Differential Geometry of Space Curves: Forgotten Chapters

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Evolutes, involutes, and osculating circles of curves belong to the main notions of planar differential geometry, going back to Christiaan Huygens in the seventeenth century. They have many interesting properties, including the surprising Tait–Kneser theorem: the osculating circles of a curve with monotonic curvature are nested (see [2, 10] or [8, Chapter 10]). Here, we consider three kinds of evolutes and involutes of space curves, all of which were studied in the early days of differential geometry. They possess many familiar properties of evolutes and involutes of plane curves, but they also have some unexpected features. One of our goals, then, is to describe their intricate interrelations by surveying these properties and surprises.

Since the terminology is not canonical, we follow [6] and [9] and define the evolute as the locus of centers of osculating spheres, or equivalently, the curve whose osculating planes are the normal planes of the given curve.[1]

We also discuss a natural modification of the construction in which the normal planes are replaced by the rectifying planes. For the resulting curve we use a somewhat awkward term, pseudo-evolute, and this is the second kind of space evolute that we consider. Properties of pseudo-evolutes offer further surprises.

In the plane, involutes are constructed by wrapping an unstretchable string around a curve. The same construction in space provides yet another definition of involute, and one defines the evolute of a curve as the result of the converse operation. We call this third version Monge evolutes and Monge involutes, after the French mathematician Gaspard Monge (1746–1818).

Although much of the material we present is not new and can be found in such classic books as [1, 5, 13], we believe that a modern and unified treatment—complemented with several novel observations and results and illustrated with the help of computer graphics—may be useful, since the geometry of space curves remains highly relevant in modern mathematics.

For example, it is closely related to the theory of completely integrable systems: the filament (aka binormal, smoke ring, local induction) equation is a completely integrable evolution of space curves, equivalent to the nonlinear Schrödinger equation [11]. Another important application is the study of curved origami, which can be informally described as folding paper along curves (as opposed to straight lines) [4, 7].

In what follows, we present some—but not all—of the calculations behind the geometric statements. They involve only elementary calculus but in some cases are cumbersome. Readers are encouraged to perform the missing calculations on their own.

Textbook Material

The following facts are undoubtedly known to the majority of readers, but we prefer to provide a brief survey of them to establish the settings, terminology, and notation.

Evolutes and involutes of plane curves. The evolute of a plane curve is the envelope of its normal lines. Equivalently, it is the locus of the centers of osculating circles of the curve; see Figure 1. The singular points of the evolute correspond to the vertices of the curve, that is, to the critical points of the curvature.

If a curve e is the evolute of a curve ξ, then ξ is an involute of e. An involute of a curve e can be constructed as follows: fix one end of an unstretchable string at a point of e, wrap the string about the curve, and move the free end, keeping the string tight. In this way, one obtains a one-parameter family of involutes: the length of the string is a parameter.

Equivalently, one can roll a straight line along the curve e. Then the trajectory of each point of the line is an involute of e.

Frenet apparatus. By a space curve $\xi = \xi(t)$, or $x = x(t)$, $y = y(t)$, $z = z(t)$, we mean a smooth map from $\mathbb{R}$ to $\mathbb{R}^3$. We consider generic curves, the precise meaning of which varies depending on the situation, but it always describes an open dense set in the space of curves.

Specifically, our curves are free from inflection points, that is, points where the curvature vanishes. If a curve has a singular point, then generically, it is a semicubic cusp, expressed in local coordinates by $\xi(t) = \left(t^2, t^3, t^4\right)$. The tangent line at such cusp point is well defined, and the curvature is infinite.

Given a unit-speed curve ξ, the Frenet frame

$$\left(\mathbf{t} = \mathbf{t}(t), \mathbf{n} = \mathbf{n}(t), \mathbf{b} = \mathbf{b}(t)\right)$$

[1] In [1, 5], the term *evolute* means something else, and Uribe-Vargas, in his detailed study of the evolute in our sense [14], prefers the term *focal curve*.

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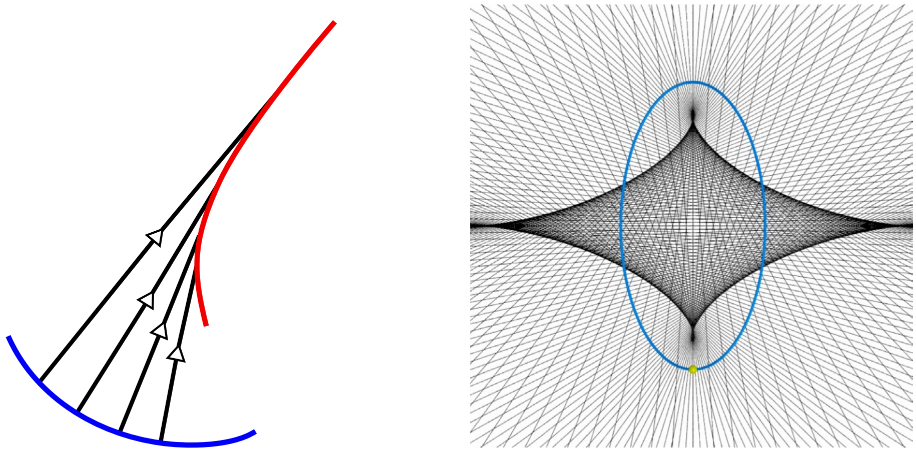

Figure 1. Left: the red curve is the evolute of the blue one, and the blue curve is an involute of the red one. Right: the evolute of an ellipse. The four cusps of the evolute correspond to the two minima and two maxima of the curvature of the ellipse.

consists of the tangent vector $\mathbf{t} = \xi'$; the (principal) normal $\mathbf{n}$, which is the unit vector in the direction of ξ''; and the binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. The dependence of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ on t is described by the Frenet formulas

$$\begin{aligned} \mathbf{t}' &= k\mathbf{n}\,, \\ \mathbf{n}' &= k\mathbf{t} + \tau\mathbf{b}\,, \\ \mathbf{b}' &= -\tau\mathbf{n}\,, \end{aligned}$$

where k and τ are the curvature and the torsion of the curve ξ. We denote the radius of curvature $1/k$ by r.

There are well-known formulas for the curvature and torsion:

$$k = \|\xi''\|, \quad \tau = \frac{\det(\xi', \xi'', \xi''')}{\|\xi''\|^2}.$$

If the parametrization is not of unit speed, then the formulas become slightly more complicated:

$$k = \frac{\|\xi' \times \xi''\|}{\|\xi'\|^3}, \quad \tau = \frac{\det(\xi', \xi'', \xi''')}{\|\xi' \times \xi''\|^2}. \tag{1}$$

The plane spanned by $\mathbf{t}$ and $\mathbf{n}$ is called the osculating plane of ξ, the plane spanned by $\mathbf{n}$ and $\mathbf{b}$ is the normal plane, and the plane spanned by $\mathbf{t}$ and $\mathbf{b}$ is the rectifying plane.

Developable Surfaces. A surface in space is called developable if it is locally isometric to a plane. An informal description of a developable surface is that it is a surface that can be made by bending, but not folding, a piece of paper. Certainly, we have in mind an "ideal paper" that is ideally bendable (without any bounds on the curvature), incompressible, and unstretchable, i.e., the length of any curve drawn on the paper remains unchanged in the process of bending.

The theory of developable surfaces was developed (pardon the unintended pun) in the late 1700s by Euler and Monge. Let us translate the main results into modern language.

A generic (in particular, nowhere planar) developable surface is ruled, that is, every point belongs to a unique straight line that is fully contained in the surface. Moreover, the tangent plane is constant along each line; this property distinguishes developable surfaces in the class of all ruled surfaces.

Generically these lines, known as rulings, are tangent to a certain curve, called the regression edge of the surface. There the surface is not smooth: all sections by planes transverse to the regression edge have cusps.

The latter property (also a characteristic one) provides a universal method for constructing developable surfaces: take an arbitrary (generic) curve ξ in space, possibly with cusps, and consider the union of all its tangent lines. This union is a developable surface with regression edge ξ, and all (generic) developable surfaces can be obtained in this way. This surface is called the tangent developable of ξ; see Figure 2.

The tangent developable deforms isometrically if the curve is deformed without changing its curvature. In particular, flattening the curve transforms its tangent developable into two identical sheets on the convex side of the curve; see [13, Section 4-5] for details.

There are two types of nongeneric developable surfaces: cylinders and cones over an arbitrary curve.

Another universal construction of developable surfaces uses an arbitrary (generic) one-parameter family of planes. For such a family there exists a unique developable surface tangent to all the planes, called the envelope of the family. These planes are the osculating planes of the regression edge. For each plane, tangency occurs along a whole line, which is a tangent of the regression edge.

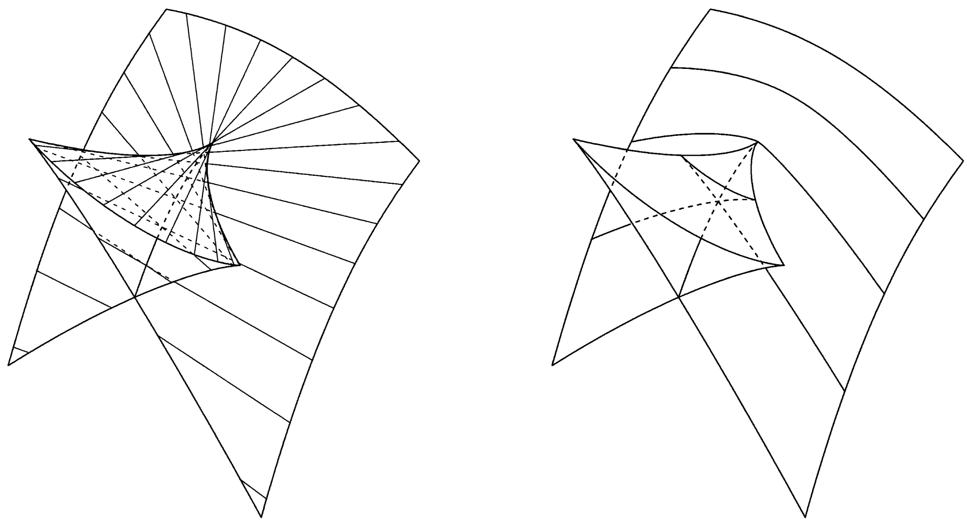

Figure 2. Tangent developable of a curve $x = at^2$, $y = bt^3$, $z = ct^4$. Left: the rulings of the surface; right: the sections of the surface by parallel planes.

The last construction provides a convenient analytic description of everything mentioned above. Namely, if $F(x, y, z; t) = 0$ is the equation of the family of planes (with parameter t), then we can form three systems of equations:

$$\begin{cases} F(x, y, z; t) = 0, \\ F'(x, y, z; t) = 0; \end{cases}$$
$$\begin{cases} F(x, y, z; t) = 0, \\ F'(x, y, z; t) = 0, \\ F''(x, y, z; t) = 0; \end{cases} \tag{2}$$
$$\begin{cases} F(x, y, z; t) = 0, \\ F'(x, y, z; t) = 0, \\ F''(x, y, z; t) = 0, \\ F'''(x, y, z; t) = 0 \end{cases}$$

(the primes denote the partial derivative with respect to t).

The solutions of the first system describe the intersection lines of two infinitesimally close planes. Therefore, if we exclude t from this system, we get the equation of a ruled surface.

The solutions of the second system correspond to the intersection points of three infinitesimally close planes, that is, two infinitesimally close rulings of the surface. Thus if we solve the second system with respect to x, y, z, we obtain a parametric equation of the regression edge.

Finally, three close rulings meet only in the cuspidal points of the regression edge, so the solutions of the last system provide coordinates of those points.

Notice that all of this may be repeated for the case in which $F(x, y, z; t) = 0$ is a family of surfaces, not necessarily planes.

Evolutes

Let $\xi = \xi(t)$ be a (generic) curve in space. There arise three families of planes: the osculating planes, the normal planes, and the rectifying planes. Each of them has an envelope (the regression edge of the developable surface tangent to the family). The first of these three cases is not interesting: the envelope is the curve ξ itself.

The envelope of the family of normal planes is called the normal developable, and its regression edge is what we call the evolute of the curve; this is similar to the definition of the evolute of a planar curve as the envelope of the family of its normals.

Equation of the evolute. Let $\xi = \xi(t) = \big(x(t), y(t), z(t)\big)$ be a curve parametrized by arc length, and let $e = e(t)$ be its evolute.

Proposition 1. $e = \xi + r\mathbf{n} + \frac{r'}{\tau}\mathbf{b}$.-

Proof. The proof is based on formulas (2). We denote the coordinates of points in $\mathbb{R}^3$ by $P = (X, Y, Z)$, and we use the dot product to make the formulas more compact.

The family of normal planes to ξ is described by

$$F(P;t) = \xi'(t) \cdot (P - \xi(t)) = 0\,.$$

The derivatives of F are

$$F'(P;t) = \xi''(t) \cdot (P - \xi(t)) - 1, F''(P;t) = \quad \xi'''(t) \cdot (P - \xi(t))$$

(we used the equalities $\xi'(t) \cdot \xi'(t) = 1$ and $\xi''(t) \cdot \xi'(t) = 0$).

The resulting (middle) system of equations (2) becomes

$$\begin{cases} \xi'(t) \cdot (P - \xi(t)) = 0, \\ \xi''(t) \cdot (P - \xi(t)) = 1, \\ \xi'''(t) \cdot (P - \xi(t)) = 0. \end{cases}$$

Solving this system for P gives the evolute $e(t)$.

The Frenet formulas imply

$$\begin{aligned} \xi' &= \mathbf{t}, \\ \xi'' &= k\mathbf{n}, \\ \xi''' &= -k^2\mathbf{t} + k'\mathbf{n} + k\tau\mathbf{b}, \end{aligned}$$

whence

$$P - \xi = \frac{1}{k}\mathbf{n} - \frac{k'}{k^2}\frac{1}{\tau}\mathbf{b},$$

and finally,

$$e = \xi + \frac{1}{k}\mathbf{n} - \frac{k'}{k^2}\frac{1}{\tau}\mathbf{b} = \xi + r\mathbf{n} + \frac{r'}{\tau}\mathbf{b},$$

as required □

In this calculation, we assume that $\tau \neq 0$; otherwise, the evolute escapes to infinity.

Osculating circles and osculating spheres. Let $\xi = \xi(t)$ be a curve, and let $\xi_0 = \xi(t_0)$ be a point on this curve. For $t_1 < t_2 < t_3$ close to t_0, we denote by C_{t_1,t_2,t_3} a circle passing through $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$; for $t_1 < t_2 < t_3 < t_4$ close to t_0, we denote by S_{t_1,t_2,t_3,t_4} a sphere passing through $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$, $\xi(t_4)$.

In the generic case, both C_{t_1,t_2,t_3} and S_{t_1,t_2,t_3,t_4} are well defined. Moreover, both have limits as all t_i go to t_0. This limit circle and limit sphere are called the osculating circle and the osculating sphere of the curve ξ at the point ξ_0.

It is well known (and obvious) that the osculating sphere can be described as the unique sphere that has a tangency of order ≥ 3 with ξ at ξ_0. The osculating circle is the intersection circle of the osculating sphere with the osculating plane; it has tangency of order ≥ 2 with ξ at the point ξ_0.

For our purposes, a more convenient description of the (center of the) osculating sphere is the following. We take three parameter values $t_1 < t_2 < t_3$ close to t_0 and consider the three normal planes to the curve ξ at points $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$. These three planes have a common point, and this point approaches the center of the osculating sphere at the point $\xi_0 = \xi(t_0)$ when t_1, t_2, and t_3 approach t_0.

Evolutes and osculating spheres. We have the following proposition.

Proposition 2. *The evolute of a curve is the locus of the centers of its osculating spheres.*

Proof. If $F(x, y, z, t) = 0$ is the equation of the normal plane at $\xi(t)$, then on the one hand, the solution set of the second system in (2) consists of the intersection points of triples of infinitesimally close normal planes, and on the other hand, it describes the regression edge of the normal developable, i.e., the evolute. □

Propositions 1 and 2 imply the following.

Corollary 3. *Let R be the radius of the osculating sphere of ξ at $\xi(t)$. Then*

$$R^2 = r^2 + \left(\frac{r'}{\tau}\right)^2.$$

Singularities of the evolute. From Proposition 1, one has

$$\begin{aligned} e' &= \mathbf{t} + r'\mathbf{n} - kr\mathbf{t} + \tau r\mathbf{b} + \left(\frac{r'}{\tau}\right)'\mathbf{b} - \frac{r'}{\tau}\tau\mathbf{n} \\ &= \left(r\tau + \left(\frac{r'}{\tau}\right)'\right)\mathbf{b}. \end{aligned}$$

Thus, analogously to the two-dimensional case, the tangent to the evolute is always parallel to the binormal. It follows that the cusps of the evolute occur when

$$\sigma := r\tau + \left(\frac{r'}{\tau}\right)' = 0.$$

In particular, $\sigma \equiv 0$ is the condition for a curve to be spherical; indeed, the evolute of a spherical curve is a point, the center of the sphere.

On the other hand, from Corollary 3,

$$(R^2)' = 2r'r + 2\frac{r'}{\tau}\left(\frac{r'}{\tau}\right)',$$

and so $(R^2)' = 2\frac{r'}{\tau}\sigma$. This shows that at every cusp of the evolute, R^2 has zero derivative; thus generically, R achieves a maximum or a minimum.

But there is also the possibility that $r' = 0$ and $\sigma \neq 0$, so although R is maximal or minimal, the evolute has no cusp at this point. By Corollary 3, in this case $r = R$, that is, the osculating circle is a great circle of the osculating sphere, and the center of the osculating sphere is contained in the osculating plane of the curve. This is one of the essential differences between the evolutes of planar and space curves.

Remark 4. The quantity

$$\frac{k^3\tau^2\sigma}{R^{5/2}}$$

is called the conformal torsion and is one of the two invariants of space curves in conformal geometry (the other being the conformal curvature); see, e.g., [3]. In conformal geometry, spheres are "flat," and the conformal torsion measures the deviation of the curve from its osculating sphere.

Example 1 (A closed curve whose evolute has no cusps). The evolute of a closed convex plane curve has cusps, in fact, at least four of them, according to the 4-vertex theorem (see, e.g., [10] or [8, Chapter 10]). But the evolute of a closed space curve with nonvanishing curvature and torsion may be free of cusps, as shown in Figure 3.

Example 2 (Evolute of a curve with a cusp). It is well known that for a planar curve with a generic cusp (such as a semi-cubic parabola), its evolute passes through the cusp and has no cusp at this point. The situation for a space curve is

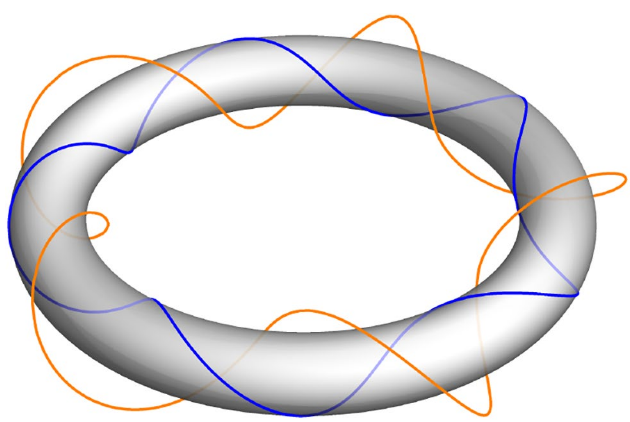

Figure 3. A cusp-free evolute (orange) of a closed space curve (blue). The blue curve is given by $t \mapsto ((l + m\cos 5t)\cos t, (l + m\cos 5t)\sin t, -m\sin 5t)$ with $l = 1$ and $m = 0.15$.

entirely different. Here is a parametric equation of a curve with a generic cusp and its evolute (see Figure 4, left). The curve has equation

$$x = t^2, \quad y = t^3, \quad z = t^4,$$

while the evolute has equation

$$\begin{aligned} x &= \frac{9}{2}t^4 + 20t^6, \\ y &= -8t^3 - 32t^5, \\ z &= \frac{1}{2} + \frac{9}{2}t^2 + 15t^4. \end{aligned}$$

We see that our curve has a cusp at the point (0, 0, 0), and the evolute has a cusp at the point (1/2, 0, 0).

Example 3 (An elliptical helix). The evolute of the standard (circular) helix is just another helix. For the elliptical helix $x = a\cos t$, $y = b\sin t$, $z = ct$, the evolute has four cusps for each turn (see Figure 4, right). The heavy dots on the evolute mark its noncuspidal points with $r' = 0$ and $\sigma \neq 0$, which correspond to maxima or minima of R.

Interior and exterior points of a curve. Since at a generic point a curve has an odd degree of tangency with the osculating sphere, a neighborhood of such a point is contained either in the interior or the exterior of that sphere. It remains unclear how to visualize the difference between "interior" and "exterior" points. Just imagine that you

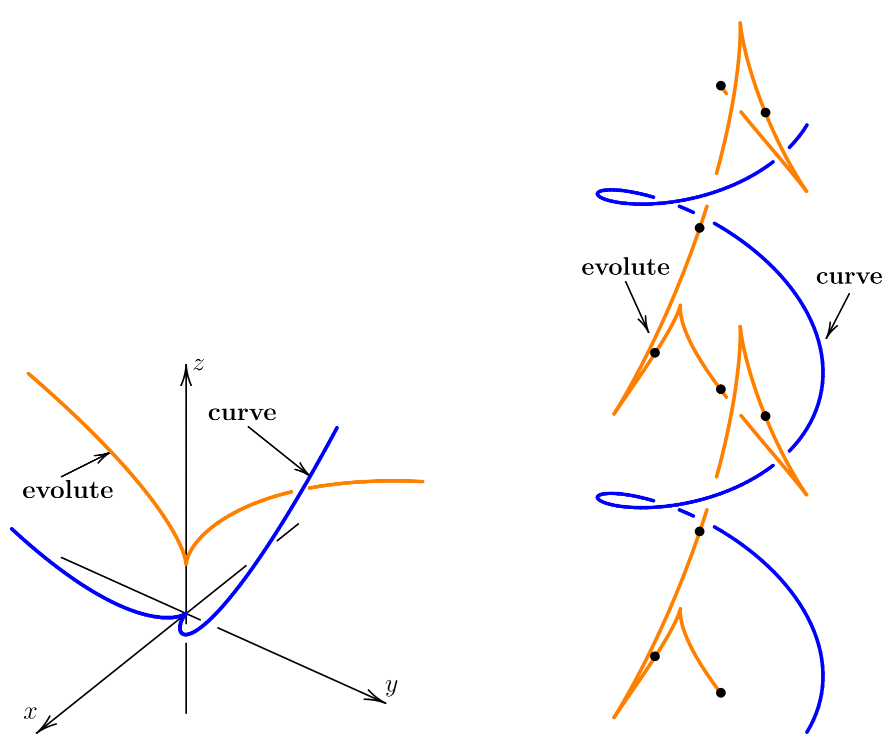

Figure 4. Left: the evolute of a curve with a cusp. Right: the evolute of an elliptical helix.

have a rigid curve, say a twisted bicycle spoke. Can you tell the interior points from the exterior ones?

The type of a point, that is, whether it is interior or exterior, changes when the curve has a higher-order contact with its osculating sphere, and this happens when the evolute has a singularity, that is, when $\sigma = 0$. A calculation, which we do not reproduce here, shows that a point is interior if and only if $\sigma\tau > 0$.

Here is a geometric interpretation: a curve of positive torsion locally lies inside its osculating sphere if the center of the osculating sphere moves in the direction of the binormal to the curve; it lies outside its osculating sphere if the center of the sphere moves in the direction opposite to the binormal.

Curvature and torsion of the evolute. As we know, $e' = \sigma\mathbf{b}$. Using the Frenet formulas, we calculate the next two derivatives of e:

$$e'' = \sigma'\mathbf{b} - \sigma\tau\mathbf{n}\,,$$
$$e''' = \sigma k\tau\mathbf{t} - (2\sigma'\tau + \sigma\tau')\mathbf{n} + (\sigma'' - \sigma\tau^2)\mathbf{b}.$$

The formulas for e' and e'' show that the Frenet frame of the evolute has the form $(\mathbf{t}_e, \mathbf{n}_e, \mathbf{b}_e) = (\pm\mathbf{b}, \pm\mathbf{n}, \pm\mathbf{t})$.

Formulas (1) imply the following result.

Proposition 5. *The curvature and the torsion of the evolute are related to the curvature and the torsion of the initial curve by the formulas*

$$k_e = \frac{|\tau|}{|\sigma|}, \quad \tau_e = \frac{k}{\sigma}\,,$$

where

$$\sigma = r\tau + \left(\frac{r'}{\tau}\right)' \text{and } r = \frac{1}{k}.$$

Assume that the evolute has no cusps, that is, $\sigma \neq 0$. Without loss of generality, $\sigma > 0$. The magical cancellations that occur on integrating the curvature or the torsion of the evolute, namely

$$\int_a^b k_e\|e'\|\,dt = \int_a^b \frac{|\tau|}{|\sigma|}|\sigma|\,dt = \int_a^b |\tau|\,dt\,,$$
$$\int_a^b \tau_e\|e'\|\,dt = \int_a^b \frac{k}{\sigma}|\sigma|\,dt = \int_a^b \operatorname{sgn}(\sigma)k\,dt\,,$$

imply the following corollary.

Corollary 6. *The total curvature of the evolute is equal to the total absolute torsion of the curve. The total torsion of the evolute is equal to the total curvature of the curve taken with the sign of* σ.

Curves congruent to their evolutes. In the plane, logarithmic spirals and cycloids are examples of curves congruent to their evolutes. For a logarithmic spiral of slope angle 45°, that is, the spiral $(e^t\cos t, e^t\sin t)$, the congruence sends every point to the center of its osculating circle.

We will say that this congruence is compatible with the parametrization. For other logarithmic spirals, the congruence involves a scaling of the parameter, and for the cycloid, a parameter shift (cusps correspond to vertices and vice versa).

The 45° logarithmic spirals are the only planar curves congruent to their evolutes in a parameter-compatible way. What about space curves?

Proposition 7. *The only space curves congruent to their evolutes* (*with the congruence compatible with the parametrization*) *are circular helices of slope angle* 45° *and helices of slope* 45° *on paraboloids of revolution. The latter helices project down to circle involutes.*

A (generalized) helix (see Figure 5) is a curve that forms a constant angle with a given direction; see [13, Section 1-9]. On the one hand, a helix is uniquely determined by its orthogonal projection along the axis and by the slope angle. On the other hand, on every surface there is a unique helix for a given axis, slope angle, and starting point (as long as the tangent plane to the surface has enough slope). The helices in the proposition above have remarkable projections and lie on remarkable surfaces.

Proof of Proposition 7. If a curve ξ is congruent to its evolute e in a parameter-compatible way, then the arc-length parameter of ξ is also an arc-length parameter of e.

Due to the equation $e' = \sigma\mathbf{b}$, this is equivalent to $|\sigma| = 1$. By Proposition 5, one then has $k_e = |\tau|$, $\tau_e = k$.

Congruent curves have the same curvature and the same or opposite torsion; thus $k = |\tau|$ is a necessary condition for a parameter-compatible congruence between ξ and e. This condition is also sufficient, since a space curve is determined by its curvature and torsion (as functions of an arc-length parameter) uniquely up to congruence.

Note that the reflection in a plane preserves k but changes the sign of τ; therefore, it changes the sign of σ. Hence, without loss of generality, we may assume that $\sigma = 1$ and thus $k = \tau$. Substituting $k = \tau$ into the formula for σ, we obtain

$$\sigma = 1 + (r'r)' = 1 + \left(\frac{r^2}{2}\right)''.$$

Thus $\sigma = 1$ if and only if $r^2 = at + b$. If $a = 0$, then $k = \tau = \text{const}$, and one obtains helices of equal curvature and torsion. If $a \neq 0$, then after a time shift and, if needed, time reversal, one has $r^2 = t/c^2$, that is, $k = \tau = ct^{-1/2}$.

Curves with a constant ratio of curvature and torsion are generalized helices; see [13, Section 1-9]. Those with equal curvature and torsion have slope angle 45°. If a helix forms an angle α with its axis, then the orthogonal projection multiplies its length element by $\sin\alpha$ and divides its curvature by $\sin^2\alpha$ [13, Section 1-9]. Because of this, it suffices to find all plane curves whose curvature is proportional to the $-\frac{1}{2}$ power of the arc-length parameter.

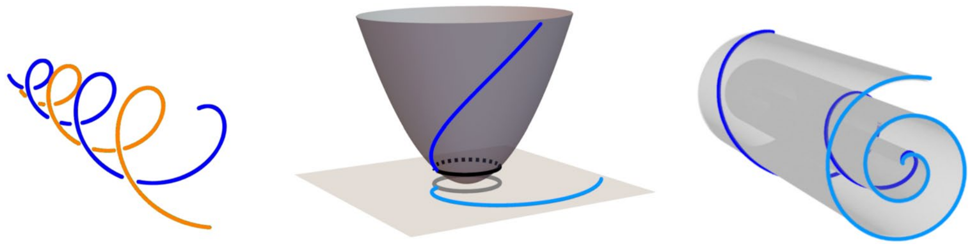

Figure 5. A curve whose curvature and torsion are both equal to $ct^{-1/2}$ (blue) together with its congruent evolute (orange). The curves are circular helices of slope angle 45° and helices of slope 45° on paraboloids of revolution.

This problem has a direct solution (see the proof of the "fundamental theorem of plane curves"), and the curves are the circle involutes.

The involute of a circle of radius R has the parametrization

$$\gamma(t) = R(\cos t + t\sin t, \sin t - t\cos t)\,.$$

Observe that $\|\gamma'(t)\| = Rt$, so that the arc-length parameter of γ is $Rt^2/2$. The corresponding helix is given by

$$\xi(t) = R\big(\cos t + t\sin t, \sin t - t\cos t, t^2/2\big)\,;$$

it lies on the paraboloid $x^2 + y^2 = 2R^2 z + R$. □

It is tempting to apply Corollary 6 twice: the curvature density and the torsion density of the second evolute coincide with those of the initial curve (under the nonrestrictive assumption that the torsion is positive). This is equivalent to the fact that the Frenet frame of the second evolute is parallel to the Frenet frame of the initial curve (which also follows from the identity $(\mathbf{t}_e, \mathbf{n}_e, \mathbf{b}_e) = (\pm\mathbf{b}, \pm\mathbf{n}, \pm\mathbf{t})$, which we mentioned earlier).

Two simultaneously parametrized curves with parallel tangents at the corresponding points are called Combescure transformations of each other; see [12] for a detailed study.

A curve is congruent to its second evolute in a parametrization-compatible way if and only if an arc-length parameter for the curve is also an arc-length parameter for the second evolute. This condition is equivalent to a complicated differential equation:

$$\frac{1}{r\tau}\left(\frac{r'}{\tau}\right)' + \left(\frac{1}{\sigma\tau}\left(\frac{\sigma}{\tau}\right)'\right)' = 0\,. \tag{3}$$

However, if the curvature k is constant, then $\sigma = \tau/k$, and equation (3) is satisfied. It follows that a space curve of constant curvature is congruent to its second evolute. See [9] for curves that are homothetic to their second evolutes.

Involutes. By definition, the curve ξ is an involute of a curve e if e is the evolute of ξ. A generic curve has a two-parameter family of involutes: they are the curves orthogonal to the family of osculating planes of the given curve.

Let S be the tangent developable of e, and let H be the osculating plane of e at some point. Let us roll the plane H along the surface S without slipping and twisting. The instantaneous motion of the plane H is a rotation about a line in H. Therefore, the velocity of each point of the plane H is perpendicular to H, and the involutes of e are the trajectories of the points of H in the process of rolling.

This description shows that the involutes of a curve are equidistant: the distances between the points of the rolling plane do not change. It also shows that each involute has a cusp every time it reaches S.

Assume that e is a generic smooth closed curve with nonvanishing torsion. After rolling the osculating plane H all the way around e, this plane returns to the original position. A self-map of H arises, which we call the monodromy.

If we orient the osculating planes of a curve by its binormals, then the monodromy is an isometry of H that preserves this orientation. An orientation-preserving isometry of the plane is either the identity, a rotation, or a translation. The rotation angle of the monodromy is computed in Corollary 9 below. Since a rotation has a unique fixed point, we conclude that a generic smooth closed curve e has a unique closed involute.

This is in contrast to the planar case: for the involute of a closed plane curve to be closed, the curve must have zero alternating perimeter (the sign changes after every cusp), and if this alternating perimeter vanishes, then all involutes are closed; see [10] or [8, Chapter 10].

Consider the trace e_H of the curve e in the plane H as this plane rolls along e. This trace is a planar development of the curve e: its curvature, as a function of the arc length, is the same as that of e. (Imagine that a curve is made of wire that is hard to bend but easy to twist. Then one can flatten this curve without changing its curvature.)

While the curve e is closed, e_H in general is not. The endpoints of e_H and the tangent directions therein are two contact elements in the plane H.

Proposition 8. *The monodromy is the orientation-preserving isometry of H that sends the initial contact element of the curve e_H to its terminal contact element.*

Proof. When one surface is rolled without slipping and twisting along a curve on another surface, this curve and its trace on the rolling surface have the same geodesic curvatures at corresponding points. This describes the relation between the curves e and e_H.

An instantaneous displacement of the plane H when it is rolled along the curve e is the parallel translation distance

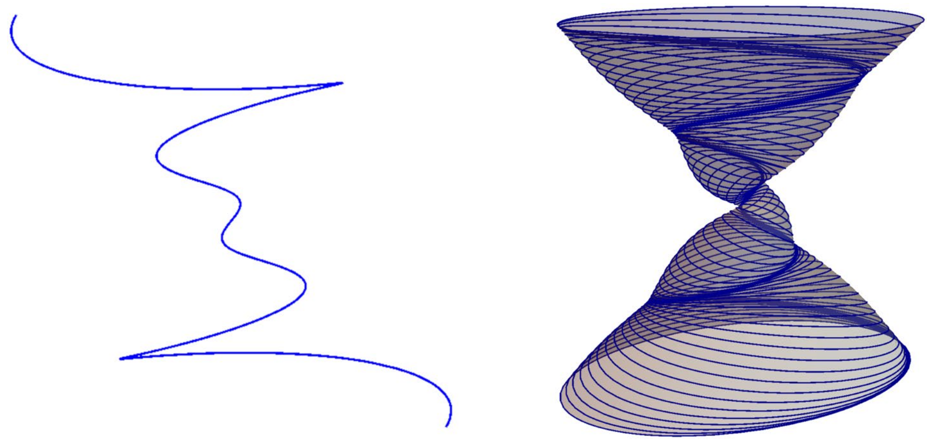

Figure 6. Osculating circles of a space curve.

dt in the tangent direction $e'(t)$, combined with the rotation through the angle $k(t)\,dt$. Integrated along e, this defines the motion described in the statement. □

Corollary 9. *The rotation angle of the monodromy is equal to the total curvature of the curve.*

Assume that a curve has a closed involute. When are all involutes also closed? That is, when is the monodromy the identity map?

Corollary 10. *If a curve has at least one closed involute and the total torsion of this involute is an integer multiple of 2π, then all involutes are closed.*

This follows from Corollary 6: the total curvature of the evolute is equal to the total absolute torsion of the curve. The torsion of the involute does not change its sign if the curve has no cusps.

An example of a closed space curve whose planar development is also closed is a curve of constant curvature 1 and length $2k\pi$ (the development is a circle traversed k times).

Osculating circles. As we mentioned earlier, the osculating circles of a plane curve with monotonic curvature are nested. What about the osculating circles of a space curve? In particular, can the osculating circles at neighboring points be linked?

The osculating circles of the curve in Figure 6 seem to be unlinked. The next proposition confirms that locally, this is always the case.

Proposition 11. *Let ξ be a curve with nonvanishing torsion. Then its osculating circle at every point is disjoint from its osculating planes at sufficiently close points. In particular, the osculating circles at close points are not linked.*

Proof. We view the curve as the regression edge of the envelope of the 1-parameter family of planes $z = a(t)x + b(t)y + c(t)$. We assume that for $t = 0$, the plane is $z = 0$, that $\xi(0) = (0, 0, 0)$, and that the tangent line to ξ at the origin is the x-axis. This means that

$$a(0) = b(0) = c(0) = a'(0) = c'(0) = c''(0) = 0\,.$$

Thus

$$a(t) = at^2 + O\left(t^3\right),$$
$$b(t) = bt + O\left(t^2\right),$$
$$c(t) = ct^3 + O\left(t^4\right).$$

We will ignore the "big O" terms in what follows: this will not affect the result, but it will make the formulas less awkward.

We find the equation of the curve using the middle equation (2):

$$\xi(t) = \left(-\frac{3ct}{a}, \frac{3ct^2}{b}, ct^3\right).$$

The curvature of this curve at the origin is $2a^2/3bc$, and the radius of curvature is $3bc/2a^2$.

The intersection of the osculating planes of the curve $\xi(t)$, that is, the planes $z = a(t)x + b(t)y + c(t)$, with the plane $z = 0$ are the lines $a(t)x + b(t)y + c(t) = 0$ (when $t = 0$, it is the x-axis), that is, $at^2x + bty + ct^3 = 0$.

The envelope of this family of lines is the parabola

$$x = \frac{2ct}{a}, \quad y = \frac{ct^2}{b}\,.$$

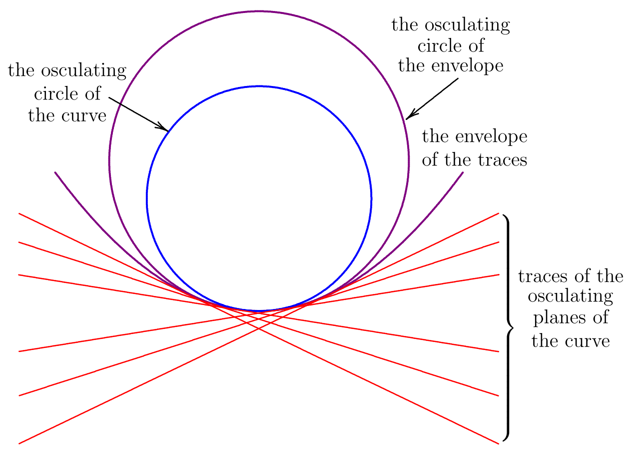

Figure 7. Data for the proof of Proposition 11.

The curvature of this curve at the origin is $a^2/2bc$, and its radius of curvature is $2bc/a^2$, which is greater than $3bc/2a^2$.

It follows that the osculating circle of the curve ξ at the origin lies inside the parabola. Therefore, the tangent lines to the parabola, that is, the intersections of the osculating planes of $\xi(t)$ with its osculating plane at $\xi(0)$, are disjoint from the osculating circle at $\xi(0)$. This implies the result; see Figure 7. $\square$

Pseudo-evolutes

Definition and first properties. A generic space curve ξ determines a one-parameter family of rectifying planes. This is the family of tangent planes to a certain developable surface S, called a rectifying developable, and this surface has a regression edge. We call this regression edge the pseudo-evolute of the initial curve. In other words, the pseudo-evolute of a curve ξ is a curve whose osculating planes are the rectifying planes of ξ.

Proposition 12. *The curve ξ is a geodesic on S.*

Proof. According to one of many descriptions of a geodesic on a surface in space, it is a curve whose principal normals are normal to the surface. For ξ on S, this clearly holds: the principal normals of ξ are perpendicular to the rectifying planes of ξ, that is, to the tangent planes of S. $\square$

Informally, this means that a piece of paper with a straight line drawn on it may be attached (without crumblings and foldings) in a unique way to a given space curve in such a way that the line follows the curve (see [7]). This surface is unique because the normals to this surface are determined: they are the normals of ξ. This gives another description of the pseudo-evolute of a (generic) curve ξ: it is the regression edge of the unique developable surface S that contains ξ as a geodesic.

Equation of the pseudo-evolute. The equation of the rectifying plane to the curve ξ at the point $\xi(t)$ is

$$(P - \xi(t)) \cdot \mathbf{n}(t) = 0\,.$$

The first and second derivatives of this equation with respect to t are

$$(P - \xi(t)) \cdot (-k\mathbf{t}(t) + \tau\mathbf{b}(t)) = 0\,,$$
$$k + (P - \xi(t)) \cdot \left(-k'\mathbf{t} + \tau'\mathbf{b} - \left(k^2 + \tau^2\right)\mathbf{n}\right) = 0\,.$$

These three equations form the system that has the solution

$$(P - \xi(t)) \cdot \mathbf{t}(t) = \frac{k\tau}{k'\tau - k\tau'}\,,$$
$$(P - \xi(t)) \cdot \mathbf{b}(t) = \frac{k^2}{k'\tau - k\tau'}\,.$$

We thereby obtain the equation of the pseudo-evolute $\varepsilon(t)$ of the curve $\xi(t)$:

Proposition 13. *The pseudo-evolute ε of the curve ξ has the equation*

$$\varepsilon = \xi + \frac{k}{k'\tau - k\tau'}(\tau\mathbf{t} + k\mathbf{b})\,.$$

Escapes to infinity and cusps.

Proposition 14. *Let ξ be a generic curve with curvature k and torsion τ. Then:*

1. *The pseudo-evolute of ξ escapes to infinity when the first derivative of the ratio τ/k vanishes.*
2. *The pseudo-evolute of ξ has cusps when the second derivative of τ/k vanishes.*

Proof. Statement 1 follows from Proposition 13: ε escapes to infinity when $k'\tau - k\tau' = 0$.

Statement 2 requires some computation. From the formula in Proposition 13, we obtain

$$\varepsilon' = \mathbf{t} + \left(\frac{k}{k'\tau - k\tau'}\right)'(\tau\mathbf{t} + k\mathbf{b}) + \frac{k}{k'\tau - k\tau'}(\tau\mathbf{t} + k\mathbf{b})'\,. \tag{4}$$

By Frenet's formulas, $(\tau\mathbf{t} + k\mathbf{b})' = (\tau'\mathbf{t} + k'\mathbf{b})$, so the right-hand side of (4) is a linear combination of $\mathbf{t}$ and $\mathbf{b}$. The coefficients of $\mathbf{t}$ and $\mathbf{b}$ are

$$\tau\left(\frac{2k'}{k'\tau - k\tau'} - \frac{k(k'\tau - k\tau')'}{(k'\tau - k\tau')^2}\right)$$

and

$$k\left(\frac{2k'}{k'\tau - k\tau'} - \frac{k(k'\tau - k\tau')'}{(k'\tau - k\tau')^2}\right).$$

Hence $\varepsilon' = 0$ if

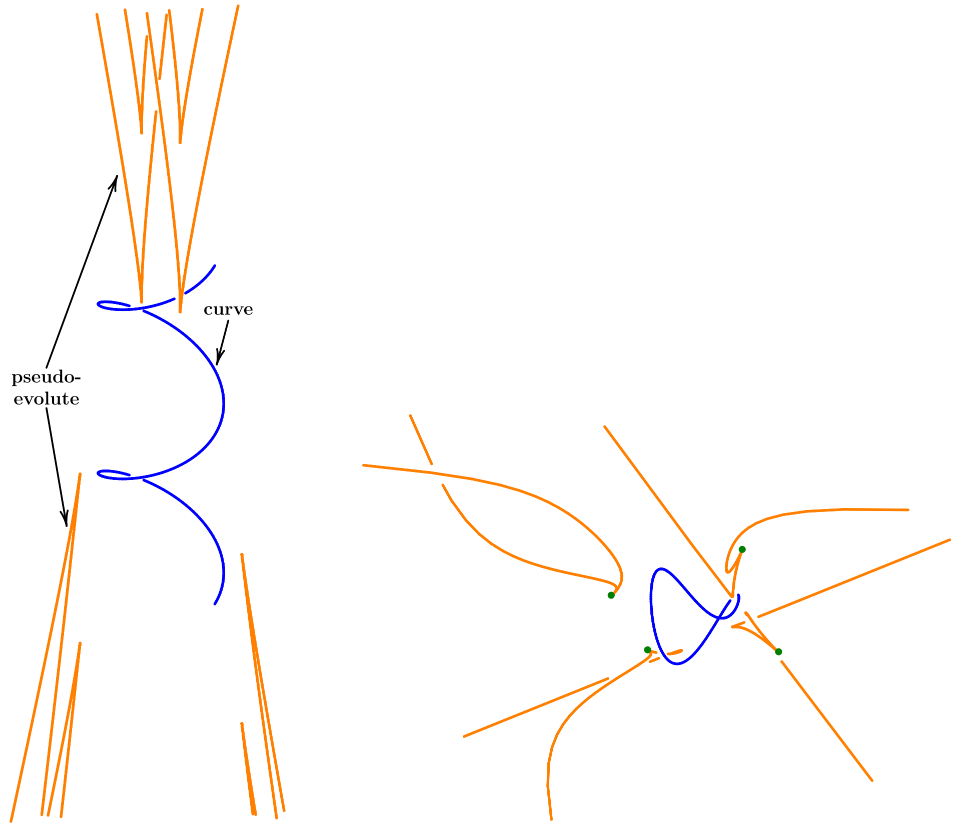

Figure 8. Left: the pseudo-evolute of an elliptical helix. Right: the pseudo-evolute of a closed smooth curve. Its four visible cusps are marked by green dots, and it escapes four times to infinity.

$$\frac{2k'}{k'\tau - k\tau'} - \frac{k(k'\tau - k\tau')'}{(k'\tau - k\tau')^2} = 0\,,$$

and the last expression is

$$\frac{k^3}{(k'\tau - k\tau')^2}\left(\frac{\tau}{k}\right)''.$$

□

Thus the derivative $(\tau/k)'$ plays, for pseudo-evolutes, a role similar to that of the curvature for evolutes of planar curves.

The function τ/k has the following geometric meaning. The tangent indicatrix of a space curve $\xi(t)$ is the curve $\xi'(t)$ on the unit sphere (recall that ξ is parametrized by arc length). The geodesic curvature of the tangent indicatrix equals τ/k.

Proposition 14 has the following consequence [13, Section 2-4, Problem 5].

Corollary 15. *The rectifying developable is a cone if and only if τ/k is a linear function, and it is a cylinder if and only if τ/k is a constant.*

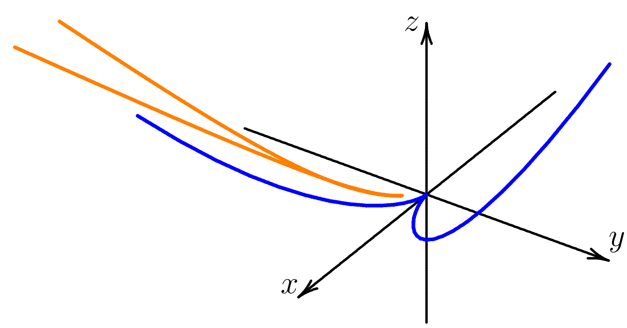

Figure 9. The pseudo-evolute of a curve with a cusp.

Examples. Figure 8 shows two examples: the pseudo-evolutes of the same elliptical helix as in Figure 4, and that of the curve $x = \cos t$, $y = \sin t$, $z = \frac{1}{2}\sin 2t$.

Pseudo-evolute of a curve with a cusp. Our definition works in this case, but we are puzzled by the geometry of the result; see Figure 9.

Let us consider the standard example of a curve with a cusp: $x = t^2$, $y = t^3$, $z = t^4$.

After writing the equations of the rectifying planes and finding the regression edge of their envelope by solving the middle system in (2), we obtain a curve with a degenerate cusp of the type (t^2, t^4, t^5), situated at the point with coordinates $\frac{3}{350}(16, 0, 9)$; see Figure 9.

It is not clear to us how our geometric interpretation of the pseudo-evolute (that the pseudo-evolute of a curve is the regression edge of the developable surface that contains the curve as a geodesic) works in this case.

Pseudo-involutes. Every curve has a two-parameter family of pseudo-involutes, namely the curves whose pseudo-evolute is the initial curve; indeed, a curve η is the regression edge of a developable surface, the union of tangent lines of η, and the geodesics of this surface are pseudo-involutes of η.

Pseudo-involutes may have cusps: it happens every time the involute reaches η. (This makes pseudo-involutes similar to involutes of planar curves, but there is a big difference: pseudo-involutes of η do not need to be perpendicular to η at the cuspidal points.)

A pseudo-involute may be a smooth closed curve, as shown in the right-hand side of Figure 8. Then it is a closed geodesic on a developable surface, and its equidistant curves on this surface are also closed geodesics. Hence such a curve is included in a 1-parameter family of smooth closed pseudo-involutes.

An annoying question. We see that pseudo-evolutes have many properties similar to those of evolutes. Still, their geometric meaning remains enigmatic. Evolutes are the loci of centers of osculating spheres. And pseudo-evolutes are the loci—of what?[2]

Monge Evolutes

Monge involutes. Attach one end of an unstretchable string to a space curve η, pull the string tight, and wrap it around η. The velocity of the free end of the string is always orthogonal to the string (it is unstretchable). The trajectory of the free end is the curve ξ, a Monge involute of η. Changing the length of the string yields a 1-parameter family of these involutes.

Differentiating the formula

$$\xi(t) = \eta(t) + (\ell - t)\mathbf{t}_\eta, \tag{5}$$

where ℓ is the length of the string attached at the point $\eta(0)$, yields

$$\xi' = (\ell - t)k_\eta \mathbf{n}_\eta \,. \tag{6}$$

It follows that if the curvature of η never vanishes, then the cusps of ξ lie on η, which happens when the free end of the string lands on the curve.

Monge evolutes. If ξ is a Monge involute of η, then by definition, η is a Monge evolute of ξ. Does every curve have such an evolute, and if so, how many?

If η is an evolute of ξ, then $\eta - \xi$ is the free part of the string wrapped around η. Since the string is tangent to η, one has

$$\eta = \xi + y\mathbf{n} + z\mathbf{b}\,,$$

for some functions y and z of the parameter ($\mathbf{n}$ and $\mathbf{b}$ denote the normal and binormal of ξ). Differentiate with respect to the arc-length parameter of ξ and apply the Frenet formulas:

$$\eta' = (1 - ky)\mathbf{t} + (y' - \tau z)\mathbf{n} + (z' + \tau y)\mathbf{b}\,.$$

By assumption, η' must be parallel to $\eta - \xi = y\mathbf{n} + z\mathbf{b}$, which leads to the system of equations

$$1 - ky = 0, \quad \frac{y' - \tau z}{y} = \frac{z' + \tau y}{z}.$$

This system has the one-parameter family of solutions

$$y = r, \quad z = -r \tan \int \tau \, dt.$$

This leads to the following result.

Proposition 16. *The Monge evolutes of an arc-length-parametrized curve ξ are given by*

$$\eta = \xi + r\mathbf{n} - r \tan\alpha\, \mathbf{b}, \quad \alpha' = \tau\,.$$

In particular, spatial Monge evolutes of a plane curve are geodesics on the cylinder over the plane evolute of this curve.

Corollary 17. *The corresponding points of two different evolutes are seen from the corresponding point of the curve under a constant angle.*

Singularities of Monge evolutes. Proposition 16 implies that

$$\|\eta - \xi\| = \frac{1}{k|\cos\alpha|}\,.$$

On the other hand, due to (5), one has

$$\|\eta - \xi\| = \int_{t_0}^{t} \|\eta'\| \, dt.$$

Therefore,

$$\|\eta'\| = \left(\frac{1}{k|\cos\alpha|}\right)'.$$

It follows that the cusps of a Monge evolute correspond to the critical points of the function $k\cos\alpha$. This generalizes a property of plane evolutes: their cusps correspond to the critical points of the curvature k, the vertices of the curve.

Proposition 16 also makes it possible to determine when an evolute escapes to infinity: this happens if either $k = 0$ or $\alpha = \frac{\pi}{2} + k\pi$.

Closed Monge evolutes and Monge involutes. Proposition 16 has the following consequence.

[2]Spoiler alert: the answer will be given at the very end of the article.

Proposition 18. *Monge evolutes of a closed space curve are closed if and only if the total torsion of the curve is an integer multiple of π. In particular, Monge evolutes of a centrally symmetric curve are closed.*

This statement is due to the obvious fact that a central symmetry reverses the sign of the torsion (Figure 10).

As for closed Monge involutes, the situation is exactly the same as in the plane:

Proposition 19. *Monge involutes of a closed space curve are closed if and only if the curve has zero length* (*the length element changes sign after each cusp*).

Interrelations between evolutes. As promised, we describe how the three kinds of evolutes interact with one another.

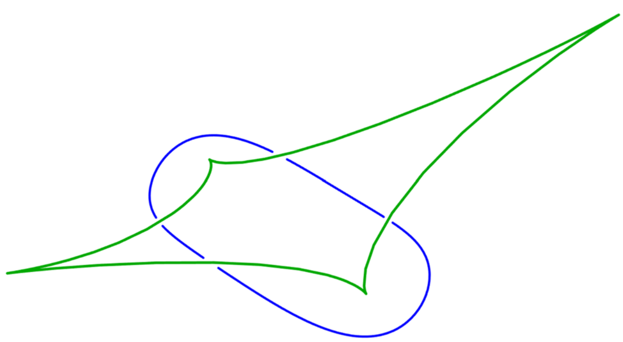

Figure 10. A Monge evolute (green) of a centrally symmetric curve.

The line $\{\xi + r\mathbf{n} + \lambda\mathbf{b} \mid \lambda \in \mathbb{R}\}$ is called the polar line of the curve ξ at the corresponding point. This is the line that goes through the center $\xi + r\mathbf{n}$ of the osculating circle and is orthogonal to the osculating plane of ξ. The center of the osculating sphere lies on the polar line.

Recall the formula $e' = \sigma\mathbf{b}$. It implies that the polar line is tangent to the evolute of ξ. It follows that the polar lines are the rulings of the normal developable of the curve ξ.

Proposition 16 implies that each point of a Monge evolute lies on some polar line, and every point of a polar line lies on a unique Monge evolute. It follows that the normal developable surface of the curve ξ is foliated by its Monge evolutes.

A Monge evolute of ξ meets the evolute of ξ at the points where $r' = r\tau\tan\alpha$. The tangent to the Monge evolute at those points coincides with the binormal of ξ, that is, the Monge evolute is tangent to the evolute (Figure 11).

As a result, we have the following.

Proposition 20. *The evolute is the envelope of Monge evolutes.*

We conclude with yet another relation between the different types of evolutes [13, Section 2-4, Exercise 3].

Proposition 21. *The pseudo-evolute of any space curve is the evolute of any of its Monge involutes.*

Proof. Let η be a space curve, and let ξ be a Monge evolute of η. One has to show that the normal planes of η are the rectifying planes of ξ. These planes are parallel because their normals are parallel; see (6). And they coincide, because $\xi - \eta$ is tangent to η and normal to ξ and thus is contained in both planes. $\square$

In particular, this provides an answer to the above "annoying question": the pseudo-evolute of a curve is the

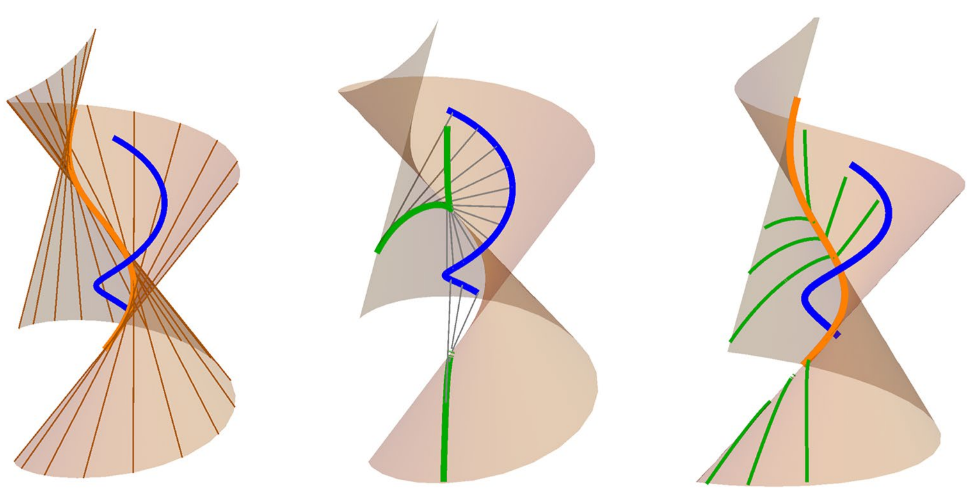

Figure 11. The evolute (orange) and some Monge evolutes (green) of a curve ξ (blue) together with its normal developable (orange surface). *Left:* The polar lines are tangent to the evolute and lie on the normal developable of ξ. *Middle:* The tangents of the Monge evolute intersect ξ orthogonally. *Right:* The Monge evolutes also foliate the normal developable of ξ.

locus of the centers of the osculating spheres of its Monge evolute.

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