DISJOINTNESS OF A SIMPLE MATRIX LIE GROUP AND ITS LIE ALGEBRA

MICHAEL LARSEN

Abstract. Let G be a connected closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is simple as a Lie group and which acts irreducibly on $\mathbb{C}^n$. Regarding both G and its Lie algebra $\mathfrak{g}$ as subsets of $M_n(\mathbb{C})$, we have $G \cap \mathfrak{g} \neq \emptyset$ if and only if G is a classical group and $\mathbb{C}^n$ is a minuscule representation.

To Pham Huu Tiep on his sixtieth birthday, with admiration

Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$ and $\rho\colon G \to \mathrm{GL}(V)$ a faithful, irreducible, complex representation of G. Let $\rho_*\colon \mathfrak{g} \to \mathfrak{gl}(V)$ denote the corresponding Lie algebra representation. Using the embeddings $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(V) \hookrightarrow \mathrm{End}(V)$ and $\mathfrak{g} \overset{\rho_*}{\hookrightarrow} \mathfrak{gl}(V) = \mathrm{End}(V)$, it makes sense to ask if $G \cap \mathfrak{g}$, by which we abbreviate $\rho(G) \cap \rho_*(\mathfrak{g})$, is empty. As we will see, the answer is usually yes, but the exceptions form a class which has received attention before (see [Ser]).

Recall [Bou2, VIII, §7.3] that V is *minuscule* if it is non-trivial and its weights with respect to a choice of maximal torus form a single orbit under the action of the Weyl group. The minuscule representations are as follows [Ser, Appendix]: for type A_r, all exterior powers of the natural representation; for B_r, the spin representation; for C_r, the natural representation; for D_r, the natural representation and the semispin representations; for E_6, the two 27-dimensional irreducibles; and for E_7, the unique 56-dimensional irreducible.

The main result of this paper is as follows:

Theorem 1. *Under the above hypotheses, $G \cap \mathfrak{g}$ is non-empty if and only if G is of classical type and V is minuscule.*

Here, by classical type, we mean type A, B, C, or D. The rest of the paper proves the theorem. The main difficulty is to show necessity of the conditions on (G, V).

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Let $g \in G$ and $x \in \mathfrak{g}$ satisfy $\rho(g) = \rho_*(x)$. The Jordan decomposition $g = tu$ in G gives the Jordan decomposition $\rho(g) = \rho(t)\rho(u)$ in $\mathrm{GL}(V)$. In $M_n(\mathbb{C})$, we can write $\rho(u) = 1+N$, where N is nilpotent and commutes with

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$\rho(t)$, so $\rho(t) + \rho(t)N$ is the Jordan-Chevalley decomposition of $\rho_*(x) = \rho(g)$ in $\mathrm{End}(V)$. By [Hum, Chap. 4, Lemma A], it follows that $\rho(t) \in \rho_*(\mathfrak{g})$. Without loss of generality, therefore, we may assume that g is semisimple, and redefining x if necessary, we still have $\rho(g) = \rho_*(x)$. Let T be a maximal torus of G containing g, and let $\mathfrak{t}$ denote its Lie algebra.

Now, $\rho_*(x)$ commutes with $\rho(t)$ for all $t \in T$. As ρ_* is injective, $\mathrm{ad}(t)(x) = x$ for all $t \in T$, so x lies in the weight-0 space of the adjoint representation of G. In other words, $x \in \mathfrak{t}$.

As usual, we identify the character group $X^*(T)$ with a subgroup of $\mathfrak{t}^*$ by means of the diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\exp} & T \\ {\scriptstyle t^*}\downarrow & & \downarrow{\scriptstyle \chi} \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times \end{array}$$

relating a character χ and its corresponding vector t^*. We choose a basis $e_1, \ldots, e_n$ of V consisting of weight vectors for T for characters $\chi_1, \ldots, \chi_n \in X^*(T)$ and let $t_1^*, \ldots, t_n^*$ denote the elements of $\mathfrak{t}^*$ corresponding to $\chi_1, \ldots, \chi_n$ respectively. If $a_1, \ldots, a_n \in \mathbb{Z}$ and $\chi_1^{a_1} \cdots \chi_n^{a_n} = 1$, then $a_1 t_1^* + \cdots + a_n t_n^* = 0$.

Lemma 2. *If χ_i, χ_j, χ_k are three characters of V such that $\chi_i \chi_k = \chi_j^2$, then $\chi_i(g) = \chi_j(g) = \chi_k(g)$.*

Proof. As $\chi_i \chi_j^{-2} \chi_k = 1$, we have $t_i^* - 2t_j^* + t_k^* = 0$. As $\rho(g) = \rho_*(x)$, we have

$$\chi_i(g) = t_i^*(x),\ \chi_j(g) = t_j^*(x),\ \chi_k(g) = t_k^*(x),$$

so we have

$$4\chi_i(g)\chi_k(g) = 4\chi_j(g)^2 = 4t_j^*(x)^2 = (t_i^*(x) + t_k^*(x))^2 = (\chi_i(g) + \chi_k(g))^2,$$

so $\chi_i(g) = \chi_k(g)$. Thus,

$$\chi_j(g) = t_j^*(x) = \frac{t_i^*(x) + t_k^*(x)}{2} = \frac{\chi_i(g) + \chi_k(g)}{2} = \chi_i(g).$$

□

We remark that this argument just makes explicit the fact that over $\mathbb{C}$, $a + c = 2b$ and $ac = b^2$ imply $a = b = c$. In fact, this is true over every field, even in characteristic 2.

Lemma 3. *If $\chi_i \chi_j = \chi_k \chi_l$ for characters of V, then*

$$(z - \chi_i(g))(z - \chi_j(g)) = (z - \chi_k(g))(z - \chi_l(g)). \tag{1}$$

Proof. The character equality implies

$$t_i^* + t_j^* = t_k^* + t_l^*, \tag{2}$$

so

$$\chi_i(g) + \chi_j(g) = t_i^*(x) + t_j^*(x) = t_k^*(x) + t_l^*(x) = \chi_k(g) + \chi_l(g).$$

As $\chi_i(g)\chi_j(g) = \chi_k(g)\chi_l(g)$, equation (1) follows immediately. □

Lemma 4. *If V is self-dual, then $\chi_i(g)^2 + 1 = 0$ for all χ_i.*

Proof. Since V is self-dual, for each χ_i, there exists χ_j such that $\chi_i\chi_j = 1$. Then $t_i^* + t_j^* = 0$, and

$$\chi_i(g) = t_i^*(x) = -t_j^*(x) = -\chi_j(g) = -\chi_i(g)^{-1}.$$

Thus $\chi_i(g)^2 + 1 = 0$. □

Proposition 5. *If V is not a minuscule representation, then $G \cap \mathfrak{g} = \emptyset$.*

Proof. We fix a maximal torus and a Weyl chamber. Let $\varpi_1, \ldots, \varpi_r$ denote the fundamental characters, and let $\lambda = \sum_i a_i\varpi_i$ denote the dominant weight of V. Let α denote the highest root in the dual root system. Let $s = \langle\lambda, \alpha\rangle$. By [Bou2, VIII, §7, Prop. 3(i)], the length of the $\alpha^\vee$-string of weights of V through λ is $s + 1$. If $s \geq 2$, then by Lemma 2, $\alpha^\vee(g) = 1$. By symmetry, the same is true for all short roots.

The root lattice for any irreducible root system is generated by short roots. Indeed, if every short root were orthogonal to every long root, the span of the short roots would be a proper Weyl-invariant subspace of the span of the root system, which is impossible by [Bou1, VI, §1 Prop. 5]. Therefore, in an irreducible but not simply laced root system, we may take two such roots to belong to a root subsystem of type B_2 or G_2. In these rank-2 systems, every long root is a sum of two short roots, and since the long roots form a single Weyl-orbit, it follows that all roots in the original system are sums of short roots.

Therefore, $\chi(g) = 1$ for all χ in the root lattice, so $\rho(g)$ is an invertible scalar matrix. However, $\rho_*(x)$ lies in $\mathfrak{sl}(V)$ and therefore has trace 0, so this is impossible. Therefore, $s \leq 1$. Now, $\langle\varpi_i, \alpha\rangle \geq 1$ with equality if and only if ϖ_i is a minuscule weight [Ser, Appendix]. Therefore the proposition holds whenever λ is a fundamental weight. On the other hand, if $\sum_i a_i \geq 2$, then

$$\langle\lambda, \alpha\rangle = \sum_i a_i\langle\varpi_i, \alpha\rangle \geq \sum_i a_i \geq 2,$$

and the proposition again holds. □

Proposition 6. *If G is of type E_6 and $\dim V = 27$, then $G \cap \mathfrak{g} = \emptyset$.*

Proof. The linear span of the set of differences $t_i^* - t_j^*$ is a Weyl-subrepresentation of $\mathfrak{t}^*$, so by [Bou1, VI, §1, Prop. 5], it is all of $\mathfrak{t}^*$. If $\{t_i^*(x) \mid 1 \leq i \leq 27\}$ has only 1 element, it follows that $x = 0$. This implies $\rho(g) = 0$, which is impossible. If $\{t_i^*(x) \mid 1 \leq i \leq 27\}$ has exactly 2 elements, then there exists a linear function on $\mathfrak{t}$ (namely, x, regarded as an element of $(\mathfrak{t}^*)^*$) which takes two values, a and b, $a < b$, on every element t_i^* and therefore a value in $[a, b]$ on every point of the convex hull of $\{t_i^* \mid 1 \leq i \leq 27\}$ in $X^*(T) \otimes \mathbb{R} \subset \mathfrak{t}^*$. This convex hull is the polytope denoted 2_{21} in Coxeter's notation, and each of its codimension 1 faces is either a simplex or a hyperoctahedron of dimension 5 [Gre, Example 8.5.16]. It therefore has 6 or 10 vertices. Therefore, the intersection of $\{t_i^* \mid 1 \leq i \leq 27\}$ with any affine hyperplane lying entirely on one side of the set has at most 10 points, and the union of two such intersections cannot have more than 20 points. Therefore, $\{t_i^*(x) \mid 1 \leq i \leq 27\}$ has at least 3 values.

Endowing $X^*(T) \otimes \mathbb{R}$ with the Weyl-invariant inner product for which the roots have length $\sqrt{2}$, we see that the inner product of any weight with any element of the root lattice is integral, while the inner product of any weight of V with itself is $4/3$ [Bou1, Planches]. Therefore, the inner product $\langle t_i^*, t_j^* \rangle$ of any two distinct weights of V is congruent to $1/3 \pmod 1$ and contained in the interval $[-4/3, 4/3)$. The only possible values are therefore $1/3$ and $-2/3$. In the former case, we say that t_i^* and t_j^* are *skew*. In the latter case, we say they are *incident*. (The terminology is motivated by the correspondence [Gre, Example 8.6.4] between the weights of V and the 27-lines on a cubic surface.) For each weight, the number of incident weights is 10 [Gre, Lemma 10.1.6].

We claim that Lemma 3 implies that $\{t_1^*(x), \ldots, t_{27}^*(x)\}$ has at most 2 elements. We prove this by contradiction, assuming that $t_i^*(x)$, $t_j^*(x)$, and $t_k^*(x)$ are pairwise distinct. There are four possibilities regarding $\{t_i^*, t_j^*, t_k^*\}$ to consider.

Case 1. No pair of the weights is incident. By [Gre, Exercise 8.1.11] and [Gre, Lemma 9.2.7], the Weyl group of E_6 acts transitively on triples of pairwise skew weights. By [Gre, Lemma 10.1.9], we may assume without loss of generality that t_i^*, t_j^*, and t_k^* belong to a 6-element set of pairwise skew weights and therefore, by [Gre, Theorem 10.2.1], to one half of a Schläfli double six. In the other half, there are three weights which are linked to each of t_i^*, t_j^*, and t_k^*; choose one of them and denote it t_l^*. We claim that $-t_m^* - t_i^*$, $-t_m^* - t_j^*$, and $-t_m^* - t_k^*$ are all weights of V. Indeed, they lie in the dual lattice to the root lattice of E_6, and each has the same length as a weight of V. This implies that each is either a weight or the negative of a weight [CS, Chapter 4, (122)]. They are actually weights since their inner products with every weight is congruent to $1/3 \pmod 1$ rather than $2/3$. However,

$$(-t_m^* - t_i^*) + t_i^* = (-t_m^* - t_j^*) + t_j^* = (-t_m^* - t_k^*) + t_k^*.$$

By Lemma 3, this implies that $t_i^*(x)$, $t_j^*(x)$, and $t_k^*(x)$ are all roots of a common quadratic polynomial, which is absurd.

Case 2. Exactly one pair of the weights is incident. We may assume t_i^* and t_j^* are incident. We have

$$(t_i^* + t_j^* - t_k^*)^2 = \frac{4}{3},$$

By the same reasoning as in Case 1, $t_i^* + t_j^* - t_k^*$ is a weight t_l^*, so (2) holds. However, $t_k^*(x)$ is not a root of $(z - t_i^*(x))(z - t_j^*(x))$, contrary to assumption.

Case 3. Exactly two pairs of the weights are incident. Assume that t_i^* and t_j^* are skew, and both are incident to t_k^*. By [Gre, Proposition 10.2.7(iii)], the number of weights incident to both t_i^* and t_j^* (including t_k^*) is 5, so the number of weights incident to at least one of the two is 15, including t_k^*. Therefore, among weights not in $\{t_i^*, t_j^*, t_k^*\}$, the number skew to both t_i^* and t_j^* is 10, and the number incident to t_k^* is 8. We conclude that there exists t_m^* which is skew to all three. Therefore, we may choose two elements of $\{t_i^*, t_j^*, t_k^*\}$ together with t_m^* and obtain three weights, all taking different values in x, containing at most a single incident pair. This is impossible by Cases 1 and 2.

Case 4. All pairs of the weights are incident. The number of incident pairs with one element in $\{t_i^*, t_j^*, t_k^*\}$ and one element in its complement is 24, so on average each weight in the complement is incident to one element of $\{t_i^*, t_j^*, t_k^*\}$. Let t_m^* be a weight incident to at most one element of $\{t_i^*, t_j^*, t_k^*\}$. Without loss of generality, we may assume that t_i^*, t_j^*, and t_m^* all take different values on x, and there are at most two incident pairs among them. This is impossible by Cases 1–3.

This proves the claim and therefore the proposition.

□

Proposition 7. *If G is of type E_7 and* $\dim V = 56$, *then* $G \cap \mathfrak{g} = \emptyset$.

Proof. There is a unique irreducible 56-dimensional representation of E_7, and it is therefore self-dual. By Lemma 4, we may assume that $t_j^*(x) = \chi_j(g) \in \{\pm i\}$ for $1 \le j \le 56$. This implies that the set of t_j^* is a union of two hyperplanes, which must therefore bound the convex hull of $\{t_j^* \mid 1 \le j \le 56\}$, which is the convex polytope 3_{21} in Coxeter's notation. However, the codimension 1 faces of 3_{21} each have 7 or 12 vertices [Gre, Exercise 8.5.17], so this is impossible. □

This concludes the proof of the only if direction of Theorem 1.

Lemma 8. *Let $m \ge 2$ and $j \in [1, m-1]$ be integers, and let $V = \wedge^j \mathbb{C}^m$. Let G denote the image of* $\mathrm{SL}_m(\mathbb{C})$ *in* $\mathrm{GL}_n(V)$ *via the exterior jth power map. Then* $G \cap \mathfrak{g} \neq \emptyset$.

Proof. Choose $a \in \mathbb{C}$ such that $a^m = \frac{j}{j-m}$, and let $b = a^{1-m}$. Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{C}^m$. We fix an ordered basis of V consisting of the vectors of the form $e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $i_1 > \cdots > i_j$, taken in lexicographic order. Thus, the image of $\mathrm{diag}(a, \ldots, a, a^{1-m}) \in \mathrm{SL}_m(\mathbb{C})$ in $\mathrm{GL}(V)$ is

$$\mathrm{diag}(\underbrace{a^j, \ldots, a^j}_{\binom{m-1}{j}}, \underbrace{a^{j-m}, \ldots, a^{j-m}}_{\binom{m-1}{j-1}}) = \frac{a^j}{j}\mathrm{diag}(\underbrace{j, \ldots, j}_{\binom{m-1}{j}}, \underbrace{j-m, \ldots, j-m}_{\binom{m-1}{j-1}}) \in \mathfrak{g}.$$

□

Lemma 9. *Let V be a representation of G, T a maximal torus of G, and $\mathfrak{t}$ the Lie algebra of T. If $x \in \mathfrak{t}$ satisfies $t_j^*(x) \in \{\pm i\}$ for all j, then $\rho_*(x) \in G \cap \mathfrak{g}$.*

Proof. Choose a basis of V whose jth vector belongs to the (1-dimensional) χ_j-weight space of V. In terms of this basis, $\rho_*(x)$ is given by the matrix $\mathrm{diag}(t_1^*(x), \ldots, t_n^*(x))$. Inside the group D of invertible diagonal matrices, the image $\rho(T)$ is a closed subgroup [Bor, Corollary 1.4(a)] and it is diagonal, so it is the intersection of the kernels of characters of D which vanish on it [Bor, Proposition 8.2(c)]. An element of $X^*(D)$ vanishes on $\rho(T)$ if and only if it lies in the kernel of the natural homomorphism $X^*(D) \to X^*(T)$. Thus, if A is the set of n-tuples $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\chi_1^{a_1} \cdots \chi_n^{a_n} = 1$, then

$$\rho(T) = \{\mathrm{diag}(c_1, \ldots, c_n) \in D \mid c_1^{a_1} \cdots c_n^{a_n} = 1\, \forall (a_1, \ldots, a_n) \in A\}.$$

For all $(a_1, \ldots, a_n) \in A$, we have $\sum_j a_j t_j^* = 0$, so $\sum_j a_j t_j^*(x) = 0$. Writing $t_j^*(x) = \epsilon_j i$, we have $\sum_j a_j \epsilon_j = 0$, so

$$\prod_j t_j^*(x)^{a_j} = \prod_j (\epsilon_j i)^{a_j} = \prod_j (i^{\epsilon_j})^{a_j} = i^{\sum_j a_j \epsilon_j} = 1.$$

This implies $\rho_*(x) \in \rho(T)$.

□

For minuscule representations when G is of type B, C, or D, such an x always exists; it suffices to find an element of $X^*(T) \otimes \mathbb{R}$ whose inner product with each weight of V is ± 1. Using the notation of [Bou1, Planches], these can be chosen as follows. For type B_r and highest weight ϖ_r, we take $(1, 0, \ldots, 0)$. For type C_r and highest weight ϖ_1, we take $(1/2, 0, \ldots, 0)$. For type D_r and highest weight ϖ_1, we take $(1/2, 0, \ldots, 0)$. For type D_r and highest weight ϖ_{r-1} or ϖ_r, we take $(1, 0, \ldots, 0)$.

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Email address: `mjlarsen@indiana.edu`

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.