(X)) \cap [X]$ for smooth X; and for proper morphisms $f : Z \to X$, a commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(Z) \arrow[r, "c_*"] \arrow[d, "f_*"] & H_*(Z) \arrow[d, "f_*"] \\
\mathcal{F}(X) \arrow[r, "c_*"] & H_*(X)
\end{tikzcd}$$

If $W \subset X$ is a constructible subset, the class $c_{\mathrm{SM}}(W) := c_*(1\!\!1_W) \in H_*(X)$ is called the *Chern-Schwartz-MacPherson (CSM)* class of W.

MacPherson's definition of the transformation c_* uses Mather classes, and a constructible function Eu_X on X, called the *local Euler obstruction*. The original definition of the local Euler obstruction in [Mac74] uses transcendental methods (the analytic topology). Later, Gonzalez-Sprinberg and Verdier [GS81], found an algebraic definition, thus extending MacPherson's transformation to one with values in the Chow group $A_*(X)$. More recently, Ohmoto [Ohm06] generalized this to the equivariant context. We recall some properties of Eu_X, see [BSS09, Thm. 8.1.1].

Lemma 3.2. *(a) The local Euler obstruction* Eu_X *is constant along the strata of any Whitney stratification.*

(b) $\mathrm{Eu}_X(x) = 1$ *if X is nonsingular at x.*

(c) If $X = X_1 \times X_2$ as analytic varieties, then $\mathrm{Eu}_{X_1 \times X_2}(x_1, x_2) = \mathrm{Eu}_{X_1}(x_1) \cdot \mathrm{Eu}_{X_2}(x_2)$.

We could not find a precise reference for the Proposition below, although we believe it to be known to experts.

Proposition 3.3. *Let $f : Z \to X$ be a smooth morphism of nonsingular complex varieties, and let $Y \subset X$ be a closed subvariety. Then for any $z \in f^{-1}(Y)$, we have* $\mathrm{Eu}_{f^{-1}(Y)}(z) = \mathrm{Eu}_Y(f(z))$, *i.e. as constructible functions* $f^* \mathrm{Eu}_Y = \mathrm{Eu}_{f^{-1}(Y)}$.

Proof. Let $z \in f^{-1}(Y)$ and let $d := \dim Z - \dim X$. Since $f : Z \to X$ is a smooth morphism of relative dimension d, [Sta, Lemma 29.36.20] implies that there exists an open affine neighborhood U of z, an open affine neighborhood V of $f(z)$ such that $f(U) \subset V$, and a commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "f"] & U \arrow[l] \arrow[r, "\eta"] \arrow[d, "f_{|U}"'] & \mathbb{A}^d_V \arrow[dl] \\
X & V \arrow[l] &
\end{tikzcd}$$

where η is étale. From the definition, the local Euler obstruction only depends on the local behavior in the *analytic* topology, and this implies that $\mathrm{Eu}_{f^{-1}(Y)}(z) = \mathrm{Eu}_{f^{-1}(Y \cap V)}(z)$. From the diagram above it follows that η provides a local isomorphism in analytic topology between $f^{-1}(V \cap Y)$ and $(V \cap Y) \times \mathbb{A}^d$. The claim follows from the product formula in (c), along with part (b). $\square$

Following MacPherson's definition [Mac74], the Mather class and the MacPherson transformation are related by $c_{\mathrm{Ma}}(X) = c_*(\mathrm{Eu}_X)$. In terms of CSM classes, this can be expressed as $c_{\mathrm{Ma}}(X) = \sum_i e_i c_{\mathrm{SM}}(S_i)$. For φ a constructible function on X, let $s(\varphi)$ denote its *Segre-MacPherson (SM)* class,

$$s(\varphi) = \frac{c_*(\varphi)}{c(TX)}.$$

(Note that this is different from the Segre-Mather class defined in [Yok90].) The following Verdier-Riemann-Roch (VRR) type theorem was proved by Yokura [Yok99].

Theorem 3.4. *Let $f : Z \to X$ be a smooth morphism of complex algebraic varieties. Then for any constructible function $\varphi \in \mathcal{F}(X)$, $f^*s(\varphi) = s(f^*(\varphi))$. Equivalently, if T_f denotes the relative tangent bundle of f, we have an equality in $A^*(X)$,*

$$c_*(f^*(\varphi)) = c(T_f) \cap f^*(c_*(\varphi)).$$

Proposition 3.3 states that if $f : Z \to X$ is a smooth morphism, then $f^*(\mathrm{Eu}_Y) = \mathrm{Eu}_{f^{-1}(Y)}$. If one takes $\varphi = \mathrm{Eu}_Y$, this implies that in terms of Mather classes

$$c_{\mathrm{Ma}}(f^{-1}(Y)) = c(T_f) \cap f^*(c_{\mathrm{Ma}}(Y)) \quad \in A_*(Z). \tag{7}$$

The results from this section can be extended to the case when all varieties have a torus T action, and all morphisms are T-equivariant. The local Euler obstruction is the same, but one uses an equivariant Whitney stratification, and Ohmoto's equivariant version of MacPherson's transformation c_* [Ohm06]; see also [AMSS22b].

4. Preliminaries on flag manifolds and cominuscule spaces

4.1 Preliminaries

References for this section are [Kum02, Bri05]. Let G be a complex semisimple Lie group; fix a maximal torus T, and a pair of Borel subgroups, B and B^-, satisfying $B \cap B^- = T$, in G.

Let R (resp. R^+, R^-, Δ) denote the set of (resp. positive, negative, simple) roots. We have a partial order on R, given by $\alpha < \beta$ if $\alpha \neq \beta$ and $\beta - \alpha$ is a non-negative combination of positive roots. The Weyl group $W := N_G(T)/T$ associated to (G, T) is a Coxeter group generated by the simple reflections $s_i := s_{\alpha_i}$, for $\alpha_i \in \Delta$. Denote by $\ell : W \to \mathbb{N}$ the length function and by w_0 the longest element.

Recall that the parabolic subgroups $P \supset B$ are in bijection with the subsets $S \subset \Delta$. We denote by R_P^+ the subset of R^+ consisting of roots whose support is contained in S. The Weyl group W_P of P is generated by the simple reflections s_i, for $\alpha_i \in S$. Denote by w_P the longest element in W_P, and let W^P be the set of *minimal length representatives* for the cosets in W/W_P. The coset wW_P has a unique minimal length representative $w^P \in W^P$; we set $\ell(wW_P) := \ell(w^P)$.

The space G/P is a projective manifold of dimension $\ell(w_0 W_P)$. For $w \in W^P$, the B-orbit $X_w^{P,\circ} = BwP/P$, and the B^--orbit $(X^P)^{w,\circ} = B^- wP/P$, are opposite Schubert cells for w, and there are isomorphisms, $X_w^{P,\circ} \simeq \mathbb{C}^{\ell(w)}$ and $(X^P)^{w,\circ} \simeq \mathbb{C}^{\dim G/P - \ell(w)}$. The *Schubert varieties* X_w^P and $(X^P)^w$ are the closures of the Schubert cells $X_w^{P,\circ}$ and $(X^P)^{w,\circ}$ respectively.

Every P-representation V determines a G-equivariant vector bundle, $G \times^P V \to G/P$. The points of $G \times^P V$ are equivalence classes $[g, v]$, for pairs $(g, v) \in G \times V$ such that $(g, v) \simeq (gp^{-1}, pv)$, and the G-action on $G \times^P V$ is given by left multiplication, $g.[g', v] := [gg', v]$. The main examples considered in this note are the following. If $P = B$ is a Borel subgroup, we will take $V := \mathbb{C}_\lambda$, the one dimensional B-module of character λ. The resulting line bundle is $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$.[1] Let $\mathfrak{p}$ and $\mathfrak{g}$ be the Lie algebras of P and G respectively, and let $\mathfrak{u}_P$ be the unipotent radical of $\mathfrak{p}$. The spaces $\mathfrak{p}$, $\mathfrak{g}$, and $\mathfrak{u}_P$ are P-stable under the adjoint action. We have $T(G/P) = G \times^P \mathfrak{g}/\mathfrak{p}$, and $T^*(G/P) = G \times^P \mathfrak{u}_P$.

A simple root α is called *cominuscule* if it appears with coefficient 1 in the highest root of R. Let α be cominuscule, and let P be the parabolic subgroup corresponding to $S = \Delta\backslash\{\alpha\}$. Then G/P is called a cominuscule space. We present a complete list of cominuscule spaces (see e.g. [BCMP18]):

- the Grassmannian $\mathrm{Gr}(k, n)$ (type A);
- the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$ (type C);
- the connected components of the orthogonal Grassmannian $\mathrm{OG}(n, 2n)$ (type D);
- Quadrics: odd dimensional in type B, and even dimensional in type D;
- the Cayley plane (type E_6), and the Freudenthal variety (type E_7).

Further, the orthogonal Grassmannian $\mathrm{OG}(n-1, 2n-1)$ (a cominuscule space of type B) is isomorphic to a connected component of $\mathrm{OG}(n, 2n)$, hence can be identified with a cominuscule space of type D, see [FP98, p. 68] or e.g. [IMN16, § 3.4].

4.2 The conormal space of a cominuscule Schubert variety

For s_i a simple reflection, let P_i denote corresponding the minimal parabolic subgroup. Let X_w^P be a Schubert variety in a cominuscule space G/P, let $\underline{w} = (s_{i_1}, \cdots, s_{i_k})$ be a reduced word for w, and let $B_{\underline{w}} := P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_k}/B$ be the corresponding Bott-Samelson variety, see [BK05, Ch.2]. Following [Sin21, Lemma 2.1], the subspace $\mathfrak{u}_w = \mathfrak{u}_P \cap w^{-1}(\mathfrak{u}_B)$ is B-stable under the adjoint action, hence we have a vector bundle $\mathcal{E}_{\underline{w}} := P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_k} \times^B \mathfrak{u}_w \to B_{\underline{w}}$. The following was proved in [Sin21, Thm. A].

Theorem 4.1. *Consider the vector bundle $\mathcal{U}_w := \overline{BwB} \times^B \mathfrak{u}_w \to X_w^B$ and the natural map $\theta'_{\underline{w}} : \mathcal{E}_{\underline{w}} \to \mathcal{U}_w$. Let $\pi' : \mathcal{U}_w \to T^*(G/P)$ be the composition*

$$\overline{BwB} \times^B \mathfrak{u}_w \hookrightarrow \overline{BwB} \times^B \mathfrak{u}_P \hookrightarrow G \times^B \mathfrak{u}_P \to G \times^P \mathfrak{u}_P = T^*(G/P). \tag{8}$$

The composite map, $\pi' \circ \theta'_{\underline{w}} : \mathcal{E}_{\underline{w}} \to T^*_{X_w^P}(G/P)$, is a resolution of singularities.

5. Mather classes of Schubert varieties

In this section we prove the formula calculating the Mather classes of cominuscule Schubert varieties, and illustrate the calculation with two examples.

5.1 The formula for Mather classes

Let $\mathcal{T}_w^* := \overline{BwB} \times^B (\mathfrak{u}_P/\mathfrak{u}_w)$. Following equation (8), we have an exact sequence of homogeneous vector bundles on X_w^B,

$$0 \longrightarrow \mathcal{U}_w \longrightarrow (\pi^*(T^*(G/P))_{|X_w^B} \longrightarrow \mathcal{T}_w^* \longrightarrow 0 \tag{9}$$

[1]With this definition $c_1(\mathcal{L}_{-\omega_i}) = [(X^B)^{s_i}]$, where ω_i is the fundamental weight.

Observe that $\mathcal{T}_w^*$ restricted to the open Schubert cell in X_w^B is the cotangent bundle of this cell (explaining the choice of notation). Following Theorem 4.1, the diagram

$$\begin{array}{ccc} \mathcal{U}_w & \xrightarrow{\pi'} & T^*_{X_w^P}(G/P) \\ \downarrow & & \downarrow \\ X_w^B & \xrightarrow{\pi} & X_w^P \end{array} \tag{10}$$

satisfies the hypotheses in Lemma 2.1 with $X' = X_w^B$ and $\mathcal{U}_w$ a subbundle of $\pi^* T^*(G/P)_{|X_w^P}$. This allows us to calculate the Mather class of X_w^P.

Recall from the introduction that $I(w) = \{\alpha > 0 \mid w(\alpha) < 0\}$ denotes the inversion set of w.

Theorem 5.1. *Let $w \in W^P$ be a minimal length representative, and let $\mathcal{T}_w$ denote the dual of the bundle $\mathcal{T}_w^*$. Then $c^T(\mathcal{T}_w) = \prod_{\alpha \in I(w)} c^T(\mathcal{L}_{-\alpha})$, and the equivariant Mather class of X_w^P is*

$$c_{\mathrm{Ma}}^T(X_w^P) = \pi_*(c^T(\mathcal{T}_w) \cap [X_w^B]). \tag{11}$$

Proof. It follows from the T-module decomposition, $(\mathfrak{u}_P/\mathfrak{u}_w)^* = \oplus_{\alpha \in I(w)} \mathfrak{g}_{-\alpha}$, that the total Chern class of the homogeneous vector bundle $\mathcal{T}_w$ has the same localization at T-fixed points $e_v \in G/B$ as the Chern class of the vector bundle $\bigoplus_{\alpha \in I(w)} \mathcal{L}_{-\alpha}$, see Section 4.1. The result for $c^T(\mathcal{T}_w)$ now follows from Whitney's formula.

Next, observe that Lemma 2.1 applied to the diagram (10) yields

$$s^T(T^*_{X_w^P}(G/P)) = \pi_*(s^T(\mathcal{U}_w) \cap [X_w^B]) = \pi_*(c^T(\mathcal{U}_w)^{-1} \cap [X_w^B]). \tag{12}$$

It follows from the equation (6) version of Theorem 3.1 that

$$\begin{aligned} c_{\mathrm{Ma}}^{T,\vee}(X_w^P) &= c^T(T^*(G/P)) \cap s^T(T^*_{X_w^P}(G/P)) \\ &= c^T(T^*(G/P)) \cap \pi_*(c^T(\mathcal{U}_w)^{-1} \cap [X_w^B]) = \pi_*(c^T(\mathcal{T}_w^*)) \cap [X_w^B]), \end{aligned}$$

where the last equality follows from the projection formula. The proof ends by changing the signs in each homogeneous component; this corresponds to taking the Chern classes of the dual bundle $\mathcal{T}_w$. □

Another algorithm to calculate Mather classes, in the case of Grassmannians, was found by B. Jones [Jon10]. He used Zelevinsky's small resolutions for Schubert varieties [Zel83], and equivariant localization, to calculate the *Kazhdan-Lusztig class* of a Schubert variety. As explained in Section 8 below (see also [AMSS22b, § 8.3]) this coincides with the Mather class. Sankaran and Vanchinathan [SV94] found Schubert varieties in the Lagrangian Grassmannian which do not admit small resolutions; Perrin [Per07] characterized the minuscule Schubert varieties with this property.

5.2 Examples

Recall the equivariant Chevalley formula, cf. [BM15, Thm. 8.1], see also [Kum02, Cor. 11.3.17 and Thm. 11.1.7]. For λ be a weight, we have

$$c_1^T(\mathcal{L}_\lambda) \cap [X_w^B] = w(\lambda)[X_w^B] + \sum \langle -\lambda, \alpha^\vee \rangle [X_{ws_\alpha}^B],$$

where the sum is over all positive roots α such that $\ell(ws_\alpha) = \ell(w) - 1$. Repeated applications to equation (11) yields an expression

$$c^T(\mathcal{T}_w) \cap [X_w^B] = \sum_{v \leqslant w} a_{w,v}[X_v^B],$$

with $a_{w,v} \in A_T^*(pt)$. In particular, $a_{w,w} = \prod_{\alpha \in I(w)}(1 - w(\alpha))$. Recall that

$$\pi_*[X_v^B] = \begin{cases} [X_v^P] & \text{if } v \in W^P, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the equivariant Mather class of X_w equals $\sum\limits_{v \leqslant w; v \in W^P} a_{w,v}[X_v^P]$.

Example 5.2. Let $G = \mathrm{SL}_4$ and the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ (notation as in [Bou02]). The maximal parabolic P associated to α_2 gives the Grassmannian $G/P = \mathrm{Gr}(2,4)$. One may identify the elements of W^P with Young diagrams, see e.g. [BCMP18, § 3]. For instance, in the table below, the shaded portion corresponds to the Schubert divisor $X^P_{s_1s_3s_2} \subset \mathrm{Gr}(2,4)$, with inversion set $I(s_1s_3s_2) = \{\alpha_2, \varepsilon_2 - \varepsilon_4, \varepsilon_1 - \varepsilon_3\}$. The Schubert divisor is the smallest example of a singular Schubert variety in $\mathrm{Gr}(2,4)$: it is a 3 dimensional quadric singular at the point $1.P$.

α_2	$\varepsilon_2-\varepsilon_4$
$\varepsilon_1-\varepsilon_3$	$\varepsilon_1-\varepsilon_4$

2	3
1	2

The Chevalley formula in $A^*(G/B)$ gives

$$c(\mathcal{T}_w) \cap [X_{132}] = [X_{132}] + 3[X_{32}] + 4[X_{31}] + 3[X_3] + 3[X_{12}] + 8[X_2] + 3[X_1] + 6[X_{id}].$$

(To ease notation, we omit the B superscript and the s's in the reduced words.) Pushing forward to $\mathrm{Gr}(2,4)$, we obtain the Mather class:

$$c_{\mathrm{Ma}}\left(\ydiagram{2,1}\right) = \ydiagram{2,1} + 3\ \ydiagram{2} + 3\ \ydiagram{1,1} + 8\ \ydiagram{1} + 6\ \emptyset. \tag{13}$$

The equivariant calculation is more involved, and we present only the final answer:

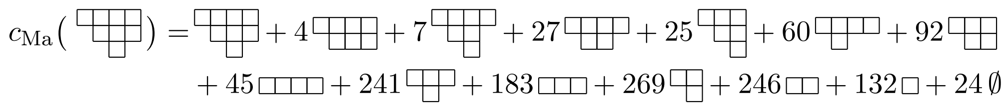
$$\begin{aligned} c^T_{\mathrm{Ma}}(X^P_{s_1s_3s_2}) = &(1+\alpha_1)(1+\alpha_3)(1+\alpha_1+\alpha_2+\alpha_3)\ \ydiagram{2,1} \\ &+ (1+\alpha_3)(3+\alpha_1+2\alpha_2+2\alpha_3+3)\ \ydiagram{2} \\ &+ (1+\alpha_1)(3+2\alpha_1+2\alpha_2+\alpha_3+3)\ \ydiagram{1,1} \\ &+ (8+2\alpha_1+4\alpha_2+2\alpha_3)\ \ydiagram{1} + 6\ \emptyset. \end{aligned} \tag{14}$$

Example 5.3. We now consider the Lagrangian Grassmannian $\mathrm{LG}(4,8)$. The Schubert subvarieties are indexed by strict partitions included in the $(4,3,2,1)$ staircase. We refer the reader to [BCMP18, § 3] for further combinatorial details.

α_4	$\varepsilon_3+\varepsilon_4$	$\varepsilon_2+\varepsilon_4$	$\varepsilon_1+\varepsilon_4$
	$2\varepsilon_3$	$\varepsilon_2+\varepsilon_3$	$\varepsilon_1+\varepsilon_3$
		$2\varepsilon_2$	$\varepsilon_1+\varepsilon_2$
			$2\varepsilon_1$

4	3	2	1
	4	3	2
		4	3
			4

Take $\lambda = (4,3,1)$. The reduced word is $w = s_4 s_2 s_3 s_4 s_1 s_2 s_3 s_4$, and the inversion set consists of the entries in the shaded boxes of the first diagram. We compute



Remark 5.4. The bundle $\mathcal{T}_w$ is *not* globally generated, even when restricted to X_w^B. Indeed, let $G/P = \mathrm{Gr}(2,4)$, $w = s_1 s_3 s_2 \in W^P$, and take $u = s_3$. Then $X_u^B \subset X_w^B$ and $c(\mathcal{T}_w) \cap [X_u^B] = [X_{s_3}^B] - [X_{id}^B]$, thus its Chern class is not effective. Still, examples suggest that the Mather classes are effective; cf. Section 8 below.

Remark 5.5. Let $Q \subset P$ be some parabolic subgroup containing B. Recall the map $\pi : G/Q \to G/P$, and let ω_π be the pull back of the cotangent bundle $T^*(G/P) \to G/P$ along π. Consider the diagram,

$$T^*(G/Q) \xhookleftarrow{\rho_\pi} G/Q \times_{G/P} T^*(G/P) \xrightarrow{\omega_\pi} T^*(G/P)$$

Here ρ_π is defined by $\rho_\pi(x,\xi) = (x, \xi \circ d\pi(x))$, where $d\pi(x) : T_x(G/Q) \to T_{\pi(x)}(G/P)$ is the differential of π at x. Since π is a smooth morphism, ω_π is smooth by base change, and ρ_π is a closed embedding; see e.g. [HTT08, p.65]. For $w \in W^P$,

$$\rho_\pi \omega_\pi^{-1}(T^*_{X_w^P}(G/P)) = T^*_{\pi^{-1}(X_w^P)}(G/Q) \tag{15}$$

is the conormal space of the pull-back Schubert variety $\pi^{-1}(X_w^P)$, see e.g. [KT84, Lemma 3] for a special case, or [Dim04, Prop. 4.3.3] for more general cases. Using Theorem 3.1 and equation (6), one can recover equation (7) in the case $f = \pi$:

$$c^T_{\mathrm{Ma}}(\pi^{-1}(X_w^P)) = c^T(T_\pi) \cap \pi^*(c^T_{\mathrm{Ma}}(X_w^P)). \tag{16}$$

Example 5.6. Let $\pi : \mathrm{Fl}(4) \to \mathrm{Gr}(2,4)$. Consider the divisor $X^P_{s_1 s_3 s_2} \subset \mathrm{Gr}(2,4)$. Then $\pi^{-1} X^P_{s_1 s_3 s_2} = X^B_{s_1 s_2 s_3 s_2 s_1}$. Using equations (13) and (16) we obtain:

$$\begin{aligned} c_{\mathrm{Ma}}(X^B_{12321}) = {} & [X_{12321}] + 3[X_{2321}] + 3[X_{1231}] + 10[X_{231}] + 28[X_{31}] + 2[X_{1232}] \\ & + 8[X_{232}] + 4[X_{123}] + 16[X_{23}] + 28[X_3] + 2[X_{3121}] + 4[X_{321}] + 8[X_{121}] \\ & + 16[X_{21}] + 28[X_1] + 4[X_{312}] + 12[X_{32}] + 12[X_{12}] + 32[X_2] + 24[X_{id}]. \end{aligned}$$

In Example 8.8 below we will calculate the Mather class of the pull-back divisor from the Lagrangian Grassmannian $\mathrm{LG}(2,4)$ and its relation to KL classes.

Remark 5.7. Our 'homological' indexing conventions for Schubert varieties gives the expected stability property for Mather classes. If i is the embedding of a (Lagrangian, orthogonal) Grassmannian into a larger Grassmannian of the same type, then $i_* c_{\mathrm{Ma}}(X_\lambda) = c_{\mathrm{Ma}}(X_\lambda) \in A_*(\mathrm{Gr}(k', n'))$.

6. A cohomological formula for the local Euler obstruction

The goal of this section is to prove Theorem 6.1, which gives a formula for the local Euler obstruction function for cominuscule Schubert varieties. Different formulae in the classical Lie types were also obtained by Boe and Fu [BF97].

The CSM classes $c_{\mathrm{SM}}(X_u^{P,\circ})$ ($u \in W^P$) form a basis of the Chow group $A_*(G/P)$. Consider the expansion of the Mather class into CSM classes of Schubert cells:

$$c_{\mathrm{Ma}}(X_w^P) = \sum_v e_{w,v} c_{\mathrm{SM}}(X_v^{P,\circ}).$$

The coefficient $e_{w,v} = \mathrm{Eu}_{X_w^P}(e_v)$ is the local Euler obstruction of X_w at the point e_v.

Consider the *Segre-MacPherson* classes of the opposite Schubert cells,

$$s_{\mathrm{M}}((X^P)^{v,\circ}) = \frac{c_{\mathrm{SM}}((X^P)^{v,\circ})}{c(T(G/P))}.$$

The Chern class in the quotient is invertible because $c(T(G/P)) = 1 + \kappa$, where κ is a nilpotent element. Following [AMSS22b, Thm. 7.1], the Segre-MacPherson classes are (Poincaré) dual to the CSM classes, i.e.,

$$\langle c_{\mathrm{SM}}(X_u^{P,\circ}), s_{\mathrm{M}}((X^P)^{v,\circ})\rangle = \int_{G/P} c_{\mathrm{SM}}(X_u^{P,\circ}) \cdot s_{\mathrm{M}}((X^P)^{v,\circ}) = \delta_{u,v}. \tag{17}$$

Combined with Theorem 5.1, this duality implies a cohomological formula for the Euler obstruction coefficients $e_{w,v}$; we record this next.

Recall that for $w \in W$ and CSM class $c_{\mathrm{SM}}((X^B)^{w,\circ}) = \sum c_i$, with $c_i \in A_i(G/B)$, the *dual CSM class* is defined by $c_{\mathrm{SM}}^{\vee}((X^B)^{w,\circ}) := \sum (-1)^{\ell(w)-i} c_i$.

Theorem 6.1. *Let $v, w \in W^P$ and assume that $v \leqslant w$. Then the local Euler obstruction coefficient $e_{w,v}$ is given by*

$$e_{w,v} = \sum \int_{G/B} c(\mathcal{T}_w) \cdot [X_w^B] \cdot c_{\mathrm{SM}}^{\vee}((X^B)^{u,\circ}),$$

where the sum is over $u \in W; v \leqslant u \leqslant w$ such that $uW_P = vW_P$.

Proof. By the duality equation (17), and the projection formula, the Euler obstruction coefficients are given by

$$e_{w,v} = \int_{G/P} \pi_*(c(\mathcal{T}_w) \cdot [X_w^B]) \cdot s_{\mathrm{M}}((X^P)^{v,\circ}) = \int_{G/B} c(\mathcal{T}_w) \cdot [X_w^B] \cdot \pi^* s_{\mathrm{M}}((X^P)^{v,\circ}).$$

The Verdier-Riemann-Roch formula implies that

$$\pi^* s_{\mathrm{M}}((X^P)^{v,\circ}) = s_{\mathrm{M}}(\pi^{-1}((X^P)^{v,\circ})) = \sum s_{\mathrm{M}}((X^B)^{u,\circ}),$$

where the sum is over all $u \geqslant v$ such that $uW_P = vW_P$. Next, it is proved in [AMSS22b, Thm. 7.5] that for any $u \in W$, we have $s_{\mathrm{M}}((X^B)^{u,\circ}) = c_{\mathrm{SM}}^{\vee}((X^B)^{u,\circ})$. Since the class $c_{\mathrm{SM}}^{\vee}((X^B)^{u,\circ})$ is supported on the Schubert variety $X^{B,u}$, it follows that the product $[X_w^B] \cdot [X^{B,u}] = 0$ unless $u \leqslant w$. This finishes the proof. $\square$

By utilizing formulae for the CSM classes obtained in [AM16], one may calculate the integrals from Theorem 6.1 in small examples.

Example 6.2. We continue with the example $G/P = \mathrm{Gr}(2,4)$ and $X_w^P = X_{s_1 s_3 s_2}^P$ the Schubert divisor. Recall from Example 5.2:

$$c(\mathcal{T}_w) \cap [X_{132}^B] = [X_{132}^B] + 3[X_{32}^B] + 4[X_{31}^B] + 3[X_3^B] + 3[X_{12}^B] + 8[X_2^B] + 3[X_1^B] + 6[X_{id}^B].$$

We take $v = s_3 s_2$; then $u = v$ and we obtain the Schubert expansion of $c_{\mathrm{SM}}^{\vee}(X^{B,s_3s_2,\circ})$

$$\begin{aligned}[X^{B,32}] - [X^{B,321}] - 2[X^{B,312}] + [X^{B,3121}] + 4[X^{B,2312}] - 2[X^{B,23121}] - [X^{B,232}] \\ +[X^{B,2321}] + 3[X^{B,1232}] - 2[X^{B,12321}] + [X^{B,123121}] - 3[X^{B,12312}].\end{aligned}$$

Since $\int_{G/B}[X_{v_1}^B] \cdot [X^{B,v_2}] = \delta_{v_1,v_2}$ we obtain $e_{s_1s_3s_2,s_3s_2} = -2 + 3 = 1$. Of course, this was expected, as the Schubert divisor is only singular at the base point $1.P$.

Example 6.3. Consider the Lagrangian Grassmannian $\mathrm{LG}(2,4)$. This is isomorphic to a three dimensional smooth quadric in $\mathbb{P}^4$. The set W^P indexing the Schubert varieties is in bijection with the strict partitions in the 2×2 rectangle: $(0),(1),(2)$ and $(2,1)$. The only singular Schubert variety is the divisor $X^P_{(2)}$, with singularity at the point $X^P_\emptyset = 1.P$. One calculates that

$$e_{(2),(2)} = e_{(2),(1)} = 1; \quad e_{(2),(0)} = 0.$$

This verifies examples from [BF97, p. 456]. Using the isomorphism of $\mathrm{LG}(2,4)$ to the 3-dimensional quadric, it also verifies one instance of [BF97, Eq.(6.3.3)].

7. Localization of conormal spaces

The goal of this section is to use Theorem 5.1 to obtain formulae for the localization of conormal spaces.

Denote by $\iota : G/P \hookrightarrow T^*(G/P)$ the zero section, and recall that $\mathbb{C}^*$ acts on the cotangent bundle $T^*(G/B)$ with character $\hbar^{-1}$. By equation (5),

$$\iota^*[T^*_{X^P_w}(G/P)]_{T\times\mathbb{C}^*} = (-1)^{\ell(w)} c^T_{\mathrm{Ma}}(X^P_w)^\hbar = (-1)^{\ell(w)}\pi_*((c^T(\mathcal{T}_w)\cap[X^B_w])^\hbar).$$

This is a class in $A_0^{T\times\mathbb{C}^*}(G/P)$ and one may check that:

$$(-1)^{\ell(w)}(c^T(\mathcal{T}_w)\cap[X^B_w])^\hbar = c^{T\times\mathbb{C}^*}_{\ell(w)}(\mathcal{T}^*_w)\cap[X^B_w]. \tag{18}$$

The fibre of $\mathcal{T}^*_w$ over the fixed point e_v is $\bigoplus_{\alpha\in I(w)}\mathfrak{g}_{v(\alpha)}\otimes\mathbb{C}_{-\hbar}$, therefore:

$$(c^{T\times\mathbb{C}^*}_{\ell(w)}(\mathcal{T}^*_w))|_v = \prod_{\alpha\in I(w)}(-\hbar+v(\alpha)). \tag{19}$$

Combining equations (18) and (19), we deduce the following lemma.

Lemma 7.1. *Let $v\leqslant w$. Then the following holds in $A^{\dim G/B}_{T\times\mathbb{C}^*}(pt)$:*

$$(-1)^{\ell(w)}((c(\mathcal{T}_w)\cap[X^B_w])^\hbar)|_v = \prod_{\alpha\in I(w)}(-\hbar+v(\alpha))\cdot[X^B_w]|_v.$$

We note that since $\mathbb{C}^*$ acts trivially on G/B, the $T\times\mathbb{C}^*$-localization $[X^B_w]|_v$ is the same as the T-equivariant localization. A formula for the latter can be found in [Kum02, Thm. 11.1.7]; see also [AJS94, App. D] or [Bil99].

For $v\in W$, let $[e_v]$ denote the T-equivariant class of the fixed point e_v. We need the following localization formula, see for example, [AF22, Thm. 5.2.1].

Lemma 7.2. *Consider $u\in W^P$ and $\kappa\in H^*_T(G/B)$. Then, in an appropriate localization of $A^*_T(G/B)$, we have*

$$\pi_*(\kappa)|_{uW_P} = \sum_{\substack{v\in W\\ v=u(mod W_P)}}\frac{[e_{uW_P}]|_{uW_P}}{[e_v]|_v}\kappa|_v.$$

Theorem 7.3. *Let $u,w\in W^P$ such that $u\leqslant w$, and let $\mathbb{C}^*$ act on $T^*(G/P)$ by character $\hbar^{-1}$. Then the $T\times\mathbb{C}^*$-localization of the conormal space $T^*_{X^P_w}(G/P)$ is:*

$$[T^*_{X^P_w}(G/P)]|_{uW_P} = \sum_{\substack{v\leqslant w\\ vW_P=uW_P}}\frac{\prod_{\alpha\in I(w)}(-\hbar+v(\alpha))}{\prod_{\alpha\in R^+_P}v(-\alpha)}[X^B_w]|_v.$$

Proof. Let $\kappa = (-1)^{\ell(w)}(c^T(\mathcal{T}_w) \cap [X_w^B])^\hbar$, regarded as a cohomology class in $A_T^*(G/B)[\hbar]$. By Section 7, the left hand side equals $\pi_*(\kappa)|_{uW_P}$. Since $\mathbb{C}^*$ acts trivially on G/B, we have $A^*_{T\times\mathbb{C}^*}(G/B) = A_T^*(G/B)[\hbar]$, and further, the projection π_* is $A^*_{T\times\mathbb{C}^*}(pt)$-linear. Observe that $[e_v]|_v = \prod_{\alpha\in R^+} v(-\alpha)$ and $[e_{uW_P}]|_{uW_P} = \prod_{\alpha\in R^+\setminus R_P^+} u(-\alpha)$. Applying Lemmas 7.1 and 7.2, we obtain

$$\begin{aligned}
[T^*_{X_w^P}(G/P)]|_{uW_P} &= \sum_{\substack{v\leqslant w\\ vW_P=uW_P}} \frac{\prod_{\alpha\in I(w)}(-\hbar+v(\alpha))\cdot\prod_{\alpha\in R^+\setminus R_P^+}u(-\alpha)}{\prod_{\alpha\in R^+}v(-\alpha)}[X_w^B]|_v \\
&= \sum_{\substack{v\leqslant w\\ vW_P=uW_P}} \frac{\prod_{\alpha\in I(w)}(-\hbar+v(\alpha))}{\prod_{\alpha\in R_P^+}v(-\alpha)}[X_w^B]|_v .
\end{aligned}$$

The second equality follows from the observation that since $vW_P = uW_P$, we have

$$\left\{v(-\alpha) \mid \alpha\in R^+\setminus R_P^+\right\} = \left\{u(-\alpha) \mid \alpha\in R^+\setminus R_P^+\right\}.$$

□

Example 7.4. To illustrate our formula, we take $u = w$. The only v satisfying the requirements is $v = w$. Then

$$\iota_w^*[T^*_{X_w^P}(G/P)]_{T\times\mathbb{C}^*} = \frac{\prod_{\alpha\in I(w)}(-\hbar+v(\alpha))\cdot[X_w^B]|_w}{\prod_{\alpha\in R_P^+}v(-\alpha)} = \prod_{\alpha\in I(w)}(-\hbar+w(\alpha))\cdot[X_w^P]|_w\,,$$

where the last equality follows from standard manipulations of (equivariant) Euler classes. (In this case $T^*_{X_w^P}(G/P)$ is smooth at e_w, and one could have calculated this localization directly.)

In [LS21], Lakshmibai and Singh identified certain conormal spaces as open subsets of affine Schubert varieties. It would be interesting to obtain localization formulae for the conormal spaces using localization for affine Schubert varieties.

8. Positivity and unimodality of Mather classes

In this section we investigate positivity properties for the Euler obstruction and for the Mather class of a Schubert variety. We also present a unimodality and log concavity conjecture for Mather classes, similar to the one for CSM classes; cf. [AMSS22a, § 8.3].

8.1 Kazhdan-Lusztig classes, Mather classes, and Euler obstructions

Unless otherwise specified, in this section P is an arbitrary parabolic subgroup. We refer to [AMSS22b, § 8.3] for more details about the material below.

Let $IH(X_w^P)$ denote the characteristic cycle of the intersection homology of the Schubert variety X_w^P. This is an effective, conic, Lagrangian cycle in the cotangent bundle $T^*(G/P)$, and its irreducible components are conormal spaces of Schubert varieties; see e.g. [HTT08, Thm. E.3.6]. Therefore there is a expansion

$$[IH(X_w^P)]_{T\times\mathbb{C}^*} = \sum_v m_{w,v}[T^*_{X_v^P}(G/P)]_{T\times\mathbb{C}^*} \in A^{T\times\mathbb{C}^*}_{\dim G/P}(T^*(G/P)). \tag{20}$$

Define the *Kazhdan-Lusztig (KL) class* $KL_w^P \in A_*^T(G/P)$ to be the $\hbar = 1$ dehomogenization of

$(-1)^{\ell(w)}(\iota^P)^*[IH(X_w^P)]$. By equation (5) this is the same as

$$KL_w^P = \sum_v (-1)^{\ell(w)-\ell(v)} m_{w,v} c_{\mathrm{Ma}}^T(X_v^P) \in A_*^T(G/P). \tag{21}$$

From the proof of the Kazhdan-Lusztig conjectures [BK81, BB81] it follows that

$$KL_w^P = \sum_v P_{w,v}(1) c_{\mathrm{SM}}^T(X_v^{P,\circ}), \tag{22}$$

where $P_{w,v}$ is the parabolic Kazhdan-Lusztig polynomial; see [AMSS22b, Prop. 8.7]. (Observe that (22) may be taken as the definition of the KL class). Consider the expansion of the Mather class into (equivariant) CSM classes of Schubert cells:

$$c_{\mathrm{Ma}}^T(X_w^P) = \sum_v e_{w,v} c_{\mathrm{SM}}^T(X_v^{P,\circ}), \tag{23}$$

where $e_{w,v} = \mathrm{Eu}_{X_w^P}(e_v)$ is the local Euler obstruction evaluated at the point e_v. Combining equations (21) to (23), we see that the characteristic cycle $IH(X_w^P)$ is irreducible if and only if the local Euler obstruction satisfies

$$e_{w,v} = P_{w,v}(1). \tag{24}$$

In particular, irreducibility of the characteristic cycle implies that the Euler obstruction is strictly positive. In general, the Euler obstruction may be negative; for instance if C is a cone over a nonsingular plane curve of degree d with vertex O, then $\mathrm{Eu}_O(C) = 2d - d^2$, cf. [Mac74]. Note that for general spaces the Euler obstruction may be negative. For instance, if C is a cone over a nonsingular plane curve of degree d with vertex O, then $\mathrm{Eu}_O(C) = 2d - d^2$, cf. [Mac74]. As seen in Example 6.3, the Euler obstruction may be equal to 0 even for cominuscule spaces. Based on computer evidence however, we propose the following conjecture, which is known to hold in types A, B, and D, see Proposition 8.2.

Conjecture 8.1. *Suppose G/P is a cominuscule space, and consider $v \leqslant w \in W^P$. Then the local Euler obstruction $\mathrm{Eu}_{X_w^P}(e_v) \geqslant 0$.*

Proposition 8.2 (cf. [BFL90, BF97, LR22]). *Let $X = G/P$ be a cominuscule space of classical Lie type. Then $e_{w,v} \geqslant 0$ for all $v, w \in W^P$. More precisely:*

- *If $X = G/P$ is a cominuscule space of type A or D, then $e_{w,v} > 0$ for all $v \leqslant w$.*
- *If X is the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$, or X is an odd-dimensional quadric, then $e_{w,v} \geqslant 0$ for all v, w.*

Proof. The strict positivity in types A and D follows from (24) because the Schubert varieties in cominuscule spaces of these types have irreducible characteristic cycles. This is proved by Bressler, Finkelberg and Lunts [BFL90] in type A, and by Boe and Fu [BF97] in type D (they also reprove the statement for type A). For the odd quadrics (type B), Boe and Fu calculated the Euler obstructions explicitly - see [BF97, § 6.3], especially equations (6.3.3) and (6.3.5) - and found them to be nonnegative. Finally, after a version of this paper appeared on arχiv, the non-negativity of the Euler obstructions for $\mathrm{LG}(n, 2n)$ was proved in [LR22]. This finishes the proof. □

Remark 8.3. In the upcoming work [MS22] the authors will prove Conjecture 8.1 in the remaining Lie types E_6, E_7, see also Remark 8.7 below. It is tempting to conjecture that the Euler obstructions are non-negative for Schubert varieties in *any* homogeneous space G/P. While one may reasonably expect that calculations of Euler obstructions of singularities related to cominuscule Schubert varieties - such as those in [GK21, AIJK21], see also [MNS22] - are possible, the problem of calculating Euler obstructions for arbitrary Schubert varieties seems to be wide open. We are not aware of any example of a Schubert variety with a negative Euler obstruction.

8.2 Mather Classes and Positivity

In [Jon10, Rmk. 5.7], B. Jones conjectured that all Mather classes for Grassmannians are nonnegative. Based on substantial experimentation for all cominuscule spaces we propose the following conjecture:

CONJECTURE 8.4 (Strong Positivity conjecture of Mather classes). *Let X_w^P be a Schubert variety in a cominuscule space G/P. Consider the Schubert expansion*

$$c_{\mathrm{Ma}}(X_w^P) = \sum a_v [X_v^P]. \tag{25}$$

Then $a_v > 0$ for any $v \leqslant w$. More generally, consider the Schubert expansion

$$c_{\mathrm{Ma}}^T(X_w^P) = \sum a_v(t) [X_v^P]_T.$$

Then $a_v(t) = a_v(\alpha_1, \ldots, \alpha_r) \in A_T^*(pt)$ is a polynomial in positive simple roots $\alpha_1, \ldots, \alpha_r$ with non-negative coefficients.

We will refer to the situation when $a_v \geqslant 0$ simply as the 'Positivity conjecture', and emphasize 'Strong' whenever we can claim it. Particular instances of this conjecture are proved in Proposition 8.6 below. A similar positivity conjecture was given in [AM09, AM16] for the CSM classes and it was proved in the non-equivariant case in [Huh16] for Grassmannians and [AMSS22b] in general.

LEMMA 8.5. *Let $X = G/P$, and let $v, w \in W^P$ such that $v \leqslant w$.*

- *(a) If Conjecture 8.1 holds for X, then the non-equivariant positivity Conjecture 8.4 holds, i.e. in equation (25), the coefficients $a_v \geqslant 0$.*
- *(b) If $e_{w,v} > 0$ for all $v \in W^P$ such that $v \leqslant w$, then the non-equivariant strong positivity Conjecture 8.4 holds for X.*
- *(c) Conjecture 8.1 holds if the intersection homology characteristic cycle $IH(X_w^P)$ is irreducible.*

Proof. Parts (a) and (b) follow from the equation (23) (for non-equivariant classes), using that the non-equivariant CSM classes of Schubert cells are nonnegative [Huh16, AMSS22b], and that the initial term of $c_{\mathrm{SM}}(X_v^{P,\circ})$ is $[X_v^P]$. Part (c) follows from equation (24), using that the Kazhdan-Lusztig polynomials $P_{w,v}$ ($v \leqslant w$) have non-negative integer coefficients, and have constant term equal to 1. □

PROPOSITION 8.6. *Let $X = G/P$ be a cominuscule space of Lie type A–D, and let $\pi : G/B \to G/P$ be the natural projection.*

- *(a) Strong positivity (Conjecture 8.4) holds in the non-equivariant case for all Schubert varieties in X of Lie types A and D; weak positivity holds in the non-equivariant case for the Lagrangian Grassmannian (type C) and for the odd quadrics (type B).*
- *(b) Let $w \in W^P$. Then the Mather class $c_{\mathrm{Ma}}(\pi^{-1}(X_w^P)) \in A_*(G/B)$ has the same (strong/weak) positivity property as $c_{\mathrm{Ma}}(X_w^P)$ from part (a).*

Proof. Part (a) follows from Lemma 8.5 and Proposition 8.2. Part (b) follows from part (a) because the Euler obstructions for the pull backs $\pi^{-1}(X_w^P)$ coincide with those for X_w^P by Proposition 3.3. $\square$

Kazhdan and Lusztig [KL80] conjectured the irreducibility of characteristic cycles of the IH sheaf in type A. However, Kashiwara and Tanisaki [KT84], then Kashiwara and Saito [KS97] found counterexamples for the full flag manifolds of Lie type B and type A respectively; see also [Bra02, Wil15]. Boe and Fu [BF97] found the decompositions of the characteristic cycles of the Schubert varieties in cominuscule spaces of Lie types B, C. Next, we use the methods of this paper to recover an example of Kashiwara and Tanisaki of a reducible IH characteristic cycle.

Remark 8.7. As noted in the introduction, the authors wrote a Sage program and checked the irreducibility for the IH characteristic cycles of Schubert varieties of cominuscule G/P of Lie type E_6 and E_7. Together with Proposition 8.2, this provides a proof of Conjecture 8.1. Furthermore, these results also imply that the strong positivity from Proposition 8.6(a) holds in Lie types E_6, E_7. The details will appear in [MS22].

Example 8.8. Consider the Lagrangian Grassmannian $\mathrm{LG} := \mathrm{LG}(2,4)$ and the Schubert divisor $X^P_{1,2} \subset \mathrm{LG}(2,4)$ (s_2 corresponds to the long simple root). Let $\mathrm{SF} := \mathrm{SF}(1,2;4)$ be the complete flag manifold of type C_2; it parametrizes flags $F_1 \subset F_2 \subset \mathbb{C}^4$ where F_i is isotropic with respect to a symplectic form. Let $\pi : \mathrm{SF} \to \mathrm{LG}$ be the projection. The preimage $\pi^{-1}(X^P_{1,2})$ is the Schubert divisor $X^B_{1,2,1} \subset \mathrm{SF}$. A calculation of the Kazhdan-Lusztig polynomials using e.g. SAGE shows that $P_{121,v} = 1$ for any $v \leqslant s_1 s_2 s_1$. Thus the non-equivariant KL class of $X^B_{1,2,1}$ is:

$$KL^B_{1,2,1} = c_{\mathrm{SM}}(X^{B,\circ}_{1,2,1}) + c_{\mathrm{SM}}(X^{B,\circ}_{1,2}) + c_{\mathrm{SM}}(X^{B,\circ}_{2,1}) + c_{\mathrm{SM}}(X^{B,\circ}_{2}) + c_{\mathrm{SM}}(X^{B,\circ}_{1}) + c_{\mathrm{SM}}(X^{B}_{id}).$$

Using the local Euler obstructions calculated in Example 6.3, we obtain:

$$c_{\mathrm{Ma}}(X^P_{1,2}) = c_{\mathrm{SM}}(X^{P,\circ}_{1,2}) + c_{\mathrm{SM}}(X^{P,\circ}_{2}).$$

Using equation (16) and the Verdier-Riemann-Roch Theorem 3.4, we have,

$$\begin{aligned} c_{\mathrm{Ma}}(X^B_{1,2,1}) &= c_{\mathrm{SM}}(\pi^{-1}X^{P,\circ}_{1,2}) + c_{\mathrm{SM}}(\pi^{-1}X^{P,\circ}_{2}) \\ &= c_{\mathrm{SM}}(X^{B,\circ}_{1,2,1}) + c_{\mathrm{SM}}(X^{B,\circ}_{1,2}) + c_{\mathrm{SM}}(X^{B,\circ}_{2,1}) + c_{\mathrm{SM}}(X^{B,\circ}_{2}). \end{aligned}$$

Using that $c_{\mathrm{Ma}}(X^B_1) = c_{\mathrm{SM}}(X^{B,\circ}_1) + c_{\mathrm{SM}}(X^{B,\circ}_{id})$ (as $X^B_1 \simeq \mathbb{P}^1$), we deduce:

$$KL^B_{1,2,1} = c_{\mathrm{Ma}}(X^B_{1,2,1}) + c_{\mathrm{Ma}}(X^B_1).$$

By Theorem 3.1 and the definition of the KL class, this shows that the IH characteristic cycle $IH(X^B_{1,2,1}) \subset T^*(G/B)$ satisfies

$$IH(X^B_{1,2,1}) = [T^*_{X^B_{1,2,1}}(\mathrm{SF})] + [T^*_{X^B_1}(\mathrm{SF})],$$

in accordance to [KT84, p. 194] (after identifying the B_2 and C_2 flag manifolds).

8.3 Unimodality and log concavity of Mather polynomials

For $w \in W^P$, consider the Schubert expansion $c_{\mathrm{Ma}}(X^P_w) = \sum a_{w,v}[X^P_v]$. The *Mather polynomial* associated to w is

$$M_w(x) = \sum a_{w,v} x^{\ell(v)}.$$

For instance, the Mather polynomial of the Schubert variety $X_{(4,3,1)} \subset \mathrm{LG}(4,8)$ from Example 5.3 is

$$M_{(4,3,1)}(x) = x^8 + 11x^7 + 52x^6 + 152x^5 + 286x^4 + 452x^3 + 246x^2 + 132x + 24.$$

If we choose an embedding $\iota : X_w^P \hookrightarrow \mathbb{P}^N$ in a projective space, then

$$\iota_* c_{\mathrm{Ma}}(X_w^P) = \sum a_{w,v} \deg(\iota(X_v^P))[\mathbb{P}^{\ell(v)}].$$

Then one may regard the Mather polynomial $M_w(x)$ as the polynomial obtained from $\iota_* c_{\mathrm{Ma}}(X_w^P)$, normalized by dividing each term by the degree of the corresponding Schubert class in the given embedding.

Following [Sta89], we say that a polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is *unimodal* if $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_k \geqslant a_{k+1} \geqslant \ldots \geqslant a_n$ for some index k. It is *log concave* if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for all i (by convention $a_{-i} = a_{n+i} = 0$ for all $i \geqslant 1$). If one assumes that the polynomial has strictly positive coefficients, then any log concave polynomial is also unimodal.

Substantial amount of calculations in all Lie types supports the following:

Conjecture 8.9. *Let $X = G/P$ be a cominuscule space and $w \in W^P$.*

(a) The Mather polynomial M_w has strictly positive coefficients and it is unimodal.

(b) Assume in addition that $G/P = \mathrm{Gr}(k,n)$. Then M_w is log concave.

Note that the positivity statement follows from the conjectural positivity in Conjecture 8.4. The log concavity fails outside type A. For instance, the Mather polynomial of the 5 dimensional quadric $\mathrm{OG}(1,7)$ is $x^5 + 5x^4 + 11x^3 + 26x^2 + 18x + 6$. (This Mather class is the same as the total Chern class of $T(\mathrm{OG}(1,7))$.) Similarly, the Mather classes of $\mathrm{LG}(5,10)$ and of $\mathrm{OG}(4,8)$ are not log concave.

The unimodality and log concavity properties of characteristic classes of singular varieties seem to be new and unexplored phenomena. For instance, in analogy to the Mather polynomial one may define two flavors of a *CSM polynomial*: one obtained from the CSM class of a Schubert cell, and the other from the CSM class of a Schubert variety. This is conjectured to satisfy an analog of Conjecture 8.9; more details are discussed in [AMSS22a, § 8]. Log concavity has also been conjectured for certain coefficients of motivic Chern classes of Schubert cells [FRW20, § 6.2]. It would be interesting to know whether these phenomena fit into the (Hodge-Riemann and Hard Lefschetz) framework from [Huh18] or [HMMSD22].

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Leonardo C. Mihalcea lmihalce@math.vt.edu
Dept. of Mathematics, Virginia Tech, 460 McBryde Hall, 225 Stanger St, Blacksburg VA 24061

Rahul Singh rahul.sharpeye@gmail.com
Tutor Intelligence, 283 Franklin St, Boston MA 02110