Quantum K theory rings of partial flag manifolds

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In this paper we use three-dimensional gauged linear sigma models to make physical predictions for Whitney-type presentations of equivariant quantum K theory rings of partial flag manifolds, as quantum products of universal subbundles and various ratios, extending previous work for Grassmannians. Physically, these arise as OPEs of Wilson lines for certain Chern-Simons levels. We also include a simplified method for computing Chern-Simons levels pertinent to standard quantum K theory.

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Contents

1 Introduction

Recently there has been interest in computations of quantum K theory rings in mathematics from three-dimensional supersymmetric gauge theories, see e.g. [1, section 2.4], [2–7]. The basic idea is that the quantum K theory ring of a Fano variety arises as the OPE ring of Wilson lines in the three-dimensional gauge theory, which is defined on a three-manifold which has the form of a circle bundle. The Wilson lines are the holonomies around the S^1 fibers, and after reducing to two dimensions, the computations of the Wilson line OPEs reduce to ordinary two-dimensional OPEs, computable using standard GLSM Coulomb branch methods [8].

In this paper we will apply these ideas to construct Whitney-type presentations of equivariant quantum K theory rings of partial flag manifolds, as quantum products of universal subbundles of the partial flag manifold and various quotients. To be clear, we are not the first to study equivariant quantum K theory of flag manifolds, and in fact, the equivariant K theory of the total space of the cotangent bundle of the full (complete) flag manifold arises in integrable systems, see e.g. [1, 9–12], [13, section 4.2]. Quantum K theory of the zero section can be obtained as a limit. See also [14, 15] for other presentations of the equivariant quantum K theory rings of partial flag manifolds. The purpose of this paper is to explore novel presentations – we do not claim any novelty in determining the rings themselves.

We begin in section 2 with a review of the basics of these computations, which we apply in section 3 to the case of Grassmannians $G(k, n)$. A presentation of the quantum K theory ring of $G(k, n)$ in terms of quantum products of bundles (the λ_y class relation) was previously derived on physical grounds in [5], and later rigorously proven in [6]. We review here a much more efficient version of that analysis, which will form the prototype of our discussion of flag manifolds.

In section 4 we turn to the main content of this paper, namely proposals for presentations of the equivariant quantum K theory ring of general partial flag manifolds. We apply Coulomb branch methods to derive a formal expression in terms of Chern roots of bundles, which we symmetrize into the form of a relation between λ_y classes of universal subbundles and various quotients thereof. Our main result is the ring relation (4.39), namely

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\widetilde{\mathcal{R}}_i) \;=\; \lambda_y(\mathcal{S}_{i+1}) \;+\; q_i y^{k_{i+1}-k_i} \left(\det \widetilde{\mathcal{R}}_i\right) \star \lambda_y(\mathcal{S}_{i-1}), \tag{1.1}$$

where the $\mathcal{S}_i$ are the universal subbundles on the flag manifold, and $\widetilde{\mathcal{R}}_i$ is defined in (4.36). This relation can alternatively be expressed without $\widetilde{\mathcal{R}}_i$ as (4.42), namely

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \;=\; \lambda_y(\mathcal{S}_{i+1}) \;-\; y^{k_{i+1}-k_i} \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \left(\lambda_y(\mathcal{S}_i) - \lambda_y(\mathcal{S}_{i-1})\right). \tag{1.2}$$

There are other presentations of the quantum K theory rings of flag manifolds in the literature. We already mentioned the presentation from [5, 6] for Grassmannians, which is a special case of (1.2); in these references we also found a 'Coulomb branch presentation', closely related to one found earlier by Gorbounov and Korff [16] in relation to Bethe Ansatz. For the complete flag manifolds, a presentation based on integrable systems techniques is stated in [9]; a similar presentation has been recently proved in [17,18] based on the relation between the (equivariant) K theory of complete flag manifolds and the K-theory of semi-infinite flag manifolds. Both these presentations are symmetric in the variables involved, and generalize the usual 'Borel presentation' for the ordinary K theory of flag manifolds. The presentation given by the relations (1.2) has a different source: it generalizes the presentation arising from the flag manifold seen as a tower of Grassmann bundles. As such, the relations in (1.2) are not symmetric. We refer the reader to Section 5.1.2 for an example of the presentation from [9, 17] for the quantum ring of Fl(3).

The result above has the correct classical limit in torus-equivariant K theory, and also correctly specializes to rigorous mathematics results [6] for equivariant quantum K theory of Grassmannians. As another consistency check, in subsection 4.6 we compare predicted quantum cohomology ring relations to existing results in the literature. As another consistency check, in subsection 4.7 we also demonstrate that this ring relation is consistent with duality between the flag manifold $F(k_1, \cdots, k_s, N)$ and its dual $F(N-k_s, \cdots, N-k_1, N)$, and can be most efficiently expressed, formally, by relating honest bundles to 'quantum' bundles, extending a similar result for quantum K theory of Grassmannians described in subsection 3.4.

Finally, as further consistency tests, in section 5 we compare the predictions for general partial flag manifolds to existing results in the literature, namely for incidence varieties (flag manifolds of the form $F(1, N-1, N)$) and full flag manifolds. In particular, a rigorous proof of our assertions for quantum K theory for the special case of incidence varieties will appear in [19], as we review in section 5.1.

This paper focuses on quantum K theory rings, and not quantum K theory invariants. In principle, in small quantum K theory, if we know both the ring structure (structure constants) as well as a pairing, then we can compute all of the correlation functions. In principle, this can be accomplished in physics using supersymmetric localization. We will not pursue this direction here, however.

In passing, we also mention that there exists (unrelated) work on quantum sheaf cohomology on partial flag manifolds, see [20], and, separately, there has been work in the physics community on I functions for vector bundles over Grassmannians and flag manifolds, relevant for the study of Wilson lines, see e.g. [21].

2 Review

In this section we review the computation of quantum K theory rings from physical Coulomb branch relations in three-dimensional gauged linear sigma models. One way to do this is to compute for the total space of the cotangent bundle in a three-dimensional $N = 4$ theory, and break to $N = 2$; here, we compute directly in a three-dimensional $N = 2$ theory. We also discuss a (to our knowledge) novel method for computing Chern-Simons levels relevant to quantum K theory, which replaces Casimirs of nonabelian groups by computations on Coulomb branches.

Briefly, the idea of [1–5] is that for e.g. Fano varieties realized in GLSMs, the quantum K theory ring can be computed using GLSM Coulomb branches in a manner closely analogous to quantum cohomology rings [8]. One starts with a three-dimensional GLSM, compactifies on a circle of radius R, and then considers the compactified two-dimensional theory obtained by summing the tower of Kaluza-Klein modes. For GLSMs for Fano spaces, Coulomb branch computations in the compactified theory yield quantum K theory relations, just as Coulomb branch computations in an ordinary two-dimensional GLSM yield quantum cohomology rings, as in [8]. For such spaces, both the quantum cohomology ring relations and quantum K theory ring relations can be obtained in the same way – from the critical locus of a quantum-corrected twisted effective superpotential. The difference is the form of the superpotential, which in the original two-dimensional theory involved ordinary logarithms, and in the compactified theory involve dilogarithms.

For quantum K theory, the one-loop effective twisted superpotential obtained by regularizing the infinite sum of KK modes has the form[1] [5, eq. (2.1)], [22, equ'n (2.33)], [23, section 2.2.2]:

$$\begin{aligned} \mathcal{W} \;=\; & \frac{1}{2}\gamma^{ab}(\ln X_a)\left[(\ln X_b)+1\right] \;+\; \gamma^{ai}(\ln X_a)\left(\ln T_i^{-1}\right) \\ & + \sum_a (\ln q_a)(\ln X_a) \;+\; \sum_a \left(i\pi \sum_{\mu \text{ pos}'} \alpha^a_\mu \right)(\ln X_a) \\ & + \sum_i \left[\mathrm{Li}_2\left(X^{\rho_i}/T_i\right) \;+\; \frac{1}{4}\left(\ln\left(X^{\rho_i}/T_i\right)\right)^2 \right], \end{aligned} \tag{2.1}$$

where Li_2 denote the dilogarithm function, the summation over i is over all the matter fields, T_i is the flavor symmetry fugacity and

$$X^{\rho_i} = \prod_a X_a^{Q^i_a}. \tag{2.2}$$

[1] A previous version of this paper listed the first term in the superpotential as $(1/2)\gamma^{ab}(\ln X_a)(\ln X_b)$. The difference is discussed in e.g. [24, section 2.2.1], [25, section 2], [26, equ'n (2.10)]. This reflects spin-structure dependence when the theory is formulated on general three-manifolds, as discussed in [24, appendix C]. In this paper, the underlying three-manifold will always be taken to be $S^1 \times \mathbb{R}^2$. We have also neglected constant contributions to the superpotential written in other references, as they play no role here.

The fields X_a are determined by the Coulomb branch σ fields, in the form

$$X_a \;=\; \exp\left(2\pi R \sigma_a\right), \tag{2.3}$$

where R is the radius of the circle on which the three-dimensional theory was compactified. The reader should note that the σ_a, and hence the X_a, are effectively constrained to avoid 'excluded loci,' and also that even after the gauge group is broken to an abelian subgroup along the Coulomb branch, the Weyl group still acts to interchange the σ_a and X_a. (See e.g. [28, 29] for further explanation of these points.)

For our computations, the superpotential above can be equivalently replaced by

$$\begin{aligned} \mathcal{W} \;=\; & \frac{1}{2}\gamma^{ab}(\ln X_a)(\ln X_b) \;+\; \gamma^{ai}(\ln X_a)\left(\ln T_i^{-1}\right) \\ & + \sum_a (\ln q_a)(\ln X_a) \;+\; \sum_a \left(i\pi \sum_{\mu \text{ pos}'} \alpha_\mu^a \right)(\ln X_a) \\ & + \sum_i \left[\mathrm{Li}_2\left(X^{\rho_i}/T_i\right) \;+\; \frac{1}{4}\left(\ln\left(X^{\rho_i}/T_i\right)\right)^2 \right], \end{aligned} \tag{2.4}$$

which differs just in the first term. This second expression omits a term of the form

$$\frac{1}{2}\sum_{a,b} \gamma^{ab} \ln X_a, \tag{2.5}$$

as the sole effect of this in the equations of motion is to generate a constant term which changes the value of q, which we absorb into a redefinition. We will work with the superpotential (2.4) in the remainder of this paper.

For the case of gauge group $U(k)$, it can be shown that

$$i\pi \sum_{\mu \text{ pos}'} \alpha_\mu^a = i\pi(k-1) \tag{2.6}$$

for all a, so the effect will be to multiply q by the phase $(-)^{k-1}$.

It remains to describe the Chern-Simons levels γ^{ab} and γ^{ai}. We will take these to be given by $U(1)_{-1/2}$ quantization for chirals, following e.g. [25, section 2.2], which, as has been discussed elsewhere (see e.g. [5]), is the choice that reproduces mathematics results for quantum K theory in simple cases, and we will check also works for partial flag manifolds. An efficient way to describe this computation is to work at generic points along the Coulomb branch, where the nonabelian gauge symmetry is Higgsed to an abelian gauge symmetry. This means in part that we replace the original gauge group by its Cartan torus, $U(1)^r$ for r the rank. In addition, we must also keep track of the (massive) W bosons, which we can describe as fields of R charge 2 and $U(1)^r$ charges given by the root vectors. Then, to reproduce mathematical results for quantum K theory, we take the Chern-Simons levels to be[2]

$$\gamma^{ab} \; = \; \frac{1}{2} \sum_i (R_i - 1) Q_i^a Q_i^b, \tag{2.7}$$

where the sum is over all matter fields (including the W bosons), R_i is the R-charge of the matter field (0 for an ordinary field, 2 for a W-boson), and Q_i^a is the charge of the field under the ath $U(1)$. We will apply the same expression to Chern-Simons levels for flavor symmetries.

This expression can also be written in terms of Casimirs. For example, for a simple Lie group,

$$\sum_{i=1}^{\dim R} Q_i^a Q_i^b \; = \; \frac{1}{2} T_2(R) \, \delta^{ab}, \tag{2.8}$$

where for T_R^α matrices representing the Lie algebra generators in representation R,

$$\mathrm{Tr} \, T_R^\alpha T_R^\beta \; = \; \frac{1}{2} T_2(R) \, \delta^{\alpha\beta}. \tag{2.9}$$

The expression above uses the fact that the Q_i^a are, simultaneously, the components of the weight vectors for the representation R, and also the $U(1)^r$ charges of the physical fields in representation R, along the Coulomb branch. (See also e.g. [27, sections 13.2.4, 13.4.2] for related computations.) As normalization conventions for Casimirs can be ambiguous, in this paper we will refer specifically to expression (2.7) to compute levels.

We will apply the same expression to Chern-Simons levels for flavor symmetries.

3 Warmup: Grassmannians

In this section we will describe the computation of the equivariant quantum K theory ring of a Grassmannian $G(k, n)$, to prototype the methods we will apply to partial flag manifolds. Now, as noted earlier, quantum K theory rings of Grassmannians and flag manifolds have been computed previously in physics in e.g. the integrable systems literature. Here, we review an improved version of the computations in [5], as a warmup before developing new presentations of the quantum K theory rings of partial flag manifolds. Specifically, we will compute pertinent Chern-Simons levels in the spirit of the abelian/nonabelian correspondence (rather than Casimirs or via three-dimensiona $N = 4$ constructions, as was done previously), and also give a direct derivation of the λ_y class relations which were arrived at considerably less directly in [5].

[2]As an aside, one way to check the R-charge dependence is to describe the ordinary $\mathbb{P}^n$ model as a degree one hyperplane in $\mathbb{P}^{n+1}$. There, the GLSM superfield multiplying the hyperplane in the superpotential has nonzero R-charge, and the expression given for levels can be shown to be invariant.

3.1 Physical realization, Chern-Simons levels and equations of motion

The GLSM describing a Grassmannian $G(k,n)$ is a $U(k)$ gauge theory with n fundamentals. Proceeding as in section 2, we will compute the equivariant quantum K theory ring by computing Coulomb branch relations (equations of motion) resulting from the twisted one-loop effective superpotential of the S^1-reduced three-dimensional GLSM.

Since the starting point is a three-dimensional theory, one must pick Chern-Simons levels, and as discussed earlier, for purposes of computing quantum K theory rings, specific values of the levels are relevant. Those levels were computed previously in [5] using group theory, and can also be understood by starting with a three-dimensional $N=4$ theory and breaking to $N=2$, see e.g. [9]. Here, we will instead compute them differently, in the spirit of the abelian/nonabelian correspondence, using equation (2.7), and then as a consistency check compare to those obtained in [5].

First, consider the contribution from the fundamentals. Each fundamental has R-charge 0, and the ath element of a single fundamental has charge $+1$ under the ath $U(1)$, and 0 under the others. Thus, for the fundamentals,

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{c=1}^{k}(R_{ic}-1)Q^a_{ic}Q^b_{ic} \;=\; -\frac{1}{2}\sum_{i=1}^{n}\sum_{c=1}^{k}\delta^a_c\delta^b_c \;=\; -\frac{n}{2}\delta^{ab}. \tag{3.1}$$

Next, consider the W-bosons. The W-boson $W_{\mu\nu}$ (for $\mu\neq\nu$) has charge

$$Q(W_{\mu\nu})^a \;=\; -\delta^a_\mu+\delta^a_\nu, \tag{3.2}$$

and R-charge 2, so

$$\begin{aligned}
&\frac{1}{2}\sum_{\mu\neq\nu}(R_{\mu\nu}-1)Q(W_{\mu\nu})^aQ(W_{\mu\nu})^b \\
&= \frac{1}{2}\sum_{\mu\neq\nu}\left(-\delta^a_\mu+\delta^a_\nu\right)\left(-\delta^b_\mu+\delta^b_\nu\right), && (3.3)\\
&= \frac{1}{2}\sum_{\mu\neq\nu}\left(\delta^a_\mu\delta^b_\mu+\delta^a_\nu\delta^b_\nu-\delta^a_\mu\delta^b_\nu-\delta^b_\mu\delta^a_\nu\right), && (3.4)\\
&= \frac{1}{2}\left(2(k-1)\delta^{ab}\;-2\left(1-\delta^{ab}\right)\right), && (3.5)\\
&= k\delta^{ab}-1. && (3.6)
\end{aligned}$$

Putting this together, we find that equation (2.7) gives

$$\gamma^{ab} \;=\; \frac{1}{2}\sum_i(R_i-1)Q^a_iQ^b_i \;=\; -\frac{n}{2}\delta^{ab}\;+\;\left(k\delta^{ab}-1\right). \tag{3.7}$$

Now, let us compare to computations in our previous paper [5], where the level was computed using group theory. (See also [7].) There, from [5, equ'n (2.38)], the terms

$$\frac{1}{2}\gamma^{ab} (\ln X_a) (\ln X_b) \tag{3.8}$$

are computed as

$$\frac{1}{2}\gamma_{SU(k)} \sum_a (\ln X_a)^2 \; + \; \frac{\gamma_{U(1)} - \gamma_{SU(k)}}{2k} \left(\sum_a \ln X_a \right)^2, \tag{3.9}$$

or in other words,

$$\gamma^{ab} \; = \; \gamma_{SU(k)} \delta^{ab} \; + \; \frac{\gamma_{U(1)} - \gamma_{SU(k)}}{k}. \tag{3.10}$$

From [5, equ'n (2.36)-(2.37)],

$$\gamma_{U(1)} \; = \; -n/2, \quad \gamma_{SU(k)} \; = \; -\frac{n}{2} T_2(\text{fund}) - \frac{1}{2} T_2(\text{adj}) \; = \; k - n/2, \tag{3.11}$$

hence

$$\gamma_{SU(k)} \delta^{ab} \; + \; \frac{\gamma_{U(1)} - \gamma_{SU(k)}}{k} \; = \; (k - n/2)\delta^{ab} \; + \; \frac{[-n/2 - (k - n/2)]}{k}, \tag{3.12}$$

$$= \; (k - n/2)\delta^{ab} \; - \; 1, \tag{3.13}$$

which matches (3.7), as expected.

To write down the twisted one-loop effective superpotential for the equivariant quantum K theory of the Grassmannian $G(k,n)$, we also need the gauge-flavor Chern-Simons levels. These can be computed using the same formula (2.7), interpreting the global symmetry group as if it were a gauge symmetry. Specifically, we now interpret the n fundamentals of $U(k)$ as a single bifundamental in the $(\mathbf{k}, \overline{\mathbf{n}})$ of $U(k) \times U(n)$, of R charge 0. As any W bosons are charged only under one factor, not both, they do not contribute, and so equation (2.7) specializes to

$$\gamma^{ai} \; = \; \frac{1}{2} \sum_{bj} (R-1) Q^a_{bj} Q^i_{bj}, \tag{3.14}$$

$$= \; \frac{1}{2} \sum_{bj} (-1) \delta^a_b \delta^i_j, \tag{3.15}$$

$$= \; -\frac{1}{2}. \tag{3.16}$$

The effective one-loop twisted superpotential (2.4) then becomes

$$\mathcal{W} = \frac{1}{2}\left(k - \frac{n}{2}\right) \sum_a (\ln X_a)^2 - \frac{1}{2}\left(\sum_a \ln X_a\right)^2 - \frac{1}{2}\left(\sum_{a=1}^{k} \ln X_a\right)\left(\sum_i \ln T_i^{-1}\right) + \left(\ln(-)^{k-1} q\right) \sum_a \ln X_a + \sum_{i=1}^{n}\sum_{a=1}^{k}\left[\mathrm{Li}_2(X_a/T_i) + \frac{1}{4}\left(\ln(X_a/T_i)\right)^2\right], \tag{3.17}$$

$$= \frac{k}{2}\sum_a (\ln X_a)^2 - \frac{1}{2}\left(\sum_a \ln X_a\right)^2 + \left(\ln(-)^{k-1} q\right)\sum_a \ln X_a + \sum_{i=1}^{n}\sum_{a=1}^{k} \mathrm{Li}_2(X_a/T_i) + \frac{k}{4}\sum_i (\ln T_i)^2 . \tag{3.18}$$

The equations of motion (Coulomb branch equations) can be calculated from

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \ln X_a}\right) = 1. \tag{3.19}$$

The resulting equations of motion are

$$(-)^{k-1} q X_a^k = \left(\prod_b X_b\right)\left(\prod_i (1 - X_a/T_i)\right), \tag{3.20}$$

which matches [6, equ'n (35)], and is the equivariant extension of [5, equ'n (2.40)]. These are also the same as the Bethe ansatz equations of [16, equ'n (4.17)]. The reader should bear in mind that the X_a are constrained to be distinct (as coincident X_a lie within the 'excluded locus'), and that there is a residual S_k of the original gauge group $U(k)$ acting that interchanges the X_a, as explained in e.g. [28, 29]. We will not belabor these points in this analysis.

3.2 Characteristic polynomial and symmetrization

Now, for purposes of comparing to mathematical results, the equations of motion (3.20) need to be reworked into a more symmetric form in order to compare to mathematics results. A proposal for a presentation in terms of λ_y classes was given in [5, 6], through a combination of physics arguments in shifted variables and rigorous mathematics proofs using unrelated methods. We shall describe next how to obtain that presentation directly, in a fashion that will generalize to flag manifolds.

To that end, we first rewrite the equations of motion (3.20) in the form

$$\begin{aligned} & (-)^{k-1} q \, (X_a)^k \, e_n(T) \\ & \quad = \; (-)^n e_k(X) \prod_i (X_a - T_i) \, , \\ & \quad = \; (-)^n e_k(X) \left[(X_a)^n \; - \; e_1(T) \, (X_a)^{n-1} \; + \; e_2(T) \, (X_a)^{n-2} \; + \; \cdots \; + \; (-)^n e_n(T) \right] , \end{aligned} \tag{3.21}$$

where $e_\ell(x)$ denotes the ℓth elementary symmetric polynomial in indeterminates $\{x_a\}$. We can rearrange this to the form

$$\sum_{\ell=0}^{n} (-)^\ell \xi^{n-\ell} \left[e_k(X) \, e_\ell(T) \; + \; q \, e_n(T) \, e_{\ell-n+k}(0) \right] \; = \; 0, \tag{3.22}$$

for $\xi = X_a$, in the convention that $e_\ell(0) = \delta_{\ell,0}$. In the previous work [5], this was referred to as the characteristic polynomial.

Remark 3.1. *The Coulomb branch equations* (3.19) *from this note and those in [5, 6] coincide. However, the symmetrization procedure in this paper differs from the one in* loc.cit.*, and it gives a* different *characteristic polynomial. For example, the non-equivariant polynomial of* $G(2,4)$ *from [5, 6] is*

$$\begin{aligned} \xi^4 + (X_1X_2 - X_1 - X_2 - 3)\xi^3 + (-3X_1X_2 + 3X_1 + 3X_2 + 3)\xi^2 + \\ (3X_1X_2 + q - 3X_1 - 3X_2 - 1)\xi - X_1X_2 + X_1 + X_2, \end{aligned}$$

while the polynomial from (3.22) *above is (after making* $T_i = 1, 1 \leq i \leq 4$*):*

$$\xi^4 X_1X_2 - 4\xi^3 X_1X_2 + \xi^2(6X_1X_2 + q) - 4\xi X_1X_2 + X_1X_2.$$

Since the characteristic polynomial is nth order, it has n roots, which include the k values of X_a, as well as $n-k$ additional roots we shall label $\overline{X}_a$. We let w denote the combined collection $\{X_a, \overline{X}_{a'}\}$.

Now, we can simplify the characteristic polynomial using Vieta's formula, which says that for any order n polynomial $P(x)$

$$P(x) \; = \; a_n x^n \; + \; a_{n-1} x^{n-1} \; + \; \cdots \; + \; a_1 x \; + \; a_0, \tag{3.23}$$

with roots $r_1, \cdots, r_n$, the coefficients a_ℓ are related to the roots by

$$e_\ell(r) \; = \; (-)^\ell \, \frac{a_{n-\ell}}{a_n}, \tag{3.24}$$

where $r = \{r_i\}$ denotes the collection of roots. In the case of the characteristic polynomial (3.22),

$$a_{n-\ell} \; = \; (-)^\ell \left[e_k(X) \, e_\ell(T) \; + \; q \, e_n(T) \, e_{\ell-n+k}(0) \right] , \tag{3.25}$$

so Vieta's formula implies

$$e_k(X) e_\ell(w) \;=\; e_k(X)\, e_\ell(T) \;+\; q\, e_n(T)\, e_{\ell-n+k}(0). \tag{3.26}$$

Now, let us simplify (3.26). First, if $\ell \neq n-k$, then (3.26) immediately implies

$$e_\ell(w) \;=\; e_\ell(T). \tag{3.27}$$

For $\ell = n-k$, equation (3.26) implies

$$e_k(X)\, e_{n-k}(w) \;=\; e_k(X)\, e_{n-k}(T) \;+\; q\, e_n(T). \tag{3.28}$$

We also know that

$$e_\ell(w) \;=\; \sum_{r=0}^{n-k} e_{\ell-r}(X)\, e_r(\overline{X}), \tag{3.29}$$

for $\ell = 0, \cdots, n$. (This follows from general properties of symmetric polynomials, plus the fact that $e_r(\overline{X}) = 0$ for $r > n-k$, as there are only $n-k$ $\overline{X}$ indeterminates.) In particular,

$$e_n(w) \;=\; \sum_{r=0}^{n-k} e_{n-r}(X)\, e_r(\overline{X}), \tag{3.30}$$

but $e_{n-r}(X) = 0$ for $n-r > k$ (meaning, $r < n-k$) as there are only k X indeterminates. Thus, using the above and also (3.27), we can write

$$e_n(T) \;=\; e_n(w) \;=\; e_k(X)\, e_{n-k}(\overline{X}). \tag{3.31}$$

Thus, equation (3.28) implies

$$e_{n-k}(w) \;=\; e_{n-k}(T) \;+\; q\, e_{n-k}(\overline{X}). \tag{3.32}$$

Putting these together, we have

$$e_\ell(w) \;=\; \left\{ \begin{array}{cc} e_\ell(T) & \ell \neq n-k, \\ e_{n-k}(T) + q\, e_{n-k}(\overline{X}) & \ell = n-k. \end{array} \right. \tag{3.33}$$

For later use, from (3.32) we have

$$\sum_{r=0}^{n-k} e_{n-k-r}(X)\, e_r(\overline{X}) \;=\; e_{n-k}(w) \;=\; e_{n-k}(T) \;+\; q\, e_{n-k}(\overline{X}), \tag{3.34}$$

hence

$$\sum_{r=0}^{n-k-1} e_{n-k-r}(X)\, e_r(\overline{X}) \;+\; e_{n-k}(\overline{X}) \;=\; e_{n-k}(T) \;+\; q\, e_{n-k}(\overline{X}), \tag{3.35}$$

which can be rearrangd to

$$\sum_{r=0}^{n-k-1} e_{n-k-r}(X)\, e_r(\overline{X}) \;+\; (1-q)e_{n-k}(\overline{X}) \;=\; e_{n-k}(T). \tag{3.36}$$

Returning to (3.26), applying (3.31) and cancelling out a common factor of $e_k(X)$, we have

$$e_\ell(w) \;=\; e_\ell(T) \;+\; q\, e_{n-k}(\overline{X}) e_{\ell-n+k}(0). \tag{3.37}$$

Applying (3.29) we then have

$$\sum_{r=0}^{n-k} e_{\ell-r}(X) e_r(\overline{X}) \;=\; e_\ell(T) \;+\; q\, e_{n-k}(\overline{X})\, e_{\ell-n+k}(0). \tag{3.38}$$

Equation (3.38) is the key result from Vieta's equation. Next, we solve it algebraically.

To that end, note that equation (3.38) is the degree-ℓ piece of

$$\left(\sum_{r=0}^{k} y^r e_r(X)\right)\left(\sum_{t=0}^{n-k} y^t e_t(\overline{X})\right) \;=\; \sum_{r=0}^{n} y^r e_r(T) \;+\; q y^{n-k} e_{n-k}(\overline{X}), \tag{3.39}$$

which implies

$$\sum_{r=0}^{n-k} y^r e_r(\overline{X}) \;=\; \left(\sum_{r=0}^{n} y^r e_r(T)\right)\left(\sum_{t=0}^{\infty} (-)^t y^t h_t(X)\right) \;+\; q y^{n-k} e_{n-k}(\overline{X}) \left(\sum_{t=0}^{\infty} (-)^t y^t h_t(X)\right), \tag{3.40}$$

for $h_t(X)$ the complete homogeneous symmetric polynomial of degree t in $\{X\}$. We read off that for $\ell < n-k$,

$$e_\ell(\overline{X}) \;=\; \sum_{r=0}^{n-k} (-)^r e_{\ell-r}(T)\, h_r(X), \tag{3.41}$$

and for $\ell = n-k$,

$$e_{n-k}(\overline{X}) \;=\; \sum_{r=0}^{n-k} (-)^r e_{n-k-r}(T)\, h_r(X) \;+\; q e_{n-k}(\overline{X}), \tag{3.42}$$

hence

$$e_{n-k}(\overline{X}) \;=\; (1-q)^{-1} \sum_{r=0}^{n-k} (-)^r e_{n-k-r}(T)\, h_r(X). \tag{3.43}$$

To simplify the expression above, define a $(n-k)$-element collection $\{\hat{X}\}$ by

$$e_\ell(\hat{X}) \;=\; \sum_{r=0}^{n-k} (-)^r e_{\ell-r}(T)\, h_r(X), \tag{3.44}$$

then we can write

$$e_\ell(\overline{X}) \;=\; \left\{ \begin{array}{cc} e_\ell(\hat{X}) & \ell < n-k, \\ (1-q)^{-1} e_{n-k}(\hat{X}) & \ell = n-k. \end{array} \right. \tag{3.45}$$

3.3 Interpretation in terms of bundles: λ_y class presentation

In this section we will interpret (3.38) in terms of λ_y classes. (We emphasize at the start that any interpretation may be slightly ambiguous; we will utilize comparisons to mathematics to justify our proposal.)

First, for reasons described earlier and elsewhere, we associate the $\{X\}$ with Chern roots of the universal subbundle $\mathcal{S}$, meaning that

$$e_\ell(X) \;\sim\; \wedge^\ell \mathcal{S}. \tag{3.46}$$

In addition, it is natural to associate the $\{\hat{X}\}$ with Chern roots of the universal quotient bundle $\mathbb{C}^n/\mathcal{S}$, meaning

$$e_\ell(\hat{X}) \;\sim\; \wedge^\ell (\mathbb{C}^n/\mathcal{S}). \tag{3.47}$$

Classically, this follows from the defining property (3.44) of the $\{\hat{X}\}$. In more detail, this follows from the short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}^n/\mathcal{S} \longrightarrow 0, \tag{3.48}$$

which implies

$$c(\mathcal{S})\, c(\mathbb{C}^n/\mathcal{S}) \;=\; c(\mathbb{C}^n), \tag{3.49}$$

and in K theory,

$$\lambda_y(\mathcal{S})\, \lambda_y(\mathbb{C}^n/\mathcal{S}) \;=\; \lambda_y(\mathbb{C}^n) \tag{3.50}$$

(where $\lambda_y(\mathcal{E}) = 1 + y\mathcal{E} + y^2 \wedge^2 \mathcal{E} + \cdots$), and hence the relation (3.44), after algebra. (This association will be justified by the fact that this will correctly reproduce rigorous results for quantum K theory ring presentations [6] and also quantum cohomology rings.)

We formally associate the $\{\overline{X}\}$ with Chern roots of a bundle $\widetilde{\mathcal{Q}}$ of rank $n-k$. From the results of the last section, we can identify

$$e_\ell(\overline{X}) \;\sim\; \wedge^\ell \widetilde{\mathcal{Q}} \;=\; \left\{ \begin{array}{cc} \wedge^\ell (\mathbb{C}^n/\mathcal{S}) & \ell < n-k, \\ (1-q)^{-1} \wedge^\ell (\mathbb{C}^n/\mathcal{S}) & \ell = n-k. \end{array} \right. \tag{3.51}$$

The reader should note that this means that for $q \neq 0$, $\widetilde{\mathcal{Q}}$ is not a classical bundle, but instead appears to be more nearly some sort of quantum exterior product which we shall not attempt to define carefully here.

With this dictionary in mind, we can now interpret (3.38), namely

$$\sum_{r=0}^{n-k} e_{\ell-r}(X) e_r(\overline{X}) \; = \; e_\ell(T) \; + \; q\, e_{n-k}(\overline{X})\, e_{\ell-n+k}(0), \tag{3.52}$$

as

$$\sum_{r=0}^{n-k} \wedge^{\ell-r}(\mathcal{S}) \star \wedge^r \widetilde{\mathcal{Q}} \; = \; \wedge^\ell(\mathbb{C}^n) \; + \; q\delta_{\ell,n-k} \det \widetilde{\mathcal{Q}}, \tag{3.53}$$

or more elegantly,

$$\lambda_y(\mathcal{S}) \star \lambda_y(\widetilde{\mathcal{Q}}) \; = \; \lambda_y(\mathbb{C}^n) \; + \; y^{n-k} q \det \widetilde{\mathcal{Q}}. \tag{3.54}$$

Equation (3.54) is the λ_y class relation defining the T-equivariant quantum K theory ring of the Grassmannian $G(k,n)$.

We can give an alternate presentation that does not involve $\widetilde{\mathcal{Q}}$ as follows. Returning to (3.38) and again using the dictionary (3.51), we have

$$\begin{aligned} \sum_{r=0}^{n-k-1} \wedge^{\ell-r}(\mathcal{S}) \star \wedge^r(\mathbb{C}^n/\mathcal{S}) \; + \; \frac{1}{1-q} \wedge^{\ell-(n-k)} \mathcal{S} \star \det(\mathbb{C}^n/\mathcal{S}) & \\ = \; \wedge^\ell \mathbb{C}^n \; + \; \frac{1}{1-q} \det(\mathbb{C}^n/\mathcal{S}) \delta_{\ell,n-k}, & \end{aligned} \tag{3.55}$$

which can be rearranged to the form

$$\sum_{r=0}^{n-k} \wedge^{\ell-r} \mathcal{S} \star \wedge^r(\mathbb{C}^n/\mathcal{S}) \; = \; \wedge^\ell \mathbb{C}^n \; - \; \frac{q}{1-q} \det(\mathbb{C}^n/\mathcal{S}) \star \left(\wedge^{\ell-n+k} \mathcal{S} \; - \; \mathcal{O}\delta_{\ell,n-k} \right). \tag{3.56}$$

Adding factors of y, this becomes the λ_y relation

$$\lambda_y(\mathcal{S}) \star \lambda_y(\mathbb{C}^n/\mathcal{S}) \; = \; \lambda_y(\mathbb{C}^n) \; - \; y^{n-k} \frac{q}{1-q} \det(\mathbb{C}^n/\mathcal{S}) \star (\lambda_y(\mathcal{S}) - 1). \tag{3.57}$$

Equation (3.57) is a second form of the λ_y class relation defining the T-equivariant quantum K theory ring of the Grassmannian $G(k,n)$. It was argued, less efficiently, in [5], and proven rigorously using different methods in [6]. Our analysis of equivariant quantum K theory rings of flag manifolds, though more complex, will be closely analogous to that we have demonstrated here for Grassmannians.

3.4 Duality: $G(k,n) \cong G(n-k,n)$

In this section we discuss how our presentation of the quantum K theory ring of the Grassmannian behaves under the duality of Grassmannians that relates $G(k,n)$ to the dual

$G(n-k,n)$. This relates the universal subbundle $\mathcal{S}$, quotient bundle $\mathcal{Q} = \mathbb{C}^n/\mathcal{S}$ and vector space $\mathbb{C}^n$ of the Grassmannian $G(k,n)$ to the $\mathcal{S}'$, $\mathcal{Q}'$ and $(\mathbb{C}^n)'$ of the dual Grassmannian $G(n-k,n)$ classically as

$$\mathcal{S}' \;=\; \mathcal{Q}^*, \;\; \mathcal{Q}' \;=\; \mathcal{S}^*, \;\; (\mathbb{C}^n)' \;=\; (\mathbb{C}^n)^*. \tag{3.58}$$

First, we recall that this mathematical duality is realized physically as an IR duality of the gauge theories: $G(k,n)$ is realized by a $U(k)$ gauge theory with n fundamentals and Chern-Simons levels

$$\gamma^{ab} = -\frac{n}{2}\delta^{ab} + (k\delta^{ab} - 1) \quad \text{and} \quad \gamma^{ai} = -\frac{1}{2}\,, \tag{3.59}$$

while the dual Grassmannian $G(n-k,n)$ is realized by a $U(n-k)$ gauge theory with n fundamentals and Chern-Simons levels

$$\gamma^{ab} = -\frac{n}{2}\delta^{ab} + ((n-k)\delta^{ab} - 1) \quad \text{and} \quad \gamma^{ai} = -\frac{1}{2} \tag{3.60}$$

(replacing k by $n-k$ in Equation (3.59)).

Now, we turn to the mathematics. If we let quantities on the dual Grassmannian be denoted with a prime ($\prime$), then the λ_y relation (3.57) on the dual Grassmannian is

$$\lambda_{y'}(\mathcal{S}') \star \lambda_{y'}((\mathbb{C}^n)^*/\mathcal{S}') \;=\; \lambda_{y'}((\mathbb{C}^n)^*) \;-\; (y')^k \frac{q'}{1-q'} \det((\mathbb{C}^n)^*/\mathcal{S}') \star (\lambda_{y'}(\mathcal{S}') - 1)\,. \tag{3.61}$$

Reinterpreting the quantities above on the original Grassmannian, this is the relation

$$\lambda_{y'}(\mathcal{Q}^*) \star \lambda_{y'}(\mathcal{S}^*) \;=\; \lambda_{y'}((\mathbb{C}^n)^*) \;-\; (y')^k \frac{q'}{1-q'} \det(\mathcal{S}^*) \star (\lambda_{y'}(\mathcal{Q}^*) - 1)\,. \tag{3.62}$$

As a consistency check, note the classical limit is correct, by virtue of the dual of the sequence defining the universal subbundle $\mathcal{S}$. We will not use this relation in this paper, but include it for completeness.

4 Partial flag manifolds

In this section we compute the equivariant quantum K theory rings of partial flag manifolds. Although the details are considerably more complicated, the underlying logic is the same as that presented in section 3 for Grassmannians. We should add that our focus is on deriving novel presentations; the underlying Coulomb branch description of the quantum K theory ring has been previously obtained from the corresponding Bethe ansatz, for example, see e.g. [9], and other presentations of the quantum K theory ring have been described in mathematics in e.g. [12, 14, 15].

4.1 Physical realization and twisted effective superpotential

The flag manifold $F(k_1, \ldots, k_s; n)$ is realized physically as a $U(k_1) \times U(k_2) \times \ldots \times U(k_s)$ gauge theory with matter fields which are bifundamentals in the $(\mathbf{k_i}, \overline{\mathbf{k_{i+1}}})$ representation of $U(k_i) \times U(k_{i+1})$ for $i = 1, \ldots, s-1$ and n fundamentals of $U(k_s)$, see e.g. [30]. We will use $\Phi^{a_i, a_{i+1}}$ $(a_i = 1, \ldots, k_i; a_{i+1} = 1, \ldots, k_{i+1})$ to denote the bifundamental of $U(k_i) \times U(k_{i+1})$. We will also use the notation $\Phi^{a_s, j}$ $(j = 1, \ldots, n)$ to denote the fundamental of $U(k_s)$. We will use the convention $k_0 = 0$.

Generically along the Coulomb branch, each $U(k_i)$ is broken to $U(1)^{k_i}$, giving rise to $k_i(k_i - 1)$ W-bosons W^{m_i, n_i} $(m_i, n_i = 1, \ldots, k_i)$. The fields charged under the b_i-th $U(1)$ factor of $U(1)^{k_i}$ are $\Phi^{a_i, a_{i+1}}$, Φ^{a_{i-1}, a_i} and W^{m_i, n_i}, with charges as listed below:

Field	Charge
$\Phi^{a_i, a_{i+1}}$	$Q_{b_i}^{a_i, a_{i+1}} = +\delta_{b_i}^{a_i}$
Φ^{a_{i-1}, a_i}	$Q_{b_i}^{a_{i-1}, a_i} = -\delta_{b_i}^{a_i}$
W^{m_i, n_i}	$Q_{b_i}^{m_i, n_i} = -\delta_{m_i}^{b_i} + \delta_{n_i}^{b_i}$

Also, associated to the b_i-th $U(1)$ factor is a sigma field we denote $\sigma_{b_i}^{(i)}$, encoded in $X_{b_i}^{(i)}$.

Then, the effective twisted superpotential (2.4) is given by

$$\mathcal{W} \;=\; \sum_{i=1}^{s} \mathcal{W}_i, \tag{4.1}$$

where

$$\begin{aligned} \mathcal{W}_i \;=\;& \frac{1}{2} \sum_{j=1}^{s} \sum_{a_i=1}^{k_i} \sum_{b_j=1}^{k_j} \gamma^{a_i b_j} \left(\ln X_{a_i}^{(i)}\right) \left(\ln X_{b_j}^{(j)}\right) \;+\; \left(\ln \left((-)^{k_i - 1} q_i\right)\right) \sum_{a_i=1}^{k_i} \ln X_{a_i}^{(i)} \\ &+ \sum_{a_i=1}^{k_i} \sum_{a_{s+1}=1}^{n} \gamma^{a_i a_{s+1}} \left(\ln X_{a_i}^{(i)}\right) \left(\ln T_{a_{s+1}}^{-1}\right) \\ &+ \sum_{a_i=1}^{k_i} \sum_{a_{i+1}=1}^{k_{i+1}} \left[\mathrm{Li}_2 \left(X_{a_i}^{(i)} / X_{a_{i+1}}^{(i+1)} \right) \;+\; \frac{1}{4} \left(\ln \left(X_{a_i}^{(i)} / X_{a_{i+1}}^{(i+1)} \right) \right)^2 \right], \end{aligned} \tag{4.2}$$

in conventions where $k_0 = 0$, $k_{s+1} = n$, and $X_{a_{s+1}}^{(s+1)} = T_{a_{s+1}}$.

It remains to compute the Chern-Simons levels, using equation (2.7).

First, for the first gauge factor, we have

$$\gamma^{a_1 b_j} \;=\; -\frac{1}{2}\sum_{c_1,c_2} Q_{a_1}^{c_1,c_2} Q_{b_j}^{c_1,c_2} \;+\; \frac{1}{2}\sum_{m_1\neq n_1=1}^{k_1} Q_{a_1}^{m_1,n_1} Q_{b_j}^{m_1,n_1}, \tag{4.3}$$

$$\;=\; \delta_{1,j}\left(-\frac{k_2}{2}\delta^{a_1 b_j} \;+\; k_1\delta^{a_1 b_j} - 1\right) \;+\; \frac{1}{2}\delta_{2,j}. \tag{4.4}$$

Then, for $1 < i, j < s$, the only nonzero levels $k^{a_i b_j}$ have $j \in \{i-1, i, i+1\}$. For those values,

$$\begin{aligned}
\gamma^{a_i b_j} &= -\frac{1}{2}\sum_{c_i}\sum_{c_{i+1}} Q_{a_i}^{c_i,c_{i+1}} Q_{b_j}^{c_j,c_{j+1}} - \frac{1}{2}\sum_{c_{i-1}}\sum_{c_i} Q_{a_i}^{c_{i-1},c_i} Q_{b_j}^{c_{j-1},c_j} + \frac{1}{2}\sum_{m_i\neq n_i=1}^{k_i} Q_{a_i}^{m_i,n_i} Q_{b_j}^{m_i,n_i}, \\
&= \delta_{i,j}\left(-\frac{k_{i+1}}{2}\delta^{a_i b_i} - \frac{k_{i-1}}{2}\delta^{a_i b_i} + k_i\delta^{a_i b_i} - 1\right) \;+\; \frac{1}{2}\left(\delta_{j,i+1} \;+\; \delta_{j,i-1}\right), \\
&= \delta_{i,j}\left[\left(k_i - \frac{k_{i-1}}{2} - \frac{k_{i+1}}{2}\right)\delta^{a_i b_i} - 1\right] \;+\; \frac{1}{4}\left(\delta_{j,i+1} \;+\; \delta_{j,i-1} \;+\; \delta_{i,j+1} \;+\; \delta_{i,j-1}\right),
\end{aligned} \tag{4.5}$$

where in the last line we have written the last two terms in an explicitly symmetric fashion. Note that the levels $\gamma^{a_1 b_i}$ are a special case if we define $k_{-1} = 0$.

For the last gauge factor, we have

$$\gamma^{a_s b_i} \;=\; -\frac{1}{2}\sum_{c_{s-1},c_s} Q_{a_s}^{c_{s-1},c_2} Q_{b_i}^{c_{s-1},c_s} \;+\; \frac{1}{2}\sum_{m_s\neq n_i=1}^{k_1} Q_{a_s}^{m_s,n_i} Q_{b_i}^{m_s,n_i}, \tag{4.6}$$

$$\;=\; \delta_{s,i}\left(-\frac{k_{s-1}}{2}\delta^{a_s b_i} \;+\; k_s\delta^{a_s b_i} - 1\right). \tag{4.7}$$

Finally, the gauge-flavor Chern-Simons level can be computed in the same fashion as in section 3:

$$\gamma^{a_i j} \;=\; -\frac{1}{2}\delta_{i,s}. \tag{4.8}$$

Next, we plug these levels into the expression (4.1) and simplify to find that the full superpotential is

$$\begin{aligned}
\mathcal{W} \;=\; & \frac{1}{2}\sum_{i=1}^{s}(k_i - 1)\sum_{a_i=1}^{k_i}\left(\ln X_{a_i}^{(i)}\right)^2 \;-\; \sum_{i=1}^{s}\sum_{1\leq a_i<b_i\leq k_i}\left(\ln X_{a_i}^{(i)}\right)\left(\ln X_{b_i}^{(i)}\right) \\
& + \sum_{i=1}^{s}\left(\ln\left((-)^{k_i-1}q_i\right)\right)\sum_{a_i=1}^{k_i}\left(\ln X_{a_i}^{(i)}\right) \\
& + \sum_{i=1}^{s}\sum_{a_i=1}^{k_i}\sum_{a_{i+1}=1}^{k_{i+1}} \mathrm{Li}_2\left(X_{a_i}^{(i)}/X_{a_{i+1}}^{(i+1)}\right)
\end{aligned} \tag{4.9}$$

in the conventions that $k_0 = 0$, $k_{s+1} = n$, and $X^{(s+1)}_{a_{s+1}} = T_{a_{s+1}}$.

4.2 Coulomb branch equations

As discussed earlier, the Coulomb branch equations can be calculated from derivatives of the superpotential (3.19), which we repeat below:

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \ln X^{(i)}_{a_i}}\right) = 1. \tag{4.10}$$

(They can also be obtained as a limit of e.g. Bethe ansatz computations, see e.g. [9]; our ultimate goal is proposals for new ring presentations, in this section we are merely reviewing routes through the physics.) Computationally, to evaluate the expression above, it is helpful to collect all of the terms involving just the $X^{(i)}$ (associated with the ith gauge group factor), which are given below:

$$\begin{aligned}\frac{1}{2}(k_i-1)\sum_{a_i=1}^{k_i}\left(\ln X^{(i)}_{a_i}\right)^2 - \sum_{1\le a_i<b_i\le k_i}\left(\ln X^{(i)}_{a_i}\right)\left(\ln X^{(i)}_{b_i}\right) + \left(\ln\left((-)^{k_i-1}q_i\right)\right)\sum_{a_i=1}^{k_i}\left(\ln X^{(i)}_{a_i}\right)\\ + \sum_{a_i=1}^{k_i}\sum_{a_{i+1}=1}^{k_{i+1}}\mathrm{Li}_2\left(X^{(i)}_{a_i}/X^{(i+1)}_{a_{i+1}}\right) + \sum_{a_{i-1}=1}^{k_{i-1}}\sum_{a_i=1}^{k_i}\mathrm{Li}_2\left(X^{(i-1)}_{a_{i-1}}/X^{(i)}_{a_i}\right)\end{aligned} \tag{4.11}$$

in the conventions that $k_0 = 0$, $k_{s+1} = n$, and $X^{(s+1)}_{a_{s+1}} = T_{a_{s+1}}$.

Plugging into (3.19), we have

$$(-)^{k_i-1}q_i\left(X^{(i)}_{a_i}\right)^{k_i}\prod_{b_{i-1}=1}^{k_{i-1}}\left(1-\frac{X^{(i-1)}_{b_{i-1}}}{X^{(i)}_{a_i}}\right) = \left(\prod_{b_i=1}^{k_i}X^{(i)}_{b_i}\right)\prod_{b_{i+1}=1}^{k_{i+1}}\left(1-\frac{X^{(i)}_{a_i}}{X^{(i+1)}_{b_{i+1}}}\right), \tag{4.12}$$

for $a_i = 1, \ldots, k_i$ and $i = 1, \ldots, s$. For the first component, $i = 1$, since $k_{i-1} = 0$, the left hand side of (4.12) is simply $(-)^{k_i-1}q_i\left(X^{(i)}_{a_i}\right)^{k_i}$.

4.3 Characteristic polynomials and symmetrization

Next we will symmetrize (4.12) to obtain characteristic polynomials and then use Vieta relations to form the physics relations for the quantum K-theory ring, much as we did for Grassmannians in section 3.2.

First, after some rearrangement, (4.12) becomes

$$(-)^{k_i-1} e_{k_{i+1}}\left(X^{(i+1)}\right) q_i \left(X^{(i)}_{a_i}\right)^{k_i-k_{i-1}} \prod_{b_{i-1}=1}^{k_{i-1}} \left(X^{(i)}_{a_i} - X^{(i-1)}_{b_{i-1}}\right)$$

$$= (-)^{k_{i+1}} e_{k_i}\left(X^{(i)}\right) \prod_{b_{i+1}=1}^{k_{i+1}} \left(X^{(i)}_{a_i} - X^{(i+1)}_{b_{i+1}}\right) \tag{4.13}$$

for $a = 1, \cdots, k_i$, where $e_i(x)$ is the i-th elementary symmetric polynomial in the indeterminates $\{x_a\}$.

Now we use the expansion

$$\prod_{j=1}^{n}(\xi - x_j) = \xi^n - e_1(x)\xi^{n-1} + e_2(x)\xi^{n-2} + \cdots + (-)^n e_n(x), \tag{4.14}$$

to write (4.13) as

$$\begin{aligned} &(-)^{k_i-1} e_{k_{i+1}}\left(X^{(i+1)}\right) q_i \left[\xi^{k_i} - e_1\left(X^{(i-1)}\right)\xi^{k_i-1} + \cdots + (-)^{k_{i-1}} e_{k_{i-1}}\left(X^{(i-1)}\right)\xi^{k_i-k_{i-1}}\right] \\ &\quad = (-)^{k_{i+1}} e_{k_i}\left(X^{(i)}\right)\left[\xi^{k_{i+1}} - e_1\left(X^{(i+1)}\right)\xi^{k_{i+1}-1} + \cdots + (-)^{k_{i+1}} e_{k_{i+1}}\left(X^{(i+1)}\right)\right]. \end{aligned} \tag{4.15}$$

for

$$\xi \;=\; X^{(i)}_{a_i}, \tag{4.16}$$

or equivalently, after rearrangement,

$$\sum_{\ell=0}^{k_{i+1}} (-)^{\ell} \xi^{k_{i+1}-\ell}\left[e_{k_i}\left(X^{(i)}\right) e_\ell\left(X^{(i+1)}\right) + q_i e_{k_{i+1}}\left(X^{(i+1)}\right) e_{\ell-k_{i+1}+k_i}\left(X^{(i-1)}\right)\right] \;=\; 0, \tag{4.17}$$

in conventions for which $e_\ell(x) = 0$ for $\ell < 0$. We call this the characteristic polynomial equation, which is of order k_{i+1}. We denote the k_{i+1} roots of this equation by w's. Of those k_{i+1} roots, k_i are the $X^{(i)}_a$, and the remainder are denoted $\bar{X}^{(i)}_a$. Then from Vieta's formula (3.24) we have the quantum K-theory ring relations,

$$e_{k_i}\left(X^{(i)}\right) e_\ell(w) \;=\; e_{k_i}\left(X^{(i)}\right) e_\ell\left(X^{(i+1)}\right) \;+\; q_i e_{k_{i+1}}\left(X^{(i+1)}\right) e_{\ell-k_{i+1}+k_i}\left(X^{(i-1)}\right), \tag{4.18}$$

for $\ell = 0, \cdots, k_{i+1}$, where

$$e_\ell(w) = \sum_{r=0}^{k_{i+1}-k_i} e_{\ell-r}\left(X^{(i)}\right) e_r\left(\bar{X}^{(i)}\right). \tag{4.19}$$

This expression can be simplified. First note that for $\ell < k_{i+1} - k_i$ or $\ell > k_{i+1} - k_i + k_{i-1}$, from (4.18) we have

$$e_\ell(w) = e_\ell\left(X^{(i+1)}\right), \tag{4.20}$$

where in our notation, $e_\ell\left(X^{(i-1)}\right) = 0$ if $\ell < 0$ or $\ell > k_{i-1}$ (since there are k_{i-1} $X^{(i-1)}$'s).

When $\ell = k_{i+1} - k_i$, equation (4.18) implies

$$e_{k_i}\left(X^{(i)}\right) e_{k_{i+1}-k_i}(w) \; = \; e_{k_i}\left(X^{(i)}\right) e_{k_{i+1}-k_i}\left(X^{(i+1)}\right) \; + \; q_i \, e_{k_{i+1}}\left(X^{(i+1)}\right). \tag{4.21}$$

To eliminate $e_{k_{i+1}}\left(X^{(i+1)}\right)$ in this equation, we take $\ell = k_{i+1}$ in (4.19) which, using (4.20), implies

$$e_{k_i}\left(X^{(i)}\right) e_{k_{i+1}-k_i}\left(\bar{X}^{(i)}\right) \; = \; e_{k_{i+1}}\left(X^{(i+1)}\right) \tag{4.22}$$

(as the only nonzero contribution to the sum is from the case $s = k_{i+1} - k_i$). Therefore, plugging this into (4.21),

$$e_{k_{i+1}-k_i}(w) \; = \; e_{k_{i+1}-k_i}\left(X^{(i+1)}\right) \; + \; q_i \, e_{k_{i+1}-k_i}\left(\bar{X}^{(i)}\right). \tag{4.23}$$

Applying (4.19), we can write this as

$$\sum_{r=0}^{k_{i+1}-k_i-1} e_{k_{i+1}-k_i-r}\left(X^{(i)}\right) e_r\left(\overline{X}^{(i)}\right) \; + \; (1-q_i) \, e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) \; = \; e_{k_{i+1}-k_i}\left(X^{(i+1)}\right). \tag{4.24}$$

Now, we are ready to simplify and derive the quantum K theory relations. Using (4.22) to simplify (4.18), we have

$$e_\ell(w) \; = \; e_\ell\left(X^{(i+1)}\right) \; + \; q_i \, e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) e_{\ell-k_{k+1}+k_i}\left(X^{(i-1)}\right). \tag{4.25}$$

Using (4.19), this becomes

$$\sum_{r=0}^{k_{i+1}-k_i} e_{\ell-r}\left(X^{(i)}\right) e_r\left(\overline{X}^{(i)}\right) \; = \; e_\ell\left(X^{(i+1)}\right) \; + \; q_i \, e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) e_{\ell-k_{i+1}+k_i}\left(X^{(i-1)}\right). \tag{4.26}$$

This is the the proposed quantum K theory relation, and is also the key algebraic relation needed to relate to λ_y class presentations in the next section.

Proceeding as for Grassmannians, let us solve this equation algebraically for the $e_\ell(\overline{X}^{(i)})$. To do so, we rewrite the expression above as the degree-ℓ part of

$$\begin{aligned}
&\left(\sum_{r=0}^{k_i} y^r e_r\left(X^{(i)}\right)\right)\left(\sum_{t=0}^{k_{i+1}-k_i} y^t e_t\left(\overline{X}^{(i)}\right)\right) \qquad (4.27)\\
&= \; \left(\sum_{r=0}^{k_{i+1}} y^r e_r\left(X^{(i+1)}\right)\right) \; + \; q_i \, y^{k_{i+1}-k_i} e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right)\left(\sum_{r=0}^{k_{i-1}} y^r e_r\left(X^{(i-1)}\right)\right),
\end{aligned}$$

hence

$$\left(\sum_{r=0}^{k_{i+1}-k_i} y^r e_r\left(\overline{X}^{(i)}\right)\right) = \left(\sum_{r=0}^{k_{i+1}} y^r e_r\left(X^{(i+1)}\right)\right)\left(\sum_{t=0}^{\infty}(-)^t y^t h_t\left(X^{(i)}\right)\right) + q_i\, y^{k_{i+1}-k_i}\, e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right)\left(\sum_{r=0}^{k_{i-1}} y^r e_r\left(X^{(i-1)}\right)\right)\left(\sum_{t=0}^{\infty}(-)^t y^t h_t\left(X^{(i)}\right)\right). \tag{4.28}$$

To simplify the expression above, we define $\{\hat{X}^{(i)}\}$ by

$$e_\ell\left(\hat{X}^{(i)}\right) = \sum_{r=0}^{k_{i+1}}(-)^r e_{\ell-r}\left(X^{(i+1)}\right) h_r\left(X^{(i)}\right), \tag{4.29}$$

then we see that for $\ell < k_{i+1} - k_i$,

$$e_\ell\left(\overline{X}^{(i)}\right) = e_\ell\left(\hat{X}^{(i)}\right), \tag{4.30}$$

and for $\ell = k_{i+1} - k_i$,

$$e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) = e_{k_{i+1}-k_i}\left(\hat{X}^{(i)}\right) + q_i\, e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) e_0\left(\hat{X}^{(i-1)}\right), \tag{4.31}$$

hence

$$e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) = (1-q_i)^{-1} e_{k_{i+1}-k_i}\left(\hat{X}^{(i)}\right). \tag{4.32}$$

In summary,

$$e_\ell\left(\overline{X}^{(i)}\right) = \begin{cases} e_\ell\left(\hat{X}^{(i)}\right) & \ell < k_{i+1}-k_i, \\ (1-q_i)^{-1} e_{k_{i+1}-k_i}\left(\hat{X}^{(i)}\right) & \ell = k_{i+1}-k_i. \end{cases} \tag{4.33}$$

4.4 Interpretation in terms of bundles: λ_y class presentation

In this section we interpret (4.26) in terms of bundles, and utilize it to generate the λ_y class presentations of the quantum K theory ring. Although the details are more elaborate, the underlying logic is the same as that presented for Grassmannians in section 3.3. (Also, just as there, we emphasize that any interpretation may be slightly ambiguous; we will utilize comparisons to mathematics to justify our proposal.)

For simplicity, we focus on three successive steps k_{i-1}, k_i, k_{i+1} in the flag of a flag manifold $F(k_1, k_2, \cdots, k_s, n)$. These correspond to three bundles, $\mathcal{S}_{i-1} \subset \mathcal{S}_i \subset \mathcal{S}_{i+1}$, of ranks k_{i-1}, k_i, k_{i+1}. We denote the quotient bundle by $\mathcal{S}_{i+1}/\mathcal{S}_i$, which is of rank $k_{i+1} - k_i$. The

$\{X^{(i)}\}$ are associated with the Chern roots of the universal subbundle $\mathcal{S}_i$. In addition, it is natural to associate the $\{\hat{X}^{(i)}\}$ with the Chern roots of the universal quotient bundle $\mathcal{S}_{i+1}/\mathcal{S}_i$. Classically, this follows from the defining property (4.29) of the $\{\hat{X}^{(i)}\}$. In more detail, this is a consequence of the short exact sequence

$$0 \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_{i+1} \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \longrightarrow 0, \tag{4.34}$$

which implies

$$c(\mathcal{S}_i)\, c(\mathcal{S}_{i+1}/\mathcal{S}_i) \;=\; c(\mathcal{S}_{i+1}), \tag{4.35}$$

and hence the relation (4.29), after algebra. (This association will be justified by the fact that this will correctly reproduce rigorous results for quantum K theory rings of e.g. incidence varieties and also quantum cohomology rings.)

We formally associate the $\{\bar{X}^{(i)}\}$ with a bundle $\widetilde{\mathcal{R}}_i$ of rank $k_{i+1} - k_i$. From the results of the last section, we can identify

$$e_\ell\left(\overline{X}^{(i)}\right) \;\leftrightarrow\; \wedge^\ell \widetilde{\mathcal{R}}_i \;=\; \left\{ \begin{array}{cc} \wedge^\ell(\mathcal{S}_{i+1}/\mathcal{S}_i) & \ell < k_{i+1} - k_i, \\ (1-q_i)^{-1} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) & \ell = k_{i+1} - k_i. \end{array} \right. \tag{4.36}$$

The reader should note that this means that although for $q_i = 0$, $\widetilde{\mathcal{R}}_i = \mathcal{S}_{i+1}/\mathcal{S}_i$, for nonzero q_i, $\widetilde{\mathcal{R}}_i$ is not a classical bundle, but instead appears to be more nearly some sort of quantum exterior product (which we shall not try to define here, aside from the statement above).

Now, we are ready to interpret the (symmetrized) quantum K theory relation (4.26), namely

$$\sum_{r=0}^{k_{i+1}-k_i} e_{\ell-r}\left(X^{(i)}\right) e_r\left(\overline{X}^{(i)}\right) \;=\; e_\ell\left(X^{(i+1)}\right) \;+\; q_i e_{k_{i+1}-k_i}\left(\overline{X}^{(i)}\right) e_{\ell-k_{i+1}+k_i}\left(X^{(i-1)}\right). \tag{4.37}$$

Interpreting ordinary products in the algebraic relation above as quantum products $\star$, and also using the dictionary derived above for $e_\ell(\overline{X})$, we have

$$\sum_{r=0}^{k_{i+1}-k_i} \wedge^{\ell-r}(\mathcal{S}_i) \star \wedge^r(\widetilde{\mathcal{R}}_i) \;=\; \wedge^\ell(\mathcal{S}_{i+1}) \;+\; q_i\left(\det \widetilde{\mathcal{R}}_i\right) \star \wedge^{\ell-k_{i+1}+k_i}(\mathcal{S}_{i-1}), \tag{4.38}$$

or more elegantly,

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\widetilde{\mathcal{R}}_i) \;=\; \lambda_y(\mathcal{S}_{i+1}) \;+\; q_i y^{k_{i+1}-k_i}\left(\det \widetilde{\mathcal{R}}_i\right) \star \lambda_y(\mathcal{S}_{i-1}). \tag{4.39}$$

Equation (4.39) is our proposal for the relation defining the T-equivariant quantum K theory ring of a general partial flag manifold.

Alternatively, we can write this without using the $\widetilde{\mathcal{R}}_i$. Returning to the relation (4.26) and using the dictionary (4.36), we have

$$\sum_{r=0}^{k_{i+1}-k_i-1} \wedge^{\ell-r}\mathcal{S}_i \,\star\, \wedge^r(\mathcal{S}_{i+1}/\mathcal{S}_i) \,+\, \frac{1}{1-q_i}\det(\mathcal{S}_{i+1}/\mathcal{S}_i) \,\star\, \wedge^{\ell-k_{i+1}+k_i}\mathcal{S}_i$$
$$= \wedge^\ell \mathcal{S}_{i+1} \,+\, \frac{q_i}{1-q_i}\det(\mathcal{S}_{i+1}/\mathcal{S}_i)\star\wedge^{\ell-k_{i+1}+k_i}\mathcal{S}_{i-1}, \tag{4.40}$$

which after a little algebra implies

$$\sum_{r=0}^{k_{i+1}-k_i} \wedge^{\ell-r}\mathcal{S}_i \,\star\, \wedge^r(\mathcal{S}_{i+1}/\mathcal{S}_i) \tag{4.41}$$
$$= \wedge^\ell \mathcal{S}_{i+1} \,-\, \frac{q_i}{1-q_i}\det(\mathcal{S}_{i+1}/\mathcal{S}_i)\star\left(\wedge^{\ell-k_{i+1}+k_i}\mathcal{S}_i \,-\, \wedge^{\ell-k_{i+1}+k_i}\mathcal{S}_{i-1}\right).$$

By multiplying factors of y^ℓ, we can rewrite it in terms of λ_y classes, as

$$\lambda_y(\mathcal{S}_i)\star\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \,=\, \lambda_y(\mathcal{S}_{i+1}) \,-\, y^{k_{i+1}-k_i}\frac{q_i}{1-q_i}\det(\mathcal{S}_{i+1}/\mathcal{S}_i)\star(\lambda_y(\mathcal{S}_i)-\lambda_y(\mathcal{S}_{i-1}))\,. \tag{4.42}$$

This is a second form of our claimed presentation of the quantum K theory relations in terms of λ_y classes. We remind the reader that in our conventions, $k_0 = 0$, $k_{s+1} = n$, $\mathcal{S}_0 = 0$, $\mathcal{S}_{s+1} = \mathbb{C}^n$, and the equivariant structure is encoded implicitly in $\mathbb{C}^n$.

As a consistency check of (4.42), the reader may note that this immediately reduces to the result [6, theorem 1.1] for the T-equivariant quantum K theory ring of the Grassmannian $G(k, n)$, namely

$$\lambda_y(\mathcal{S})\star\lambda_y(\mathcal{Q}) \,=\, \lambda_y(\mathbb{C}^n) - y^{n-k}\frac{q}{1-q}\det\mathcal{Q}\star(\lambda_y(\mathcal{S})-1)\,. \tag{4.43}$$

As another consistency check of (4.42), we observe that the classical ($q_i \to 0$) limit can be immediately derived from the short exact sequence

$$0 \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_{i+1} \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \longrightarrow 0. \tag{4.44}$$

Remark 4.1. *The equation* (4.42) *suggests the following interpretation. Realize the partial flag manifold $F(k_1, \ldots, k_s, n)$ as the Grassmann bundle $Gr(k_i - k_{i-1}, \mathcal{S}_{i+1}/\mathcal{S}_{i-1}) \to F(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_s, n)$. This is equipped with the tautological sequence*

$$0 \longrightarrow \mathcal{S}_i/\mathcal{S}_{i-1} \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_{i-1} \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \longrightarrow 0.$$

We may formally divide (4.42) *by $\lambda_y(\mathcal{S}_{i-1})$ to obtain*

$$\frac{\lambda_y(\mathcal{S}_i)}{\lambda_y(\mathcal{S}_{i-1})}\star\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) = \frac{\lambda_y(\mathcal{S}_{i+1})}{\lambda_y(\mathcal{S}_{i-1})} - y^{k_{i+1}-k_i}\frac{q_i}{1-q_i}\det(\mathcal{S}_{i+1}/\mathcal{S}_i)\star\left(\frac{\lambda_y(\mathcal{S}_i)}{\lambda_y(\mathcal{S}_{i-1})}-1\right). \tag{4.45}$$

This holds in the classical K theory ring (i.e., when $q_i = 0$), since $\lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) = \frac{\lambda_y(\mathcal{S}_i)}{\lambda_y(\mathcal{S}_{i-1})}$ and $\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}) = \frac{\lambda_y(\mathcal{S}_{i+1})}{\lambda_y(\mathcal{S}_{i-1})}$, and because of the usual K-theoretic Whitney relations. The identity (4.45) *may be interpreted as a relative version of the quantum relations* (4.43) *on the Grassmannian $G(k_i - k_{i-1}, \mathbb{C}^{k_{i+1}-k_{i-1}})$.*

For later use, it may be helpful to specialize the relation (4.41). First, for the case $\ell = k_{i+1}$, equation (4.41) reduces to

$$\det \mathcal{S}_i \star \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \; = \; (1-q_i) \det \mathcal{S}_{i+1}. \tag{4.46}$$

Similarly, by multiplying factors of $\det \mathcal{S}_i$ for $\ell < k_{i+1}$ and using the relation above, one gets

$$\begin{aligned} &\left(\wedge^{k_{i+1}} \mathcal{S}_{i+1}\right) \star \left(\wedge^{\ell - k_{i+1} + k_i} \mathcal{S}_i - q_i \wedge^{\ell - k_{i+1} + k_i} \mathcal{S}_{i-1}\right) \\ &= \; \left(\wedge^{k_i} \mathcal{S}_i\right) \star \left[\wedge^{\ell} \mathcal{S}_{i+1} - \sum_{s=0}^{k_{i+1}-k_i-1} \wedge^{\ell-s} \mathcal{S}_i \star \wedge^s (\mathcal{S}_{i+1}/\mathcal{S}_i) \right]. \end{aligned} \tag{4.47}$$

An equivalent formulation of the λ_y class relations is in terms of corresponding K-theoretic Chern roots. We can write equation (4.42) as

$$\begin{aligned} &\left[\prod_{j=1}^{k_i} (1 + y x_j) \right] \cdot \left[\prod_{j=1}^{k_{i+1}-k_i} (1 + y v_j) \right] \\ &= \prod_{j=1}^{k_{i+1}} (1 + y z_j) \; - \; \frac{q_i}{1-q_i} y^{k_{i+1}-k_i} \left[\prod_{j=1}^{k_{i+1}-k_i} v_j \right] \left[\prod_{j=1}^{k_i} (1 + y x_j) \; - \; \prod_{j=1}^{k_{i-1}} (1 + y u_j) \right], \end{aligned} \tag{4.48}$$

where

$$x \sim \mathcal{S}_i, \quad v \sim \mathcal{S}_{i+1}/\mathcal{S}_i, \quad z \sim \mathcal{S}_{i+1}, \quad u \sim \mathcal{S}_{i-1}. \tag{4.49}$$

4.5 Shifted variables

We can also use the shifted Wilson line basis discussed in [5, 6]. The shifted variables z are defined by $z_a \equiv 1 - X_a$. Defining

$$c^{(i+1)} = \prod_{b_{i+1}=1}^{k_{i+1}} \left(1 - z^{(i+1)}_{b_{i+1}}\right), \tag{4.50}$$

we can rewrite (4.12) in terms of shifted variables as

$$(-)^{k_i - 1} q_i \left(1 - z^{(i)}_{a_i}\right)^{k_i - k_{i-1} - 1} c^{(i+1)} \prod_{b_{i-1}=1}^{k_{i-1}} \left(z^{(i-1)}_{b_{i-1}} - z^{(i)}_{a_i} \right) = \left(\prod_{b_i \neq a_i} \left(1 - z^{(i)}_{b_i}\right) \right) \prod_{b_{i+1}=1}^{k_{i+1}} \left(z^{(i)}_{a_i} - z^{(i+1)}_{b_{i+1}} \right). \tag{4.51}$$

Following the same argument in [6], we may rewrite this in the form

$$\left(z_{a_i}^{(i)}\right)^{k_{i+1}} + \sum_{r=0}^{k_{i+1}-1} (-1)^{k_{i+1}-r} \left(z_{a_i}^{(i)}\right)^r g_{n-r}^{(i)} \left(z^{(i-1)}, z^{(i)}, z^{(i+1)}, q_i\right). \tag{4.52}$$

where $g_r^{(i)}$ are symmetric polynomials in $z_1^{(i)}, \ldots, z_{k_i}^{(i)}$ and $z_1^{(i+1)}, \ldots, z_{k_{i+1}}^{(i+1)}$. To state the formula for the this polynomial, we make the following definitions borrowed from [6]. Set

$$\begin{aligned} c^{(i)} &= \prod_{a_i=1}^{k_i} \left(1 - z_{a_i}^{(i)}\right) = \sum_{s\geq 0} (-1)^s e_s \left(z^{(i)}\right), && (4.53) \\ c_{\leq j}^{(i)} &= \sum_{r=0}^{j} (-1)^r e_r \left(z^{(i)}\right), && (4.54) \\ c_{\geq j}^{(i)} &= (-1)^j \left(c^{(i)} - c_{\leq j-1}^{(i)}\right). && (4.55) \end{aligned}$$

Similarly, one defines $c^{(i+1)}, c_{\leq j}^{(i+1)}, c_{\geq j}^{(i+1)}$. Set

$$\begin{aligned} &c'_{\geq \ell}(z^{(i)}, z^{(i+1)}) && (4.56) \\ &= e_\ell\left(z^{(i+1)}\right) + e_{\ell-1}\left(z^{(i+1)}\right) c_{\geq 2}^{(i)} + e_{\ell-2}\left(z^{(i+1)}\right) c_{\geq 3}^{(i)} + \cdots + e_{\ell-k_i+1}(z^{(i+1)}) c_{\geq k_i}^{(i)} \end{aligned}$$

for $k_i \geq 2$ and $c'_{\geq \ell}(z^{(i)}, z^{(i+1)}) = e_\ell(z^{(i+1)})$ when $k_i = 1$.

Define the matrices

$$E = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ -e_1(z^{(i)}) & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k_i-1}(z^{(i)}) & -e_{k_i-2}(z^{(i)}) & \ldots & -1 \end{pmatrix}; \quad C_{\geq k_{i+1}-k_i+2}^{(i+1)} = \begin{pmatrix} c_{\geq k_{i+1}-k_i+2}^{(i+1)} \\ \vdots \\ c_{\geq k_{i+1}}^{(i+1)} \\ 0 \end{pmatrix}. \tag{4.57}$$

Then the polynomial coefficients $g_\ell^{(i)}$ are given by

$$g_\ell^{(i)} = \begin{cases} c'_{\geq \ell}\left(z^{(i)}, z^{(i+1)}\right) & 1 \leq \ell \leq k_{i+1} - k_i \\ c'_{\geq \ell}\left(z^{(i)}, z^{(i+1)}\right) + \left(E \cdot C_{\geq k_{i+1}-k_i+2}^{(i+1)}\right)_\ell + (-1)^{k_{i+1}+k_i} q_i c^{(i+1)} \alpha_\ell^{(i)} & k_{i+1} - k_i + 1 \leq \ell \leq k_{i+1} \end{cases} \tag{4.58}$$

where

$$\alpha_\ell^{(i)} = \sum_{r=0}^{k_{i-1}} \binom{k_i - k_{i-1} - 1}{k_{i+1} - k_{i-1} + r - \ell} e_r\left(z^{(i-1)}\right). \tag{4.59}$$

Define a characteristic polynomial $f^{(i)}\left(\xi, z^{(i-1)}, z^{(i)}, z^{(i+1)}, q_i\right)$ by

$$f^{(i)}\left(\xi, z^{(i-1)}, z^{(i)}, z^{(i+1)}, q_i\right) = \xi^{k_{i+1}} + \sum_{r=0}^{k_{i+1}-1} (-1)^{k_{i+1}-r} \xi^r g_{k_{i+1}-r}^{(i)}\left(z^{(i-1)}, z^i, z^{(i+1)}, q_i\right). \tag{4.60}$$

We have $f^{(i)} = 0$ whenever $\xi = z_{a_i}^{(i)}$ for $a_i = 1, \ldots, k_i$. Since $f^{(i)}$ is a degree k_{i+1} polynomial in ξ, the equation $f^{(i)} = 0$ has k_{i+1} roots, which include all $z_{a_i}^{(i)}$'s. Let $\{z^{(i)}, \hat{z}^{(i)}\} = \{z_1^{(i)}, \cdots, z_{k_i}^{(i)}; \hat{z}_{k_i+1}^{(i)}, \cdots, \hat{z}_{k_{i+1}}^{(i)}\}$ denote the k_{i+1} roots of (4.60). From Vieta's formula, we then have the relations

$$\sum_{r+t=\ell} e_r\left(z^{(i)}\right) e_t\left(\hat{z}^{(i)}\right) = g_\ell^{(i)}\left(z^{(i-1)}, z^{(i)}, z^{(i+1)}, q_i\right). \tag{4.61}$$

We conjecture that these relations define the T-equivariant quantum K theory ring of the partial flag manifold, generalizing the corresponding presentation for Grassmannians (denoted there the "Coulomb branch presentation") given in [6, section 10].

4.6 Consistency test: quantum cohomology

In this section, we take the two-dimensional limit of the quantum K theory, to recover a prediction for T-equivariant quantum cohomology QH_T^*, which can be checked against results in the literature, see for example [30–33].

To that end, we take the theory to be defined on a 3-manifold $S^1 \times \Sigma$ for some Riemann surface Σ, where S^1 has diameter L. Then, as discussed in e.g. [5], the quantities appearing in this section can be expanded as follows:

$$\begin{aligned} q_i = L^{k_{i+1}-k_{i-1}} q_{i,2d}, \qquad z = 1 - X &= -L\sigma - \frac{L^2}{2}\sigma^2 - \cdots \\ \hat{z} &= -L\hat{\sigma} - \cdots \\ T_i = \exp(Lt_i) &= 1 + Lt_i + \frac{L^2}{2}t_i^2 + \cdots . \end{aligned}$$

We will first compute directly by expanding the λ_y relations, then we will separately back up to the twisted effective superpotential, and repeat the same computation there.

4.6.1 Expansion of the λ_y class

In section 4.5, we gave results for quantum K theory relations in terms of shifted variables. In this section we describe their two-dimensinoal limits. To that end, we first observe that in the limit described above,

$$\begin{aligned} c_{\geq j}^{(i)} &\mapsto (-L)^j e_j(\sigma^{(i)}) + \mathcal{O}(L^j), \\ c'_{\geq \ell}\left(z^{(i)}, z^{(i+1)}\right) &\mapsto (-L)^\ell e_\ell(\sigma^{(i+1)}) + \mathcal{O}(L^\ell), \\ (E \cdot C)_\ell &\mapsto \mathcal{O}(L^\ell), \end{aligned}$$

where $e_0(\sigma^{(i)}) = 1$. The first nonzero coefficient in

$$\alpha_\ell^{(i)} \ = \ \sum_{r=0}^{k_{i-1}} \binom{k_i - k_{i-1} - 1}{k_{i+1} - k_{i-1} + r - \ell} e_r\left(z^{(i-1)}\right) \tag{4.62}$$

is at

$$r \ = \ \ell + k_{i-1} - k_{i+1}. \tag{4.63}$$

As a result, the two-dimensional limit of $\alpha_\ell^{(i)}$ is

$$\alpha_\ell^{(i)} \mapsto (-L)^{\ell+k_{i-1}-k_{i+1}} e_{\ell+k_{i-1}-k_{i+1}}\left(\sigma^{(i-1)}\right) + \mathcal{O}(L^{\ell+k_{i-1}-k_{i+1}}). \tag{4.64}$$

Using the computations above, the two-dimensional limit of the polynomial coefficients $g_\ell^{(i)}$ is given by

$$\begin{aligned} &g_{\ell,2d}^{(i)} \\ &= \begin{cases} (-L)^\ell e_\ell\left(\sigma^{(i+1)}\right) + \mathcal{O}(L^\ell) & 1 \le \ell \le k_{i+1} - k_i \\ (-L)^\ell e_\ell\left(\sigma^{(i+1)}\right) + (-1)^{\ell+k_i+k_{i-1}} q_{i,2d} \cdot e_{\ell+k_{i-1}-k_{i+1}}\left(\sigma^{(i-1)}\right) + \mathcal{O}(L^\ell) & k_{i+1} - k_i < \ell \le k_{i+1}. \end{cases} \end{aligned} \tag{4.65}$$

Finally, we have

$$\sum_{r+t=\ell} e_s\left(z^{(i)}\right) e_t\left(\hat{z}^{(i)}\right) \ \mapsto \ (-L)^\ell \sum_{r+t=\ell} e_r\left(\sigma^{(i)}\right) e_t\left(\hat{\sigma}^{(i)}\right). \tag{4.66}$$

Applying the computations above to (4.61), this gives the quantum cohomology ring relations

$$\begin{aligned} &\sum_{s+t=\ell} e_s\left(\sigma^{(i)}\right) e_t\left(\hat{\sigma}^{(i)}\right) \\ &\quad = \begin{cases} e_\ell\left(\sigma^{(i+1)}\right) & 1 \le \ell \le k_{i+1} - k_i, \\ e_\ell\left(\sigma^{(i+1)}\right) + (-1)^{k_i-k_{i-1}} q_{i,2d} \cdot e_{\ell+k_{i-1}-k_{i+1}}\left(\sigma^{(i-1)}\right) & k_{i+1} - k_i + 1 \le \ell \le k_{i+1}, \end{cases} \end{aligned} \tag{4.67}$$

which can be written more succinctly in terms of T-equivariant total Chern classes c^T as follows:

$$c^T(\mathcal{S}_i) \star c^T\left(\mathcal{S}_{i+1}/\mathcal{S}_i\right) \ = \ c^T\left(\mathcal{S}_{i+1}\right) \ + \ (-)^{k_i-k_{i-1}} q_i c^T\left(\mathcal{S}_{i-1}\right). \tag{4.68}$$

4.6.2 Expansion of the twisted effective superpotential

As an alternative procedure, we will check in this subsection that we can also get the same result from the twisted effective superpotential. To that end, it can be shown that

$$\widetilde{W}_{3d} \mapsto L\widetilde{W}_{2d} + \mathcal{O}(L), \tag{4.69}$$

which implies

$$\widetilde{W}_{2d} \;=\; \sum_{i=1}^{N}\sum_{a=1}^{k_i}\Sigma_a^{(i)}\Bigg[-t_i \;+\; i\pi\left(k_i-1\right) \;-\; \sum_{r=1}^{k_{i+1}}\left(\log\left(\Sigma_a^{(i)}-\Sigma_r^{(i+1)}\right)-1\right) \\ +\sum_{r=1}^{k_{i-1}}\left(\log\left(\Sigma_r^{(i-1)}-\Sigma_a^{(i)}\right)-1\right)\Bigg]. \tag{4.70}$$

The vacuum equations are

$$\prod_{r=1}^{k_{i+1}}\left(\sigma_a^{(i)}-\sigma_r^{(i+1)}\right) \;=\; (-1)^{k_i-1}q_{i,2d}\prod_{r=1}^{k_{i-1}}\left(\sigma_r^{(i-1)}-\sigma_a^{(i)}\right).$$

The symmetrization of the above equation is straightforward, and one finds the characteristic polynomial

$$\left(\sigma_{a_i}^{(i)}\right)^{k_{i+1}} + \sum_{r=0}^{k_{i+1}-1}\left(\sigma_{a_i}^{(i)}\right)^{r} g_{n-r,2d}^{(i)}\left(\sigma^{(i-1)},\sigma^{(i+1)},q_{i,2d}\right) \;=\; 0, \tag{4.71}$$

where $g_{\ell,2d}^{(i)}$ is defined in equation (4.65). As a result, the Vieta relations of the above equation will be equivalent to equation (4.67), and so again we recover the quantum cohomology (4.68).

4.6.3 Comparison of quantum cohomology prediction to the literature

In this section we will argue that the non-equivariant specialization of the result above matches the non-equivariant quantum cohomology of [33, theorem 0.2], specialized to partial flag manifolds, as in [33, example 2.2]. The reader should note that the results in [33] are given in terms of universal quotient bundles, which we convert to universal subbundles here.

In our notation, the presentation of [33, theorem 0.2] is

$$\mathbb{C}[\sigma_j^{(i)}, q_i]^W/I_q, \tag{4.72}$$

where W is the Weyl group of $U(k_1)\times\cdots\times U(k_s)$

$$\omega \;=\; \prod_{i=1}^{s}\prod_{j<k}\left(\sigma_j^{(i)}-\sigma_k^{(i)}\right), \tag{4.73}$$

and the ideal is

$$I_q \;=\; \langle f\in\mathbb{C}[\sigma_j^{(i)},q_i]^W \,|\, \omega f\in J\rangle, \tag{4.74}$$

for

$$J \;=\; \langle K_{ij}, i=1,\cdots s, j=1,\cdots,k_i\rangle, \tag{4.75}$$

where

$$K_{ij} \;=\; \prod_{k=1}^{k_{i+1}} \left(\sigma_k^{(i+1)} - \sigma_j^{(i)} \right) \;+\; (-)^{k_i} q_i \prod_{k=1}^{k_{i-1}} \left(\sigma_j^{(i)} - \sigma_k^{(i-1)} \right), \tag{4.76}$$

in conventions in which $k_0 = 0$, $k_{s+1} = n$, and $\sigma^{(0)} = \sigma^{(s+1)} = 0$.

Next, we compute

$$\sum_{j=1}^{k_i} K_{ij} (-)^j \left(\sigma_j^{(i)} \right)^a \prod_{\ell<k, \ell \neq j, k \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right) \;=\; A + B \tag{4.77}$$

for $0 \leq a \leq k_i - 1$ and where we define

$$A \;=\; \sum_{j=1}^{k_i} (-)^j \left(\sigma_j^{(i)} \right)^a \left(\prod_{k=1}^{k_{i+1}} \left(\sigma_k^{(i+1)} - \sigma_j^{(i)} \right) \right) \left(\prod_{\ell<k, \ell \neq j, \ell \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right) \right), \tag{4.78}$$

$$B \;=\; \sum_{j=1}^{k_i} (-)^j \left(\sigma_j^{(i)} \right)^a \left((-)^{k_i} q_i \prod_{k=1}^{k_{i-1}} \left(\sigma_j^{(i)} - \sigma_k^{(i-1)} \right) \right) \left(\prod_{\ell<k, \ell \neq j, \ell \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right) \right).$$

Focusing momentarily on A, one can show that

$$\begin{aligned} A \;&=\; \sum_{j=1}^{k_i} \left[\sum_{m=0}^{k_{i+1}} (-)^{m+j} e_{k_{i+1}-m} \left(\sigma^{(i+1)} \right) \left(\sigma_j^{(i)} \right)^{a+m} \right] \prod_{\ell<k, \ell \neq j, k \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right), \\ &=\; \sum_{m=0}^{k_{i+1}} e_{k_{i+1}-m} \left(\sigma^{(i+1)} \right) \sum_{j=1}^{k_i} (-)^{m+j} \left(\sigma_j^{(i)} \right)^{a+m} \prod_{\ell<k, \ell \neq j, k \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right). \end{aligned} \tag{4.79}$$

Next, we use the Jacobi-Trudi formula

$$s_\lambda \prod_{i<j} (x_i - x_j) \;=\; \det \left[\begin{array}{ccc} x_1^{r-1+\lambda_1} & \cdots & x_r^{r-1+\lambda_1} \\ \vdots & & \vdots \\ x_1^{1+\lambda_r} & \cdots & x_r^{1+\lambda_r} \\ x_1^{\lambda_r} & \cdots & x_r^{\lambda_r} \end{array} \right] \tag{4.80}$$

for the Schur polynomial $s_\lambda(x_1, \cdots, x_r)$ of a Young tableau λ. This implies

$$\frac{\sum_{j=1}^{k_i} (-)^{m+j} \left(\sigma_j^{(i)} \right)^{a+m} \prod_{\ell<k, \ell \neq j, k \neq j} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right)}{\prod_{\ell<k} \left(\sigma_\ell^{(i)} - \sigma_k^{(i)} \right)} \;=\; (-)^{m-1} h_{a-k_i+m+1} \left(\sigma^{(i)} \right)$$

by using the Jacobi-Trudi formula for $\lambda = (a - k_i + m + 1, 0, \cdots, 0)$, for which $s_\lambda = h_{a-k_i+m+1}$.

Putting this together, we have, for $1 \leq i \leq s$,

$$\frac{A}{\prod_{\ell<k}\left(\sigma_\ell^{(i)} - \sigma_k^{(i)}\right)} \;=\; \sum_{m=0}^{k_{i+1}} (-)^{m-1} e_{k_{i+1}-m}\left(\sigma^{(i+1)}\right) h_{a-k_i+m+1}\left(\sigma^{(i)}\right). \tag{4.81}$$

Proceeding similarly, one can show

$$\frac{B}{\prod_{\ell<k}\left(\sigma_\ell^{(i)} - \sigma_k^{(i)}\right)} \;=\; q_i \sum_{m=0}^{k_{i-1}} (-)^{k_i+k_{i-1}-m-1} e_{k_{i-1}-m}\left(\sigma^{(i-1)}\right) h_{m+a+1-k_i}\left(\sigma^{(i)}\right) \tag{4.82}$$

which is valid for all $1 \leq i \leq s$.

Let $\alpha = a + k_{i+1} - k_i + 1$, we have the following relations

$$F_\alpha^{(i)} = -\sum_{m=0}^{k_{i+1}} (-)^m e_{k_{i+1}-m}(\sigma^{(i+1)}) h_{m+\alpha-k_{i+1}}(\sigma^{(i)}) \\ - q_i \sum_{m=0}^{k_{i-1}} (-)^{k_i+k_{i-1}-m} e_{k_{i-1}-m}(\sigma^{(i-1)}) h_{m+\alpha-k_{i+1}}(\sigma^{(i)}) \tag{4.83}$$

for $1 \leq i \leq s$ and $\alpha = k_{i+1} - k_i + 1, \ldots, k_{i+1}$.

Next, we compare to the quantum cohomology presentation we derived from our predicted λ_y relations in quantum K theory in section 4.6.1, for which the relations are

$$c(\mathcal{S}_i) \star c(\mathcal{S}_{i+1}/\mathcal{S}_i) \;=\; c(\mathcal{S}_{i+1}) + (-)^{k_i-k_{i-1}} q_i c(\mathcal{S}_{i-1}), \tag{4.84}$$

which in terms of symmetric polynomials in the σ's can be written

$$\sum_{r=0}^{\ell} e_r\left(\sigma^{(i)}\right) e_{\ell-r}\left(\hat{\sigma}^{(i)}\right) \;=\; e_\ell\left(\sigma^{(i+1)}\right) \;+\; (-)^{k_i-k_{i-1}} q_i e_{\ell-k_{i+1}+k_{i-1}}\left(\sigma^{(i-1)}\right) \tag{4.85}$$

for $\ell = 0, 1, \ldots, k_{i+1}$.

To that end, we multiply equation (4.85) by y^ℓ and sum over ℓ from 0 to k_{i+1}, and obtain

$$\left(\sum_{r=0}^{k_i} y^r e_r\left(\sigma^{(i)}\right)\right)\left(\sum_{t=0}^{k_{i+1}-k_i} y^t e_t\left(\hat{\sigma}^{(i)}\right)\right) \\ = \left(\sum_{\ell=0}^{k_{i+1}} y^\ell e_\ell\left(\sigma^{(i+1)}\right)\right) + (-)^{k_i-k_{i-1}} q_i \left(\sum_{\ell=0}^{k_{i+1}} y^\ell e_{\ell-k_{i+1}+k_{i-1}}\left(\sigma^{(i-1)}\right)\right) \tag{4.86}$$

hence,

$$\left(\sum_{\ell=0}^{k_{i+1}-k_i} y^\ell e_\ell\left(\hat{\sigma}^{(i)}\right)\right) = \left(\sum_{t=0}^{k_{i+1}} y^t e_t\left(\sigma^{(i+1)}\right)\right)\left(\sum_{r=0}^{\infty}(-)^r y^r h_r\left(\sigma^{(i)}\right)\right)$$
$$+ (-)^{k_i-k_{i-1}} q_i \left(\sum_{t=0}^{k_{i+1}} y^t e_{t-k_{i+1}+k_{i-1}}\left(\sigma^{(i-1)}\right)\right)\left(\sum_{r=0}^{\infty}(-)^r y^r h_r\left(\sigma^{(i)}\right)\right) \quad (4.87)$$

Now consider y^α terms for $k_{i+1} - k_i < \alpha \leq k_{i+1}$, we have

$$0 = \sum_{t=0}^{k_{i+1}} (-)^{\alpha-t} e_t\left(\sigma^{(i+1)}\right) h_{\alpha-t}\left(\sigma^{(i)}\right)$$
$$+ (-)^{k_i-k_{i-1}} q_i \sum_{t=0} (-)^{\alpha-t} e_{t-k_{i+1}+k_{i-1}}\left(\sigma^{(i-1)}\right) h_{\alpha-t}\left(\sigma^{(i)}\right). \quad (4.88)$$

After making change of summation variables and getting rid of a total sign $(-)^{\alpha-k_{i+1}}$, we obtain

$$0 = \sum_{m=0}^{k_{i+1}} (-)^m e_{k_{i+1}-m}\left(\sigma^{(i+1)}\right) h_{m+\alpha-k_{i+1}}\left(\sigma^{(i)}\right)$$
$$+ (-)^{k_i+k_{i-1}} q_i \sum_{m=0}^{k_{i+1}} (-)^m e_{k_{i-1}-m}\left(\sigma^{(i-1)}\right) h_{m+\alpha-k_{i+1}}\left(\sigma^{(i)}\right) \quad (4.89)$$

which agrees with equation (4.83).

This demonstrates that the non-equivariant specialization of our prediction (4.68) for quantum cohomology rings of partial flag manifolds, holds. In principle, to prove this rigorously, we would also need to demonstrate that no additional relations are needed. This follows from the graded Nakayama lemma, as in [34], see Lemma 4.1 and Thm. 4.2; in turn, it generalizes to the equivariant version the statement from [35, Prop. 11]. More precisely, one needs to show that if one takes the classical specialization $q_i = 0$ of the given relations, then they form a complete set of relations. The latter fact follows from known presentations of partial flag manifolds when realized as towers of Grassmann bundles, see e.g. [36, Ex. 14.6.6]. We leave the details to the reader.

4.7 Duality

In this section we will apply the duality of flag manifolds relating $F(k_1, \cdots, k_s, N)$ to $F(N - k_s, N - k_{s-1}, \cdots, N - k_1, N)$, to write the λ_y class relations in terms of universal quotient bundles rather than universal subbundles.

Define $\mathcal{Q}_i = \mathbb{C}^N/S_i$, then we have the short exact sequences

$$0 \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_{i+1} \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \longrightarrow 0, \tag{4.90}$$

$$0 \longrightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \longrightarrow \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i+1} \longrightarrow 0. \tag{4.91}$$

Using $'$ to denote bundles on the dual flag manifold, it is a classical result that $\mathcal{S}_i$ on the flag manifold $F(k_1, \cdots, k_s, N)$ is isomorphic to $\mathcal{Q}'^*_{s+1-i}$ on the dual flag manifold $F(N - k_s, \cdots, N - k_1, N)$. Similarly, $\mathcal{S}_{i+1}/\mathcal{S}_i$ on the flag manifold $F(k_1, \cdots, k_s, N)$ is isomorphic to

$$\mathcal{Q}'^*_{s-i}/\mathcal{Q}'^*_{s+1-i} \cong \left(\mathcal{S}'_{s+1-i}/\mathcal{S}'_{s-1}\right)^* \tag{4.92}$$

on the dual flag manifold, where in the isomorphism we have used the dual of the sequence (4.91). This duality is also reflected in the two short exact sequences above, as the sequence (4.90) maps to

$$0 \longrightarrow \mathcal{Q}'^*_{s+1-i} \longrightarrow \mathcal{Q}'^*_{s-i} \longrightarrow \mathcal{Q}'^*_{s-i}/\mathcal{Q}'^*_{s+1-i} \cong \left(\mathcal{S}'_{s+1-i}/\mathcal{S}'_{s-i}\right)^* \longrightarrow 0, \tag{4.93}$$

which is just the dual of sequence (4.91).

Now that we have seen how the duality relates the classical λ_y relations, it remains to describe the quantum version.

Given the λ_y relations in the form (4.42), namely

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) = \lambda_y(\mathcal{S}_{i+1}) - y^{k_{i+1}-k_i} \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star (\lambda_y(\mathcal{S}_i) - \lambda_y(\mathcal{S}_{i-1})),$$

if we use primes ($'$) to denote bundles on the dual flag manifold, we have immediately

$$\begin{aligned} &\lambda_{y'}(\mathcal{S}'_i) \star \lambda_{y'}(\mathcal{S}'_{i+1}/\mathcal{S}'_i) \\ &= \lambda_{y'}(\mathcal{S}'_{i+1}) - (y')^{(N-k_{s-i})-(N-k_{s+1-i})} \frac{q'_i}{1-q'_i} \det(\mathcal{S}'_{i+1}/\mathcal{S}'_i) \star \left(\lambda_{y'}(\mathcal{S}'_i) - \lambda_{y'}(\mathcal{S}'_{i-1})\right). \end{aligned}$$

Interpreting $\mathcal{S}'_i = \mathcal{Q}^*_{s+1-i}$ as above, and writing $j = s+1-i$, we can rewrite this in the form

$$\begin{aligned} &\lambda_{y'}(\mathcal{Q}^*_j) \star \lambda_{y'}\left((\mathcal{S}_j/\mathcal{S}_{j-1})^*\right) \\ &= \lambda_{y'}(\mathcal{Q}^*_{j-1}) - (y')^{k_j-k_{j-1}} \frac{q'_{s+1-j}}{1-q'_{s+1-j}} \det\left((\mathcal{S}_j/\mathcal{S}_{j-1})^*\right) \star \left(\lambda_{y'}(\mathcal{Q}^*_j) - \lambda_{y'}(\mathcal{Q}^*_{j+1})\right) \end{aligned} \tag{4.94}$$

on the original flag manifold. We will not use this relation in this paper, but include it for completeness.

5 Examples

5.1 Incidence varieties, meaning, flag manifolds $F(1, n-1, n)$

5.1.1 Overview of physics

Now let us specialize to incidence varieties, which are flag manifolds $F(1, n-1, n)$. As described earlier in section 4.1, these are realized physically by a $U(1) \times U(n-1)$ gauge theory with one bifundamental (charge $+1$ under $U(1)$ and antifundamental under $U(n-1)$), and n fundamentals of $U(n-1)$.

Generically on the Coulomb branch, the $U(n-1)$ gauge symmetry is broken to a $U(1)^{n-1}$ subgroup, with $(n-1)(n-2)$ W-bosons. Specializing equation (4.9), the superpotential in this case is

$$\begin{aligned} \mathcal{W} \;=\;& \frac{1}{2}(n-2) \sum_{a=1}^{n-1} \left(\ln X_a^{(2)} \right)^2 \;-\; \sum_{i=1}^{2} \sum_{1 \leq a_i < b_i \leq k_i} \left(\ln X_{a_i}^{(i)} \right) \left(\ln X_{b_i}^{(i)} \right) \\ &+\; (\ln q_1) \left(\ln X^{(1)} \right) \;+\; \left(\ln \left((-)^{n-2} q_2 \right) \right) \sum_{a=1}^{n-1} \left(\ln X_a^{(2)} \right) \\ &+\; \sum_{a=1}^{n-1} \mathrm{Li}_2 \left(X^{(1)} / X_a^{(2)} \right) \;+\; \sum_{a=1}^{n-1} \sum_{i=1}^{n} \mathrm{Li}_2 \left(X_a^{(2)} / T_i \right). \end{aligned} \tag{5.1}$$

The Coulomb branch equations (4.12), derived from derivatives of the superpotential $\mathcal{W}$, specialize to

$$q_1 X^{(1)} \;=\; X^{(1)} \prod_{b=1}^{n-1} \left(1 - \frac{X^{(1)}}{X_b^{(2)}} \right), \tag{5.2}$$

$$(-)^{n-2} q_2 \left(X_a^{(2)} \right)^{n-1} \left(1 - \frac{X^{(1)}}{X_a^{(2)}} \right) \;=\; \left(\prod_{b_2=1}^{n-1} X_{b_2}^{(2)} \right) \prod_{b_3=1}^{n} \left(1 - \frac{X_a^{(2)}}{T_{b_3}} \right), \tag{5.3}$$

for $a = 1, \cdots, n-1$, which after cleaning up the algebra can be slightly rewritten

$$q_1 \;=\; \prod_{b=1}^{n-1} \left(1 - \frac{X^{(1)}}{X_b^{(2)}} \right), \tag{5.4}$$

$$(-)^{n-2} q_2 \left(X_a^{(2)} \right)^{n-2} \left(X_a^{(2)} - X^{(1)} \right) \;=\; \left(\prod_{b_2=1}^{n-1} X_{b_2}^{(2)} \right) \prod_{b_3=1}^{n} \left(1 - \frac{X_a^{(2)}}{T_{b_3}} \right). \tag{5.5}$$

After symmetrizing, these equations become (specializing (4.26)):

$$\sum_{r=0}^{n-2} e_{\ell-r}\left(X^{(1)}\right) e_r\left(\overline{X}^{(1)}\right) \;=\; e_\ell\left(X^{(2)}\right) \;+\; \left\{ \begin{array}{cc} q_1 e_{n-2}\left(\overline{X}^{(1)}\right) & \ell = n-2, \\ 0 & \text{else}, \end{array} \right. \tag{5.6}$$

for $\ell = 0, \cdots, n-1$, and

$$\sum_{r=0}^{1} e_{\ell-r}\left(X^{(2)}\right) e_r\left(\overline{X}^{(2)}\right) \;=\; e_\ell(T) \;+\; q_2\, e_1\left(\overline{X}^{(2)}\right) e_{\ell-1}\left(X^{(1)}\right), \tag{5.7}$$

for $\ell = 0, \cdots, n$.

Next, we turn to λ_y class presentations. We interpret $e_\ell(X^{(i)})$ as $\wedge^\ell \mathcal{S}_i$, and following the dictionary (4.36), we interpret

$$e_\ell\left(\overline{X}^{(1)}\right) \;\leftrightarrow\; \left\{ \begin{array}{cc} \wedge^\ell\left(\mathcal{S}_2/\mathcal{S}_1\right) & \ell < n-2, \\ (1-q_1)^{-1}\det(\mathcal{S}_2/\mathcal{S}_1) & \ell = n-2, \end{array} \right. \tag{5.8}$$

$$e_\ell\left(\overline{X}^{(2)}\right) \;\leftrightarrow\; \left\{ \begin{array}{cc} 1 & \ell = 0, \\ (1-q_2)^{-1}\,\mathbb{C}^n/\mathcal{S}_2 & \ell = 1. \end{array} \right. \tag{5.9}$$

Then, equations (5.6), (5.7) are interpreted as

$$\begin{aligned} &\sum_{r=0}^{n-3} \wedge^{\ell-r}\mathcal{S}_1 \,\star\, \wedge^r(\mathcal{S}_2/\mathcal{S}_1) \;+\; \frac{1}{1-q_1}\,\wedge^{\ell-(n-2)}\,\mathcal{S}_1 \,\star\, \det(\mathcal{S}_2/\mathcal{S}_1) \\ &\qquad\qquad = \;\wedge^\ell\mathcal{S}_2 \;+\; \left\{ \begin{array}{cc} q_1(1-q_1)^{-1}\det(\mathcal{S}_2/\mathcal{S}_1) & \ell = n-2, \\ 0 & \text{else}, \end{array} \right. \end{aligned} \tag{5.10}$$

and

$$\wedge^\ell\mathcal{S}_2 + \frac{1}{1-q_2}\,\wedge^{\ell-1}\,\mathcal{S}_2 \,\star\, \mathbb{C}^n/\mathcal{S}_2 \;=\; \wedge^\ell\mathbb{C}^n \;+\; \frac{q_2}{1-q_2}\mathbb{C}^n/\mathcal{S}_2 \,\star\, \wedge^{\ell-1}\mathcal{S}_1, \tag{5.11}$$

respectively.

It is straightforward to rewrite these equations as the coefficients of y^ℓ in the relations

$$\lambda_y(\mathcal{S}_1) \star \lambda_y(\mathcal{S}_2/\mathcal{S}_1) \;=\; \lambda_y(\mathcal{S}_2) \;-\; y^{n-2}\frac{q_1}{1-q_1}\det(\mathcal{S}_2/\mathcal{S}_1) \star (\lambda_y(\mathcal{S}_1) - 1)\,, \tag{5.12}$$

$$\lambda_y(\mathcal{S}_2) \star \lambda_y(\mathbb{C}^n/\mathcal{S}_2) \;=\; \lambda_y(\mathbb{C}^n) \;-\; y\frac{q_2}{1-q_2}\det(\mathbb{C}^n/\mathcal{S}_2) \star (\lambda_y(\mathcal{S}_2) - \lambda_y(\mathcal{S}_1))\,, \tag{5.13}$$

where $\mathcal{S}_1$ has rank $k_1 = 1$ and $\mathcal{S}_2$ has rank $k_2 = n-1$, so that, for example,

$$\det\left(\mathbb{C}^n/\mathcal{S}_2\right) \;=\; \mathbb{C}^n/\mathcal{S}_2 \tag{5.14}$$

classically.

These are the specializations of the λ_y class relation (4.42) of section 4.4 for the T-equivariant quantum K theory ring relations, namely

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \;=\; \lambda_y(\mathcal{S}_{i+1}) \;-\; y^{k_{i+1}-k_i} \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star (\lambda_y(\mathcal{S}_i) - \lambda_y(\mathcal{S}_{i-1})) \,. \quad (5.15)$$

These λ_y class relations for incidence varieties have also been proven rigorously, using independent methods from [37]. The proof will appear in [19].

5.1.2 $F(1,2,3)$

In this section we will give explicitly the T-equivariant quantum K theory relations for the special case $F(1,2,3)$.

Specializing the earlier analysis for incidence varieties, the superpotential for $F(1,2,3)$ is

$$\begin{aligned} \mathcal{W} \;=\; & \frac{1}{2}\sum_{a=1}^{2} \left(\ln X_a^{(2)}\right)^2 \;-\; \left(\ln X_1^{(2)}\right)\left(\ln X_2^{(2)}\right) \\ & + \left(\ln q_1\right)\left(\ln X^{(1)}\right) \;+\; \left(\ln((-)q_2)\right)\sum_{a=1}^{2}\left(\ln X_a^{(2)}\right) \\ & + \sum_{a=1}^{2} \mathrm{Li}_2\left(X^{(1)}/X_a^{(2)}\right) \;+\; \sum_{a=1}^{2}\sum_{i=1}^{3} \mathrm{Li}_2\left(X_a^{(2)}/T_i\right) . \end{aligned} \quad (5.16)$$

The Coulomb branch equations (4.12), derived from derivatives of the superpotential $\mathcal{W}$, specialize to

$$q_1 \;=\; \prod_{b=1}^{2}\left(1 - \frac{X^{(1)}}{X_b^{(2)}}\right), \quad (5.17)$$

$$(-)q_2\left(X_a^{(2)}\right)\left(X_a^{(2)} - X^{(1)}\right) \;=\; \left(\prod_{b_2=1}^{2} X_{b_2}^{(2)}\right)\prod_{b_3=1}^{3}\left(1 - \frac{X_a^{(2)}}{T_{b_3}}\right). \quad (5.18)$$

After symmetrizing, these equations become (specializing (4.26)):

$$\sum_{r=0}^{1} e_{\ell-r}\left(X^{(1)}\right) e_r\left(\overline{X}^{(1)}\right) \;=\; e_\ell\left(X^{(2)}\right) \;+\; \begin{cases} q_1 e_1\left(\overline{X}^{(1)}\right) & \ell = 1, \\ 0 & \text{else}, \end{cases} \quad (5.19)$$

for $\ell = 0, \cdots, 2$, and

$$\sum_{r=0}^{1} e_{\ell-r}\left(X^{(2)}\right) e_r\left(\overline{X}^{(2)}\right) \;=\; e_\ell(T) \;+\; q_2\, e_1\left(\overline{X}^{(2)}\right) e_{\ell-1}\left(X^{(1)}\right), \quad (5.20)$$

for $\ell = 0, \cdots, 3$. Following the dictionary (4.36), we interpret

$$e_\ell\left(\overline{X}^{(1)}\right) \quad\leftrightarrow\quad \begin{cases} \wedge^\ell \left(\mathcal{S}_2/\mathcal{S}_1\right) & \ell < n-2, \\ (1-q_1)^{-1}\det(\mathcal{S}_2/\mathcal{S}_1) & \ell = n-2, \end{cases} \tag{5.21}$$

$$e_\ell\left(\overline{X}^{(2)}\right) \quad\leftrightarrow\quad \begin{cases} 1 & \ell = 0, \\ (1-q_2)^{-1}\,\mathbb{C}^n/\mathcal{S}_2 & \ell = 1. \end{cases} \tag{5.22}$$

This leads to the λ_y class relations, which, specializing (5.12), (5.13), take the form

$$\lambda_y(\mathcal{S}_1) \star \lambda_y(\mathcal{S}_2/\mathcal{S}_1) \;=\; \lambda_y(\mathcal{S}_2) \;-\; y\frac{q_1}{1-q_1}\det(\mathcal{S}_2/\mathcal{S}_1)\star\left(\lambda_y(\mathcal{S}_1)-1\right), \tag{5.23}$$

$$\lambda_y(\mathcal{S}_2) \star \lambda_y(\mathbb{C}^3/\mathcal{S}_2) \;=\; \lambda_y(\mathbb{C}^3) \;-\; y\frac{q_2}{1-q_2}\det(\mathbb{C}^3/\mathcal{S}_2)\star\left(\lambda_y(\mathcal{S}_2)-\lambda_y(\mathcal{S}_1)\right), \tag{5.24}$$

where $\mathcal{S}_1$ has rank 1 and $\mathcal{S}_2$ has rank 2.

We check these relations against known calculations in the quantum K ring utilizing Schubert classes. To reduce from the amount of notation, we will work in the non-equivariant situation.

Fix the standard basis e_1, e_2, e_3 of $\mathbb{C}^3$. For a permutation $w \in S_3$ let $\ell(w)$ be the number of inversions of w. Denote by X^w to be the Schubert variety of complex codimension $\ell(w)$ and which contains the torus fixed point $\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \mathbb{C}^3$. (See the companion paper [19] for more details.) Denote by $\mathcal{O}^w$ the class in K theory given by the structure sheaf of X^w. Denote by $s_1 = (12)$ and $s_2 = (23)$ the simple reflections in S_3. To further simplify notation, we will denote by $\mathcal{O}^1, \mathcal{O}^{12}$ etc the Schubert classes associated to $s_1, s_1 s_2$ etc. With this notation

$$\begin{aligned} \mathcal{S}_1 &= 1 - \mathcal{O}^1; \\ \mathcal{S}_2/\mathcal{S}_1 &= 1 + \mathcal{O}^1 - \mathcal{O}^2 - \mathcal{O}^{1,2}; \\ \mathcal{S}_2 &= 2 - \mathcal{O}^2 - \mathcal{O}^{1,2}; \\ \wedge^2\mathcal{S}_2 &= 1 - \mathcal{O}^2; \\ \mathbb{C}^3/\mathcal{S}_2 &= 1 + \mathcal{O}^2 + \mathcal{O}^{1,2}. \end{aligned} \tag{5.25}$$

The relevant multiplications in QK(Fl(3)) are given by (see e.g. [37, Theorem 5.1], [38])

$$\begin{aligned} \mathcal{O}^1 \circ \mathcal{O}^1 &= \mathcal{O}^{2,1} - q_1\mathcal{O}^2 + q_1; \\ \mathcal{O}^1 \circ \mathcal{O}^2 &= \mathcal{O}^{1,2} + \mathcal{O}^{2,1} - \mathcal{O}^{1,2,1}; \\ \mathcal{O}^1 \circ \mathcal{O}^{1,2} &= \mathcal{O}^{1,2,1}; \\ \mathcal{O}^2 \circ \mathcal{O}^2 &= \mathcal{O}^{1,2} - q_2\mathcal{O}^1 + q_2; \\ \mathcal{O}^2 \circ \mathcal{O}^{1,2} &= q_2\mathcal{O}^1; \\ \mathcal{O}^{1,2} \circ \mathcal{O}^{1,2} &= q_2\mathcal{O}^{2,1}. \end{aligned}$$

It is straightforward to check that, with the dictionary above, these products are equivalent to (the non-equivariant versions of) the λ_y class relations.

Let us now specialize the quantum cohomology relations (4.68) to this case, and compare to rigorous mathematics results. The T-equivariant quantum cohomology ring $QH^*_T(\mathrm{Fl}(3))$ has the relations[3]

$$c^T(\mathcal{S}_1) \star c^T(\mathcal{S}_2/\mathcal{S}_1) \;=\; c^T(\mathcal{S}_2) + (-)q_1, \tag{5.27}$$

$$c^T(\mathcal{S}_2) \star c^T\left(\mathbb{C}^3/\mathcal{S}_2\right) \;=\; c^T(\mathbb{C}^3) + (-)q_2\, c^T(\mathcal{S}_1), \tag{5.28}$$

where c^T denotes the T-equivariant total Chern class, where the maximal torus T consists of diagonal matrices in $GL(3)$, and the T-module $\mathbb{C}^3$ has a weight space decomposition $\mathbb{C}^3 = \mathbb{C}_{t_1} \oplus \mathbb{C}_{t_2} \oplus \mathbb{C}_{t_3}$, where T acts on $\mathbb{C}_{t_i} \simeq \mathbb{C}$ with weight t_i. For example,

$$c^T(\mathbb{C}^3) \;=\; (1+t_1)(1+t_2)(1+t_3). \tag{5.29}$$

We let $[X^w] \in H_T^{2\ell(w)}(\mathrm{Fl}(3))$ denote the equivariant fundamental class indexed by w (given by a reduced decomposition). Then

$$\begin{aligned}
c^T(\mathcal{S}_1) &= 1 + t_1 - [X^1], \\
c^T(\mathcal{S}_2) &= (1+t_1)(1+t_2) - (1+t_1)[X^2] + [X^{1,2}], \\
c^T(\mathcal{S}_2/\mathcal{S}_1) &= (1+t_2) + [X^1] - [X^2], \\
c^T(\mathbb{C}^3/\mathcal{S}_2) &= 1 + t_3 + [X^2],
\end{aligned}$$

(The reader may recall that in taking the two-dimensional limit, $T_i = \exp(Lt_i)$.)

To check the relevant multiplications in $\mathrm{QH}^*_T(\mathrm{Fl}(3))$ one may use the Chevalley formulae proved in [39]. We obtain:

$$\begin{aligned}
[X^1] \star [X^1] &= [X^{2,1}] + (t_1 - t_2)[X^1] + q_1, \\
[X^1] \star [X^2] &= [X^{2,1}] + [X^{1,2}], \\
[X^2] \star [X^2] &= [X^{1,2}] + (t_2 - t_3)[X^2] + q_2, \\
[X^1] \star [X^{1,2}] &= (t_1 - t_2)[X^{1,2}] + [X^{1,2,1}], \\
[X^2] \star [X^{1,2}] &= (t_1 - t_3)[X^{1,2}] + q_2[X^1].
\end{aligned}$$

[3] To simplify notation, we are writing results for T-equivariant cohomology for T a maximal torus of $GL(n)$. However, the overall scaling is a gauge symmetry. To convert to T-equivariant parameters for T a maximal torus of $SL(n)$, one can take the t parameters to obey

$$\sum_{i=1}^{n} t_i \;=\; 0. \tag{5.26}$$

We have checked that our results correctly reproduce mathematical results for T-equivariant cohomology for T a maximal torus of $SL(n)$ and $n = 3$.

It can be shown that the products above correctly reproduce the relations (5.27), (5.28) arising from physics. Note that $\{t_1 - t_2, t_2 - t_3\}$ form a basis of the positive simple roots for the root system associated to GL_3.

As another consistency check, it can also be shown that the non-equivariant version of the quantum cohomology presentation above matches that in [33, theorem 0.2] for $F(1,2,3)$.

Now, let us compare to the presentation of the quantum K theory ring of $F(1,2,3)$ in [9, theorem 5.4] and [17, theorem 3]. For $F(1,2,3)$, our presentation can be summarized as follows:

$$\mathcal{S}_1 + \mathcal{S}_2/\mathcal{S}_1 = \mathcal{S}_2 \tag{5.30}$$

$$\mathcal{S}_2 + \mathbb{C}^3/\mathcal{S}_2 = \mathbb{C}^3 \tag{5.31}$$

$$\mathcal{S}_2/\mathcal{S}_1 \star \mathcal{S}_1 = (1 - q_1) \wedge^2 \mathcal{S}_2 \tag{5.32}$$

$$\mathbb{C}^3/\mathcal{S}_2 \star (\mathcal{S}_2 - q_2\mathcal{S}_1) = (1 - q_2)(\wedge^2\mathbb{C}^3 - \wedge^2\mathcal{S}_2) \tag{5.33}$$

$$\mathbb{C}^3/\mathcal{S}_2 \star \wedge^2\mathcal{S}_2 = (1 - q_2) \wedge^3 \mathbb{C}^3 \tag{5.34}$$

To compare to [9, theorem 5.4] and [17, theorem 3], we define

$$\mathfrak{p}_1 = \mathcal{S}_1 = (1 - x_1)(1 - q_1),$$

$$(1 - q_1)\mathfrak{p}_2 = \mathcal{S}_2/\mathcal{S}_1 = (1 - q_2)(1 - x_2),$$
$$(1 - q_2)\mathfrak{p}_3 = \mathbb{C}^3/\mathcal{S}_2 = 1 - x_3.$$

Then by 5.30 and 5.31,

$$\mathfrak{p}_1 + \mathfrak{p}_2(1 - q_1) + \mathfrak{p}_3(1 - q_2) = \mathbb{C}^3; \tag{5.35}$$

by 5.30, 5.32 and 5.33,

$$\mathfrak{p}_1 \star \mathfrak{p}_2 + \mathfrak{p}_2 \star \mathfrak{p}_3(1 - q_1) + \mathfrak{p}_3 \star \mathfrak{p}_1(1 - q_2) = \wedge^2\mathbb{C}^3; \tag{5.36}$$

and by 5.32 and 5.34,

$$\mathfrak{p}_1 \star \mathfrak{p}_2 \star \mathfrak{p}_3 = \wedge^3\mathbb{C}^3, \tag{5.37}$$

recovering the presentation in [9, Theorem 5.4], where $z_i^{\#} = q_i$ for $i = 1, 2$, and $e_r(\mathfrak{a}_1, \ldots, \mathfrak{a}_3) = \wedge^r\mathbb{C}^3$ for $r = 1, 2, 3$. Similarly,

$$(1 - x_1)(1 - q_1) + (1 - x_2)(1 - q_2) + (1 - x_3) = \mathbb{C}^3,$$

$$(1 - x_1)(1 - x_2)(1 - q_2) + (1 - x_1)(1 - x_3)(1 - q_1) + (1 - x_2)(1 - x_3) = \wedge^2\mathbb{C}^3,$$
$$(1 - x_1)(1 - x_2)(1 - x_3) = \wedge^3\mathbb{C}^3.$$

recovering the presentation in [17, theorem 3].

5.2 Full flag manifolds

5.2.1 Overview

Consider the full flag manifold $F(1,2,3,\cdots,n)$, (meaning that $k_i = i$ for each i, and all steps appear,) and let S_k be the kth universal subbundle, which has rank k. As described earlier in section 4.1, these are realized physically by a $U(1)\times U(2)\times\cdots\times U(n-1)$ gauge theory with a set of bifundamentals.

Generically on the Coulomb branch, each $U(k)$ gauge symmetry factor is broken to a $U(1)^k$ subgroup, with $k(k-1)$ W-bosons. Specializing equation (4.9), the superpotential in this case is

$$\begin{aligned}\mathcal{W} &= \frac{1}{2}\sum_{i=1}^{n-1}(i-1)\sum_{a_i=1}^{i}\left(\ln X_{a_i}^{(i)}\right)^2 - \sum_{i=1}^{n-1}\sum_{1\leq a_i<b_i\leq i}\left(\ln X_{a_i}^{(i)}\right)\left(\ln X_{b_i}^{(i)}\right)\\ &\quad + \sum_{i=1}^{n-1}\left(\ln\left((-)^{i-1}q_i\right)\right)\sum_{a_i=1}^{i}\left(\ln X_{a_i}^{(i)}\right)\\ &\quad + \sum_{i=1}^{n-1}\sum_{a_i=1}^{i}\sum_{a_{i+1}=1}^{i+1}\mathrm{Li}_2\left(X_{a_i}^{(i)}/X_{a_{i+1}}^{(i+1)}\right),\end{aligned} \tag{5.38}$$

where $k_0=0$, $k_n=n$, $X_{a_n}^{(n)}=T_{a_n}$, and we have used the fact that there are $s=n-1$ gauge factors and each rank $k_i=i$.

The Coulomb branch equations (4.12), derived from derivatives of the superpotential $\mathcal{W}$, specialize to

$$(-)^{i-1}q_i\left(X_{a_i}^{(i)}\right)^i\prod_{b_{i-1}=1}^{i-1}\left(1-\frac{X_{b_{i-1}}^{(i-1)}}{X_{a_i}^{(i)}}\right) = \left(\prod_{b_i=1}^{i}x_{b_i}^{(i)}\right)\prod_{b_{i+1}=1}^{i+1}\left(1-\frac{X_{a_i}^{(i)}}{X_{b_{i+1}}^{(i+1)}}\right), \tag{5.39}$$

for $a_i=1,\cdots,i$ and $i=1,\cdots,n-1$.

After symmetrizing, these equations become (specializing (4.26))

$$\sum_{r=0}^{1}e_{\ell-r}\left(X^{(i)}\right)e_r\left(\overline{X}^{(i)}\right) = e_\ell\left(X^{(i+1)}\right) + q_i\,e_1\left(\overline{X}^{(i)}\right)e_{\ell-1}\left(X^{(i-1)}\right), \tag{5.40}$$

in conventions in which $X^{(n)}=T$ and $X^{(0)}=0$.

We interpret $e_\ell(X^{(i)})$ as $\wedge^\ell\mathcal{S}_i$, identifying the components $X_a^{(i)}$ with Chern roots of $\mathcal{S}_i$, and following the dictionary (4.36), we interpret

$$e_\ell\left(\overline{X}^{(i)}\right) \leftrightarrow \begin{cases}\mathcal{O} & \ell=0,\\ (1-q_i)^{-1}\mathcal{S}_{i+1}/\mathcal{S}_i & \ell=1.\end{cases} \tag{5.41}$$

(Since $\mathcal{S}_{i+1}/\mathcal{S}_i$ has rank one, there are no higher exterior powers, and $\mathcal{S}_{i+1}/\mathcal{S}_i = \det \mathcal{S}_{i+1}/\mathcal{S}_i$.) Then, equation (5.40) becomes

$$\wedge^\ell \mathcal{S}_i \, + \, \frac{1}{1-q_i} \wedge^{\ell-1} \mathcal{S}_i \star (\mathcal{S}_{i+1}/\mathcal{S}_i) \; = \; \wedge^\ell \mathcal{S}_{i+1} \, + \, \frac{q_i}{1-q_i} (\mathcal{S}_{i+1}/\mathcal{S}_i) \, \star \wedge^{\ell-1} \mathcal{S}_{i-1}. \tag{5.42}$$

which can be rearranged algebraically to become

$$\sum_{s=0}^{1} \wedge^{\ell-s} \mathcal{S}_i \, \star \, \wedge^s (\mathcal{S}_{i+1}/\mathcal{S}_i) \; = \; \wedge^\ell \mathcal{S}_{i+1} \, - \, \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \left(\wedge^{\ell-1} \mathcal{S}_i - \wedge^{\ell-1} \mathcal{S}_{i-1} \right). \tag{5.43}$$

Adding powers of y, this can be encoded in the λ_y class expression

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \; = \; \lambda_y(\mathcal{S}_{i+1}) \, - \, y \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star (\lambda_y(\mathcal{S}_i) - \lambda_y(\mathcal{S}_{i-1})), \tag{5.44}$$

which is precisely the specialization of the λ_y class relation (4.42).

A different presentation may also be helpful. As discussed earlier in section 4.4, the λ_y class relations imply (4.46), (4.47), of which the relation (4.47) specializes in the present case to

$$\left(\wedge^k \mathcal{S}_k \right) \otimes \left(\wedge^\ell \mathcal{S}_{k+1} - \wedge^\ell \mathcal{S}_k \right) \; = \; \left(\wedge^{\ell-1} \mathcal{S}_k - q_k \wedge^{\ell-1} \mathcal{S}_{k-1} \right) \otimes \wedge^{k+1} \mathcal{S}_{k+1}. \tag{5.45}$$

The relation (4.46) is redundant, as it corresponds to the special case that $\ell = 1$.

5.2.2 $F(1,2,3,4)$

As we have already considered $F(1,2,3)$ in section 5.1.2, we turn here to the next full flag manifold, namely, $F(1,2,3,4)$ with tautological subbundles

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{S}_4 = \mathbb{C}^4, \tag{5.46}$$

of ranks $0, 1, 2, 3, 4$ respectively.

Specializing equation (4.9), the superpotential in this case is

$$\begin{aligned} \mathcal{W} \;\; = \;\; & \frac{1}{2} \sum_{i=1}^{3} (i-1) \sum_{a_i=1}^{i} \left(\ln X_{a_i}^{(i)} \right)^2 \, - \, \sum_{i=1}^{3} \sum_{1 \leq a_i < b_i \leq i} \left(\ln X_{a_i}^{(i)} \right) \left(\ln X_{b_i}^{(i)} \right) \\ & + \sum_{i=1}^{3} \left(\ln \left((-)^{i-1} q_i \right) \right) \sum_{a_i=1}^{i} \left(\ln X_{a_i}^{(i)} \right) \\ & + \sum_{i=1}^{3} \sum_{a_i=1}^{i} \sum_{a_{i+1}=1}^{i+1} \mathrm{Li}_2 \left(X_{a_i}^{(i)} / X_{a_{i+1}}^{(i+1)} \right). \end{aligned} \tag{5.47}$$

The Coulomb branch equations (4.12), derived from derivatives of the superpotential $\mathcal{W}$, specialize to

$$(-)^{i-1} q_i \left(X^{(i)}_{a_i}\right)^i \prod_{b_{i-1}=1}^{i-1} \left(1 - \frac{X^{(i-1)}_{b_{i-1}}}{X^{(i)}_{a_i}} \right) \;=\; \left(\prod_{b_i=1}^{i} x^{(i)}_{b_i} \right) \prod_{b_{i+1}=1}^{i+1} \left(1 - \frac{X^{(i)}_{a_i}}{X^{(i+1)}_{b_{i+1}}} \right), \tag{5.48}$$

for $a_i = 1, \cdots, i$ and $i = 1, \cdots, 3$. After symmetrizing, these equations become (specializing (4.26))

$$\sum_{r=0}^{1} e_{\ell-r}\left(X^{(i)}\right) e_r\left(\overline{X}^{(i)}\right) \;=\; e_\ell\left(X^{(i+1)}\right) \;+\; q_i\, e_1\left(\overline{X}^{(i)}\right) e_{\ell-1}\left(X^{(i-1)}\right), \tag{5.49}$$

and after interpreting as before, we are led to λ_y class relations ('quantum Whitney relations') given by

$$\lambda_y(\mathcal{S}_i) \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \;=\; \lambda_y(\mathcal{S}_{i+1}) \;-\; y \frac{q_i}{1-q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \left(\lambda_y(\mathcal{S}_i) - \lambda_y(\mathcal{S}_{i-1}) \right), \tag{5.50}$$

for $1 \leq i \leq 3$, i.e.,

$$\begin{aligned}
\lambda_y(\mathcal{S}_1) \star \lambda_y(\mathcal{S}_2/\mathcal{S}_1) &= \lambda_y(\mathcal{S}_2) \;-\; y\frac{q_1}{1-q_1} \det(\mathcal{S}_2/\mathcal{S}_1) \star \left(\lambda_y(\mathcal{S}_1) - 1 \right), \\
\lambda_y(\mathcal{S}_2) \star \lambda_y(\mathcal{S}_3/\mathcal{S}_2) &= \lambda_y(\mathcal{S}_3) \;-\; y\frac{q_2}{1-q_2} \det(\mathcal{S}_3/\mathcal{S}_2) \star \left(\lambda_y(\mathcal{S}_2) - \lambda_y(\mathcal{S}_1) \right), \\
\lambda_y(\mathcal{S}_3) \star \lambda_y(\mathbb{C}^4/\mathcal{S}_3) &= \lambda_y(\mathbb{C}^4) \;-\; y\frac{q_3}{1-q_3} \det(\mathbb{C}^4/\mathcal{S}_3) \star \left(\lambda_y(\mathcal{S}_3) - \lambda_y(\mathcal{S}_2) \right).
\end{aligned}$$

This is the specialization of the λ_y class relation (4.42).

Now, let us compare to existing[4] mathematics results, which are phrased in terms of Schubert classes. (As for $F(1,2,3)$, we will only compare to non-equivariant quantum K theory.) The Schubert classes will be denoted by $\mathcal{O}^w$ with $w \in S_4$, and the same definitions as those for $\mathrm{Fl}(3) = F(1,2,3)$ are utilized. For the convenience of the reader, we list the expansions of the Schubert classes of each exterior power of the bundles $\mathcal{S}_i$.

$$\begin{aligned}
\mathcal{S}_1 &= 1 - \mathcal{O}^1, \\
\mathcal{S}_2 &= 2 - \mathcal{O}^2 - \mathcal{O}^{1,2}, \\
\wedge^2 \mathcal{S}_2 &= 1 - \mathcal{O}^2, \\
\mathcal{S}_3 &= 3 - \mathcal{O}^3 - \mathcal{O}^{2,3} - \mathcal{O}^{1,2,3}, \\
\wedge^2 \mathcal{S}_3 &= 3 - 2\mathcal{O}^3 - \mathcal{O}^{2,3}, \\
\wedge^3 \mathcal{S}_3 &= 1 - \mathcal{O}^3, \\
\mathcal{S}_i/\mathcal{S}_{i-1} &= \mathcal{S}_i - \mathcal{S}_{i-1}, \quad 1 \leq i \leq 4.
\end{aligned} \tag{5.51}$$

[4]Conjectural formulas for the Chevalley coefficients appeared in [40], and were later proven in [18].

The relevant multiplications of Schubert classes are:

$$\begin{aligned}
\mathcal{O}^1 \circ \mathcal{O}^1 &= \mathcal{O}^{2,1} - q_1\mathcal{O}^2 + q_1, \\
\mathcal{O}^1 \circ \mathcal{O}^2 &= \mathcal{O}^{1,2} + \mathcal{O}^{2,1} - \mathcal{O}^{1,2,1}, \\
\mathcal{O}^1 \circ \mathcal{O}^3 &= \mathcal{O}^{1,3}, \\
\mathcal{O}^1 \circ \mathcal{O}^{1,2} &= \mathcal{O}^{1,2,1}, \\
\mathcal{O}^1 \circ \mathcal{O}^{2,3} &= \mathcal{O}^{1,2,3} + \mathcal{O}^{2,3,1} - \mathcal{O}^{1,2,3,1}, \\
\mathcal{O}^1 \circ \mathcal{O}^{1,2,3} &= \mathcal{O}^{1,2,3,1}, \\
\mathcal{O}^2 \circ \mathcal{O}^2 &= \mathcal{O}^{1,2} + \mathcal{O}^{3,2} - \mathcal{O}^{3,1,2} - q_2\mathcal{O}^1 + q_2\mathcal{O}^{1,3} - q_2\mathcal{O}^3 + q_2, \\
\mathcal{O}^2 \circ \mathcal{O}^3 &= \mathcal{O}^{2,3} + \mathcal{O}^{3,2} - \mathcal{O}^{2,3,2}, \\
\mathcal{O}^2 \circ \mathcal{O}^{1,2} &= \mathcal{O}^{3,1,2} - q_2\mathcal{O}^{1,3} + q_2\mathcal{O}^1, \\
\mathcal{O}^2 \circ \mathcal{O}^{2,3} &= \mathcal{O}^{1,2,3} + \mathcal{O}^{2,3,2} - \mathcal{O}^{1,2,3,2}, \\
\mathcal{O}^2 \circ \mathcal{O}^{1,2,3} &= \mathcal{O}^{1,2,3,2}, \\
\mathcal{O}^3 \circ \mathcal{O}^3 &= \mathcal{O}^{2,3} + q_3 - q_3\mathcal{O}^2, \\
\mathcal{O}^3 \circ \mathcal{O}^{1,2} &= \mathcal{O}^{3,1,2} + \mathcal{O}^{1,2,3} - \mathcal{O}^{1,2,3,2}, \\
\mathcal{O}^3 \circ \mathcal{O}^{2,3} &= \mathcal{O}^{1,2,3} + q_3\mathcal{O}^2 - q_3\mathcal{O}^{1,2}, \\
\mathcal{O}^3 \circ \mathcal{O}^{1,2,3} &= q_3\mathcal{O}^{1,2}, \\
\mathcal{O}^{1,2} \circ \mathcal{O}^{1,2} &= \mathcal{O}^{2,3,1,2} + q_2\mathcal{O}^{2,1} - q_2\mathcal{O}^{2,3,1}, \\
\mathcal{O}^{1,2} \circ \mathcal{O}^{2,3} &= \mathcal{O}^{1,2,3,2} + \mathcal{O}^{2,3,1,2} - \mathcal{O}^{1,2,3,1,2}, \\
\mathcal{O}^{1,2} \circ \mathcal{O}^{1,2,3} &= \mathcal{O}^{1,2,3,1,2}, \\
\mathcal{O}^{2,3} \circ \mathcal{O}^{2,3} &= q_3\mathcal{O}^{3,2} + q_3\mathcal{O}^{1,2} - q_3\mathcal{O}^{3,1,2}, \\
\mathcal{O}^{2,3} \circ \mathcal{O}^{1,2,3} &= q_3\mathcal{O}^{3,1,2}, \\
\mathcal{O}^{1,2,3} \circ \mathcal{O}^{1,2,3} &= q_3\mathcal{O}^{2,3,1,2}.
\end{aligned}$$

It is straightforward, albeit tedious, to check that the multiplications above are consistent with the (nonequivariant version of the) λ_y class relations for T-equivariant quantum K theory.

Let us now specialize the quantum cohomology relations (4.68) to this case, and compare to rigorous mathematics results. The T-equivariant quantum cohomology ring $QH_T^*(\mathrm{Fl}(4))$ has the relations

$$c^T(\mathcal{S}_1) \star c^T\left(\mathcal{S}_2/\mathcal{S}_1\right) \;=\; c^T\left(\mathcal{S}_2\right) \,+\, (-)q_1, \tag{5.52}$$
$$c^T(\mathcal{S}_2) \star c^T\left(\mathcal{S}_3/\mathcal{S}_2\right) \;=\; c^T\left(\mathcal{S}_3\right) \,+\, (-)q_2\, c^T\left(\mathcal{S}_1\right), \tag{5.53}$$
$$c^T(\mathcal{S}_3) \star c^T\left(\mathbb{C}^4/\mathcal{S}_3\right) \;=\; c^T\left(\mathbb{C}^4\right) \,+\, (-)q_3\, c^T\left(\mathcal{S}_2\right), \tag{5.54}$$

where c^T denotes the T-equivariant total Chern class.

Next, we compare to the literature. In the same notation as section 5.1.2, we let $[X^w] \in H_T^{2\ell(w)}(\mathrm{Fl}(3))$ denote the equivariant fundamental class indexed by w (given by a reduced decomposition). The maximal torus T consists of diagonal matrices in GL_4, and the T-module $\mathbb{C}^4$ has a weight space decomposition $\mathbb{C}^4 = \mathbb{C}_{t_1} \oplus \mathbb{C}_{t_2} \oplus \mathbb{C}_{t_3} \oplus \mathbb{C}_{t_4}$ where T acts on $\mathbb{C}_{t_i} \simeq \mathbb{C}$ with weight t_i. Then

$$
\begin{aligned}
c^T(\mathcal{S}_1) &= (1+t_1) - [X^1], \\
c^T(\mathcal{S}_2) &= [X^{1,2}] - (1+t_1)[X^2] + (1+t_1)(1+t_2), \\
c^T(\mathcal{S}_3) &= -[X^{1,2,3}] - (1+t_1)(1+t_2)[X^3] + (1+t_1)[X^{2,3}] \\
&\quad + (1+t_1)(1+t_2)(1+t_3), \\
c^T(\mathcal{S}_2/\mathcal{S}_1) &= (1+t_2) + [X^1] - [X^2], \\
c^T(\mathcal{S}_3/\mathcal{S}_2) &= (1+t_3) + [X^2] - [X^3], \\
c^T(\mathbb{C}^4/\mathcal{S}_3) &= (1+t_4) + [X^3].
\end{aligned}
$$

To check the relevant multiplications in $\mathrm{QH}^*_T(\mathrm{Fl}(3))$ one may use the Chevalley formulae proved in [39]. We obtain

$$
\begin{aligned}
[X^1] \star [X^1] &= [X^{2,1}] + (t_1 - t_2)[X^1] + q_1, \\
[X^1] \star [X^2] &= [X^{2,1}] + [X^{1,2}], \\
[X^1] \star [X^3] &= [X^{3,1}], \\
[X^2] \star [X^2] &= [X^{1,2}] + [X^{3,2}] + (t_2 - t_3)[X^2] + q_2, \\
[X^2] \star [X^3] &= [X^{2,3}] + [X^{3,2}], \\
[X^3] \star [X^3] &= [X^{2,3}] + (t_3 - t_4)[X^3] + q_3, \\
[X^1] \star [X^{1,2}] &= (t_1 - t_2)[X^{1,2}] + [X^{1,2,1}], \\
[X^2] \star [X^{1,2}] &= (t_1 - t_3)[X^{1,2}] + q_2[X^1] + [X^{3,1,2}], \\
[X^3] \star [X^{1,2}] &= [X^{3,1,2}] + [X^{1,2,3}], \\
[X^3] \star [X^{2,3}] &= (t_2 - t_4)[X^{2,3}] + q_3[X^2] + [X^{1,2,3}], \\
[X^3] \star [X^{1,2,3}] &= q_3[X^{1,2}] + (t_1 - t_4)[X^{1,2,3}].
\end{aligned}
$$

It is straightforward to check that these products do indeed reproduce the relations (5.52), (5.53), and (5.54). Note that $\{t_1 - t_2, t_2 - t_3, t_3 - t_4\}$ form a basis of the positive simple roots for the root system associated to GL_4.

6 Conclusions

In this paper we have used Coulomb branch methods in GLSMs to make predictions for quantum K theory ring relations in partial flag manifolds.

As remarked in section 4.4, the form of our result for partial flag manifolds was suggestive of an alternative expression in terms of a relative Grassmannian, suggesting that there might exist a notion of "vertical quantum K theory." We leave this for future work.

In discussions of duality between Grassmannians and flag manifolds and their duals in sections 3.4 and 4.7, we found an elegant expression for certain dual bundles in terms of elements of K theory which differ from honest bundles by factors of q. This suggests that there might exist a "quantum duality" map, a map $\mathcal{E} \mapsto \mathcal{E}^*$ for a 'quantum' dual operation $*$. We leave this for future work.

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